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Recommended Citation

Christensen, J., Darweesh, A., & Ólafsson, G. (2019). Coorbits for projective representations with an application to Bergman spaces. *Monatshefte fur Mathematik, 189* (3), 385-420. https://doi.org/10.1007/s00605-019-01296-4

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arXiv:1704.02522v2 [math.FA] 26 Mar 2019

COORBITS FOR PROJECTIVE REPRESENTATIONS WITH AN APPLICATION TO BERGMAN SPACES

JENS GERLACH CHRISTENSEN, AMER H. DARWEESH, AND GESTUR ÓLAFSSON

Abstract. Representation theory of locally compact topological groups is a powerful tool to analyze Banach spaces of functions and distributions. It provides a unified framework for constructing function spaces and to study several generalizations of the wavelet transform. Recently representation theory has been used to provide atomic decompositions for a large collection of classical Banach spaces. But in some natural situations, including Bergman spaces on bounded domains, representations are too restrictive. The proper tools are projective representations. In this paper we extend known techniques from representation theory to also include projective representations. This leads naturally to twisted convolution on groups avoiding the usual central extension of the group. As our main application we obtain atomic decompositions of Bergman spaces on the unit ball through the holomorphic discrete series for the group of isometries of the ball.

1. Introduction

With the rise of continuous wavelet theory it was discovered that representation theory could be used to obtain atomic decompositions for some classical Banach spaces. This area of harmonic analysis is called coorbit theory and it was initiated by Feichtinger and Gröchenig [21, 22, 23, 25]. Several interesting generalizations were later presented in [9, 35, 24, 36, 13, 14, 15, 12]. All these examples use irreducible integrable representations in order to construct atomic decompositions. This allows one to choose atoms in an appropriate minimal Banach space. As has been remarked recently, assuming integrability and irreducibility is not needed, and in fact often the restriction of irreducibility and integrability as well as the criteria for selecting atoms turns out to be too restrictive. Therefore, the first and last author suggested the use of Fréchet spaces [7, 4] in coorbit theory, and recently the idea has been used in several cases [6, 8, 5, 11].

There are many situations in which projective representations arise more naturally than representations. Typical examples are the modulation spaces and the short time Fourier transform which stems from modulation and translation, as well as the holomorphic discrete series representations on Bergman spaces on bounded symmetric domains. The idea of using projective representations in coorbit theory has earlier been explored in [9] under the assumptions of irreducibility, integrability, and continuity of the multiplier. As mentioned, our aim is to present and apply a coorbit theory without these restrictions. The first two restrictions were removed in the thesis of the second author [16]. The continuity assumption used in both [9]

The research was partially supported by NSF grant DMS 1321794. The first and last named authors would also like to thank AMS for it's support during the MRC program *Lie Group Representations, Discretization, and Gelfand Pairs* June 5–June 11, 2016 .

and [16] means that those approaches apply only to some special cases like Abelian groups or simply connected groups. For many simply connected groups this creates new obstacles due to them having infinite center. As a consequence, the papers [9] and [16] cannot be used to describe Bergman spaces on the unit ball in $\mathbb{C}^n$ as coorbits for the group $\mathrm{SU}(n,1)$, or more generally, Bergman spaces on bounded domains in $\mathbb{C}^n$.

This is also the reason that we had to work with finite covering groups of $\mathrm{SU}(n,1)$ and require rationality of the representation parameter in [5]. Bargmann and Mackey have shown that for general locally compact groups the multiplier can be chosen continuous in a neighborhood of the identity, and for Lie groups the multiplier can even be chosen smooth in such a neighborhood. In this paper we show that these facts are sufficient for obtaining a working coorbit theory for projective representations. Furthermore, we demonstrate the benefits of the theory by applying it to the case of Bergman spaces on the unit ball. This finishes the work initiated in [5], since it allows us to remove the rationality restriction on the representation parameter, and thereby we can provide atomic decompositions for the entire scale of Bergman spaces. The approach extends to Bergman spaces on general bounded symmetric domains, which will appear in a forthcoming paper by the first and last authors.

We would like to also point out another difference between the present paper and [9] and [16]. In those papers the atomic decompositions are obtained by applying results in [22] and [4], respectively, to the Mackey obstruction group (which will be introduced in the next section). In this paper we avoid this difficulty and work directly on the group. Here the reader should keep in mind the modulation spaces of Feichtinger [18, 20] (see also the book [27]). Those spaces arise from translation and modulation on functions or distributions on $\mathbb{R}^n$. But those two actions do not commute and lead in a natural way to the Schrödingar representation of the reduced Heisenberg group. But the theory is usually carried out without any mention of the compact center of the reduced Heiseinberg group, and only the action of $\mathbb{R}^{2n}$, the corresponding cocylce $e^{-ix\cdot y}$, and the twisted convolution are used. For modulation spaces on general Abelian groups see [19, 26].

2. Projective Representations

In this section we review known results on measurable and locally continuous projective representations, and, following Mackey, we will construct a representation of an extension of a locally compact second countable group from a given projective representation. We use [38] as a standard reference even if the results are mostly due to Mackey and Bargmann.

We assume that G is a locally compact second countable group equipped with a fixed left invariant Haar measure which we denote by dx. We denote by $\mathbb{T} := \{t \in \mathbb{C} \mid |t| = 1\}$ the one dimensional torus with normalized Haar measure dt.

Definition 2.1. Let $\mathcal{S}$ be a locally convex Hausdorff topological vector space over $\mathbb{C}$, and denote by $\mathcal{S}^*$ its conjugate dual. A projective representation of G is a mapping $\rho : G \to \mathrm{GL}(\mathcal{S})$, the space of continuous linear maps $\mathcal{S} \to \mathcal{S}$ with a continuous inverse, that satisfies the following three conditions:

(1) $\rho(1) = \mathrm{id}$.

(2) There is a Borel function (called a *multiplier* or a *cocycle*) $\sigma : G \times G \to \mathbb{T}$, which satisfies the condition
$$\rho(xy) = \sigma(x, y)\rho(x)\rho(y).$$
(3) For every $v \in \mathcal{S}$ and every $\lambda \in S^*$ the mapping
$$x \mapsto \langle \lambda, \rho(x)v\rangle := \lambda(\rho(x)v)$$
is a Borel function.

If there is a neighbourhood around e on which the function $x \mapsto \langle \lambda, \rho(x)v\rangle$ is continuous for all $\lambda \in \mathcal{S}^*$, we say that ρ is locally weakly continuous. A projective representation with multiplier σ is said to be a *σ-representation.* We call the multiplier σ locally continuous if it is continuous on a neighbourhood of $e \times e$.

Notice, that for a σ-representation ρ and $x \in G$ we have
$$\rho(x^{-1}) = \overline{\sigma(x, x^{-1})}\rho(x)^{-1} = \overline{\sigma(x^{-1}, x)}\rho(x)^{-1}\,. \tag{2.1}$$
Let $x, y, z \in G$. The following are straightforward consequences about the cocycle σ:

(1) $\sigma(x, 1) = \sigma(1, x) = 1$,
(2) $\sigma(x, y)^{-1} = \overline{\sigma(x, y)}$,
(3) $\sigma(x, x^{-1}) = \sigma(x^{-1}, x)$,
(4) $\sigma(xy, z)\sigma(x, y) = \sigma(x, yz)\sigma(y, z)$.

Following Bargmann, we say that two cocycles σ and τ are *similar* if there exists a Borel function $a : G \to \mathbb{T}$ such that
$$\tau(x, y) = \frac{a(xy)}{a(x)a(y)}\sigma(x, y). \tag{2.2}$$
We note that if τ is similar to σ via the Borel function a and ρ is a σ-representation, then $\eta(x) := a(x)\rho(x)$ is a τ-representations.

Theorem 2.2. *Every multiplier is similar to a multiplier which is continuous on some open neighborhood of $(e, e) \in G\times G$. If G is a Lie group, then every multiplier is similar to a multiplier that is smooth in an open neighborhood of (e, e). If G is a connected and simply connected Lie group, then every multiplier is similar to a multiplier which is analytic on the entire group $G \times G$.*

Proof. The first statement is [38, Corollary 7.6]. The second statement is [38, Lemma 7.20] and the last segment is [38, Corollary 7.30]. □

For projective representations we define irreducibility, cyclicity, admissible vectors, unitarity and square integrability in the same way as for representations. We summarize these notions in the following definition.

Definition 2.3. Let $(\rho, \mathcal{S})$ be a projective representation, then

(1) A subspace W of $\mathcal{S}$ is ρ-invariant if $\rho(x)W \subseteq W$ for all $x \in G$.
(2) $(\rho, \mathcal{S})$ is irreducible if the only closed ρ-invariant subspaces are $\{0\}$ and $\mathcal{S}$ itself.
(3) A vector $u \in \mathcal{S}$ is a ρ-cyclic if $\operatorname{span}\{\rho(x)u \mid x \in G\}$ is dense in $\mathcal{S}$. If such a cyclic vector exists, we say that $(\rho, \mathcal{S})$ is a cyclic projective representation.
(4) $(\rho, \mathcal{H})$ is unitary if $\mathcal{H}$ is a Hilbert space, and $\rho(x)$ is a unitary operator for every $x \in G$.

(5) If $\mathcal{H}$ is a Hilbert space and $(\rho, \mathcal{H})$ is irreducible and unitary it is called square-integrable if there is a nonzero vector $u \in \mathcal{H}$ such that

$$\int_G |\langle u, \rho(x)u\rangle|^2 \, dx < \infty.$$

In this case u is called a *ρ-admissible vector*.

In the following lemma we define the dual projective representation on the conjugate dual of a Fréchet space.

Let $\mathcal{S}$ be a Fréchet space and denote by $\mathcal{S}^*$ be the conjugate dual of $\mathcal{S}$ equipped with the weak*-topology. This implies that $\mathcal{S}^*$ is a locally convex vector space and $(\mathcal{S}^*)^* = \mathcal{S}$.

Lemma 2.4. *Let $(\rho, \mathcal{S})$ be a σ-representation of G on a Fréchet space $\mathcal{S}$, and let $\mathcal{S}^*$ be the conjugate dual of $\mathcal{S}$ equipped with the weak*-topology. The mapping ρ^*, which is given by*

$$\langle \rho^*(x)\lambda, v\rangle := \langle \lambda, \rho(x)^{-1}v\rangle$$

for all $\lambda \in \mathcal{S}^*$ and all $v \in \mathcal{S}$, defines a σ-representation of G on the space $\mathcal{S}^*$. Finally $x \mapsto \langle \rho^*(x)\lambda, u\rangle$ is continuous around e if σ is locally continuous and ρ is locally weakly continuous.

Proof. As $\mathcal{S}^*$ is equipped with the weak* topology, its conjugate dual is $\mathcal{S}$. We denote this dual pairing by $\langle\langle \cdot, \cdot \rangle\rangle$. For $v \in (\mathcal{S}^*)^* = \mathcal{S}$ and $\lambda \in \mathcal{S}^*$ we have $\langle\langle v, \lambda\rangle\rangle = \overline{\langle \lambda, v\rangle}$. Therefore the following calculation shows that ρ^* has cocycle σ

$$\begin{aligned}
\langle \rho^*(xy)\lambda, v\rangle &= \langle \lambda, \rho(xy)^{-1}v\rangle \\
&= \langle \lambda, (\sigma(x,y)\rho(x)\rho(y))^{-1}v\rangle \\
&= \langle \lambda, \overline{\sigma(x,y)}\rho(y)^{-1}\rho(x)^{-1}v\rangle \\
&= \langle \sigma(x,y)\lambda, \rho(y)^{-1}\rho(x)^{-1}v\rangle \\
&= \langle \sigma(x,y)\rho^*(x)\rho^*(y)\lambda, v\rangle
\end{aligned}$$

Hence, $\rho^*(xy) = \sigma(x,y)\rho^*(x)\rho^*(y)$. Furthermore, the equation (2.1) implies that

$$\langle\langle v, \rho^*(x)\lambda\rangle\rangle = \overline{\langle \rho^*(x)\lambda, v\rangle} = \overline{\langle \lambda, \rho(x)^{-1}u\rangle} = \sigma(x^{-1}, x)\overline{\langle \lambda, \rho(x^{-1})u\rangle}\,.$$

Since $x \mapsto x^{-1}$ is continuous and σ and ρ are Borel it follows that $x \mapsto \langle\langle v, \rho^*(x)\lambda\rangle\rangle$ is Borel. Moreover, the mapping is locally continuous if σ locally continuous and ρ is locally weakly continous. □

For any projective representation ρ of G we can construct an actual representation of a new group related to G which is called the Mackey obstruction group of G (see p. 269f in [32]). We refer to [38, Chap. VII] for a detailed discussion.

We first gather some facts about the Mackey group. Let σ be a cocycle on $G \times G$. As a set, the Mackey group that corresponds to G is the group $G_\sigma := G \times \mathbb{T}$, with multiplication given by

$$(x,t)(y,s) = (xy, \overline{\sigma(x,y)}\; ts)\,.$$

The inverse of $(x,t) \in G_\sigma$ is given by

$$(x,t)^{-1} = (x^{-1}, \sigma(x, x^{-1})\bar{t}) = (x^{-1}, \sigma(x^{-1}, x)\bar{t}).$$

Note that this is a *central* extension of $\mathbb{T}$ by G as all the elements (e, s), $s \in \mathbb{T}$, are central in G_σ. The product of the Borel algebras of G and $\mathbb{T}$ defines a σ-algebra on G_σ and the product measure $dxdt$ is left invariant.

Theorem 2.5. *Let the notation be as above. Then the following holds:*

1. *There exists a unique topology on G_σ, called the Weyl topology, that generates the product σ-algebra and at the same time makes G_σ into a locally compact Hausdorff topological group.*
2. *If the multiplier σ is continuous around (e,e), then there exists a neighborhood U of e in G such that the Weyl topology on $U \times \mathbb{T}$ corresponds to the product topology on $U \times \mathbb{T}$.*
3. *Two extensions G_σ and G_τ are isomorphic if and only if the multipliers σ and τ are similar. If the similarity is given by the function a as in (2.2) then the isomorphism $G_\sigma \to G_\tau$ is given be $(x,t) \mapsto (x, \overline{a(x)}t)$.*
4. *If G is a connected Lie group, then there exists a unique analytic structure on G_σ compatible with the Weyl topology. The maps $t \to (e,t)$ and $(x,t) \to x$ are analytic.*
5. *There exists a smooth, respectively analytic, multiplier τ similar to σ if and only if there exists a smooth, respectively analytic, map $\kappa : G \to G_\sigma$ such that $p_1(\kappa(x)) = x$ where $p_1(x,t) = x$.*

Proof. (1) and (3) are [38, Theorem 7.8]. (2) is [38, Corollary 7.10]. (4) is [38, Theorem 7.21]. Finally, (5) is [38, Corollary 7.23]. □

Next we discuss the construction of a representation of G_σ from a σ-representation ρ. Define

$$\rho_\sigma(x,t) := t\rho(x)\,.$$

Then a simple calculation shows that ρ_σ is a homomorphism. If τ is similar to σ, $\eta(x) = a(x)\rho(x)$ is the corresponding canonical τ-representation and $\varphi(x,t) = (x, \overline{a(x)}t)$ is the natural isomorphism $G_\sigma \simeq G_\tau$, then

$$\eta_\tau(\varphi(x,t)) = \overline{a(x)}t\eta(x) = \overline{a(x)}a(x)t\rho(x) = \rho_\sigma(x,t)$$

as $a(x) \in \mathbb{T}$. Hence $\eta_\tau \circ \varphi = \rho_\sigma$.

The following is well known for representations on a separable Hilbert space, see [38, Theorem 7.16].

Theorem 2.6. *Assume that the multiplier σ is continuous in a neighborhood of (e,e), and that G is connected. Then*

$$G_\sigma \to \mathbb{C}\,, \quad (x,t) \mapsto \langle \lambda, \rho_\sigma(x,t)u \rangle = \overline{t}\langle \lambda, \rho(x)u \rangle \tag{2.3}$$

is a Borel function for all $\lambda \in \mathcal{S}^*$ and $u \in \mathcal{S}$. Furthermore the following holds:

1. *The map in (2.3) is Borel.*
2. *The map in (2.3) is continuous if and only if $G \to \mathbb{C}$, $x \mapsto \langle \lambda, \rho(x)u \rangle$ is continuous in an open neighborhood around e.*
3. *Assume that G is a Lie group. The map in (2.3) is smooth if and only if $G \to \mathbb{C}$, $x \mapsto \langle \lambda, \rho(x)u \rangle$ is smooth in an open neighborhood around e.*

Proof. The first statement follows from the fact that it is the product of the two Borel maps

$$(x,t) \mapsto \overline{t} \qquad \text{and} \qquad (x,t) \mapsto \langle \lambda, \rho(x)u \rangle,$$

and that the Borel σ-algebra on G_σ is the product of the Borel algebras of G and $\mathbb{T}$.

If σ is continuous around (e,e) there exists, according to Theorem 2.5, part (2), an e-neighborhood U in G such that the Weyl topology on $U \times \mathbb{T}$ agrees with the product topology. Thus the map in (2.3) is continuous on $U \times \mathbb{T}$. Let $z = (y, s) \in G_\sigma$. Then $U \times \mathbb{T}(y, 1)$ is a neighborhood around z and for $(x, t) \in U \times \mathbb{T}$ we have

$$(x, t) \mapsto \lambda(\rho_\sigma((x, t)(y, 1))u) = \lambda(\rho_\sigma((x, t))\rho_\sigma((y, 1))u)$$

which is continous in (x, t). Hence the map in (2.3) is continuous. Let us check the opposite direction and assume that the mapping (2.3) is continuous. Restrict the mapping to a neighbourhood $U \times \mathbb{T}$ in the Weyl topology for which U is open in G. The mapping from U to $U \times \mathbb{T}$ given by $x \to (x, 1)$ is then continuous, and therefore $x \mapsto \langle \lambda, \rho_\sigma(x, 1)u\rangle = \langle \lambda, \rho(x)u\rangle$ is continuous on the neighbourhood U.

Smoothness is verified in the same manner. □

Assumption 2.7. From now on we will assume that multipliers are continuous in a neighborhood of (e, e). Moreover, we will assume that for every $\lambda \in \mathcal{S}^*$ and every $u \in \mathcal{S}$, there is a neighbourhood U of e on which the mapping $x \mapsto \langle \lambda, \rho(x)u\rangle$ is continuous. Finally the group G is assumed to be connected.

The following examples show that there are plenty of examples for which this assumption is satisfied.

Example 2.8. If $\mathcal{H}$ is a Hilbert space and ρ is a unitary projective representation, then Corollary 7.10 in [38] ensures that there is a neighbourhood U on which $x \mapsto \langle \lambda, \rho(x)u\rangle$ is continuous.

We also note that the representation ρ_σ is unitary if and only if the σ-projective representation ρ is unitary. Finally, the σ-representation ρ is square-integrable if and only if ρ_σ-is square integrable. This last statement follows from the fact that $(u, \rho_\sigma(x, t)u) = \bar{t}(u, \rho(x)u)$ and that $\mathbb{T}$ is compact with measure 1. Thus $\int_{G_\sigma} |(u, \rho_\sigma(x, t)u)|^2 \, dxdt = \int_G |(u, \rho(x)u)|^2 \, dx$

We now provide a version of the Duflo-Moore theorem for square integrable projective representations. This result can be found in [1].

Theorem 2.9. *Let $(\rho, \mathcal{H})$ be a square-integrable projective representation of G.*

(1) *There exists a positive self adjoint operator A_ρ which is defined on a dense subset D of $\mathcal{H}$, such that $u \in \mathcal{H}$ is ρ-admissible if and only if $u \in D$. Moreover, the orthogonality relation*

$$\int_G (v_1, \rho(x)u_1) \, (\rho(x)u_2, v_2) \, dx = (A_\rho u_2, A_\rho u_1) \, (v_1, v_2)$$

holds for all $u_1, u_2 \in D$ and $v_1, v_2 \in \mathcal{H}$.

(2) *In addition, if G is a unimodular, then $D = \mathcal{H}$ and $A_\rho = c_\rho Id_{\mathcal{H}}$. Thus, all vectors of $\mathcal{H}$ are ρ-admissible and*

$$\int_G (v_1, \rho(x)u_1) \, (\rho(x)u_2, v_2) \, dx = c_\rho^2 (u_2, u_1) \, (v_1, v_2)$$

for all $u_1, u_2, v_1, v_2 \in \mathcal{H}$. The constant $1/c_\rho^2$ is called the formal dimension of ρ.

Example 2.10. We will now show that for a specific Gelfand triple $(\mathcal{S}, \mathcal{H}, \mathcal{S}^*)$ from the original coorbit theory [21, 22, 23, 9] the weak continuity requirement from Assumption 2.7 is automatically satisfied.

Let $(\rho, \mathcal{H})$ be a unitary σ-representation of G, and let ρ_σ be the Mackey representation of G_σ. Then $(\rho_\sigma, \mathcal{H})$ is a (strongly) continuous representation (follows from Example 2.8 and Theorem 2.6). Moreover, assume that ρ is an irreducible square integrable projective representation and assume there is a non-zero u for which $(u, \rho(\cdot)u)$ is in $L^1_w(G)$ for some submultiplicative weight $w \geq 1$, see Remark 3.2 bellow for definition. Let $\mathcal{S}$ be the Banach space

$$\mathcal{S} = \mathcal{H}^1_w = \{v \in \mathcal{H} \mid (v, \rho(\cdot)u) \in L^1_w(G)\}$$

equipped with the norm it inherits from $L^1_w(G)$. It is clear that for a cocycle σ for ρ the space $\mathcal{H}^1_w$ is isometrically isomorphic to the space

$$\{v \in \mathcal{H} \mid (v, \rho_\sigma(\cdot)u) \in L^1_w(G_\sigma)\}$$

when $w(x,t) = w(x)$.

From standard coorbit theory [22] the representation ρ_σ restricted to H^1_w is strongly continous (because left translation is continous on $L^1_w(G_\sigma)$). Therefore it follows immediately that $(x,t) \mapsto \langle \lambda, \rho_\sigma(x,t)v\rangle$ is continuous on G_σ for $\lambda \in (H^1_w)^*$ and $v \in H^1_w$. Then from Theorem 2.6 it follows that $x \mapsto \langle \lambda, \rho(x)v\rangle$ is continuous on a neighbourhood of e.

Example 2.11. Assume that G is a Lie group. We will show that the smooth vectors for a projective representation satisfy the weak continuity from Assumption 2.7.

The smooth vectors for the projective representation ρ on the Hilbert space $\mathcal{H}$ is the collection of vectors u for which $G \ni x \mapsto \rho(x)u \in \mathcal{H}$ is C^∞ on a neighbourhood of e. By arguments similar to those in [34] this is equivalent to the weak smoothness of the mapping $x \mapsto (v, \rho(x)u)$ for any $v \in \mathcal{H}$ on a neighbourhood of e. By Theorem 2.6 we then get that $(x,t) \mapsto (v, \rho_\sigma(x,t)u)$ is smooth, which tells us that u is a smooth vector for ρ_σ by [34]. This argument also works in the reverse direction, so we see that the smooth vectors for ρ are the same as the smooth vectors for ρ_σ, i.e. $\mathcal{H}^\infty_\rho = \mathcal{H}^\infty_{\rho_\sigma}$. Since $\rho_\sigma(x,t)u = t\rho(x)u$ the derivatives in t are just multiples of the identity. Therefore the usual Fréchet space topologies on $\mathcal{H}^\infty_\rho$ and $\mathcal{H}^\infty_{\rho_\sigma}$ are generated by the same differential operators, and therefore their topologies are equivalent. Therefore they also have the same dual spaces $\mathcal{H}^{-\infty}_\rho = \mathcal{H}^{-\infty}_{\rho_\sigma}$.

It is classical, see [40], that ρ_σ restricted to $\mathcal{H}_{\rho_\sigma}$ is a continuous representation. Therefore $(x,t) \mapsto \langle \lambda, \rho_\sigma(x,t)u\rangle$ is continuous on G_σ for $\lambda \in \mathcal{H}^{-\infty}_{\rho_\sigma}$ and $u \in \mathcal{H}^\infty_{\rho_\sigma}$. Therefore Theorem 2.6 ensures that $x \mapsto \langle \lambda, \rho(x)u\rangle$ is continuous on a neighbourhood of e.

Let $u \in \mathcal{S}$ be ρ-cyclic vector. We define the wavelet transform $W^\rho_u : \mathcal{S}^* \to C(G)$ by

$$W^\rho_u(\lambda)(x) := \langle \lambda, \rho(x)u\rangle.$$

In the following lemma we state the relation between a projective representation and the corresponding representation of the Mackey group. The proof is straightforward from the definition.

Lemma 2.12. *Let $(\rho, \mathcal{S})$ be a projective representation of G and let $(\rho_\sigma, \mathcal{S})$ be the corresponding representation of G_σ. Then the following are true:*

1. *The vector $u \in \mathcal{S}$ is ρ-cyclic if and only if u is ρ_σ-cyclic.*
2. *The wavelet transforms are related by $W^{\rho_\sigma}_u(\lambda)(x,t) = \overline{t}W^\rho_u(\lambda)(x)$.*

3. Banach function spaces and sequence spaces

In this section we define twisted translation and twisted convolution, and we summarize the assumptions we will place on a Banach space of functions (or BF-space for short) throughout this paper. We will also introduce a collection of sequence spaces which will be used in Sections 5 and 6 in order to formulate our results on atomic decompositions.

Definition 3.1. Let B be a Banach space over $\mathbb{C}$ of functions on G, and let σ be a cocycle on G. For a function $f \in B$, we define twisted left translation by

$$\ell_y^\sigma f(x) := \overline{\sigma(y, y^{-1}x)} f(y^{-1}x),$$

and we define twisted right translation by

$$r_y^\sigma f(x) := \sigma(x, y) f(xy).$$

If the cocycle is 1 we set $\ell_y = \ell_y^\sigma$ and $r_y = r_y^\sigma$, and we drop the use of the word twisted.

It is important to notice the following relations between the translation operators and the wavelet transform

$$\ell_y^\sigma W_u^\rho(\lambda)(x) = W_u^\rho(\rho^*(y)\lambda)(x)$$
$$r_y^\sigma W_u^\rho(\lambda)(x) = W_{\rho(y)u}^\rho(\lambda)(x).$$

We say that the BF-space B is twisted left-invariant if $\ell_y^\sigma f \in B$ for all $f \in B$ and $f \mapsto \ell_y^\sigma f$ is bounded for all $y \in G$. Analogously, we define twisted right-invariant spaces. In the sequel we will assume any BF-space to be twisted right- and left-invariant, and that twisted left and right translations by elements y in a compact set U are uniformly bounded in the sense that there is a finite constant C_U such that for any $y \in U$

$$\|\ell_y^\sigma f\| \leq C_U \|f\| \text{ and } \|r_y^\sigma f\| \leq C_U \|f\|. \tag{3.1}$$

We say that left translation, ℓ, is continuous on B if for every $f \in B$ the mapping $y \mapsto \ell_y f$ is continous $G \to B$. Right translation r is defined to be continous in a similar manner. In general we do not assume that ℓ and r are continuous on B until we need to derive atomic decompositions in sections 5 and 6. Moreover, the twisted translations ℓ^σ and r^σ are generally not continuous (not even near e), since σ is not globally continuous. A BF-space B is called solid if $|f| \leq |g|$ and $g \in B$ implies that $f \in B$. If B is a solid space, then B is twisted left or right invariant if and only if B is left or right invariant.

Remark 3.2. A weight on G is a measurable function $w : G \to (0, \infty)$. For a weight w we define the weighted Lebesgue spaces

$$L_w^p(G) = \left\{ f \text{ measurable} \,\middle|\, \|f\|_{L_w^p} := \left(\int |f(x)|^p w(x)\, dx \right)^{1/p} < \infty \right\}.$$

The spaces $L_w^p(G)$ are clearly solid Banach function spaces.

The weight w is called submultiplicative, if $w(xy) \leq w(x)w(y)$ for all $x, y \in G$. When w is submultiplicative the spaces $L_w^p(G)$ are left and right invariant for $1 \leq p \leq \infty$ and left and right translation are continuous for $1 \leq p < \infty$. Moreover, ℓ^σ and r^σ are projective representations on $L_w^p(G)$ for $1 \leq p < \infty$. The cocycle for ℓ^σ is σ while r^σ has cocycle $\overline{\sigma}$.

It is easily verified that if w is bounded on bounded sets, then condition (3.1) holds on $L^p_w(G)$ for $1 \le p < \infty$. By direct inspection of the norm, it can also be verified for $p = \infty$.

We now define sequence spaces related to a solid BF-space B. Let U be a compact neighbourhood of the identity, and let $\{x_i\}$ be a countable U-dense and well-spread collection of elements in G. Define the sequence space $\dot{b}$ by

$$\dot{b} = \left\{ \{c_i\} \subseteq \mathbb{C} \,\Big|\, \sum_i c_i 1_{x_i U} \in B \right\},$$

with norm

$$\|\{c_i\}\|_{\dot{b}} = \left\| \sum_i c_i 1_{x_i U} \right\|_B .$$

For example, if $B = L^p(G)$, then $\dot{b} = \ell^p$.

In the remainder of this paper we will need the following generalization of convolution in the presence of a cocycle. For a cocycle σ we define the twisted convolution of functions f and g on G by

$$f\#g(x) := \int_G f(y)\ell^\sigma_y g(x)\,dy = \int_G f(y)g(y^{-1}x)\overline{\sigma(y,y^{-1}x)}\,dy$$

whenever the integral exists. If the cocycle is constant this is the same as usual group convolution for which we reserve the special notation

$$f * g(x) := \int_G f(y)g(y^{-1}x)\,dy.$$

Remark 3.3. The twisted convolution always exists if f, g are in L^1 or if the product $f(y)g(y^{-1}x)$ is integrable in y. But sometimes it is necessary to consider more general cases. For example, twisted convolution might have to be defined weakly in the following way. If $0 \le \psi_n \le 1$ is an increasing sequence of compactly supported continuous functions which are identically 1 on nested compact sets C_n satisfying $\cup_n C_n = G$, then we might define the twisted convolution by

$$f\#g(x) = \lim_{n\to\infty} \int_G \psi_n(y) f(y)\ell^\sigma_y g(x)\,dy \tag{3.2}$$

whenever the limit exists. In applications one has to verify that such a definition makes sense. Notice, that if we know that the product $f(y)g(y^{-1}x)$ is integrable in y, then Lebesgue's Dominated Convergence Theorem tells us that this weak definition agrees with

$$f\#g(x) := \int_G f(y)\ell^\sigma_y g(x)\,dy.$$

Example 3.4 (Locally Compact Abelian Groups)**.** The best known example for twisted convolution is $\mathbb{R}^{2n}$ and its relation to time-frequency analysis and the Schrödinger representation of the Heisenberg group. This example can be generalized to locally compact abelian groups, see [26, 19, 30, 29, 33] and the references therein. For this we assume that $G = H \times \widehat{H}$ where H is a locally compact abelian group and $\widehat{H}$ is the dual group of continuous homomorphisms $\varphi : H \to \mathbb{T}$. The topology is the product topology and the product is defined as the product of each of the components. Define

$$\sigma((x,\varphi),(y,\psi)) = \overline{\varphi(y)}\,.$$

It is easy to see that σ is a cocycle. The following time-frequency representation is then an example of a σ-representation with continuous cocycle. Define translation by $T_x f(y) = f(x^{-1}y)$ and modulation by $M_\varphi f(y) = \varphi(y)f(y)$. Then

$$T_x M_\varphi = \overline{\varphi(x)} M_\varphi T_x.$$

Thus, with $\rho(x,\varphi) = T_x M_\varphi$ we get a projective representation, since

$$\begin{aligned}\rho(xy,\psi\varphi) &= T_{xy}M_{\psi\varphi}\\ &= T_x T_y M_\psi M_\varphi\\ &= \overline{\psi(y)} T_x M_\psi T_y M_\varphi\\ &= \sigma((x,\psi),(y,\varphi))\rho(x,\psi)\rho(y,\varphi)\,.\end{aligned}$$

The twisted convolution now becomes a simple generalization of the well known twisted convolution $\mathbb{R}^n$:

$$f\#g((x,\psi)) = \int_G f((y,\phi)g((y^{-1}x,\bar{\phi}\psi))\psi(y)\overline{\psi(x)}\,dyd\psi$$

We would like to point to the reference [19] where modulation spaces are defined for Abelian groups. In the special case where H is an abelian Lie group, then $\widehat{H}$ is also a Lie group, which could be discrete and countably infinite. Thus $H\times\widehat{H}$ is also a Lie group and the results of this article, in particular the discretization, become available for the modulation spaces.

4. Coorbit Spaces

Let $u\in\mathcal{S}$ and let B be a twisted left-invariant BF-space. Define

$$\mathrm{Co}_\rho^u B := \{\lambda\in\mathcal{S}^* \mid W_u^\rho(\lambda)\in B\}$$

with the norm

$$\|\lambda\|_{\mathrm{Co}_\rho^u B} := \|W_u^\rho(\lambda)\|_B.$$

We will now impose conditions which ensure that $\mathrm{Co}_\rho^u B$ is a Banach space. In the process we will demonstrate that these conditions ensure that that the space

$$B_u^\# := \{f\in B \mid f\#W_u^\rho(u) = f\}$$

with norm inherited from B is a reproducing kernel Banach space isometrically isomorphic to $\mathrm{Co}_\rho^u B$.

A ρ-cyclic vector $u\in\mathcal{S}$, is called a ρ-analyzing vector for $\mathcal{S}$ if the reproducing formula

$$W_u^\rho(\lambda)\#W_u^\rho(u) = W_u^\rho(\lambda)$$

holds for all $\lambda\in\mathcal{S}^*$.

Assumption 4.1. Let B be a twisted left-invariant BF-space on G. Assume there exists a nonzero ρ-analyzing vector $u\in\mathcal{S}$ satisfying the following continuity condition: The mapping

$$B\times\mathcal{S}\ni(f,v)\mapsto f\#W_v^\rho(u)(1) = \int_G f(y)W_v^\rho(u)^\vee(y)\overline{\sigma(y,y^{-1})}\,dy\in\mathbb{C}$$

is continuous.

Remark 4.2. The twisted convolution $W_u^\rho(\lambda)\# W_u^\rho(u)$ for $\lambda \in \mathcal{S}^*$ might only be defined in a weak sense as mentioned in Remark 3.3. The conditions we have placed on the Banach space B ensure that the twisted convolution $f\# W_u^\rho(u)$ exists as an integral for all $f \in B$. Therefore, for $\lambda \in \mathrm{Co}_\rho^u B$, the two definitions agree, since $W_u^\rho(\lambda) \in B$. This means that from this point on, whenever $\lambda \in \mathrm{Co}_\rho^u B$ the twisted convolution can and should be interpreted as an integral.

This observation is essential for producing atomic decompositions in section 5.

Remark 4.3. If $B = L_w^p(G)$ then the continuity condition will be a duality requirement. Specifically, Assumption 4.1 is satisfied for $B = L_w^p(G)$ if the topology on $\mathcal{S}$ is such that

$$\mathcal{S} \ni v \mapsto W_v^\rho(u)^\vee \in L^q_{w^{-q/p}}(G)$$

is continuous, where $\frac{1}{p} + \frac{1}{q} = 1$.

The main result of this section is

Theorem 4.4. *Let $(\rho, \mathcal{S})$ be a projective representation of G, and let B be a twisted left-invariant BF-space on G. Assume that $u \in \mathcal{S}$ is a ρ-analyzing vector satisfying Assumption 4.1. Then*

1. $W_u^\rho(v)\# W_u^\rho(u) = W_u^\rho(v)$ *for* $v \in \mathrm{Co}_\rho^u B$.
2. *The space* $\mathrm{Co}_\rho^u B$ *is a* ρ^**-invariant Banach space.*
3. $W_u^\rho : \mathrm{Co}_\rho^u B \to B$ *intertwines* ρ^* *and* ℓ^σ.
4. $W_u^\rho : \mathrm{Co}_\rho^u B \to B_u^\#$ *is an isometric isomorphism.*
5. $\mathrm{Co}_\rho^u B = \{\rho^*(F)u \mid F \in B_u^\#\}$ *when* $\rho^*(F)u$ *is defined by* $\langle \rho^*(F)u, v\rangle = \int F(x)\langle \rho^*(x)u, v\rangle\, dx$.

From [7] it is known that the statements are true if the cocycle σ is constant and therefore ρ is a representation. Notice that in [7] the representation ρ is assumed strongly continuous, but that requirement can be replaced by the weak continuity of $x \mapsto \langle \lambda, \rho(x)u\rangle$ for $\lambda \in \mathcal{S}^*$ and $u \in \mathcal{S}$ without modifications. We will therefore prove Theorem 4.4 by connecting it to coorbit theory for the representation ρ_σ for an appropriate choice of BF-space $\widehat{B}$ on the Mackey group. It turns out that the space

$$\widehat{B} := \{F : G \times \mathbb{T} \to \mathbb{C} \ \mid F(a,t) = \overline{t} f(a), f \in B\}$$

with norm $\|F\|_{\widehat{B}} := \|f\|_B$ is a good choice.

Lemma 4.5. *If $G \times \mathbb{T}$ and $\widehat{B}$ are defined as before, then the following relations hold.*

1. *The spaces B, $\widehat{B}$ are isometrically isomorphic via $\Lambda f(x,t) := \overline{t} f(x)$.*
2. *If the space B is twisted left-invariant, then $\widehat{B}$ is left-invariant.*
3. *For $F \in \widehat{B}$, we have $F * W_u^{\rho_\sigma}(u)(x,z) = \overline{z}\, f \# W_u^\rho(u)(x)$ when $F(a,t) = \overline{t} f(a)$.*

Proof. The first part is clear. The second part follows from the following calculations:

$$\begin{aligned}
\ell_{(a,w)}F(x,z) =& F(a^{-1}x, \overline{w} z \sigma(a,a^{-1})\overline{\sigma(a^{-1},x)}) \\
=& F(a^{-1}x, \overline{w} z \sigma(a, a^{-1}x)) \\
=& w\overline{z}\, \overline{\sigma(a,a^{-1}x)} f(a^{-1}x) \\
=& \overline{z} w\, \ell_a^\sigma f(x).
\end{aligned}$$

Therefore, B is twisted left-invariant if and only if $\widehat{B}$ is left invariant.

For the third part, we have

$$\begin{aligned} F * W_u(u)(x,z) &= \iint F(y,w)W_u^{\rho_\sigma}(u)((y,w)^{-1}(x,z))dwdy \\ &= \iint F(y,w)W_u^{\rho_\sigma}(u)(y^{-1}x, \overline{w}z\sigma(y,y^{-1})\overline{\sigma(y^{-1},x)})dwdy \\ &= \overline{z}\int f(y)W_u^{\rho}(u)(y^{-1}x)\overline{\sigma(y,y^{-1})}\sigma(y^{-1},x)\,dy \\ &= \overline{z}\, f\#W_u^{\rho}(u)(x). \end{aligned}$$

□

Lemma 4.6. *A vector u is ρ-analyzing if and only if u is ρ_σ-analyzing.*

Proof. First, we know that u is ρ_σ-cyclic if and only if u is ρ-cyclic, since $\langle \lambda, \rho(x)u\rangle = t\langle \lambda, \rho_\sigma(x,t)u\rangle$.

A straightforward calculation gives

$$W_u^{\rho_\sigma}(\lambda) * W_u^{\rho_\sigma}(u)(x,t) = \overline{t}W_u^{\rho}(\lambda)\#W_u^{\rho}(u)(x),$$

and the claim follows from this. □

The following theorem provides the connection between the coorbit theory that arises from representations [7] and the coorbit theory that arises from projective representations.

Theorem 4.7. *The triple B, ρ and u satisfy Assumption 4.1 with cocycle σ if and only if the triple $\widehat{B}$, ρ_σ and u satisfy Assumption 4.1 with a constant cocycle. Therefore $\mathrm{Co}_\rho^u B$ and $\mathrm{Co}_{\rho_\sigma}^u \widehat{B}$ are simultaneously defined. Moreover, $\mathrm{Co}_\rho^u B = \mathrm{Co}_{\rho_\sigma}^u \widehat{B}$ with the same norm.*

Proof. By Lemma 4.5 the space $\widehat{B}$ is left invariant. Next, denote the wavelet transform related to the representation ρ_σ by $W_u^{\rho_\sigma}$.

Note that if $F(x,t) = \overline{t}f(x)$, then

$$\begin{aligned} \iint F(x,z)W_v^{\rho_\sigma}(u)((x,z)^{-1})\,dzdx &= \iint F(x,z)W_v^{\rho_\sigma}(u)((x^{-1},\overline{z}\sigma(x^{-1},x)))\,dzdx \\ &= \iint \overline{z}f(x)W_v^{\rho}(u)(x^{-1})z\overline{\sigma(x,x^{-1})}\,dzdx \\ &= f\#W_v^{\rho}(u)(1). \end{aligned}$$

It follows that the continuity of $(f,v) \mapsto \int f(x)W_v^{\rho}(u)(x^{-1})\overline{\sigma(x,x^{-1})}\,dx$ on $B\times\mathcal{S}$ is equivalent to the continuity of $(F,v) \mapsto \iint F(x,z)W_v^{\rho_\sigma}(u)((x,z)^{-1})\,dzdx$.

Next, assume that $\lambda\in\mathcal{S}$. Then, by Lemma 2.12, $\lambda\in \mathrm{Co}_\rho^u B \Leftrightarrow W_u^{\rho}(\lambda)\in B \Leftrightarrow W_u^{\rho_\sigma}(\lambda)\in\widehat{B} \Leftrightarrow \lambda\in\mathrm{Co}_{\rho_\sigma}^u\widehat{B}$. □

Now we demonstrate our main result about the coorbit space constructed by the twisted convolution.

Proof of Theorem 4.4. By Theorem 4.7, the space $\widehat{B}$ and u satisfy Assumption 4.1. So we can apply Theorem 4.7 to the space $\widehat{B}$.

(1) The identity is assumed true for all analyzing vectors u and all functionals λ, and therefore it is also true for members of the coorbit space.

(2) We know that the space $\mathrm{Co}^u_{\rho_\sigma}\widehat{B} = \mathrm{Co}^u_\rho B$ is ρ^*_σ-invariant Banach space. So $W_u(\rho^*_\sigma(y,w)\phi) \in \mathrm{Co}^u_{\rho_\sigma}\widehat{B}$. On the other hand

$$W_u^{\rho_\sigma}(\rho^*_\sigma(y,w)\phi)(x,z) = \overline{z}wW_u^\rho(\rho^*(y)\phi)(x),$$

which implies that $W_u^\rho(\rho^*(y)\phi) \in B$.

(3) Using the fact that $W_u^{\rho_\sigma}$ intertwines ρ^*_σ with left translation, and $\rho^*_\sigma(x,z) = z\rho^*(x)$. We have

$$\begin{aligned} W_u^\rho(\rho^*(y)\phi)(x) =& \overline{w}zW_u^{\rho_\sigma}(\rho^*_\sigma(y,w)\phi)(x,z) = \overline{w}z\ell_{(y,w)}W_u^{\rho_\sigma}(\phi)(x,z) \\ =& \overline{\sigma(y,y^{-1})}\sigma(y^{-1},x)\ell_y W_u^\rho(\phi)(x) \\ =& \ell_y^\sigma W_u^\rho(\phi)(x). \end{aligned}$$

(4) According to (1) and (3) in Lemma 4.5 the spaces $B_u^\#$ and

$$\widehat{B}_u = \{F \in \widehat{B} \mid F * W_u^{\rho_\sigma}(u) = F\},$$

where convolution is on the group G_σ, are isometrically isomorphic. If we denote the isometrical isomorphism between B_u and $\widehat{B}_u$ by Λ, then $W_u^\rho = \Lambda^{-1}W_u^{\rho_\sigma} : \mathrm{Co}^u_\rho B \to B_u^\#$ and the result is obtained.

(5) First, $\rho^*(F)u$ is well-defined due to Assumption 4.1. Also, if $F \in B_u^\#$, then

$$W_u^\rho(\rho^*(F)u)(x) = \langle \rho^*(F)u, \rho(x)u\rangle = F\#W_u^\rho(u)(x) = F(x).$$

This shows that $\rho^*(F)u$ is in $\mathrm{Co}^u_\rho B$. If, on the other hand, $v \in \mathrm{Co}^u_\rho B$, then $W_u^\rho(v)$ is in B. Then $W_u^\rho(v)$ satisfies the twisted convolution reproducing formula $W_u^\rho(v)\#W_u^\rho(u) = W_u^\rho(v)$, which means

$$\langle v, \rho(x)u\rangle = \int W_u^\rho(v)(y)\langle \rho^*(y)u, \rho(x)u\rangle\, dy = \langle \rho^*(F)u, \rho(x)u\rangle.$$

Since u is cyclic, we see that $v = \rho^*(F)u$. □

In the following theorem, we prove that the twisted coorbit space is independent of the choice of the ρ-analyzing vector under some assumptions.

Theorem 4.8. *Assume that u_1 and u_2 both satisfy Assumption 4.1, and the following properties are true for $i,j \in \{1,2\}$*

1. *there are nonzero constants $C_{i,j}$ such that $W_{u_i}^\rho(\lambda)\#W_{u_j}^\rho(u_i) = C_{i,j}W_{u_j}^\rho(\lambda)$ for all $\lambda \in \mathcal{S}^*$*
2. *the mapping $B_{u_i} \ni f \mapsto f\#W_{u_j}^\rho(u_i) \in B$ is continuous.*

Then $\mathrm{Co}^{u_1}_\rho B = \mathrm{Co}^{u_2}_\rho B$ with equivalent norms.

Proof. We already know from [7] that this theorem is true for representations, i.e. when $\sigma = 1$. The proof thus consists of connecting the statements for ρ with similar statements for ρ_σ.

Consider the space $\widehat{B}$ and the Mackey group $G \times \mathbb{T}$. Since u_1 and u_2 are ρ-analyzing vectors for $\mathcal{S}$ that satisfy Assumption 4.1, they are also π_ρ-analyzing vectors for $\mathcal{S}$ that satisfy Assumption 4.1 (see Theorem 4.7). Also for $i,j \in \{1,2\}$ and $\lambda \in \mathcal{S}^*$, we have

$$\begin{aligned} W_{u_i}^{\rho_\sigma}(\lambda) * W_{u_j}^{\rho_\sigma}(u_i)(x,t) =& \overline{t}W_{u_i}^\rho(\lambda)\#W_{u_j}^\rho(u_i)(x) \\ =& \overline{t}C_{i,j}W_{u_j}^\rho(\lambda)(x) \\ =& C_{i,j}W_{u_j}^{\rho_\sigma}(\lambda)(x,t) \end{aligned}$$

Moreover, the mapping $\widehat{B}_{u_i} \ni F \mapsto F * W^{\rho_\sigma}_{uj}(u_i) \in \widehat{B}$ is continuous, indeed,

$$\|F * W^{\rho_\sigma}_{uj}(u_i)\|_{\widehat{B}} = \|f \# W^{\rho}_{u_j}(u_i)\|_B \leq C\|f\|_B = C\|F\|_{\widehat{B}}\,.$$

Therefore, by Theorem 2.7 in [7], $\mathrm{Co}^{u_1}_{\rho_\sigma}\widehat{B} = \mathrm{Co}^{u_2}_{\rho_\sigma}\widehat{B}$. Since $\mathrm{Co}^{u_i}_{\rho_\sigma}\widehat{B} = \mathrm{Co}^{u_i}_{\rho}B$, the result is obtained. □

5. Atomic decompositions and frames

The atomic decompositions we provide build on sampling on the reproducing kernel space $B^{\#}_u$. To do so, we need to show that the reproducing kernel does not vary too much under translation by elements of a fixed compact neighbourhood. The special form of the reproducing kernel allow us to estimate such local variations using smoothness of the analyzing vector u. Assume that G is a connected Lie group with Lie algebra $\mathfrak{g}$, and that ρ is a projective representation of G on the Fréchet space $\mathcal{S}$.

Definition 5.1. A vector $u \in \mathcal{S}$ is called ρ-weakly differentiable if for all $X \in \mathfrak{g}$ and for all $\lambda \in \mathcal{S}^*$ the mapping $t \mapsto \langle \lambda, \rho(\exp(tX))u\rangle$ is differentiable and there is a $u_X \in \mathcal{S}$ satisfying

$$\langle \lambda, u_X\rangle = \frac{d}{dt}\Big|_{t=0} \langle \lambda, \rho(\exp(tX))u\rangle.$$

We then define $\rho(X)u = u_X$. If for any $\lambda \in \mathcal{S}$ the mapping $x \mapsto \langle \lambda, \rho(x)u\rangle$ is differentiable up to order n, we say that the vector u is ρ-weakly differentiable up to order n.

Similarly, a vector $\lambda \in \mathcal{S}$ is called ρ^*-weakly differentiable if for all $X \in \mathfrak{g}$ and for all $u \in \mathcal{S}$ the mapping $t \mapsto \langle \rho^*(\exp(tX))\lambda, u\rangle$ is differentiable. In this case we denote by $\rho^*(X)\lambda$ the distribution in $\mathcal{S}^*$ satisfying

$$\langle \rho^*(X)\lambda, u\rangle = \frac{d}{dt}\Big|_{t=0} \langle \rho^*(\exp(tX))\lambda, u\rangle.$$

If for any $u \in \mathcal{S}$ the mapping $x \mapsto \langle \rho^*(x)\lambda, u\rangle$ is differentiable up to order n, we say that the vector λ is ρ^*-weakly differentiable up to order n.

Remark 5.2. As pointed out by one of the referees, the existence of $\rho(X)u$ in the definition will be guaranteed by the Banach-Steinhaus theorem if the space $\mathcal{S}$ is quasi-reflexive (and hence $\mathcal{S}^*$ is barreled). We often work with smooth vectors for the projective representation ρ in which case the mapping $x \mapsto \rho(x)u$ is strongly differentiable. Therefore the existence of $\rho(X)u$ is guaranteed for most of our applications.

Remembering that $\mathcal{S}^*$ is equipped by the weak* topology, we know that $(\mathcal{S}^*)^* = \mathcal{S}$ is barreled and therefore the existence of $\rho^*(X)u$ is automatic.

Assumption 5.3. Let B be a solid BF-space on G on which left and right translation are continuous, and assume that B, ρ and u satisfy Assumption 4.1. Assume, moreover, that u is ρ-weakly and ρ^*-weakly differentiable up to order $n = \dim(G)$, and that for all finite subsets with n elements, $\{Y_1, Y_2, \ldots, Y_n\} \subseteq \mathfrak{g}$ the mappings

$$f \mapsto f * |W^{\rho}_u(\rho^*(Y_1)\rho^*(Y_2)\cdots\rho^*(Y_n)u)| \text{ and } f \mapsto f * |W^{\rho}_{\rho(Y_1)\rho(Y_2)\cdots\rho(Y_n)u}(u)|$$

are bounded on the solid, left and right invariant BF-space B.

Notice, that all convolutions in this section and in Appendix A are expected to be defined as proper integrals. We are no longer allowing weak definitions as in Remark 3.3, or at least we have to show that the weak definition agrees with an integral.

Theorem 5.4. *Let u be a vector satisfying Assumption 5.3. Let $X_1, \dots, X_n$ be a fixed basis for $\mathfrak{g}$ and define $U_\epsilon = \{e^{t_1 X_1} \cdots e^{t_n X_n} \mid -\epsilon \leq t_k \leq \epsilon\}$. In the following we always choose a cocycle σ and $\epsilon > 0$ small enough so that σ is C^∞ on a neighbourhood containing $U_\epsilon \times U_\epsilon$.*

(1) *Given a U_ϵ-dense and well-spread sequence $\{x_i\} \subseteq G$ and a U_ϵ-BUPU $\{\psi_i\}$, the operators*

$$T_1 f = \sum_i f(x_i)\sigma(x, x^{-1}x_i)\psi_i \# W_u^\rho(u)$$

$$T_2 f = \sum_i \lambda_i(f)\ell^\sigma_{x_i} W_u^\rho(u)$$

$$T_3 f = \sum_i c_i f(x_i)\ell^\sigma_{x_i} W_u^\rho(u)$$

where $\lambda_i(f) = \int f(y)\psi_i(y)\overline{\sigma(y, y^{-1}x_i)}\,dy$ and $c_i = \int \psi_i(y)\,dy$, are well defined from B_u^σ to B_u^σ.

(2) *There is an ϵ small enough for which the operators T_1, T_2, T_3 are invertible for any U_ϵ-dense and well-spread sequence $\{x_i\} \subseteq G$ and any U_ϵ-BUPU $\{\psi_i\}$. In this case the family $\{\rho^*(x_i)u\}$ is a Banach frame for $\mathrm{Co}_\rho^u B$ with respect to the sequence space $\dot{b}$, and the families $\{\lambda_i \circ T_2^{-1} \circ W_u^\rho, \rho^*(x_i)u\}$ and $\{c_i T_3^{-1} \circ W_u^\rho, \rho^*(x_i)u\}$ are atomic decompositions for $\mathrm{Co}_\rho^u B$ with respect to the sequence space $\dot{b}$. In particular, $\gamma \in \mathrm{Co}_\rho^u B$ can be reconstructed by*

$$\gamma = (W_u^\rho)^{-1} T_1^{-1}\Big(\sum_i W_u^\rho(\gamma)(x_i)\psi_i \# W_u^\rho(u)\Big)$$

$$\gamma = \sum_i \lambda_i\big(T_2^{-1} W_u^\rho(\gamma)\big)\rho^*(x_i)u$$

$$\gamma = \sum_i c_i\ (T_3^{-1} W_u^\rho(\gamma))\,\rho^*(x_i)u$$

with convergence in $\mathcal{S}^*$. The convergence is in $\mathrm{Co}_\rho^u B$ if $C_c(G)$ is dense in B.

These results can be proven by applying Theorems 3.4, 3.6 and 3.7 from [4] to the representation ρ_σ of the Mackey group G_σ. This requires us to define an appropriate solid BF-space on G_σ. There are many more or less natural choices for such BF-spaces and two different approaches have been presented in [9] and [16]. The more natural choice is found in [16] where a minimal extension, naturally isomorphic to the original choice, is considered. The drawback is, that the space is not solid. In this paper we wish to avoid making this choice and to work directly on the group G and the Banach space B. One of the advantages is that the atoms are obtained via sampling at points in the group G and in a reproducing kernel subspace of B. This makes the apprach presented here natural. In [9, 16] the atoms are instead obtained from sampling at points in the extended group G_σ on an reproducing kernel subspace of functions on G_σ. Moreover, the other two approaches rely on the topology of G_σ being the product topology of $G \times \mathbb{T}$, a property which can only be ensured if the multiplier is continuous. This property is utilized when choosing the sample points in G_σ (see for example the proof of Theorem 6.1 on p. 1305 of [9]). The present approach avoids this assumption on σ. The calculations differ non-trivially from [4] by the occurrence of the cocycle, and the details are carried out in the appendix.

The result above is focused on using differentiable vectors as atoms. It should be mentioned that this is not strictly necessary, since the results in the appendix can be used for vectors that are not necessarily differentiable. It is possible to obtain atoms for non-differentiable vectors, as long as it can be shown that convolution with local oscillations of the kernel $W_u^\rho(u)$ is bounded on the space B (see Corollary A.2) . We intentionally left this to the appendix, since smooth vectors suffice for our intended application.

6. Bergman Spaces on the Unit Ball

Recently, in [5], the first and the third authors together with K. Gröchenig, obtained atomic decompositions for Bergman spaces on the unit ball in $\mathbb{C}^n$ through a finite covering group of the group $\mathrm{SU}(n,1)$ with the restriction that the representation parameter $s > n$ had to be rational. As a special case, atomic decompositions of Bergman spaces through the group $\mathrm{SU}(n,1)$ is valid for integer values of the parameter $s > n$. Overcoming the restriction on the parameter was one reason for introducing a coorbit theory for projective representations.

We dedicate this chapter to generating Banach frames and atomic decompositions of Bergman spaces on the unit ball via the group $\mathrm{SU}(n,1)$. For more references we encourage the reader to see [3, 17, 28, 31, 39, 41].

6.1. Facts about Bergman Spaces on the Unit Ball.

In this section we collect facts about Bergman spaces on the unit ball. Let $\mathbb{C}^n$ be equipped with the usual inner product $(z,w) = z_1\overline{w_1} + z_2\overline{w_2} + ... + z_n\overline{w_n}$ and define the unit ball by

$$\mathbb{B}^n := \left\{ z \in \mathbb{C}^n \mid |z|^2 := |z_1|^2 + |z_2|^2 + ... + |z_n|^2 < 1 \right\}.$$

Let $\mathrm{d}v$ denote the normalized volume measure on the unit ball upon identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. For $\alpha > -1$, define the measure $\mathrm{d}v_\alpha(z) := C_\alpha(1-|z|^2)^\alpha \mathrm{d}v(z)$, where $C_\alpha = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}$ makes $\mathrm{d}v_\alpha$ a probability measure. Notice that the measure $\mathrm{d}v_\alpha$ is finite measure on $\mathbb{B}^n$ if and only if $\alpha > -1$.

We define the α-weighted L^p space on the unit ball as

$$L^p_\alpha(\mathbb{B}^n) = \{ f : \mathbb{B}^n \to \mathbb{C} \mid \int_{\mathbb{B}^n} |f(z)|^p \mathrm{d}v_\alpha(z) < \infty \}$$

with norm

$$\|f\|_{L^p_\alpha} = \left(\int_{\mathbb{B}^n} |f(z)|^p \, \mathrm{d}v_\alpha(z) \right)^{1/p},$$

where $1 \le p < \infty$. For $\alpha > -1$, we define the weighted Bergman spaces on the unit ball to be

$$A^p_\alpha(\mathbb{B}^n) := L^p_\alpha(\mathbb{B}^n) \cap \mathcal{O}(\mathbb{B}^n)$$

with norm inherited from $L^p_\alpha(\mathbb{B}^n)$, where $\mathcal{O}(\mathbb{B}^n)$ is the space of holomorphic functions on the unit ball. We have the condition $\alpha > -1$ to construct a non-trivial Bergman spaces, in fact, if $\alpha \le -1$, then the only holomorphic function in $L^p_\alpha(\mathbb{B}^n)$ is the zero function.

As we have seen in the special case on the unit disc, Bergman spaces are closed subspaces of $L^p_\alpha(\mathbb{B}^n)$, *i.e.*, Bergman spaces are Banach spaces. In the case $p = 2$, the space $A^2_\alpha(\mathbb{B}^n)$ is a Hilbert space with the inner product

$$(f, g)_\alpha = \int_{\mathbb{B}^n} f(z)\overline{g(z)}\,\mathrm{d}v_\alpha(z).$$

The orthogonal projection of $L^2_\alpha(\mathbb{B}^n)$ on the space $A^2_\alpha(\mathbb{B}^n)$ is given by

$$P_\alpha f(z) = \int_{\mathbb{B}^n} f(w)K_\alpha(z, w)\,\mathrm{d}v_\alpha(w),$$

where

$$K_\alpha(z, w) = \frac{1}{(1 - (z, w))^{n+1+\alpha}}$$

is the reproducing kernel for $A^2_\alpha(\mathbb{B}^n)$.

The group $\mathrm{SU}(n, 1)$ is defined to be the group of all $(n + 1) \times (n + 1)$-matrices x of determinant 1 for which $x^*J_{(n,1)}x = J_{(n,1)}$, where

$$J_{(n,1)} = \begin{pmatrix} -I_n & 0 \\ 0 & 1 \end{pmatrix}.$$

We always write $x \in \mathrm{SU}(n, 1)$ in the block form

$$x = \begin{pmatrix} A & b \\ c^t & d \end{pmatrix},$$

where A is an $n \times n$ matrix, and b, c are vectors in $\mathbb{C}^n$, and $d \in \mathbb{C}$. Simple calculations show that

$$x^{-1} = \begin{pmatrix} A^* & -\overline{c} \\ -\overline{b}^t & \overline{d} \end{pmatrix}.$$

The identity $xx^{-1} = I$ implies

$$|d|^2 - |b|^2 = 1 \tag{6.1}$$

Form now on, we write $G = \mathrm{SU}(n, 1)$. This group acts transitively on $\mathbb{B}^n$ by

$$x \cdot z = (Az + b)((c, \overline{z}) + d)^{-1}.$$

If we define the subgroup K of G as

$$K = \left\{ \begin{pmatrix} k & 0 \\ 0 & \overline{\det(k)} \end{pmatrix} \,\middle|\, k \in U(n) \right\},$$

then the stabilizer of the origin $0 \in \mathbb{C}^n$ is K and $\mathbb{B}^n \simeq G/K$. It follows that there is a one to one correspondence between the K-right invariant functions on G and the functions on $\mathbb{B}^n$ via

$$\widetilde{f}(x) = f(x \cdot 0).$$

This correspondence relates the G-invariant measure on $\mathbb{B}^n$, which is given by $dv_{-n-1}(z)$, to the measure on the group G. The compactness of K ensures that we can normalize the measure on G so that, for any K-right invariant function $\widetilde{f}$ on G, we have

$$\int_G \widetilde{f}(x)\,dx = \int_{\mathbb{B}^n} f(z)\,dv_{-n-1}(z). \tag{6.2}$$

Define the weighted L^p_α spaces on G by

$$L^p_\alpha(G) = \left\{ F : G \to \mathbb{C} \;\middle|\; \|F\|_{L^p_\alpha(G)} := \left(C_\alpha \int_G |F(x)|^p \, (1 - |x \cdot o|^2)^\alpha \, dx \right)^{1/p} < \infty \right\}$$

If we denote by $L^p_\alpha(G)^K$ the space of K-right invariant functions in the space $L^p_\alpha(G)$, then it is easy to see that $L^p_\alpha(\mathbb{B}^n)$ and $L^p_{\alpha+n+1}(G)^K$ are isometric. That is,

$$\|f\|_{L^p_\alpha(\mathbb{B}^n)} = \|\widetilde{f}\|_{L^p_{\alpha+n+1}(G)}. \tag{6.3}$$

For $s > n$, the action of G on $\mathbb{B}^n$ defines an irreducible unitary projective representation of G on the space $\mathcal{H}_s = A^2_{s-n-1}$ by

$$\rho_s(x) f(z) = (-(z, b) + \overline{d})^{-s} f(x^{-1} \cdot z), \tag{6.4}$$

which also defines a representation for the universal covering group of G. From now on we assume that a cocycle σ has been chosen for the projective representation ρ_s. Note, that since ρ_s is a local representation, the cocycle can be chosen equal to one on a neighbourhood of $e \times e$ (see also Theorem 7.1 in [2]).

We denote the twisted wavelet transform on $\mathcal{H}_s$ by

$$W^{\rho_s}_u(\lambda)(x) = (\lambda, \rho_s(x)u)_{\mathcal{H}_s}.$$

Let $\mathcal{P}_k$ be the space of all homogeneous polynomials of degree k on $\mathbb{C}^n$. In the following theorem we summarize some properties of the space of smooth vectors for ρ_s and its conjugate dual space, which will be the candidate Fréchet space $\mathcal{S}$ for constructing the coorbits of $L^p_{\alpha+n+1-sp/2}(G)$.

Theorem 6.1. *Let $s > n$ and let $(\rho_s, \mathcal{H}_s)$ be the projective representation of G which is defined in 6.4. The following are true:*

(1) *Every polynomial is a smooth vector for ρ_s.*
(2) *Every smooth vector for ρ_s is bounded.*
(3) *Assume $v \in \mathcal{H}_s$, then $v \in \mathcal{H}^\infty_s$ if and only if $v = \sum_k v_k$, $v_k \in \mathcal{P}_k$, and for all $N \in \mathbb{N}$ there exists a constant $C_N > 0$ such that $\|v_k\|_{\mathcal{H}_s} \le C_N(1+k)^{-N}$.*
(4) *A vector $\phi \in \mathcal{H}^{-\infty}_s$ if and only if $\phi = \sum_k \phi_k$, $\phi_k \in \mathcal{P}_k$, and there exist $N \in \mathbb{N}$ and $C > 0$ such that $\|\phi_k\|_{\mathcal{H}_s} \le C(1+k)^N$. Moreover, the dual pairing is given by*

$$\langle \phi, v \rangle_s = \sum_k (\phi_k, v_k)_{\mathcal{H}_s}.$$

Proof. The proof is done by noting that ρ_s is a unitary representation of the universal covering group of G, so the smooth vectors are the same for both, where the smooth vectors for ρ_s, as a representation, satisfy all the above properties as proved in [3] and [5, Lemma 2.10]. □

6.2. **Bergman Spaces as Coorbits spaces.** In this section we collect several facts from [5] and use them with some modifications to provide atomic decompositions of Bergman spaces through projective representations of the group $\mathrm{SU}(n, 1)$.

As before, we assume $G = \mathrm{SU}(n, 1)$ and $(\mathcal{H}^\infty_s, \rho_s)$ is the smooth projective representation obtained by restricting $(\mathcal{H}_s, \rho_s)$ to $\mathcal{H}^\infty_s$. In this section we show that Bergman spaces are twisted convolutive coorbits of weighted L^p spaces, which allows us to discretize Bergman spaces using the full group $\mathrm{SU}(n, 1)$. For this goal we need the following results which was already proved for the linear representation in [5]. The same proof will work (with minor differences) for the projective representation case. For completeness we will provide a full proof for each of these results. The first lemma corresponds to Lemma 3.15 in [5].

Lemma 6.2. *Assume u and v are smooth vectors for ρ_s. There is a constant C depending on u and v such that*

$$|W_u^{\rho_s}(v)(x)| \leq C(1-|x\cdot o|^2)^{s/2}\left(1-\log(1-|x\cdot o|^2)\right).$$

Moreover, the constant C can be chosen uniform in $|\alpha|$ for $v = z^\alpha$.

The next result is Proposition 3.16(i) in [5] extended to irrational s.

Proposition 6.3. *Let $\alpha > -1$, $1 \leq p < \infty$, and $s > n$ be chosen. Assume that u and v are smooth vectors for ρ_s. Then $W_u^{\rho_s}(v) \in L_t^p(G)$ for $t + ps/2 > n$.*

We will now verify that any smooth (appropriately normalized) vector u is analyzing. First we will verify the reproducing formula for vectors in the Hilbert space. Assume that $v, w \in \mathcal{H}$. Since ρ_s is square integrable and G is unimodular, every vector is ρ_s-admissible, *i.e.*, u is in the domain of the operator A_ρ, which is given in Theorem 2.9. Note, that A_ρ is a multiple of the identity since G is unimodular. By the orthogonality relation in Theorem 2.9, we have

$$\int_G (v,\rho(x)u)_{\mathcal{H}_s}\,(\rho(x)u,w)_{\mathcal{H}_s}\,dx = c_\rho^2\|u\|^2_{\mathcal{H}_s}\,(v,w)_{\mathcal{H}_s}.$$

Letting $w = \rho_s(x)u$ it follows that

$$\begin{aligned}
W_u^{\rho_s}(v)\#W_u^{\rho_s}(u)(x) &= \int_G (v,\rho(y)u)_{\mathcal{H}_s}\,(u,\rho(y^{-1}x)u)_{\mathcal{H}_s}\,\overline{\sigma(y,y^{-1}x)}\,dy\\
&= \int_G (v,\rho(y)u)_{\mathcal{H}_s}\,(u,\rho(y^{-1})\rho(x)u)_{\mathcal{H}_s}\,\overline{\sigma(y^{-1},x)}\,\overline{\sigma(y,y^{-1}x)}\,dy\\
&= \int_G (v,\rho(y)u)_{\mathcal{H}_s}\,(u,\rho(y)^{-1}\rho(x)u)_{\mathcal{H}_s}\,\sigma(y,y^{-1})\,\overline{\sigma(y,y^{-1})}\,dy\\
&= \int_G (v,\rho(y)u)_{\mathcal{H}_s}\,(\rho(y)u,\rho(x)u)_{\mathcal{H}_s}\,dy\\
&= c_\rho^2(v,\rho(x)u)_{\mathcal{H}_s}\,(u,u)_{\mathcal{H}_s}\\
&= CW_u^{\rho_s}(v)(x)
\end{aligned}$$

for all $v \in \mathcal{H}$.

We will now define the twisted convolution in a weak sense as described in Remark 3.3. Let $0 \leq \psi_n \leq 1$ be an increasing sequence of compactly supported smooth functions on G that are identically 1 on growing compact sets C_n for which $\cup_n C_n = G$. Also assume that derivatives of order N of $x \mapsto \ell_x^\sigma \psi_n$ at the origin are uniformly bounded in n, i.e. for $Y_1, \cdots, Y_N \in \mathfrak{g}$ we have

$$\sup_n \|\ell^\sigma(Y_1)\cdots\ell^\sigma(Y_N)\psi_n\|_\infty < \infty,$$

where $\ell^\sigma(X)f(x) = \frac{d}{dt}\Big|_{t=0}\ell^\sigma_{e^{tX}}f(x)$ are strong derivatives. Now define the twisted convolution of two functions f, g by

$$f\#g(x) = \lim_{n\to\infty}\int_G \psi_n(y)f(y)\ell_y^\sigma g(x)\,dy \tag{6.5}$$

whenever the limit exists.

Lemma 6.4. *Let u be a non-zero smooth vector for ρ_s for which $\|u\|_{\mathcal{H}_s} = c_\rho^{-1}$. Then for $\phi \in \mathcal{H}_s^{-\infty}$ the twisted convolution $W_u^{\rho_s}(\phi) \# W_u^{\rho_s}(u)$ exists and equals $W_u^{\rho_s}(\phi)$.*

Proof. We get that

$$\begin{aligned} W_u^{\rho_s}(\phi)\#W_u^{\rho_s}(u)(x) &= \lim_{n\to\infty} \int \psi_n(y) W_u^{\rho_s}(\phi)(y)\overline{\sigma(y,y^{-1}x)} W_u^{\rho_s}(u)(y^{-1}x)\, dy \\ &= \lim_{n\to\infty} \int \psi_n(y)\langle \phi, \rho_s(y)u\rangle \overline{\sigma(y,y^{-1}x)} \langle u, \rho_s(y^{-1}x)u\rangle\, dy \\ &= \lim_{n\to\infty} \int \psi_n(y)\langle \phi, \rho_s(y)u\rangle \langle \rho_s(y)u, \rho_s(x)u\rangle\, dy. \end{aligned}$$

Define the smooth compactly supported function

$$\Psi_n(y) = \psi_n(y)\langle \rho_s(x)u, \rho_s(y)u\rangle,$$

then

$$W_u^{\rho_s}(\phi)\#W_u^{\rho_s}(u)(x) = \lim_{n\to\infty} \int \langle \phi, \Psi_n(y)\rho_s(y)u\rangle\, dy.$$

Since Ψ_n is smooth and compactly supported the vector

$$\rho_s(\Psi_n)u := \int \Psi_n(y)\rho_s(y)u, dy$$

is a smooth vector, and

$$W_u^{\rho_s}(\phi)\#W_u^{\rho_s}(u)(x) = \lim_{n\to\infty} \langle \phi, \rho_s(\Psi_n)u\rangle.$$

To show that this limit exists and equals $\langle \phi, \rho_s(x)u\rangle$, we need to verify that $\rho_s(\Psi_n)u$ converges to $\rho_s(x)u$ in the topology of the smooth vectors $\mathcal{H}_s^\infty$. Denote $\rho(x)u$ by v, then we have to show that for

$$\Psi_n(y) = \psi_n(y)\langle v, \rho_s(y)u\rangle,$$

the vectors $\rho_s(\Psi_n)u$ converge to v in $\mathcal{H}_s^\infty$. Let us first verify convergence in $\mathcal{H}_s$.

$$\|\rho_s(\Psi_n)u - v\|_{\mathcal{H}_s}^2 = \|\rho_s(\Psi_n)u\|_{\mathcal{H}_s}^2 + \|v\|_{\mathcal{H}_s}^2 - (\langle \rho_s(\Psi_n)u, v\rangle - \langle v, \rho_s(\Psi_n)u\rangle).$$

By the Lebesgue dominated convergence theorem and square integrability of the representation, we have that

$$\lim_{n\to\infty} \langle \rho_s(\Psi_n)u, v\rangle = \lim_{n\to\infty} \int \psi_n(y)|\langle \rho_s(y)u, v\rangle|^2\, dy = \|v\|_{\mathcal{H}_s}^2.$$

Therefore we just need to check that

$$\lim_{n\to\infty} \|\rho_s(\Psi_n)u\|^2 = \|v\|_{\mathcal{H}_s}^2.$$

Notice that by Fubini we have

$$\|\rho_s(\Psi_n)u\|_{\mathcal{H}_s}^2 = \int\int \psi_n(x)\psi_n(y)\langle v, \rho_s(y)u\rangle\langle \rho_s(y)u, \rho_s(x)u\rangle\langle \rho_s(x)u, v\rangle\, dx\, dy,$$

and we will be able to apply the Lebesgue dominated convergence theorem if we can show that the function $|\langle v, \rho_s(y)u\rangle\langle \rho_s(y)u, \rho_s(x)u\rangle\langle \rho_s(x)u, v\rangle|$ is integrable. This is the same as showing that the integral

$$\int\int |W_u^{\rho_s}(v)(y)||W_u^{\rho_s}(u)(y^{-1}x)||W_u^{\rho_s}(v)(x)|\, dx\, dy$$

is finite. From Proposition 6.3 it is known that $W_u^{\rho_s}(v) \in L^2(G)$ and $W_u^{\rho_s}(u) \in L^p$ for some $1 < p < 2$, so the Kunze-Stein phenomenon [10] tells us that the integral is finite.

We can now repeat the argument with derivatives of the vector $\rho_s(\Psi_n)u$ to show it converges to derivatives of v. If $X \in \mathfrak{g}$ then

$$\rho_s(X)\rho_s(\Psi_n)u = \rho_s(X\Psi_n)\rho_s.$$

Since $\Psi_n = \psi_n W_u^{\rho_s}(v)$ we get that

$$\rho_s(X)\rho_s(\Psi_n)u = (\ell^\sigma(X)\psi_n(x))W_u^{\rho_s}(v) + \psi_n(x)W_u^{\rho_s}(\rho_s(X)v).$$

As before, it follows from the Lebesgue dominated convergence theorem and the assumption that $\ell^\sigma(X)\psi_n$ is uniformly bounded in n, that $\rho_s(X)\rho_s(\Psi_n)u$ converges to $\rho_s(X)v)$ in $\mathcal{H}_{\rho_s}$. This argument can be repeated to show that $\rho_s(\Psi_n)u$ converges to v in $\mathcal{H}_{\rho_s}^\infty$.

The argument also shows that the twisted convolution $W_u^{\rho_s}(\phi)\#W_u^{\rho_s}(u)$ exists and does not depend on the particular choice of the sequence of functions ψ_n. □

Now we are ready to show that the twisted coorbits of the spaces $L^p_{\alpha+n+1-sp/2}(G)$ generated by any nonzero smooth vector $u \in \mathcal{H}_s^\infty$ are well defined nonzero spaces under the assumptions in the following theorem. This result uses techniques found in the proof of Proposition 3.16 in [5], but needs to be verified in our situation due to the occurence of the cocycle.

Theorem 6.5. *Let $1 \leq p < \infty$, and $s > n$. Assume that $-1 < \alpha < p(s-n) - 1$. For a nonzero smooth vector $u \in \mathcal{H}_s^\infty$, the coorbit space $\mathrm{Co}_{\rho_s}^u L^p_{\alpha+n+1-sp/2}(G)$ is a nonzero well defined Banach space.*

Proof. Let us show that any nonzero smooth vector $u \in \mathcal{H}_s^\infty$ satisfies Assumption 4.1. First, u is ρ-cyclic because $\mathcal{H}_s^\infty$ is an irreducible projective representation. By the previous lemma it is also analyzing. All that is left to show is that the mapping

$$(f, v) \mapsto \int_G f(x) W_u^{\rho_s}(x^{-1})\overline{\sigma(x, x^{-1})}\, dx$$

is continuous on $L^P_{\alpha+n+1-sp/2}(G)$. By Remark 4.3, it is enough to show that

$$W_u^{\rho_s}(v) \in (L^p_{\alpha+n+1-sp/2}(G))^* = L^q_{sq/2-(\alpha+n+1)q/p}(G),$$

where $1/p + 1/q = 1$. This is done by Proposition 6.3, because

$$sq/2 - (\alpha + n + 1)q/p + sp/2 > n,$$

whenever $-1 < \alpha < p(s-n) - 1$. Therefore, the space $\mathrm{Co}_{\rho_s}^u L^p_{\alpha+n+1-sp/2}(G))$ is well defined. Finally, note that $W_u^{\rho_s}(u) \in L^p_{\alpha+n+1-sp/2}(G)$ again by Proposition 6.3, hence it is nonzero Banach space. □

Our goal now is to describe Bergman spaces as twisted coorbits generated by any nonzero smooth vector $u \in \mathcal{H}_s^\infty$. First we describe Bergman spaces as twisted coorbits by the special ρ-analyzing vector $u = 1_{\mathbb{B}^n}$, then we show that this coorbit is independent of the choice of u.

Theorem 6.6. *Let $\alpha > -1$, $1 \leq p < \infty$, and $u = 1_{\mathbb{B}^n}$. The Bergman space $A^p_\alpha(\mathbb{B}^n)$ is the twisted coorbit space of $L^p_{\alpha+n+1-sp/2}(G)$ that corresponds to the projective representation $(\mathcal{H}_s^\infty, \rho_s)$. i.e., $A^p_\alpha(\mathbb{B}^n) = \mathrm{Co}_{\rho_s}^u L^p_{\alpha+n+1-sp/2}(G)$ for $\alpha < p(s-n)-1$.*

Proof. As in [5, Theorem 3.6], the space $A^p_\alpha(\mathbb{B}^n) \subset \mathcal{H}_s^{-\infty}$ for all $p \geq 1$, which is still valid in the case of smooth vectors for ρ_s. The reason is that the smooth vectors for the projective representation ρ_s are the same as the smooth vectors of the representation of the universal covering of G. First we recall that every element in $\mathcal{H}_s^{-\infty}$ can be realized as a holomorphic function $f = \sum_k f_k$ in $\mathcal{H}_s^{-\infty}$ (for details see [5]). We then have to show that the function f is in $L^p_\alpha(\mathbb{B}^n)$ if and only if $W_u^{\rho_s}(f) \in L^p_{\alpha+n+1-sp/2}(G)$.

To this end, assume $x = \begin{pmatrix} A & b \\ c^t & d \end{pmatrix}$. By definition of the dual pairing and the fact that the polynomials f_k belong to all Bergman spaces, we have

$$\begin{aligned} |W_u^{\rho_s}(f)(x)| &= \left|\sum_k W_u^{\rho_s}(f_k)(x)\right| \\ &= \left|\sum_k (\overline{d})^{-s} f_k(bd^{-1})\right| \\ &= |d|^{-s}\left|\sum_k f_k(x\cdot o)\right| \\ &= |d|^{-s}|f(x\cdot o)|. \end{aligned}$$

As we have seen before, $|d|^{-s} = (1-|x\cdot o|^2)^{s/2}$. It follows that

$$|f(x\cdot o)| = (1-|x\cdot o|^2)^{-s/2}|W_u^{\rho_s}(f)(x)|$$

by the isometry in (6.3). We conclude that $f \in L^p_\alpha(\mathbb{B}^n)$ if and only if $W_u^{\rho_s}(f) \in L^p_{\alpha+n+1-sp/2}(G)$. □

To prove our main result in this section, which says that Bergman spaces are twisted coorbits for weighted L^p spaces generated by any smooth vector, we need the following theorem. It will be used in the subsequent section to generate a Banach frame and atomic decomposition for Bergman spaces. The following Lemma makes the transition from the weakly defined twisted convolution to a proper integral which is needed for providing frames and atoms.

Lemma 6.7. *Let $1 \leq p < \infty$, $-1 < \alpha < p(s-n)-1$, and let v and u be smooth vectors. When f is in $L^p_{\alpha+n+1-sp/2}(G)$ the twisted convolution $f\#W_u^{\rho_s}(v)$ is a proper integral, i.e.*

$$f\#W_u^{\rho_s}(v)(x) = \int_G f(y)\ell_y^\sigma W_u^{\rho_s}(v)(x)\,dy.$$

Proof. Just an application of Lebesgue's dominated convergence theorem and the fact that $W_u^{\rho_s}(v)$ is in the dual of $L^p_{\alpha+n+1-sp/2}(G)$ for the specified parameters. □

Theorem 6.8. *Let $1 \leq p < \infty$, $-1 < \alpha < p(s-n)-1$, and let v and u be smooth vectors. The convolution operator $f \mapsto f * |W_u^{\rho_s}(v)|$ is continuous on $L^p_{\alpha+n+1-sp/2}(G)$. In particular, $f \mapsto f\#W_u^{\rho_s}(v)$ is continuous on $L^p_{\alpha+n+1-sp/2}(G)$.*

In [5] this result was proved for a solvable and simply connected subgroup of G in Corollary 3.10. The proof below is similar, but we include it for completeness.

Proof. Let $F \in L^p_{\alpha+n+1-sp/2}(G)$ and define

$$\widetilde{f}(x) := \int_K F(xk)\mathrm{d}k.$$

Then $\widetilde{f}$ is K-right invariant function on G. Therefore, there is a corresponding $f \in L^p_{\alpha-sp/2}(\mathbb{B}^n)$. Now, for $\epsilon > 0$ small enough such that $-(s-\epsilon)p/2 < \alpha - sp/2 + 1 < p\,((s-\epsilon)/2 - n) - 1$ whenever $-sp/2 < \alpha - sp/2 + 1 < p\,(s/2 - n) - 1$, we have

$$\begin{aligned}|F| * |W_u^{\rho_s}(v)|(x) =& \int_G |F(y)|\, |W_u^{\rho_s}(v)(y^{-1}x)|\, dy \\ \le& C \int_G |F(y)|(1 - |y^{-1}x \cdot o|^2)^{s/2} |1 - \log(1 - |y^{-1}x \cdot o|^2)|\, dy \\ \le& C \int_G |F(y)|(1 - |y^{-1}x \cdot o|^2)^{(s-\epsilon)/2}\, dy \\ =& C \int_{G/K} |\widetilde{f}(y)|(1 - |y^{-1}x \cdot o|^2)^{(s-\epsilon)/2}\, dy.\end{aligned}$$

If we assume that $x = \begin{pmatrix} A_x & b_x \\ c_x^t & d_x \end{pmatrix}$, $y = \begin{pmatrix} A_y & b_y \\ c_y^t & d_y \end{pmatrix}$, $w = x \cdot o = b_x d_x^{-1}$, and $z = y \cdot o = b_y d_y^{-1}$, then

$$d_{y^{-1}x} = \overline{d}_y d_x (1 - (w, z))$$

and

$$|d_x|^{-(s-\epsilon)} = (1 - |x \cdot o|^2)^{(s-\epsilon)/2}.$$

Therefore,

$$\begin{aligned}(1 - |y^{-1}x \cdot o|^2)^{(s-\epsilon)/2} =& |d_{y^{-1}x}|^{-(s-\epsilon)} \\ =& (1 - |x \cdot o|^2)^{(s-\epsilon)/2}\, (1 - |y \cdot o|^2)^{(s-\epsilon)/2} (1 - (x \cdot o, y \cdot o))^{-(s-\epsilon)}.\end{aligned}$$

Thus,

$$\begin{aligned}|F| * |W_u^{\rho_s}(v)|(x) =& C \int_{G/K} |\widetilde{f}(y)|\, \frac{(1 - |x \cdot o|^2)^{(s-\epsilon)/2}\, (1 - |y \cdot o|^2)^{(s-\epsilon)/2}}{|1 - (x \cdot o, y \cdot o)|^{(s-\epsilon)}}\, dy \\ =& C(1 - |w|^2)^{(s-\epsilon)/2} \int_{\mathbb{B}^n} |f(z)|\, \frac{(1 - |z|^2)^{(s-\epsilon)/2-n-1}}{|1 - (w, z)|^{(s-\epsilon)}}\, dz.\end{aligned}$$

According to [41, Theorem 2.10], the operator S which is given by

$$Sf(z) = (1 - |w|^2)^{(s-\epsilon)/2} \int_{\mathbb{B}^n} |f(z)|\, \frac{(1 - |z|^2)^{(s-\epsilon)/2-n-1}}{|1 - (w, z)|^{(s-\epsilon)}}\, dz$$

is continuous on $L^p_{\alpha-sp/2}(\mathbb{B}^n)$ whenever

$$-(s-\epsilon)p/2 < \alpha - sp/2 + 1 < p\,((s-\epsilon)/2 - n) - 1,$$

which is equivalent to $-1 < \alpha < p(s-n) - 1$. Since

$$\|f\|_{L^p_{\alpha-sp/2}(\mathbb{B}^n)} = \|\widetilde{f}\|_{L^p_{\alpha+n+1-sp/2}(G/K)} = \|F\|_{L^p_{\alpha+n+1-sp/2}(G)},$$

the operator $F \mapsto F * |W_u^{\rho_s}(v)|$ is continuous on $L^p_{\alpha+n+1-sp/2}(G)$. The second part is clear from the relation $|F \# W_u^{\rho_s}(v)(x)| \le |F| * |W_u^{\rho_s}(v)(x)|$. □

We conclude our section with the following main result which extends [5, Proposition 3.16(v)] to the projective representation for irrational s.

Theorem 6.9. *Let $1 \le p < \infty$ and $-1 < \alpha < p(s-n)-1$, and let $v \in \mathcal{H}_s^\infty$ be a nonzero smooth vector. The Bergman space $A^p_\alpha(\mathbb{B}^n)$ is the twisted coorbit space of $L^p_{\alpha+n+1-sp/2}(G)$ via the projective representation $(\mathcal{H}_s^\infty, \rho_s)$. That is, $A^p_\alpha(\mathbb{B}^n) = \mathrm{Co}^v_{\rho_s} L^p_{\alpha+n+1-sp/2}(G)$ for $\alpha < p(s-n)-1$.*

Proof. Assume $u = 1_{\mathbb{B}^n}$. By Theorem 6.8, we have $A^p_\alpha(\mathbb{B}^n) = \mathrm{Co}^u_\rho L^p_{\alpha+n+1-sp/2}(G)$. We will show that the twisted coorbit $\mathrm{Co}^v_{\rho_s} L^p_{\alpha+n+1-sp/2}(G)$ does not depend on the analyzing vector v, by applying Theorem 4.8. First, according to Theorem 6.8, the operators $f \mapsto f \# W^{\rho_s}_u(v)$ and $f \mapsto f \# W^{\rho_s}_v(u)$ are continuous on $L^p_{\alpha+n+1-sp/2}(G)$. Next, we show that $W^{\rho_s}_u(\phi) \# W^{\rho_s}_v(u) = C W^{\rho_s}_v(\phi)$ for all $\phi \in \mathcal{H}_s^{-\infty}$. For $f \in \mathcal{H}_s^\infty$, we can use the orthogonality relation in Theorem 2.9 to get $W^{\rho_s}_u(f) \# W^{\rho_s}_v(u) = C W^{\rho_s}_v(f)$. To extend this relation to the dual of the smooth vectors, it is enough to show that

$$\phi \mapsto \int_G \langle \phi, \rho(x)u \rangle \; \langle \rho(x)v, u \rangle \; dx$$

is weakly continuous. Same argument, as in the proof of Theorem 6.5, can be made to show our claim. Therefore, the twisted coorbit spaces $\mathrm{Co}^v_{\rho_s} L^p_{\alpha+n+1-sp/2}(G)$ are all equal to the space $A^p_\alpha(\mathbb{B}^n)$. □

6.3. **New atomic decompositions and frames for Bergman Spaces.** In this section we generate a wavelet frame and an atomic decomposition of Bergman spaces depending on the coorbit theory, where this discretization would work for all projective representations with $s > n$, including the non-integrable cases. Also, we have more freedom in choosing the wavelet u. That is we show that any nonzero smooth vector can be used to generate a Banach frame and an atomic decomposition for Bergman spaces. This result removes the restriction of $s > n$ being rational in Theorem 3.17 from [5]. Also, it completely avoids the use of covering groups of G that was present in that paper, essentially generating atoms from points in G rather than points on a cover.

Theorem 6.10. *Assume that $1 \le p < \infty$, $s > n$, and $-1 < \alpha < p(s-n)-1$. For a nonzero smooth vector u for ρ_s, we can choose ϵ small enough such that for every U_ϵ-well spread set $\{x_i\}_{i \in I}$ in G the following hold.*

1. *(Twisted wavelet frame) The family $\{\rho_s(x_i)u : i \in I\}$ is a Banach frame for $A^p_\alpha(\mathbb{B}^n)$ with respect to the sequence space $\ell^p_{\alpha+n+1-ps/2}(I)$. That is, there exist constants $A, B > 0$ such that for all $f \in A^p_\alpha(\mathbb{B}^n)$ we have*
$$A\|f\|_{A^p_\alpha(\mathbb{B}^n)} \le \|\{\langle f, \rho_s(x_i)u\rangle\}\|_{\ell^p_{\alpha+n+1-sp/2}(I)} \le B\|f\|_{A^p_\alpha(\mathbb{B}^n)},$$
and f can be reconstructed by
$$f = (W^{\rho_s}_u)^{-1} S_1^{-1}\Big(\sum_i W^{\rho_s}_u(f)(x_i)\psi_i \# W^{\rho_s}_u(u)\Big)$$
where $\{\psi_i\}$ is any U_ϵ-BUPU with $\mathrm{supp}\psi_i \subset x_i U_\epsilon$.
2. *(Atomic decomposition) There exists a family of functionals $\{\gamma_i\}_{i\in I}$ on $A^p_\alpha(\mathbb{B}^n)$ such that the family $\{\gamma_i, \rho_s(x_i)u\}$ forms an atomic decomposition for $A^p_\alpha(\mathbb{B}^n)$ with respect to the sequence space $\ell^p_{\alpha+n+1-ps/2}(I)$, so that any $f \in A^p_\alpha(\mathbb{B}^n)$ can be reconstructed by*
$$f = \sum_i \gamma_i(f)\,\rho_s(x_i)u.$$

Proof. We show that the assumptions of Theorem 5.4 are satisfied. Under the conditions on p and s, the twisted coorbit of $L^p_{\alpha+n+1-ps/2}(G)$ is well defined and u satisfies Assumption 4.1 as we have seen in Theorem 6.5, and it is equal to $A^p_\alpha(\mathbb{B}^n)$. Since u is smooth vector for $\mathcal{H}_s$, and $\mathcal{H}_s^\infty$ is continuously embedded in its dual $\mathcal{H}_s^{-\infty}$, the vector u is ρ - and ρ^*-weakly differentiable. According to Theorem 6.8, the mappings

$$f \mapsto f * |W^{\rho_s}_{\rho(E_\alpha)u}(u)| \quad \text{and} \quad f \mapsto f * |W^{\rho_s}_u(\rho^*(E_\alpha)u)|$$

are continuous on $L^p_{\alpha+n+1-ps/2}(G)$. Therefore, we can choose ϵ small enough so that the family $\{\rho_s(x_i)u\}$ forms a frame and an atomic decomposition for $A^p_\alpha(\mathbb{B}^n)$ with reconstruction operators that are given in Theorem 5.4. □

Appendix A. Decompositions of reproducing kernel spaces for twisted convolution

From now on we let $\phi(x) = W^\rho_u(u)$ for some fixed $u \in \mathcal{S}$. Then $B^\sigma_u = \{f \in B \mid f = f\#\phi\}$. Given a compact neighbourhood U of the identity in G, a U-dense and well-spread sequence $\{x_i\} \subseteq G$ and a U-BUPU $\{\psi_i\}$, we formally define the operators

$$T_1 f = \sum_i f(x_i)\sigma(x, x^{-1}x_i)\psi_i\#\phi$$
$$T_2 f = \sum_i \lambda_i(f)\ell^\sigma_{x_i}\phi$$
$$T_3 f = \sum_i c_i f(x_i)\ell^\sigma_{x_i}\phi$$

where $\lambda_i(f) = \int f(y)\psi_i(y)\overline{\sigma(y, y^{-1}x_i)}\, dy$ and $c_i = \int \psi_i(y)\, dy$. The following results will establish when these operators are well defined on B^σ_u.

Define the local oscillations

$$\operatorname{osc}^{r^\sigma}_U f(x) = \sup_{y\in U} |r^\sigma_y f(x) - f(x)| \quad \text{and} \quad \operatorname{osc}^{\ell^\sigma}_U f(x) = \sup_{y\in U} |\ell^\sigma_y f(x) - f(x)|.$$

Proposition A.1. *If $f \in B^\sigma_u$ then*

$$|T_1 f(x) - f(x)| \le |f| * \operatorname{osc}^{r^\sigma}_{U^{-1}}\phi(x)$$
$$|T_2 f(x) - f(x)| \le |f| * \operatorname{osc}^{\ell^\sigma}_{U^{-1}}\phi(x)$$
$$|T_3 f(x) - f(x)| \le |f| * \operatorname{osc}^{r^\sigma}_{U^{-1}}\phi * (|\phi| + \operatorname{osc}^{\ell^\sigma}_{U^{-1}}\phi)(x) + |f| * \operatorname{osc}^{\ell^\sigma}_{U^{-1}}\phi(x).$$

Proof. We see that

$$|\sum_i f(x_i)\sigma(x, x^{-1}x_i)\psi_i(x) - f(x)| \le \sum_i |f(x_i)\sigma(x, x^{-1}x_i) - f(x)|\psi_i(x)$$

and for $x \in x_i U$ we get that $x_i \in xU^{-1}$, so

$$|f(x_i)\sigma(x, x^{-1}x_i) - f(x)| \le \sup_{y\in U^{-1}} |f(xy)\sigma(x, y) - f(x)| = \operatorname{osc}^{r^\sigma}_{U^{-1}} f(x).$$

Next, if $f = f\#\phi$ we get

$$|r^\sigma_y f(x) - f(x)| \le \int |f(z)||\phi(z^{-1}xy)\overline{\sigma(z, z^{-1}xy)}\sigma(x, y) - \phi(z^{-1}x)\overline{\sigma(z, z^{-1})}|\, dz.$$

Since $\sigma(z, z^{-1}xy) = \sigma(z, z^{-1}x)\sigma(x, y)\overline{\sigma(z^{-1}x, y)}$ we get that the integral above reduces to

$$\int |f(z)||r_y^\sigma \phi(z^{-1}x) - \phi(z^{-1}x)|\, dz.$$

Taking supremum we get the desired result.

We have that

$$\begin{aligned} f(x) - T_2 f(x) &= \int f(y)\ell_y^\sigma \phi(x)\, dy - \int \sum_i f(y)\psi_i(y)\overline{\sigma(y, y^{-1}x_i)}\, dy \ell_{x_i}^\sigma \phi(x) \\ &= \int \sum_i f(y)\psi_i(y)[\ell_y^\sigma \phi(x) - \overline{\sigma(y, y^{-1}x_i)}\ell_{x_i}^\sigma \phi(x)]\, dy \\ &= \int \sum_i f(y)\psi_i(y)\ell_y^\sigma[\phi(x) - \ell_{y^{-1}x_i}^\sigma \phi(x)]\, dy. \end{aligned}$$

When $y \in x_i U$ for a compact neighbourhood U of the identity, then $y^{-1}x_i \in U^{-1}$. Thus we get

$$\begin{aligned} |f(x) - T_2 f(x)| &\leq \int \sum_i |f(y)|\psi_i(y)|\ell_y^\sigma \phi(x) - \ell_{y^{-1}x_i}^\sigma \phi(x)|\, dy \\ &\leq \int \sum_i |f(y)|\psi_i(y)|\ell_y^\sigma \mathrm{osc}_{U^{-1}}^{\ell^\sigma} \phi|\, dy \\ &\leq \int |f(y)|\ell_y \mathrm{osc}_{U^{-1}}^{\ell^\sigma} \phi\, dy. \end{aligned}$$

This shows the claim.

$$|T_3 f(x) - f(x)| \leq \int \sum_i \psi_i(y)|f(x_i)\ell_{x_i}^\sigma \phi(x) - f(y)\ell_y^\sigma \phi(x)|\, dy$$

Let us rewrite part of the integrand when $y \in x_i U$

$$\begin{aligned} &|f(x_i)\ell_{x_i}^\sigma \phi(x) - f(y)\ell_y^\sigma \phi(x)| \\ &= |f(x_i)\sigma(y, y^{-1}x_i)\ell_y^\sigma \ell_{y^{-1}x_i}^\sigma \phi(x) - f(y)\ell_y^\sigma \phi(x)| \\ &\leq |[f(x_i)\sigma(y, y^{-1}x_i) - f(y)]\ell_y^\sigma \ell_{y^{-1}x_i}^\sigma \phi(x)| + |f(y)||\ell_y^\sigma \ell_{y^{-1}x_i}^\sigma \phi(x) \;- \ell_y^\sigma \phi(x)| \\ &= |[r_{y^{-1}x_i}^\sigma f(y) - f(y)]\ell_y^\sigma \ell_{y^{-1}x_i}^\sigma \phi(x)| + |f(y)||\ell_y^\sigma \ell_{y^{-1}x_i}^\sigma \phi(x) \;- \ell_y^\sigma \phi(x)| \\ &\leq \mathrm{osc}_{U^{-1}}^{r^\sigma} f(y)[|\ell_y^\sigma \ell_{y^{-1}x_i}^\sigma \phi(x) - \ell_y^\sigma \phi(x)| + |\ell_y^\sigma \phi(x)|] + |f(y)||\ell_y^\sigma \mathrm{osc}_{U^{-1}}^{\ell^\sigma} \phi(x)| \\ &\leq \mathrm{osc}_{U^{-1}}^{r^\sigma} f(y)[|\ell_y^\sigma \mathrm{osc}^{\ell^\sigma}{}_{U^{-1}} \phi(x)| + |\ell_y^\sigma \phi(x)|] + |f(y)||\ell_y^\sigma \mathrm{osc}_{U^{-1}}^{\ell^\sigma} \phi(x)|. \end{aligned}$$

As before the oscillation of f can be transferred onto the kernel, and the final result is obtained. □

From this we obtain

Corollary A.2. *Let B be a solid BF-space and assume that $f \mapsto f * |\phi|$, $f \mapsto f * \mathrm{osc}_{U^{-1}}^{\ell^\sigma} \phi$ and $f \mapsto f * \mathrm{osc}_{U^{-1}}^{r^\sigma} \phi$ are bounded on B, then T_1, T_2, T_3 are well-defined bounded operators on $B_u^\#$.*

Moreover, if there are constants C_U for which

$$\|f * \mathrm{osc}_{U^{-1}}^{\ell^\sigma} \phi\| \leq C_U \|f\| \text{ and } \|f * \mathrm{osc}_{U^{-1}}^{r^\sigma} \phi\| \leq C_U \|f\|$$

and $\lim_{U \to \{e\}} C_U = 0$, then there is a U small enough as well as U-dense $\{x_i\}$ such that the operators are invertible on $B_u^\#$.

We will now use the special form of ϕ to find oscillation estimates via derivatives. We have defined $\phi(x) = \langle u, \rho(x)u\rangle$, and from this and Remark 3.2 we get

$$\operatorname{osc}_U^{r^\sigma} \phi(x) = \sup_{y\in U} |\langle \rho^*(x^{-1})u, \rho(y)u - u\rangle|,$$

and

$$\operatorname{osc}_U^{\ell^\sigma} \phi(x) = \sup_{y\in U} |\langle \rho^*(y)u - u, \rho(x)u\rangle|.$$

In light of this is seems possible to evaluate the oscillation by a certain level of smoothness of the vector u, and this is exactly the approach we will take. We let $v \in \mathcal{S}$ and $\lambda \in \mathcal{S}^*$ be arbitrary elements and define the functions $H(y) = \langle \lambda, \rho(y)u\rangle$ and $K(y) = \langle \rho^*(y)u, v\rangle$. We will now investigate the local oscillations of H and K in terms of derivatives, but first we need to introduce some notation.

If f is a function on G and X is in $\mathfrak{g}$, then define

$$Xf(y) = \frac{d}{dt}\Big|_{t=0} f(y\exp(tX)).$$

We now fix a basis $X_1, \dots, X_n$ for the Lie algebra $\mathfrak{g}$, and for a multi-index α we define

$$X^\alpha f = X_1^{\alpha_1} \cdots X_n^{\alpha_n} f.$$

We will investigate oscillations of H and K on the specific neighbourhood

$$U_\epsilon = \{\exp(t_1X_1)\cdots\exp(t_nX_n) \mid -\epsilon \le t_k \le \epsilon\}.$$

Remember that we choose the cocycle σ and $\epsilon > 0$ such that σ is C^∞ on a neighbourhood containing $U_\epsilon \times U_\epsilon$. According to Lemma 2.5 in [4] there is a constant C_ϵ such that

$$\sup_{y\in U_\epsilon} |H(y) - H(e)| \le C_\epsilon \sum_{\substack{1\le|\alpha|\le n\\ |\delta|=|\alpha|}} \int_{[-\epsilon,\epsilon]^{|\delta|}} |X^\alpha H(\tau_\delta(t_1,\dots,t_n))|(dt_1)^{\delta_1}\dots(dt_n)^{\delta_n},$$

and

$$\sup_{y\in U_\epsilon} |K(y) - K(e)| \le C_\epsilon \sum_{\substack{1\le|\alpha|\le n\\ |\delta|=|\alpha|}} \int_{[-\epsilon,\epsilon]^{|\delta|}} |X^\alpha K(\tau_\delta(t_1,\dots,t_n))|(dt_1)^{\delta_1}\dots(dt_n)^{\delta_n},$$

where $\tau_\delta(t_1,\dots,t_n) = \exp(\delta_1t_1X_1)\cdots\exp(\delta_nt_nX_n)$ for a multi-index δ. Due to the special form of H

$$XH(y) = \frac{d}{dt}\Big|_{t=0}\langle \lambda, \rho(y\exp(tX))u\rangle = \frac{d}{dt}\Big|_{t=0}\langle \lambda, \rho(y)\rho(\exp(tX))u\rangle\overline{\sigma(y,\exp(tX))}.$$

Therefore $X^\delta H(y)$ can be expressed as a sum

$$\sum_{|\gamma|\le|\delta|} \langle \lambda, \rho(y)\rho(X_n)^{\gamma_n}\cdots\rho(X_1)^{\gamma_1}u\rangle g_\gamma(y),$$

where g_γ is an appropriate derivative of the cocycle σ of order $|\gamma|$. Notice, that the g_γ's do not depend on the vectors v and λ used to define H and K. If y is in the compact set U_ϵ the functions g_γ are uniformly bounded, and therefore there is a constant D_ϵ such that

$$\sup_{y\in U_\epsilon}|H(y)-H(e)| \\ \leq D_\epsilon \sum_{\substack{1\leq|\alpha|\leq n\\ |\delta|=|\alpha|}} \int_{[-\epsilon,\epsilon]^{|\delta|}} |\langle \lambda, \rho(\tau_\delta(\mathbf{t}))\rho(X_n)^{\delta_n}\cdots\rho(X_1)^{\delta_1}u)|(dt_1)^{\delta_1}\dots(dt_n)^{\delta_n},$$

when we write $\mathbf{t}=(t_1,t_2,\dots,t_n)$. From this we get that

$$\operatorname{osc}_U^{r^\sigma}\phi(x) \\ \leq D_\epsilon \sum_{\substack{1\leq|\alpha|\leq n\\ |\delta|=|\alpha|}} \int_{[-\epsilon,\epsilon]^{|\delta|}} |\langle u, \rho(x\tau_\delta(\mathbf{t}))\rho(X_n)^{\delta_n}\cdots\rho(X_1)^{\delta_1}u)|(dt_1)^{\delta_1}\dots(dt_n)^{\delta_n}.$$

Treating K the same way we get

$$\operatorname{osc}_U^{\ell^\sigma}\phi(x) \\ \leq D_\epsilon \sum_{\substack{1\leq|\alpha|\leq n\\ |\delta|=|\alpha|}} \int_{[-\epsilon,\epsilon]^{|\delta|}} |\langle \rho^*(X_n)^{\delta_n}\cdots\rho^*(X_1)^{\delta_1}u, \rho(\tau_\delta(\mathbf{t})^{-1}x)u)|(dt_1)^{\delta_1}\dots(dt_n)^{\delta_n}.$$

Lemma A.3. *Assume that B is a solid BF-space on which left and right translations are continuous. If $f\mapsto f*|\langle u,\rho(\cdot)\rho(X_n)^{\delta_n}\cdots\rho(X_1)^{\delta_1}u\rangle|$ and $f\mapsto f*|\langle \rho^*(X_n)^{\delta_n}\cdots\rho^*(X_1)^{\delta_1}u,\rho(\cdot)u\rangle|$ are bounded on B for all $|\delta|\leq\dim(G)$, then*

$$\|f*\operatorname{osc}_U^{\ell^\sigma}\phi\|_B\leq C_\epsilon\|f\|_B$$

and

$$\|f*\operatorname{osc}_U^{r^\sigma}\phi\|_B\leq C_\epsilon\|f\|_B.$$

Moreover, $\lim_{\epsilon\to 0}C_\epsilon=0$.

Proof. Write $\mathbf{t}=(t_1,\dots,t_n)$. Notice that

$$|f*\operatorname{osc}_U^{\ell^\sigma}\phi(x)| \\ \leq D_\epsilon \sum_{\substack{1\leq|\alpha|\leq n\\ |\delta|=|\alpha|}} \int_{[-\epsilon,\epsilon]^{|\delta|}} |\ell_{\tau_\delta(\mathbf{t})}f|*|W_{\rho^*(X_n)^{\delta_n}\cdots\rho^*(X_1)^{\delta_1}u}(u)(x))|(dt_1)^{\delta_1}\dots(dt_n)^{\delta_n}.$$

Since left translation is continuous on B the right hand side defines a function in B by Theorem 3.29 in [37], and by solidity $f*\operatorname{osc}_U^{\ell^\sigma}\phi(x)$ is also in B. Moreover, $\|f*\operatorname{osc}_U^{\ell^\sigma}\phi\|\leq C_\epsilon\|f\|$, where C_ϵ is equal to D_ϵ multiplied by a polynomial in ϵ with no constant term. Since D_ϵ is uniform in ϵ we see that $\lim_{\epsilon\to 0}C_\epsilon=0$.

The proof for convolution with right oscillations follows in a similar manner. □

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