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Paley–Wiener theorem for line bundles over compact symmetric spaces and new estimates for the Heckman–Opdam Hypergeometric Functions

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Paley–Wiener type theorems describe the image of a given space of functions, often compactly supported functions, under an integral transform, usually a Fourier transform on a group or homogeneous space. In this article we proved a Paley–Wiener theorem for smooth sections f of homogeneous line bundles on a compact Riemannian symmetric space U/K . It characterizes f with small support in terms of holomorphic extendability and exponential growth of their χ -spherical Fourier transforms, where χ is a character of K .

An important tool in our proof is a generalization of Opdam’s estimate for the hypergeometric functions associated to multiplicity functions that are not necessarily positive. At the same time the radius of the domain where this estimate is valid is increased. This is done in an appendix.

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1 Introduction

1.1 Paley–Wiener Theorem

The classical Paley–Wiener theorem identifies the space of smooth compactly supported functions on \mathbb{R}^n with certain classes of holomorphic functions on \mathbb{C}^n of exponential growth via the Fourier transform on \mathbb{R}^n . The exponent is determined by the size of the support, see [22, Thm. 7.3.1].

There are several generalizations of this theorem to settings where \mathbb{R}^n is replaced by a Lie group or a homogeneous space. Of all of those generalizations, the Riemannian symmetric spaces are best understood, in particular, those of the noncompact type due to the work of Helgason [16, 17] and Gangolli [10] for smooth functions, and Eguchi, Hashizume and Okamoto [9] and Dadok [7] for distributions. The case of semisimple Lie groups was due to Arthur [2], the case of general reductive symmetric spaces was done by van den Ban and Schlichtkrull [37, 38], and the case of hyperbolic spaces was treated by Andersen [1]. More recently the Paley–Wiener type theorems have been extended to the case of the Heckman–Opdam hypergeometric Fourier transform and the compact settings, [3, 4, 11, 26–28, 30–32] and even infinite dimensional Lie groups [33]. We refer to [8] and references therein for overview and further discussions. In the compact case every smooth function has a compact support, so the Paley–Wiener theorem is a local statement and only valid for functions supported in sufficiently small balls around the base point.

The first part of this article is inspired by [30] which deals with functions on compact symmetric spaces. We aim to generalize the results of [30] to line bundles over symmetric spaces of compact type. Similar results of noncompact type was treated in [36].

Consider a Riemannian symmetric space $\mathbf{Y} = U/K$ of compact type. Let $\mathbf{X} = G/K_o$ be the noncompact dual symmetric space, K_o the connected component of K containing the identity element (note that K is connected if U is assumed to be simply connected). Let χ be a character of K . A homogeneous line bundle over \mathbf{Y} is then given by the fiber-product $U \times_{\chi} \mathbb{C} =: \mathcal{L}_{\chi}$. The space of smooth sections on \mathcal{L}_{χ} is isomorphic to the space

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$C^\infty(\mathbf{Y}; \mathcal{L}_\chi)$ of all smooth functions $f : U \rightarrow \mathbb{C}$ such that

$$f(uk) = \chi(k)^{-1} f(u) \quad \text{for all } k \in K \text{ and all } u \in U. \quad (1.1)$$

Similarly, one defines the space of L^2 -sections, $L^2(\mathbf{Y}; \mathcal{L}_\chi)$. There is a natural unitary representation of U on $L^2(\mathbf{Y}; \mathcal{L}_\chi)$, the *regular representation*, given by left translation

$$\lambda(g)f(u) = f(g^{-1}u).$$

An irreducible representation (π_μ, V_μ) of U is said to be χ -spherical if there exists a nonzero vector $e_\mu \in V_\mu$ such that

$$\pi_\mu(k)e_\mu = \chi(k)e_\mu \quad \text{for all } k \in K. \quad (1.2)$$

As U is compact the regular representation decomposes into a direct sum of irreducible representations. The representations that are contained in this sum are the χ -spherical representations and each is contained with multiplicity one, see [35]. The classification of the χ -spherical representations in terms of their highest weights can also be found in [35], see also [15, Cor. 5.2.9]. In this article we give a slightly simpler description of this set, see Proposition 8.

The replacement for the K -biinvariant functions studied in [30] are the χ bicovariant functions

$$f(k_1 u k_2) = \chi(k_1 k_2)^{-1} f(u) \quad \text{for all } k_1, k_2 \in K \text{ and all } u \in U. \quad (1.3)$$

The space of those functions is denoted by $C^\infty(U//K; \mathcal{L}_\chi)$. The replacement for the spherical functions on \mathbf{Y} are the χ -spherical functions ψ_μ given by

$$\psi_\mu(u) = \langle e_\mu, \pi_\mu(u)e_\mu \rangle \quad (1.4)$$

and the χ -spherical Fourier transform is given by

$$\widehat{f}(\mu) = \langle f, \psi_\mu \rangle = \int_U f(u) \psi_\mu(u^{-1}) du. \quad (1.5)$$

We investigate χ -bicovariant sections f supported in a closed metric ball $\overline{B}_r(x_o)$ of sufficiently small radius $r > 0$ around a base point x_o . We show the χ -spherical Fourier transform of f extends to a holomorphic function of exponential type r (it is exactly the size of the support of f). The image of χ -spherical Fourier transform is certain space of holomorphic functions, see the Paley–Wiener theorem (Theorem 18).

The proof of the Paley–Wiener theorem uses new estimates for the Heckman–Opdam hypergeometric functions. This is the main content of the second part of this article.

1.2 The Hypergeometric Functions

The Heckman–Opdam hypergeometric functions were introduced by a series of joint work of Heckman and Opdam [13, 14, 24, 25]. They are joint eigenfunctions of a commuting algebra of differential operators associated to a root system and a multiplicity parameter (which is a Weyl group invariant function on the root system). The multiplicities can be arbitrary complex numbers. The hypergeometric functions are holomorphic, Weyl group invariant, and normalized by the value one at the identity. When the root multiplicities do correspond to those of a Riemannian symmetric space, the hypergeometric functions are simply the restrictions of spherical functions to a Cartan subspace. We refer to [15, Part I, Chapter 4] and [36, Section 3.2] for an introduction to the Harish-Chandra asymptotic expansion for the χ -spherical functions on G .

The χ -spherical functions on G were identified as hypergeometric functions in [15], but whose multiplicity parameters are not necessarily positive. So far uniform exponential estimates on the growth behavior of hypergeometric functions have only been proven with the condition that all multiplicities are positive [26], and hence not applicable in our situation. We therefore extend such results. Our estimate

1. works for the hypergeometric functions whose multiplicity parameters are allowed to be certain negative numbers

2. the domain of the hypergeometric functions where this new estimate is valid is increased (double the size of the original domain in [26]).

This is of crucial importance to obtain the right exponential type growth of f under the χ -spherical Fourier transform (see the proof of part (1) of Theorem 18). The new estimate is proved in Proposition 30, Appendix A. The technique to prove it is inspired by ideas from [26].

An alternative method to attack this problem is to apply some suitable shift operators in order to move negative multiplicities to positive ones. This allows us to use many well-known results of the hypergeometric functions associated with positive multiplicities. This method was investigated in [21].

1.3 Plan of the Article

In Section 2 we introduce basic notations and structure theory on Riemannian symmetric spaces. In Section 3 and Section 4 we discuss harmonic analysis related to line bundles over compact symmetric spaces, including the theory of highest weights for χ -spherical representations, elementary spherical functions of type χ , and χ -spherical Fourier transform. In Section 5 we define the relevant Paley–Wiener space and state the main theorem (Theorem 18), to prove which we need some tools of differential operators (Sections 4.2) and hypergeometric functions (Appendix A). Sections 5 and 6 contains the main body of the proof. In Section 5 we show the χ -spherical Fourier transform maps into the Paley–Wiener space. In Section 6.2 we prove the bijection of the χ -spherical Fourier transform. The proof of injectivity is obvious, while the proof of surjectivity is by reduction to the Paley–Wiener theorem for the group case, recalled in Section 6.1.

2 Notation and Preliminaries

The material in this section is standard. We refer to [18] for references. We will often need [19, 20] too. We use the notation from the introduction mostly without reference.

2.1 Symmetric spaces

We recall some standard notations and facts related to symmetric spaces.

A Riemannian symmetric space of the compact type can be realized as $\mathbf{Y} = U/K$ where U is a connected semisimple compact Lie group and $K \subseteq U$ is a closed symmetric subgroup. Thus, there exists a nontrivial involution $\theta : U \rightarrow U$ such that

$$U^\theta \subseteq K \subseteq U^\theta = \{u \in U \mid \theta(u) = u\}.$$

We fix the base point $x_o = eK$ and write $a \cdot (bK) = (ab)K$ for the action of U on \mathbf{Y} . Assume U is simply connected. Then \mathbf{Y} is simply connected. It is not necessary to assume that \mathbf{Y} is simply connected, but it makes several arguments simpler. In particular, the classification of the χ -spherical representations is simpler. Note that the spherical harmonic analysis on a general compact symmetric space can be reduced to the simply connected case (see [29, p.4860] and [30]).

Let \mathfrak{u} be the Lie algebra of U . Then θ induces an involution on \mathfrak{u} , also denoted by θ . Decompose $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{q}$ into ± 1 -eigenspaces of θ , where $\mathfrak{k} = \mathfrak{u}^\theta$ is the Lie algebra of K . We identify \mathfrak{q} with the tangent space $T_{x_o}(\mathbf{Y})$ of \mathbf{Y} at x_o by

$$D_X f(x_o) = \left. \frac{d}{dt} \right|_{t=0} f(\text{Exp}(tX))$$

where $\text{Exp}(Y) = \exp(Y) \cdot x_o$. Then $T\mathbf{Y} \cong \mathfrak{q} \times_K U$. Any positive K -invariant bilinear form on \mathfrak{q} defines a Riemannian metric on \mathbf{Y} . As an example, we let $\langle \cdot, \cdot \rangle$ be the inner product on \mathfrak{u} defined by

$$\langle X, Y \rangle = -\text{Tr}(\text{ad}(X)\text{ad}(Y)).$$

This inner product is K -invariant and defines a Riemannian metric on \mathbf{Y} . The inner product on \mathfrak{u} gives an inner product on the dual space \mathfrak{u}^* in a canonical way, and by hermitian extension they induce U -invariant inner

products on $u_{\mathbb{C}} = u \otimes_{\mathbb{R}} \mathbb{C}$ and the complex dual space $u_{\mathbb{C}}^*$, all denoted by the same symbol. We write $\|\lambda\| = \sqrt{\langle \lambda, \lambda \rangle}$ for the corresponding norm. Similar notations will be used for other Lie algebras and vector spaces. The \mathbb{C} -bilinear extension to $u_{\mathbb{C}}^*$ will be denoted by $\lambda, \mu \mapsto \langle \lambda, \mu \rangle$.

A maximal abelian subspace of \mathfrak{q} is called a Cartan subspace for \mathbf{Y} (or (U, K)). All Cartan subspaces are K -conjugate and their common dimension is called the rank of \mathbf{Y} . From now on we fix a Cartan subspace \mathfrak{b} . Let $n = \dim \mathfrak{b}$. We fix a Cartan subalgebra \mathfrak{h} of \mathfrak{u} containing \mathfrak{b} . Then \mathfrak{h} is θ -stable and

$$\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus \mathfrak{b}. \quad (2.1)$$

Let $B = \exp(\mathfrak{b})$ be the analytic subgroup of U with Lie algebra \mathfrak{b} . The subspace $B \cdot x_o \simeq B/B \cap K$ is a Cartan subgroup of \mathbf{Y} . Note that $B \cap K$ is finite.

Since U is compact, it admits a finite dimensional faithful unitary representation. Thus $U \subset U(p) \subset GL(p, \mathbb{C})$ for some p . As $u \subset u(p)$ it follows that $u \cap iu = \{0\}$ and hence $u_{\mathbb{C}} \simeq u \oplus iu$. Let $U_{\mathbb{C}}$ denote the analytic subgroup of $GL(p, \mathbb{C})$ with Lie algebra $u_{\mathbb{C}}$. Note that $U_{\mathbb{C}}$ is simply connected. Let $\mathfrak{s} = i\mathfrak{q}$ and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. Note that \mathfrak{g} is a Lie algebra. Denote by G the analytic subgroup of $GL(p, \mathbb{C})$ (and $U_{\mathbb{C}}$) with Lie algebra \mathfrak{g} . Then $G = K \exp \mathfrak{s}$ and $\mathbf{X} = G/K$ is a Riemannian symmetric space of the noncompact type. It is called the Riemannian dual of \mathbf{Y} . The Riemannian structure on \mathbf{X} is again determined by the inner product $\langle X, Y \rangle = \text{Tr ad}(X)\text{ad}(Y)$ on \mathfrak{s} .

We will from now on view \mathbf{X} and \mathbf{Y} as real forms of the complex homogeneous space $U_{\mathbb{C}}/K_{\mathbb{C}}$ where $K_{\mathbb{C}} = \exp \mathfrak{k}_{\mathbb{C}} \subset U_{\mathbb{C}}$. Again $x_o = eK_{\mathbb{C}}$ is the common base point.

The involution θ extends to a holomorphic involution on $U_{\mathbb{C}}$, also denoted by θ . We also write θ for the restriction to G and note that θ is a Cartan involution on G .

Again, a maximal abelian subspace of \mathfrak{s} is called a Cartan subspace of \mathbf{X} . The Cartan subspaces of \mathbf{X} are conjugate under K . We fix from now on the Cartan subspace $\mathfrak{a} = i\mathfrak{b}$ where \mathfrak{b} is the fixed Cartan subspace for \mathbf{Y} . Then $A \simeq \mathfrak{a}$ is simply connected. We have $\mathfrak{a}_{\mathbb{C}} = \mathfrak{b}_{\mathbb{C}} = \mathfrak{a} \oplus \mathfrak{b}$ and $A_{\mathbb{C}} = \exp(\mathfrak{a}_{\mathbb{C}}) = AB$. We denote by $\log : A_{\mathbb{C}} \rightarrow \mathfrak{a}_{\mathbb{C}}$ the multivariable inverse of $\exp|_{\mathfrak{a}_{\mathbb{C}}}$. It is a single valued isomorphism $\log : A \rightarrow \mathfrak{a}$.

2.2 Line bundles on hermitian symmetric spaces

We will from now on assume that U is simple and that there exists nontrivial line bundles over \mathbf{Y} although some of our statements are true in general. Let $\chi : K \rightarrow \mathbb{T}$ be a character. Then the homogeneous line bundle $\mathcal{L}_{\chi} \rightarrow \mathbf{Y}$ is defined as

$$\mathcal{L}_{\chi} = U \times_{\chi} \mathbb{C} = (U \times \mathbb{C})/K$$

where \mathbb{C} denotes the complex numbers with the action $k \cdot z = \chi(k)^{-1} z$ and K acts on $U \times \mathbb{C}_{\chi}$ by $(u, z) \cdot k = (uk, k \cdot z)$. The existence of a homogeneous line bundle over \mathbf{Y} is equivalent to the existence of a nontrivial character which in turn is equivalent to $\dim Z(K) = 1$, where $Z(K)$ is the center of K . Let \mathfrak{z} denote the center of \mathfrak{k} and $\mathfrak{k}_1 = [\mathfrak{k}, \mathfrak{k}]$. Then $\dim \mathfrak{z} = 1$ and $\mathfrak{k} = \mathfrak{z} \oplus \mathfrak{k}_1$. Let K_1 denote the analytic subgroup of K (and U) with Lie algebra \mathfrak{k}_1 . Then K_1 is closed and $K = Z(K)_o K_1$. The spaces \mathbf{X} and \mathbf{Y} are hermitian symmetric spaces of the noncompact and compact type respectively. They are complex homogeneous spaces where the complex structure is given by the adjoint action of a central element in \mathfrak{k} with eigenvalues 0 and $\pm i$.

Up to coverings the irreducible spaces with $\dim \mathfrak{z} = 1$ are given in the following table (cf. [18, p.516, 518]). Here $n = \dim \mathfrak{b} = \dim \mathfrak{a}$ is the rank of \mathbf{X} and \mathbf{Y} and $d = \dim_{\mathbb{R}} \mathbf{Y} = \dim_{\mathbb{R}} \mathbf{X}$. The conditions listed in the last column are given to prevent coincidence between different classes due to lower dimensional isomorphisms.

The hermitian symmetric spaces							
	class	G	U	K	n	d	
1	<i>AIII</i>	$SU(p, q)$	$SU(p+q)$	$S(U(p) \times U(q))$	p	$2pq$	$q \geq p \geq 1$
2	<i>BDI</i>	$SO_o(p, q)$	$SO(p+q)$	$SO(p) \times SO(q)$	p	pq	$p = 2, q \geq 5$
3	<i>DIII</i>	$SO^*(2j)$	$SO(2j)$	$U(j)$	$\lfloor \frac{1}{2} j \rfloor$	$j(j-1)$	$j \geq 5$
4	<i>CI</i>	$Sp(j, \mathbb{R})$	$Sp(j)$	$U(j)$	j	$j(j+1)$	$j \geq 2$
5	<i>EIII</i>	$\mathfrak{e}_6(-14)$	$\mathfrak{e}_6(-78)$	$\mathfrak{so}(10) + \mathbb{R}$	2	32	
6	<i>EVII</i>	$\mathfrak{e}_7(-25)$	$\mathfrak{e}_7(-133)$	$\mathfrak{e}_6 + \mathbb{R}$	3	54	

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Remark 1. Here are some comments on the classification (2.2). (1) In Case 1, when $p = 1$, the space $\mathfrak{g} = \mathfrak{su}(1, q)$ is labeled as the class *AIV* in the Satake diagram in [18, p.531, 532].

(2) In Case 1,

$$\mathbf{Y} = \mathrm{SU}(p+q)/\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$$

is the complex Grassmann manifold of p -dimensional subspaces in \mathbb{C}^{p+q} . In Case 2,

$$\mathbf{Y} = \mathrm{SO}(p+q)/(\mathrm{SO}(p) \times \mathrm{SO}(q))$$

is a covering of $\mathrm{SO}(p+q)/\mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(q))$, the real Grassmann manifold of p -dimensional subspaces in \mathbb{R}^{p+q} .

We recall the parametrization of the group of characters given in [35]. Fix $Z \in \mathfrak{z}$ as in [35, p.283, (3.1)]. Thus $\exp(tZ) \in Z(K)$ for all $t \in \mathbb{R}$, and $\exp(tZ) \in K_1$ if and only if $t \in 2\pi\mathbb{Z}$.

Proposition 2 (H. Schlichtkrull). *Let $l \in \mathbb{Z}$. Define $\chi_l : K \rightarrow \mathbb{T}$ by*

$$\chi_l(\exp(tZ)k) = e^{ilt}, \quad t \in \mathbb{R} \text{ and } k \in K_1.$$

Then χ_l is a well defined character on K . If χ is a character on K , then there is a unique $l \in \mathbb{Z}$ such that $\chi = \chi_l$.

Proof. See Proposition 3.4 in [35] and its following comment. \square

Since all one dimensional representations χ of K have this form, hereafter, we parametrize $\chi = \chi_l$ for $l \in \mathbb{Z}$. If $l = 0$, then χ_0 is trivial.

2.3 Root structures and the Weyl group

For $\alpha \in \mathfrak{h}_{\mathbb{C}}^*$ let

$$\mathfrak{u}_{\mathbb{C}, \alpha} = \{X \in \mathfrak{u}_{\mathbb{C}} \mid (\forall H \in \mathfrak{h}_{\mathbb{C}}) [H, X] = \alpha(H)X\}.$$

If $\alpha \neq 0$ and $\mathfrak{u}_{\mathbb{C}, \alpha} \neq \{0\}$ then α is said to be a root. We write $\Delta = \Delta(\mathfrak{u}, \mathfrak{h})$ for the set of roots. Similarly we define $\mathfrak{u}_{\mathbb{C}, \beta}$ for $\beta \in \mathfrak{b}_{\mathbb{C}}^*$ and write $\Sigma = \Sigma(\mathfrak{u}, \mathfrak{b})$ for the set of (restricted) roots. Note that $\Delta \subset i\mathfrak{h}^*$, $\Sigma \subset i\mathfrak{b}^*$, and $\Sigma = \Delta|_{\mathfrak{b}} \setminus \{0\}$. The numbers

$$m_{\beta} = \dim_{\mathbb{C}} \mathfrak{u}_{\mathbb{C}, \beta} = \#\{\alpha \in \Delta \mid \alpha|_{\mathfrak{b}} = \beta\}, \quad \beta \in \Sigma$$

are called multiplicities. Also note that $\mathfrak{u}_{\mathbb{C}, \alpha} \cap \mathfrak{u} = \{0\}$ for all $\alpha \in \Delta \cup \Sigma$.

Similarly, we can define the roots of \mathfrak{a} in \mathfrak{g} and we have $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$. We have $\mathfrak{u}_{\mathbb{C}, \beta} = \mathfrak{g}_{\beta} \oplus i\mathfrak{g}_{\beta}$ and $m_{\beta} = \dim_{\mathbb{R}} \mathfrak{g}_{\beta}$ for all $\beta \in \Sigma$. Working with roots it is therefore more convenient to work with \mathfrak{g} and \mathfrak{a} rather than the pair \mathfrak{u} and \mathfrak{b} .

An element $X \in \mathfrak{a}$ is called regular if $\alpha(X) \neq 0$ for all $\alpha \in \Sigma$. The subset $\mathfrak{a}^{\mathrm{reg}} \subset \mathfrak{a}$ is dense and is a finite union of open cones called Weyl chambers. We fix a Weyl chamber \mathfrak{a}^+ and let

$$\Sigma^+ := \{\alpha \in \Sigma \mid (\forall H \in \mathfrak{a}^+) \alpha(H) > 0\}.$$

We choose a positive system Δ^+ in Δ such that if $\alpha \in \Delta^+$ and $\alpha|_{\mathfrak{a}} \neq 0$ then $\alpha|_{\mathfrak{a}} \in \Sigma^+$. Let $\Delta_0 = \{\alpha \in \Delta \mid \alpha|_{\mathfrak{a}} = 0\}$ and $\Delta_0^+ = \Delta_0 \cap \Delta^+$. Let

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_{\alpha} \alpha \quad \text{and} \quad \rho_{\mathfrak{h}} = \frac{1}{2} \sum_{\beta \in \Delta^+} \beta. \quad (2.3)$$

Let $\rho_0 = \frac{1}{2} \sum_{\alpha \in \Delta_0^+} \alpha$. Then $\rho_0 \in i(\mathfrak{h} \cap \mathfrak{k})^*$ and

$$\rho_{\mathfrak{h}}|_{\mathfrak{a}} = \rho \quad \text{and} \quad \rho_{\mathfrak{h}} = \rho + \rho_0. \quad (2.4)$$

If $\alpha \in \Sigma$ then it can happen that either $\alpha/2 \in \Sigma$ or $2\alpha \in \Sigma$, but not both. A root $\alpha \in \Sigma$ is said to be unmultiplicable if $2\alpha \notin \Sigma$ and indivisible if $\alpha/2 \notin \Sigma$. Denote by

$$\Sigma_* = \{\alpha \in \Sigma \mid 2\alpha \notin \Sigma\} \quad \text{and} \quad \Sigma_i = \{\alpha \in \Sigma \mid \frac{1}{2}\alpha \notin \Sigma\}. \quad (2.5)$$

Both Σ_* and Σ_i are reduced root systems. Set $\Sigma_*^+ = \Sigma_* \cap \Sigma^+$. Note that \mathbf{Y} is irreducible if and only if Σ_i is irreducible. Let $\Pi = \{\beta_j\}_{j=1}^n$ be the fundamental system of simple roots in Σ_*^+ . For any $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $\alpha \in \mathfrak{a}^*$ with $\alpha \neq 0$, define

$$\lambda_\alpha := \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}.$$

We will use similar notation for $i\mathfrak{h}^*$ without comments. Note that $2\lambda_\alpha = \lambda_{\alpha/2}$. Define $\omega_j \in \mathfrak{a}^*$, $j = 1, \dots, n$, by

$$(\omega_j)_{\beta_i} = \delta_{i,j}, \quad 1 \leq i, j \leq n. \quad (2.6)$$

The weights ω_j are the class 1 fundamental weights for $(\mathfrak{u}, \mathfrak{k})$ and $(\mathfrak{g}, \mathfrak{k})$. We let

$$\Lambda_0^+ = \{\lambda \in \mathfrak{a}^* \mid (\forall \alpha \in \Sigma^+) \lambda_\alpha \in \mathbb{Z}^+\} = \sum_{j=1}^n \mathbb{Z}^+ \omega_j. \quad (2.7)$$

For $\alpha \in \Sigma$ define the reflection $r_\alpha : \mathfrak{a}^* \rightarrow \mathfrak{a}^*$ by

$$r_\alpha(\lambda) := \lambda - 2\lambda_\alpha \alpha, \quad \forall \lambda \in \mathfrak{a}^*.$$

The group $W = W(\Sigma)$ generated by r_α , $\alpha \in \Sigma$, is finite and called the Weyl group associated to Σ . Note that $W = W(\Sigma_*) = W(\Sigma_i)$ with the obvious notation. The W -action extends to \mathfrak{a} by duality, and then to $\mathfrak{a}_{\mathbb{C}}$ and $\mathfrak{a}_{\mathbb{C}}^*$ by \mathbb{C} -linearity, and to A and $A_{\mathbb{C}}$ by $w \cdot \exp(H) = \exp(w(H))$. This action can be written as $w \cdot b = kbk^{-1}$ where $k \in N_K(\mathfrak{a})$ is such that $\text{Ad}(k)|_{\mathfrak{a}} = w$. Here $N_K(\mathfrak{a}) = \{k \in K \mid \text{Ad}(k)\mathfrak{a} = \mathfrak{a}\}$ is the normalizer of \mathfrak{a} in K . The group W then acts on functions f on any of these spaces by $(w \cdot f)(x) := f(w^{-1} \cdot x)$, $w \in W$. We recall that $W \cdot \mathfrak{a}^+ = \mathfrak{a}^{\text{reg}}$.

We now describe the root structures for the special case of irreducible hermitian symmetric spaces in more details. Denote by $\{\varepsilon_1, \dots, \varepsilon_n\}$ the standard orthonormal base for $\mathfrak{a}^* = i\mathfrak{b}^*$. We then have the following description of the root system Σ , see Moore [23, Thm. 5.2], [18, p.528, 532]:

Theorem 3. *There are two possibilities for the root system Σ^+ :*

$$\text{Case I: } \Sigma^+ = \{\varepsilon_j \pm \varepsilon_i \mid (1 \leq i < j \leq n), 2\varepsilon_j \mid (1 \leq j \leq n)\}$$

$$\text{Case II: } \Sigma^+ = \{\varepsilon_j \mid (1 \leq j \leq n), \varepsilon_j \pm \varepsilon_i \mid (1 \leq i < j \leq n), 2\varepsilon_j \mid (1 \leq j \leq n)\}.$$

Remark 4. In Case II, Σ is of type BC_n . Since \mathbf{Y} is irreducible, $\Sigma = \mathcal{O}_S \cup \mathcal{O}_M \cup \mathcal{O}_L$ is a disjoint union of three W -orbits in Σ corresponding to short, medium, and long roots, respectively. Let $\mathcal{O}_S^+ = \mathcal{O}_S \cap \Sigma^+$, $\mathcal{O}_M^+ = \mathcal{O}_M \cap \Sigma^+$, and $\mathcal{O}_L^+ = \mathcal{O}_L \cap \Sigma^+$. Thus,

$$\mathcal{O}_S^+ = \{\varepsilon_j \mid (1 \leq j \leq n)\}, \quad \mathcal{O}_M^+ = \{\varepsilon_j \pm \varepsilon_i \mid (1 \leq i < j \leq n)\}, \quad \mathcal{O}_L^+ = \{2\varepsilon_j \mid (1 \leq j \leq n)\}.$$

Adopt the notation

$$m = (m_\alpha) = (m_S, m_M, m_L)$$

for root multiplicities of short, medium, and long roots, respectively, where

$$m_S = m_{\varepsilon_i}, \quad m_M = m_{\varepsilon_j \pm \varepsilon_i} \quad (i \neq j), \quad m_L = m_{2\varepsilon_i}.$$

In Case I, Σ is actually of type C_n . We consider it as being of type BC_n with $m_S = 0$. In this way, the root system Σ is of type BC_n in both cases.

The individual cases from Table (2.2) are listed in the following table:

Hermitian symmetric spaces U/K : root structures and multiplicities				
	U	K	Σ	(m_S, m_M, m_L)
1	$SU(p+q)$	$S(U_p \times U_q)$	Case I $p = q$	$(0, 2, 1)$
			Case II $q > p \geq 2$	$(2(q-p), 2, 1)$
			Case II $q > p = 1$	$(2(q-1), 0, 1)$
2	$SO(2+q)$	$SO(2) \times SO(q)$	Case I	$(0, q-2, 1)$
3	$SO(2j)$	$U(j)$	Case I j is even	$(0, 4, 1)$
			Case II j is odd	$(4, 4, 1)$
4	$Sp(j)$	$U(j)$	Case I	$(0, 1, 1)$
5	$\mathfrak{e}_6(-78)$	$\mathfrak{so}(10) + \mathbb{R}$	Case II	$(8, 6, 1)$
6	$\mathfrak{e}_7(-133)$	$\mathfrak{e}_6 + \mathbb{R}$	Case I	$(0, 8, 1)$

2.4 Basic structure theory

Let $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$. Then \mathfrak{n} is a nilpotent Lie algebra and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Let $N = \exp \mathfrak{n}$ be the analytic subgroup of G with Lie algebra \mathfrak{n} . Then N is nilpotent, simply connected and closed. The group G is analytically diffeomorphic to $K \times A \times N$ via the multiplication map $K \times A \times N \ni (k, a, n) \mapsto kan \in G$. The inverse is denoted by $x \mapsto (\kappa(x), a(x), n(x))$. We write $H(x) = \log(a(x))$. This is the Iwasawa decomposition of G . Furthermore $K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}$ is open and dense in $G_{\mathbb{C}}$.

For the compact group U we have the Cartan decomposition $U = KBK$. We have then the corresponding integral formulas for the Lie group U .

Lemma 5. *Let $\delta(\exp H) = \prod_{\alpha \in \Sigma^+} |\sin(\frac{1}{i}\alpha(H))|^{m_\alpha}$ for $H \in \mathfrak{b}$. Then there exists a constant $c > 0$ such that for all $f \in L^1(\mathbf{Y})$*

$$\int_{\mathbf{Y}} f(y) dy = c \int_K \int_B f(ka \cdot x_o) \delta(a) da dk.$$

3 Spherical Fourier Analysis on \mathbf{Y}

In this section we recall the classification of χ_l -spherical representations of U (and G) due to H. Schlichtkrull [35]. We then recall the Plancherel formula for $L^2(\mathbf{Y}; \mathcal{L}_l)$ where $\mathcal{L}_l = \mathcal{L}_{\chi_l}$. Next, we discuss the χ_l -spherical functions and the decomposition of $L^2(\mathbf{Y}; \mathcal{L}_l)$.

3.1 The χ_l -spherical representations

Let $\Lambda^+(U) \subset i\mathfrak{h}^*$ be the semi-lattice of highest weights of irreducible representations of U . As we are assuming that U is simply connected we have

$$\Lambda^+(U) = \{ \lambda \in \mathfrak{h}_{\mathbb{C}}^* \mid (\forall \alpha \in \Delta^+) 2\lambda_\alpha \in \mathbb{Z}^+ \}.$$

For $\lambda \in \Lambda^+(U)$ choose an irreducible unitary representation (π_λ, V_λ) of U . For $l \in \mathbb{Z}$ let

$$V_\lambda^{\chi_l} = V_\lambda^l = \{ v \in V_\lambda \mid (\forall k \in K) \pi_\lambda(k)v = \chi_l(k)v \}. \tag{3.1}$$

Note that $V_\lambda^0 = V_\lambda^K = \{ v \in V_\lambda \mid (\forall k \in K) \pi_\lambda(k)v = v \}$ is the space of K -fixed vectors. The representation (π_λ, V_λ) is said to be χ_l -spherical if $V_\lambda^l \neq \{0\}$ and spherical if it is χ_0 -spherical. If (π_λ, V_λ) is χ_l -spherical then $\dim V_\lambda^l = 1$. Denote by $\Lambda_l^+(U)$ the set of highest weights of χ_l -spherical representations of U . Let

$$\Lambda_l^+ := \{ \mu \in \mathfrak{a}^* \mid \mu = \lambda|_{\mathfrak{a}}, \lambda \in \Lambda_l^+(U) \}. \tag{3.2}$$

According to [19, p.535, 538] we have that if $\lambda \in \Lambda_0^+(U)$ then $\lambda|_{\mathfrak{h} \cap \mathfrak{k}} = 0$ and the set Λ_0^+ is exactly the set introduced in (2.7).

According to [35], we can decompose $\mathfrak{h} \cap \mathfrak{k}$ as

$$\mathfrak{h} \cap \mathfrak{k} = (\mathfrak{h} \cap \mathfrak{k}_1) \oplus \mathbb{R}X$$

where X is defined as in [35, p.285, (4.4)] so that

1. $e^{tX} \in K_1$ if and only if $t \in 2\pi i\mathbb{Z}$,
2. $Z - X \in \mathfrak{k}_1$ where Z is the same as in Proposition 2 (see Lemma 4.3 in [35]).

Note that $X = 0$ in Case I. For $\lambda \in i\mathfrak{h}^*$ we write accordingly $\lambda = (\mu, \lambda_1, \mu_0)$ where $\mu = \lambda|_{\mathfrak{a}}$, $\lambda_1 = \lambda|_{\mathfrak{k}_1 \cap \mathfrak{h}}$ and $\mu_0 = \lambda(iX)$. If $X = 0$ then we write $\mu_0 = 0$. When χ_l is fixed for some $l \in \mathbb{Z}$, μ_0 is then fixed. If $\lambda \in \Lambda_l^+(U)$ then $\lambda|_{\mathfrak{h} \cap \mathfrak{k}_1} = 0$. Hence, λ is uniquely determined by its restriction μ . Thus there is a bijective correspondence $\Lambda_l^+(U) \cong \Lambda_l^+$ via $\lambda = (\mu, 0, \mu_0) \mapsto \mu$. For convenience, we sometimes write μ and μ_0 for the vectors $(\mu, 0, 0)$ and $(0, 0, \mu_0)$ in $\mathfrak{h}_{\mathbb{C}}^*$, respectively. Thus, $\mu + \mu_0$ stands for the vector $(\mu, 0, 0) + (0, 0, \mu_0)$ which is exactly $(\mu, 0, \mu_0)$. We identify $\mathfrak{a}_{\mathbb{C}}^*$ with \mathbb{C}^n by $\mu = (\mu_1, \dots, \mu_n)$ with $\mu_j = \mu_{\varepsilon_j}$. Recall that $\mathfrak{a}^* = i\mathfrak{b}^*$.

Theorem 6 (H. Schlichtkrull). *Let U be a compact simply connected semisimple Lie group and K the fixed point group of an involution of U . Let $l \in \mathbb{Z}$. The set Λ_l^+ of highest restricted weights of irreducible χ_l -spherical representations of U is given by*

$$\Lambda_l^+ = \left\{ \mu \in \mathfrak{a}^* \mid \begin{array}{l} \mu_j - \mu_i \in 2\mathbb{Z}^+ \ (1 \leq i < j \leq n) \\ \mu_1 \in |l| + 2\mathbb{Z}^+ \end{array} \right\}. \quad (3.3)$$

Moreover, $\mu_0 = 0$ in Case I, and $\mu_0 = -l$ in Case II.

Proof. See the proofs¹ of Proposition 7.1 and Theorem 7.2 in [35]. □

Remark 7. The description of Λ_l^+ implies that in Case I if l is even then the χ_l -spherical representation π_μ is also spherical. In fact it was shown in [35, Thm. 7.2] that if l is even π_μ must also contain the character χ_0 .

A simpler description of the set Λ_l^+ is given by the following proposition. For that let

$$\rho_s = \frac{1}{2} \sum_{\alpha \in \mathcal{O}_s^+} \alpha = \frac{1}{2} \sum_{j=1}^n \varepsilon_j. \quad (3.4)$$

Proposition 8. *For $l \in \mathbb{Z}$ we have $\Lambda_l^+ = \Lambda_0^+ + 2|l|\rho_s$.*

Proof. Let $\mu \in \Lambda_l^+$. We want to show that $(\mu - 2|l|\rho_s)_\alpha = \mu_\alpha - 2|l|(\rho_s)_\alpha \in \mathbb{Z}^+$ for all $\alpha \in \Sigma^+$. Let $r = |l|$. We have

$$\mu_{\varepsilon_j} - 2r(\rho_s)_{\varepsilon_j} = \mu_j - r \in 2\mathbb{Z}^+ \quad (3.5)$$

and

$$\mu_{\varepsilon_j \pm \varepsilon_i} - 2r(\rho_s)_{\varepsilon_j \pm \varepsilon_i} = \frac{1}{2}(\mu_j \pm \mu_i - (r \pm r)) \in \mathbb{Z}^+ \quad (3.6)$$

according to (3.3). Finally, again by (3.3), we have

$$\mu_{2\varepsilon_j} - 2r(\rho_s)_{2\varepsilon_j} = \frac{1}{2}(\mu_j - r) \in \mathbb{Z}^+. \quad (3.7)$$

Thus $\mu - 2|l|\rho_s \in \Lambda_0^+$.

On the other hand, if $\mu^0 \in \Lambda_0^+$ define $\mu = \mu^0 + 2|l|\rho_s$. Then (3.5), (3.6) and (3.7) together with (3.3) show that $\mu \in \Lambda_l^+$. □

Recall that the fundamental spherical weights ω_j are defined by (2.6). Then $\Lambda_0^+ = \mathbb{Z}^+\omega_1 \oplus \dots \oplus \mathbb{Z}^+\omega_n$. Hence

$$(\mathbb{Z}^+)^n \simeq \Lambda_l^+, \quad (k_1, \dots, k_n) \mapsto k_1\omega_1 + \dots + k_n\omega_n + 2|l|\rho_s. \quad (3.8)$$

¹ We get $\mu_0 = -l$ for Case II, which has a minus sign different from what Theorem 7.2 in [35] stated.

3.2 The Fourier transform

In this section we recall the basic facts about Fourier analysis on $L^2(\mathbf{Y}; \mathcal{L}_l)$. Let $\lambda = (\mu, 0, \mu_0) \in \Lambda_l^+(U)$. As l , and hence μ_0 will be fixed most of the time, the only variable is the first coordinate μ and sometimes (μ, l) . We therefore simply write μ instead of λ .

Let $L^2(\mathbf{Y}; \mathcal{L}_l)$ be the space of L^2 -sections of the line bundles \mathcal{L}_l , $L^2(U//K; \mathcal{L}_l)$ the space of elements in $L^2(\mathbf{Y}; \mathcal{L}_l)$ such that $f(k_1 u k_2) = \chi_l(k_1 k_2)^{-1} f(u)$ for all $k_1, k_2 \in K$ and $u \in U$. Finally $C^\infty(U//K; \mathcal{L}_l)$ is the space of smooth elements in $L^2(U//K; \mathcal{L}_l)$.

Let $d(\mu) := \dim V_\mu$. Then $\mu \mapsto d(\mu)$ is a polynomial map. Fix $e_{\mu, l} \in V_\mu^l$ of length one. We will mostly write e_μ for $e_{\mu, l}$ as l will be fixed.

We normalize the invariant measure on all compact groups so that the total measure is one. Define

$$P_{\mu, l}(u) := \int_K \chi_l(k)^{-1} \pi_\mu(k) u \, dk. \quad (3.9)$$

Then $P_{\mu, l}$ is the orthogonal projection: $V_\mu \rightarrow V_\mu^l$. In particular, $P_{\mu, l}(u) = \langle u, e_\mu \rangle e_\mu$. If $f \in L^2(\mathbf{Y}; \mathcal{L}_l)$ then $\pi_\mu(f) = \pi_\mu(f) P_{\mu, l}$, where, as usually,

$$\pi_\mu(f) = \int_U f(u) \pi_\mu(u) \, du.$$

It is therefore natural to define the vector valued Fourier transform of f to be

$$\widehat{f}(\mu, l) = \widehat{f}(\mu) = \pi_\mu(f) e_\mu. \quad (3.10)$$

Note that

$$\text{Tr}(\pi_\mu(f)) = \langle \pi_\mu(f) e_\mu, e_\mu \rangle = \int_U f(u) \langle \pi_\mu(u) e_\mu, e_\mu \rangle \, du = \langle f, \psi_{\mu, l} \rangle.$$

The function

$$\psi_{\mu, l}(u) = \langle e_\mu, \pi_\mu(u) e_\mu \rangle, \quad u \in U \quad (3.11)$$

is the (μ, l) , or χ_l , *spherical function* on U which we will discuss in more details in the next section. Furthermore, if $f \in L^2(U//K; \mathcal{L}_l)$ then $\pi_\mu(f) e_\mu$ is again a scalar multiple of e_μ and so

$$\pi_\mu(f) e_\mu = \langle \pi_\mu(f) e_\mu, e_\mu \rangle e_\mu = \langle f, \psi_{\mu, l} \rangle e_\mu.$$

The χ_l -spherical function is the unique element in $C^\infty(U//K; \mathcal{L}_l)$ of type μ such that $\psi_{\mu, l}(e) = 1$. Furthermore $\{d(\mu)^{1/2} \psi_{\mu, l}\}$ is an orthogonal basis for $L^2(U//K; \mathcal{L}_l)$. Note that

$$\overline{\psi_{\mu, l}(u)} = \langle \pi_\mu(u) e_\mu, e_\mu \rangle = \langle e_\mu, \pi_\mu(u^{-1}) e_\mu \rangle = \psi_{\mu, l}(u^{-1}).$$

Let

$$\ell_d^2(\Lambda_l^+) = \{(a(\mu))_{\mu \in \Lambda_l^+} \mid \sum_{\mu \in \Lambda_l^+} d(\mu) |a(\mu)|^2 < \infty\}. \quad (3.12)$$

Then $\ell_d^2(\Lambda_l^+)$ is an Hilbert space with inner product $(a, b) = \sum d(\mu) a(\mu) \overline{b(\mu)}$. The χ_l -spherical Fourier transform $\mathcal{S}_l : L^2(U//K, \mathcal{L}_l) \rightarrow \ell_d^2(\Lambda_l^+)$ defined by

$$\mathcal{S}_l(f)(\mu) = \int_U f(u) \psi_{\mu, l}(u^{-1}) \, du = \langle f, \psi_{\mu, l} \rangle \quad (3.13)$$

is an unitary isomorphism. We collect the main facts in the following theorem:

Theorem 9 (The Plancherel Theorem). *Assume that \mathbf{Y} is simply connected. For $\mu \in \Lambda_l^+(U)$ and $v \in V_\mu$ let $f_{\mu,v}(x) = \langle v, \pi_\mu(x) e_\mu \rangle$, $v \in V_\mu$ and $L_\mu^2(\mathbf{Y}; \mathcal{L}_l) = \{f_{\mu,v} \mid v \in V_\mu\}$. Then the following hold true:*

(1) *If $f \in L^2(\mathbf{Y}; \mathcal{L}_l)$ then*

$$\|f\|^2 = \sum_{\mu \in \Lambda_l^+} d(\mu) \|\widehat{f}\|_{V_\mu}^2$$

$$f(x) = \sum_{\mu \in \Lambda_l^+} d(\mu) \langle \widehat{f}(\mu), \pi_\mu(x) e_\mu \rangle$$

where the convergence is in the L^2 -norm topology. The convergence is uniform if f is smooth.

(2) $L^2(\mathbf{Y}; \mathcal{L}_l) \simeq \bigoplus_{\mu \in \Lambda_l^+} L_\mu^2(\mathbf{Y}; \mathcal{L}_l)$.

(3) *If $f \in L^2(U//K; \mathcal{L}_l)$ then*

$$\|f\|^2 = \sum_{\mu \in \Lambda_l^+} d(\mu) |\mathcal{S}_l(f)(\mu)|^2$$

$$f = \sum_{\mu \in \Lambda_l^+} d(\mu) \mathcal{S}_l(f)(\mu) \psi_{\mu,l}$$

where the sum is understood in the L^2 -norm sense and uniformly if f is smooth.

(4) $L^2(U//K; \mathcal{L}_l) \simeq \ell_d^2(\Lambda_l^+)$.

4 The χ_l -Spherical Functions

The χ_l -spherical functions on U were already introduced in the last section. We now discuss them in more details and present the results needed for the proof of the Paley–Wiener Theorem in Section 5. The standard reference for the material in this section is [15], see also [36]. Our assumptions are the same as in the last section. In particular $\mathbf{Y} = U/K$ is an irreducible hermitian symmetric space with U simple and simply connected.

4.1 The χ_l -spherical functions on G

Let us start by recalling the definition of a χ_l -spherical function.

Definition 10. Let H be a locally compact Hausdorff group and $L \subset H$ a compact subgroup. Let $\chi_l : L \rightarrow \mathbb{T}$ be a continuous homomorphism. A continuous function $\varphi : H \rightarrow \mathbb{C}$ is an (*elementary*) *spherical function of type χ_l* if ψ is not identically 0 and

$$\int_L \varphi(akb) \chi_l(k) dk = \varphi(a) \varphi(b), \quad \text{for all } a, b \in H. \quad (4.1)$$

We will mostly say that φ is a χ_l -spherical function or a spherical function of type χ_l .

Lemma 11. *Let φ be a spherical function of type χ_l . Then*

$$\varphi(k_1 h k_2) = \chi_l(k_1 k_2)^{-1} \varphi(h), \quad \text{for all } h \in H \text{ and } k_1, k_2 \in L.$$

Furthermore, $\varphi(e) = 1$.

Proof. Let $b \in H$ be so that $\varphi(b) \neq 0$. Let $a \in H$ and $m \in L$. Then

$$\varphi(am) = \frac{1}{\varphi(b)} \int_L \varphi(amkb) \chi_l(k) dk = \frac{1}{\varphi(b)} \int_L \varphi(akb) \chi_l(m^{-1}k) dk = \chi_l(m)^{-1} \varphi(a).$$

One can show that $\varphi(ma) = \chi_l(m)^{-1} \varphi(a)$ in the same way by applying (4.1) to

$$\frac{1}{\varphi(b)} \int_L \varphi(bka) \chi_l(k m^{-1}) dk.$$

That $\varphi(e) = 1$ follows from (4.1) by taking $b = e$. □

As $G_{\mathbb{C}} = U_{\mathbb{C}}$ is simply connected it follows that π_{μ} extends to an irreducible holomorphic representation of $G_{\mathbb{C}}$ which we also denote by π_{μ} . Thus χ_l extends to a homomorphism of $K_{\mathbb{C}}$ also denoted by χ_l .

Lemma 12. *Let $\mu \in \Lambda_l^+$. Then $\psi_{\mu,l}$ is a spherical function of type χ_l . It extends to a holomorphic function on $U_{\mathbb{C}}$. The extension is given by*

$$\check{\psi}_{\mu,l}(g) = \langle \pi_{\mu}(g^{-1})e_{\mu}, e_{\mu} \rangle, \quad g \in U_{\mathbb{C}}.$$

Furthermore, the holomorphic extension satisfies $\check{\psi}_{\mu,l}(k_1 g k_2) = \chi_l(k_1 k_2)^{-1} \check{\psi}_{\mu,l}(g)$ for all $k_1, k_2 \in K_{\mathbb{C}}$ and $g \in U_{\mathbb{C}}$.

Proof. This is standard, but let us show that $\psi_{\mu,l}$ satisfies (4.1). For that we note that

$$\int_K \chi_l(k)^{-1} \pi_{\mu}(k) \pi_{\mu}(b) e_{\mu} dk = \langle \pi_{\mu}(b) e_{\mu}, e_{\mu} \rangle e_{\mu}, \quad b \in U.$$

Hence, for $a, b \in U$,

$$\begin{aligned} \int_K \psi_{\mu,l}(akb) \chi_l(k) dk &= \int_K \langle \pi_{\mu}(a^{-1})e_{\mu}, \chi_l(k)^{-1} \pi_{\mu}(k) \pi_{\mu}(b) e_{\mu} \rangle dk \\ &= \langle e_{\mu}, \pi_{\mu}(a) e_{\mu} \rangle \overline{\langle \pi_{\mu}(b) e_{\mu}, e_{\mu} \rangle} \\ &= \psi_{\mu,l}(a) \psi_{\mu,l}(b). \end{aligned}$$

□

For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ define $\varphi_{\lambda,l} : G \rightarrow \mathbb{C}$ by

$$\varphi_{\lambda,l}(g) = \int_K a(g^{-1}k)^{\lambda-\rho} \chi_l(\kappa(g^{-1}k)k^{-1}) dk. \quad (4.2)$$

Remark 13. Notice that the formula (4.2) differs from the one in [15, p.82, (5.4.1)] by an inverse sign. The definition (4.2) for $\varphi_{\lambda,l}$ is equivalent to

$$\varphi_{\lambda,l}(g) = \int_K a(gk)^{-\lambda-\rho} \chi_l(\kappa(gk)^{-1}k) dk.$$

When $l = 0$, $\varphi_{\lambda}(g) = \varphi_{\lambda,0}(g)$ is the Harish-Chandra spherical function on G .

For the following theorem see [15, p.82, Prop. 5.4.1], [36, Prop. 3.3, Cor. 3.7], and Remark 13:

Theorem 14. *The function $\varphi_{\lambda,l}$ is a spherical function of type χ_l on G . If ψ is a spherical function of type χ_l then there exists $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ such that $\psi = \varphi_{\lambda,l}$. Furthermore the following hold true:*

1. $\varphi_{\lambda,l}(g)$ is real analytic in $g \in G$, and holomorphic in $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$.
2. $\varphi_{\lambda,l} = \varphi_{\mu,l}$ if and only if there is a $w \in W$ such that $\lambda = w\mu$.
3. $\varphi_{\lambda,l}(w \cdot a) = \varphi_{\lambda,l}(a)$, $\forall w \in W$.
4. $\varphi_{\lambda,l}(a) = \varphi_{\lambda,-l}(a) = \varphi_{-\lambda,-l}(a) = \varphi_{-\lambda,l}(a)$, $\forall a \in A$.
5. $\varphi_{\lambda,l}(g) = \varphi_{-\lambda,-l}(g^{-1}) = \varphi_{\lambda,-l}(g^{-1}) = \varphi_{-\lambda,l}(g)$, $\forall g \in G$.

The following lemma gives one of the main steps to analytically continue the χ_l -spherical Fourier transform.

Lemma 15. *Let $\mu \in \Lambda_l^+$. Then $\varphi_{\mu+\rho,l}$ extends to a holomorphic function on $G_{\mathbb{C}}$, denoted again by $\varphi_{\mu+\rho,l}$, and $\check{\psi}_{\mu,l} = \varphi_{\mu+\rho,l}$.*

Proof. As G is totally real in $G_{\mathbb{C}}$ it is enough to show that $\check{\psi}_{\mu,l}|_G = \varphi_{\mu+\rho,l}$. Let $u_{\mu} \in V_{\mu,l}$ be a nonzero highest weight vector. Then $V_{\mu,l}$ is generated by $\pi_{\mu}(K_{\mathbb{C}})u_{\mu}$. As $K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}$ is dense it follows that $\langle u_{\mu}, e_{\mu} \rangle \neq 0$.

Choose u_μ so that $\langle u_\mu, e_\mu \rangle = 1$. Then $P_{\mu,l}u_\mu = e_\mu$. Thus for $g \in G$:

$$\begin{aligned} \tilde{\psi}_{\mu,l}(g) &= \langle \pi_\mu(g^{-1})e_\mu, e_\mu \rangle \\ &= \int_K \langle \chi_l(k)^{-1} \pi_\mu(g^{-1}k)u_\mu, e_\mu \rangle dk \\ &= \int_K a(g^{-1}k)^\mu \chi_l(\kappa(g^{-1}k)k^{-1}) dk \\ &= \varphi_{\mu+\rho,l}(g). \end{aligned}$$

□

4.2 Holomorphic extension and estimates for $\varphi_{\lambda,l}$

We refer to Chapter 5 in [15] and Section 2 in [36] for detailed discussion about invariant differential operators on \mathcal{L}_l . Here we just recall what we need. Let $\mathbb{D}_l(\mathbf{X}) \simeq \mathbb{D}_l(\mathbf{Y})$ be the algebra of invariant differential operators $D : C^\infty(\mathbf{X}; \mathcal{L}_l) \rightarrow C^\infty(\mathbf{X}; \mathcal{L}_l)$. Let $U(\mathfrak{g})^K$ be the Ad(K)-invariant elements in the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{u}_{\mathbb{C}}$. Then there exists a surjective map $u \mapsto D_u$ of $U(\mathfrak{g})^K$ onto $\mathbb{D}_l(\mathbf{X})$. We denote by $\gamma_l : U(\mathfrak{g})^K \rightarrow S(\mathfrak{a})^W$ the Harish-Chandra homomorphism. Here $S(\mathfrak{a})^W$ is the commutative algebra of W -invariant polynomials on $\mathfrak{a}_{\mathbb{C}}^*$. Then γ_l induces an algebra isomorphism $\mathbb{D}_l(\mathbf{X}) \simeq S(\mathfrak{a})^W$, see [15, Thm. 5.1.10]. Define a homomorphism $\zeta_{\lambda,l} : \mathbb{D}_l(\mathbf{X}) \rightarrow \mathbb{C}$ by $\zeta_{\lambda,l}(D) = \gamma_l(D)(\lambda)$. We also write $\zeta_l(D; \lambda)$ for $\zeta_{\lambda,l}(D)$. We then have:

Lemma 16 (Theorem 3.2, [36]). *Let $D \in \mathbb{D}_l(\mathbf{X})$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Then*

$$D\varphi_{\lambda,l} = \zeta_l(D; \lambda)\varphi_{\lambda,l}. \quad (4.3)$$

If $\varphi \in C^\infty(U//K; \mathcal{L}_l)$ is a solution of the system of differential equations (4.3) and $\varphi(e) = 1$, then $\varphi = \varphi_{\lambda,l}$.

For $l \in \mathbb{Z}$ define K -biinvariant functions η_l^\pm on U such that their restriction to B is given by

$$\eta_l^+ := \prod_{\alpha \in \mathcal{O}_S^+} \left(\frac{e^\alpha + e^{-\alpha}}{2} \right)^{2|l|}, \quad \eta_l^- := \prod_{\alpha \in \mathcal{O}_S^+} \left(\frac{e^\alpha + e^{-\alpha}}{2} \right)^{-2|l|}. \quad (4.4)$$

Note that η_l^- is only well defined on the set where $e^{\alpha(\log b)} + e^{-\alpha(\log b)} \neq 0$. We often write η_l for η_l^+ . We will also view those functions as K -invariant functions on U/K .

Recall that a multiplicity function m is a W -invariant functions $m : \Sigma \rightarrow \mathbb{C}$. In our case it can only take three values $m = (m_S, m_M, m_L)^2$. It is possible that one or more of those numbers is zero. For $l \in \mathbb{Z}$ define multiplicity functions

$$m_+(l) = (m_S - 2|l|, m_M, m_L + 2|l|) \quad (4.5)$$

$$m_-(l) = (m_S + 2|l|, m_M, m_L - 2|l|). \quad (4.6)$$

We will also write $m(l)$ for $m_+(l)$.

Note that the radial part of the Laplace-Beltrami operator on \mathbf{X} (acting on χ_l -covariant functions) is exactly the operator, see [19]:

$$L(l) = L(m(l)) := \sum_{j=1}^n \partial_j^2 + \sum_{\alpha \in \Sigma^+} m(l)_\alpha \frac{1 + e^{-2\alpha}}{1 - e^{-2\alpha}} \partial_\alpha$$

associated with the root system Σ and the multiplicity $m(l)$. This operator is actually defined on $A_{\mathbb{C}}^{\text{reg}} = \exp(\mathfrak{a}_{\mathbb{C}}^{\text{reg}})$. We write $\rho(l) = \rho(m(l))$. It was shown that

$$\zeta_l(L(l); \lambda) = (\lambda, \lambda) - (\rho(l), \rho(l)).$$

² Our multiplicity notation is different from the one used by Heckman and Opdam. The root system R they use is related to our 2Σ , and the multiplicity function k in Heckman and Opdam's work is related to our m by $k_{2\alpha} = \frac{1}{2}m_\alpha$.

Theorem 17. *Let*

$$\Omega = \{X \in \mathfrak{b} \mid (\forall \alpha \in \Sigma) |\alpha(X)| < \pi\}. \quad (4.7)$$

The function $A \times \mathfrak{a}_{\mathbb{C}}^* \rightarrow \mathbb{C}$, $(a, \lambda) \mapsto \varphi_{\lambda,l}(a)$, extends to a holomorphic function $(b, \lambda) \mapsto \varphi_{\lambda,l}(b)$ on $A \exp(\Omega) \times \mathfrak{a}_{\mathbb{C}}^*$. The extension satisfies the symmetry conditions in Theorem 14. Furthermore there exists a constant $C > 0$ such that for $X \in \Omega$ we have

$$|\varphi_{\lambda,l}(\exp X)| \leq C e^{\|X\| \|\operatorname{Re}(\lambda)\|}.$$

Proof. This is done in the Appendix. The estimate for $\varphi_{\lambda,l}$ follows from Remark 28 and Proposition 30. \square

Let $\varepsilon > 0$ and $\Omega_{\varepsilon} = \{X \in \mathfrak{b} \mid (\forall \alpha \in \Sigma) |\alpha(X)| \leq \pi - \varepsilon\}$. Then $\Omega = \cup_{\varepsilon > 0} \Omega_{\varepsilon}$. Let $Y_j \in \mathfrak{a}$ be such that $\varepsilon_i(Y_j) = \delta_{i,j}$. For $Y \in \mathfrak{a}_{\mathbb{C}}$ write

$$Y = \sum_{j=1}^n y_j Y_j = \sum_{j=1}^n \varepsilon_j(Y) Y_j.$$

We have

$$\mathfrak{a} + \Omega_{\varepsilon} = \{Y \in \mathfrak{a}_{\mathbb{C}} \mid |\operatorname{Im} y_j| \leq \frac{1}{2}(\pi - \varepsilon)\}.$$

5 The Paley–Wiener Theorem

Let $\|\cdot\|$ be the norm on \mathfrak{u} with respect to the inner product $\langle \cdot, \cdot \rangle$ defined earlier using the Cartan–Killing form. For $r > 0$, let $B_r(0) = \{X \in \mathfrak{q} : \|X\| < r\}$ be the open ball in \mathfrak{q} centered at 0 with radius r . Let $B_r(x_o) = \operatorname{Exp}(B_r(0))$. We will fix $R > 0$ such that R is smaller than the injectivity radius and such that $B_R(0) \cap \mathfrak{b} \subset \Omega$. In particular $\operatorname{Exp} : B_r(0) \rightarrow B_r(x_o)$ is a diffeomorphism and $\varphi_{\lambda,l}$ is well defined on the closure of $B_r(x_o)$ for all $0 < r < R$. Let $\overline{B}_r(0)$ be the closed ball in \mathfrak{q} with radius r and $\overline{B}_r(x_o) = \operatorname{Exp}(\overline{B}_r(0))$ the closure of $B_r(x_o)$. Finally we let $C_r^{\infty}(U//K; \mathcal{L}_l)$ be the space of functions in $C^{\infty}(U//K; \mathcal{L}_l)$ with support in $\overline{B}_r(x_o)$. As R is smaller than the injectivity radius it follows that for $0 < r < R$ we have

$$C_r^{\infty}(B)^W \cong C_r^{\infty}(B \cdot o)^W \quad \text{and} \quad C_r^{\infty}(U//K; \mathcal{L}_l) \xrightarrow{\cong} \eta_l \cdot C_r^{\infty}(B)^W. \quad (5.1)$$

For $r > 0$ denote by $\operatorname{PW}_r(\mathfrak{b}_{\mathbb{C}}^*)$ the space of holomorphic functions φ on $\mathfrak{b}_{\mathbb{C}}^*$ of exponential type r . Thus a holomorphic function F on $\mathfrak{b}_{\mathbb{C}}^*$ is in $\operatorname{PW}_r(\mathfrak{b}_{\mathbb{C}}^*)$ if and only if for every $k \in \mathbb{N}$, there is a constant C_k such that

$$|\varphi(\lambda)| \leq C_k (1 + \|\lambda\|)^{-k} e^{r \|\operatorname{Re} \lambda\|}, \quad \forall \lambda \in \mathfrak{b}_{\mathbb{C}}^*.$$

There are two natural actions of the Weyl group. The first one is the usual conjugation of the variable, and the second is the ρ -shifted affine action $R(w)F(\lambda) = F(w^{-1}(\lambda + \rho) - \rho)$. Let

$$\operatorname{PW}_r(\mathfrak{b}_{\mathbb{C}}^*)^{R(W)} = \{F \in \operatorname{PW}_r(\mathfrak{b}_{\mathbb{C}}^*) \mid (\forall w \in W) R(w)F = F\}.$$

We note that this is the same Paley–Wiener space as in [30].

Similarly one defines the space $\operatorname{PW}_r(\mathfrak{b}_{\mathbb{C}}^*)^W$ where we now use the standard action of the Weyl group. We note that those spaces are isomorphic via the map

$$F \mapsto \Psi(F) : \lambda \mapsto F(\lambda - \rho)$$

with inverse

$$G \mapsto \Psi^{-1}(G) : \lambda \mapsto G(\lambda + \rho).$$

To see that $\Psi(F)$ is W -invariant a simple calculation gives:

$$\Psi(F)(w\lambda) = F(w\lambda - \rho) = F(w(\lambda - \rho + \rho) - \rho) = F(\lambda - \rho) = \Psi(F)(\lambda).$$

Similarly for $\Psi^{-1}(G)$.

We will use this isomorphism in the following to connect results from [33] on restriction of Paley–Wiener spaces to subspaces without comment that one has sometimes to use the above isomorphism.

In the following theorem it would be natural to expect that one can take R to be the injectivity radius. But the application of Carleson’s theorem in the part (3) leads to a constant that might be smaller, see the remark following Corollary 19.

Theorem 18 (Paley–Wiener Theorem). *There exists a constant R , smaller or equal to the injectivity radius, such that for all $0 < r < R$ the (extended) χ_l -spherical Fourier transform \mathcal{S}_l gives a linear bijection*

$$\mathcal{S}_l : C_r^\infty(U//K; \mathcal{L}_l) \xrightarrow{\cong} \text{PW}_r(\mathfrak{b}_\mathbb{C}^*)^{R(W)}. \quad (5.2)$$

More precisely,

1. If $f \in C_r^\infty(U//K; \mathcal{L}_l)$, then $\mathcal{S}_l(f) : \Lambda_l^+ \rightarrow \mathbb{C}$ extends to a function in $\text{PW}_r(\mathfrak{b}_\mathbb{C}^*)^{R(W)}$;
2. Let $\varphi \in \text{PW}_r(\mathfrak{b}_\mathbb{C}^*)^{R(W)}$. There exists a unique $f \in C_r^\infty(U//K; \mathcal{L}_l)$ such that

$$\mathcal{S}_l(f)(\mu) = \varphi(\mu), \quad \forall \mu \in \Lambda_l^+;$$

3. The functions in $\text{PW}_r(\mathfrak{b}_\mathbb{C}^*)^{R(W)}$ are uniquely determined by their values on Λ_l^+ .

Corollary 19. *Let $l, k \in \mathbb{Z}$ and $0 < r < R$. Then*

$$\mathcal{S}_k^{-1} \circ \mathcal{S}_l : C_r^\infty(U//K; \mathcal{L}_l) \cong C_r^\infty(U//K; \mathcal{L}_k)$$

is a linear isomorphism.

Proof. This follows from Theorem 18 applied to l and k . □

Remark 20. 1) As remarked in [30, Remark 4.3] one can use different R in (1), (2) and (3), and then take the minimum of those constants for the map (5.2) to be a bijection. The constant R does not depend on the line bundle parameter l .

2) In [30] the authors used for Ω the domain where $|\alpha(X)| \leq \pi/2$. This is because [30] used the Opdam estimates [26] which were shown for this domain.

3) We note that $C_r^\infty(\mathbf{Y})^K = C_r^\infty(U//K; \mathcal{L}_0)$ so this case is included in the corollary.

The hard part of Theorem 18 is (2) so we start with (1) and (3) and leave (2) for the next section. The proof follows closely [30].

Proof. (Part (1)) Let $\mu \in \Lambda_l^+$ and $f \in C_r^\infty(U//K; \mathcal{L}_l)$. It is easy to see that the function $u \mapsto f(u)\overline{\psi_{\mu,l}(u)}$ is K -biinvariant. Using Lemma 5 and the fact that $-1 \in W$, we get

$$\begin{aligned} \mathcal{S}_l(f)(\mu) &= \frac{1}{|W|} \int_B f(b)\delta(b)\overline{\psi_{\mu,l}(b)} db \\ &= \frac{1}{|W|} \int_B [f(b)\delta(b)]\psi_{\mu,l}(b) db \\ &= \frac{1}{|W|} \int_B [f(b)\delta(b)]\varphi_{\mu+\rho,l}(b) db. \end{aligned}$$

As $\text{supp}(f|_{B \cdot x_o}) \subseteq \exp(\Omega) \cdot x_o$ we can, using Theorem 17 define the holomorphic extension of $\mathcal{S}_l(f)$ to $\mathfrak{a}_\mathbb{C}^*$ by

$$\lambda \mapsto \mathcal{S}_l(f)(\lambda) = \frac{1}{|W|} \int_B [f(b)\delta(b)]\varphi_{\lambda+\rho,l}(b) db.$$

Then, by the W -invariance of $\varphi_{\lambda,l}$ it follows that $\mathcal{S}_l(f)(w(\lambda + \rho) - \rho) = \mathcal{S}_l(f)(\lambda)$.

As $b \mapsto f(b)\delta(b)$ is in $C_r^\infty(B)^W$ it follows again from Theorem 17 that

$$|\mathcal{S}_l(f)(\lambda)| \leq C e^{r \|\operatorname{Re} \lambda\|}.$$

The polynomial estimate follows by applying $D \in \mathcal{D}_l(\mathbf{Y})$ to $\varphi_{\lambda,l}$ and noticing that

$$\zeta_l(D; \lambda) \mathcal{S}_l(f)(\lambda) = \int_U f(y) D \varphi_{\lambda,l}(y^{-1}) dy = \int_U D^* f(y) \varphi_{\lambda,l}(y^{-1}) dy = \mathcal{S}_l(D^* f)(\lambda)$$

where D^* is the adjoint of D . □

Proof. (Part (3)) This follows from a generalization of Carleson's theorem for higher dimensions, see [30, Lem. 7.1]. Here we use the fundamental weights $\omega_1, \dots, \omega_n$ and (3.8) to view $\mathcal{S}_l(f)$ as a function on \mathbb{C}^n . Denote by $\|\lambda\|_0$ the standard norm on \mathbb{C}^n . Then there exists a constant $C > 0$ such that $\|\lambda\| \leq C \|\lambda\|_0$. It follows that there exists a $C_1 > 0$ such that

$$|\mathcal{S}_l f(\sum \lambda_j \omega_j + 2|l| \rho_s)| \leq C_1 e^{(rC) \|\lambda\|_0}.$$

Hence, if $rC < \pi$ and $\mathcal{S}_l f(\sum k_j \omega_j + 2|l| \rho_s) = 0$ for all $k_j \in \mathbb{Z}^+$ then Carleson's theorem implies that $\mathcal{S}_l f = 0$ and hence the extension is unique. □

6 The Surjectivity

In this section we prove part (2) of Theorem 18. The proof is reduced to the Paley-Wiener theorem for central functions on compact Lie group originally proved by F. Gonzalez in [11]. The reduction depends on a surjectivity criterion for restriction of Paley-Wiener spaces.

6.1 The Paley-Wiener theorem for central functions on U

This is a special case of the Paley-Wiener theorem for compact symmetric spaces as $U \simeq U \times U/K$ where K is the diagonal in $U \times U$, i.e. $K = \{(u, u) \mid u \in U\} \simeq U$. The corresponding involution is $\tau(a, b) = (b, a)$ and the action of $U \times U$ on U is $(a, b) \cdot u = aub^{-1}$. In particular we have

$$C^\infty(U)^U = \{f \in C^\infty(U) \mid (\forall u, k \in U) f(kuk^{-1}) = f(u)\}.$$

The spherical functions on $U = U \times U/K$ are the normalized trace functions

$$\xi_\mu(u) = \frac{1}{d(\mu)} \operatorname{Tr}(\pi_\mu(u^{-1})).$$

The noncompact dual is $G/K = U_{\mathbb{C}}/K$. The role of \mathfrak{a} is played by the Cartan subalgebra $i\mathfrak{h}$, W by $W_{\mathfrak{h}}$, the Weyl group associated to Δ , and Λ_0^+ by $\Lambda^+(U)$, the semi-lattice of all highest weights of irreducible representations of U . Finally, the spherical functions on $U_{\mathbb{C}}/K$ are given by, see [19, Thm. 5.7, Chapter IV]:

$$\varphi_\lambda(a) = \frac{\pi(\rho(\mathfrak{h}))}{\pi(\lambda)} \frac{\sum_{w \in W} (\det w) a^{w\lambda}}{\sum_{w \in W} (\det w) a^{w\rho_{\mathfrak{h}}}}, \quad \text{where } \pi(\mu) = \prod_{\alpha \in \Delta^+} \langle \alpha, \mu \rangle.$$

The result of [11] was used by [30] to prove the surjectivity part of the Paley-Wiener theorem for K -invariant functions on U/K . The simple reformulation corresponding to results of [30], i.e., using the normalized trace function ξ_μ instead of the character $\operatorname{Tr} \circ \pi_\mu$, was done in [33, Lem. 5.4]. The formulation of Gonzalez theorem, in the form we need it, is then:

Theorem 21. *Denote by $R > 0$ the injectivity radius and let $0 < r < R$. Let $\operatorname{PW}_r(\mathfrak{h}_{\mathbb{C}}^*)^{R(W_{\mathfrak{h}})}$ be the space of holomorphic functions on $\mathfrak{h}_{\mathbb{C}}$ of exponential growth r and such that $F(w(\lambda + \rho_{\mathfrak{h}}) - \rho_{\mathfrak{h}}) = F(\lambda)$ for all $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ and $w \in W_{\mathfrak{h}}$. Then the spherical Fourier transform*

$$\tilde{f}(\lambda) = \int_U f(u) \varphi_{\lambda+\rho}(u^{-1}) du$$

is a surjective linear map $C_r^\infty(U)^U \rightarrow \operatorname{PW}_r(\mathfrak{h}_{\mathbb{C}}^)^{R(W_{\mathfrak{h}})}$.*

6.2 The Surjectivity of the χ_l -Spherical Fourier Transform

It is easy to see that if $F \in \text{PW}_r(\mathfrak{h}_{\mathbb{C}}^*)^{R(W)}$ then the function

$$f(u) = \sum_{\mu \in \Lambda_l^+} d(\mu) F(\mu) \psi_{\mu,l}(u)$$

is smooth and $\mathcal{S}_l(f) = F$. The hard part is to see that $\text{supp}(f) \subseteq \overline{B}_r(x_o)$. To use Theorem 21 we define

$$Q_l(f)(u) = \int_K f(uk) \chi_l(k) dk = \int_K \chi_l(k) f(ku) dk, \quad f \in C_r^\infty(U)^U.$$

Lemma 22. *Assume that $f \in C_r^\infty(U)^U$. Then $Q_l(f) \in C_r^\infty(U//K; \mathcal{L}_l)$. Moreover, if $F \in \text{PW}_r(\mathfrak{h}_{\mathbb{C}}^*)^{R(W_{\mathfrak{h}})}$ and $f = \sum_{\mu \in \Lambda^+(U)} d(\mu)^2 F(\mu) \xi_{\mu}$, then*

$$Q_l(f)(u) = \sum_{\mu \in \Lambda_l^+} d((\mu, 0, \mu_0)) F((\mu, 0, \mu_0)) \psi_{\mu,l}(u). \quad (6.1)$$

Proof. If $f = \sum_{\mu \in \Lambda^+(U)} d(\mu)^2 F(\mu) \xi_{\mu}$ then f is the inverse Fourier transform of F . Because of the rapid decay of F it follows that

$$Q_l(f) = \sum_{\mu \in \Lambda^+(U)} d(\mu)^2 F(\mu) Q_l(\xi_{\mu}).$$

Note that the square in $d(\mu)^2$ comes from the fact that the representation that we are in fact using in $V_{\mu} \otimes V_{\mu}^*$ has dimension $d(\mu)^2$.

Recall that $P_{\mu,l}v = \int_K \chi_l(k)^{-1} \pi_{\mu}(k)v dk$ is the orthogonal projection $V_{\mu} \rightarrow V_{\mu}^l$. In particular, if π_{μ} is not χ_l -spherical, then $P_{\mu,l}(V_{\mu}) = 0$. Fix an orthonormal basis of V_{μ} , say $v_1, \dots, v_{d(\mu)}$. In case V_{μ} is χ_l -spherical, we assume that $v_1 = e_{\mu,l}$. Then

$$\begin{aligned} d(\mu) \int_K \xi_{\mu}(uk) \chi_l(k) dk &= \sum_{j=1}^{d(\mu)} \langle v_j, \int_K \chi_l(k^{-1}) \pi_{\mu}(u) \pi_{\mu}(k) v_j dk \rangle \\ &= \sum_{i=1}^{d(\mu)} \langle v_j, \pi_{\mu}(u) P_{\mu,l} v_j \rangle \\ &= \begin{cases} 0 & \text{if } \pi_{\mu} \text{ is not } \chi_l \text{ spherical} \\ \psi_{\mu_{\mathfrak{b}},l} & \text{if } \pi_{\mu} \text{ is } \chi_l \text{ spherical} \end{cases} \end{aligned}$$

where $\mu_{\mathfrak{b}}$ is the projection of $\mu \in i\mathfrak{h}^*$ onto $i\mathfrak{b}^*$. The claim (6.1) now follows from our description of $\Lambda_l^+(U) = \{(\mu, 0, \mu_0) \mid \mu \in \Lambda_l^+\}$. The claim that $Q_l(f)$ is supported in a ball of radius r follows as Lemma 9.3 in [30]. \square

From now on, for $(\mu, 0, \mu_0) \in \Lambda_l^+(U)$, we fix the projection μ_0 . As mentioned earlier, we simply write $(0, 0, \mu_0)$ as μ_0 , and $(\mu, 0, \mu_0)$ as $\mu + \mu_0$. Recall that $\rho_0 = \rho_{\mathfrak{h}}|_{\mathfrak{a}^+}$ and $\rho_{\mathfrak{h}} = \rho + \rho_0$. Also recall that $\Delta_0 = \{\alpha \in \Delta \mid \alpha|_{\mathfrak{a}} = 0\}$. Let

$$\begin{aligned} \widetilde{W} &= \{w \in W_{\mathfrak{h}} \mid w(\mathfrak{a}_{\mathbb{C}}) = \mathfrak{a}_{\mathbb{C}}\} \\ W_0 &= \{w \in W_{\mathfrak{h}} \mid w|_{\mathfrak{a}} = \text{Id}\}. \end{aligned}$$

Then $\widetilde{W} \subseteq W_{\mathfrak{h}}$ and $W_0 \subseteq \widetilde{W}$. We have to consider the extra factors μ_0 and ρ_0 when we analyze the actions of the Weyl groups. This is a new aspect that did not occur in the case of K -biinvariant functions in [30].

Let us now make observations on restriction of Paley–Wiener spaces to subspaces and then obtain a surjectivity criteria for such a restriction. Notice that the map

$$F \mapsto \Psi_{\mathfrak{h}}(F), \quad \Psi_{\mathfrak{h}}(F)(\lambda) = F(\lambda - \rho_{\mathfrak{h}})$$

is an isomorphism from $\text{PW}_r(\mathfrak{h}_{\mathbb{C}}^*)^{R(W_{\mathfrak{h}})}$ onto $\text{PW}_r(\mathfrak{h}_{\mathbb{C}}^*)^{W_{\mathfrak{h}}}$ with the inverse $\Psi_{\mathfrak{h}}^{-1}(F)(\lambda) = F(\lambda + \rho_{\mathfrak{h}})$. The map

$$F \longmapsto \Phi(F), \quad \Phi(F)(\lambda) = F(\lambda + \mu_0 + \rho_0)$$

is an isomorphism of $\text{PW}_r(\mathfrak{h}_{\mathbb{C}}^*)$ onto itself.

Lemma 23. *We have $w\mu_0 = \mu_0$, i.e. $w(0, 0, \mu_0) = (0, 0, \mu_0)$, for all $w \in \widetilde{W}$.*

Proof. Since $\mu_0 = \lambda(iX)$ (see Section 3.1), it remains to show that $wX = X$ for $w \in \widetilde{W}$. Recall [35, (4.4)] for the definition of X . Note that $\widetilde{W}|_{\mathfrak{a}} = W$, and elements of W are permutations and sign changes. The permutations of ε_j 's are given by products of reflections $r_{\frac{1}{2}(\varepsilon_i - \varepsilon_j)}$'s. Let $\beta \in \Delta$ be given by $\beta = \beta^+ + \beta^-$ where $\beta^- = \frac{1}{2}(\varepsilon_i - \varepsilon_j) \in \mathfrak{a}^*$ and $\beta^+ \in (\mathfrak{a}^{\perp})^*$. Let $\bar{\beta} = -\beta^+ + \beta^-$. By Moore's theorem [23], there is only one root length in Δ . This implies that $\langle \beta, \bar{\beta} \rangle = 0$. It then follows that

$$r_{\beta} r_{\bar{\beta}}|_{\mathfrak{a}} = r_{\frac{1}{2}(\varepsilon_i - \varepsilon_j)}$$

and thus $r_{\beta} r_{\bar{\beta}} \in \widetilde{W}$. Notice that both β and $\bar{\beta}$ are compact roots in Δ . So $r_{\beta} r_{\bar{\beta}}$ preserves compact roots and noncompact roots, respectively. Therefore, $(r_{\beta} r_{\bar{\beta}})X = X$. It is easy to see that any $w \in \widetilde{W}$ with $w|_{\mathfrak{a}} = r_{\frac{1}{2}(\varepsilon_i - \varepsilon_j)}$ also satisfies $wX = X$. On the other hand, the sign changes of ε_j 's are given by products of r_{ε_j} 's. But then $r_{\varepsilon_j}X = X$ since $X \perp \mathfrak{a}$. This implies that any $w \in \widetilde{W}$ with $w|_{\mathfrak{a}} = r_{\varepsilon_j}$ satisfies $wX = X$. Hence we have shown that $wX = X$ for any $w \in \widetilde{W}$. \square

Lemma 24. *Let $G \in \text{PW}_r(\mathfrak{h}_{\mathbb{C}}^*)^{W_{\mathfrak{h}}}$. Then $\Phi(G)|_{\mathfrak{a}_{\mathbb{C}}^*} = G(\cdot + \mu_0 + \rho_0)$ is W -invariant, i.e. if $w \in W$ and $\mu \in \mathfrak{a}_{\mathbb{C}}^*$, then*

$$G(w\mu + \mu_0 + \rho_0) = G(\mu + \mu_0 + \rho_0).$$

Proof. Let $w \in W$. Let $\tilde{w} \in \widetilde{W}$ be such that $\tilde{w}|_{\mathfrak{a}} = w$. Since G is $W_{\mathfrak{h}}$ -invariant, it is thus \widetilde{W} -invariant. In view of Lemma 23 we get

$$G(w\mu + \mu_0 + \rho_0) = G(\mu + \tilde{w}^{-1}\mu_0 + \tilde{w}^{-1}\rho_0) = G(\mu + \mu_0 + \tilde{w}^{-1}\rho_0).$$

Since $\tilde{w}^{-1}\Delta_0 = \Delta_0$, we can choose $w_0 \in W_0$ such that $w_0(\tilde{w}^{-1}\Delta_0^+) = \Delta_0^+$, in particular, choose w_0 such that $w_0\tilde{w}^{-1}(\rho_0) = \rho_0$. Moreover, $w_0\mu = \mu$. It follows that

$$G(w\mu + \mu_0 + \rho_0) = G(w_0(\mu + \mu_0 + \tilde{w}^{-1}\rho_0)) = G(\mu + \mu_0 + \rho_0).$$

\square

Let $G \in \text{PW}_r(\mathfrak{h}_{\mathbb{C}}^*)$. Let $k = |W_{\mathfrak{h}}|$. Let P_1, \dots, P_k be a basis for $S(\mathfrak{h})$ over $S(\mathfrak{h})^{W_{\mathfrak{h}}}$. Here, $S(\mathfrak{h})$ is the symmetric algebra of \mathbb{C} -valued polynomials on $\mathfrak{h}_{\mathbb{C}}^*$, and $S(\mathfrak{h})^{W_{\mathfrak{h}}}$ consists of $W_{\mathfrak{h}}$ -invariant elements in $S(\mathfrak{h})$. According to Rais [34], published proof due to L. Clozel and P. Delorme, see [5], there exist $G_1, \dots, G_k \in \text{PW}_r(\mathfrak{h}_{\mathbb{C}}^*)^{W_{\mathfrak{h}}}$ such that

$$G = P_1 G_1 + \dots + P_k G_k.$$

We are now ready to prove that the χ_l -spherical Fourier transform \mathcal{S}_l (5.2) is surjective.

Proof. (Part (2) of Theorem 18) Let $F \in \text{PW}_r(\mathfrak{h}_{\mathbb{C}}^*)^{R(W)}$. Then $\Psi(F) \in \text{PW}_r(\mathfrak{h}_{\mathbb{C}}^*)^W$. It follows from Cowling [6] that there exists $E \in \text{PW}_r(\mathfrak{h}_{\mathbb{C}}^*)$ such that $\Phi(E)|_{\mathfrak{a}_{\mathbb{C}}^*} = \Psi(F)(\cdot + \mu_0)$, i.e.

$$E(\mu + \mu_0 + \rho_0) = \Phi(E)|_{\mathfrak{a}_{\mathbb{C}}^*}(\mu) = \Psi(F)(\mu + \mu_0), \quad \mu \in \mathfrak{a}_{\mathbb{C}}^*.$$

By the above results of Rais, there exist polynomials $P_j \in S(\mathfrak{h})$ and $G_j \in \text{PW}_r(\mathfrak{h}_{\mathbb{C}}^*)^{W_b}$ such that $E = \sum_{j=1}^k P_j G_j$. Hence

$$\Psi(F)(\cdot + \mu_0) = \sum_{j=1}^k \Phi(P_j)|_{\mathfrak{a}_{\mathbb{C}}^*} \Phi(G_j)|_{\mathfrak{a}_{\mathbb{C}}^*}. \quad (6.2)$$

Taking the average of (6.2) over W gives that

$$\begin{aligned} \Psi(F)(\mu + \mu_0) &= \sum_{j=1}^k \left(\frac{1}{|W|} \sum_{w \in W} \Phi(P_j)|_{\mathfrak{a}_{\mathbb{C}}^*}(w\mu) \Phi(G_j)|_{\mathfrak{a}_{\mathbb{C}}^*}(w\mu) \right) \\ &= \sum_{j=1}^k \underbrace{\left(\frac{1}{|W|} \sum_{w \in W} \Phi(P_j)|_{\mathfrak{a}_{\mathbb{C}}^*}(w\mu) \right)}_{=: q_j(\mu)} \Phi(G_j)|_{\mathfrak{a}_{\mathbb{C}}^*}(\mu) \end{aligned}$$

where by Lemma 24

$$\Phi(G_j)|_{\mathfrak{a}_{\mathbb{C}}^*}(w\mu) = G_j(w\mu + \mu_0 + \rho_0) = G_j(\mu + \mu_0 + \rho_0) = \Phi(G_j)|_{\mathfrak{a}_{\mathbb{C}}^*}(\mu), \quad w \in W.$$

Note that $q_j \in S(\mathfrak{a})^W$. Let $D_j \in \mathcal{D}_l(\mathbf{Y})$ be such that $q_j(\lambda) = \overline{\zeta_l(D_j^*, \lambda)}$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, see the discussion at the beginning of Section 4.2.

By the Paley–Wiener theorem for $C_r^\infty(U)^U$, there exists $\varphi_j \in C_r^\infty(U)^U$ with spherical Fourier transform $\Psi_{\mathfrak{h}}^{-1} G_j$. Then $f_j = Q_l(\varphi_j) \in C_r^\infty(U//K; \mathcal{L}_l)$ has the χ_l -spherical Fourier transform:

$$\mathfrak{S}_l(f_j)(\mu + \mu_0) = \Phi(G_j)|_{\mathfrak{a}_{\mathbb{C}}^*}(\mu + \rho).$$

It follows that F is the χ_l -spherical Fourier transform of $f := D_1 f_1 + \dots + D_k f_k$. The surjectivity now follows from the fact that differentiation does not increase supports and hence $f \in C_r^\infty(U//K; \mathcal{L}_l)$. \square

A Estimates for the Heckman–Opdam Hypergeometric Functions

We first review some facts on the theory of Heckman–Opdam hypergeometric functions. We refer to [15] for notations and basic definitions. We do not assume that Σ corresponds to a hermitian symmetric space. Recall that a multiplicity function $m : \Sigma \rightarrow \mathbb{C}$ is a W -invariant function. It is said to be positive if $m(\alpha) \geq 0$ for all α . The set of multiplicity functions is denoted by \mathcal{M} and the subset of positive multiplicity functions is denoted by \mathcal{M}^+ . The Harish-Chandra series corresponding to a multiplicity function m is denoted by $\Phi(\lambda, m; a)$ and the c -function is denoted by $c(\lambda, m)$.

Definition 25. The function

$$F(\lambda, m; a) = \sum_{w \in W} c(w\lambda, m) \Phi(w\lambda, m; a)$$

is called the *hypergeometric function* on A associated with the triple $(\mathfrak{a}, \Sigma, m)$.

Theorem 26. *Let*

$$\mathcal{M}_{\geq} = \{m \in \mathcal{M} \mid (\forall \alpha \in \Sigma_*) \ m_{\alpha} + m_{\alpha/2} \geq 0, \ m_{\alpha} \geq 0\}.$$

Then the following hold:

- (1) *There exists an open set \mathcal{M}_{reg} containing \mathcal{M}_{\geq} and an open W -invariant set $\mathcal{V} \subset A_{\mathbb{C}}$ containing A such that $F(\lambda, m; a)$ is holomorphic on $\mathfrak{a}_{\mathbb{C}}^* \times \mathcal{M}_{reg} \times \mathcal{V}$, and satisfies*

$$F(w\lambda, m; a) = F(\lambda, m; a), \quad \forall w \in W$$

$$F(\lambda, m; w a) = F(\lambda, m; a), \quad \forall w \in W$$

with $(\lambda, m; a) \in \mathfrak{a}_{\mathbb{C}}^ \times \mathcal{M}_{reg} \times \mathcal{V}$.*

(2) One can take \mathcal{V} in (1) to be $\exp(\mathfrak{a} + \Omega)$, where Ω is defined as in (4.7).

Proof. For (1) see [15, Thm. 4.4.2 and Remark 4.4.3]. See also [26, Thm. 3.15]. For (2) see Remark 3.17 in [3]. \square

Proposition 27. *Let the notation be as above. Let $l \in \mathbb{Z}$ and let η_l^\pm be as in (4.4). Then the following hold:*

(1) *The multiplicity functions $m_\pm(l)$ are in \mathcal{M}_\geq .*

(2) *For $\lambda \in \mathfrak{a}_\mathbb{C}^*$,*

$$\varphi_{\lambda, l}|_A = \eta_l^\pm F(\lambda, m_\pm(l); \cdot)$$

where the \pm sign indicates that both possibilities are valid.

Proof. (1) follows from the definition of $m_\pm(l)$ (cf. (4.5) and (4.6)), and (2) is [15, p.76, Thm. 5.2.2]. \square

Remark 28. For $X \in \Omega$ and $\lambda \in \mathfrak{a}_\mathbb{C}^*$ we have

$$\varphi_{\lambda, l}(\exp X) = \eta_l(\exp X)F(\lambda, m(l); \exp X).$$

Since $\alpha(\mathfrak{b}) \subset i\mathbb{R}$ for $\alpha \in \Sigma$,

$$0 < |\eta_l(\exp X)| = \prod_{\alpha \in \Theta_s^+} |\cos \operatorname{Im} \alpha(X)|^{2|l|} \leq 1.$$

Thus η_l is holomorphic on $A(\exp \Omega)$, W -invariant on $A(\exp \Omega)$, and bounded on $\exp \Omega$.

In the main part showing that the χ_l -spherical Fourier transform maps the space $C_r^\infty(U//K; \mathcal{L}_i)$ into the Paley–Wiener space, a good control over the growth of the hypergeometric functions was needed. Proposition 6.1 in [26] gives a uniform estimate both in $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and in $Z \in \mathfrak{a} + \overline{\Omega}/2$ (recall the difference in the notation) in case all multiplicities are positive. But we need similar estimates where some multiplicities are allowed to be negative. In the following we will generalize Opdam’s results to multiplicities in \mathcal{M}_\geq . We also point out, that Opdam’s estimates holds for $Y \in \overline{\Omega}/2$, but our estimates, with possibly a different constant in the exponential growth, holds for $Y \in \Omega$. Our proof is based on ideas from [26] but uses a different regrouping of terms as we will point out later.

For $m \in \mathcal{M}_{\text{reg}}$ let

$$\tilde{\rho} = \tilde{\rho}(m) = \frac{1}{2} \sum_{\alpha \in \Sigma^+} |m_\alpha| \alpha.$$

Proposition 29. *Let $m \in \mathcal{M}_\geq$. Let F be the hypergeometric function associated with Σ and m . Let $\varepsilon > 0$. Then there is a constant $C = C_\varepsilon > 0$ depending on ε such that*

$$|F(\lambda, m; \exp Z)| \leq |W|^{\frac{1}{2}} \exp\left(-\min_{w \in W} \operatorname{Im}(w\lambda(Y))\right) + \frac{C}{2} \max_{w \in W} w \tilde{\rho}(Y) + \max_{w \in W} \operatorname{Re}(w\lambda(X))$$

where $Z = X + iY$ with $X, Y \in \mathfrak{a}$ and $|\alpha(Y)| \leq \pi - \varepsilon$ for all $\alpha \in \Sigma$.

Proof. Let $\phi_w(\exp Z) = G(\lambda, m, w^{-1}Z)$ where G is the nonsymmetric hypergeometric function defined as in [26, Thm. 3.15], so that

$$F(\lambda, m; \exp Z) = |W|^{-1} \sum_{w \in W} G(\lambda, m, w^{-1}Z). \tag{A.1}$$

In the following we will often write ϕ_w instead of $\phi_w(\exp Z)$. By Definition 3.1 and Lemma 3.2 in [26] we have

$$\partial_\xi \phi_w = -\frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha(\xi) \left[\frac{1 + e^{-2\alpha(Z)}}{1 - e^{-2\alpha(Z)}} (\phi_w - \phi_{r_\alpha w}) - \operatorname{sgn}(w^{-1}\alpha) \phi_{r_\alpha w} \right] + (w\lambda, \xi) \phi_w.$$

Here, $\text{sgn}(\alpha) = 1$ if $\alpha \in \Sigma^+$, and $\text{sgn}(\alpha) = -1$ if $\alpha \in -\Sigma^+$. We get by taking complex conjugates,

$$\partial_{\xi} \bar{\phi}_w = -\frac{1}{2} \sum_{\alpha \in \Sigma^+} m_{\alpha} \alpha(\bar{\xi}) \left[\frac{1 + e^{-2\alpha(\bar{Z})}}{1 - e^{-2\alpha(\bar{Z})}} (\bar{\phi}_w - \bar{\phi}_{r_{\alpha} w}) - \text{sgn}(w^{-1}\alpha) \bar{\phi}_{r_{\alpha} w} \right] + (w\bar{\lambda}, \bar{\xi}) \bar{\phi}_w.$$

It follows that

$$\begin{aligned} & \partial_{\xi} \sum_w |\phi_w|^2 \\ &= \sum_w [(\partial_{\xi} \phi_w) \bar{\phi}_w + \phi_w (\partial_{\xi} \bar{\phi}_w)] \\ &= -\frac{1}{2} \sum_{\alpha \in \Sigma^+, w} [m_{\alpha} \alpha(\xi) \left(\frac{1 + e^{-2\alpha(Z)}}{1 - e^{-2\alpha(Z)}} (\phi_w - \phi_{r_{\alpha} w}) \bar{\phi}_w - \text{sgn}(w^{-1}\alpha) \phi_{r_{\alpha} w} \bar{\phi}_w \right) \\ & \quad + m_{\alpha} \alpha(\bar{\xi}) \left(\frac{1 + e^{-2\alpha(\bar{Z})}}{1 - e^{-2\alpha(\bar{Z})}} (\bar{\phi}_w - \bar{\phi}_{r_{\alpha} w}) \phi_w - \text{sgn}(w^{-1}\alpha) \bar{\phi}_{r_{\alpha} w} \phi_w \right)] \\ & \quad + 2 \sum_w \text{Re}(w\lambda(\xi)) |\phi_w|^2. \end{aligned}$$

For fixed α , we add the terms with index w and $r_{\alpha} w$. Then

$$\begin{aligned} & \partial_{\xi} \sum_w |\phi_w|^2 \\ &= -\frac{1}{4} \sum_{\alpha \in \Sigma^+, w} m_{\alpha} \left[\alpha(\xi) \frac{1 + e^{-2\alpha(Z)}}{1 - e^{-2\alpha(Z)}} + \alpha(\bar{\xi}) \frac{1 + e^{-2\alpha(\bar{Z})}}{1 - e^{-2\alpha(\bar{Z})}} \right] |\phi_w - \phi_{r_{\alpha} w}|^2 \\ & \quad + \sum_{\alpha \in \Sigma^+, w} m_{\alpha} \text{sgn}(w^{-1}\alpha) \text{Im}(\alpha(\xi)) \text{Im}(\bar{\phi}_w \phi_{r_{\alpha} w}) + 2 \sum_w \text{Re}(w\lambda(\xi)) |\phi_w|^2. \end{aligned}$$

Observe that

$$|1 - e^{-2\alpha(Z)}|^2 = (1 - e^{-2\alpha(Z)}) \overline{1 - e^{-2\alpha(Z)}} = (1 - e^{-2\alpha(Z)})(1 - e^{-2\alpha(\bar{Z})}),$$

which gives

$$\begin{aligned} & \alpha(\xi) \frac{1 + e^{-2\alpha(Z)}}{1 - e^{-2\alpha(Z)}} + \alpha(\bar{\xi}) \frac{1 + e^{-2\alpha(\bar{Z})}}{1 - e^{-2\alpha(\bar{Z})}} \\ &= \frac{\alpha(\xi)(1 + e^{-2\alpha(Z)})(1 - e^{-2\alpha(\bar{Z})}) + \alpha(\bar{\xi})(1 + e^{-2\alpha(\bar{Z})})(1 - e^{-2\alpha(Z)})}{|1 - e^{-2\alpha(Z)}|^2}. \end{aligned}$$

Let $\varepsilon > 0$ and write $Z = X + iY$ with $X, Y \in \mathfrak{a}$ and $|\alpha(Y)| \leq \pi - \varepsilon$, for all $\alpha \in \Sigma$. Let $\alpha(X) = t \in \mathbb{R}$ and $\alpha(Y) = s \in \mathbb{R}$. Then $\alpha(Z) = t + is$. We have

$$(1 + e^{-2\alpha(Z)})(1 - e^{-2\alpha(\bar{Z})}) = 1 - e^{-4t} - 2ie^{-2t} \sin(2s).$$

Similarly,

$$(1 + e^{-2\alpha(\bar{Z})})(1 - e^{-2\alpha(Z)}) = 1 - e^{-4t} + 2ie^{-2t} \sin(2s).$$

A simple calculation shows that

$$\begin{aligned} & \alpha(\xi)(1 + e^{-2\alpha(Z)})(1 - e^{-2\alpha(\bar{Z})}) + \alpha(\bar{\xi})(1 + e^{-2\alpha(\bar{Z})})(1 - e^{-2\alpha(Z)}) \\ &= 2\text{Re}(\alpha(\xi))(1 - e^{-4\alpha(X)}) + 4\text{Im}(\alpha(\xi))e^{-2\alpha(X)} \sin(2\alpha(Y)). \end{aligned}$$

Hence,

$$\begin{aligned} & \partial_\xi \sum_w |\phi_w|^2 \tag{A.2} \\ &= -\frac{1}{2} \sum_{\alpha \in \Sigma^+, w} m_\alpha \left[\frac{\operatorname{Re}(\alpha(\xi))(1 - e^{-4\alpha(X)}) + \frac{2\operatorname{Im}(\alpha(\xi))\sin(2\alpha(Y))}{e^{2\alpha(X)}}}{|1 - e^{-2\alpha(Z)}|^2} \right] |\phi_w - \phi_{r_\alpha w}|^2 \\ &+ \sum_{\alpha \in \Sigma^+, w} m_\alpha \operatorname{sgn}(w^{-1}\alpha) \operatorname{Im}(\alpha(\xi)) \operatorname{Im}(\bar{\phi}_w \phi_{r_\alpha w}) + 2 \sum_w \operatorname{Re}(w\lambda(\xi)) |\phi_w|^2. \end{aligned}$$

We first take $X, \xi \in \mathfrak{a}^{\text{reg}}$ such that they are in the same Weyl chamber. Let $\mu \in \{w\operatorname{Re}\lambda\}_{w \in W}$ be such that $\mu(\xi) = \max_w \operatorname{Re}(w\lambda)(\xi)$. Then $(w\operatorname{Re}\lambda - \mu)(\xi) \leq 0$. The formula (A.2) gives

$$\begin{aligned} & \partial_\xi (e^{-2\mu(X)} \sum_{w \in W} |\phi_w(\exp Z)|^2) \\ &= -\frac{1}{2} \sum_{\alpha \in \Sigma^+, w} m_\alpha \frac{\alpha(\xi)(1 - e^{-4\alpha(X)})}{|1 - e^{-2\alpha(Z)}|^2} |\phi_w - \phi_{r_\alpha w}|^2 e^{-2\mu(X)} \tag{A.3} \end{aligned}$$

$$+ 2 \sum_{w \in W} (w\operatorname{Re}\lambda - \mu)(\xi) |\phi_w|^2 e^{-2\mu(X)}, \tag{A.4}$$

Observe that the term (A.4) is clearly nonpositive. In the term (A.3), the factor

$$|\phi_w - \phi_{r_\alpha w}|^2 e^{-2\mu(X)} \geq 0.$$

We let $m_{\alpha/2} = 0$ if $\alpha/2$ is not a root. Consider

$$\begin{aligned} & \sum_{\alpha \in \Sigma^+, w} m_\alpha \frac{\alpha(\xi)(1 - e^{-4\alpha(X)})}{|1 - e^{-2\alpha(Z)}|^2} |\phi_w - \phi_{r_\alpha w}|^2 e^{-2\mu(X)} \\ &= \sum_{\alpha \in \Sigma_*^+, w} \left[m_\alpha \frac{\alpha(\xi)(1 - e^{-4\alpha(X)})}{|1 - e^{-2\alpha(Z)}|^2} + m_{\alpha/2} \frac{\frac{1}{2}\alpha(\xi)(1 - e^{-2\alpha(X)})}{|1 - e^{-\alpha(Z)}|^2} \right] |\phi_w - \phi_{r_\alpha w}|^2 e^{-2\mu(X)} \\ &= \sum_{\alpha \in \Sigma_*^+, w} \alpha(\xi) \frac{1 - e^{-2\alpha(X)}}{|1 - e^{-\alpha(Z)}|^2} \left[m_\alpha \frac{1 + e^{-2\alpha(X)}}{|1 + e^{-\alpha(Z)}|^2} + \frac{1}{2} m_{\alpha/2} \right] |\phi_w - \phi_{r_\alpha w}|^2 e^{-2\mu(X)}. \tag{A.5} \end{aligned}$$

Since X, ξ are in the same Weyl chamber, $\alpha(\xi)(1 - e^{-2\alpha(X)}) \geq 0$ for all $\alpha \in \Sigma^+$. Since $m_{\alpha/2} \geq -m_\alpha$ and $m_\alpha \geq 0$ for all $\alpha \in \Sigma_*^+$, then

$$m_\alpha \frac{1 + e^{-2\alpha(X)}}{|1 + e^{-\alpha(Z)}|^2} + \frac{1}{2} m_{\alpha/2} \geq m_\alpha \frac{1 + e^{-2\alpha(X)}}{|1 + e^{-\alpha(Z)}|^2} - \frac{1}{2} m_\alpha = m_\alpha \left[\frac{1 + e^{-2\alpha(X)}}{|1 + e^{-\alpha(Z)}|^2} - \frac{1}{2} \right] \geq 0.$$

The reason is as follows:

$$\frac{1 + e^{-2\alpha(X)}}{|1 + e^{-\alpha(Z)}|^2} - \frac{1}{2} = \frac{2(1 + e^{-2t}) - |1 + e^{-t}e^{-is}|^2}{2|1 + e^{-t}e^{-is}|^2} \geq 0$$

if and only if the numerator is nonnegative as follows from

$$\begin{aligned} 2(1 + e^{-2t}) - |1 + e^{-t}e^{-is}|^2 &= 1 + e^{-2t} - 2e^{-t} \cos(s/2) \\ &\geq 1 + e^{-2t} - 2e^{-t} \\ &= (1 - e^{-t})^2 \geq 0. \end{aligned}$$

It follows that (A.5) is nonnegative. Thus the term (A.3) is nonpositive and hence

$$\partial_\xi (e^{-2\mu(X)} \sum_{w \in W} |\phi_w(\exp Z)|^2) \leq 0.$$

This implies

$$\begin{aligned} e^{-2 \max_w \operatorname{Re}(w\lambda(X))} \sum_w |\phi_w(\exp Z)|^2 &\leq e^{-2 \max_w \operatorname{Re}(w\lambda(0))} \sum_w |\phi_w(\exp(0 + iY))|^2 \\ &= \sum_w |\phi_w(\exp(iY))|^2 \end{aligned}$$

if $X \in \mathfrak{a}^{\text{reg}}$, and by continuity this holds for all $X \in \mathfrak{a}$. Note that

$$|\phi_e(\exp Z)| = |G(\lambda, m, e^{-1}Z)| = |G(\lambda, m, Z)|$$

and $|\phi_e(\exp Z)|^2 \leq \sum_w |\phi_w(\exp Z)|^2$ which implies $|\phi_e(\exp Z)| \leq (\sum_w |\phi_w(\exp Z)|^2)^{1/2}$. Hence, we have

$$|G(\lambda, m, X + iY)| \leq e^{\max_w \operatorname{Re}(w\lambda(X))} \left(\sum_w |\phi_w(\exp(iY))|^2 \right)^{1/2}. \quad (\text{A.6})$$

Substituting $Y = 0$ yields

$$|G(\lambda, m, X)| \leq |W|^{1/2} e^{\max_w \operatorname{Re}(w\lambda(X))},$$

where we use the fact that $G(\lambda, m, 0) = 1$ (cf. [26, Thm. 3.15]).

Next we take $\eta, Y \in \mathfrak{a}^{\text{reg}}$ such that η and Y belongs to the same Well chamber and $|\alpha(Y)| \leq \pi - \varepsilon$ for all $\alpha \in \Sigma$. Let $\xi = i\eta$. Then

$$\operatorname{Re}(w\lambda(\xi)) = -\operatorname{Im}(w\lambda(\eta)) \quad \text{and} \quad \operatorname{Im}(\alpha(\xi)) = \operatorname{Re}(\alpha(\eta)).$$

Take $\mu \in \{w\operatorname{Im}\lambda\}_{w \in W}$ such that $-\operatorname{Im}(w\lambda(\eta)) \leq -\mu(\eta)$ for all $w \in W$. This is to say, $\mu = \min_w \operatorname{Im}(w\lambda)$. Observe that

$$\sum_{\alpha \in \Sigma^+} |m_\alpha| |\alpha(\eta)| \leq \max_w \sum_{\alpha \in \Sigma^+} |m_\alpha| \alpha(w\eta) = 2 \max_w (w\tilde{\rho}, \eta).$$

We have

$$\begin{aligned} \left| \sum_{\alpha \in \Sigma^+, w} m_\alpha \operatorname{sgn}(w^{-1}\alpha) \operatorname{Im}(\alpha(\xi)) \operatorname{Im}(\overline{\phi_w} \phi_{r_\alpha w}) \right| &\leq \sum_{\alpha \in \Sigma^+, w} |m_\alpha| |\alpha(\eta)| |\phi_w| |\phi_{r_\alpha w}| \\ &\leq 2 \max_w (w\tilde{\rho}, \eta) \sum_w |\phi_w|^2. \end{aligned}$$

Choose $\nu \in \{w\tilde{\rho}\}_{w \in W}$ such that $(\nu, \eta) = \max_w (w\tilde{\rho}, \eta)$. Let $C > 2$ be a constant to be determined and let

$$H(iY) = e^{2\mu(Y)} e^{-C\nu(Y)} \sum_w |\phi_w(\exp(iY))|^2.$$

Now using the formula (A.2) we obtain

$$\begin{aligned} (\partial_\xi H)(iY) &= - \sum_{\alpha \in \Sigma^+, w} m_\alpha \frac{\alpha(\eta) \sin 2\alpha(Y)}{|1 - e^{-2\alpha(iY)}|^2} |\phi_w - \phi_{r_\alpha w}|^2 \cdot e^{(2\mu - C\nu)(Y)} \quad (\text{I}) \end{aligned}$$

$$- (C - 2)(\nu, \eta) \sum_w |\phi_w|^2 e^{(2\mu - C\nu)(Y)} \quad (\text{II})$$

$$+ \left[\sum_{\alpha \in \Sigma^+, w} m_\alpha \operatorname{sgn}(w^{-1}\alpha) \operatorname{Im}(\alpha(\xi)) \operatorname{Im}(\overline{\phi_w} \phi_{r_\alpha w}) - 2(\nu, \eta) \sum_w |\phi_w|^2 \right] e^{(2\mu - C\nu)(Y)} \quad (\text{III})$$

$$+ \left[2 \sum_w (\mu - w\operatorname{Im}\lambda)(\eta) |\phi_w|^2 \right] e^{(2\mu - C\nu)(Y)}, \quad (\text{IV})$$

where (III) and (IV) are clearly nonpositive. In the original proof in [26] $\alpha(\eta) \sin 2\alpha(Y) \geq 0$ and $m_\alpha \geq 0$ for all $\alpha \in \Sigma^+$. Hence (I) is nonpositive. Then the author takes $C = 2$ to let (II) vanish. In fact any $C \geq 2$ would be good enough. Our assumption allows for $\pi \leq |2\alpha(Y)| \leq 2\pi - 2\epsilon$ which would imply that $\alpha(\eta) \sin 2\alpha(Y) \leq 0$ and so (I) ≥ 0 . Similar problems arise since m_α might be nonpositive for some $\alpha \in \Sigma^+$. Thus the rest of the proof differs from the proof in [26] by grouping (I) and (II) together and then choosing the constant C such that the sum becomes nonpositive.

As earlier we set $m_{\alpha/2} = 0$ if $\alpha/2$ is not a root. For (I) observe that

$$\begin{aligned} & \sum_{\alpha \in \Sigma^+, w \in W} m_\alpha \frac{\alpha(\eta) \sin 2\alpha(Y)}{|1 - e^{-2\alpha(iY)}|^2} |\phi_w - \phi_{r_\alpha w}|^2 \cdot e^{(2\mu - C\nu)(Y)} \\ &= \sum_{\alpha \in \Sigma_*^+, w} \left[m_\alpha \frac{\alpha(\eta) \sin 2\alpha(Y)}{|1 - e^{-2\alpha(iY)}|^2} + \frac{1}{2} m_{\alpha/2} \frac{\alpha(\eta) \sin(\alpha(Y))}{|1 - e^{-\alpha(iY)}|^2} \right] |\phi_w - \phi_{r_\alpha w}|^2 \cdot e^{(2\mu - C\nu)(Y)}. \end{aligned}$$

Fix a root $\alpha \in \Sigma_*^+$. Let $s_\alpha = s := \alpha(Y)$. Then

$$|1 - e^{-2\alpha(iY)}|^2 = 4 \sin^2(s).$$

Using that $m_\alpha + m_{\alpha/2} \geq 0$ and $m_\alpha \geq 0$ for all $\alpha \in \Sigma_*^+$ we get:

$$\begin{aligned} m_\alpha \frac{\alpha(\eta) \sin(2s)}{|1 - e^{-2is}|^2} + \frac{m_{\alpha/2} \alpha(\eta) \sin(s)}{2 |1 - e^{-is}|^2} &\geq m_\alpha \frac{\alpha(\eta) \sin(2s)}{4 \sin^2(s)} - \frac{1}{2} m_\alpha \frac{\alpha(\eta) \sin(s)}{4 \sin^2(s/2)} \\ &= \frac{m_\alpha \alpha(\eta)}{4} \left[\frac{\sin(2s)}{\sin^2(s)} - \frac{1}{2} \frac{\sin(s)}{\sin^2(s/2)} \right] \\ &= -\frac{m_\alpha \alpha(\eta)}{4} \tan(s/2), \end{aligned}$$

where we use the formulas $\sin(2s) = 2 \sin(s) \cos(s)$ and $\cos(2s) = \cos^2(s) - \sin^2(s)$. Since η, Y belong to the same chamber, we see that $\alpha(\eta) \tan(s/2) \geq 0$. Since $|s| = |\alpha(Y)| \leq \pi - \epsilon$, then there is a constant $C_1 = C_1(\epsilon) > 0$ depending on ϵ such that

$$\max_{\alpha \in \Sigma_*^+} |\tan(s_\alpha/2)| \leq C_1.$$

Also,

$$\sum_{w \in W} |\phi_w - \phi_{r_\alpha w}|^2 \leq \sum_{w \in W} (|\phi_w| + |\phi_{r_\alpha w}|)^2 = \sum_{w \in W} 4 |\phi_w|^2.$$

It follows that

$$\begin{aligned} \text{(I)} &\leq C_1 \sum_{\alpha \in \Sigma_*^+, w} m_\alpha |\alpha(\eta)| |\phi_w|^2 e^{(2\mu - C\nu)(Y)} \\ &\leq 2C_1 \max_w(w\tilde{\rho}, \eta) \sum_w |\phi_w|^2 e^{(2\mu - C\nu)(Y)}. \end{aligned}$$

Hence (I)+(II) ≤ 0 if we take $C \geq 2 + 2C_1$ (C thus depends on ϵ). It follows that $(\partial_\xi H)(iY) \leq 0$. Therefore, $H(iY) \leq H(0) = |W|$. So

$$\sum_w |\phi_w(\exp(iY))|^2 \leq |W| e^{(C\nu - 2\mu)(Y)}.$$

Together with (A.6), we get

$$|G(\lambda, m, Z)| \leq |W|^{\frac{1}{2}} \exp\left(-\min_{w \in W} \text{Im}(w\lambda(Y))\right) + \frac{C}{2} \max_{w \in W} w\tilde{\rho}(Y) + \max_{w \in W} \text{Re}(w\lambda(X)). \quad (\text{A.7})$$

The estimate (A.7) for $G(\lambda, m, Z)$ is Weyl group invariant and it thus holds for all terms that are summed up for $F(\lambda, m; \exp Z)$ (cf. (A.1)). We have therefore proved the desired estimate for F . \square

Since $\tilde{\rho}(\cdot)$ (as $\tilde{\rho}$ is independent of λ) and $|W|$ are constants, we restate Proposition 29 as

Proposition 30. *Let $m \in \mathcal{M}_{\geq}$. Let $\varepsilon > 0$. Then there exists a constant $C = C_{\varepsilon}$ such that*

$$|F(\lambda, m; \exp(X + iY))| \leq C \exp(\max_{w \in W} \operatorname{Re} w\lambda(X) - \min_{w \in W} \operatorname{Im} w\lambda(Y)),$$

for all $X \in \Omega_{\varepsilon}$, $Y \in \mathfrak{b}$, and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

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