2013

A semigroup/Laplace transform approach to approximating flows

Ladorian Nichele Latin
Louisiana State University and Agricultural and Mechanical College

A Semigroup/Laplace Transform Approach To Approximating Flows

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
Ladorian Latin
B.S., Northwestern State University, 2006
M.S., Louisiana State University, 2008
December 2013

Acknowledgments

I thank God for allowing me to make it this to this point in my life. My experiences in graduate school drew me closer to Him, which made my journey worth everything. I thank my dissertation advisor, Dr. Frank Neubrander. The Lord knows that we have had our moments, but our pairing must have been ordained by God. I wouldn't have made it this far with any other advisor.

I thank my family and friends. I appreciate all of the prayers and support given to me over the years. I am especially thankful for my grandmother, Ms. Doris Douglas, whom I was named after, and my mother Mrs. Cynthia Latin. This dissertation is dedicated to them.

Table of Contents

Acknowledgments 1

Abstract 3

1 Flows and Semigroups 1
1.1 Semigroups Induced By Flows 1
1.2 Strongly Continuous Semigroups 8
1.3 Semigroups Induced By Autonomous Flows 32
1.4 Semigroups Induced By Non-Autonomous Flows 46

2 Approximating Solutions of Non-Autonomous Systems of Linear Ordinary Differential Equations With Semigroup Techniques 49
2.1 The Resolvent Approach 54
2.2 Lie-Trotter Type Approximation Methods 62
2.3 Peaceman-Rachford Approximation Method 69

References 72

Vita 74

Abstract

It is well known that all flows in a state space Ω induce a semigroup of linear operators on an appropriately chosen vector space of functions (observables) from Ω into a vector space Z (observations). After choosing appropriate continuity assumptions on the flow, the associated semigroup will be strongly continuous and will have a linear, infinitesimal generator $\mathscr{A}$. The purpose of this dissertation is to explore approximation methods for linear semigroups and/or Laplace transform inversion methods in order to reconstruct the flow starting with the linear generator $\mathscr{A}$. In preparing for these investigations, we collect some of the essential approximation theorems of semigroup theory and improve a recent generalization of the Trotter-Kato Theorem due to McAllister, Neubrander, Riser, and Zhuang. Moreover, we show that rational Laplace transform inversions of order m are exact for all polynomials of degree less than m. We will demonstrate that the flow can be efficiently reconstructed whenever the generator $\mathscr{A}$ of the induced semigroup has a resolvent that can be efficiently computed or approximated. We demonstrate this for flows $\sigma(t,\omega)$ solving nonlinear first order ordinary differential equations $x'(t) = a(x(t))$, $x(s) = \omega$ and the induced generator $(\mathscr{A}f)(s) = a(s)f'(s)$ and for flows $\gamma(t,s,\omega)$ solving non-autonomous linear first order ordinary differential equations $u'(t) = a(t)u(t)$, $u(s) = \omega$ and the induced generator $(\mathscr{A}f)(s) = f'(s)+a(s)f(s)$. As a by-product of our investigation, we find a numerically efficient way to compute the inverse of increasing real-valued functions. Finally, we explore whether linear semigroup approximation methods can be used efficiently to approximate solutions of non-autonomous Cauchy problems $u'(t) = A(t)u(t)$, $u(s) = x$ in terms of the generator $(\tilde{\mathscr{A}}f)(s) = f'(s) + A^*(s)f(s)$ of the induced linear operator semigroup. As we will see, the Lie-Trotter approach suggested by G. Nickel seems to be the only efficient way to find the solutions of the non-autonomous problems in terms of the semigroup generated by $\tilde{\mathscr{A}}$.

Chapter 1
Flows and Semigroups

1.1 Semigroups Induced By Flows

To give a framework for our discussion of non-autonomous systems of linear ordinary differential equations, we will discuss first how all (linear or nonlinear) evolution processes lead to the consideration of linear operator semigroups on vector spaces of functions. In order to study evolution equations,(that is, equations that describe processes that change over time) we introduce the following notation. Assuming that time can be identified with the real number line, let

$$H = \{(t, s) \in \mathbb{R}^2 : t \geq s \geq 0\}$$

and let Ω be the set of all possible states of an evolutionary system. Then the evolution of the system with the initial state $\omega \in \Omega$ given at time $s \geq 0$ can be described as a map

$$\gamma : H \times \Omega \to \Omega$$

that assigns to the initial state ω existing at time $s \geq 0$ the state $\gamma(t, s, \omega)$ that the system attains at time $t \geq s \geq 0$. Since γ describes the time propagation of $\omega \in \Omega$ starting at time $s \geq 0$, the map

$$t \mapsto \gamma(t, s, \omega),$$

for $t \geq s \geq 0$, is called a *flow* or orbit with initial value $\gamma(s, s, \omega) = \omega$.

Now let $t \geq r \geq s \geq 0$ and let a system be in an initial state ω at time $s \geq 0$. Then the system is in the state

$$\omega' := \gamma(r, s, \omega)$$

at time $r \geq s$ and at the final state

$$\gamma(t, s, \omega)$$

at time $t \geq r$. If the system is *deterministic*, it follows that

$$\gamma(t, r, \omega') = \gamma(t, s, \omega)$$

or

$$\gamma(t, r, \gamma(r, s, \omega)) = \gamma(t, s, \omega).$$

Therefore, every evolutionary flow satisfies

$$\begin{array}{ll} \text{(i)} & \gamma(t, r, \gamma(r, s, \omega)) = \gamma(t, s, \omega), \\ \text{(ii)} & \gamma(s, s, \omega) = \omega \end{array} \tag{1.1}$$

for all $t, s \geq 0$ and $\omega \in \Omega$. This is called "Huygens' principle of scientific determinism." It was first formulated by Jacques Hadamard in his 1923 treatise "Lectures on Cauchy's Problem" where he writes:

"The action of phenomena produced at the instant $t = 0$ on the state of matter at the later time $t = t_0$ takes place by the meditation of every intermediate instant $t = t'$ i.e. (assuming $0 < t' < t_0$) in order to find out what takes place for $t = t_0$, we can deduce from the state at $t = 0$ the state at $t = t'$ and, from the latter, the required state at $t = t_0$. [Huygens' principle] is what philosophers (...) call one of the "laws of thought": that is, an unavoidable law of our reason, which we could by no means conceive as not existing and without which we could not think. If today we discover Assyrian inscriptions, we cannot dream of supposing that, at any instant between the time when they were made and the time of their discovery, those inscriptions could have ceased to exist and all trace of them have disappeared. [Huygens' principle] must therefore be considered as a truism, which does not mean that it cannot interest us; for the geometer does not dislike truisms."

So far we have seen that every deterministic system leads to a flow. Now, to show that every flow leads to the consideration of a linear operator semigroup on a vector space of functions, it is convenient to consider autonomous flows first. This special type of flow is easier to deal with and will be covered first before we look at the general set-up.

Time-shift invariant deterministic systems will be referred to as autonomous. These autonomous systems are special because it is assumed that the evolution $\gamma(t, s, \omega)$ of a system does not depend on the time instants t, s (points of time), but only on the time span, i.e., the quantity $t - s$. A flow γ is *autonomous* provided that

$$\gamma(t + r, s + r, \omega) = \gamma(t, s, \omega) \tag{1.2}$$

for all $\omega \in \Omega$, $t \geq s \geq 0$ and $r \geq 0$. Because of the time-shift invariance, an autonomous flow is completely determined by

$$\sigma(t, \omega) := \gamma(t, 0, \omega)$$

where $t \geq 0$ and $\omega \in \Omega$. It follows that

$$\begin{aligned} \sigma(t, \sigma(s, \omega)) &= \gamma(t, 0, \gamma(s, 0, \omega)) = \gamma(t + s, s, \gamma(s, 0, \omega)) \\ &= \gamma(t + s, 0, \omega) = \sigma(t + s, \omega) \end{aligned}$$

since γ is autonomous. Thus every autonomous flow σ satisfies

$$\begin{aligned} \sigma(t, \sigma(s, \omega)) &= \sigma(t+s, \omega), \\ \sigma(0, \omega) &= \omega. \end{aligned} \tag{1.3}$$

Next, we will show that every autonomous flow σ leads to a linear *operator semigroup*, i.e. a family $\mathcal{T} = \{T(t)\}_{t \geq 0}$ of linear operators defined on a vector space $\mathcal{N}$ satisfying

$$\begin{aligned} T(t)T(s)g &= T(t+s)g, \\ T(0)g &= g \end{aligned} \tag{1.4}$$

for all $g \in \mathcal{N}$ and $t, s \geq 0$. To construct the space $\mathcal{N}$ we assume that the states $\sigma(t, \omega)$ of an autonomous system σ can be observed through functions $g : \Omega \to Z$ where Z is a vector space of observations capturing essential features of the physical states. In other words, we assume that the observation functions $g : \Omega \to Z$ are contained in some (complex) vector space

$$\mathcal{N} := \mathcal{G}(\Omega, Z)$$

of functions from Ω into Z. In this manner, every autonomous flow defines a linear semigroup $T(t)$ on $\mathcal{N}$ by

$$T(t)g : \omega \mapsto g(\sigma(t, \omega)) \tag{1.5}$$

for $g \in \mathcal{N}$. To see that 1.5 is a semigroup, observe that

$$(T(t)T(r)g)[\omega] = (T(t)z)[\omega] = z(\sigma(t, \omega)),$$

where

$$z[\omega] = (T(r)g)[\omega] = g(\sigma(r, \omega)).$$

Then

$$(T(t)T(r)g)[\omega] = g(\sigma(r, \sigma(t, \omega))) = g(\sigma(t+r, \omega)) = T(t+r)g[\omega].$$

Also,

$$T(0)g[\omega] = g(\gamma(0, \omega)) = g(\omega).$$

We summarize these observations with the following proposition that goes back to Sophus Lie (see Chapter 4 in [13]; see also [5], [6], [7], [17], [9], Section 3.28 and Epilogue Section B).

Proposition 1.1. *Let Ω be a set, let $\sigma(t, \omega)$ be an autonomous flow where $t \geq 0$ and $\omega \in \Omega$, and let $\mathcal{N} := \mathcal{G}(\Omega, Z)$ be a vector space of functions from Ω into Z. Then the operators*

$$T(t)g : \omega \to g(\sigma(t, \omega))$$

$(t \geq 0)$ define a semigroup of linear operators on $\mathcal{N}$.

As it is shown in the next proposition, examples of autonomous flows are given by solutions $\sigma(t,\omega) = x(t)$ of linear or nonlinear autonomous initial value problems

$$x'(t) = a(x(t)),\; x(0) = \omega.$$

Proposition 1.2. *Let X be a Banach space and let $a\colon D(a) \to X$ be an operator with domain $D(a) \subset X$ and let $x\colon [0,\infty) \to D(a)$. If, for all $\omega \in D(a)$ and $s \geq 0$, the intial value problem*

$$x'(t) = a(x(t)),\; x(s) = \omega, \tag{1.6}$$

has a unique solution $x(t)$ for all $t \geq s$, then the flow is autonomous; i.e, (1.2) is valid.

Proof. Let $\omega \in D(a)$, $s \geq 0$, and let $x(\cdot)$ be the unique solution of (1.6). Let $r \geq 0$ and

$$x(t) := \tilde{x}(t-r),$$

where $\tilde{x}$ is some differentiable continuation of $x(\cdot)$ from $[s,\infty)$ to $(-\infty,\infty)$. Then

$$x(s+r) = \tilde{x}(s) = x(s) = \omega$$

and for $t \geq s + r$,

$$x'(t) = \tilde{x}'(t-r).$$

Since $t - r \geq s$, it follows that

$$x'(t) = \tilde{x}'(t-r) = x'(t-r) = a(x(t-r)) = a(\tilde{x}(t-r)) = a(x(t)).$$

By the uniqueness of the solution,

$$x(t) = \sigma(t, s+r, \omega)$$

for all $t \geq s + r$. Now let $t' \geq s$. Then $t' + r \geq s + r$ and therefore,

$$x(t'+r) = \sigma(t'+r, s+r, \omega)$$

and

$$x(t'+r) = \tilde{x}(t') = x(t') = \sigma(t', s, \omega).$$

Thus,

$$\sigma(t+r, s+r, \omega) = \sigma(t, s, \omega)$$

for all $0 \leq s \leq t$, $\omega \in D(a)$ and $r \geq 0$.

□

Proposition 1.3. *Let $X = \mathbb{R}$ and consider the first order ordinary differential equation*

$$x'(t) = a(x(t)),\ x(0) = \omega.$$

If $\dfrac{1}{a(\cdot)}$ *is integrable, and if*

$$A(t) := \int_0^t \frac{1}{a(r)} dr$$

is invertible, then the solution $x(\cdot)$ is given by the autonomous flow

$$\sigma(t, \omega) = x(t) = A^{-1}(t + A(\omega)) \tag{1.7}$$

for all $t \geq 0$ such that

$$t + A(\omega) \in Range(A).$$

Moreover, let $\mathcal{N} := \mathcal{G}(\mathbb{R}, Z)$ be a vector space of functions from $\mathbb{R}$ into Z where Z is some vector space of observations and where $\omega \to g(A^{-1}(t + A(\omega)) \in \mathcal{N}$ for all $g \in \mathcal{N}$ and $t \geq 0$. Then

$$T(t)g(\omega) := g(\sigma(t, \omega)) = g(A^{-1}(t + A(\omega))) \tag{1.8}$$

is the linear semigroup associated with (1.7).

Proof. Note that if the function a takes on values in $X = \mathbb{R}$ and if a satisfies the previous conditions, then any solution $x(\cdot)$ of (1.6) for which $a(x(\cdot))$ is not identically zero on an open interval satifies

$$A(x(t)) - A(\omega) = \int_0^t \frac{x'(r)}{a(x(r))} dr = \int_0^t 1\, dr = t.$$

Therefore,

$$A(x(t)) = A(\omega) + t$$

or

$$\sigma(t, \omega) = x(t) = A^{-1}(t + A(\omega))$$

if $t + A(\omega) \in \text{Range}(A) =$ Domain (A^{-1}). By Proposition 1.2, the flow

$$\sigma(t, \omega) = \gamma(t, 0, \omega)$$

is autonomous. It follows from (1.5) that

$$T(t)g(\omega) := g(A^{-1}(t + A(\omega)))$$

defines a semigroup on $\mathcal{N} := \mathcal{G}(\mathbb{R}, Z)$ given that $T(t)g \in \mathcal{N}$ for all $g \in \mathcal{N}$ and $t \geq 0$. □

The uniqueness assumption is crucial in order to ensure that a solution $\sigma(t,\omega) = x(t)$ of (1.6) defines a flow. To see this, consider $a(x) = 2\sqrt{x}$ and

$$x'(t) = 2\sqrt{x(t)},\ x(0) = \omega. \tag{1.9}$$

For $\omega = 0$, the solution of (1.9) are not unique. For any $\delta \geq 0$, the zero solution as well as the function

$$x(t) := \begin{cases} 0, & \text{if } 0 \leq t \leq \delta \\ (t-a)^2, & \text{if } t > \delta \end{cases}$$

are solutions of (1.9) with $x(0) = 0$. Moreover,

$$x(t) = (t + \sqrt{\omega})^2 \text{ with } t \geq 0$$

solves (1.9) with $x(0) = \omega \geq 0$. Now define

$$\sigma^*(t,\omega) := \begin{cases} (t+\sqrt{\omega})^2, & \text{if } t \geq 0, \omega > 0 \\ (t-1)^2 \chi_{[0,1]}(t), & \text{if } t \geq 0, \omega = 0. \end{cases}$$

Then, for each $\omega \geq 0$, $\sigma^*(t,\omega)$ solves (1.9) but $\sigma^*(t,\omega)$ is not an autonomous flow since $\sigma^*\left(\frac{3}{4}, \sigma^*\left(\frac{3}{4}, 0\right)\right) = 1 \neq \sigma^*\left(\frac{3}{4} + \frac{3}{4}, 0\right) = 0$.

If the flow is not autonomous, then in order to construct a linear semigroup $\{T(t)\}_{t\geq 0}$, our observation functions, $g \in \mathcal{N} = \mathcal{G}(\Omega, Z)$ must be time dependent; i.e., our observations g_t may change according to functions

$$x \in \mathcal{M} := \mathcal{F}([0,\infty), \mathcal{N}), x(t) = g_t \in \mathcal{N} = \mathcal{G}(\Omega, Z) \tag{1.10}$$

where $\mathcal{F}$ is some vector space of functions from $[0,\infty)$ into $\mathcal{N}$. The flow γ induces a family $\{T(t)\}_{t\geq 0}$ of linear maps (operators) on $\mathcal{M}$, where

$$T(t)x(s) : \omega \mapsto x(t+s)[\gamma(t+s,s,\omega)]. \tag{1.11}$$

The fact that all flows induce linear semigroups on an appropriately chosen state space is well known. However, we could not find the explicit form of the semigroup (1.11) anywhere in the literature. If the non-autonomous flow is given by an evolution family

$$\zeta(t,s,\omega) = U(t,s)\omega$$

solving the non-autonomous Cauchy problem

$$u'(t) = A(t)u(t),\ u(s) = \omega,$$

with linear generator $A(t)\,(t \geq 0)$, then the construction (1.11) can be found in [17]; for a similar construction, see [22]. In this generality, (1.11) was introduced in [19] and [24].

Proposition 1.4. *Let Ω be a set, $\gamma(t,s,\omega)$ be a non-autonomous flow with $t \geq s \geq 0$, $\omega \in \Omega$, and let $\mathcal{M} := \mathcal{F}([0,\infty),\mathcal{N})$. Then the operators*

$$T(t)x(s) : \omega \mapsto x(t+s)[\gamma(t+s,s,\omega)].$$

define a semigroup of linear operators on $\mathcal{M}$.

Proof. To see that $\mathcal{T} = \{T(t)\}_{t\geq 0}$ is a linear semigroup, observe that

$$T(t)T(r)x(s)[\omega] = T(t)z(s)[\omega] = z(t+s)[\zeta(t+s),s,\omega)],$$

where $z(s)[\omega] = T(r)x(s)[\omega] = x(r+s)[\zeta(r+s,s,\omega)]$. Then,

$$\begin{aligned} T(t)T(r)x(s)[\omega] &= x(r+t+s)[\zeta(r+t+s,t+s,\zeta(t+s,s,\omega))] \\ &= x(t+r+s)[\zeta(t+r+s,s,\omega)] = T(t+r)x(s)[\omega]. \end{aligned}$$

□

Proposition 1.4 shows that every flow satisfying (1.1) leads to a semigroup of linear operators on a vector space X. If the space X is a Banach space, if the semigroup operators are bounded linear operators from X into X, and if the maps $t \to T(t)x$ are continuous from $\mathbb{R}^+$ into X for all $x \in X$, then we will see in the following section that one can associate a linear "generator" $\mathscr{A}$ with domain and range in X to the semigroup $T(t)$. Since the flow uniquely determines the semigroup and the semigroup uniquely determines its generator $\mathscr{A}$, it is reasonable to expect that the generator $\mathscr{A}$ carries all pertinent information about the flow $\gamma(t,s,\omega)$. In particular, one can expect that one should be able to reconstruct the flow starting with the generator $\mathscr{A}$. For this reason, in Section 1.2, we will concentrate on semigroup results that allow us to construct the semigroup $\{T(t)\}_{t\geq 0}$ and the flow $\gamma(t,s,\omega)$ generated by $\mathscr{A}$ in terms of the resolvent. In particular, we consider

(a) autonomous flows $\sigma(t,\omega)$ given by the solution $x(t)$ of non-linear first order ordinary differential equations $x'(t) = a(x(t))$, $x(0) = \omega$, and

(b) non-autonomous flows $\gamma(t,s,\omega)$ given by the solution of non-autonomous initial value problems $x'(t) = A(t)x(t)$, $x(s) = \omega$ for some continuous function $A : t \to A(t) \in M_{2\times 2}(\mathbb{R})$.

In case (a), the flow semigroup $\{T(t)\}_{t\geq 0}$ is given by

$$T(t)g(\omega) = g(\sigma(t,\omega))$$

for $g \in C_0([0,\infty),\mathbb{C})$ and the associated generator is

$$(\mathscr{A}g)(\omega) := a(\omega)g'(\omega). \tag{1.12}$$

For completeness, we mention that if a non-autonomous flow $\gamma(t,s,\omega)$ is given by the solution of non-autonomous, nonlinear, first order problems

$$x'(t) = a(t,x(t)),\ x(s) = \omega,$$

then the flow semigroup $\{T(t)\}_{t\geq 0}$ is given by

$$T(t)f(s,\omega) := f(t+s, \gamma(t+s,s,\omega))$$

for $f : ([0,\infty), \mathbb{R}^+) \to \mathbb{C}$ with associated generator

$$(\mathscr{A}f)(s,\omega) := f_s(s,\omega) + a(s,\omega)f_\omega(s,\omega)$$

where f_s, f_ω are the partial derivatives of f with respect to s and ω, respectively. In case (b), the flow semigroup $\{T(t)\}_{t\geq 0}$ is given by

$$T(t)f(s) = f(t+s)U(t+s,s)$$

where $U(t,s)$ are the evolution operators generated by the matrix $A(t)$ and where f is in a Banach space of functions with values in $M_{2\times 2}(\mathbb{C})$. The associated generator is of the form

$$(\mathscr{A}f)(s) := f'(s) + f(s)A(s). \tag{1.13}$$

As we will see, the flows $\sigma(t,\omega)$ (for 1.12) and $\gamma(t,s,\omega)$ (for 1.13) can be approximated efficiently through either Laplace transform inversion methods or approximation methods for the semigroup $\{T(t)\}_{t\geq 0}$ generated by $\mathscr{A}$.

1.2 Strongly Continuous Semigroups

Let X be a Banach space. Then the space of all bounded linear operators T from X into X will be denoted by $\mathcal{L}(X)$ with norm

$$||T|| = \sup_{||x||\leq 1} ||Tx||.$$

Definition 1.5. A family $\mathcal{T} = \{T(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ is called a semigroup if

(i) $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$,

(ii) $T(0) = I$.

It is called a *strongly continuous* semigroup (or C_0 - semigroup) if

(iii) $\lim_{t\to t_0} T(t)x = T(t_0)x$ for all $x \in X$ and $t_0 \geq 0$.

The following lemma and its proof are standard; see, for example [19].

Lemma 1.6. *A family $\mathcal{T} = \{T(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ is strongly continuous if and only if $\lim_{t\to 0+} T(t)x = x$ for all $x \in X$ and $t_0 \geq 0$. A strongly continuous semigroup $\mathcal{T}$ is always of type (M,ω), i.e., there exists $M \geq 1$ and $\omega \in \mathbb{R}$ such that $||T(t)|| \leq Me^{\omega t}$ for all $t \geq 0$.*

Proof. It is clear that if $\mathcal{T}$ is strongly continuous, then

$$\lim_{t\to 0+} T(t)x = T(0)x = x$$

for all $x \in X$. Conversely, assume that $\lim_{t\to 0+} T(t)x = x$ for all $x \in X$. First, we will show that there exists $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$||T(t)|| \leq Me^{\omega t}$$

for all $t \geq 0$. We want to show that there exists $\mu > 0$ such that

$$||T(t)|| \leq M$$

for all $t \in [0, \mu]$. We will assume that assertion is false, i.e., there exists a sequence $t_n \to 0$ with $||T(t_n)|| \geq n$. Then, by the Principle of Uniform Boundedness, there exists $x \in X$ such that

$$||T(t_n)x|| \to \infty$$

as $t_n \to 0$ which contradicts the assumption that

$$\lim_{t\to 0+} T(t)x = x.$$

Now let $t > 0$. Then

$$t = n\mu + \varepsilon$$

for some $n \in \mathbb{N}_0$ and $0 \leq \varepsilon < \mu$. Thus,

$$||T(t)|| = ||T(\mu)^n T(\varepsilon)|| \leq M^{n+1} = M * M^n = Me^{n \ln M} = Me^{n\mu\omega} \leq M^{et\omega},$$

where $\omega := \frac{1}{\mu} \ln M$. Now let $t_0 > 0$ and $t_0 + h > 0$. Since

$$||T(t_0 + h)x - T(t_0)x|| \leq ||T(t_0)||||T(h)x - x||$$

for $h > 0$ and

$$||T(t_0 + h)x - T(t_0)x|| \leq ||T(t_0 + h)||||x - T(-h)x||$$

for $h < 0$, it follows that $\lim_{h\to 0+} T(h)x = x$ implies strong continuity. □

Example 1.7. Consider the differential equation (1.6) for the case $a(x) = 1$; i.e., consider

$$x'(t) = 1,\ x(0) = \omega \geq 0.$$

Then the autonomous flow solving this differential equation is given by $\sigma(t, \omega) = t + \omega$ and the associated semigroup (1.8) is given by shift operators

$$T(t)g : \omega \to g(t + \omega) \tag{1.14}$$

where g is a function in a function vector space $\mathcal{N} := \mathcal{N}([0,\infty),\mathbb{C})$. Now, if $\Omega = \mathbb{R}_+$ and $\mathcal{N} = C_0[0,\infty)$ (the space of all continuous functions $[0,\infty) \to \mathbb{C}$ vanishing at infinity), then (1.14) is a strongly continuous semigroup. To see this, let $g \in C_0[0,\infty)$ and let $\varepsilon > 0$. Then there exists $N > 0$ such that

$$\sup_{x \geq N} |g(x)| \leq \frac{\varepsilon}{3}$$

and $0 < \delta < 1$ such that

$$|g(x) - g(y)| \leq \frac{\varepsilon}{3}$$

whenever $x, y \in [0, N+1]$ and $|x - y| < \delta$. Thus,

$$||T(t)g - g|| \leq \sup_{x \in [0,N]} |g(t+x) - g(x)| +$$
$$\sup_{x \geq N} |g(t+x) - g(x)| \leq \sup_{x \in [0,N]} |g(t+x) - g(x)| + 2 \sup_{x > N} |g(x)| \leq \varepsilon$$

if $0 < t < \delta$. However, if one choses $\mathcal{N}$ to be $C_b[0,\infty)$ (the space of all bounded continuous functions from $[0,\infty)$ into $\mathbb{C}$), then the shift semigroup (1.14) is not strongly continuous. To see this let $g(x) = e^{ix^2}$. Then

$$\begin{aligned} ||T(t+h)g - T(t)g|| &= \sup_{t \geq 0} |e^{i(t+h)^2} - e^{it^2}| = \sup_{t \geq 0} |e^{i(t^2+2ht+h^2)} - e^{it^2}| \\ &= \sup_{t \geq 0} |e^{i(2ht+h^2)} - 1| = 2 \end{aligned}$$

for all $t \geq 0$ and $h > 0$. Therefore, $t \to T(t)g$ is nowhere continuous and, thus by Pettis' Theorem (see [1], p.7), not measurable since the range of $T(\cdot)g$ contains no countably dense subset. This is because all the elements in the range $\{T(t)g, t \geq 0\}$ have distance 2 from each other.

Further examples of semigroups of the form $T(t)g(x) = g(\sigma(t,x))$ generated by flows solving first order ordinary differential equations of the form

$$x'(t) = a(x(t)),\ x(0) = \omega$$

will be given in Section 1.3.

Definition 1.8. The *infinitesimal generator* $\mathscr{A}$ of a strongly continuous semigroup $\mathcal{T} = \{T(t)\}_{t \geq 0}$ on a Banach space X is the operator

$$\mathscr{A}x := \lim_{h \to 0+} \frac{T(h)x - x}{h}$$

defined for x in the domain

$$D(\mathscr{A}) := \left\{ x \in X |\lim_{h \to 0+} \frac{T(h)x - x}{h} \text{ exists} \right\}.$$

The following result summarizes the fundamental properties of strongly continuous semigroups (see also [19]).

Proposition 1.9. *Let $(\mathscr{A}, D(\mathscr{A}))$ be the generator of a strongly continuous semigroup $\mathcal{T} = \{T(t)\}_{t\geq 0}$. Then the following are valid.*

(i) The semigroup $\mathcal{T}$ commutes with $\mathscr{A}$ on $D(\mathscr{A})$ and, for all $t \geq 0$,

$$T(t)x - x = \begin{cases} \mathscr{A} \int_0^t T(s)x\, ds & \text{if } x \in X, \\ \int_0^t T(s)\mathscr{A}x\, ds & \text{if } x \in D(\mathscr{A}). \end{cases} \tag{1.15}$$

(ii) $D(\mathscr{A})$ is dense in X and $(\mathscr{A}, D(\mathscr{A}))$ is a closed, linear operator.

Proof. (i) Since $T(t)$ commutes with $\frac{1}{h}(T(h) - I)$ for $t \geq 0$ and $h > 0$, it follows that $x \in D(\mathscr{A})$ implies $T(t)x \in D(\mathscr{A})$ and $\mathscr{A}T(t)x = T(t)\mathscr{A}x$. Now, for $x \in X$,

$$\begin{aligned} \frac{1}{h}(T(h)\int_0^t T(s)x\ ds - \int_0^t T(s)x\ ds) &= \frac{1}{h}\int_0^t T(s+h)x\ ds - \frac{1}{h}\int_0^t T(s)x\ ds \\ &= \frac{1}{h}\int_t^{t+h} T(s)x\ ds - \frac{1}{h}\int_0^h T(s)x\ ds \to T(t)x - x \text{ as } h \downarrow 0. \end{aligned}$$

Therefore, $\int_0^t T(s)x\ ds \in D(\mathscr{A})$ and

$$T(t)x - x = \mathscr{A}\int_0^t T(s)x\ ds$$

for all $x \in X$. If $x \in D(\mathscr{A})$, then $\frac{1}{h}T(s)(T(h) - I)x \to T(s)\mathscr{A}x$ uniformly for $s \in [0, t]$. So,

$$\begin{aligned} \mathscr{A}\int_0^t T(s)x\ ds &= \lim_{h\downarrow 0}\frac{1}{h}(T(h) - I)\int_0^t T(s)x\ ds \\ &= \int_0^t \lim_{h\downarrow 0}\frac{1}{h}(T(h) - I)T(s)x\ ds = \int_0^t \mathscr{A}T(s)x\ ds. \end{aligned}$$

(ii) It is easy to verify that $\mathscr{A}$ is a linear operator. Now, suppose $x_n \to x$ and $\mathscr{A}x_n \to y$. Since $T(\cdot)\mathscr{A}x_n \to T(\cdot)y$ uniformly on $[0, t]$, statement (i) implies that

$$T(t)x - x = \lim_{n\to\infty}(T(t)x_n - x_n) = \lim_{n\to\infty}\int_0^t T(s)\mathscr{A}x_n\ ds = \int_0^t T(s)y\ ds.$$

Dividing both sides by t and taking the limit as $t \downarrow 0$, we can conclude that $x \in D(\mathscr{A})$ and $\mathscr{A}x = y$. So, A is closed. Finally, since $\frac{1}{t}\int_0^t T(s)x\ ds \in D(\mathscr{A})$ for all $x \in X$ and $\lim_{t\downarrow 0}\frac{1}{t}\int_0^t T(s)x\ ds = x$, the domain $D(\mathscr{A})$ is dense in X. □

By (1.15), the function $t \mapsto T(t)x$ is continuously differentiable for $x \in D(\mathscr{A})$ and

$$\frac{d}{dt}T(t)x = T(t)\mathscr{A}x = \mathscr{A}T(t)x \ (t \geq 0). \tag{1.16}$$

Thus, if $x \in D(\mathscr{A})$, then $u(t) := T(t)x$ is a *classical solution* of the *abstract Cauchy problem*

$$\text{(ACP)} \quad \begin{cases} u'(t) &= \mathscr{A}u(t), \quad (t \geq 0) \\ u(0) &= x. \end{cases}$$

A continuous function $u : [0, \infty) \to X$ with $\int_0^t u(s)\ ds \in D(\mathscr{A})$ for all $t \geq 0$ and

$$u(t) - x = A\int_0^t u(s)\ ds$$

is called a *mild solution* of (ACP). Note that if u is a mild solution, then

$$\frac{1}{h}(u(t+h) - u(t)) = A\frac{1}{h}\int_t^{t+h} u(s)\,ds$$

for all arbitrarily small $h > 0$. Since

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} u(s)\,ds = u(t)$$

and $\mathscr{A}$ is closed, it follows that $u'(t) = Au(t)$ for every mild solution for which $u'(t)$ exists. Also, if u is a classical solution of (ACP) and $\mathscr{A}$ is closed, then u is a mild solution of (ACP). Since $u(t) \in D(\mathscr{A})$ for all $t \geq 0$ and u is continuous, the Riemann sums[1] $\sum_\pi u(\xi_i)(t_i - t_{i-1})$ are in $D(\mathscr{A})$, u is Riemann integrable with

$$\int_0^t u(s)\ ds = \lim_{|\pi|\to 0}\sum_\pi u(\xi_i)(t_i - t_{i-1}).$$

Similarly, the continuity of $\mathscr{A}u$ yields

$$\int_0^t \mathscr{A}u(s)\ ds = \lim_{|\pi|\to 0} Au(\xi_i)(t_i - t_{i-1}).$$

The fact that $\mathscr{A}$ is closed implies that $\int_0^t u(s)\ ds \in D(\mathscr{A})$ and

$$\mathscr{A}\int_0^t u(s)\ ds = \int_0^t \mathscr{A}u(s)\ ds = \int_0^t u'(s)\ ds = u(t) - u(0). \tag{1.17}$$

In Example 1.7 we saw that the shift semigroup $T(t)g(x) = f(x+t)$ is strongly continuous on $C_0[0, \infty)$. Next we discuss properties of its generator $\mathscr{A}$.

[1]As usual in the context of Riemann sums, π denotes a partition $0 = t_1 < ... < t_n = t$ of $[0, t]$ with partition size $|\pi| = \max_(t_i - t_{i-1})$ and where $\xi_i \in [t_{i-1}, t_i]$ are arbitrary intermediate points.

Example 1.10. Let $X = C_0[0,\infty)$, $T(t)g(x) := g(t+x)$ and let $\mathscr{A}$ be the generator of the strongly continuous semigroup $\{T(t)\}_{t\geq 0}$. Then

$$D(\mathscr{A}) = C_0^1[0,\infty),$$

the space of all functions $g \in C_0[0,\infty)$ which are continuously differentiable, $g' \in C_0[0,\infty)$ and $\mathscr{A}g = g'$. To see this, let $g \in D(\mathscr{A})$. Then, by definition,

$$f = \lim_{h\to 0} \frac{T(h)g - g}{h} \in C_0[0,\infty).$$

Also, since uniform convergence implies pointwise convergence,

$$f(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = g'(x).$$

Thus, $D(\mathscr{A}) \subset C_0^1[0,\infty)$. Conversely, let $g \in C_0^1[0,\infty)$. Then

$$\begin{aligned}
||\frac{T(h)g - g}{h} - g'|| &= \sup_{x\geq 0} |\frac{g(h+x) - g(x)}{h} - g'(x)| \\
&= \sup_{x\geq 0} |\frac{1}{h}\int_x^{x+h} g'(s)\,ds - g'(x)| \\
&= \sup_{x\geq 0} |\frac{1}{h}\int_x^{x+h} (g'(s) - g'(x))\,ds|.
\end{aligned}$$

Now let $\varepsilon > 0$. Since $g' \in C_0[0,\infty)$, there exists $N > 0$ such that $\sup_{x\geq N} |g'(x)| \leq \frac{\varepsilon}{3}$ and $0 < \delta < 1$ such that $|g'(x) - g'(y)| \leq \frac{\varepsilon}{3}$ whenever $x, y \in [0, N+1]$ and $|x-y| < \delta$. Therefore,

$$\begin{aligned}
&\sup_{x\geq 0} |\frac{1}{h}\int_x^{x+h} (g'(s) - g'(x))\,ds| \\
&\leq \sup_{x\in[0,N)} \frac{1}{h}\int_x^{x+h} |g'(s) - g'(x)|\,ds + \sup_{x\geq N} \frac{1}{h}\int_x^{x+h} |g'(s)|\,ds \\
&+ \sup_{x\geq N} \frac{1}{h}\int_x^{x+h} |g(x)|\,dx \leq \varepsilon.
\end{aligned}$$

Thus, $\frac{T(h)g-g}{h} \to g'$ as $h \to 0$ and, therefore, $C_0^1[0,\infty) \subset D(\mathscr{A})$.

In general it is often difficult to determine the domain $D(\mathscr{A})$ exactly. In these cases the concept of a *core* is useful.

Definition 1.11. A subspace D of the domain $D(\mathscr{A})$ of a linear operator $A : D(\mathscr{A}) \subset X \to X$ is called a *core* for A if D is dense in $D(\mathscr{A})$ for the *graph norm*

$$||x||_{\mathscr{A}} := ||x|| + ||\mathscr{A}x||.$$

We now state a useful criterion for subspaces to be a core for the generator (see [19]).

Proposition 1.12. *Let $(\mathscr{A}, D(\mathscr{A}))$ be the generator of a strongly continuous semigroup $\mathcal{T} = \{T(t)\}_{t\geq 0}$ on a Banach space X. A subspace D of $D(\mathscr{A})$ that is $||\cdot||$-dense in X and invariant under the semigroup is always a core for A.*

Proof. For every $x \in D(\mathscr{A})$ we can find a sequence $(x_n)_{n\in\mathbb{N}} \subset D$ such that $\lim_{n\to\infty} x_n = x$. For each n the map $s \mapsto T(s)x_n \in D$ is continuous for the graph norm $||\cdot||_{\mathscr{A}}$, because the maps $s \to T(s)x_n$ and $s \to \mathscr{A}T(s)x_n = T(s)\mathscr{A}x_n$ are continuous in the X-norm. Thus,

$$\int_0^t T(s)x_n\, ds,$$

being a Riemann integral, belongs to the $||\cdot||_{\mathscr{A}}$ closure of D. Similiarly, the $||\cdot||_{\mathscr{A}}$ - continuity of $s \mapsto T(s)x$ for $x \in D(\mathscr{A})$ implies that

$$||\frac{1}{t}\int_0^t T(s)x\, ds - x||_{\mathscr{A}} \to 0$$

as $t \downarrow 0$ and

$$\begin{aligned} ||\frac{1}{t}\int_0^t T(s)x_n\, ds - \frac{1}{t}\int_0^t T(s)x\, ds||_{\mathscr{A}} &= ||\frac{1}{t}\int_0^t T(s)(x_n - x)\, ds||_X \\ + \frac{1}{t}||T(t)(x_n - x) + (x_n - x)||_X \to 0 \end{aligned}$$

as $n \to \infty$ for each $t > 0$. This proves that for every $\varepsilon > 0$ we can find $t > 0$ and $n \in \mathbb{N}$ such that

$$||\frac{1}{t}\int_0^t T(s)x_n\, ds - x||_{\mathscr{A}} < \varepsilon.$$

Hence, $x \in \bar{D}^{||\cdot||_{\mathscr{A}}}$. □

Now we will recall some terminology from the spectral theory of closed linear operators. For a closed operator $(\mathscr{A}, D(\mathscr{A}))$ the set

$$\rho(\mathscr{A}) := \{\lambda \in \mathbb{C} \mid \lambda I - \mathscr{A} \text{ is bijective}\}$$

is called the *resolvent set* of $\mathscr{A}$. By the closed graph theorem, the *resolvent*

$$R(\lambda, \mathscr{A}) := (\lambda I - \mathscr{A})^{-1} : X \to D(\mathscr{A})$$

is a bounded linear operator on X. It is easy to show that resolvents satisfy the so-called *resolvent equation*

$$R(\lambda, \mathscr{A}) - R(\mu, \mathscr{A}) = (\mu - \lambda)R(\lambda, \mathscr{A})R(\mu, \mathscr{A}) \text{ for all } \lambda, \mu \in \rho(\mathscr{A}).^2 \tag{1.18}$$

The following theorem (see [1] or [19]) gives a first hint towards the intrinsic connection between Laplace transform theory and semigroup theory.

[2] The resolvent equation is an extension of the partial fraction decomposition $\frac{1}{(\lambda-x)(\mu-x)} = \frac{1}{\mu-\lambda}\left[\frac{1}{\lambda-x} - \frac{1}{\mu-x}\right]$ for $\lambda, \mu, x \in \mathbb{C}$.

Theorem 1.13. *Let $\mathcal{T} = \{T(t)\}_{t\geq 0}$ be a strongly continuous family of bounded linear operators. Then the following are equivalent.*

(i) $\mathcal{T}$ is a strongly continuous semigroup.

(ii) There exists $s_0 \in \mathbb{R}^+$ such that the Laplace transform $\int_0^\infty e^{-\lambda t}T(t)x\ dt$ of $t \to T(t)x$ exists for all $\lambda > \lambda_0$ and $x \in X$ and there exists a linear operator $(\mathscr{A}, D(\mathscr{A}))$ such that $(\lambda_0, \infty) \subset \rho(\mathscr{A})$ and, for all $x \in X$ and $\lambda > \lambda_0$,

$$R(\lambda, \mathscr{A})x = \int_0^\infty e^{-\lambda t}T(t)x\ dt. \tag{1.19}$$

Moreover, if (ii) is valid, then there exists $M \geq 1$, $\omega \in \mathbb{R}$ such that $(\mathscr{A}, D(\mathscr{A}))$ is the generator of the semigroup $\mathscr{T}$ of type (M, ω) and $||R(\lambda, A)^n|| \leq \frac{M}{(\lambda-\omega)^n}$ for all $\lambda > \omega$ and $n \in \mathbb{N}$.

As a corollary to Theorem 1.13, we obtain the following elementary result.

Corollary 1.14. *Let X be a Banach space and $\mathscr{A} \in \mathcal{L}(X)$. Then $\mathscr{A}$ generates the semigroup given by*

$$T(t) := \sum_{j=0}^{\infty} \frac{\mathscr{A}^j}{j!} t^j.$$

Since the power series converges for all $t \in \mathbb{C}$, the map $t \to T(t)$ is an entire function from $\mathbb{C}$ into $\mathcal{L}(X)$.

Proof. It is known that the power series $e^{t\mathscr{A}} = \sum_{j=0}^{\infty} \frac{\mathscr{A}^j}{j!} t^j$ converges uniformly for t in compact subsets of $\mathbb{C}$ and that the map $t \to T(t)$ is an entire function. Since $|| \sum_{j=0}^{n} \frac{A^j}{j!} t^j|| \leq e^{\mathrm{Re} t ||\mathscr{A}||}$ for all $n \in \mathbb{N}$ and $t \in \mathbb{C}$, the dominated convergence theorem implies, for $|\lambda| > ||\mathscr{A}||$,

$$\sum_{j=0}^{\infty} \int_0^\infty e^{-\lambda s} \frac{\mathscr{A}^j}{j!} s^j\ ds = \int_0^\infty e^{-\lambda s} \sum_{j=0}^{\infty} \frac{\mathscr{A}^j}{j!} s^j\ ds.$$

For $|\lambda| > ||\mathscr{A}||$, the resolvent $R(\lambda, A)$ can be represented with the Neumann series $\sum_{j=0}^{\infty} \frac{1}{\lambda^{j+1}} \mathscr{A}^j$. Thus,

$$\begin{aligned} R(\lambda, \mathscr{A}) &= \sum_{j=0}^{\infty} \mathscr{A}^j \frac{1}{\lambda^{n+1}} = \sum_{j=0}^{\infty} \mathscr{A}^j \int_0^\infty e^{-\lambda s} \frac{s^j}{j!}\ ds \\ &= \int_0^\infty e^{-\lambda s} \sum_{j=0}^{\infty} \frac{(s\mathscr{A})^j}{j!}\ ds = \int_0^\infty e^{-\lambda s} T(s)\ ds. \end{aligned}$$

for $\lambda > ||\mathscr{A}||$. Proposition 1.13 yields that $\mathcal{T}$ is the strongly continuous semigroup with generator $\mathscr{A}$. □

Now we would like to compute the semigroup $t \to T(t)x$ as the inverse Laplace transform of $\lambda \to R(\lambda, \mathscr{A})x$. One of the commonly used tools to do this is either the complex inversion formula

$$T(t)x = \frac{1}{2\pi i} \int_{\omega + i\mathbb{R}} e^{\lambda t} R(\lambda, \mathscr{A})x \, d\lambda \quad (\omega > 0, x \in D(\mathscr{A})) \tag{1.20}$$

or the Post-Widder inversion (Backward Euler scheme)

$$T(t)x = \lim_{n\to\infty} \left(I - \frac{t}{n}\mathscr{A} \right)^{-n} x = \lim_{n\to\infty} \left(\frac{n}{t}\right)^n R\left(\frac{\lambda}{t}, \mathscr{A}\right)^n x \tag{1.21}$$

(see [9], Corollary 5.5 and Theorem 5.14). Unfortunately, for our purposes, both inversions are not very efficient. First of all, along the path $\Gamma = \omega + i\mathbb{R}$ the function $e^{\lambda t}$ is rapidly oscillating even for moderate values of $t > 0$. Thus, the numerical usefulness of the complex inversion formula is limited unless, with the help of Cauchy's Integral Formula, the path of integration can be shifted into a path like $\tilde{\Gamma} = \tilde{\Gamma}_+ \cup \tilde{\Gamma}_-$, where $\tilde{\Gamma}_\pm = \{\lambda = -r \pm \sqrt{r}\}, \quad r \in \mathbb{R}_+$ (see [25]). However, since the spectra of the flow generators

$$(\mathscr{A} g)(\omega) := a(\omega)g'(\omega)$$

(see 1.12) or

$$(\mathscr{A} g)(s) := g'(s) + g(s)A(s)$$

(see 1.13) often contain the left half plane (or a vertical strip), this is not a workable approach for our purposes. Also, the Post-Widder Inversion (Backward Euler Scheme) is not practical since (a) it is very slow to converge (in general, the rate of convergence is like $\frac{1}{\sqrt{n}}$) and (b) since it requires the computation of the powers of the resolvents which turns out to be an entirely non-trivial task for operators of the form $(\mathscr{A} f)(s) = f'(s) + f(s)A(s)$. Thus, in order to proceed, we need inversion formulas that are better suited for our purposes.

In order to compute the semigroup $t \to T(t)x$ as the inverse Laplace transform of $\lambda \to R(\lambda, \mathscr{A})x$, let us assume assume that there is a rational function

$$r_q(z) = \frac{c_1}{b_1 - z} + \ldots + \frac{c_q}{b_q - z} = \frac{P(z)}{Q(z)}$$

($b_i, c_i \in \mathbb{C}$, Re $b_i > 0$) that approximates e^z of order m; i.e., the first m-terms in the Taylor expansion of r around 0 coincide with those of the exponential series or, equivalently, there exists a constant C such that

$$|r_q(z) - e^z| \leq C|z|^{m+1}$$

for $|z|$ sufficiently small. Then for $a \in \mathbb{C}$, and ta of sufficiently small modulus,

$$\begin{aligned}||r(ta)x - e^{ta}x|| &= ||c_1\left(b_1 - ta\right)^{-1}x + \ldots + c_q\left(b_q - ta\right)^{-1}x - e^{ta}x|| \\ &= ||\frac{c_1}{t}R\left(\frac{b_1}{t}, a\right)x + \ldots + \frac{c_q}{t}R\left(\frac{b_q}{t}, a\right)x - e^{ta}x|| \\ &\leq C_q t^{m+1}||a^{m+1}x||.\end{aligned}$$

Using functional calculus methods, it is reasonable to expect that the estimate $C_q t^m||a^{m+1}x||$ extends to generators $\mathscr{A}$ of strongly continuous semigroups. In fact, this was done in a series of papers by P. Jara, F. Neubrander, K. Özer, and L. Windsperger (see [27], [14], [15], and [21]) based on ground-breaking work of Hersh and Kato [12] and Brenner and Thomée [3].

Theorem 1.15. *Let $\mathscr{A}$ be the generator of a C_0-semigroup $T(t)$ with $||T(t)|| \leq M$. Let $p = q - 1$, $m = p + q$, and*

$$P(z) = \sum_{j=0}^{p} \frac{(m-j)!p!}{m!j!(p-j)!}z^j,$$

$$Q(z) = \sum_{j=0}^{q} \frac{(m-j)!q!}{m!j!(q-j)!}(-z)^j.$$

Then Q has q distinct roots $b_1, \ldots, b_q$ with $Re\ b_i > 0$ such that

$$r_q(z) = \frac{P(z)}{Q(z)} = \frac{c_1}{b_1 - z} + \ldots + \frac{c_q}{b_q - z}$$

where $c_i = \dfrac{m!}{p!}\dfrac{P(b_i)}{\prod_{j+i}(b_j - b_i)}$ is a rational approximation of the exponential of order $m = p + q = 2q - 1$. Moreover, for $x \in D(\mathscr{A}^{2q})$,

$$||\sum_{j=1}^{q} \frac{c_j}{t}R\left(\frac{b_j}{t}, \mathscr{A}\right)x - T(t)x|| \leq C_q t^{2q}||\mathscr{A}^{2q}x||,$$

where $C_q = \dfrac{M\sqrt{2\pi}}{(2q-1)!}\left[\dfrac{(2q-2)!(2q)!}{(4q-1)!}\right]^{1/2} = \dfrac{M\sqrt{2\pi}}{m!}\dfrac{(m-1)!(m+1)!}{(2m+1)!}$.

Theorem 1.15 is not only important in order to approximate semigroups in terms of the resolvents. Using the "Transference Principle" as formulated in [15] and [20], Theorem 1.15 can be reformulated in terms of Laplace transforms.

Corollary 1.16. *Let q, c_i, b_i and C_q be as in Theorem 1.15, let $f \in C_b^{2q}([0,\infty), X)$ and*

$$\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\, dt \quad Re\ \lambda > 0).$$

Then,

$$||\sum_{j=1}^{q} \frac{c_j}{t} \hat{f}\left(\frac{b_j}{t}\right) - f(t)|| \leq C_q t^{2q} ||f^{(2q)}||_\infty. \tag{1.22}$$

Proof. We prove the statement for

$$f \in C_0^{2q}([0,\infty), X) = \left\{f : f^{(j)} \in C_0([0,\infty), X) \text{ for all } 0 \leq j \leq 2q\right\}.$$

For a proof for $f \in C_b^{2q}([0,\infty), x)$ see [20]. On $C_0([0,\infty), X)$, the shift operators

$$T(t)f(\omega) := f(t+\omega)$$

define a strongly continuous semigroup of contraction; i.e., of type $(1,0)$ (see 1.7). Thus,

$$\begin{aligned} R(\lambda, \mathscr{A})f(\omega) &= \int_0^\infty e^{-\lambda t} T(t) f(\omega)\, dt \\ &= \int_0^\infty e^{-\lambda t} f(t+\omega) \end{aligned}$$

and, therefore, $T(t)f(0) = f(t)$ and

$$R(\lambda, \mathscr{A})f(0) = \int_0^\infty e^{-\lambda t} f(t)\, dt = \hat{f}(\lambda).$$

Thus, by Theorem 1.15,

$$\begin{aligned} ||\sum_{j=1}^{q} \frac{c_j}{t} \hat{f}\left(\frac{b_j}{t}\right) - f(t)||_X &= ||\sum_{j=1}^{q} \frac{c_j}{t} R\left(\frac{b_j}{t}, \mathscr{A}\right) f(0) - T(t)f(0)||_X \\ &\leq \sup_{\omega \geq 0} ||\sum_{j=1}^{q} \frac{c_j}{t} R\left(\frac{b_j}{t}, \mathscr{A}\right) f(\omega) - T(t)f(\omega)||_X \\ &= ||\sum_{j=1}^{q} \frac{c_j}{t} R\left(\frac{b_j}{t}, \mathscr{A}\right) f - T(t)f|| \\ &\leq C_q t^{2q} ||\mathscr{A}^{2q}|| = C_q t^{2q} ||f^{(2q)}||_\infty. \end{aligned}$$

□

Since $f^{(2q)} = 0$ for polynomials f with degree at most $2q-1$, the previous corollary indicates that the inversion formula (1.22) is exact for such polynomials. However, since polynomials are not in $C_b^{2q}([0,\infty), X)$, a direct proof is required for this statement.

Proposition 1.17. *Let $b_i, c_i \in \mathbb{C}$, $Re\ b_i > 0$, be such that*

$$r(z) = \frac{P(z)}{Q(z)} = \frac{c_1}{b_1 - z} + \ldots + \frac{c_q}{b_q - z}$$

is a rational approximation of the exponential of order m. Then, for all polynomials f of degree at most m, the inversion formula (1.22) is exact; i.e.,

$$\sum_{j=1}^{q} \frac{c_j}{t} \hat{f}\left(\frac{b_j}{t}\right) = f(t)$$

for all $t > 0$.

Proof. Since r is a rational approximation of the exponential of order m, it follows that the first m Taylor coefficients of r coincide with those of the exponential series or, equivalently,

$$\frac{r^{(n)}(0)}{n!} = \frac{c_1}{b_1^{n+1}} + \ldots + \frac{c_q}{b_q^{n+1}} = \frac{1}{n!}$$

for all $0 \leq n \leq m$. This shows that

$$\sum_{j=1}^{q} \frac{c_j n!}{b_j^{n+1}} = 1$$

for all $0 \leq n \leq m$. Now let $f(t) = \sum_{n=0}^{m} a_n t^n$. Then $\hat{f}(\lambda) = \sum_{n=0}^{m} a_n \frac{n!}{\lambda^{n+1}}$ and therefore

$$\begin{aligned} \sum_{j=1}^{q} \frac{c_j}{t} \hat{f}\left(\frac{b_j}{t}\right) &= \sum_{j=1}^{q} \sum_{n=0}^{m} \frac{c_j a_n n! t^n}{b_j^{n+1}} = \sum_{n=0}^{m} \sum_{j=1}^{q} \frac{c_j a_n n! t^n}{b_j^{n+1}} \\ &= \sum_{n=0}^{m} a^n t^n \sum_{j=1}^{q} \frac{c_j n!}{b_j^{n+1}} = \sum_{n=0}^{m} a_n t^n = f(t). \end{aligned}$$

□

Remark 1.18. The constant

$$C_q = \frac{\sqrt{2\pi}}{(2q-1)!} \left[\frac{(2q-2)!(2q)!}{(4q-1)!}\right]^{\frac{1}{2}}$$

given in Theorem 1.15 and Corollary 1.16 converges rapidly towards 0 as $q \to \infty$. In fact, using Mathematica one can easily see that for $5 \leq q \leq 50$ one has that $C_q \approx 10^{7-3q-0.01q^2}$. That is, for $5 \leq q \leq 50$, there exists a constant C close to 1 such that

$$\left|\left|\sum_{j=1}^{q} \frac{c_j}{t} R\left(\frac{b_j}{t}, \mathscr{A}\right) x - T(t)x\right|\right| \leq C \cdot 10^{7-3q-0.01q^2} t^{2q} ||\mathscr{A}^{2q} x||$$

or, for $f \in C_b^{(2q)}([0, \infty), X)$,

$$||\sum_{j=1}^{q} \frac{c_j}{t} \hat{f}\left(\frac{b_j}{t}\right) - f(t)|| \leq C \cdot 10^{7-3q-0.01q^2} t^{2q} ||f^{(2q)}||_\infty.$$

Unfortunately, computing the resolvent $R(\lambda, \mathscr{A})$ of flow generators

$$(\mathscr{A} g)(\omega) = g'(\omega) + g(\omega)A(\omega)$$

explicitly is usually quite difficult or even impossible (see Section 2.1). Therefore, approximation and/or perturbation methods must be employed. We introduce the Dyson-Phillips Bounded Perturbation Theorem now which will be further investigated in Chapter 2, Section 2 in the context of semigroups induced by flows $\zeta(t, s, \omega)$ solving second order ordinary differential equations

$$x'(t) = A(t)x(t),\ x(s) = \omega \in \mathbb{R},\ t \geq s \geq 0.$$

Theorem 1.19 (Dyson-Phillips Bounded Perturbation Theorem)**.** *Let $(\mathscr{A}_0, D(\mathscr{A}_0))$ be the generator of a strongly continuous semigroup $(T_0(t))_{t\geq 0}$ on a Banach space X of type (M, ω). If $B \in \mathcal{L}(X)$, then the operator*

$$\mathscr{A} := \mathscr{A}_0 + B \text{ with } D(\mathscr{A}) := D(\mathscr{A}_0)$$

generates a strongly continuous semigroup $\mathcal{H} = \{H(t)\}$ of type $(M, \omega + M||B||)$ given by

$$H(t) = \sum_{j=0}^{\infty} S_j(t), \tag{1.23}$$

where $S_0(t) := T_0(t)$ and

$$S_{j+1}(t) := \int_0^t T_0(t-s) B S_j(s)\, ds\ (j \geq 0).$$

Proof. For $\lambda > \omega + M||B||$, Proposition 1.13 implies that

$$||R(\lambda, \mathscr{A}_0)B|| \leq \frac{M}{\lambda - \omega}||B|| < 1.$$

Then, it follows from

$$\lambda - (\mathscr{A}_0 + B) = (\lambda - A_0)[I - R(\lambda, \mathscr{A}_0)B]$$

that $R(\lambda, \mathscr{A}_0 + B)$ exists and is given by

$$R(\lambda, \mathscr{A}_0 + B) = [I - R(\lambda, \mathscr{A}_0)B]^{-1} R(\lambda, \mathscr{A}_0) = \sum_{j=0}^{\infty} R_j(\lambda) \tag{1.24}$$

where $R_j(\lambda) := (R(\lambda, \mathscr{A}_0)B)^j R(\lambda, \mathscr{A}_0)$. The operators $R_j(\lambda)$ can be computed recursively via

$$R_0(\lambda) = R(\lambda, \mathscr{A}_0) \text{ and } R_{j+1}(\lambda) = R(\lambda, \mathscr{A}_0)BR_j(\lambda).$$

Since $R_0(\lambda)x = R(\lambda, \mathscr{A}_0)x$ is the Laplace transform of $S_0(t)x := T(t)x$ and $B \in \mathcal{L}(X)$, it follows inductively that $R_{j+1}(\lambda)x$ is the Laplace transform of

$$S_{j+1}(t)x := \int_0^t T(t-s)BS_j(s)x\,ds.$$

A simple induction shows that

$$||S_j(t)|| \leq M^{j+1}||B||^j \frac{t^j}{j!} e^{\omega t}$$

and that the operator families $S_j(t)_{t\geq 0} \subset \mathcal{L}(X)$ are strongly continuous for $j \in \mathbb{N}$. Hence, the operator families $H_n(t)_{t\geq 0} \subset \mathcal{L}(X)$ defined by

$$H_n(t) : x \mapsto \sum_{j=0}^{n} S_j(t)x$$

are strongly continuous ($n \in \mathbb{N}$). Moreover, since the series

$$\sum_{j=0}^{\infty} M^{j+1}||B||^j \frac{T^j}{j!} e^{\omega T}$$

converges for arbitrary $T > 0$, it follows from

$$||H_n(t)x - H_m(t)x|| = ||\sum_{j=m+1}^{n} S_j(t)x|| \leq \sum_{j=m+1}^{n} M^{j+1}||B||^j \frac{T^j}{j!} e^{\omega T}||x||$$

for $t \in [0, T]$, and $x \in X$ that $H_n(t)$ converges uniformly to $H(t)x := \sum_{j=0}^{\infty} S_j(t)x$. Therefore, the operator family $\mathcal{H} = H(t)_{t\geq 0} \subset \mathcal{L}(X)$ is strongly continuous with $||H(t)|| \leq Me^{\omega + M||B||)t}$. For $\lambda > \omega + M||B||$ it follows from the Lebesgue dominated convergence theorem that the Laplace transforms $\int_0^\infty e^{-\lambda t} H_n(t)xdt = \sum_{j=0}^{n} R_j(\lambda)x$ converge to $\int_0^t e^{-\lambda t} H(t)x\,dt$. By (1.24),

$$R(\lambda, A_0 + B)x = \int_0^\infty e^{-\lambda t} H(t)x\,dt \text{ for all } x \in X.$$

Thus, by Theorem 1.13, $\mathcal{H}$ is a strongly continuous semigroup with generator $\mathscr{A} = \mathscr{A}_0 + B$. □

The following Laplace transform result is needed for the proof of the Trotter-Kato Theorem below. For a proof, see [1, Theorem 3.1.7]).

Proposition 1.20. *Let $f_n \in C(\mathbb{R}^+, X)$ with $\|f_n(t)\| \leq Me^{\omega t}$ for some $M > 0$, $\omega \in \mathbb{R}$, and all $n \in \mathcal{N}$ and $t \geq 0$. The following are equivalent.*

(i) The Laplace transforms $\hat{f}_n(\lambda)$ converge for all $\lambda \in (\omega, \infty)$ and the sequence $(f_n)_{n\in\mathcal{N}}$ is equicontinuous.

(ii) The functions f_n converge uniformly on compact subsets of $\mathbb{R}^+$.

Moreover, if (ii) holds, then $\hat{f}(\lambda) = \lim_{n\to\infty} \hat{f}_n(\lambda)$ for all $\lambda > \omega$, where $f(t) := \lim_{n\to\infty} f_n(t)$.

The following theorems are the basic approximation results of semigroup theory. For a proof of these stabilized versions of the classical results, see [16]. Following a suggestion of Professor A. Bobwroski, the versions stated below are again slightly more general than those in [16] and the necessary modifications of the proof given in [16] are included here.

Theorem 1.21. *(Trotter–Kato). Let $(\mathscr{A}_n, D(\mathscr{A}_n))$ generate C_0–semigroups $T_n(t)$ and let $(\mathscr{A}, D(\mathscr{A}))$ be densely defined with $D(\mathscr{A}) \subset D(\mathscr{A}_n)(n \in \mathbb{N})$. Suppose that there are $W_n \in \mathcal{L}(X)$ with $W_n x \to x$ for all $x \in X$ and $M \geq 1$, $\omega \geq 0$, such that the stability condition*

$$\|W_n T_n(t)\| \leq Me^{\omega t}$$

is valid for all $t \geq 0$ and $n \in \mathbb{N}$. If $\mathscr{A}_n x \to Ax$ for all $x \in D(\mathscr{A})$ and if $(\lambda - \mathscr{A})D(\mathscr{A})$ is dense in X for all $\lambda > \omega$, then $\mathscr{A}$ is closable and the closure generates a C_0–semigroup given by

$$T(t)x = \lim_{n\to\infty} W_n T_n(t)x \quad (x \in X),$$

where the limit is uniform for t in compact subsets of $\mathbb{R}^+$.

Proof. Let $\lambda > \omega$. The stability condition implies that the operators $R_n(\lambda)$ defined by

$$R_n(\lambda) : x \mapsto \int_0^\infty e^{-\lambda t} W_n T_n(t)x \; dt$$

are in $\mathcal{L}(X)$ with $\|R_n(\lambda)\| \leq \frac{M}{\lambda - \omega}$. Since $W_n \in \mathcal{L}(X)$ one obtains for $x \in D(\mathscr{A}_n)$ and $t \geq 0$,

$$W_n T_n(t)x - W_n x = W_n \int_0^t T_n(\tau) A_n x \; d\tau = \int_0^t W_n T_n(\tau) \mathscr{A}_n x \; d\tau. \tag{1.25}$$

Thus, for $x \in D(A_n)$ and $\lambda > \omega$, this identity together with integration by parts and the stability assumption $\|W_nT_n(t)\| \le Me^{\omega t}$ leads to

$$\begin{aligned}
\lambda R_n(\lambda)x &= \int_0^\infty \lambda e^{-\lambda t} W_n T_n(t)x\; dt \\
&= \int_0^\infty \lambda e^{-\lambda t} W_n x\; dt + \int_0^\infty \lambda e^{-\lambda t} \int_0^t W_n T_n(\tau)\mathscr{A}_n x\; d\tau\; dt \\
&= W_n x - e^{-\lambda t} \int_0^t W_n T_n(\tau)\mathscr{A}_n x\; d\tau \Big|_0^\infty + \int_0^\infty e^{-\lambda t} W_n T_n(t)\mathscr{A}_n x\; dt \\
&= W_n x + R_n(\lambda)\mathscr{A}_n x.
\end{aligned}$$

Therefore, for each $n \in \mathbb{N}$ and all $x \in D(\mathscr{A}_n)$,

$$R_n(\lambda)(\lambda - \mathscr{A}_n)x = W_n x. \tag{1.26}$$

Let $x \in (\lambda I - \mathscr{A})D(\mathscr{A})$. Then $x = (\lambda I - \mathscr{A})y$ for some $y \in D(\mathscr{A})$ and, by (1.26) and by the fact that $\|R_n(\lambda)\| \le \frac{M}{\lambda - \omega}$ for all $n \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}
\|R_n(\lambda)x - R_m(\lambda)x\| &= \|R_n(\lambda)(\lambda - \mathscr{A})y - R_m(\lambda)(\lambda - \mathscr{A})y\| \\
&\le \|R_n(\lambda)(\lambda - \mathscr{A})y - R_n(\lambda)(\lambda - \mathscr{A}_n)y\| \\
&+ \|R_n(\lambda)(\lambda - \mathscr{A}_n)y - R_m(\lambda)(\lambda - \mathscr{A}_m)y\| \\
&+ \|R_m(\lambda)(\lambda - \mathscr{A}_m)y - R_m(\lambda)(\lambda - \mathscr{A})y\| \\
&\le \|R_n(\lambda)(\mathscr{A}_n y - \mathscr{A}y)\| + \|W_n y - W_m y\| \\
&+ \|R_m(\lambda)(\mathscr{A}_m y - \mathscr{A}y)\| \le \varepsilon
\end{aligned}$$

for all $n, m \ge n_0$. Therefore, $\lim_{n\to\infty} R_n(\lambda)x$ exists for all x in the dense set $(\lambda - \mathscr{A})D(\mathscr{A})$ uniformly for λ on compact intervals of (ω, ∞). Since $\|R_n(\lambda)\| \le \frac{M}{\lambda - \omega}$ it follows that for all $\lambda > \omega$ there exists $R(\lambda) \in \mathcal{L}(X)$ such that $R(\lambda)x = \lim_{n\to\infty} R_n(\lambda)x$ for all $x \in X$. Moreover, it follows from

$$\begin{aligned}
&\|R_n(\lambda)(\lambda - \mathscr{A}_n)x - R(\lambda)(\lambda - \mathscr{A})x\| \\
&\qquad \le \|R_n(\lambda)\|\|\mathscr{A}_n x - \mathscr{A}x\| + \|R_n(\lambda)(\lambda - \mathscr{A})x - R(\lambda)(\lambda - \mathscr{A})x\|
\end{aligned}$$

that $W_n x = R_n(\lambda)(\lambda - \mathscr{A}_n)x$ converges to $R(\lambda)(\lambda - \mathscr{A})x$ for all $x \in D(\mathscr{A})$. Thus, for all $\lambda > \omega$ and $x \in D(\mathscr{A})$,

$$R(\lambda)(\lambda - \mathscr{A})x = x \tag{1.27}$$

To apply Proposition 1.20, it only remains to be shown that the sequence $(W_nT_n(\cdot)x)_n$ is equicontinuous for every $x \in X$. Let $\varepsilon > 0$ and $t \ge 0$. Since $D(\mathscr{A})$ is dense in X, there exists for any $x \in X$ an element $y \in D$ such that $\|x - y\| < \frac{\varepsilon}{3Me^{\omega(t+1)}}$. Moreover, since $\mathscr{A}_n y$ is convergent, there exists a constant $C > 0$ such that $\|\mathscr{A}_n y\| \le C$ for all $n \in \mathbb{N}$. Let $\omega > 0$ (the proof can easiliy be modified for $\omega = 0$). The exponential function $s \mapsto e^{s\omega}$ is continuous at t and so there is a $\delta \in (0,1)$ such that $|e^{\omega t} - e^{\omega s}| < \frac{\varepsilon\omega}{3MC}$ for all $s \geq 0$ with $|t-s| < \delta$. Now, if $s \geq 0$ and $|t-s| < \delta$, then $s < t+1$ and (1.25) implies that

$$
\begin{aligned}
&\|W_nT_n(t)x - W_nT_n(s)x\| \\
&\quad\leq \|W_nT_n(t)(x-y)\| + \|W_nT_n(s)(x-y)\| + \|W_nT_n(t)y - W_nT_n(s)y\| \\
&\quad\leq M(e^{\omega t} + e^{\omega s})\|x-y\| + \|\int_0^t W_nT_n(\tau)A_ny\, d\tau - \int_0^s W_nT_n(\tau)A_ny\, d\tau\| \\
&\quad\leq 2Me^{\omega(t+1)}\|x-y\| + |\int_s^t MCe^{\omega\tau}\, d\tau| < \frac{2\varepsilon}{3} + \frac{MC}{\omega}|e^{\omega t} - e^{\omega s}| < \varepsilon
\end{aligned}
$$

Thus, $W_nT_n(\cdot)x$ is equicontinuous. Now, Proposition 1.20 yields that the functions $(W_nT_n(\cdot)x)$ converge uniformly on compact intervals of $\mathbb{R}^+$ and

$$\int_0^\infty e^{-\lambda t}T(t)x\, dt = R(\lambda)x$$

for all $x \in X$, where $T(t)x := \lim_{n\to\infty} W_nT_n(t)x$. Since

$$\|T(t)x\| \leq \|T(t)x - W_nT_n(t)x\| + \|W_nT_nx\|$$

for all $n \in \mathbb{N}$ and $\|W_nT_n(t)\| \leq Me^{\omega t}$, it follows that $T(t) \in \mathcal{L}(X)$ with $\|T(t)\| \leq Me^{\omega t}$. Moreover, $t \mapsto W_nT_n(t)x$ is continuous for all $x \in X$, $n \in \mathbb{N}$, and $W_nT_n(\cdot)x \to T(\cdot)x$ uniformly on compact intervals. This implies that $t \mapsto T(t)x$ is continuous for every $x \in X$. Hence, the family $\{T(t)\}_{t\geq 0}$ is strongly continuous. Let $x \in D(\mathscr{A})$. It follows from

$$(\lambda - \nu)R(\nu)R(\lambda)(\lambda x - Ax) = (\lambda - \nu)R(\nu)x$$

and from

$$(R(\nu) - R(\lambda))(\lambda x - Ax) = R(\nu)(\lambda x - Ax) + R(\nu)(\lambda - \nu)x - x = (\lambda - \nu)R(\nu)x$$

that the bounded operators $R(\nu) - R(\lambda)$ and $(\lambda - \nu)R(\nu)R(\lambda)$ coincide on the dense set $(\lambda - A)D(A)$. Thus, the resolvent equation

$$R(\nu) - R(\lambda) = (\lambda - \nu)R(\nu)R(\lambda)$$

holds for all $\nu > \omega$. In particular, the resolvent equation implies that the kernels and ranges of the operators $R(\lambda)$ are independent of $\lambda > \omega$. We show next that $\mathrm{Ker}(R(\lambda)) = \{0\}$. Clearly, $T(0)x = \lim W_nT_n(0)x = x$. Thus, by the uniqueness property of the Laplace transform,

$$R(\lambda)x = \int_0^\infty e^{-\lambda t}T(t)x\, dt = 0$$

for (one and thus all) $\lambda > \omega$ implies $T(t)x = 0$ for all $t \geq 0$ and therefore $x = 0$. Thus, by Proposition B.6 in [1], there exits a closed linear operator B such that

$$R(\lambda) = R(\lambda, B).$$

Thus, B generates the C_0- semigroup $\{T(t)\}$ .Moreover since

$$x = R(\lambda, B)(\lambda x - \mathscr{A} x)$$

for all $x \in D(\mathscr{A})$ that $D(\mathscr{A}) \subset D(B)$ and $Bx = \mathscr{A} x$ on D, and that B is the (unique) closure of A. □

The following version of the stabilized Chernoff Product Formula is more general that the one given in [16]. The proof is identical to the one given in [16] when using the previous version of the Trotter-Kato Theorem.

Theorem 1.22. *(Chernoff Product Formula). Let $(\mathscr{A}, D(\mathscr{A}))$ be densely defined and assume that $(\lambda - \mathscr{A})D(\mathscr{A})$ is dense in X for all $\lambda > \omega$ for some $\omega \geq 0$. Let $\mathcal{V} = \{V(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ satisfy $V(0) = I$ and let $\mathcal{W} = \{W(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ be strongly continuous at $t = 0$ with $W(0) = I$. If*

(a) there exists $M \geq 1$ such that $\|W(t)V(t)^n\| \leq Me^{\omega nt}$ for all $n \in \mathbb{N}$ and $t \geq 0$,

(b) $\mathscr{A} x = \lim_{t\to 0+} \frac{V(t)x - x}{t}$ for all $x \in D(\mathscr{A})$,

then $\mathscr{A}$ is closable and its closure generates a C_0- semigroup

$$T(t)x = \lim_{n\to\infty} W\left(\frac{t}{n}\right) V\left(\frac{t}{n}\right)^n x, \quad (x \in X),$$

where the limit is uniform in t on compact subsets of $\mathbb{R}^+$.

Since semigroups induced by flows (see Section 1.1) are contraction semigroups, for which the resolvent of the generator is hard to compute, it is appropriate to mention the Lumer-Phillips Theorem since it does not require explicit knowledge of the resolvent. The following consequence of the Chernoff Product Formula is needed for the proof of the Lumer-Phillips Theorem.

Corollary 1.23. *(Hille-Yosida). Let $(\mathscr{A}, D(\mathscr{A}))$ be a linear operator on a Banach space X and $M \geq 1$. Then the following statments are equivalent.*

(i) $(\mathscr{A}, D(\mathscr{A}))$ generates a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ of type $(M, 0)$.

(ii) $(\mathscr{A}, D(\mathscr{A}))$ is densely defined with $(0, \infty) \subset \rho(\mathscr{A})$ and the Backward-Euler scheme $\mathcal{V} = \{V(t)\}_{t\geq 0}$ with $V(t) := \frac{1}{t}R(\frac{1}{t}, \mathscr{A})$ for $t > 0$ and $V(0) := I$ is stable with $||V(t)^n|| \leq M$. Moreover, if (i) is valid, then

$$T(t)x = \lim_{n\to\infty} V\left(\frac{t}{n}\right)^n x = \lim_{n\to\infty} \left(I - \frac{t}{n}\mathscr{A}\right)^{-n} x$$

for all $x \in X$, where the limit is uniform for t in compact subsets of $\mathbb{R}^+$.

Proof. $(i) \Rightarrow (ii)$. By Lemma 1.9, $\mathscr{A}$ is densely defined. Let $\{T(t)\}_{t\geq 0}$ be bounded by $M \geq 1$. Then, $||V(0)^n|| = ||I|| \leq M$ for all $n \in \mathbb{N}$. Moreover, Thoorem 1.13 implies that $(0, \infty) \subset \rho(\mathscr{A})$ and that, for all $t > 0, ||V(t)^n|| = ||\frac{1}{t^n} R\left(\frac{1}{t}, \mathscr{A}\right)^n|| \leq M$.

$(ii) \Rightarrow (i)$. In order to apply Theorem 1.22 (Chernoff Product Formula), it suffices to show consistency. Since $\lambda \mapsto R(\lambda, \mathscr{A})$ is an analytic function, it is clear that $\mathcal{V}$ is strongly continuous on $(0, \infty)$. By hypothesis, $||V(t)^n|| = ||\frac{1}{t^n} R\left(\frac{1}{t}, \mathscr{A}\right)^n|| \leq M$ for all $t \in (0, \infty)$. To show the continuity of $t \mapsto V(t)x$ at $t = 0$, assume first that $x \in D(\mathscr{A})$. Then, as $t \to 0$,

$$||V(t)x - x|| = ||\frac{1}{t} R\left(\frac{1}{t}, \mathscr{A}\right) x - x|| = ||R\left(\frac{1}{t}, \mathscr{A}\right) \mathscr{A}x|| \leq Mt||\mathscr{A}x|| \to 0.$$

Since $D(A)$ is dense in X and $||V(t)|| \leq M$, it follows that $V(0)x = x$ for all $x \in X$. Now let $x \in D(\mathscr{A})$. Then,

$$\frac{V(t)x - x}{t} = \frac{\frac{1}{t} R\left(\frac{1}{t}, \mathscr{A}\right) x - x}{t} = \frac{1}{t} R\left(\frac{1}{t}, \mathscr{A}\right) \mathscr{A}x = V(t)\mathscr{A}x \to \mathscr{A}x \text{ as } t \to 0.$$

By the Chernoff Product Formula, $(\mathscr{A}, D(\mathscr{A}))$ generates the semigroup given by $T(t)x = \lim_{n\to\infty} V\left(\frac{t}{n}\right)^n x$ for all $x \in X$. Clearly, since $||V\left(\frac{t}{n}\right)^n|| \leq M$ for all $t \geq 0$ and $n \in \mathbb{N}$, it follows that $||T(t)|| \leq M$. □

Definition 1.24. A linear operator $(\mathscr{A}, D(\mathscr{A}))$ on a Banach space$X$ is called *dissipative* if

$$||(\lambda - \mathscr{A})x|| \geq \lambda||x|| \tag{1.28}$$

for all $\lambda > 0$ and $x \in D(\mathscr{A})$.

Theorem 1.25. *(Lumer-Phillips). Let $\mathscr{A}$ be a densely defined operator on X. Then $\mathscr{A}$ generates a C_0 - semigroup of contractions on X if and only if*

1. *$\mathscr{A}$ is dissipative and*
2. *$(\lambda - \mathscr{A})D(\mathscr{A}) = X$ for some (or all) $\lambda > 0$.*

Proof. Let $\mathscr{A}$ be the generator of a C_0 - semigroup of contractions. Then assertion (2) is valid by the Hille-Yosida theorem (Corollary1.23). Moreover, the Hille-Yosida theorem combined with the the definition of dissipative implies assertion (1). In order to prove the converse implication note that by the definition of dissipative we have

$$||(\lambda - \mathscr{A})x|| \geq \lambda||x|| \tag{1.29}$$

for all $x \in D(\mathscr{A}), \lambda > 0$. Since $(\lambda_0 - \mathscr{A})D(\mathscr{A}) = X$ for some $\lambda_0 > 0$, it follows from (1.29) that $\lambda_0 - \mathscr{A}$ is invertible and that $||R(\lambda_0, \mathscr{A})|| \leq \lambda_0^{-1}$. We show that this property is valid for all $\lambda > 0$. In fact, let $\Gamma := \rho(\mathscr{A}) \cap (0, \infty)$. Then $\Gamma \neq \emptyset$ and therefore $\mathscr{A}$ is closed. Furthermore, let $(\lambda_n) \subset \Gamma$ such that $\lim_{n\to\infty} \lambda_n = \lambda > 0$. Now

$$dist(\lambda_n, \sigma(\mathscr{A})) \geq ||R(\lambda_n, \mathscr{A})||^{-1} \geq \lambda_n$$

for all $n \in \mathbb{N}$ and it follows that $\lambda \in \Gamma$. This shows that Γ is closed in $(0, \infty)$. Since Γ is obviously open, it follows that $\Gamma = (0, \infty)$ and therefore $(0, \infty) \subset \rho(\mathscr{A})$. The inequality (1.29) implies that $||R(\lambda, \mathscr{A})|| \leq \lambda^{-1}$ for all $\lambda > 0$ and the Hille-Yosida theorem implies the assertion. □

Another important consequence of the Chernoff Product Formula (Theorem 1.22) is the following Lie-Trotter Product Formula. As we will see in Chapter 2, this product formula is very useful in the cases where

(a) $\mathscr{A}$ and $\mathscr{B}$ generate strongly continuous contraction semigroups $\{T(t)\}_{t\geq 0}$ and $\{S(t)\}_{t\geq 0}$ that can be computed explicitly.

(b) The sum $\mathscr{A} + \mathscr{B}$ generates a strongly continuous semigroup $\{\mathcal{C}(t)\}_{t\geq 0}$, but the resolvent $R(\lambda, \mathscr{A} + \mathscr{B})$ cannot be computed explicitly so that $\{\mathcal{C}(t)\}_{t\geq 0}$, cannot be obtained by computing the inverse Laplace transform of $R(\lambda, \mathscr{A} + \mathscr{B})$.

Corollary 1.26. *(Lie-Trotter Product Formula). Let $\{T(t)\}_{t\geq 0}$ and $\{S(t)\}_{t\geq 0}$ be strongly continuous semigroups on X satisfying the stability condition*

$$|| \left[T\left(\frac{t}{n}\right) S\left(\frac{t}{n}\right) \right]^n || \leq Me^{\omega t} \text{ for all } t \geq 0, n \in \mathbb{N}, \tag{1.30}$$

and for constants $M \geq 1$, $\omega \in \mathbb{R}$. Consider the "sum" $\mathscr{A} + \mathscr{B}$ on $D := D(\mathscr{A}) \cap D(\mathscr{B})$ of the generators $(\mathscr{A}, D(\mathscr{A}))$ of $\{T(t)\}_{t\geq 0}$ and $(\mathscr{B}, D(\mathscr{B}))$ of $\{S(t)\}_{t\geq 0}$, and assume that D and $(\lambda_0 - \mathscr{A} - \mathscr{B})D$ are dense in X for some $\lambda_0 > \omega$. Then $\mathcal{C} := \overline{\mathscr{A} + \mathscr{B}}$ generates a strongly continuous semigroup $\{U(t)\}_{t\geq 0}$ given by the Trotter product formula

$$U(t)x = \lim_{n\to\infty} \left[T\left(\frac{t}{n}\right) S\left(\frac{t}{n}\right) \right]^n x,\ x \in X \tag{1.31}$$

where the limit is uniform for t in compact intervals.

Proof. For $t \geq 0$, define $V(t) := T(t)S(t)$. Then for all $x \in D$,

$$\lim_{t\to 0+} \frac{V(t)x - x}{t} = \lim_{t\to 0+} T(t)\frac{S(t)x - x}{t} + \lim_{t\to 0+} \frac{T(t)x - x}{t} = \mathscr{B}x + \mathscr{A}x.$$

Thus, applying this to Theorem 1.22 proves the statement. □

In order to analyze the rate of convergence of the Lie-Trotter product formula, we need the following elementary lemmas.

Lemma 1.27. *Let $\mathscr{A}$ be the generator of a C_0- semigroup $\{T(t)\}_{t\geq 0}$. Then, for all $x \in D(\mathscr{A}^{n+1})$,*

$$T(t)x = x + t\mathscr{A}x + \ldots + \frac{t^n}{n!}\mathscr{A}^n x + \int_0^t \frac{(t-s)^n}{n!}T(s)\mathscr{A}^{n+1}x\,ds.$$

Proof. By Lemma 1.9, for all $x \in D(\mathscr{A})$,

$$T(t)x = x + \int_0^t T(s)\mathscr{A}x\,ds.$$

So the base case, $n = 0$, is true. Let us assume it is valid for some $n \in \mathbb{N}$. Then, for $x \in D(\mathscr{A}^{n+2})$,

$$T(t)x = x + t\mathscr{A}x + \ldots + \frac{t^n}{n!}\mathscr{A}^n x + \int_0^t \frac{(t-s)^n}{n!}T(s)y\,ds$$

where $y = \mathscr{A}^{n+1}x \in D(\mathscr{A})$. Thus,

$$\begin{aligned}T(t)x &= x + t\mathscr{A}x + \ldots + \frac{t^n}{n!}\mathscr{A}^n x + \int_0^t \frac{(t-s)^n}{n!}[y + \int_0^s T(r)\mathscr{A}y\,dr]\,ds\\ &= x + t\mathscr{A}x + \ldots + \frac{t^n}{n!}\mathscr{A}^n x + \int_0^t \frac{(t-s)^n}{n!}y\,ds + \int_0^t \frac{(t-s)^n}{n!}\int_0^s T(r)\mathscr{A}y\,drds.\end{aligned}$$

Now,

$$\begin{aligned}\int_0^t \frac{(t-s)^n}{n!}\int_0^s f(r)\,drds &= \int_0^t\int_0^s \frac{(t-s)^n}{n!}f(r)\,dr\,ds = \int_0^t\int_r^t \frac{(t-s)^n}{n!}f(r)\,ds\,dr\\ &= \int_0^t f(r)\int_r^t \frac{(t-s)^n}{n!}\,ds\,dr = \int_0^t \frac{(t-r)^{n+1}}{(n+1)!}f(r)\,dr\\ &= \int_0^t \frac{(t-s)^{n+1}}{(n+1)!}f(s)\,ds.\end{aligned}$$

Therefore,

$$T(t)x = x + t\mathscr{A}x + \ldots + \frac{t^n}{n!}\mathscr{A}^n x + \frac{t^{n+1}}{(n+1)!}\mathscr{A}^{n+1}x + \int_0^t \frac{(t-s)^{n+1}}{(n+1)!}T(r)\mathscr{A}^{n+2}x\,ds.$$

□

We need the following non-commutative version of the classical binomial formula $u^n - v^n = (u-v)\sum_{m=0}^{n-1} u^{n-1-m}v^m$.

Lemma 1.28. *(Binomial Formula for Non-Commutative Operators). Let U, V be operators. Then.*

$$U^n - V^n = U^{n-1}[U - V] + \sum_{m=1}^{n-1} U^{n-m-1}[U - V]V^m.$$

Moreover, if $U(t)$ and $V(t)$ are bounded linear operators satisfying $||U(t)|| \leq e^{\omega t}$ and $||V(t)|| \leq e^{\omega t}$, then

$$||U(t)^n - V(t)^n|| \leq ne^{n\omega t}||U(t) - V(t)||$$

for all $n \in \mathbb{N}$.

Proof.

$$\begin{aligned} &U^{n-1}[U - V] + \sum_{m=1}^{n-1} U^{n-m-1}[U - V]V^m = U^n - U^{n-1}V + U^{n-2}[U - V]V \\ &+ \sum_{m=2}^{n-1} U^{n-m-1}[U - V]V^m = U^n - U^{n-2}V^2 + U^{n-3}[U - V]V^2 \\ &+ \sum_{m=3}^{n-1} U^{n-m-1}[U - V]V^m = U^n - U^{n-3}V^3 + \sum_{m=3}^{n-1} U^{n-m-1}[U - V]V^m \\ &= \dots \\ &= U^n - V^n. \end{aligned}$$

It follows from the binomial formula above that

$$||U(t)^n - V(t)^n|| \leq e^{n\omega t}||U(t) - V(t)|| + e^{n\omega t} \sum_{m=1}^{n-1} ||U(t) - V(t)|| = ne^{n\omega t}||U(t) - V(t)||.$$

□

Proposition 1.29. *Let $\mathscr{A}$ be the generator of a contraction $\mathcal{C}_0-$ semigroup $\{T(t)\}_{t\geq 0}$ and let $\mathscr{B}$ be a bounded linear operator generating a semigroup $\{S(t)\}_{t\geq 0}$ with $||S(t)|| \leq e^{\omega t}$ and define $V(t) = S(t)T(t)$ or $V(t) = T(t)S(t)$. Then $\mathcal{C} = \mathscr{A} + \mathscr{B}$ generates a C_0- semigroup $U(t)$ given by*

$$U(t)x = \lim_{n\to\infty} \left(V\left(\frac{t}{n}\right)\right)^n x$$

for all $x \in X$. Moreover, $||U(t)|| \leq e^{\omega t}$ and for all $x \in D(\mathcal{C}^2) \cap D(\mathscr{A}^2)$ and $T > 0$ there exists a constant $C_{x,T} > 0$ such that

$$||U(t)x - \left(V\left(\frac{t}{n}\right)\right)^n x|| \leq C_{x,T}\frac{t^2}{n}e^{\omega t}$$

for all $t \in [0, T]$.

Proof. Let $V(t) = S(t)T(t)$. (The proof of the case $V(t) = T(t)S(t)$ is similar by using Lemmas 2.6 and 2.7 in [10].) By Lemma 1.28,

$$||U(t)^n - V(t)^n|| \leq ne^{n\omega t}||U(t) - V(t)||.$$

Thus,

$$\begin{aligned} ||U(t) - V\left(\frac{t}{n}\right)^n|| &= ||U\left(\frac{t}{n}\right)^n - V\left(\frac{t}{n}\right)^n|| \\ &\leq ne^{\omega t}||U\left(\frac{t}{n}\right) - V\left(\frac{t}{n}\right)||. \end{aligned}$$

To estimate $||U(t) - V(t)||$, observe that it follows from Lemma 1.27 that

$$\begin{aligned} V(t)x &= S(t)T(t)x = S(t)y \text{ (where y = T(t)x)} \\ &= y + tBy + \int_0^t (t-s)S(s)\mathscr{B}^2 y\, ds \\ &= T(t)x + t\mathscr{B}T(t)x + \int_0^t (t-s)S(s)\mathscr{B}^2 T(t)x\, ds \\ &= x + t\mathscr{A}x + \int_0^t (t-s)T(s)\mathscr{A}^2 x\, ds + t\mathscr{B}[x + \int_0^t T(s)\mathscr{A}x\, ds] \\ &+ \int_0^t (t-s)S(s)\mathscr{B}^2 T(t)x\, ds \\ &= x + t(\mathscr{A} + \mathscr{B})x + t\mathscr{B}\int_0^t T(s)\mathscr{A}x\, ds + \int_0^t (t-s)S(s)\mathscr{B}^2 T(t)x\, ds. \end{aligned}$$

Moreover, since

$$U(t)x = x + t(\mathscr{A} + \mathscr{B})x + \int_0^t (t-s)U(s)(\mathscr{A} + \mathscr{B})^2 x,$$

it follows that

$$||U(t)x - V(t)x|| \leq (||(\mathscr{A} + \mathscr{B})^2 x|| + ||\mathscr{B}||^2||x||)\int_0^t (t-s)e^{\omega s}\, ds + t^2||\mathscr{B}||||\mathscr{A}x||.$$

Since $\int_0^t (t-s)e^{\omega s}\, ds = -\frac{1}{\omega^2} - \frac{t}{\omega} + \frac{1}{\omega^2}e^{\omega t} = t^2 g(t)$, where $g(t) = \sum_{j=2}^\infty \dfrac{\omega^{j-2}t^{j-2}}{j!}$ is an entire function, it follows that there exists a constant $C_{x,T} > 0$ such that

$$||U(t)x - V(t)x|| \leq C_{x,T}t^2.$$

Thus, by Lemma 1.28,

$$||U(t)x - V\left(\frac{t}{n}\right)^n x|| \leq ne^{\omega t}||U\left(\frac{t}{n}\right)x - V\left(\frac{t}{n}\right)x|| \leq ne^{\omega t}C_{x,T}\frac{t^2}{n^2} = C_{x,T}e^{\omega t}\frac{t^n}{n}.$$

□

The following consequence of the Chernoff Product Formula is a variant of the Lie-Trotter Product Formula that is also known as "Strang-Splitting". This approximation converges like $\frac{1}{n^2}$ for sufficiently smooth $x \in X$.

Corollary 1.30. *(Lie-Trotter-Kato Product Formula). Let $\{T(t)\}_{t\geq 0}$, $\{S(t)\}_{t\geq 0}$ be strongly continuous semigroups on X satisfying the stability condition*

$$\|[T\left(\frac{t}{2n}\right)S\left(\frac{t}{n}\right)T\left(\frac{t}{2n}\right)]^n\| \leq Me^{\omega t} \text{ for all } t \geq 0, n \in \mathbb{N}, \tag{1.32}$$

and some constants $M \geq 1$, $\omega \in \mathbb{R}$. Consider the "sum" $\mathscr{A} + \mathscr{B}$ on $D := D(\mathscr{A}) \cap D(\mathscr{B})$ of the generators $(\mathscr{A}, D(\mathscr{A}))$ of $\{T(t)\}_{t\geq 0}$ and $(\mathscr{B}, D(\mathscr{B}))$ of $\{S(t)\}_{t\geq 0}$, and assume that D and $(\lambda_0 - \mathscr{A} - \mathscr{B})D$ are dense in X for some $\lambda_0 > \omega$. Then $\mathcal{C} := \overline{\mathscr{A} + \mathscr{B}}$ generates a strongly continuous semigroup $\{U(t)\}_{t\geq 0}$ given by the Lie-Trotter-Kato product formula

$$U(t)x = \lim_{n\to\infty}\left[T\left(\frac{t}{2n}\right)S\left(\frac{t}{n}\right)T\left(\frac{t}{2n}\right)\right]^n x,\ x \in X. \tag{1.33}$$

Moreover, if $\mathscr{A}$ generates a contraction semigroup $\{T(t)\}_{t\geq 0}$ and $\mathscr{B}$ is a bounded linear operator generating a semigroup $\{S(t)\}_{t\geq 0}$ with $||S(t)|| \leq e^{\omega t}$, then $||V(t)|| \leq e^{\omega t}$ and for all $x \in D(\mathcal{C}^3) \cap D(\mathscr{A}^3)$ and $T > 0$, there exists a constant $C_{x,T} > 0$ such that

$$||U(t)x - \left[T\left(\frac{t}{2n}\right)S\left(\frac{t}{n}\right)T\left(\frac{t}{2n}\right)\right]^n x|| \leq C_{x,T}\frac{t^3}{n^2}e^{\omega t}.$$

Proof. For $t \geq 0$, define $V(t) := T\left(\frac{t}{2}\right)S(t)T\left(\frac{t}{2}\right)$. Then for all $x \in D$,

$$\begin{aligned}
&\lim_{t\to 0+}\frac{T\left(\frac{t}{2}\right)S(t)T\left(\frac{t}{2}\right)x - x}{t} \\
&= \lim_{t\to 0+} T\left(\frac{t}{2}\right)S(t)\frac{T\left(\frac{t}{2}\right)x - x}{t} + \lim_{t\to 0+} T\left(\frac{t}{2}\right)\frac{S(t)x - x}{t} + \lim_{t\to 0+}\frac{T\left(\frac{t}{2}\right)x - x}{t} \\
&= \frac{\mathscr{A}}{2}x + \mathscr{B}x + \frac{\mathscr{A}}{2}x = \mathscr{A}x + \mathscr{B}x.
\end{aligned}$$

Thus, applying this to Theorem 1.22 proves the statement 1.33. Similar to the proof of Proposition 1.29, it can be shown that for $x \in D(\mathcal{C}^3) \cap D(\mathscr{A}^3)$ and $T > 0$ there exists a constant $C_{x,T} > 0$ such that

$$||U(t)x - V(t)x|| \leq C_{x,T}t^3$$

for all $t \in [0, T]$ (see [10] for details). Thus, by Lemma 1.28,

$$||U(t)x - V\left(\frac{t}{n}\right)^n x|| \leq C_{x,T}e^{\omega t}\frac{t^3}{n^2}$$

for all $x \in D(\mathcal{C}^3) \cap D(\mathscr{A}^3)$ and $t \in [0, T]$. □

We have seen in the previous corollary that the Lie-Trotter-Kato approximations converge like $\frac{1}{n^2}$ for sufficiently smooth initial data. Unfortunately, it is impossible to get higher order convergence unless one considers groups (see [2], [11]).

Corollary 1.31. *(Lie-Trotter-Yoshida). Let $\{T(t)\}_{t\geq 0}$, $\{S(t)\}_{t\geq 0}$ be strongly continuous groups on X. Define*

$$V(t) = S\left(\frac{\gamma_1 t}{2}\right) T(\gamma_1 t)\, S\left(\frac{(\gamma_1+\gamma_2)t}{2}\right) T(\gamma_2 t)\, S\left(\frac{(\gamma_1+\gamma_2)t}{2}\right) T(\gamma_1 t)\, S\left(\frac{\gamma_1 t}{2}\right)$$

where $\gamma_1 = \dfrac{1}{2-2^{\frac{1}{3}}}$ and $\gamma_2 = -\dfrac{2^{\frac{1}{3}}}{2-2^{\frac{1}{3}}}$. Assume that $V(t)$ satisfies the stability condition

$$||V\left(\frac{t}{n}\right)|| \leq Me^{\omega t}$$

for all $t \geq 0, n \in \mathbb{N}$ and for some constants $M \geq 1$, $\omega \in \mathbb{R}$. Consider the "sum" $\mathscr{A} + \mathscr{B}$ on $D := D(\mathscr{A}) \cap D(\mathscr{B})$ of the generators $(\mathscr{A}, D(\mathscr{A}))$ of $\{T(t)\}_{t\geq 0}$ and $(\mathscr{B}, D(\mathscr{B}))$ of $\{S(t)\}_{t\geq 0}$, and assume that D and $(\lambda_0 - \mathscr{A} - \mathscr{B})D$ are dense in X for some $\lambda_0 > \omega$. Then $C := \overline{\mathscr{A} + \mathscr{B}}$ generates a strongly continuous semigroup $\{U(t)\}_{t\geq 0}$ given by the Lie-Trotter-Yoshida product formula

$$U(t)x = \lim_{n\to\infty} V\left(\frac{t}{n}\right)^n x,\ x \in X.$$

Moreover, if $\mathscr{A}$ generates a contraction semigroup $\{T(t)\}_{t\geq 0}$ and $\mathscr{B}$ is a bounded linear operator generating a semigroup $\{S(t)\}_{t\geq 0}$ with $||S(t)|| \leq e^{\omega t}$, then $||V(t)|| \leq e^{\omega t}$ and for all $x \in D(\mathcal{C})^5 \cap D(\mathscr{A}^5)$ and $T > 0$, there exists a constant $C_{x,T} > 0$ such that

$$||U(t)x - V\left(\frac{t}{n}\right)^n x|| \leq C_{x,T}\frac{t^5}{n^4}e^{\omega t}.$$

Proof. The proof is similar to the proofs of Corollaries 1.26 and 1.30. Now,

$$\begin{aligned} V'(0) &= \frac{\gamma_1}{2}\mathscr{B} + \gamma_1\mathscr{A} + \frac{\gamma_1+\gamma_2}{2}\mathscr{B} + \gamma_2\mathscr{A} + \frac{\gamma_1+\gamma_2}{2}\mathscr{B} + \gamma_1\mathscr{A} + \frac{\gamma_1}{2}\mathscr{B} \\ &= (2\gamma_1+\gamma_2)(\mathscr{A}+\mathscr{B})x = \mathscr{A}x + \mathscr{B}x. \end{aligned}$$

For the error estimates, we refer to [4]. □

1.3 Semigroups Induced By Autonomous Flows

This section is an extension of Examples 1.7 and 1.10, where we considered the shift semigroup

$$T(t)g(x) = g(x+t) = g(\sigma(t,x))$$

induced by the flow $\sigma(t,x) = t + x$ solving the ordinary differential equation $x'(t) = 1$, $x(0) = \omega$. Here we consider the case $\Omega \subseteq \mathbb{R}$ and

$$x'(t) = a(x(t)), x(0) = \omega \in \Omega. \tag{1.34}$$

In this case, the induced linear semigroup $\mathcal{T} = \{T(t)\}_{t\geq 0}$ is given by

$$T(t)g(\omega) := g(\sigma(t,\omega)) \tag{1.35}$$

with $\sigma(t,\omega) := A^{-1}(t + A(\omega))$, where $A(t) := \int \frac{1}{a(t)}\,dt, t + A(\omega) \in \text{Range}(A)$, and $g \in \mathcal{N} = \mathcal{G}(\Omega,\mathbb{R})$ (a vector space of functions from $\Omega \to \mathbb{R}$). This was shown in Proposition 1.3. Formally, the generator $\mathscr{A}$ of $\mathcal{T} = \{T(t)\}_{t\geq 0}$ is given by

$$(\mathscr{A}g)(\omega) := a(\omega)g'(\omega), \tag{1.36}$$

where $\omega \in \Omega$.

If we assume that a is continuous, $\omega \geq 0$, and that $\mathscr{A}$ generates a strongly contionous contraction semigroup on $C_0[0,\infty)$, then we know that for all $f \in C_0[0,\infty)$ there exists $g \in D(\mathscr{A})$ such that $(\lambda - \mathscr{A})g = f$ or

$$\lambda g(\omega) - a(\omega)g'(\omega) = f(\omega)$$

for all $\omega \geq 0$. Using an integrating factor and the fact that $g \in D(\mathscr{A}) \subset C_0[0,\infty)$, we conclude that $R(\lambda,\mathscr{A})g = f$, where

$$g(\omega) = \int_\omega^\infty \frac{1}{a(s)} e^{-\lambda[A(s)-A(\omega)]} f(s)\,ds. \tag{1.37}$$

Thus, the computation of $R(\lambda,\mathscr{A})g$ does not require knowledge of the function A^{-1}. Since

$$R(\lambda,\mathscr{A})g(\omega) = \int_\omega^\infty \frac{1}{a(s)} e^{-\lambda[A(s)-A(\omega)]} f(s)\,ds$$

and

$$T(t)g(\omega) = g(\sigma(t,\omega) = g(A^{-1}(t + A(\omega)),$$

it follows that

$$g(A^{-1}(t + A(\omega)) \ \ (\omega \geq 0)$$

is the inverse Laplace transform of $R(\lambda,\mathscr{A})g(\omega)$. If we assume that $A(0) = 0$ and that $\mathscr{A}$ generates a C_0 - semigroup on $C_0[0,\infty)$ we see that

$$g(A^{-1}(t)) = T(t)g(0)$$

is the inverse Laplace transform of

$$\lambda \to \int_0^\infty \frac{1}{a(s)} e^{-\lambda A(s)} f(s)\,ds.$$

This leads to the following observation that seems to be new.

Proposition 1.32. *Let $A \in C^1[0,\infty)$ be an increasing function with $0 \le A(0) = M$ and $\lim_{t\to\infty} A(t) = N$. If*

$$r(\lambda) = \int_0^\infty e^{-\lambda A(s)} s A'(s)\, ds$$

exists, then $A^{-1}(t)$ is the inverse Laplace transform of $r(\lambda)$.

Proof. The statement follows immediately from

$$\int_0^\infty e^{-\lambda A(s)} s A'(s)\, ds = \int_M^N e^{-\lambda u} A^{-1}(t)\, dt = \int_0^\infty e^{-\lambda t} A^{-1}(t)\, dt$$

when we define $A^{-1}(t) := 0$ for $t \notin [M, N)$. □

Example 1.33. Let $A(s) = s^5 + s^3 + s$. Then $A(0) = 0, A(\infty) = \infty, A$ is invertible and

$$r(\lambda) = \int_0^\infty s(5s^4 + 3s^2 + 1)e^{-\lambda(s^5+s^3+s)}\, ds$$

exists for all $\lambda > 0$. Let A^{-1} be the inverse of the function $A(s) = s^5 + s^3 + s$. Then

$$r(\lambda) = \int_0^\infty e^{-\lambda t} A^{-1}(t)\, dt.$$

Thus, $A^{-1}(t)$ is the inverse Laplace transform of

$$r(\lambda) = \int_0^\infty s(5s^4 + 3s^2 + 1)e^{-\lambda(s^5+s^3+s)}\, ds.$$

Here are the numerical results when using Corollary 1.16 for $q = 5, 10, 15$ where $A_q^{-1}(t) = \sum_{j=1}^q \frac{c_j}{t} r\left(\frac{b_i}{t}\right)$ is an approximation of $A^{-1}(t)$.

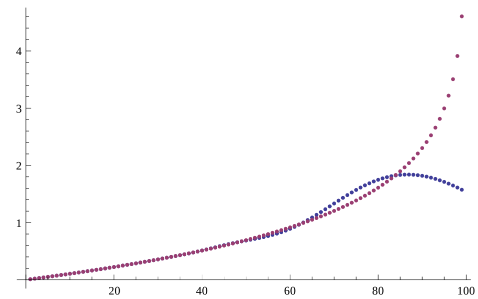
Table 1.1: Approximating A^{-1}

q	t	$A^{-1}(t)$	$A_q^{-1}(t) - A^{-1}(t)$
5	$\frac{21}{32}$	$\frac{1}{2}$	2×10^{-5}
5	3	1	2×10^{-3}
5	42	2	3×10^{-2}
10	$\frac{21}{32}$	$\frac{1}{2}$	3×10^{-8}
10	3	1	3×10^{-5}
10	42	2	5×10^{-3}
15	$\frac{21}{32}$	$\frac{1}{2}$	5×10^{-13}
15	3	1	2×10^{-7}
15	42	2	2×10^{-4}

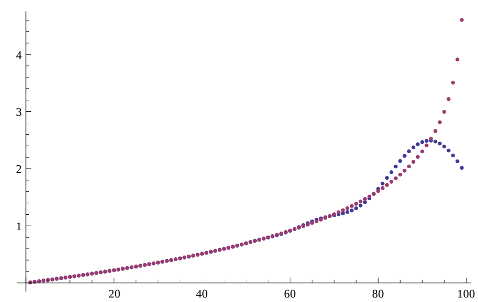
Example 1.34. Let $A(s) = 1 - e^{-s}$. Then

$$r(\lambda) = -\int_0^\infty e^{-\lambda(1-e^{-s})} s e^{-s}\, ds = -\int_0^1 \ln(u-1) e^{-\lambda u}\, du$$

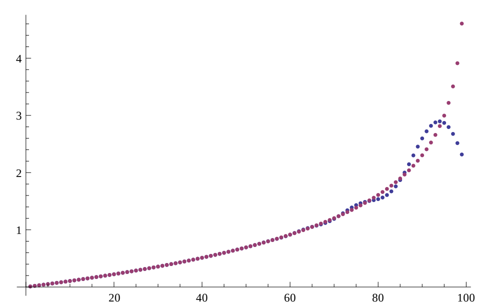
exists for all $\lambda \in \mathbb{C}$ and $A^{-1}(s) = -\ln(1-s)$ is the inverse Laplace transform of $r(\lambda)$ for all $0 \le s < 1$. Here are the graphical results when using Corollary 1.16 for $q = 5, 10, 15, 20$.
For $q = 5$ we have the following graph (where 100 corresponds to the number 1):



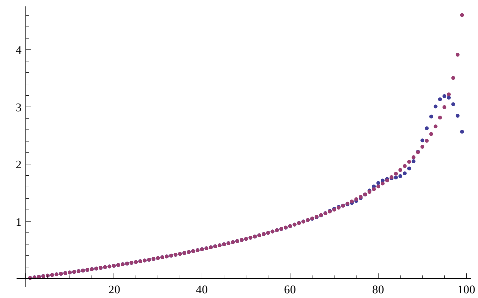
For $q = 10$:



For $q = 15$:



For $q = 20$:



We will now examine specific cases where we will try to approximate the nonlinear flow solving (1.34) in terms of the resolvent of the generator $\mathscr{A}$ of the associated linear semigroup. That is, we will approximate the flow $\sigma(t,\omega)$ given by the solutions $x(t)$ of the nonlinear initial value problem

$$x'(t) = a(x(t)),\ x(0) = \omega$$

in terms of the resolvent of the linear operators $(\mathscr{A} f)(x) = a(x)f'(x)$.

Proposition 1.35. *Let $\sigma(t,\omega)$ with $t \geq 0$ be the solution of $x'(t) = a(x(t)), x(0) = \omega \geq 0$. If $\sigma(t,\omega) \geq 0, \sigma(t,\omega) \to \infty$ as $\omega \to \infty$ uniformly for $t \geq 0$ and $\sigma(t,\omega) \to \omega$ as $t \to 0$ uniformly for all ω in compact intervals, then*

$$T(t)g(\omega) := g(\sigma(t,\omega))$$

is a strongly continuous contraction semigroup on $C_0[0,\infty)$ and

$$D = \left\{g \in C_0[0,\infty) \cap C^1(0,\infty) : \omega \to a(\omega)g'(\omega) \in C_0[0,\infty)\right\}$$

is a subset of the domain of the generator $\mathscr{A}$ of $\{T(t)\}_{t\geq 0}$ and $(\mathscr{A} g)(\omega) = a(\omega)g'(\omega)$ for all $g \in D$.

Proof. If $g \in C_0[0,\infty)$, then $T(t)g : \omega \to g(\sigma(t,\omega)) \in C_0[0,\omega))$ since $0 \leq \sigma(t,\omega) \to \infty$ as $\omega \to \infty$. Clearly, $||T(t)g|| \leq ||g||$ for all $g \in C_0[0,\infty)$. To see that $\{T(t)\}_{t\geq 0}$ is strongly continous, let $g \in C_0[0,\infty)$ and let N be such that $|g(\omega)| \leq \frac{\varepsilon}{3}$ for all $\omega \geq N$. Since $\sigma(t,\omega) \to \infty$ as $\omega \to \infty$ uniformly in $t \geq 0$, there exists $N' \geq N$ such that $|\sigma(t,\omega)| \geq N$ for all $\omega \geq N' \geq N$ and all $t \geq 0$. Thus, $|g(\sigma(t,\omega))| \leq \dfrac{\varepsilon}{3}$ and $|g(\omega)| \leq \dfrac{\varepsilon}{3}$ for all $\omega \geq N'$ and all $t \geq 0$. This shows that

$$||T(t)g - g|| = \sup_{\omega\geq 0} |g(\sigma(t,\omega)) - g(\omega)| \leq \sup_{\omega\in[0,N']} |g(\sigma(t,\omega) - g(\omega)| + \frac{2\varepsilon}{3}.$$

Since g is uniformly continuous on $[0, N'+1]$ there exists $\delta > 0$ such that $|g(z) - g(\omega)| < \frac{\varepsilon}{3}$ for all $z, \omega \in [0, N'+1]$ with $|z-\omega| < \delta$. Since $|\sigma(t,\omega) - \omega| \to 0$ as $t \to 0$ uniformly for ω in compact intervals, there exists $0 < \delta < 1$ such that $|\sigma(t,\omega) - \omega| \leq \delta$ for all $\omega \in [0, N']$ and $0 < t < \delta$. Thus

$$|g(\sigma(t,\omega)) - g(\omega)| \leq \frac{\varepsilon}{3}$$

for all $0 < t < \delta$ and all $\omega \in [0, N']$. This shows that $||T(t)g - g|| \leq \varepsilon$ for all $0 \leq t < \delta$. Hence, $\{T(t)\}_{t\geq 0}$ is a strongly continuous contraction semigroup on $C_0[0,\infty)$. Now let

$$g \in D = \left\{ g \in C_0[0,\infty) \cap C^1(0,\infty) : \omega \to a(\omega)g'(\omega) \in C_0[0,\infty) \right\}.$$

Then

$$\begin{aligned}
(\mathscr{A} g)(\omega) - a(\omega)f'(\omega) &= \lim_{t\to 0} \frac{T(t)g(\omega) - g(\omega)}{t} - a(\omega)g'(\omega) \\
&= \lim_{t\to 0} \frac{g(\sigma(t,\omega) - g(\omega)}{t} - a(\omega)g'(\omega) \\
&= \lim_{t\to 0} \frac{1}{t} \int_0^t \sigma'(s,\omega)g'(\sigma(s,\omega))\, ds - a(\omega)g'(\omega) \\
&= \lim_{t\to 0} \frac{1}{t} \int_0^t [a(\sigma(s,\omega)g'(\sigma(s,\omega)) - a(\omega)g'(\omega)]\, ds.
\end{aligned}$$

Let $N > 0$ be such that $|a(\omega)g'(\omega)| \leq \frac{\varepsilon}{3}$ for all $\omega \geq N$. Since $\sigma(t,\omega) \to \infty$ uniformly in $t \geq 0$ and since there exists $N' \geq N$ such that $|\sigma(t,\omega)| \geq N$ for all $\omega \geq N' \geq N$ and all $t \geq 0$ it follows that $|a(\sigma(s,\omega))g'(s,\omega)| \leq \frac{\varepsilon}{3}$ and $|a(\omega)g'(\omega)| \leq \frac{\varepsilon}{3}$ for all $s \geq 0$ and $\omega \geq N'$. Thus,

$$\sup_{\omega\geq 0} |(\mathscr{A} g)(\omega) - a(\omega)g'(\omega)| \leq \sup_{\omega\in[0,N']} |(\mathscr{A} g)(\omega) - a(\omega)g'(\omega)| + \frac{2\varepsilon}{3}.$$

Since $\omega \to a(\omega)g'(\omega)$ is uniformly continuous on $[0, N'+1]$, there exists $\delta > 0$ such that

$$|a(z)g'(z) - a(\omega)g'(\omega)| \leq \frac{\varepsilon}{3}$$

for all $\omega, z \in [0, N'+1]$ with $|z - \omega| < \delta$. Since $\sigma(t,\omega) \to \omega$ as $t \to 0$ uniformly for ω in compact intervals, there exists $0 < \delta < 1$ such that $|\sigma(s,\omega) - \omega| \leq \delta$ for all $0 \leq s < \delta$ and $\omega \in [0, N']$. Thus,

$$|(\mathscr{A} g)(\omega) - a(\omega)g'(\omega)| \leq \lim_{t\to 0} \frac{1}{t} \int_0^t |a(\sigma(s,\omega))g'(\sigma(s,\omega)) - a(\omega)g'(\omega)|\, ds \leq \frac{\varepsilon}{3}$$

for all $0 < t < \delta$ and $\omega \in [0, N']$. This shows that $D \subset D(\mathscr{A})$ and $(\mathscr{A} g)(\omega) = a(\omega)g'(\omega)$ for all $g \in D$. □

Example 1.36. (a) We consider $x'(t) = a(x(t))$, $x(0) = \omega$, for $a(x) = x$. Then the flow is given by

$$\sigma(t,\omega) = \omega e^t.$$

Since $0 \leq \omega \leq \omega e^t \to \infty$, it follows that $\sigma(t,\omega) \to \infty$ as $\omega \to \infty$ uniformly in $t \geq 0$. Let $\omega \in [0, N]$. Then $|\sigma(t,\omega) - \omega| \leq N|e^t - 1| \to 0$ as $t \to 0$. Thus, $\sigma(t,\omega) \to \omega$ as $t \to 0$ uniformly for all ω in compact intervals. Thus, by Proposition 1.35,

$$T(t)g(\omega) = g(\omega e^t)$$

defines a strongly continuous contraction semigroup on $C_0[0,\infty)$ with generator $\mathscr{A}$ given by

$$(\mathscr{A}g)(\omega) = \omega g'(\omega)$$

for all $g \in D := \{g \in C_0[0,\infty) \cap C^1(0,\infty) : \omega \to \omega g'(\omega) \in C_0[0,\infty)\}$. Moreover, for $\mathrm{Re}(\lambda) > 0$ and $g \in C_0[0,\infty)$, consider the first order linear ode

$$(\lambda I - \mathscr{A})f(x) = \lambda f(x) - xf'(x) = g(x)$$

for $g \in D$. Using integrating factors, one sees that the equation is equivalent to

$$(x^{-\lambda}f(x))' = -x^{-(\lambda+1)}g(x)$$

or

$$f(x) = x^{\lambda}\int_x^{\infty} \frac{1}{s^{\lambda+1}} g(s)\, ds + cx^{\lambda}.$$

To ensure that $f(x) \to 0$ as $x \to \infty$ we take $c = 0$. Then $f \in C_0[0,\infty) \cap C^1(0,\infty)$ and $x \to xf'(x) = \lambda f(x) - g(x) \in C_0[0,\infty)$. Thus, $f \in D, D = D(\mathscr{A})$, and

$$R(\lambda,\mathscr{A})g(x) = x^{\lambda}\int_x^{\infty} \frac{1}{s^{\lambda+1}} g(s)\, ds = r(\lambda, x).$$

By Theorem 1.15, for all $g \in C_0[0,\infty)$,

$$g(\sigma(t,\omega)) = T(t)g(\omega) = \lim_{q\to\infty}\sum_{j=1}^{q} \frac{c_j}{t} r\left(\frac{b_j}{t}, \omega\right),$$

where the limit is uniform in $\omega \geq 0$ and b_j and c_j are as in Theorem 1.15. In particular, if g is invertible, then

$$\sigma(t,\omega) = g^{-1}(T(t)g(\omega)) = g^{-1}\left(\lim_{q\to\infty}\sum_{j=1}^{q} \frac{c_j}{t} r\left(\frac{b_j}{t}, \omega\right)\right) = \lim_{q\to\infty} g^{-1}\left(\sum_{j=1}^{q} \frac{c_j}{t} r\left(\frac{b_j}{t}, \omega\right)\right).$$

Here are the results for $g(s) = \dfrac{1}{s+1}$and $g^{-1}(s) = \dfrac{1}{s} - 1$, where the error is $g^{-1}(u(t,\omega)) - \omega e^t$. .

Table 1.2: $a(\omega) = \omega$

q	t	ω	Error	q	t	ω	Error
10	1/2	1	10^{-22}	10	1/2	10	10^{-22}
10	1	1	10^{-16}	10	1	10	10^{-16}
10	2	1	10^{-10}	10	2	10	10^{-11}
10	4	1	10^{-5}	10	4	10	10^{-6}
10	8	1	10^{-3}	10	8	10	10^{-3}
20	1/2	1	10^{-44}	20	1/2	10	10^{-46}
20	1	1	10^{-32}	20	1	10	10^{-35}
20	2	1	10^{-21}	20	2	10	10^{-25}
20	4	1	10^{-13}	20	4	10	10^{-16}
20	8	1	10^{-4}	20	8	10	10^{-7}
30	1/2	1	10^{-66}	30	1/2	10	10^{-71}
30	1	1	10^{-48}	30	1	10	10^{-54}
30	2	1	10^{-34}	30	2	10	10^{-39}
30	4	1	10^{-21}	30	4	10	10^{-26}
30	8	1	10^{-10}	30	8	10	10^{-14}

(b) We consider $x'(t) = a(x(t)), x(0) = \omega \geq 0, t \geq 0$, for $a(x) = 2\sqrt{x}$ (see also (1.9) and the results thereafter). Then the flow solving the initial value problem is given by the polynomial $\sigma(t,\omega) = (t + \sqrt{\omega})^2 = t^2 + 2t\sqrt{\omega} + \omega$. Clearly, $0 \leq \omega \leq \sigma(t,\omega) \to \infty$ as $\omega \to \infty$ uniformly in $t \geq 0$. Let $t \in [0,1]$ and $\omega \in [0,N]$. Then

$$|\sigma(t,\omega) - \omega| = t|t + 2\sqrt{\omega}| \leq t(1 + 2\sqrt{N})$$

and therefore, $\sigma(t,\omega) \to \omega$ as $t \to 0$ uniformly for ω in compact intervals. Thus, by Proposition 1.35,

$$T(t)g(\omega) = g\left((t + \sqrt{\omega})^2\right)$$

is a strongly continuous contraction semigroup on $C_0[0,\infty)$ with generator $\mathscr{A}$ given by

$$(\mathscr{A}g)(\omega) = 2\sqrt{\omega}g'(\omega)$$

for all $g \in D$, where

$$D := \left\{g \in C_0[0,\infty) \cap C^1(0,\infty) : \omega \to 2\sqrt{\omega}g'(\omega) \in C_0[0,\infty)\right\}.$$

Moreover, for $\mathrm{Re}(\lambda) > 0$ and $g \in C_0[0,\infty)$, consider the first order linear ordinary differential equation

$$(\lambda I - \mathscr{A})f(x) = \lambda f(x) - 2\sqrt{x}f'(x) = g(x)$$

for $g \in D$. Using integrating factors, we obtain

$$(e^{-\lambda\sqrt{x}}f(x))' = -e^{-\lambda\sqrt{x}}\frac{g(x)}{2\sqrt{x}}$$

or

$$f(x) = e^{\lambda\sqrt{x}} \int_x^\infty e^{-\lambda\sqrt{s}} \frac{f(s^2)}{2\sqrt{s}}\, ds + ce^{\lambda\sqrt{x}}.$$

To ensure that $f(x) \to 0$ as $x \to \infty$, we take $c = 0$. Setting $t = \sqrt{s}$ or $dt = \dfrac{1}{2\sqrt{s}}\, ds$ we obtain

$$f(x) = e^{\lambda\sqrt{x}} \int_{\sqrt{x}}^\infty e^{-\lambda t} g(t^2)\, dt = \int_0^\infty e^{-\lambda u} g((u+\sqrt{x})^2)\, du = R(\lambda, \mathscr{A})g(x) = r(\lambda, x).$$

Since $f \in C_0[0,\infty) \cap C'(0,\infty)$ and $x \to 2\sqrt{x}f'(x) = \lambda f(x) - g(x) \in C_0[0,\infty)$, it follows that $D = D(\mathscr{A})$. As in Example 1.36 a, if g is invertible, then

$$\sigma(t,\omega) = \lim_{q\to\infty} g^{-1}\left(\sum_{j=1}^{q} \frac{c_j}{t} r\left(\frac{b_j}{t}, \omega\right)\right).$$

Observe the integral defining $R(\lambda, \mathscr{A})g(x)$ for $\mathrm{Re}(\lambda) > 0$ exists also if one chooses $g_0(x) = g_0^{-1}(x) = x$ (even though $g_0 \notin C_0[0,\infty)$. Since $\sigma(t,\omega)$ is a polynomial in t one can expect by Proposition 1.17 that for $q \geq 2$,

$$\sigma(t,\omega) = \sum_{j=1}^{q} \frac{c_j}{t} r\left(\frac{b_j}{t}, \omega\right),$$

where

$$r(\lambda,\omega) = \int_0^\infty e^{-\lambda u} g_0((u+\sqrt{x})^2)\, du = \int_0^\infty e^{-\lambda u}(u+\sqrt{x})^2\, du.$$

A confirmation of this can be seen in the following table.

Table 1.3: $a(\omega) = 2\sqrt{\omega}$

q	t	ω	Error
10	1/2	1	0
10	1	1	0
10	10	1	0
15	1/2	1	0
15	1	1	0
15	10	1	0
20	1/2	1	0
20	1	1	0
20	10	1	0

(c) Consider $x'(t) = a(x(t)), x(0) = \omega \geq 0, t \geq 0$ for $a(x) = \dfrac{1}{x}$. Then the flow solving this intial value problem is

$$\sigma(t,\omega) = \sqrt{2t+\omega^2}.$$

Then $\sigma(t,\omega) \to \infty$ as $\omega \to \infty$ uniformly in $t \geq 0$ (since $\omega \leq \sigma(t,\omega)$). Moreover, for $t > 0$ and all $\omega \geq 0$,

$$\sigma(t,\omega) - \omega = \frac{2t}{\sqrt{2t+\omega^2}+\omega} \leq \frac{2t}{\sqrt{2t}} = \sqrt{2t}.$$

Thus, $\sigma(t,\omega) \to \omega$ as $t \to 0$ uniformly for ω in compact subsets of $\mathbb{R}^+$. Hence, by Proposition 1.35,

$$T(t)g(\omega) := g(\sqrt{2t+\omega^2})$$

is a strongly continuous contraction semigroup on $C_0[0,\infty)$. As in the examples above, one can show that the generator $\mathscr{A}$ is given by

$$(\mathscr{A}g)(x) = \frac{1}{x}g'(x)$$

with domain

$$D(\mathscr{A}) := \left\{ g \in C_0[0,\infty) \cap C^1(0,\infty) : x \to \frac{1}{x}g'(x) \in C_0[0,\infty) \right\}$$

and resolvent

$$R(\lambda,\mathscr{A})g(x) = e^{\lambda\frac{x^2}{2}} \int_x^\infty e^{-\lambda\frac{s^2}{2}} s g(s)\, ds = r(\lambda,x).$$

Thus, as seen in the previous examples, if $g \in C_0[0,\infty)$ is invertible, then

$$\sigma(t,\omega) = \lim_{q\to\infty} g^{-1}\left(\sum_{j=1}^{q} \frac{c_j}{t} r\left(\frac{b_j}{t},\omega\right) \right).$$

However, since the integral defining $R(\lambda,\mathscr{A})$ with $\mathrm{Re}(\lambda) > 0$ exists also for the identity map $g_0(x) = x$, one can expect that

$$\sigma(t,\omega) = \lim_{q\to\infty} \sum_{j=1}^{q} \frac{c_j}{t} r\left(\frac{b_j}{t},\omega\right),$$

where $r(\lambda,x) = e^{\lambda\frac{x^2}{2}} \int_x^\infty e^{-\lambda\frac{s^2}{2}} s^2\, ds$.

Table 1.4: $a(\omega) = \dfrac{1}{\omega}$

q	t	ω	Error	q	t	ω	Error	q	t	ω	Error
10	1	1	10^{-10}	10	1	10	10^{-41}	10	1	20	10^{-53}
10	10	1	10^{-7}	10	10	10	10^{-22}	10	10	20	10^{-33}
10	40	1	10^{-5}	10	40	10	10^{-14}	10	40	20	10^{-2}
20	1	1	10^{-19}	20	1	10	10^{-82}	10	1	20	10^{-105}
20	10	1	10^{-8}	20	10	10	10^{-44}	10	10	20	10^{-66}
20	40	1	10^{-7}	20	40	10	10^{-27}	10	40	20	10^{-44}
30	1	1	10^{-28}	30	1	10	10^{-122}	10	1	20	10^{-157}
30	10	1	10^{-11}	30	10	10	10^{-66}	10	10	20	10^{-99}
30	40	1	10^{-7}	30	40	10	10^{-39}	10	40	20	10^{-66}

(d) Consider $x'(t) = a(x(t)), x(0) = \omega \geq 0, t \geq 0$ for $a(x) = x^2$. Then the flow is given by

$$\sigma(t,\omega) := \begin{cases} \dfrac{\omega}{1-t\omega} & \text{if } t\omega < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Clearly, $\sigma(t,\omega) \geq 0$ and $\omega \leq \sigma(t,\omega)$. Thus, $\sigma(t,\omega) \to \infty$ as $\omega \to \infty$ uniformly in $t \geq 0$. Let $\omega \in [0,N]$ and $0 < t < \dfrac{1}{N}$. Then, $t < \dfrac{1}{\omega}$ for all $\omega \in [0,N]$ and

$$|\sigma(t,\omega) - \omega| = \frac{\omega}{1-t\omega} - \omega = \frac{t\omega^2}{1-t\omega} \leq \frac{t\omega^2}{1-tN} \leq \frac{tN^2}{1-tN}$$

for all $\omega \in [0,N]$ and $0 < t < \dfrac{1}{N}$. Thus, $\sigma(t,\omega) \to \omega$ as $t \to 0$ uniformly for $\omega \in [0,N]$. Thus, by Proposition 1.35, $T(t)g(\omega) := g(\sigma(t,\omega))$ defines a strongly continuous contraction semigroup with generator $(\mathscr{A}g)(\omega) = \omega^2 g'(\omega)$ for all

$$g \in D = \left\{g \in C_0[0,\infty) \cap C^1(0,\infty) : \omega \to \omega^2 g'(\omega) \in C_0[0,\infty)\right\} \subset D(\mathscr{A}).$$

Moreover, for $\mathrm{Re}(\lambda) > 0$,

$$\begin{aligned} R(\lambda,\mathscr{A})g(\omega) &= \int_0^\infty e^{-\lambda t} T(t)g(\omega)\,dt = \int_0^\infty e^{-\lambda t} g(\sigma(t,\omega))\,dt \\ &= \int_0^{\frac{1}{\omega}} e^{-\lambda t} g\left(\frac{\omega}{1-t\omega}\right) dt + \int_{\frac{1}{\omega}}^\infty e^{-\lambda t} g(0)\,dt \\ &= \int_0^{\frac{1}{\omega}} e^{-\lambda t} g\left(\frac{\omega}{1-t\omega}\right) dt + g(0)\frac{1}{\lambda} e^{-\frac{\lambda}{\omega}} \\ &= \int_\omega^\infty e^{-\lambda\left[\frac{1}{\omega} - \frac{1}{u}\right]} g(u) \frac{1}{u^2}\,du + g(0)\frac{1}{\lambda}e^{-\frac{\lambda}{\omega}} \\ &= e^{-\frac{\lambda}{\omega}} \int_0^{\frac{1}{\omega}} e^{\lambda s} g\left(\frac{1}{s}\right) ds + g(0)\frac{1}{\lambda}e^{-\frac{\lambda}{\omega}}. \end{aligned}$$

By Theorem 1.15, for all $g \in C_0[0,\infty)$,

$$g(\sigma(t,\omega)) = T(t)g(\omega) = \lim_{q\to\infty} \sum_{j=1}^{q} \frac{c_j}{t} r\left(\frac{b_j}{t},\omega\right),$$

where the limit is uniform in $\omega \geq 0, b_j$ and c_j are as in Theorem 1.15, and

$$r(\lambda,\omega) = R(\lambda,\mathscr{A}) = e^{-\frac{\lambda}{\omega}} \int_0^{\frac{1}{\omega}} e^{\lambda s} g\left(\frac{1}{s}\right) ds + g(0)\frac{1}{\lambda}e^{-\frac{\lambda}{\omega}}.$$

In particular, if g is invertible, then

$$\sigma(t,\omega) = \lim_{q\to\infty} g^{-1}\left(\sum_{j=1}^{q} \frac{c_j}{t} r\left(\frac{b_j}{t},\omega\right)\right).$$

Table 1.5: $a(\omega) = \omega^2$

q	t	ω	Error
10	17/10	1/2	10^{-3}
10	1/2	1	10^{-5}
10	1/10	2	10^{-9}
15	17/10	1/2	10^{4}
15	1/2	1	10^{-7}
15	1/10	2	10^{-11}
20	17/10	1/2	10^{-3}
20	1/2	1	10^{-6}
20	1/10	2	10^{-13}

(e) Let $a(x) = -x^2, t \geq 0$, and $\omega \in [0, N]$. Then the flow is given by

$$\sigma(t, \omega) = \frac{\omega}{1 + t\omega}$$

and leaves the interval $[0, N]$ invariant;i.e., if $\omega \in [0, N]$, then $\sigma(t, \omega) \in [0, N]$ for all $t \geq 0$. The induced semigroup

$$T(t)g(\omega) = g(\sigma(t, \omega))$$

is a strongly continuous semigroup on $C[0, N]$ since $|\sigma(t, \omega) - \omega| \leq t\omega^2 \leq tN^2$. The set

$$D = \left\{g \in C[0, N] \cap C^1(0, N] : \omega \to \omega^2 g'(\omega) \in C[0, N]\right\}$$

is a core for the generator

$$(\mathscr{A} g)(\omega) = -\omega^2 g'(\omega)$$

and

$$R(\lambda, \mathscr{A}) = e^{\frac{\lambda}{\omega}} \int_0^{\omega} e^{-\frac{\lambda}{u}} \frac{1}{u^2} g(u)\, du.$$

Taking $g(u) = 0$ yields the following approximation of $\sigma(t, \omega)$:

Table 1.6: $a(\omega) = -\omega^2$

q	t	ω	Error	q	t	ω	Error	q	t	ω	Error
10	1	1	10^{-11}	10	1	10	10^{-6}	10	1	20	10^{-4}
10	10	1	10^{-7}	10	10	10	10^{-4}	10	10	20	10^{-3}
10	20	1	10^{-6}	10	20	10	10^{-4}	10	20	20	10^{-3}
10	50	1	10^{-5}	10	50	10	10^{-4}	10	50	20	10^{-3}
20	1	1	10^{-23}	20	1	10	10^{-7}	20	1	20	10^{-7}
20	10	1	10^{-8}	20	10	10	10^{-5}	20	10	20	10^{-4}
20	20	1	10^{-6}	20	20	10	10^{-4}	20	20	20	10^{-4}
20	50	1	10^{-7}	20	50	10	10^{-4}	20	50	20	10^{-4}
30	1	1	10^{-34}	30	1	10	10^{-4}	30	1	20	10^{-7}
30	10	1	10^{-12}	30	10	10	10^{-6}	30	10	20	10^{-5}
30	20	1	10^{-9}	30	20	10	10^{-5}	30	20	20	10^{-4}
30	50	1	10^{-6}	30	50	10	10^{-4}	30	50	20	10^{-4}

(f) Let $a(x) = \dfrac{1}{1+x}$ and $\omega \geq 0$. Then the flow is

$$\sigma(t,\omega) = -1 + \sqrt{2t + (1+\omega)^2}$$

and the induced semigroup

$$T(t)g(\omega) = g\left(-1 + \sqrt{2t + (1+\omega)^2}\right)$$

is a strongly continuous semigroup on $C_0[0,\infty)$. This follows from Proposition 1.35 and the fact that $\omega \leq \sigma(t,\omega)$ for all $\omega \geq 0$ and $t \geq 0$ and

$$|\sigma(t,\omega) - \omega| = \frac{2t}{(\omega+1) + \sqrt{2t + (\omega+1)^2}} \leq \sqrt{2t}$$

for all $\omega \geq 0$ and $t \geq 0$. The set

$$D = \left\{ g \in C_0[0,\infty) \cap C^1(0,\infty) : \omega \to \frac{1}{1+\omega} g'(\omega) \in C_0[0,\infty) \right\}$$

is a core of the generator

$$(\mathscr{A} g)(\omega) = \frac{1}{1+\omega} g'(\omega)$$

and

$$R(\lambda, \mathscr{A}) f(s) = e^{\lambda \frac{(\omega+1)^2}{2}} \int_\omega^\infty e^{-\lambda \frac{(s+1)^2}{2}} (s+1) f(s)\, ds.$$

Using $g(s) = \dfrac{1}{s+1}$ with $g^{-1}(s) = \dfrac{1}{s} - 1$ yields the following approximation results:

Table 1.7: $a(\omega) = \dfrac{1}{1+\omega}$

q	t	ω	Error	q	t	ω	Error
10	1	1	10^{-15}	10	1	10	10^{-41}
10	10	1	10^{-7}	10	10	10	10^{-22}
10	30	1	10^{-5}	10	30	10	10^{-15}
10	50	1	10^{-4}	10	50	10	10^{-12}
20	1	1	10^{-31}	20	1	10	10^{-83}
20	10	1	10^{-11}	20	10	10	10^{-45}
20	30	1	10^{-6}	20	30	10	10^{-30}
20	50	1	10^{-7}	20	50	10	10^{-24}
30	1	1	10^{-46}	30	1	10	10^{-125}
30	10	1	10^{-16}	30	10	10	10^{-68}
30	30	1	10^{-9}	30	30	10	10^{-45}
30	50	1	10^{-7}	30	50	10	10^{-37}

1.4 Semigroups Induced By Non-Autonomous Flows

In the previous section we saw how the resolvent of the linear operator

$$(\mathscr{A}g)(\omega) = a'(\omega)g(\omega)$$

can be used, via Laplace transform methods, to approximate the solution $\sigma(t,\omega)$ of the nonlinear initial value problem

$$x'(t) = a(x(t)),\ x(0) = \omega.$$

In this section we will explore how the solutions $\gamma(t,s,\omega)$ of the non-autonomous Cauchy problem

$$u'(t) = a(t)u(t)\ u(s) = \omega \geq 0,\ t \geq s \geq 0$$

can be approximated with inverse Laplace transform methods by studying the resolvent of the generator

$$(\mathscr{A}f)(t) = f'(t) + a(t)f(t)$$

of the induced linear semigroup

$$T(t)f(t) = f(t+s)[\zeta(t+s,s)].$$

In Chapter 2 we will study the linear case

$$u'(t) = A(t)u(t),\ u(s) = \omega \in \mathbb{R}^2$$

where $A = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$. In order to prepare for the next chapter, let $a : [0,\infty) \to \mathbb{R}$ be integrable. Then for all $\omega \in \mathbb{R}$ and $t \geq s \geq 0$, the initial value problem

$$u'(t) = a(t)u(t),\ u(s) = \omega$$

has a unique solution (by separation of variables) given by

$$u(t) = U(t,s)(\omega) = \gamma(t,s,\omega) = \omega e^{\int_s^t a(r)\,dr}.$$

Let $\mathcal{F}_1$ be a vector space of functions from $[0,\infty)$ and let $\mathcal{F}_2$ be a vector space of functions from $[0,\infty)\times[0,\infty)$ into $\mathbb{R}$. Then the non-autonomous flow $\zeta(t,s,\omega)$ induces a linear semigroup $\{T(t)_1\}_{t\geq 0}$ on $\mathcal{F}_1([0,\infty),\mathbb{R})$ by

$$T_1(t)f(s) := e^{\int_s^{t+s} a(r)\,dr} f(t+s)$$

and a linear semigroup $\{T_2(t)\}_{t\geq 0}$ on $F_2([0,\infty)\times[0,\infty))$ by

$$T_2(t)f(s,\omega) := f\left(\omega + s, \omega e^{\int_s^{t+s} a(r)\,dr} f(t+s)\right).$$

The (formal) generator of $\{T_1(t)\}_{t\geq 0}$ on $\mathcal{F}_1$ is

$$(\mathscr{A}_1 f)(s) = f'(s) + a(s)f(s)$$

and the (formal) generator of $\{T_2(t)\}_{t\geq 0}$ on $\mathcal{F}_2$ is

$$(\mathscr{A}_2 f)(s,\omega) = f_s(s,\omega) + \omega a(s) f_\omega(s,\omega).$$

Since $\{T_1(t)\}_{t\geq 0}$ is structurally easier that $\{T_2(t)\}_{t\geq 0}$, we concentrate henceforth only on

$$T(t)f(s) = e^{\int_s^{t+s} a(r)\,dr} f(t+s)$$

with generator

$$(\mathscr{A} f)(s) = f'(s) + a(s)f(s).$$

Proposition 1.37. *Let $a : (0,N) \to \mathbb{R}$ be integrable such that the antiderivative A of a is continuous on $[0,N]$. Then*

$$T(t)f(s) = \begin{cases} e^{\int_s^{t+s} a(r)\,dr} f(t+s) & 0 \leq t+s \leq N \\ 0 & \textit{otherwise} \end{cases}$$

is a strongly continuous semigroup on $C_0[0,N] := \{f \in C[0,N] : f(N) = 0\}$ with generator

$$(\mathscr{A} f)(s) = f'(s) + a(s)f(s)$$

for $f \in D \subset D(\mathscr{A})$, where

$$D := \{f \in C_0[0,N] \cap C_1(0,N) : s \to f'(s) + a(s)f(s) \in C_0[0,N]\}.$$

Moreover, for all $\lambda \in \mathbb{C}$,

$$R(\lambda,\mathscr{A})f(s) = \int_0^\infty e^{-\lambda t} T(t)f(s)\,ds = \int_0^{N-s} e^{-\lambda t} f(t+s) e^{\int_s^{t+s} a(r)\,dr}\,dt.$$

Proof. Clearly, $T(t)$ maps $C_0[0,N]$ into itself and $||T(t)f|| \leq M||f||$, where $M = \sup_{0\leq t+s\leq N} e^{A(t+s)-A(s)}$. The strong continuity follows from the equi-continuity of A and f on $[0,N]$ and the estimate

$$||T(t)f - f|| \leq \sup_{s\in[0,N]} |e^{A(t+s)-A(s)} - 1||f(t+s)| + |f(t+s) - f(s)|.$$

□

Example 1.38. Consider $a(s) = 2s$. Then $A(s) = s^2$ and the induced semigroup on $C_0[0,N]$ is

$$T(t)f(s) = \begin{cases} e^{(t+s)^2-s^2} f(t+s) & \text{if } 0 \leq t+s \leq N \\ 0 & \text{otherwise} \end{cases}$$

Then

$$R(\lambda, \mathscr{A})f(s) = \int_0^{N-s} e^{-\lambda u} f(u+s) e^{(u+s)^2-s^2}\, du = r(\lambda, s).$$

By Theorem 1.15,

$$T(t)f(s) = f(t+s)e^{(t+s)^2-s^2} = \lim_{q\to\infty} \sum_{j=1}^{q} \frac{c_j}{t} r\left(\frac{b_j}{t}, s\right).$$

Taking $f(s) = N - s$, it follows that

$$e^{(t+s)^2-s^2} = \frac{1}{N-t+s} \lim_{q\to\infty} \sum_{j=1}^{q} \frac{c_j}{t} r\left(\frac{b_j}{t}, s\right),$$

where $r(\lambda, s) = \int_0^{N-s} e^{-\lambda u}(N-u-s)e^{(u+s)^2-s^2}\, du$.

Chapter 2
Approximating Solutions of Non-Autonomous Systems of Linear Ordinary Differential Equations With Semigroup Techniques

Consider the initial value problem

$$u'(t) = A(t)u(t), u(s) = \omega \in \mathbb{C}^n \tag{2.1}$$

for $t \geq s \geq 0$, where $A : [0, \infty) \to M_{n\times n}(\mathbb{C})$ is such that (2.1) has a unique solution

$$u(t) = \gamma(t, s, \omega) = X(t)X^{-1}(s)\omega \tag{2.2}$$

for $t \geq s \geq 0$ and $\omega \in \mathbb{C}^n$ and where $X(t)$ is the fundamental solution matrix of the system (2.1) (i.e., each column of the matrix $X(t)$ is an independent solution of (2.1)). Now choose, for example,

$$\mathcal{N} = \mathcal{L}(\mathbb{C}^n, \mathbb{C}^n) \cong M_{n\times n} \text{ and } \mathcal{M} = \mathcal{F}([0, \infty), M_{n\times n})$$

for some appropriate vector space $\mathcal{F}$ of functions from $[0, \infty)$ into $M_{n\times n}(\mathbb{C})$. Then by (1.11) and the comments thereafter, the semigroup $\{T(t)\}_{t\geq 0}$ associated to (2.2) is given by $T(t) : \mathcal{M} \to \mathcal{M}$

$$T(t)x(s) = x(t + s)U(t + s, s) \tag{2.3}$$

where the linear operators

$$U(t, s) := X(t)X^{-1}(s)$$

satisfy

(i) $U(s, s) = I \qquad (s \geq 0)$

(ii) $U(t,r)U(r,s) = U(t,s) \quad (t \geq r \geq s)$

and the (formal) generator of $\{T(t)\}_{t\geq 0}$ is given by

$$(\mathscr{A} f)(s) = f'(s) + f(s)A(s).$$

To see that $\{T(t)\}_{t\geq 0}$ is a linear semigroup, note that

$$T(0)f(s) = f(s)U(s,s) = f(s)$$

(i.e., $T(0) = I$) and that

$$T(t_1)T(t_2)f(s) = T(t_1)h(s),$$

where $h(s) = T(t_2)f(s) = f(t_2+s)U(t_2+s,s)$. Thus, for $t_1, t_2 > 0$ and all $s \geq 0$,

$$\begin{aligned} T(t_1)T(t_2)f(s) &= T(t_1)h(s) = h(t_1+s)U(t_1+s,s) \\ &= f(t_2+t_1+s)U(t_2+t_1+s,t_1+s)U(t_1+s,s) \\ &= f(t_1+t_2+s)U(t_1+t_2+s,s) \\ &= T(t_1+t_2)f(s). \end{aligned}$$

In order to find some examples of equations

$$u'(t) = A(t)u(t), A(t) \in M_{2\times 2}(\mathbb{C}) \tag{2.4}$$

for which the fundamental matrix and hence, $U(t,s)$ can be found we consider second order linear differential equations of the form

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0 \tag{2.5}$$

and their reduction matrices

$$A(t) = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix}$$

as well as the "reduction of order" procedure that allows us to find the fundamental matrix if one solution of (2.1) can be found. Let us first consider the case where we can find (by any method) one function $y(t) \neq 0$ that satisfies (2.5). Then the D'Alembert Reduction Method allows us to find a second linearly independent solution $\omega(t)$ of (2.5) of the form $\omega(t) = \phi(t)y(t)$. In order for $\omega(t)$ to be a solution we recall that one needs

$$\begin{aligned} \omega''(t) + p(t)\omega'(t) + q(t)\omega(t) &= \phi''(t)y(t) + 2\phi'(t)y'(t) + p(t)\phi'(t)y(t) \\ &\quad + \phi(t)[y''(t) + p(t)y'(t) + q(t)y(t)] \\ &= \phi''(t)y(t) + p(t)\phi'(t)y(t) + 2\phi'(t)y'(t) = 0 \end{aligned}$$

Dividing the last equality by $y(t)$ yields that the derivative ϕ' of the unknown function ϕ satisfies the first order differential equation

$$\phi''(t) + \left(p(t) + 2\frac{y'(t)}{y(t)}\right)\phi'(t) = 0 \tag{2.6}$$

that can be solved by separating variables. To demonstrate D'Alembert's Reduction Method for (2.5), we consider the following elementary example.

Example 2.1. Consider

$$y''(t) - \frac{1}{t+1}y'(t) + \frac{1}{(t+1)^2}y(t) = 0, \tag{2.7}$$

where

$$A(t) = \begin{pmatrix} 0 & 1 \\ -\frac{1}{(t+1)^2} & \frac{1}{t+1} \end{pmatrix}.$$

Looking for solutions of the form $y(t) = y_0 + y_1 t + y_2 t^2 + \dots$, we find that

$$y(t) = 1 + t$$

is a solution of (2.7). In order to find a second solution $\omega(t) = \phi(t)y(t)$ we know from (2.6) that $\phi(t)$ must satisfy

$$\phi''(t) + \left(-\frac{1}{t+1} + 2\frac{1}{t+1}\right)\phi'(t) = 0$$

or

$$\phi''(t) = -\frac{1}{t+1}\phi'(t).$$

Thus, using separation of variables, we obtain $\phi'(t) = \dfrac{1}{t+1}$ or $\phi(t) = \ln(t+1)$. By the above, we have that $\omega(t) = \phi(t)y(t) = (t+1)\ln(t+1)$ is a second solution of (2.7). Thus, the fundamental matrix is given by

$$X(t) = \begin{pmatrix} t+1 & (t+1)\ln(t+1) \\ 1 & \ln(t+1)+1 \end{pmatrix}.$$

Since

$$X^{-1}(t) = \frac{1}{t+1}\begin{pmatrix} \ln(t+1)+1 & -(t+1)\ln(t+1) \\ -1 & t+1 \end{pmatrix},$$

it follows that

$$U(t,s) = X(t)X^{-1}(s) = \begin{pmatrix} \frac{t+1}{s+1} & 0 \\ 0 & 1 \end{pmatrix} + \ln\left(\frac{t+1}{s+1}\right)\begin{pmatrix} -\frac{t+1}{s+1} & t+1 \\ -\frac{1}{s+1} & 1 \end{pmatrix}.$$

Therefore, the semigroup $\{T(t)\}_{t\geq 0}$ associated to (2.7) is given by

$$\begin{aligned} T(t)f(s) &= f(t+s)U(t+s,s) \\ &= f(t+s)\left[\begin{pmatrix} \frac{t+s+1}{s+1} & 0 \\ 0 & 1 \end{pmatrix} + \ln\left(\frac{t+s+1}{s+1}\right)\begin{pmatrix} -\frac{t+s+1}{s+1} & t+s+1 \\ -\frac{1}{s+1} & 1 \end{pmatrix}\right] \\ &= f(t+s)\left[\begin{pmatrix} \frac{t}{s+1}+1 & 0 \\ 0 & 1 \end{pmatrix} + \ln\left(\frac{t}{s+1}+1\right)\begin{pmatrix} -\frac{t}{s+1}-1 & t+s+1 \\ -\frac{1}{s+1} & 1 \end{pmatrix}\right] \end{aligned}$$

with (formal) generator

$$(\mathscr{A}f)(s) = T'(0)f(s) = f'(s) + f(s)\left[\begin{pmatrix} \frac{1}{s+1} & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{s+1}\begin{pmatrix} -1 & s+1 \\ -\frac{1}{s+1} & 1 \end{pmatrix}\right]$$
$$= f'(s) + f(s)\begin{pmatrix} 0 & 1 \\ -\frac{1}{(s+1)^2} & \frac{1}{s+1} \end{pmatrix} = f'(s) + f(s)A(s).$$

Before we consider another example, let us discuss solving a system of equations

$$u'(t) = A(t)u(t), A(t) \in M_{2\times 2}(\mathbb{C}),$$

by using D'Alembert's Method of Reduction of Order. Suppose we have a system $u'(t) = A(t)u(t)$ where

$$A(t) = \begin{pmatrix} a_1(t) & a_2(t) \\ a_3(t) & a_4(t) \end{pmatrix}$$

and

$$u(t) = (u_1(t), u_2(t))$$

is a known solution. Then

$$y'(t) = \alpha(t)u(t) + z(t)$$

will produce the second solution. Here we have that the unknown function $\alpha(t)$ is a real-valued function and that $z(t) = (0, z_2(t))$ with $z_2(t)$ to be determined. Now

$$y'(t) = \alpha'(t)u(t) + u'(t)\alpha(t) + z'(t)$$
$$= \alpha'(t)u(t) + A(t)u(t)\alpha(t) + z'(t).$$

We need $y'(t) = A(t)y(t)$ or

$$y'(t) = A(t)[\alpha(t)u(t) + z(t)]$$
$$= A(t)\alpha(t)u(t) + A(t)z(t).$$

It follows that

$$y'(t) = A(t)y(t)$$

if and only if

$$\alpha'(t)u(t) + \alpha(t)A(t)u(t) + z'(t) = \alpha(t)A(t)u(t) + A(t)z(t)$$

or if

$$\alpha'(t)u(t) + z'(t) = A(t)z(t).$$

Solving for $z'(t)$, we obtain

$$z'(t) = A(t)z(t) - \alpha'(t)u(t).$$

Thus,

$$(0, z_2'(t)) = \begin{pmatrix} a_1(t) & a_2(t) \\ a_3(t) & a_4(t) \end{pmatrix} \begin{pmatrix} 0 \\ z_2(t) \end{pmatrix} - \alpha'(t)\,[u_1(t), u_2(t)]$$

or

$$\begin{cases} 0 = a_2(t)z_2(t) - \alpha'(t)u_1(t) \\ z_2'(t) = a_4(t)z_2(t) - \alpha'(t)u_2(t). \end{cases}$$

From the first equation, we see that

$$\alpha'(t) = \frac{a_2(t)z_2(t)}{u_1(t)}.$$

Then

$$z_2'(t) = \left[a_4(t) - \frac{a_2(t)u_2(t)}{u_1(t)}\right] z_2(t)$$

which is solvable using seperation of variables.

Example 2.2. Consider the system

$$\begin{cases} y_1'(t) = \frac{1}{t+1}y_1(t) - y_2(t) \\ y_2'(t) = \frac{1}{(t+1)^2}y_1(t) + \frac{2}{t+1}y_2(t) \end{cases} \tag{2.8}$$

or $u'(t) = A(t)u(t)$ for where

$$A(t) = \begin{pmatrix} \frac{1}{t+1} & -1 \\ \frac{1}{(t+1)^2} & \frac{2}{t+1} \end{pmatrix}.$$

A power series approach yields the first solution

$$u_1(t) = ((t+1)^2, -(t+1)).$$

Using D'Alembert's reduction of order method described above yields a second solution

$$u_2(t) = \left(-(t+1)^2 \ln(t+1), (t+1)[1 + \ln(t+1)]\right).$$

Therefore,

$$X(t) = \begin{pmatrix} (t+1)^2 & -(t+1)^2 \ln(t+1) \\ -(t+1) & (t+1)[1+\ln(t+1)] \end{pmatrix}$$

is the fundamental solution matrix with

$$X^{-1}(t) = \frac{1}{(t+1)^2} \begin{pmatrix} 1+\ln(t+1) & (t+1)\ln(t+1) \\ 1 & t+1 \end{pmatrix}.$$

Thus,

$$\begin{aligned} X(t)X^{-1}(s) = U(t,s) = &\frac{1}{(s+1)^2}\left[\begin{pmatrix} (t+1)^2 & 0 \\ 0 & (t+1)(s+1) \end{pmatrix}\right] \\ &+ \frac{1}{(s+1)^2}\left[(t+1)\ln\left(\frac{t+1}{s+1}\right)\begin{pmatrix} -(t+1) & -(t+1)(s+1) \\ 1 & s+1 \end{pmatrix}\right]. \end{aligned}$$

Hence, the linear semigroup induced by (2.8) is $T(t)f(s) = f(t+s)U(t+s,s)$ with generator

$$(\mathscr{A}f)(s) = T'(0)f(s) = f'(s) + f(s)U'(t+s,s) = f'(s) + f(s)A(s).$$

We will return to this example in Section 2.2 where we will show how one can use the generator $\mathscr{A}$ to construct the evolution family $U(t,s)$.

2.1 The Resolvent Approach

In this section we will investigate if the semigroup $T(t)f(s) = f(t+s)U(t+s,s)$ generated by $(\mathscr{A}f)(s) = f'(s) + f(s)A(s)$ can be obtained by inverse Laplace transform methods (as in Sections 1.3 and 1.4). Clearly, this approach requires that $R(\lambda, \mathscr{A})$ can be computed explicitly. Unfortunately, as we will see in this section, there is no explicit formula for $R(\lambda, \mathscr{A})$ (except in special cases) and perturbation/approximation theorems must be employed to approximate $\{T(t)\}_{t\geq 0}$ and/or $R(\lambda, \mathscr{A})$. First, let us consider the semigroup

$$T(t)f(s) = f(t+s)U(t+s,s)$$

more closely, where $f : [0,\infty) \to M_{2\times 2}$ is given by $f(s) = \begin{pmatrix} f_1(s) & f_2(s) \\ f_3(s) & f_4(s) \end{pmatrix}$ and $U(t,s) = \begin{pmatrix} u_1(t,s) & u_2(t,s) \\ u_3(t,s) & u_4(t,s) \end{pmatrix}$. Then,

$$T(t)f(s) = f(t+s)U(t+s,s) = \begin{pmatrix} \mathrm{I} & \mathrm{II} \\ \mathrm{III} & \mathrm{IV} \end{pmatrix}$$

where

$$\begin{aligned} \mathrm{I} &= f_1(t+s)u_1(t+s,s) + f_2(t+s)u_3(t+s,s), \\ \mathrm{II} &= f_1(t+s)u_2(t+s,s) + f_2(t+s)u_4(t+s,s), \\ \mathrm{III} &= f_3(t+s)u_1(t+s,s) + f_4(t+s)u_3(t+s,s), \\ \mathrm{IV} &= f_3(t+s)u_2(t+s,s) + f_4(t+s)u_4(t+s,s). \end{aligned}$$

Observing that the first and second rows are identical when replacing f_3 by f_1 and f_4 by f_2, it is obvious that the study of the semigroup $\{T(t)\}_{t\geq 0}$ can be related to the study of the semigroup

$$\tilde{T}(t)f(s) := U^*(t+s,s)f(s)$$

where $f : [0,\infty) \to \mathbb{C}^2$ is given by

$$f(s) := (f_1(s), f_2(s))$$

and

$$U^*(t,s) := \begin{pmatrix} u_1(t,s) & u_3(t,s) \\ u_2(t,s) & u_4(t,s) \end{pmatrix}$$

is the transpose of $U(t,s)$; that is,

$$\tilde{T}(t)f(s) = \begin{pmatrix} u_1(t+s,s) & u_3(t+s,s) \\ u_2(t+s,s) & u_4(t+s,s) \end{pmatrix} \begin{pmatrix} f_1(t+s) \\ f_2(t+s) \end{pmatrix}.$$

To confirm that $\left\{\tilde{T}(t)\right\}_{t\geq 0}$ is a semigroup, observe that $\tilde{T}(0)f(s) = f(s)$. Let $h(s) = \tilde{T}(t_2)f(s) = U^*(t_2+s,s)f(t_2+s)$. Then

$$\begin{aligned}
\tilde{T}(t_1)\tilde{T}(t_2)f(s) = \tilde{T}(t_1)h(s) &= U^*(t_1+s)h(t_1+s) \\
&= U^*(t_1+s,s)U^*(t_1+t_2+s,t_1+s)f(t_1+t_2+s) \\
&= (U(t_1+t_2+s,t_1+s)U(t_1+s,s))^*f(t_1+t_2+s) \\
&= U^*(t_1+t_2+s,s)f(t_1+t_2+s) \\
&= \tilde{T}(t_1+t_2)f(s).
\end{aligned}$$

Hence, it is sufficient to investigate the semigroup

$$\tilde{T}(t)f(s) := U^*(t+s,s)f(s)$$

with (formal) generator

$$(\tilde{\mathscr{A}}f)(s) = f'(s) + A^*(s)f(s)$$

on a Banach space $\mathcal{F}([0,\infty),\mathbb{C}^2)$, where $A^*(s) = \begin{pmatrix} a(s) & c(s) \\ b(s) & d(s) \end{pmatrix}$. The formal resolvent of $\tilde{\mathscr{A}}$ is

$$R(\lambda,\tilde{\mathscr{A}})f(s) = \int_0^\infty e^{-\lambda t}\tilde{T}(t)g(s)\,dt.$$

The problem is now to find an expression $\Box$ such that

$$R(\lambda,\tilde{\mathscr{A}})f(s) = \int_0^\infty e^{-\lambda t}\Box\,dt.$$

Then, by the uniqueness of the Laplace transform,

$$\Box = \tilde{T}(t)f(s) = U^*(t+s,s)f(t+s).$$

Lemma 2.3. *Let D_s denote the first derivative operator . Then $\lambda \in \rho(\mathscr{A})$ if and only if the operator $\lambda - \tilde{\mathscr{A}} = \begin{pmatrix} \lambda - a - D_s & -c \\ -b & \lambda - d - D_s \end{pmatrix}$ is invertible. Moreover, if $\lambda \in \rho(\tilde{\mathscr{A}})$, then*

$$R(\lambda, \tilde{\mathscr{A}}) = \begin{pmatrix} \lambda - a - D_s & -c \\ -b & \lambda - d - D_s \end{pmatrix}^{-1}.$$

Proof. Let $f = R(\lambda, \tilde{\mathscr{A}})g$. Then

$$\begin{aligned} (\lambda I - \tilde{\mathscr{A}})f = g \iff & \lambda f_1(s) - f_1'(s) - a(s)f_1(s) - c(s)f_2(s) = g_1(s) \\ & \lambda f_2(s) - f_2'(s) - b(s)f_1(s) - d(s)f_2(s) = g_2(s) \\ \iff & (\lambda - a - D_s)f_1 - cf_2 = g_1 \\ & (\lambda - d - D_s)f_2 - bf_1 = g_2 \\ \iff & \begin{pmatrix} \lambda - a - D_s & -c \\ -b & \lambda - d - D_s \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} g_1 & g_2 \end{pmatrix} \\ \iff & \begin{pmatrix} f_1 & f_2 \end{pmatrix} = \begin{pmatrix} \lambda - a - D_s & -c \\ -b & \lambda - d - D_s \end{pmatrix}^{-1} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}. \end{aligned}$$

for all $s \geq 0$ □

Lemma 2.4. *Let X_1, X_2 be Banach spaces and $A \in \mathcal{L}(X_1, X_1)$, $B \in \mathcal{L}(X_2, X_1)$, $C \in \mathcal{L}(X_1, X_2)$, and $D \in \mathcal{L}(X_2, X_2)$. If the operators A, D, $A - BD^{-1}C$, and $D - CA^{-1}B$ are invertible, then*

$$\mathcal{K} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is invertible and

$$\mathcal{K}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}.$$

Moreover, if A, C, D, $Q_1 = AC^{-1}D - B$ and $Q_2 = DB^{-1}A - C$ are invertible then $(D - CA^{-1}B)^{-1} = (DB^{-1}A - C)^{-1}DB^{-1}$ on the domain of DB^{-1}, say $\tilde{D}$, and

$$\mathcal{K}^{-1} = \begin{pmatrix} C^{-1}D & -A^{-1}B \\ -I & I \end{pmatrix} \begin{pmatrix} Q_1^{-1} & 0 \\ 0 & Q_2^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & DB^{-1} \end{pmatrix}.$$

on $X_1 \times \tilde{D}$.

Proof. To see that the given expression is a right-inverse, observe that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix} = \begin{bmatrix} J & K \\ L & M \end{bmatrix},$$

where

$$\begin{aligned} J &= A(A - BD^{-1}C)^{-1} - BD^{-1}C(A - BD^{-1}C)^{-1} \\ &= (A - BD^{-1}C)[A - BD^{-1}C]^{-1} = I, \end{aligned}$$

$$K = -B(D - CA^{-1}B)^{-1} + B(D - CA^{-1}B)^{-1} = 0,$$

$$L = C(A - BD^{-1}C)^{-1} - C(A - BD^{-1}C)^{-1} = 0,$$

and

$$\begin{aligned} M &= -CA^{-1}B(D - CA^{-1}B)^{-1} + D(D - CA^{-1}B)^{-1} \\ &= (D - CA^{-1}B)(D - CA^{-1}B)^{-1} = I. \end{aligned}$$

To see that the expression is also a left-inverse, observe that

$$\begin{bmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix},$$

where

$$\begin{aligned} W &= (A - BD^{-1}C)^{-1}A - A^{-1}B(D - CA^{-1}B)^{-1}C \\ &= (A - BD^{-1}C)^{-1}[A - (A - BD^{-1}C)A^{-1}B(D - CA^{-1}B)^{-1}C] \\ &= (A - BD^{-1}C)^{-1}[A - (B - BD^{-1}CA^{-1}B)(D - CA^{-1}B)^{-1}C] \\ &= (A - BD^{-1}C)^{-1}[A - B(I - D^{-1}CA^{-1}B)(D - CA^{-1}B)^{-1}C] \\ &= (A - BD^{-1}C)^{-1}[A - BD^{-1}(D - CA^{-1}B)(D - CA^{-1}B)^{-1}C] \\ &= (A - BD^{-1}C)^{-1}[A - BD^{-1}C] = I, \end{aligned}$$

$$\begin{aligned} X &= (A - BD^{-1}C)^{-1}B - A^{-1}B(D - CA^{-1}B)^{-1}D \\ &= (A - BD^{-1}C)^{-1}[B - (A - BD^{-1}C)A^{-1}B(D - CA^{-1}B)^{-1}D] \\ &= (A - BD^{-1}C)^{-1}[B - (B - BD^{-1}CA^{-1}B)(D - CA^{-1}B)^{-1}D] \\ &= (A - BD^{-1}C)^{-1}[B - B(I - D^{-1}CA^{-1}B)(D - CA^{-1}B)^{-1}D] \\ &= (A - BD^{-1}C)^{-1}[B - BD^{-1}(D - CA^{-1}B)(D - CA^{-1}B)^{-1}D] \\ &= (A - BD^{-1}C)^{-1}[B - BD^{-1}D] \\ &= (A - BD^{-1}C)^{-1}[B - B] = 0, \end{aligned}$$

$$
\begin{aligned}
Y &= -D^{-1}C(A - BD^{-1}C)^{-1}A + (D - CA^{-1}B)^{-1}C \\
&= (D - CA^{-1}B)^{-1}[-(D - CA^{-1}B)D^{-1}C(A - BD^{-1}C)^{-1}A + C] \\
&= (D - CA^{-1}B)^{-1}[-(C - CA^{-1}BD^{-1}C)(A - BD^{-1}C)^{-1}A + C] \\
&= (D - CA^{-1}B)^{-1}[-CA^{-1}(A - BD^{-1}C)(A - BD^{-1}C)^{-1}A + C] \\
&= (D - CA^{-1}B)^{-1}[-C + C] = 0,
\end{aligned}
$$

$$
\begin{aligned}
Z &= -D^{-1}C(A - BD^{-1}C)^{-1}B + (D - CA^{-1}B)^{-1}D \\
&= (D - CA^{-1}B)^{-1}[-(D - CA^{-1}B)D^{-1}C(A - BD^{-1}C)^{-1}B + D] \\
&= (D - CA^{-1}B)^{-1}[-(C - CA^{-1}BD^{-1}C)(A - BD^{-1}C)^{-1}B + D] \\
&= (D - CA^{-1}B)^{-1}[-CA^{-1}(A - BD^{-1}C)(A - BD^{-1}C)^{-1}B + D] \\
&= (D - CA^{-1}B)^{-1}(D - CA^{-1}B) = I.
\end{aligned}
$$

To see that

$$
\begin{aligned}
\mathcal{A}^{-1} &= \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} C^{-1}DQ_1^{-1} & -A^{-1}BQ_2^{-1}DB^{-1} \\ -Q_1^{-1} & Q_2^{-1}DB^{-1} \end{pmatrix}
\end{aligned}
$$

note that

$$
\begin{aligned}
(A - BD^{-1}C)^{-1} &= \left[\left(AC^{-1}D - B\right) D^{-1}C\right]^{-1} \\
&= C^{-1}D(AC^{-1}D - B)^{-1} \\
&= C^{-1}DQ_1^{-1}
\end{aligned}
$$

$$
\begin{aligned}
-A^{-1}B(D - CA^{-1}B)^{-1} &= -A^{-1}B(DB^{-1}A - C)^{-1}DB^{-1} \\
&= -A^{-1}BQ_2^{-1}DB^{-1}
\end{aligned}
$$

$$
\begin{aligned}
-D^{-1}C(A - BD^{-1}C)^{-1} &= -D^{-1}CC^{-1}D(AC^{-1}D - B)^{-1} \\
&= -(AC^{-1}D - B)^{-1} \\
&= -Q_1^{-1}
\end{aligned}
$$

and

$$
\begin{aligned}
(D - CA^{-1}B)^{-1} &= (BD^{-1}(DB^{-1}A - C))^{-1} \\
&= (DB^{-1}A - C)^{-1}DB^{-1} \\
&= Q_2^{-1}DB^{-1}
\end{aligned}
$$

on the domain of DB^{-1}. □

Lemma 2.5. *Let* $A(s) = \begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix}$, $\tilde{\mathscr{A}} f(s) = f'(s) + A(s)^* f(s)$ *and assume that* $\lambda - a - D_s$, $\lambda - d - D_s$, *and* $B : f \to bf$ *are invertible operators. Then* $R\left(\lambda, \tilde{\mathscr{A}}\right)$ *exists if and only if the Riccati equations*

$$q' = ad - \frac{b'd}{b} + d' - cb - \left(a + d - \frac{b'}{b}\right) q + q^2 \tag{2.9}$$

$$m' = da - \frac{c'a}{c} + a' - bc - \left(d + a - \frac{c'}{c}\right) m + m^2 \tag{2.10}$$

have a solution. Moreover, if q *solves (2.9) and* m *solves (2.10), then*

$$R\left(\lambda, \tilde{\mathscr{A}}\right) = \begin{pmatrix} -\frac{1}{b}\left(\lambda - d - D_s\right) & \left(\lambda - a - D_s\right)^{-1} c \\ -I & I \end{pmatrix} \begin{pmatrix} Q_1^{-1} & 0 \\ 0 & Q_2^{-1} \end{pmatrix}$$
$$\begin{pmatrix} I & 0 \\ 0 & -\left(\lambda - d - D_s\right)\frac{1}{c} \end{pmatrix}$$

where $p = a + d - \frac{b'}{b} - q$, $n = d + a - \frac{c'}{c} - m$,

$$Q_1 = \left(-\frac{1}{b}\right)\left(\lambda - p - D_s\right)\left(\lambda - q - D_s\right),$$

and

$$Q_2 = \left(-\frac{1}{c}\right)\left(\lambda - n - D_s\right)\left(\lambda - m - D_s\right)$$

Proof. Note that $A - BD^{-1}C = (AC^{-1}D - B)D^{-1}C$ is invertible if and only if $AC^{-1}D - B$ is invertible. Let $H : f \to hf$ and $D_s : f \to f'$. Then $(D_s \circ H)f = D_s(hf) = h'f + hf'$. Then we write $D_s h = h' + hD_s$ (which is a slight abuse of notation). Now,

$$\begin{aligned}
AC^{-1}D - B &= \left(\lambda - a - D_s\right)\left(-\frac{1}{b}\right)\left(\lambda - d - D_s\right) + c \\
&= \left[(\lambda - a)\left(-\frac{1}{b}\right) + \left(\frac{-b'}{b^2} + \frac{1}{b}D_s\right)\right]\left(\lambda - d - D_s\right) + c \\
&= \left[(\lambda - a)\left(-\frac{1}{b}\right) - \frac{b'}{b^2} + \frac{1}{b}D_s\right]\left(\lambda - d - D_s\right) + c \\
&= \left[(\lambda - a)\left(-\frac{1}{b}\right) - \frac{b'}{b^2}\right](\lambda - d) + \frac{\lambda}{b}D_s - \frac{1}{b}\left[d' + dD_s\right] - \frac{1}{b}D_s^2 + c \\
&\quad - \left[(\lambda - a)\left(-\frac{1}{b}\right) - \frac{b'}{b^2}\right] D_s
\end{aligned}$$

$$
\begin{aligned}
&= \left[(\lambda - a)\left(-\frac{1}{b}\right) - \frac{b'}{b^2}\right](\lambda - d) - \frac{1}{b}d' + c \\
&+ \left[\frac{\lambda}{b} - \frac{d}{b} - \left[(\lambda - a)\left(-\frac{1}{b}\right) - \frac{b'}{b^2}\right]\right] D_s - \frac{1}{b}D_s^2 \\
&= \left(-\frac{1}{b}\right)\left[\left((\lambda - a) + \frac{b'}{b}\right)(\lambda - d) + d' - cb\right] \\
&\left(-\frac{1}{b}\right)\left[+\left(-\lambda + d - \lambda + a - \frac{b'}{b}\right) D_s + D_s^2\right]
\end{aligned}
$$

Next we show that $AC^{-1}D - B$ can be factored in the form

$$
\begin{aligned}
AC^{-1}D - B &= \left(-\frac{1}{b}\right)(\lambda - p - D_s)(\lambda - q - D_s) \\
&= \left(-\frac{1}{b}\right)\left[\lambda^2 - \lambda(p+q) + pq + q' - (2\lambda - p - q)D_s + D_s^2\right]
\end{aligned}
$$

if the Riccati equation (2.9) is valid. The previous statement is valid if and only if

$$
\left((\lambda - a) + \frac{b'}{b}\right)(\lambda - d) + d' - cb = \lambda^2 - \lambda(p+q) + pq + q'
$$

and

$$
-2\lambda + a + d - \frac{b'}{b} = -2\lambda + p + q.
$$

This is true if and only if

$$
p + q = a + d - \frac{b'}{b}
$$

and

$$
\lambda^2 - \lambda(a+d) + ad + \frac{b'}{b}\lambda - \frac{b'd}{b} + d' - cb = \lambda^2 - \lambda(p+q) + pq + q'. \tag{2.11}
$$

Now (2.11) is valid

$$
\begin{aligned}
&\Leftrightarrow \lambda^2 - \lambda(a+d) + ad + \frac{b'}{b}\lambda - \frac{b'd}{b}\lambda + d' - cb = \lambda^2 - \lambda(a+d) + \frac{b'}{b}\lambda + pq + q' \\
&\Leftrightarrow ad - \frac{b'd}{b} + d' - cb = pq + q' \\
&\Leftrightarrow ad - \frac{b'd}{b} + d' - cb = \left(a + d - \frac{b'}{b}\right) q - q^2 + q' \\
&\Leftrightarrow q' = ad - \frac{b'd}{b} + d' - cb - \left(a + d - \frac{b'}{b}\right) q + q^2
\end{aligned}
$$

and

$$
p = a + d - \frac{b'}{b} - q.
$$

Note that $D-CA^{-1}B = (DB^{-1}A-C)A^{-1}B$ is invertible if and only if $DB^{-1}A-C$ is invertible. Now,

$$\begin{aligned}
DB^{-1}A - C &= (\lambda - d - D_s)\left(-\frac{1}{c}\right)(\lambda - a - D_s) + b \\
&= \left[(\lambda - d)\left(-\frac{1}{c}\right) + \left(\frac{-c'}{c^2} + \frac{1}{c}D_s\right)\right](\lambda - a - D_s) + b \\
&= \left[(\lambda - d)\left(-\frac{1}{c}\right) - \frac{c'}{c^2} + \frac{1}{c}D_s\right](\lambda - a - D_s) + b \\
&= \left[(\lambda - d)\left(-\frac{1}{c}\right) - \frac{c'}{c^2}\right](\lambda - a) + \frac{\lambda}{c}D_s - \frac{1}{c}\left[a' + aD_s\right] - \frac{1}{c}D_s^2 + b \\
&- \left[(\lambda - d)\left(-\frac{1}{c}\right) - \frac{c'}{c^2}\right]D_s \\
&= \left[(\lambda - d)\left(-\frac{1}{c}\right) - \frac{c'}{c^2}\right](\lambda - a) - \frac{1}{c}a' + b \\
&+ \left[\frac{\lambda}{c} - \frac{a}{c} - \left[(\lambda - d)\left(-\frac{1}{c}\right) - \frac{c'}{c^2}\right]\right]D_s - \frac{1}{c}D_s^2 \\
&= \left(-\frac{1}{c}\right)\left[\left((\lambda - d) + \frac{c'}{c}\right)(\lambda - a) + a' - bc\right] \\
&+ \left(-\frac{1}{c}\right)\left[\left(-\lambda + a - \lambda + d - \frac{c'}{c}\right)D_s + D_s^2\right]
\end{aligned}$$

Next we show that $DB^{-1}A - C$ can be factored in the form

$$\begin{aligned}
DB^{-1}A - C &= \left(-\frac{1}{c}\right)(\lambda - n - D_s)(\lambda - m - D_s) \\
&= \left(-\frac{1}{c}\right)\left[\lambda^2 - \lambda(n+m) + nm + m' - (2\lambda - n - m)D_s + D_s^2\right]
\end{aligned}$$

if the Riccati equation (2.10) is valid. The previous statement is valid if and only if

$$\left((\lambda - d) + \frac{c'}{c}\right)(\lambda - a) + a' - bc = \lambda^2 - \lambda(n+m) + nm + m'$$

and

$$-2\lambda + a + d - \frac{b'}{b} = -2\lambda + n + m.$$

This is true if and only if

$$n + m = a + d - \frac{c'}{c}$$

and

$$\lambda^2 - \lambda(d+a) + da + \frac{c'}{c}\lambda - \frac{c'a}{c} + a' - bc = \lambda^2 - \lambda(n+m) + nm + m'. \quad (2.12)$$

Now (2.12) is valid

$$\Leftrightarrow \lambda^2 - \lambda(d+a) + da + \frac{c'}{c}\lambda - \frac{c'a}{c}\lambda + a' - bc = \lambda^2 - \lambda(d+a) + \frac{c'}{c}\lambda + nm + m'$$
$$\Leftrightarrow da - \frac{c'a}{c} + a' - bc = pq + q'$$
$$\Leftrightarrow da - \frac{c'a}{c} + a' - bc = \left(d + a - \frac{c'}{c}\right) m - m^2 + m'$$
$$\Leftrightarrow m' = da - \frac{c'a}{c} + a' - bc - \left(d + a - \frac{c'}{c}\right) m + m^2$$

and $n = d + a - \frac{c'}{c} - m$.

□

In general, it is difficult or even impossible to obtain explicit solutions to Riccati equations. Therefore, Lemma 2.5 shows that in general it will be impossible to comput $R(\lambda, \tilde{\mathscr{A}})$ explicitly, where

$$(\tilde{\mathscr{A}} f)(s) := f'(s) + A^*(s) f(s).$$

This is not too surprising. After all, there is no general formula for the fundamental solution matrix $X(t)$ for (2.1). In order to approximate the semigroup

$$\tilde{T}(t) f(s) = U^*(t+s, s) f(t+s)$$

generated by $\tilde{\mathscr{A}}$, we will shift our focus to using perturbation and approximation methods.

2.2 Lie-Trotter Type Approximation Methods

In this section we will develop formulas to approximate evolution families $U(t, s)$ for (2.1) using the Lie-Trotter type approximation formulas presented in Section 1.2. The following elementary result from the theory of systems of linear ordinary differential equations will be used throughout this section.

Proposition 2.6. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *Define* $\delta := ad - bc$, $\tau := a + d$, *and take* $\gamma \in \mathbb{C}$ *such that:* $\gamma^2 = \frac{1}{4}(\tau^2 - 4\delta)$. *Then*

$$e^{tA} = m(t) A + n(t) I,$$

where

$$m(t) = \begin{cases} e^{t\frac{\tau}{2}} \frac{1}{\gamma} \sinh(t\gamma) & \text{for } \gamma \neq 0, \\ t e^{t\frac{\tau}{2}} & \text{for } \gamma = 0, \end{cases}$$

and

$$n(t) = \begin{cases} e^{t\frac{\tau}{2}}\left[\cosh(t\gamma) - \frac{\tau}{2\gamma}\sinh(t\gamma)\right] & \textit{for } \gamma \neq 0, \\ n(t) = e^{t\frac{\tau}{2}}\left(1 - t\frac{\tau}{2}\right) & \textit{for } \gamma = 0. \end{cases}$$

Proof. Note first that

$$(\lambda I - A)^{-1} = \frac{1}{(\lambda - \lambda_1)(\lambda - \lambda_2)} \begin{pmatrix} \lambda - d & b \\ c & \lambda - a \end{pmatrix},$$

where λ_1, λ_2 are the eigenvalues of A; that is, λ_1, λ_2 solve

$$0 = \det(\lambda I - A) = \lambda^2 - (a+d)\lambda + ad - bc = \lambda^2 - \tau\lambda + \delta$$

or

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2} = \frac{\tau}{2} \pm \gamma.$$

Using partial fractions we obtain (for $\gamma \neq 0$),

$$(\lambda I - A)^{-1} = \begin{pmatrix} \dfrac{A_1}{\lambda - \lambda_1} + \dfrac{B_1}{\lambda - \lambda_2} & \dfrac{A_2}{\lambda - \lambda_1} + \dfrac{B_2}{\lambda - \lambda_2} \\ \dfrac{A_3}{\lambda - \lambda_1} + \dfrac{B_3}{\lambda - \lambda_2} & \dfrac{A_4}{\lambda - \lambda_1} + \dfrac{B_4}{\lambda - \lambda_2} \end{pmatrix},$$

where $A_1 = \dfrac{\lambda_1 - d}{2\gamma}$, $B_1 = \dfrac{d - \lambda_2}{2\gamma}$, $A_2 = \dfrac{b}{2\gamma} = -B_2$, $A_3 = \dfrac{c}{2\gamma} = -B_3$, $A_4 = \dfrac{\lambda_1 - a}{2\gamma}$, $B_4 = \dfrac{a - \lambda_2}{2\gamma}$. Since e^{tA} is the inverse Laplace transform of $R(\lambda, A) = (\lambda I - A)^{-1}$, it follows that

$$\begin{aligned} e^{tA} &= \begin{pmatrix} A_1 e^{t\lambda_1} + B_1 e^{t\lambda_2} & A_2 e^{t\lambda_1} + B_2 e^{t\lambda_2} \\ A_3 e^{t\lambda_1} + B_3 e^{t\lambda_2} & A_4 e^{t\lambda_1} + B_4 e^{t\lambda_2} \end{pmatrix} \\ &= e^{t\frac{\tau}{2}} \begin{pmatrix} A_1 e^{t\gamma} + B_1 e^{-t\gamma} & A_2(e^{t\gamma} - e^{-t\gamma}) \\ A_3(e^{t\gamma} - e^{-t\gamma}) & A_4 e^{t\gamma} + B_4 e^{-t\gamma} \end{pmatrix} \\ &= e^{t\frac{\tau}{2}} \begin{pmatrix} A_1 e^{t\gamma} + B_1 e^{-t\gamma} & \frac{b}{\gamma}\sinh(t\gamma) \\ \frac{c}{\gamma}\sinh(t\gamma) & A_4 e^{t\gamma} + B_4 e^{-t\gamma} \end{pmatrix}. \end{aligned}$$

Now, since

$$\begin{aligned} A_1 &= \frac{\lambda_1 - d}{2\gamma} = \frac{\frac{\tau}{2} + \gamma - d}{2\gamma} = \frac{\tau + 2\gamma - 2d}{4\gamma} \\ &= \frac{a - d + 2\gamma}{4\gamma} = \frac{2a + 2\gamma - \tau}{4\gamma} \end{aligned}$$

and

$$B_1 = \frac{d-\lambda_2}{2\gamma} = \frac{d-\frac{\tau}{2}+\gamma}{2\gamma} = \frac{2d-\tau+2\gamma}{4\gamma}$$
$$= \frac{-a+d+2\gamma}{4\gamma} = \frac{-2a+\tau+2\gamma}{4\gamma}$$

$\left(\text{and, simliarly, } A_4 = \frac{\lambda_1 - a}{2\gamma} = \frac{2d+2\gamma-\tau}{4\gamma},\ \ B_4 = \frac{-2d+\tau+2\gamma}{4\gamma}\right)$, it follows that

$$A_1e^{t\gamma} + B_1e^{-t\gamma} = \frac{a}{\gamma}(e^{t\gamma}-e^{-t\gamma}) + (e^{t\gamma}+e^{-t\gamma}) - \frac{\tau}{2\gamma}(e^{t\gamma}-e^{-t\gamma})$$

and

$$A_4e^{t\gamma} + B_4e^{-t\gamma} = \frac{d}{\gamma}(e^{t\gamma}-e^{-t\gamma}) + (e^{t\gamma}+e^{-t\gamma}) - \frac{\tau}{2\gamma}(e^{t\gamma}-e^{-t\gamma}).$$

This shows that

$$e^{tA} = m(t)A + n(t)I,$$

where $m(t) = e^{t\frac{\tau}{2}}\frac{1}{\gamma}\sinh(t\gamma)$ and $n(t) = e^{t\frac{\tau}{2}}\left[\cosh(t\gamma) - \frac{\tau}{2\gamma}\sinh(t\gamma)\right]$. For $\gamma = 0$, we have

$$(\lambda - A)^{-1} = \frac{1}{(\lambda-\lambda_1)^2}\begin{pmatrix}\lambda - d & b\\ c & \lambda - a\end{pmatrix}$$

where $\lambda_1 = \frac{\tau}{2}$. Using partial fractions we obtain

$$(\lambda - A)^{-1} = \begin{pmatrix}\frac{1}{\lambda-\lambda_1} + \frac{\lambda_1 - d}{(\lambda-\lambda_1)^2} & \frac{b}{(\lambda-\lambda_1)^2}\\ \frac{c}{(\lambda-\lambda_1)^2} & \frac{1}{\lambda-\lambda_1} + \frac{\lambda_1 - a}{(\lambda-\lambda_1)^2}\end{pmatrix}.$$

Therefore, the inverse Laplace transform of $(\lambda - A)^{-1}$ is

$$e^{tA} = \begin{pmatrix}e^{t\lambda_1} + (\lambda_1 - d)te^{t\lambda_1} & -tbe^{t\lambda_1}\\ tce^{t\lambda_1} & e^{t\lambda_1} + (\lambda_1 - a)te^{t\lambda_1}\end{pmatrix}$$
$$= \begin{pmatrix}e^{t\frac{\tau}{2}} + (a-\frac{\tau}{2})te^{t\frac{\tau}{2}} & tbe^{t\frac{\tau}{2}}\\ tce^{t\frac{\tau}{2}} & e^{t\frac{\tau}{2}} + (d-\frac{\tau}{2})te^{t\frac{\tau}{2}}\end{pmatrix}$$
$$= te^{t\frac{\tau}{2}}A + \left(1 - \frac{t\tau}{2}e^{t\frac{\tau}{2}}\right)I$$

since $\lambda_1 - d = a - \frac{\tau}{2}$ and $\lambda_1 - a = d - \frac{\tau}{2}$. □

Next, we would like to approximate the semigroup $\{T(t)\}_{t\geq 0}$ generated by

$$\mathscr{A}f(s) = f'(s) + A^*(s)f(s)$$
$$= \mathscr{A}_0 f(s) + (Bf)(s).$$

We assume that $(\mathscr{A}_0 f)(s) = f'(s)$ generates the shift semigroup

$$T_0(t)f(s) = \begin{pmatrix} T_1(t) & 0 \\ 0 & T_2(t) \end{pmatrix} \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = \big(f_1(t+s), f_2(t+s)\big)\,. \tag{2.13}$$

We also assume that the map $A^* : s \to A^*(s)$ is continuous and bounded on $[0,\infty)$ so that the bounded operator $(Bf)(s) = A^*(s)f(s)$ generates the semigroup

$$S(t)f(s) = m(t,s)A^*(s) + n(t,s)I, \tag{2.14}$$

where

$$m(t,s) = \begin{cases} e^{t\frac{\tau(s)}{2}} \frac{1}{\gamma(s)} \sinh(t\gamma(s)) & \text{for } \gamma(s) \neq 0 \\ te^{t\frac{\tau(s)}{2}} & \text{for } \gamma(s) = 0 \end{cases}$$

and

$$n(t,s) = \begin{cases} e^{t\frac{\tau(s)}{2}} \left[\cosh(t\gamma(s)) - \frac{\tau(s)}{2\gamma(s)} \sinh(t\gamma(s))\right] & \text{for } \gamma(s) \neq 0 \\ n(t) = e^{t\frac{\tau(s)}{2}} \left(1 - t\frac{\tau(s)}{2}\right) & \text{for } \gamma(s) = 0 \end{cases}\,.$$

and where $\gamma = \gamma(s)$ and $\tau = \tau(s)$ are as in Proposition 2.6.

The following proposition is an adaptation/extension of a result of G. Nickel (see [22]), where we use Proposition 2.6 to compute $e^{tA^*(s)}$.

Proposition 2.7. *Let $X = C_0([0,\infty), \mathbb{C}^2)$ and $A^* : s \to A^*(s) \in C_b([0,\infty), \mathbb{C}^2)$. Then*

$$\tilde{\mathscr{A}} f(s) = f'(s) + A^*(s)f(s)$$

generates a strongly continuous semigroup

$$\tilde{T}(t)f(s) = U^*(t+s,s)f(t+s).$$

Moreover,
$\tilde{T}(t)f(s) = U^*(t+s,s)f(t+s) = \lim_{n\to\infty} V_{1,2}(n,t,s)f(t+s)$, *where*

$$V_1(n,t,s) = \prod_{j=1}^{n} \left[m\left(\frac{t}{n}, \frac{jt}{n} + s\right) A^*\left(\frac{jt}{n} + s\right) + n\left(\frac{t}{n}, \frac{jt}{n} + s\right) I \right] \tag{2.15}$$

$$V_2(n,t,s) = \prod_{j=1}^{n} e^{\frac{t}{n} A^*(\frac{(2j-1)t}{2n} + s)} \tag{2.16}$$

where the limits are uniform for t in compact intervals and all $s \geq 0$ and where $m(t,s)$ and $n(t,s)$ are as above. In particular, for t, s in compact intervals

$$\begin{aligned} U(t+s,s) &= \lim_{n\to\infty} V_1(n,t,s) \\ &= \lim_{n\to\infty} V_2(n,t,s), \end{aligned}$$

where $U(t,s)$ is the evolution family solving

$$u'(t) = A(t)u(t)\, u(s) = x \;\; (t \geq s \geq 0).$$

Proof. The fact that $\tilde{\mathscr{A}}$ generates a strongly continuous semigroup $\left\{\tilde{T}(t)\right\}_{t\geq 0}$ follows immediately from the Lie-Trotter Product Formula (Corollary 1.26). Moreover, using the Lie-Trotter approximation

$$V\left(\frac{t}{n}\right)^n f = \left(T_0\left(\frac{t}{n}\right)S\left(\frac{t}{n}\right)\right)^n f$$

we can obtain (2.15) as follows. First, we observe that

$$T_0(t)S(t)f(s) = T_0(t)g(s) = g(t+s),$$

where $g(s) = S(t)f(s)$. Thus,

$$T_0(t)S(t)f(s) = S(t)f(t+s) = [m(t,t+s)A^*(t+s) + n(t,t+s)I]f(t+s).$$

Moreover,

$$\left[T_0\left(\frac{t}{2}\right)S\left(\frac{t}{2}\right)\right]^2 f(s) = T_0\left(\frac{t}{2}\right)S\left(\frac{t}{2}\right)h(s)$$

where

$$\begin{aligned} h(s) &= T_0\left(\frac{t}{2}\right)S\left(\frac{t}{2}\right)f(s) \\ &= \left[m\left(\frac{t}{2},\frac{t}{2}+s\right)A^*\left(\frac{t}{2}+s\right) + n\left(\frac{t}{2},\frac{t}{2}+s\right)I\right]f\left(\frac{t}{2}+s\right). \end{aligned}$$

Thus,

$$\begin{aligned} T_0\left(\frac{t}{2}\right)S\left(\frac{t}{2}\right)h(s) &= \left[m\left(\frac{t}{2},\frac{t}{2}+s\right)A^*\left(\frac{t}{2}+s\right) + n\left(\frac{t}{2},\frac{t}{2}+s\right)I\right]h\left(\frac{t}{2}+s\right) \\ &= \left[m\left(\frac{t}{2},\frac{t}{2}+s\right)A^*\left(\frac{t}{2}+s\right) + n\left(\frac{t}{2},\frac{t}{2}+s\right)I\right] \\ &\quad\cdot\left[m\left(\frac{t}{2},t+s\right)A^*(t+s) + n\left(\frac{t}{2},t+s\right)I\right]f(t+s). \end{aligned}$$

Similarly,

$$\begin{aligned} \left[T_0\left(\frac{t}{3}\right)S\left(\frac{t}{3}\right)\right]^3 f(s) &= T_0\left(\frac{t}{3}\right)S\left(\frac{t}{3}\right)T_0\left(\frac{t}{3}\right)S\left(\frac{t}{3}\right)T_0\left(\frac{t}{3}\right)S\left(\frac{t}{3}\right)f(s) \\ &= T_0\left(\frac{t}{3}\right)S\left(\frac{t}{3}\right)T_0\left(\frac{t}{3}\right)S\left(\frac{t}{3}\right)h_1(s) \\ &= T_0\left(\frac{t}{3}\right)S\left(\frac{t}{3}\right)h_2(s) \end{aligned}$$

where $h_1(s) = T_0\left(\frac{t}{3}\right) S\left(\frac{t}{3}\right) f(s)$ and $h_2(s) = T_0\left(\frac{t}{3}\right) S\left(\frac{t}{3}\right) h_1(s)$. Then,

$$\begin{aligned}
T_0\left(\frac{t}{3}\right) S\left(\frac{t}{3}\right) h_2(s) &= \left[m\left(\frac{t}{3}, \frac{t}{3}+s\right) A^*\left(\frac{t}{3}+s\right) + n\left(\frac{t}{3}, \frac{t}{3}+s\right) I\right] h_2\left(\frac{t}{3}+s\right) \\
&= \left[m\left(\frac{t}{3}, \frac{t}{3}+s\right) A^*\left(\frac{t}{3}+s\right) + n\left(\frac{t}{3}, \frac{t}{3}+s\right) I\right] \\
&\cdot \left[m\left(\frac{t}{3}, \frac{2t}{3}+s\right) A^*\left(\frac{t}{3}+s\right) + n\left(\frac{t}{3}, \frac{2t}{3}+s\right) I\right] \\
&= \left[m\left(\frac{t}{3}, t+s\right) A^*(t+s) + n\left(\frac{t}{3}, t+s\right) I\right] f(t+s).
\end{aligned}$$

So, by the Lie-Trotter Product Formula (Corollary 1.26),

$$\begin{aligned}
&\tilde{T}(t)f(s) = U^*(t+s,s)f(t+s) \\
&= \lim_{n\to\infty} V\left(\frac{t}{n}\right)^n f(s) = \lim_{n\to\infty}\left[m\left(\frac{t}{n}, \frac{t}{n}+s\right) A^*\left(\frac{t}{n}+s\right) + n\left(\frac{t}{n}, \frac{t}{n}+s\right) I\right] \\
&\cdot \left[m\left(\frac{t}{n}, \frac{2t}{n}+s\right) A^*\left(\frac{2t}{n}+s\right) + n\left(\frac{t}{n}, \frac{2t}{n}+s\right) I\right] \\
&\vdots \\
&\cdot \left[m\left(\frac{t}{n}, t+s\right) A^*(t+s) + n\left(\frac{t}{n}, t+s\right) I\right] f(t+s) \\
&= \lim_{n\to\infty}\prod_{j=1}^{n}\left[m\left(\frac{t}{n}, \frac{jt}{n}+s\right) A^*\left(\frac{jt}{n}+s\right) + n\left(\frac{t}{n}, \frac{jt}{n}+s\right) I\right] f(t+s)
\end{aligned}$$

where the limit is uniform for $s \geq 0$ and for t in compact intervals. If $t, s \in [0, N]$ and if we choose $f_0 \in C_0([0,\infty), \mathbb{C}^2)$ such that $f_0(r) = 1$ for all $r \in [0, 2N]$, then

$$\begin{aligned}
U^*(t+s,s)f_0(t+s) &= U^*(t+s,s) \\
&= \lim_{n\to\infty}\prod_{j=1}^{n}\left[m\left(\frac{t}{n}, \frac{jt}{n}+s\right) A^*\left(\frac{jt}{n}+s\right) + n\left(\frac{t}{n}, \frac{jt}{n}+s\right) I\right].
\end{aligned}$$

The proof of (2.16) proceeds similarly using the Lie-Trotter-Kato Formula (Corollary 1.30). In this case, we have

$$\tilde{T}(t)f(s) = \lim_{n\to\infty}\left[T_0\left(\frac{t}{2n}\right) S\left(\frac{t}{n}\right) T_0\left(\frac{t}{2n}\right)\right]^n f(s).$$

For $n = 1$,

$$T_0\left(\frac{t}{2}\right) S(t) T_0\left(\frac{t}{2}\right) f(s) = T_0(t)S(t)g(s)$$

where

$$g(s) = T_0\left(\frac{t}{2}\right) f(s) = f\left(\frac{t}{2}+s\right).$$

Thus,

$$T_0\left(\frac{t}{2}\right)S(t)T_0\left(\frac{t}{2}\right)f(s)=T_0\left(\frac{t}{2}\right)h(s)$$

where

$$h(s)=S(t)g(s)=e^{tA^*(s)}g(s)=e^{tA^*(s)}f\left(\frac{t}{2}+s\right).$$

Therefore,

$$T_0\left(\frac{t}{2}\right)S(t)T_0\left(\frac{t}{2}\right)f(s)=T_0\left(\frac{t}{2}\right)h(s)=h\left(\frac{t}{2},s\right)=e^{tA^*(\frac{t}{2}+s)}f(t+s).$$

For $n=2$,

$$\left[T_0\left(\frac{t}{4}\right)S\left(\frac{t}{2}\right)T_0\left(\frac{t}{4}\right)\right]^2 f(s)=T_0\left(\frac{t}{4}\right)S\left(\frac{t}{2}\right)T_0\left(\frac{t}{4}\right)h(s),$$

where

$$h(s)=T_0\left(\frac{t}{4}\right)S\left(\frac{t}{2}\right)T_0\left(\frac{t}{4}\right)f(s)=e^{\frac{t}{2}A^*(\frac{t}{4}+s)}f\left(\frac{t}{2}+s\right).$$

Thus,

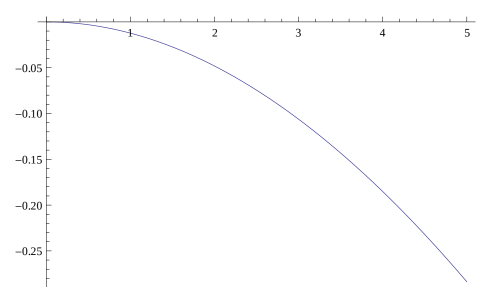
$$T_0\left(\frac{t}{4}\right)S\left(\frac{t}{2}\right)T_0\left(\frac{t}{4}\right)h(s)=e^{\frac{t}{2}A^*(\frac{t}{4}+s)}h\left(\frac{t}{2}+s\right)=e^{\frac{t}{2}A^*(\frac{t}{4}+s)}e^{\frac{t}{2}A^*(\frac{3t}{4}+s)}f(t+s).$$

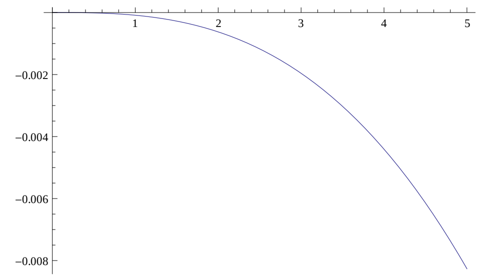
Continuing in this fashion, we obtain

$$\tilde{T}(t)f(s)=\lim_{n\to\infty}\left[\prod_{j=1}^{n}e^{\frac{t}{n}A^*(\frac{(2j-1)t}{2n}+s)}\right]f(t+s)$$

by Proposition 2.6. □

Example 2.8. Let $A(t)=\begin{pmatrix}\frac{1}{t+1} & 0\\ 0 & 1\end{pmatrix}$. We use Proposition 2.7 to approximate the evolution family $U(t+s,s)=\begin{pmatrix}\frac{t+s}{s} & 0\\ 0 & e^t\end{pmatrix}$. Below are the approximation errors using the Lie-Trotter Product Formula and the Lie-Trotter-Kato Product Formula, respectively, for $n=10, s=1,$ and $0\le t\le 5$.





If $A(t)$ and $U(t+s,s)$ are as in Example 2.1 and if we implore the Lie-Trotter Product Formula and the Lie-Trotter-Kato Product Formula as described in Proposition 2.7, respectively, for $t = 1, s = 2$, the following are the error matrices for $n = 60$, showing almost no difference between the two approaches:

$$\begin{pmatrix} -0.00934217 & 0.161037 \\ -0.00180046 & 0.0097662 \end{pmatrix}$$

$$\begin{pmatrix} -0.00914584 & 0.0158052 \\ -0.00131764 & 0.00914236 \end{pmatrix}.$$

2.3 Peaceman-Rachford Approximation Method

The results of the previous section depend heavily on the fact that the semigroup e^{tA} generated by a 2×2 matrix A can be computed explicitly. This remains true for $n \times n$ matrices but requires a serious computational effort. More precisely, repeating the arguments in the proof of Proposition 2.7, if we have

$$\tilde{\mathscr{A}} f(s) = f'(s) + A^*(s)f(s) = (\mathscr{A}_0 f)(s) + (Bf)(s),$$

where $\mathscr{A}_0 f = f'$ generates the shift semigroup $\{T_0(t)\}_{t\geq 0}$ and $(Bf)(s) = A^*(s)f(s)$ generates the semigroup $S(t)f(s) = e^{tA^*(s)}f(s)$, then the semigroup

$$\tilde{T}(t)f(s) = U^*(t+s,s)f(t+s)$$

generated by $\tilde{\mathscr{A}}$ can be approximated by

$$\left[T_0\left(\frac{t}{n}\right) S\left(\frac{t}{n}\right)\right]^n f(s) = \left[\prod_{j=1}^{n} e^{\frac{t}{n}A^*\left(s+\frac{jt}{n}\right)}\right] f(t+s) \tag{2.17}$$

or by

$$\left[T_0\left(\frac{t}{2n}\right) S\left(\frac{t}{n}\right) T_0\left(\frac{t}{2n}\right)\right]^n f(s) = \left[\prod_{j=1}^{n} e^{\frac{t}{n}A^*\left(s+\frac{(2j-1)t}{2n}\right)}\right] f\left(t+s\right). \tag{2.18}$$

It is obvious that these approximation methods are only useful in cases where $e^{tA^*(s)}$ can be "efficiently" computed (like in the 2×2 case). Otherwise, these approximations are of little practical use. However, even for a large dimension n, the resolvent $(\lambda I - A)^{-1}$ of an $n \times n$ matrix A is significantly easier to be computed than the semigroup e^{tA}. In 1955, Peaceman and Rachford (see [23]) proposed the "alternating direction implicit (ADI) method" $V\left(\frac{t}{n}\right)^n$ where

$$V(t) := \left(I - \frac{t}{2}\mathscr{A}_0\right)^{-1}\left(I - \frac{t}{2}B\right)^{-1}\left(I + \frac{t}{2}B\right)\left(I + \frac{t}{2}\mathscr{A}_0\right) \tag{2.19}$$

to approximate the semigroup $\left\{\tilde{T}(t)\right\}_{t\geq 0}$ generated by $\tilde{\mathscr{A}} = \mathscr{A}_0 + B$. Applying Theorem 4.1 (p. 76 in [28]) to our case, we obtain that for all $f \in D(\mathscr{A}_0)$,

$$V\left(\frac{t}{n}\right)^n f \to \tilde{T}(t)f$$

as $n \to \infty$ uniformly for t in compact intervals. To see if this method can be applied to our case effectively, we will now investigate the operators $V(t)$ as defined in (2.19). Clearly,

$$V(t)f(s) = \left(I - \frac{t}{2}\mathscr{A}_0\right)^{-1}\left(I - \frac{t}{2}B\right)^{-1}\left(I + \frac{t}{2}B\right)h_1(s),$$

where $h_1(s) = \left(I + \frac{t}{2}\mathscr{A}_0\right)f(s) = f(s) + \frac{t}{2}f'(s)$. Thus,

$$V(t)f(s) = \left(I - \frac{t}{2}\mathscr{A}_0\right)^{-1}\left(I - \frac{t}{2}B\right)^{-1}h_2(s),$$

where

$$\begin{aligned} h_2(s) &= \left(I + \frac{t}{2}B\right)h_1(s) = h_1(s) + \frac{t}{2}A^*(s)h_1(s) = f(s) + \frac{t}{2}f'(s) \\ &\quad + \frac{t}{2}A^*(s)\left(f(s) + \frac{t}{2}f'(s)\right) = \left(I + \frac{t}{2}A^*(s)\right)\left(f(s) + \frac{t}{2}f'(s)\right). \end{aligned}$$

It follows that

$$V(t)f(s) = \left(I - \frac{t}{2}\mathscr{A}_0\right)^{-1}h_3(s)$$

where

$$\begin{aligned} h_3(s) &= \left(I - \frac{t}{2}B\right)^{-1}h_2(s) = \left(I - \frac{t}{2}A^*(s)\right)^{-1}h_2(s) \\ &= \left(I - \frac{t}{2}A^*(s)\right)^{-1}\left(I + \frac{t}{2}A^*(s)\right)\left(f(s) + \frac{t}{2}f'(s)\right). \end{aligned}$$

Now,

$$(\lambda I - \mathscr{A}_0)^{-1}f(s) = \int_0^\infty e^{-\lambda r}T(r)f(s)\,dr = \int_0^\infty e^{-\lambda r}f(r+s)\,dr$$

and therefore,

$$
\begin{aligned}
V(t)f(s) &= \left(I - \frac{t}{2}\mathscr{A}_0\right)^{-1} h_3(s) = \frac{2}{t}\left(\frac{2}{t} - \mathscr{A}_0\right)^{-1} h_3(s) \\
&= \frac{2}{t}\int_0^\infty e^{-\frac{2r}{t}} h_3(s+r)\,dr \\
&= \frac{2}{t}e^{\frac{2s}{t}}\int_s^\infty e^{-\frac{2u}{t}}\left(I - \frac{t}{2}A^*(u)\right)^{-1}\left(I + \frac{t}{2}A^*(u)\right)\left[f(u) + \frac{t}{2}f'(u)\right] du \\
&\vdots \\
&\frac{2}{t}e^{\frac{2s}{t}}\int_s^\infty e^{-\frac{2s}{t}}\left[-I + 2\left(I - \frac{t}{2}A^*(u)\right)^{-1}\right]\left[f(u) + \frac{t}{2}f'(u)\right] du.
\end{aligned}
$$

We notice that $h(s) = V(t)f(s)$ is again continosly differentiable with

$$
h'(s) = \frac{2}{t}h(s) - \frac{2}{t}\left(I - \frac{t}{2}A^*(s)\right)^{-1}\left(I + \frac{t}{2}A^*(s)\right)\left[f(u) + \frac{t}{2}f'(u)\right].
$$

Since $V(t)f(s)$ is differentiable, it follows that the formula above can be iterated. However, it seems to be very unlikely that $V\left(\frac{t}{n}\right)^n f(s)$ can be computed efficiently for large values of n. Thus, the Lie-Trotter formulas presented in Proposition 2.7 or (2.17) and (2.18) appear to be the presently best approaches to approximate the evolution operators $U(t,s)$ corresponding to

$$
u'(t) = A(t)u(t),\ u(s) = x
$$

in terms of the semigroup generated by the operator

$$
\tilde{\mathscr{A}}f(s) := f'(s) + A^*(s)f(s).
$$

References

[1] W. Arendt, C.J.K. Batty, M. Hieber and F. Neubrander, *Vector-Valued Laplace Transforms and Cauchy Problems*, 2nd edition, Monographs in Mathematics, Birkhäuser Verlag, 2011.

[2] S. Blanes and F. Casas, *On the Necessity of Negative Coefficients for Operator Splitting Schemes of Order Higher Than Two*, Applied Numerical Mathematics, **54** (2005), 23-37.

[3] P. Brenner and V. Thomée, *On rational approximation of semigroups*, SIAM J. Numer. Anal. **16** (1979), 683-694.

[4] S. Descombes and M. Thalhammer, *An exact local error representation of exponential operator splitting methods for evolutionary problems and applications to linear Schrödinger equations in the semi-classical regime*, BIT Numerical Mathematics, **50** (2010), 729-749.

[5] J. Dorroh and J. Neuberger, *Lie generators for semigroups of transforms on a Polish space*, Electron. J. Differential Equations **1** (1993), 1-7.

[6] J. Dorroh and J. Neuberger, *A theory of strongly continuous semigroups in terms of the Lie generators*, J. Funct. Anal **136** (1996), 114-126.

[7] J. Dorroh and J. Neuberger, *Linear extensions of nonlinear Semigroups, Progress in Nonlinear Differential Equations and Their App.* **42** (2000), 96-102.

[8] K. Engel and R. Nagel, *A Short Course on Operator Semigroups*, Springer-Verlag, 2006.

[9] K.J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer-Verlag, 2000.

[10] I. Faragó and Á. Havasi, *Consistency Analysis of Operator Splitting Methods for C_0–Semigroups*, Semigroup Forum **74** (2007), 125-139.

[11] E. Hairer, Ch. Lubich and G. Wanner, *Geometric Numerical Integration. Structure-Preserving Algorithms for Ordinary Differential Equations.* Springer, Berlin, 2002.

[12] R. Hersh and T. Kato, *High-accuracy stable difference schemes for wellposed inital value problems*, SIAM J. Numer. Anal. **16** (1979), 670-682.

[13] E.L. Ince, Ordinary Differential Equations, Dover Publications, Inc., 1956.

[14] P. Jara, *Rational approximation schemes for bi-continuous semigroups*, J. Math. Anal. Appl. **344** (2008), 956-968.

[15] P. Jara, F. Neubrander, and K. Özer, *Rational inversion of the Laplace transform*, Journal of Evolution Equations, to appear.

[16] S. McAllister, F. Neubrander, A. Reiser, and Y. Zhuang, *Stabilizations of the Trotter-Kato theorem and the Chernoff Product Formula*, Semigroup Forum 86 (2013), 511-524.

[17] F. Neubrander, *Integrated Semigroups and their Applications to the Abstract Cauchy Problems*, Pacific J. Math, 135 (1988), 111-155.

[18] R.I. McLachlan and R. Quispel. *Splitting Methods.* Acta Numerica 11 (2002), 341-434.

[19] F. Neubrander, *Evolution Equations*, Lecture Notes, LSU, 2010.

[20] F. Neubrander, K. Özer, and T. Sandmaier, *Rational Approximation of Semigroups without Scaling and Squaring.* Discrete and Continuous Dynamical Systems 33 (2013), 5305 - 5317.

[21] K. Özer, *Laplace Transform Inversion and Time-Discretization Methods for Evolution Equations*, Ph.D. thesis, LSU, 2008.

[22] G. Nickel, *Evolution semigroups and product formulas for nonautonomous Cauchy problems*, Math. Nachr. 212 (2000), 101-116.

[23] D.W. Peaceman and H.H. Rachford,Jr., "The numerical solution of parabolic and elliptic differential equations", J. SIAM 3 (1955) p. 28-41.

[24] A. Reiser, *Time Discretization for Evolution Equations*; Diplomarbeit, Louisiana State University and Universität Tübingen, 2008.

[25] J.A.C. Weideman, *Optimizing Talbot's contours for the inversion of the Laplace transform.* Siam J. Numerical Analysis 44 (2006), 2342-2362.

[26] D.V. Widder, *The Laplace Transform*, Princeton University Press, 1946.

[27] L. Windsperger, Dissertation, Louisiana State University, Fall 2012.

[28] Y. Zhuang, *Classically Unstable Approximations for Linear Evolution Equations and Applications*, Dissertation, LSU, 2000.

Vita

Ladorian Nichele Latin is a native of Shreveport, LA. She completed her undergraduate studies at Northwestern State University in May 2006, earning a Bachelor of Science in Mathematics. She began her graduate studies at Louisiana State University in August 2006 and earned a Master of Science in Mathematics in December 2008. Ladorian is currently a candidate for the degree of Doctor of Philosophy in Mathematics, which will be awarded in December 2013.