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New Atomic Decompositions of Bergman Spaces on Bounded Symmetric Domains

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NEW ATOMIC DECOMPOSITIONS OF BERGMAN SPACES ON BOUNDED SYMMETRIC DOMAINS

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Abstract. We provide a large family of atoms for Bergman spaces on irreducible bounded symmetric domains. The atomic decompositions are derived using the holomorphic discrete series representations for the domain, and the approach is inspired by recent advances in wavelet and coorbit theory. Our results vastly generalize previous work by Coifman and Rochberg. Their atoms correspond to translates of a constant function at a discrete subset of the automorphism group of the domain. In this paper we show that atoms can be obtained as translates of any holomorphic function with rapidly decreasing coefficients (including polynomials). This approach also settles the relation between atomic decompositions for the bounded and unbounded realizations of the domain.

1. Introduction

This paper is concerned with providing atomic decompositions of Bergman spaces on bounded symmetric domains. The results extend similar results by Coifman and Rochberg [14] carried out for Bergman spaces on the unbounded realization of the domain and on the unit ball. Coifman and Rochberg asked if their decompositions would hold for the bounded domains, and in this paper we give a positive answer to this question. Moreover, we rectify an issue occuring in higher rank spaces which was pointed out in a remark on p. 614 in [2] (see also Remark 4 in [16]). While an extension to the bounded domains was predicted by Faraut and Koranyi in the introduction of [16] our results provide a much larger class of atoms than have previously been discovered. The usual atomic decompositions of Bergman spaces arise from a discretization of the integral reproducing formula, and atoms can thus be regarded as samples of the Bergman kernel in one of the variables. It turns out that this result can be formulated in terms of the holomorphic discrete series representation, in which case the classical atoms correspond to letting a discrete subset of the group of isometries act on a constant function. This viewpoint is extended widely in this paper where we show that atoms can be obtained by translates of any polynomial, or more generally, any analytic function with rapidly decreasing coefficients. Similar results have been obtained by Pap [21] on the unit disc (using the Blaschke group) and by the authors and their collaborators [10, 9] in the case of the unit ball.

In the last section of the article we use representation theory to explain the relation between atomic decompositions of Bergman spaces in the bounded and the unbounded realization. This issue was unresolved in [14] and therefore their Theorem 2 did not transfer directly to bounded domains for all parameters (see beginning of §2 on p. 14 in [14]). Representation

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theory lifts the construction of atomic decompositions from discretizing a reproducing formula on the domain to discretizing a convolution reproducing formula of matrix coefficients for the group of isometries. In particular, we use that the matrix coefficients have the same decay for the two realizations of the domain.

The motivation for our approach comes from coorbit theory for invariant Banach spaces of functions and distributions which was initiated by Feichtinger and Gröchenig [18, 19] and subsequently generalized in [13, 11, 7, 12, 10, 9] in order to treat a wider class of (projective) representations. In this paper we have chosen to state most results without referring to coorbit theory. The main reason for doing so is that some results can be stated with fewer restrictions than if coorbit theory were used, and also the paper would be a lot longer if we had to introduce the entire coorbit machinery.

2. Main results

In this section we state the main results of this article. The following sections will then be devoted to proving those statements. We also introduce the minimal set of notation that will be needed to formulate the statements.

Let D be a bounded symmetric domain in $\mathbb{C}^n$ containing the origin o. Let G be a connected Lie group locally isomorphic with the group of isometries of D. Then G acts transitively on D and $D = G/K$ where K is the stabilizer of o, $K = \{g \in G \mid g \cdot o = o\}$. If G is linear as we will always assume, or more generally, with finite center, then K is compact.

If nothing else is stated, let dz denote the normalized Euclidean measure on D and define $A^2(D)$ to be the Hilbert space of holomorphic functions $f : D \to \mathbb{C}$ for which

$$\|f\|_{A^2} = \left(\int_D |f(z)|^2\, dz\right)^{1/2} < \infty.$$

Thus $A^2(D) = \mathcal{O}(D) \cap L^2(D, dz)$. The space $A^2(D)$ is a reproducing kernel Hilbert space. Denote the reproducing kernel (Bergman kernel) by $K(z, w)$ so that

$$f(z) = \int_D f(w)K(z, w)\, dw$$

for $f \in A^2(D)$. The map $K(z, w)$ is holomorphic in the first variable and antiholomorphic in the second variable. Furthermore $K(w, z) = \overline{K(z, w)}$.

Let $J(x, z)$ be the complex Jacobian of the G-action on D at the point z. Then $z \mapsto J(x, z)$ is holomorphic for every $x \in G$,

$$\int_D f(x \cdot z)|J(x, z)|^2 dz = \int_D f(z) dz \quad \text{for all } f \in L^1(D), \tag{2.1}$$

and the chain rule implies that

$$J(xy, z) = J(x, y \cdot z)J(y, z). \tag{2.2}$$

In particular, since $J(e, z) = 1$ as e acts trivally, it follows that

$$J(x^{-1}, x \cdot z) = J(x, z)^{-1}. \tag{2.3}$$

Furthermore it follows that if $f \in A^2(D)$ then $J(x, z)f(x \cdot z)$ is also in $A^2(D)$ which implies by simple calculation using the reproducing property that, see Lemma 3.1,

$$J(x, z)\overline{J(x, w)}K(x \cdot z, x \cdot w) = K(z, w). \tag{2.4}$$

The genus of D is the number $g := (n + n_1)/r$ where $n = \dim_{\mathbb{C}} D$, n_1 is the complex dimension of the maximal complex subdomain of D of tube type and r is the rank of D, the dimension of a maximal totally geodesic euclidean submanifold. Lastly, let a be a structural constant which satisfies $r(r-1)\frac{a}{2} = n_1 - r$. Define the kernel $h(z,w) = K(z,w)^{-1/g}$, and by abuse of notation let $h(z) = h(z,z)$. For $x, y \in G$ also let $h(x,y) = h(x\cdot 0, y\cdot 0)$ and $h(x) = h(x,x)$. Note that by definition $h(x,y)$ is right K-invariant.

The function $h(z,z)^{\gamma-g} = K(z,z)^{1-\gamma/g}$ is integrable if and only if $\gamma > g-1$. For $\gamma > g-1$ let $c_\gamma^{-1} = \int_D h(z,z)^{\gamma-g} dz$ and $d\mu_\gamma(z) = c_\gamma h(z,z)^{\gamma-g}\, dz$. Then μ_γ is a quasi-invariant probability measure on D.

Let $L^p_\gamma(D)$ be the space of equivalence classes of measurable functions on D for which

$$\|f\|_{L^p_\gamma(D)} := \left(\int_D |f(z)|^p d\mu_\gamma(z)\right)^{1/p} < \infty.$$

Define the weighted Bergman space $A^p_\gamma(D)$ to be the subspace of holomorphic functions in $L^p_\gamma(D)$, i.e. $A^p_\gamma(D) = L^p_\gamma(D) \cap \mathcal{O}(D)$ with norm inherited from $L^p_\gamma(D)$. We note that the space of holomorphic polynomials $P[\mathbb{C}^n]$ is dense in $A^p_\gamma(D)$. The spacet $A^p_\gamma(D)$ is a reproducing kernel Banach space. Denote the reproducing kernel by $K_\gamma(z,w)$, which then satisfies

$$f(z) = \int_D f(w) K_\gamma(z,w)\, d\mu_\gamma(w)$$

for $f \in A^p_\gamma(D)$. See for example Theorem 3 in [24]. We have

$$K_\gamma(z,w) = h(z,w)^{-\gamma} = K(z,w)^{\gamma/g}.$$

Define $j_\gamma(x,z) = J(x,z)^{\gamma/g}$. Note that $j_\gamma(x,z)$ is not necessarily defined globally on the group G, but it is always defined on the universal covering group $\widetilde{G}$ of G. The function j_γ satisfies the cocycle relation $j_\gamma(xy,z) = j_\gamma(x, y\cdot z) j_\gamma(y,z)$. Furthermore

$$j_\gamma(x,z)\overline{j_\gamma(x,w)} K_\gamma(x\cdot z, x\cdot w) = K_\gamma(z,w).$$

This implies that

$$\pi_\gamma(x) f(z) = j_\gamma(x^{-1}, z) f(x^{-1}\cdot z).$$

defines a representation of $\widetilde{G}$ on A^2_γ (or a projective representation of G) and (2.1) implies that the representation is unitary. It is well known that it is also irreducible.

For a holomorphic function f on D decompose it into homogeneous polynomials f_k of degree k, i.e.

$$f = \sum_{k\geq 0} f_k.$$

Define the holomorphic functions with rapidly decreasing coefficients S_γ by

$$S_\gamma = \{f = \sum_{k\geq 0} f_k \mid \forall N,\ \exists C : \|f_k\|_{A^2_\gamma} \leq C(1+k)^{-N}\},$$

and the space of holomorphic functions with moderately growing coefficients S^*_γ by

$$S^*_\gamma = \{f = \sum_{k\geq 0} f_k \mid \exists N, C : \|f_k\|_{A^2_\gamma} \leq C(1+k)^{N}\}.$$

The space S_γ is invariant under π_γ, and π_γ is a projective representation of G on S_γ. The space S_γ^* is the dual of S_γ and the dual pairing of $f \in S_\gamma^*$ and $g \in S_\gamma$ is given by

$$\langle f, g\rangle_\gamma = \sum_{k\geq 0} \int_D f_k(z)\overline{g_k(z)}\, d\mu_\gamma(z).$$

For $\psi \in S_\gamma$ define the wavelet transform $W_\psi^\gamma : S_\gamma^* \to C(\widetilde{G})$ by

$$W_\psi^\gamma(f)(x) = \langle f, \pi_\gamma(x)\psi\rangle_\gamma.$$

Notice that the function $|W_\psi^\gamma(f)|$ defines a continuous function on G, so we will often allow ourselves to write $|W_\psi^\gamma(f)(x)|$ for $x \in G$. Also, the wavelet transform is injective if ψ is non-zero, due to the fact that the representation π_γ restricted to S_γ is irreducible. This follows since S_γ are the smooth vectors for the representation π_γ by [6], and by [5, Proposition 2.6] π_γ restricted to the smooth vectors is irreducible if and only if π_γ is irreducible on $A_\gamma^2(D)$.

Define $L_\alpha^p(G)$ as the space of equivalence classes of functions on the group G for which

$$\|f\|_{L_\alpha^p(G)} := \left(\int_G |f(x)|^p h(x)^\alpha \, d\mu_G(x)\right)^{1/p} < \infty$$

where $d\mu_G(x) = dx$ denotes the suitably normalized left-invariant Haar measure on G.

We are now ready to present the first result of this paper, which gives a wavelet characterization of the Bergman spaces.

Theorem 2.1. *A function f is in the Bergman space $A_\alpha^p(D)$ if and only if $f \in S_\gamma^*$ and $W_\psi^\gamma(f) \in L^p_{\alpha-\gamma p/2}(G)$ and either of the following two conditions are satisfied*

1. *$\gamma, \alpha > g-1$ and and ψ is a non-zero constant,*
2. *$\gamma > g-1+(r-1)\frac{a}{2}$ and $g-1-(r-1)\frac{a}{2}+p(r-1)\frac{a}{2} < \alpha < g-1-(r-1)\frac{a}{2}+p(\gamma-g+1)$ and $\psi \in S_\gamma$ is non-zero.*

Moreover, the norms $\|f\|_{A_\alpha^p}$ and $\|W_\psi^\gamma(f)\|_{L^p_{\alpha-\gamma p/2}}(G)$ are equivalent.

Remark 2.2. *(1) If the rank of the space is large, then γ has to be chosen sufficiently large in order to have the second part of the wavelet description.*

(2) We conjecture that it is possible to avoid this restriction and to show that the Bergman spaces A_α^p have a wavelet characterization for the entire range $g-1 < \alpha < p(\gamma-g+1)+g-1$ for general $\psi \in S_\gamma$.

(3) Due to the coorbit theory for projective representations in [9] *the collection*

$$\mathrm{Co}_\psi^\gamma L^p_{\alpha-\gamma p/2}(G) = \{f \in S_\gamma^* \mid W_\psi^\gamma(f) \in L^p_{\alpha-\gamma p/2}(G)\} \tag{2.5}$$

is a non-trivial Banach space for the entire range of $g-1 < \alpha < g-1+p(\gamma-g+1)$ when ψ is a polynomial. By Theorem 2.1 these spaces are Bergman spaces for α in the smaller interval from Theorem 2.1, and we expect that this is also true in the larger interval.

(4) Similarly the coorbits (2.5) are non-empty Banach spaces for $\psi \in S_\gamma$ when $g-1-(r-1)a/2 < \alpha < g-1-(r-1)a/2+p(\gamma-g+1)$. We expect this to be true for the entire range $g-1 < \alpha < g-1+p(\gamma-g+1)$. In order to improve on this result we need a better understanding of the behaviour of wavelet coefficients $W_\psi^\gamma(\phi)$ for general $\psi, \phi \in S_\gamma$ than we use in this paper (Proposition 5.1).

For a fixed countable collection of points $\{x_i\}_{i\in I}$ in G define the weighted sequence space $\ell^p_\alpha(I)$ as the sequences $\{\lambda_i\}$ for which the norm

$$\|\{\lambda_i\}\|_{\ell^p_\alpha} := \left(\sum_{i\in I} |\lambda_i|^p h(x_i)^\alpha\right)^{1/p}$$

is finite. We are now ready to state the main result of this paper which provides atomic decompositions for Bergman spaces with atoms $\pi_\gamma(x_i)\psi$ for appropriately chosen x_i in G.

Theorem 2.3. *Assume that $\gamma > g-1+(r-1)\frac{a}{2}$ and $\psi \in S_\gamma$ is non-zero. If*

$$g-1+(p-1)(r-1)\frac{a}{2} < \alpha < g-1+p(\gamma-g+1)-(r-1)\frac{a}{2},$$

there is a countable discrete collection of points $\{x_i\}_{i\in I}$ in G and associated functionals $\{\lambda_i\}$ on A^p_α such that every $f \in A^p_\alpha$ can be written

$$f = \sum_{i\in I} \lambda_i(f)\pi_\gamma(x_i)\psi$$

with $\|\{\lambda_i(f)\}\|_{\ell^p_{\alpha-\gamma p/2}} \leq \|f\|_{L^p_\alpha(D)}$. Moreover, if $\{c_i\} \in \ell^p_{\alpha-\gamma p/2}$ then

$$g := \sum_{i\in I} c_i\pi_\gamma(x_i)\psi$$

is in A^p_α and $\|g\|_{L^p_\alpha(D)} \leq \|\{\lambda_i(g)\}\|_{\ell^p_{\alpha-\gamma p/2}}$.

Remark 2.4. *Notice that the range of parameters in Theorem 2.3 is smaller than the range of parameters for which atomic decompositions have been found for tube type domains. See* [8] *in which atomic decompositions are constructed for tube type domains $T = V + i\Omega$ by taking Laplace extensions of Besov spaces of distributions supported on the cone Ω. These atoms do not arise as samples of the Bergman kernel. Also see the Arxiv preprint* [3] *where atoms are obtained for the same spaces but with atoms determined by the Bergman kernel.*

Remark 2.5. *In the case of $\psi = 1$ it is possible to choose the points x_i in a solvable subgroup of G on which π_γ is a representation. In this case the atomic decomposition becomes*

$$f(z) = \sum_{i\in I} \lambda_i(f)\left(\pi_\gamma(x_i)\psi\right)(z) = \sum_{i\in I} \lambda_i(f)\overline{j_\gamma(x_i, o)}K_\gamma(z, x_i\cdot o).$$

By choosing $\alpha = g$ and $\gamma = 2g/p$ we recover Theorem 1 of Coifman and Rochberg [14] *for*

$$1 \leq p < 1+\max\left\{\frac{2}{(r-1)a}, \frac{g-(r-1)a/2}{g-1}\right\}.$$

Namely, there are functionals $\widetilde{\lambda}_i$ such that $\|\{\widetilde{\lambda}_i(f)\}\|_{\ell^p} \leq \|f\|_{A^p(D)}$ and

$$f(z) = \sum_{i\in I} \widetilde{\lambda}_i(f)\left(\frac{K^2(z, w_i)}{K(w_i, w_i)}\right)^{1/p}$$

for $w_i = x_i\cdot o$. Here the functionals $\widetilde{\lambda}_i$ differ from λ_i by unimodular factors.

For a parameter θ we can let $\gamma = (2\alpha+\theta g)/p$. Then we get the expansion from Theorem 2 in [14] *for general bounded domains. Namely, there are functionals $\widehat{\lambda}_i$ such that $\|\{\widehat{\lambda}_i(f)\}\|_{\ell^p} \leq \|f\|_{A^p_\alpha(D)}$ and*

$$f(z) = \sum_{i\in I} \widehat{\lambda}_i(f) \left(\frac{K^2(z,w_i)}{K(w_i,w_i)}\right)^{\frac{\alpha}{gp}} \left(\frac{K(z,w_i)}{K(w_i,w_i)}\right)^{\theta/p}$$

for $w_i = x_i \cdot o$. Here the functionals $\widehat{\lambda}_i$ differ from $h(w_i)^{-\frac{\theta g}{2p}}\lambda_i$ by unimodular factors.

The special case of the unit ball in $\mathbb{C}^n$ was treated in [9] and this article follows the same strategy and we will usually use results from that article as far as the statements are stated in full generality or the generalization is trivial.

3. Background on Bergman spaces

For $z \in D$ let $x_z \in G$ be such that $x_z \cdot 0 = z$. The element x_z is not unique because $x \cdot z = (xk) \cdot 0$ for all $k \in K$. The following is well known but we give a short discussion of the proof as this will be used quite often in the following:

Lemma 3.1. *For $z, w \in D$ and $x \in G$ we have*
a) $J(x^{-1}, x\cdot z) = J(x,z)^{-1}$.
b) $J(x,z)\overline{J(x,w)}K(x\cdot z, x\cdot w) = K(z,w)$.
c) $h(x) = |J(x,0)|^{2/g}$.
d) The G-invariant measure on D is, up to a positive constant, given by

$$f \mapsto \int_D f(z)h(z)^{-g}dz = \int_D f(z)K(z,z)dz, \quad f \in C_c(D).$$

e) The function h is K-biinvariant and $h(x^{-1}) = h(x)$ for all $x \in G$.
f) If U is a compact subset of G containing the identity then there exist constant $0 < C_1 < C_2$ such that

$$C_1 h(x) \leq \sup_{u\in U} h(xu) \leq C_2 h(x) \quad \text{for all } x \in G.$$

Proof. a) We have $J(e,z) = 1$. Hence the cocylce relation gives

$$1 = J(x^{-1}x, z) = J(x^{-1}, x\cdot z)J(x,z).$$

b) Let $f \in A^2(D)$. Then $z \mapsto J(x,z)f(x\cdot z) = J(x^{-1}, x\cdot z)f(x\cdot z)$ is again in $A^2(D)$ as $z \mapsto J(x,z)$ is holomorphic and

$$\int_D |J(x,z)f(x\cdot z)|^2 dz = \int_D |f(x\cdot z)|^2 |J(x,z)|^2 dz = \|f\|^2_{A^2} < \infty$$

The reproducing property of $K(z,w)$ gives:

$$\begin{aligned} J(x,z)f(x\cdot z) &= \int_D f(x\cdot w)J(x,w)K(z,w)dw \\ &= \int_D f(x\cdot w)\overline{J(x^{-1}, x\cdot w)}K(z, x^{-1}x\cdot w)|J(x,w)|^2 dw \\ &= \int_D f(w)\overline{J(x^{-1}, w)}K(z, x^{-1}\cdot w)dw \end{aligned}$$

But this can also be calculated as

$$J(x,z)f(x\cdot z)=\int_D f(w)J(x,z)K(x\cdot z,w)dw.$$

Replacing w by $x\cdot w$ and using that $J(x^{-1},x\cdot w)=J(x,w)^{-1}$ implies the claim.

c) By (b) and by the definition of h as $h(z,z)=K(z,z)^{-1/g}$ we get

$$h(x\cdot 0,x\cdot 0)=K(x\cdot 0,x\cdot 0)^{-1/g}=|J(x,0)|^{2/g}.$$

d) This follows easily from (b).

e) This is well know, but let us give a proof. That h is K-biinvariant follows from (c) and (2.4) as $J(k,z)=1$ for all $k\in K$ and $z\in D$. As $K(z,0)=K(0,w)=1$ for all $z,w\in D$ we get from the transformation rule (2.4) that $J(x,0)J(x,x^{-1}\cdot 0)=1=J(x^{-1},0)h(x^{-1},x\cdot 0)$. Together with a) this implies that $J(x,x^{-1}\cdot 0)=h(x)^{-1}=J(x^{-1},x\cdot 0)$ and the claim follows.

f) The function $J(g,z)$ is the complex Jacobian of the G action at the point $z\in D$. The G-action extends to a smooth action on $\bar{D}$, the closure of D, which is compact. Thus $J(g,z)$ is well defined and smooth on $G\times\bar{D}$. As $U^{-1}\times\bar{D}$ is compact and $J(g,z)$ is never zero on $G\times\bar{D}$, it follows that there exist constants $0<C_1<C_2$ such that $C_1\le|J(u^{-1},z)|\le C_2$ for $u\in U$ and $z\in D$. We also have

$$h(xu)=h(u^{-1}x^{-1})=|J(u^{-1}x^{-1},0)|^{2/g}=|J(u^{-1},x^{-1}\cdot 0)|^{2/g}h(x^{-1})$$

The claim now follows as $h(x^{-1})=h(x)$. □

Remark 3.2. *The statement f) also follows from* [16, Theorem 3.8] *and the following lemma.*

Lemma 3.3. *Let $z=x\cdot o, w=y\cdot o\in D$. Then*

$$h(y^{-1}x)=|J(y^{-1}x,o)|^{2/g}=\frac{h(z)h(w)}{|h(z,w)|^2}\,.$$

Proof. We first note that $h(u,0)=h(0,v)=1$ for all $u,v\in D$. Hence by Lemma 3.1 part b:

$$\begin{aligned}1&=h(y^{-1}x\cdot o,y^{-1}y\cdot o)\\&=J(y^{-1},x\cdot o)^{1/g}\overline{J(y^{-1},y\cdot o)}^{1/g}h(z,w)\\&=|J(y^{-1},x\cdot o)|^{1/g}|J(y^{-1},y\cdot o)|^{1/g}|h(z,w)|\,.\end{aligned}$$

By part (c) in the above lemma we have

$$|J(y^{-1},x\cdot o)|^{2/g}=|J(y^{-1}x,o)||J(x,o)|^{-1}=h(y^{-1}x)/h(x)\,.$$

Then take $x=y$ to get

$$|J(y^{-1},y\cdot o)|^{2/g}=1/h(y)\,.$$

This proves the statement. □

Next we present generalized Forelli-Rudin estimates for the Bergman kernel due to Faraut and Koranyi (see Theorem 4.1, p. 80, in [16]). Denote by ∂D the boundary of the bounded domain D.

Theorem 3.4. *For $b \in \mathbb{R}$ and $c > g-1$ define*

$$J_{b,c}(z) = \int_D \frac{h(w,w)^{c-g}}{|h(z,w)|^{b+c}}\, dw\,, \quad z \in D\,.$$

Then the following holds:

1. *$J_{b,c}(z)$ is bounded on D if and only if $b < -(r-1)a/2$.*
2. *If $b > (r-1)a/2$ then*

$$J_{b,c}(z) \sim h(z,z)^{-b}, \quad \text{as } z \to \partial D\,.$$

These estimates provide the following result about boundedness for an integral operator with kernel given by the modulus of the Bergman kernel.

Theorem 3.5. *For $p > 1$ the operator T defined by*

$$Tf(z) = \int_D f(w)|h(z,w)|^{-\gamma} h(w)^{\gamma-g}\, dw$$

is bounded $L^p_\alpha(D) \to L^p_\alpha(D)$ if $\gamma > g-1+(r-1)a/2$ and

$$g-1-(r-1)a/2 + p(r-1)a/2 < \alpha < g-1-(r-1)a/2 + p(\gamma+1-g).$$

A more general result was proved by Békollé and Temgoua Kagou (see Theorem II.7 in [4]).

Proof. This involves a standard trick based on Schur's lemma. Notice that for $\epsilon > 0$ we have, by Hölder's inequality with $1/p + 1/q = 1$,

$$\begin{aligned}
\|T_\gamma f\|^p_{L^p_\alpha(D)} &\leq \int_D \left(\int_D |f(w)|h(w)^\epsilon |h(z,w)|^{-\gamma} h(w)^{-\epsilon} h(w)^{\gamma-g}\, dw\right)^p h(z)^{\alpha-g}\, dz \\
&\leq \int_D \left(\int_D |f(w)|^p h(w)^{p\epsilon} |h(z,w)|^{-\gamma} h(w)^{\gamma-g}\, dw\right) \\
&\quad \times \left(\int_D h(w)^{-q\epsilon}|h(z,w)|^{-\gamma} h(w)^{\gamma-g}\, dw\right)^{p/q} h(z)^{\alpha-g}\, dz
\end{aligned}$$

which by Theorem 3.4 is, up to a constant, less than

$$\leq \int_D \left(\int_D |f(w)|^p h(w)^{p\epsilon}|h(z,w)|^{-\gamma} h(w)^{\gamma-g}\, dw\right) \left(h(z)^{-q\epsilon}\right)^{p/q} h(z)^{\alpha-g}\, dz,$$

if $q\epsilon > (r-1)a/2$. By Tonelli's theorem this equals

$$= \int_D |f(w)|^p h(w)^{p\epsilon} h(w)^{\gamma-g} \int_D |h(z,w)|^{-\gamma} h(z)^{-p\epsilon} h(z)^{\alpha-g}\, dz\, dw,$$

which by Theorem 3.4 is less than

$$\leq \int_D |f(w)|^p h(w)^{p\epsilon} h(w)^{\gamma-g} h(w)^{-\gamma+\alpha-p\epsilon}\, dw$$

if $\gamma + p\epsilon - \alpha > (r-1)a/2$. Finally this equals

$$= \|f\|^p_{L^p_\alpha(D)}.$$

The restrictions for the parameters α and γ required by Theorem 3.4 can be rewritten to

$$g-1-(r-1)a/2 + p(r-1)a/2 < \alpha < g-1-(r-1)a/2 + p(\gamma+1-g).$$

□

The proof of the wavelet characterization in Theorem 2.1 will be carried out using ideas from coorbit theory for projective representations as presented in [9]. For this purpose we will rephrase Theorem 3.5 in the form of boundedness of convolution operators on G.

For the remainder of this paper we normalize the Haar measure on G such that if f is K-right-invariant and $\widetilde{f}(z) = f(x)$ when $z = xK$, then

$$\int_G f(x)\,dx = \int_D \widetilde{f}(z)h(z)^{-g}\,dz.$$

For functions F, G on the group G we define convolution in the usual way:

$$F * G(x) = \int F(y)G(y^{-1}x)\,dy.$$

Theorem 3.6. *The operator* $C_\gamma : L^p_{\alpha-\gamma p/2}(G) \to L^p_{\alpha-\gamma p/2}(G)$ *defined by*

$$C_\gamma F(x) = \int_G F(y)h(y^{-1}x)^{\gamma/2}\,dy = F * h^{\gamma/2}(x)$$

is bounded if $g - 1 + (p-1)(r-1)a/2 < \alpha < g - 1 + p(\gamma + 1 - g) - (r-1)a/2$.

Proof. We already know that $h(y^{-1}x) = h(x)h(y)/|h(x,y)|^2$, so C_γ takes the form

$$\begin{aligned} C_\gamma F(x) &= h(x)^{\gamma/2} \int_G F(y)h(y)^{\gamma/2}|h(x,y)|^{-\gamma}\,dy \\ &= h(x)^{\gamma/2} \int_D \widetilde{F}(w)|h(x\cdot o, w)|^{-\gamma}h(w)^{\gamma-g}\,dw \end{aligned}$$

where

$$\widetilde{F}(w) = \frac{1}{h(w)^{\gamma/2}} \int_K F(yk)\,dk$$

for $w = y \cdot o$. By Hölder's inequality

$$\int_K |F(yk)|\,dk \le \left(\int_K |F(yk)|^p\,dk\right)^{1/p} \left(\int_K 1\,dk\right)^{1/q} = \left(\int_K |F(yk)|^p\,dk\right)^{1/p},$$

so by Fubini's Theorem and the right-K-invariance of h we get

$$\begin{aligned} \|\widetilde{F}\|^p_{L^p_\alpha(D)} &= \int_D |\widetilde{F}(w)|^p h(w)^{\alpha-g}\,dw \\ &= \int_D \left|\int_K F(yk)\,dk\right|^p h(w)^{\alpha-\gamma p/2-g}\,dw \\ &= \int_G \left|\int_K F(yk)\,dk\right|^p h(y)^{\alpha-\gamma p/2}\,dy \\ &\le \int_G \left(\int_K |F(yk)|^p\,dk\right) h(y)^{\alpha-\gamma p/2}\,dy \\ &\le \int_G |F(y)|^p h(y)^{\alpha-\gamma p/2}\,dy. \end{aligned}$$

Since $C_\gamma F$ is right-K-invariant its $L^p_{\alpha-\gamma p/2}(G)$-norm can be estimated by

$$\begin{aligned}\int_G |C_\gamma F(x)|^p h(x)^{\alpha-\gamma p/2}\,dx &= \int_G \left|\int_D \widetilde{F}(w)|h(x\cdot o,w)|^{-\gamma}h(w)^{\gamma-g}\,dw\right|^p h(x)^\alpha\,dx \\ &= \int_D \left|\int_D \widetilde{F}(w)|h(z,w)|^{-\gamma}h(w)^{\gamma-g}\,dw\right|^p h(z)^{\alpha-g}\,dz \\ &= \int_D \left|T\widetilde{F}(z)\right|^p h(z)^{\alpha-g}\,dz \\ &= \|T\widetilde{F}\|^p_{L^p_\alpha(D)} \\ &\le C\|\widetilde{F}\|^p_{L^p_\alpha(D)} \\ &\le C\|F\|^p_{L^p_{\alpha-\gamma p/2}(G)}\end{aligned}$$

In the second to last estimate we used the boundedness of the operator T. This shows that C_γ is bounded on $L^P_{\alpha-\gamma p/2}(G)$ for the specified ranges of α,γ and p. □

4. Proof of the first part of Theorem 2.1

We now turn to the proof of Theorem 2.1.

Lemma 4.1. *If $\alpha > g-1$ then for $\beta > 2\alpha/p + g - 1$ we have $A^p_\alpha \subseteq A^2_\beta$.*

Proof. Lemma 2.1 in [22] states that

$$|f(z)| \le Ch(z)^{-\alpha/p}\|f\|_{A^p_\alpha}.$$

Therefore,

$$\|f\|^2_{A^2_\beta} \le C\|f\|^2_{A^p_\alpha}\int_D h(z)^{\beta-2\alpha/p-g}\,dz$$

which is finite if $\beta - 2\alpha/p > g-1$. □

Lemma 4.2. *If $\alpha, \gamma > g-1$ then $A^p_\alpha \subseteq S^*_\gamma$.*

Proof. First, if $\alpha > g-1$, then by Lemma 4.1, there is a $\beta > 2\alpha/p+g-1$ for which $A^p_\alpha \subseteq A^2_\beta$. It is known that the monomials $\psi_m(z) = \frac{1}{\|z^m\|_{A^2_\beta}}z^m$, $m = (m_1,\dots,m_n)$ and $z^m = z_1^{m_1}\cdots z_n^{m_n}$, form an orthonormal basis for A^2_β, so

$$f = \sum_{m\ge 0}\langle f,\psi_m\rangle_\beta \psi_m.$$

Let P_k denote the space of polynomials of homogeneous degree k, then $f = \sum_k f_k$ for $f_k \in P_k$, and f_k is given by

$$f_k = \sum_{|m|=k}\langle f,\psi_m\rangle_\beta\psi_m.$$

Taking the A^2_γ norm and using the triangle inequality, we get

$$\|f_k\|_{A^2_\gamma} \le \sum_{|m|=k}\|\psi_m\|_{A^2_\gamma}|\langle f,\psi_m\rangle_\beta| \le \sum_{|m|=k}\frac{\|\psi_m\|_{A^2_\gamma}}{\|\psi_m\|_{A^2_\beta}}\|f\|_{A^2_\beta}.$$

By Proposition 3.3 in [6] there are integers N_1,N_2 and a constant C, which only depend on β and γ, such that

$$\frac{\|\psi_m\|_{A^2_\gamma}}{\|\psi_m\|_{A^2_\beta}} \leq C(1+k)^{N_1-N_2}.$$

From this estimate we get

$$\begin{aligned}\|f_k\|_{A^2_\gamma} &\leq C\|f\|_{A^2_\beta} \sum_{|m|=k} (1+k)^{N_1-N_2} \\ &\leq C\|f\|_{A^2_\beta} \dim(P_k)(1+k)^{N_1-N_2}.\end{aligned}$$

The fact that there is a constant A and an integer N_3 for which

$$\dim(P_k) \leq A(1+k)^{N_3}$$

finishes the proof that $\|f_k\|_\gamma \leq C(1+k)^N$ for some integer N and for some constant C which do not depend on k. Therefore, f is in S^*_γ. □

To prove the first part of Theorem 2.1 let $\psi = 1$ and consider the representation π_γ. Then the wavelet coefficients with $\psi = 1$ and $f \in S^*_\gamma$ become

$$\begin{aligned} W^\gamma_\psi(f)(x) &= \int_D f(z)\overline{J(x^{-1}, z)}^{\gamma/g} h(z)^{\gamma-g}\, dz \\ &= \int_D f(z)K_\gamma(x \cdot o, z)J(x, o)^{\gamma/g}h(z)^{\gamma-g}\, dz \\ &= J(x, o)^{\gamma/g} f(x \cdot o). \end{aligned} \tag{4.1}$$

We see that

$$\int_G |W^\gamma_\psi(f)(x)|^p h(x)^{\alpha-\gamma p/2}\, dx = \int_G |f(x \cdot o)|^p h(x)^\alpha\, dx = \int_D |f(z)|^p h(z)^{\alpha-g}\, dz, \tag{4.2}$$

and from this it follows that if $W^\gamma_\psi(f)$ is in $L^p_{\alpha-\gamma p/2}(G)$ then $f \in A^p_\alpha$. If $f \in A^p_\alpha$, then by Lemma 4.2 f is in S^*_γ. Then equation (4.2) tells us that $W^\gamma_\psi(f)$ is in $L^p_{\alpha-\gamma p/2}(G)$. This finishes the proof of the first part of Theorem 2.1.

5. Proof of the second part of Theorem 2.1

Before we can prove the second part of the theorem, we need to better understand the growth of wavelet coefficients for vectors in S_γ. The following gives the estimates of wavelet coefficients which turn out to be a crucial part of our proof.

Proposition 5.1. *If $u, v \in S_\gamma$ and $x \in G$, then*

$$|W^\gamma_u(v)(x)| \leq Ch^{\gamma/2} * h^{\gamma/2}(x).$$

Moreover, if either u or v is a polynomial, then

$$|W^\gamma_u(v)(x)| \leq Ch^{\gamma/2}(x).$$

Remark 5.2. *Using this result and Theorem 3.4 it is possible to show that $|W_u(v)| \leq Ch^{\gamma/2-\epsilon}$ for any $\epsilon > (r-1)a/2$. We expect that this inequality can be verified for all $\epsilon > 0$. This would enable us to answer some of the questions posed in Remark 2.2.*

Proof. We first assume that $u \in S_\gamma$ and v is a polynomial. Let us calculate the wavelet coefficient for $x \in \widetilde{G}$

$$\begin{aligned} W_u^\gamma(v)(x) &= (v, \rho_\gamma(x)u) \\ &= \int_D v(z)\overline{J(x^{-1}, z)^{\gamma/g} u(x^{-1} \cdot z)}\, d\mu_\gamma(z) \\ &= J(x, o)^{\gamma/g} \int_D v(z)\overline{K(z, x \cdot o)^{\gamma/g} u(x^{-1} \cdot z)}\, d\mu_\gamma(z) \\ &= J(x, o)^{\gamma/g} \int_D v(z)\overline{K_\gamma(z, x \cdot o) u(x^{-1} \cdot z)}\, d\mu_\gamma(z). \end{aligned}$$

Since K_γ is the reproducing kernel for A^2_γ and since $\psi_m = \frac{1}{\|z^m\|_{A^2_\gamma}} z^m$ is an orthonormal basis for A^2_γ, we know that

$$K_\gamma(z, w) = \sum_{m \geq 0} \psi_m(z)\overline{\psi_m(w)}.$$

This yields

$$W_u^\gamma(v)(x) = J(x, o)^{\gamma/g} \sum_{m \geq 0} \psi_m(x \cdot o) \int_D v(z)\overline{\psi_m(z) u(x^{-1} \cdot z)}\, d\mu_\gamma(z).$$

Since v is a polynomial and $z \mapsto u(x^{-1} \cdot z)$ is holomorphic, we get that the sum over m is finite, i.e.

$$W_u^\gamma(v)(x) = J(x, o)^{\gamma/g} \sum_{m \in M} \psi_m(x \cdot o) \int_D v(z)\overline{\psi_m(z) u(x^{-1} \cdot z)}\, d\mu_\gamma(z).$$

where M is a finite index set (which can be chosen independently of x). By Lemma 1.3 and Proposition 3.3 in [6] we get that u is bounded. Therefore, there is a constant C_m for which

$$\int_D |v(z)\overline{\psi_m(z) u(x^{-1} \cdot z)}|\, d\mu_\gamma(z) \leq C_m.$$

By the finiteness of M we can therefore derive that for $x \in G$ we have

$$|W_u^\gamma(v)(x)| \leq C|J(x, o)|^{\gamma/g} = Ch^{\gamma/2}(x).$$

Now assume that $v \in S_\gamma$ is not a polynomial. For $x, y \in \widetilde{G}$ equation (4.1) gives

$$W_1(v)(y)W_u(1)(y^{-1}x) = h(y)^\gamma v(y \cdot o)\overline{\pi_\gamma(x)u(y \cdot o)}.$$

This function is well-defined for y in G, and

$$\int_G W_1(v)(y)W_u(1)(y^{-1}x)\, dy = \int_D v(z)\overline{\pi_\gamma(x)u(z)} h(z)^{\gamma - g}\, dz = W_u^\gamma(v)(x).$$

From this we get

$$|W_u^\gamma(v)| \leq |W_1^\gamma(v)| * |W_u^\gamma(1)|.$$

By unitarity $|W_1^\gamma(v)(x)| = |W_v^\gamma(1)(x^{-1})| \leq Ch^{\gamma/2}(x^{-1}) = Ch^{\gamma/2}(x)$ which concludes the proof. □

Recall that a Banach space B of functions on G is said to be solid if $f \in B$, $f \geq 0$, and g is measurable function on G with $|g| \leq f$ implies that $g \in B$. A typical examples are the spaces $L^p(G)$, $1 \leq p \leq \infty$.

Corollary 5.3. *If $f \mapsto f * h^{\gamma/2}$ is bounded on a solid Banach function space B on G, then $f \mapsto f * |W_u^\gamma(v)|$ is bounded for all u, v in S_γ.*

Proof. From the previous result we get

$$|f * |W_u^\gamma(v)|| \leq C|f| * h^{\gamma/2} * h^{\gamma/2}.$$

Since B is solid, $|f|$ is in B if f is, and then $|f| * h^{\gamma/2} * h^{\gamma/2}$ is in B by assumption. By solidity $f * |W_u^\gamma(v)|$ is in B and $\|f * |W_u^\gamma(v)|\|_B \leq C\|f\|_B$. □

So far we have not needed to introduce a projective representation of G along with its cocycle. We choose to introduce them now, since subsequent arguments are easily carried out based on established knowledge about square integrable projective representations and twisted convolution. In particular we will prove the second part of Theorem 2.1 by using twisted convolution to swap the vector $\psi = 1$ in $W_\psi^\gamma(f)$ by an arbitrary vector in S_γ.

Let ρ_γ be a projective representation of G corresponding to the representation π_γ of $\widetilde{G}$, and let σ_γ be the corresponding cocycle for ρ_γ. We retain the notation $W_\psi^\gamma : S_\gamma^* \to \mathcal{M}(G)$ for the wavelet transform

$$W_\psi^\gamma(f)(x) = \langle f, \rho_\gamma(x)\psi\rangle,$$

and note that it agrees with the previous notation up to a unimodular factor. Here $\mathcal{M}(G)$ denotes the space of Borel measurable functions on G. Also define twisted convolution of f, g by

$$f\#_\gamma g(x) = \int_G f(y)g(y^{-1}x)\overline{\sigma_\gamma(y, y^{-1}x)}\, dy$$

when the integral makes sense.

Lemma 5.4. *Let ϕ, ψ be in S_γ. If $F \in L^p_{\alpha-\gamma p/2}(G)$, then $F\#W_\psi^\gamma(\phi)$ exists as an integrable function if $g - 1 + (p-1)(r-1)a/2 < \alpha < p(\gamma - g + 1) + g - 1 - (r-1)a/2$. Moreover, $F \mapsto F\#W_\psi^\gamma(\phi)$ is bounded $L^p_{\alpha-\gamma p/2}(G) \to L^p_{\alpha-\gamma p/2}(G)$.*

Proof. We know from Proposition 5.1 that $|F| * |W_\psi^\gamma(\phi)| \leq C|F| * (h^{\gamma/2} * h^{\gamma/2})$. By Tonelli's Theorem this equals $(|F| * h^{\gamma/2}) * h^{\gamma/2}$. From Theorem 3.6 we know that $|F| * h^{\gamma/2}$ is in $L^p_{\alpha-\gamma p/2}(G)$, which means that $|F| * (h^{\gamma/2} * h^{\gamma/2})$ exists as an integrable function. This also means that $|F| * |W_\psi^\gamma(\phi)|$ exists and therefore so does $F\#W_\psi^\gamma(\phi)$. Another application of Theorem 3.6 proves the continuity statement. □

For a function F on G and a vector $X \in \mathfrak{g}$ define the derivative

$$XF(x) = \frac{d}{dt}\Big|_{t=0} F(e^{-tX}x).$$

Lemma 5.5. *There is a sequence of smooth compactly supported functions $\psi_n : G \to [0,1]$ such that $\psi_{n+1} \geq \psi_n$ and for any finite collection $X_1, X_2, \ldots, X_N \in \mathfrak{g}$ there is a constant C_N such that for every n we have $\|X_1X_2\ldots X_N\psi_n\|_\infty \leq C_N$.*

Proof. We can take $\psi_n^\vee(x) = \psi_n(x^{-1})$ to be partial sums of a partition of unity as contstructed on p. 329 in [25]. □

Proposition 5.6. *Let ψ, ϕ and η be in S_γ and assume that*

$$g - 1 + (p-1)(r-1)a/2 < \alpha < p(\gamma - g + 1) + g - 1 - (r-1)a/2.$$

If f is in S_γ^* and $W_\psi^\gamma(f)$ is in $L^p_{\alpha-\gamma p/2}(G)$, then $W_\psi^\gamma(f)\#W_\phi^\gamma(\eta)$ equals $d_\gamma^{-1}\langle \eta, \psi\rangle_{A^2_\gamma} W_\phi^\gamma(f)$ where d_γ is the formal dimension of π_γ.

Proof. First, note that if $W_\psi^\gamma(f)$ is in $L^p_{\alpha-\gamma p/2}(G)$, then by Lemma 5.4 the twisted convolution is defined in terms of an integrable function. Therefore we can employ the Lebesgue Dominated Convergence Theorem to get that

$$W_\psi^\gamma(f)\#W_\phi^\gamma(\eta)(x) = \lim_{n\to\infty}\int_G W_\psi^\gamma(f)(y)W_\phi^\gamma(\eta)(y^{-1}x)\psi_n(y)\overline{\sigma_\gamma(y,y^{-1}x)}\,dy, \tag{5.1}$$

where $\psi_n \in C_c^\infty(G)$ is a sequence of functions which are equal to one on growing compact sets whose union is G and which satisfy $0 \leq \psi_n(x) \leq 1$ for all $x \in G$. The equation (5.1) rewrites to

$$W_\psi^\gamma(f)\#W_\phi^\gamma(\psi)(x) = \lim_{n\to\infty}\int_G \psi_n(y)\langle f, \rho_\gamma(y)\psi\rangle\langle \rho_\gamma(y)\eta, \rho_\gamma(x)\phi\rangle\,dy.$$

Replace $\rho_\gamma(x)\phi$ by a general smooth vector ξ and define the smooth compactly supported function

$$\Psi_n(y) = \psi_n(y)\overline{\langle \rho_\gamma(y)\eta, \xi\rangle},$$

then

$$\int_G \psi_n(y)\langle f, \rho_\gamma(y)\psi\rangle\langle\rho_\gamma(y)\eta,\xi\rangle\,dy = \int_G \langle f, \Psi_n(y)\rho_\gamma(y)\psi\rangle\,dy.$$

Since Ψ_n is smooth and compactly supported, by Theorem 3.27 in [23] the vector

$$\rho_\gamma(\Psi_n)\psi := \int_G \Psi_n(y)\rho_\gamma(y)\psi\,dy$$

is a smooth vector, and

$$\int_G \langle f, \Psi_n(y)\rho_\gamma(y)\psi\rangle\,dy = \langle f, \rho_\gamma(\Psi_n)\psi\rangle.$$

In order to finish the proof it suffices to show that the latter expresson converges to $c\langle f,\xi\rangle$ where $c = d_\gamma^{-1}\langle\psi,\eta\rangle$. We will do so by demonstrating that the vectors $\rho_\gamma(\Psi_n)\psi$ converge to $c\xi$ in S_γ.

Let us first verify convergence in A_γ^2.

$$\|\rho_\gamma(\Psi_n)\psi - c\xi\|_{A_\gamma^2}^2 = \|\rho_\gamma(\Psi_n)\psi\|_{A_\gamma^2}^2 + |c|^2\|\xi\|_{A_\gamma^2}^2 - (\overline{c}\langle\rho_\gamma(\Psi_n)\psi,\xi\rangle - c\langle\xi,\rho_\gamma(\Psi_n)\psi\rangle).$$

By the Lebesgue dominated convergence theorem and square integrability of the representation, and the unimodularity of the group, we have that

$$\lim_{n\to\infty}\langle\rho_\gamma(\Psi_n)\psi,\xi\rangle = \lim_{n\to\infty}\int \psi_n(y)\langle\xi,\rho_\gamma(y)\eta\rangle\langle\rho_\gamma(y)\psi,\xi\rangle\,dy = \int\langle\xi,\rho_\gamma(y)\eta\rangle\langle\rho_\gamma(y)\psi,\xi\rangle\,dy = c\|\xi\|_{A_\gamma^2}^2.$$

Therefore we just need to check that

$$\lim_{n\to\infty}\|\rho_\gamma(\Psi_n)\psi\|_{A_\gamma^2}^2 = |c|^2\|\xi\|_{A_\gamma^2}^2.$$

Notice that by Fubini we have

$$\|\rho_\gamma(\Psi_n)\psi\|_{A_\gamma^2}^2 = \int_G\int_G \psi_n(x)\psi_n(y)\langle\xi,\rho_\gamma(x)\eta\rangle\langle\rho_\gamma(x)\psi,\rho_\gamma(y)\psi\rangle\langle\rho_\gamma(y)\eta,\xi\rangle\,dx\,dy,$$

and we will be able to apply the Lebesgue dominated convergence theorem if we can show that the function $|\langle\xi,\rho_\gamma(x)\eta\rangle\langle\rho_\gamma(x)\psi,\rho_\gamma(y)\psi\rangle\langle\rho_\gamma(y)\eta,\xi\rangle|$ is integrable on $G\times G$. This is the same as showing that the integral

$$\int_G\int_G |W_\eta^\gamma(\xi)(y)||W_\psi^\gamma(\psi)(y^{-1}x)||W_\eta^\gamma(\xi)(x)|\,dx\,dy$$

is finite. From Proposition 5.1 it is known that $W_\eta^\gamma(\xi) \in L^2(G)$ and $W_\psi^\gamma(\psi) \in L^p(G)$ for some $1 < p < 2$, so the Kunze-Stein phenomenon [15] tells us that the integral is finite since $|W_\eta^\gamma(\xi)| * |W_\psi^\gamma(\psi)|$ is again in $L^2(G)$. Therefore

$$\lim_{n\to\infty} \|\rho_\gamma(\Psi_n)\psi\|^2_{A^2_\gamma} = \int_G \int_G \langle \xi, \rho_\gamma(x)\eta\rangle \langle \rho_\gamma(x)\psi, \rho_\gamma(y)\psi\rangle \langle \rho_\gamma(y)\eta, \xi\rangle \, dx\, dy = |c|^2 \|\xi\|^2_{A^2_\gamma}$$

by the orthogonality relations [1, Theorem 3].

We can now repeat the argument with derivatives of the vector $\rho_\gamma(\Psi_n)\psi$ to show it converges to derivatives of $c\eta$. If $X \in \mathfrak{g}$ then define $\rho_\gamma(X) = \lim_{t\to 0} \frac{1}{t}(\rho_\gamma(e^{tX}) - I)$. We have

$$\rho_\gamma(X)\rho_\gamma(\Psi_n)\psi = W_\psi^\gamma(\eta)(y) X\psi_n(y) + W_\psi^\gamma(\rho_\gamma(X)\eta)\psi_n(y).$$

As before, it follows from the Lebesgue dominated convergence theorem and the assumption that $X\psi_n$ is uniformly bounded in n, that $\rho_\gamma(X)\rho_\gamma(\Psi_n)\psi$ converges to $c\rho_\gamma(X)\eta$ in A^2_γ. This argument can be repeated to show that $\rho_\gamma(\Psi_n)\psi$ converges to $c\eta$ in S_γ. □

Remark 5.7. *If one chooses to define the twisted convolution of $W_\psi(f)$ and $W_\phi(\eta)$ using the expression* (5.1)*, then the proof in fact shows that it equals $d_\gamma^{-1}\langle \eta, \psi\rangle_{A^2_\gamma} W_\phi^\gamma(f)$ regardless of the integrability condition. This could be a key observation for applying coorbit theory in this setting for dealing with the extended range of α.*

We have gathered all the results needed to finish the proof of Theorem 2.1. First, we know that $f \in A^p_\alpha$ if and only if $f \in S^*_\gamma$ and $W_1^\gamma(f) \in L^p_{\alpha-\gamma p/2}(G)$. Employ Proposition 5.6 to show that for an arbitrary vector $\psi \in S_\gamma$ we have $W_\psi^\gamma(f) = C W_1(f) \# W_\psi(1)$ for non-zero constant C. By Lemma 5.4 we get that

$$\|W_\psi^\gamma(f)\|_{L^p_{\alpha-\gamma p/2}(G)} = C\|W_1(f)\#W_\psi(1)\|_{L^p_{\alpha-\gamma p/2}(G)} \le C'\|W_1(f)\|_{L^p_{\alpha-\gamma p/2}(G)}.$$

Therefore $W_\psi^\gamma(f)$ is in $L^p_{\alpha-\gamma p/2}(G)$.

If, on the other hand, $W_\psi^\gamma(f)$ is in $L^p_{\alpha-\gamma p/2}(G)$, then Lemma 5.4 and Proposition 5.6 tell us that

$$\|W_1^\gamma(f)\|_{L^p_{\alpha-\gamma p/2}(G)} = D\|W_\psi(f)\#W_1(\psi)\|_{L^p_{\alpha-\gamma p/2}(G)} \le D'\|W_\psi(f)\|_{L^p_{\alpha-\gamma p/2}(G)},$$

where D is a non-zero constant. By the first part of Theorem 2.1 we get that f is in A^p_α. These calculations also show that the norms are equivalent, and the proof is done.

6. Proof of Theorem 2.3

Normalize $\psi \in S_\gamma$ such that $W_\psi^\gamma(\psi)\#W_\psi^\gamma(\psi) = W_\psi^\gamma(\psi)$. Then $L^p_{\alpha-\gamma p/2}(G)\#W_\psi^\gamma(\psi)$ is a non-zero closed Banach subspace of $L^p_{\alpha-\gamma p/2}(G)$. This follows from Corollary 5.3 which implies that $F \mapsto F\#W_\psi^\gamma(\psi)$ is a bounded projection on $L^p_{\alpha-\gamma p/2}(G)$. By Theorem 2.1 the space $L^p_{\alpha-\gamma p/2}(G)\#W_\psi^\gamma(\psi)$ is isomorphic to $A^p_\alpha(D)$ via the wavelet transform. Therefore, if $f \in A^p_\alpha(D)$ we have the following integral representation of $W_\psi^\gamma(f)$:

$$W_\psi^\gamma(f) = W_\psi^\gamma(f)\#W_\psi^\gamma(\psi).$$

The atomic decomposition in Theorem 2.3 will follow from a discretization of this integral representation. This approach is standard in coorbit theory for integrable representations [18, 19] and has been extended to non-integrable projective representations via estimates involving the smoothness of the kernel $W_\psi^\gamma(\psi)$ in [9]. We refer to these papers for details.

The first thing we note is that the space S_γ is the space of smooth vectors for the representation π_γ (see [6]). Therefore the space S_γ is invariant under the differential operator

$$\pi_\gamma(X)u = \frac{d}{dt}\Big|_{t=0} \pi_\gamma(e^{-tX})u$$

where X is in the Lie algebra $\mathfrak{g}$ of G. This means that left and right derivatives (as defined in [9]) of a wavelet coefficient $W_u^\gamma(v)$, where u and v are in S_γ, will correspond to wavelet coefficients $W_{u'}^\gamma(v')$ where $u', v' \in S_\gamma$. Using Theorem 3.6 and Corollary 5.3, while noticing that $|W_1^\gamma(1)| = h^{\gamma/2}$, we realize that $L^p_{\alpha-\gamma p/2}(G) * |W_{u'}^\gamma(v')| \subseteq L^p_{\alpha-\gamma p/2}(G)$ for the specified range of parameters when u' and v' are in S_γ. This implies that the solid Banach function space $L^p_{\alpha-\gamma p/2}(G)$ and the vector $\psi \in S_\gamma$ satisfy [9, Assumption 3], and then [9, Theorem 8] can be applied as described below.

Let $X_1, \dots, X_n$ be a fixed basis for the Lie algebra $\mathfrak{g}$ of G. For $\epsilon > 0$ define the compact neighborhood U_ϵ of the identity by

$$U_\epsilon = \{\exp(t_1X_1)\dots\exp(t_nX_n) \mid -\epsilon \leq t_k \leq \epsilon \text{ for all } k = 1, \dots, n\}.$$

We will choose the cocycle σ and an ϵ small enough such that σ is C^∞ on $U_\epsilon \times U_\epsilon$ (this is always possible due to [26, Lemma 7.20]). Assume that x_i in G satisfy that x_iU_ϵ cover G, and that there is a compact neighborhood $V_\epsilon \subseteq U_\epsilon$ such that the x_iV_ϵ are pairwise disjoint. Let $\{\psi_i\}$ be a bounded uniform partition of unity satisfying that (i) $0 \leq \psi_i \leq 1$, (ii) $\sum_i \psi_i = 1$ and (iii) $\mathrm{supp}(\psi_i) \subseteq x_iU_\epsilon$. The existence of the points x_i and partitions of unity was proved in [17]. Define the operator $S : L^p_{\alpha-\gamma p/2}(G)\#W_\psi^\gamma(\psi) \to L^p_{\alpha-\gamma p/2}(G)\#W_\psi^\gamma(\psi)$ by

$$SF = \sum_i \widetilde{\lambda}_i(F)\ell^{\sigma_\gamma}_{x_i} W_\psi^\gamma(\psi),$$

where the functionals $\widetilde{\lambda}_i$ are given by

$$\widetilde{\lambda}_i(F) = \int F(y)\psi_i(y)\overline{\sigma_\gamma(y, y^{-1}x_i)}\,dy.$$

This operator is well-defined, and it is possible to choose ϵ small enough that it is invertible. In that case F in $L^p_{\alpha-\gamma p/2}(G)\#W_\psi^\gamma(\psi)$ can be reconstructed by

$$F = \sum_i \lambda_i(S^{-1}F)\ell^{\sigma_\gamma}_{x_i} W_\psi^\gamma(\psi).$$

Notice, that since $C_c(G)$ are dense in $L^p_{\alpha-\gamma p/2}(G)$ for $1 \leq p < \infty$, this sum converges in norm (see Theorem 8 in [9]). Since $W_\psi^\gamma : A^p_\alpha(D) \to L^p_{\alpha-\gamma p/2}(G)\#W_\psi^\gamma(\psi)$ is an isomorphism we get that

$$W_\psi^\gamma(f) = \sum_i \widetilde{\lambda}_i(S^{-1}W_\psi^\gamma(f))\ell^{\sigma_\gamma}_{x_i} W_\psi^\gamma(\psi) = W_\psi^\gamma\left(\sum_i \widetilde{\lambda}_i(S^{-1}W_\psi^\gamma(f))\rho_\gamma(x_i)\psi\right).$$

for $f \in A^p_\alpha(D)$. This proves the decomposition

$$f = \sum_i \lambda_i(f)\rho_\gamma(x_i)\psi$$

when the functionals $\lambda_i : A^p_\alpha(D) \to \mathbb{C}$ are defined by $\lambda_i(f) = \widetilde{\lambda}_i(S^{-1}W^\gamma_\psi(f))$. According to Theorem 8 in [9] this is an atomic decomposition with sequence space given by the norm

$$\|\{c_i\}\| = \left(\int_G \left| \sum_i c_i 1_{x_i U_\epsilon}(x) \right|^p h^{\alpha-\gamma p/2}(x)\, dx \right)^{1/p}.$$

This norm is equivalent to the sequence norm used in Theorem 2.3, since there are constants $0 < C_1 < C_2$ such that $C_1 h(x) \leq h(xu) \leq C_2 h(x)$ when $u \in U_\epsilon$, see Lemma 3.1.

7. The unbounded case

In this section we explain how the atomic decompositions for the unbounded realization of the domain can be obtained. The idea is simply to use the Caley transform to transform information from the bounded domain to the unbounded realization. In particular, the representation theory makes the transition from the bounded to the unbounded case clear. Let $c : U \to D$ be the Cayley transform from the irreducible unbounded symmetric domain U to the irreducible bounded symmetric domain D. The Cayley transform is explicity described in Section 6 in [20]. Note that there exists an element $c \in G_{\mathbb{C}}$, where $G_{\mathbb{C}}$ is the simply connected complex Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ such that $c(z) = cz$. For the special case of the unit disc $D = \{z \in \mathbb{C} \mid |z| < 1\}$, the Cayley transform is given by $z \mapsto i\frac{z+1}{-z+1} = \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix} \cdot z$. The Bergman space $A^2(U)$ is defined as the holomorphic functions for which

$$\|f\|^2_{A^2(U)} = \int |f(z)|^2\, dz < \infty.$$

Here dz denotes the Euclidean measure on $\mathbb{C}^n$ which gives the unit cube volume 1. The Bergman space is a reproducing kernel Hilbert space with reproducing kernel denoted K_U satisfying

$$f(z) = \int_U f(w) K_U(z,w)\, dz.$$

Let $J(c,w)$ denote the complex Jacobian of the mapping $c : U \to D$, then $C : A^2(D) \to A^2(U)$ defined by $Cf(z) = J(c,z)f(cz)$ is an isometry. Also, remembering that the Bergman kernel on the bounded realization is denoted K, we have

$$J(c,z)\overline{J(c,w)}K(cz,cw) = K_U(z,w).$$

Define the functions $h_U(z,w) = K_U(z,w)^{-1/g}$ and $h_U(z) = h_U(z,z)$, then we have

$$h_U(z) = |J(c,z)|^{-2/g} h(cz).$$

Now define the weighted Bergman space to be the space of holomorphic functions on U for which

$$\|f\|^p_{A^p_\alpha(U)} = \int_U |f(z)|^p h_U(z)^{\alpha-g}\, dz < \infty.$$

Then $C_{\alpha,p} : A^p_\alpha(D) \to A^p_\alpha(U)$ given by

$$C_{\alpha,p} f(z) = J(c,z)^{\frac{2\alpha}{pg}} f(cz)$$

is an isometry with inverse

$$C^{-1}_{\alpha,p} f(w) = J(c^{-1},w)^{\frac{2\alpha}{pg}} f(c^{-1}w).$$

We can then define a projective representation τ_γ of G on $A^2_\gamma(U)$ by

$$\tau_\gamma(x) = C_{\gamma,2}\pi_\gamma(x)C^{-1}_{\gamma,2}.$$

This projective representation is irreducible, unitary and square integrable, and the smooth vectors are $T_\gamma = C_{\gamma,2}S_\gamma$ with dual $T^*_\gamma = (C^{-1}_{\gamma,2})^* S^*_\gamma$ where the adjoint is defined in terms of the weak dual pairing. Define the wavelet transform of $f \in T^*_\gamma$ and $\psi \in T_\gamma$ by

$$W^\gamma_\psi(f)(x) = \langle f, \tau_\gamma(x)\psi\rangle.$$

We can now restate our main theorems in the unbounded setting:

Theorem 7.1. *A function f is in the Bergman space $A^p_\alpha(U)$ if and only if $f \in T^*_\gamma$ and $W^\gamma_\psi(f) \in L^p_{\alpha-\gamma p/2}(G)$ and either of the following two conditions are satisfied*

(1) $\gamma, \alpha > g-1$ *and and* $\psi = C_{\gamma,2}1$,
(2) $\gamma > g-1+(r-1)\frac{a}{2}$ *and* $g-1-(r-1)\frac{a}{2}+p(r-1)\frac{a}{2} < \alpha < g-1-(r-1)\frac{a}{2}+p(\gamma-g+1)$ *and* $\psi \in T_\gamma$ *is non-zero.*

Moreover, the norms $\|f\|_{A^p_\alpha(U)}$ and $\|W^\gamma_\psi(f)\|_{L^p_{\alpha-\gamma p/2}}(G)$ are equivalent.

Also we have

Theorem 7.2. *Assume that $\gamma > g-1+(r-1)\frac{a}{2}$ and $\psi \in T_\gamma$ is non-zero. If*

$$g-1+(p-1)(r-1)\frac{a}{2} < \alpha < g-1+p(\gamma-g+1)-(r-1)\frac{a}{2},$$

there is a countable discrete collection of points $\{x_i\}_{i\in I}$ in G and associated functionals $\{\lambda_i\}$ on $A^p_\alpha(U)$ such that every $f \in A^p_\alpha(U)$ can be written

$$f = \sum_{i\in I} \lambda_i(f)\rho_\gamma(x_i)\psi$$

with $\|\{\lambda_i(f)\}\|_{\ell^p_{\alpha-\gamma p/2}} \le \|f\|_{L^p_\alpha(U)}$. Moreover, if $\{c_i\} \in \ell^p_{\alpha-\gamma p/2}$ then

$$g := \sum_{i\in I} c_i\rho_\gamma(x_i)\psi$$

is in $A^p_\alpha(U)$ and $\|g\|_{A^p_\alpha(U)} \le \|\{\lambda_i(g)\}\|_{\ell^p_{\alpha-\gamma p/2}}$.

Remark 7.3. *(1) In the case where $\psi = C_{\gamma,2}1$ we recover the results by Coifman and Rochberg with $\alpha = g$ and $\gamma = (2\alpha+\theta g)/p$. Namely*

$$\begin{aligned} f(z) &= \sum_i \lambda_i(f)\overline{J(c^{-1}x_i, o)^{\gamma/g}} K_U^{\gamma/g}(z, c^{-1}x_i\cdot o) \\ &= \sum_i \lambda_i(f)\overline{J(c^{-1}x_i, o)^{\gamma/g}} K_U^{\gamma/g}(z, c^{-1}x_i\cdot o). \end{aligned}$$

Thus representation theory makes clear the transition between atomic decomopsitions for the two realizations of the domain.

(2) One could also use the isometry $C_{\alpha,p} : A^p_\alpha(D) \to A^p_\alpha(U)$ to transfer the atomic decomposition from the bounded domain to the unbounded realization. In particular, if $F \in A^p_\alpha(U)$ and if ψ is in S_γ, then

$$C^{-1}_{\alpha,p}F = \sum_i \lambda_i(C^{-1}_{\alpha,p}F)\pi_\gamma(x_i)\psi.$$

This sum converges absolutely in $A^p_\alpha(D)$, and since $\pi_\gamma(x_i)\psi \in S_\gamma \subseteq A^p_\alpha(D)$ it follows that

$$F = \sum_i \lambda_i(C^{-1}_{\alpha,p}F)C_{\alpha,p}\pi_\gamma(x_i)\psi.$$

Finally note that this equals

$$F(z) = J(c,z)^{\frac{2\alpha}{pg}-\frac{\gamma}{g}} \sum_i \lambda_i(C^{-1}_{\alpha,p}F)\tau_\gamma(x_i)C_{\gamma,2}\psi(z).$$

Due to the factor $J(c,z)^{\frac{2\alpha}{pg}-\frac{\gamma}{g}}$ the atoms in this decomposition are not translates under τ_γ of a single function. This makes this decomposition obtained in Theorem 7.2 more attractive.

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