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ON PROPERTIES OF LINEAR CONTROL SYSTEMS ON LIE GROUPS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
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in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

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by
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Abstract

In this work we study controllability properties of linear control systems on Lie groups as introduced by Ayala and Tirao in [AT99]. A linear control system Σ on a Lie group G is defined by

$$\dot{x} = X(x) + \sum_{j=1}^{k} u_j Y_j(x), \tag{1}$$

where the drift vector field X is an infinitesimal automorphism, u_j are piecewise constant functions, and the control vectors Y_j are left-invariant vector fields. Properties for the flow of the infinitesimal automorphism X and for the reachable set defined by Σ are presented in Chapter 3. Under a condition similar to the Kalman condition which is needed for controllability of linear control systems on $\mathbb{R}^n$, Ayala and Tirao showed local controllability of the system Σ at the group identity e. An alternate proof of this result is obtained using the Lie theory of semigroups. More importantly, an extension of this result is proved. These results are contained in Chapter 4. Finally, in Chapter 5 an example on the Heisenberg Lie group is presented and its properties are proved using the theory developed.

Chapter 1
Introduction

The study of mathematical control theory began in the 1950's. Control systems are dynamical systems whose dynamical laws are not entirely fixed, but depend on parameters called controls. A suitable choice of the controls can force the system to achieve a desired goal. One of these goals may be to determine the set of original states which can be steered to a certain final state (controllability problem). For example, in the case of driving a vehicle, the controls are the accelerator, the brakes, and the steering wheel. Another way in which controls may be needed is to counter deviations from a desired path (stability problem). For example, to avoid the wandering of a ship from its set course, controls are needed in the form of touches to the rudder. Applications of control theory are found not only in the mechanics of motion, but in many other areas such as the growth processes in organisms and populations, where the controls are the added nutrients or pesticides. In economics, the operation of a company is dependent on financial controls. Another typical control-theory question is to determine the properties of the set of points achievable from a starting one when all possible choices of controls are used. This set is called *the reachable set.* In real life problems the control variables are usually subject to constraints on their magnitudes. This implies that the set of final states which can be achieved is restricted. When the reachable set from the origin is the whole state space, the system is said to have the *reachability property.* In general, it is possible to steer a system from one state to another state using several choices of controls. In such a case, a selection among those successful controls can be made to minimize some quantity, known as the cost. Some examples of cost functions are: the time taken to reach a desired goal (time optimal problem), the energy utilized, and the manufacturing costs, to name a few. The problem is then to minimize the cost function over all admissible controls. The study of these types of problems is in the scope of Optimal Control Theory. A control for which the minimum cost is attained is called an *optimal control* and the associated trajectory is called an *optimal trajectory*. A well-known tool used in optimal control theory is the so called *Pontryagin Maximum's Principle* obtained by L. S. Pontryagin and his co-workers in 1962 [PBGM62]. The Pontryagin Maximum Principle provides necessary conditions that must be satisfied by any optimal solution. These conditions are based on the topological fact that an optimal trajectory must terminate on the boundary of the reachable set determined by the starting point. An illustration is consider in the following example.

Example 1.0.1. *The Harmonic Oscillator.*

In a wide variety of dynamical problems, the goal is to avoid vibrations or oscillations about the equilibrium position. An example of this type of motion is provided by the pendulum. The pendulum is described as a weight that is suspended by a string from a fixed point and swings in a vertical plane. It is assumed that air resistance, the mass of the string, and the dimensions of the body can all be neglected, and that gravitational attraction is constant. Let θ be the angle that the pendulum makes with the vertical direction. If the angle is small, then the equation of motion can be approximated by the linear equation

$$\ddot{\theta} = -\theta.$$

An initial displacement θ_0 from the equilibrium produces an oscillatory movement of the weight about the equilibrium position. If an external force $F = mu$ is available, the goal of bringing the weight to rest in the least amount of time can be set. For *simplicity assume that* $m = 1$ *and that the magnitude of the force* $|F|$ *is bounded by* 1*. The model of the forced system becomes*

$$\ddot{\theta} = -\theta + u, \quad |u| \leq 1.$$

Given the initial and terminal conditions $\theta(t_0) = \theta_0,\ \dot{\theta}(t_0) = \theta_0',\ \theta(t_f) = 0,$ *and* $\dot{\theta}(t_f) = 0,$ *the time optimal problem is to find a control* u *such that the trajectory* $\theta(\cdot)$ *determined by* u *satisfies the boundary conditions and such that the terminal time* t_f *is minimized. Renaming* θ *and* $\dot{\theta}$ *as* x_1, x_2*, the state equations can be rewritten as the linear control system in* $\mathbb{R}^2$

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since the goal is to stop the oscillations in the minimum time possible, it is intuitively clear that a force of the maximum available size should be used at all times. This implies that the control $u \in \{-1, 1\}$*, where the changes of sign indicate a switch in the direction of the force. The switchings will be needed until the weight is brought to rest. The pattern of the switchings is not at all obvious. One might guess that the best strategy would be to apply the force in the direction opposite to the motion of the pendulum. However, it can be verified, using the maximum principle, that this is not always the optimal strategy. The Pontryagin Maximum Principle leads to the construction of the optimal trajectories for any initial state* [Hoc91].

The study of optimal control theory is important for other reasons besides optimality itself. For nonlinear systems it is often difficult to find controls that achieve a desired goal. Optimal control gives a systematic approach to search for such controls,and it is used as the basis of numerical methods.

In the early 1960's, R. Hermann [Her63] incorporated the use of *differential-geometric methods* into the study of control problems. His work was then followed by C. Lobry [Lob70] in 1970, and then by Brockett [Bro72], Krener [Kre74], Sussmann-Jurdjevic [SJ72] and others. These efforts gave birth to *geometric control theory*, where the idea is that the state space is a manifold, and the dynamics of the system are described by vector fields that depend on control parameters. Some geometric control problems are best formulated with the state space on a Lie Group. In his book V. Jurdjevic [Jur97] presents several applications and results in this area. In particular a generalization of the Pontryagin Maximum Principle is presented and used to solve various problems. For instance, Jurdjevic analyzes the problem of Dubins and uses the maximum principle to solve it.

Example 1.0.2. *The Problem of Dubins.*

This problem was first considered and solved using ad-hoc methods by Dubins in 1957 [Dub57]*. In intuitive terms, this problem can be thought of as the problem of parking a car (that is, initial and final positions and directions of the vehicle are fixed) using the shortest trajectory. The problem of Dubins can be stated as the problem of finding the curves of minimal length that connect the given initial and terminal configurations in the tangent bundle of* $\mathbb{R}^2$*, with a constraint on the curvature. This example is best modeled as an optimal control problem on the Lie group* $SO_2(\mathbb{R}) \times \mathbb{R}^2$*. The convenience of having a well-developed theory of Lie groups and Lie algebras makes the study of this and other geometric control problems accessible.*

A necessary precursor of any study of optimal control issues is the analysis of controllability, since it would be impossible to find the "best" control when the set of successful controls is empty. On the search for necessary and sufficient conditions for controllability, new ideas and problems arise such as determining when the reachable set is nonempty (accessibility property), when a point is in the interior of its reachable set (local controllability), and when the reachable set is the whole state space (controllability).

The local controllability property is linked to the time optimal problem as follows. If ξ is a time-optimal trajectory defined on $[0, T]$, then by the maximum principle, $\xi(t)$ belongs to the boundary of $\mathcal{R}(\xi(0), t)$ for each $t \in [0, T]$. Here $\mathcal{R}(\xi(0), t)$ denotes the reachable set at time t from the starting point $\xi(0)$. Therefore, each sufficient condition for $\xi(t)$ to belong to the interior of $\mathcal{R}(\xi(0), t)$ yields a necessary condition for the trajectory ξ to be time-optimal.

The goal of this work is to extend some of the standard results of Linear Control System Theory on Euclidean spaces to Linear Control Systems on Lie Groups. The material has been organized as follows: Chapter 2 introduces background material and definitions that will be needed later as well as classical results. The first sections are devoted to control theory in a vector space, while the last section contains some classic results on geometric control theory. In Chapter 3 the definition of a linear control system on a Lie group is presented as well as generalizations of some of the results presented in Chapter 2. Our main results are presented in Chapter 4, specifically Theorem 4.3.6 which deals with local controllability at the group identity using a Lie wedge approach. We also prove an extension of this result in Theorem 4.4.3. Finally, in Chapter 5 an example on the Heisenberg group is discussed.

Chapter 2
Some Concepts on Control Theory

The purpose of this chapter is to introduce some basic concepts and results that will be needed later. The first two sections deal with control theory on a vector space. In particular, Section 2.2 contains definitions and results for linear control systems on a vector space. The last section gives an overview of geometric control theory. A description of control theory on a manifold is given. The chapter ends with some well-known results of control theory on Lie groups. A general exposition is given and some results are presented without proofs. Further details can be found in the cited references.

2.1 Control Systems

Let V be an n-dimensional vector space, called the state space, and let $x \in V$ be a state vector. A *control system* Σ on V is defined by

$$\frac{dx}{dt} = f(x, u(t)), \quad x(t_0) = x_0, \tag{2.1}$$

where the control functions u belong to a class $\mathcal{U}$ of admissible controls with values in a subset U of $\mathbb{R}^m$ and f is continuously differentiable in $V \times U$.

Given a sufficiently smooth control function $u \in \mathcal{U}$, a solution of the system Σ, called a trajectory, is determined. Such a solution can be described by the transition function Φ. Specifically, $\Phi(t, t_0, x_0, u)$ denotes the state that results at time t if the system was in state x_0 at time t_0 and the control u was applied.

Definition 2.1.1. *The state z can be* ***reached from*** *the state x if and only if there is a trajectory of Σ whose initial state is x and whose final state is z, that* *is, if there exists* $u \in \mathcal{U}$ *such that* $\Phi(t_f, 0, x, u) = z$*. One also says that* x *can be* ***controlled*** *to* z*.*

The controllable set at time t_1 *is the set of initial states that can be controlled to the origin in time* t_1 *using an admissible control, that is,*

$$\mathcal{C}(t_1) = \{x_0 : \ \Phi(t_1, 0, x_0, u) = 0 \textit{ for some } u \in \mathcal{U}\}.$$

The ***controllable set*** $\mathcal{C}$ *is the set of states that can be controlled to the origin in any finite time,* i.e., $\mathcal{C} = \cup_{t_1 \geq 0} \mathcal{C}(t_1)$.

The system Σ is called **controllable at** x if z can be controlled to x for all $z \in V$. Therefore, Σ is controllable at the origin if and only if $\mathcal{C} = V$.

If all initial states can be controlled to x for all $x \in V$, then the system Σ is called **controllable**.

In the case where V is a metric space, the concept of local controllability along a trajectory can be considered. If γ is a trajectory of Σ defined on the interval $[0, T]$ from x_0 to x_1, the system Σ is **locally controllable along** γ if for each $\epsilon > 0$ there is some $\delta > 0$ such that the following property holds: For each $x, y \in V$ with $d(x, x_0) < \delta$ and $d(y, x_1) < \delta$ there is some trajectory ζ of Σ defined on $[0, T]$ with $\zeta(0) = x$ and $\zeta(T) = y$ such that $d_\infty(\zeta, \gamma) < \epsilon$. Here d_∞ denotes the uniform distance.

Let $x \in N$, where N is a neighborhood of x in V. The system Σ is **locally controllable** at x if there is an open set $O \subseteq N$ containing x such that for each $z \in O$ there is a trajectory ζ of Σ joining x and z entirely contained in N.

Let the class $\mathcal{U}$ of admissible controls be the set of integrable functions of t, with or without bounds. The following subsets of $\mathcal{U}$ will be useful.

- $\mathcal{U}_u$, the set of controls without previously set bound for their values (unbounded controls)

$$\mathcal{U}_b = \{u \in \mathcal{U} : \ |u_i(t)| \le 1, \ i = 1, 2, \cdots, m\},$$

$$\mathcal{U}_{bb} = \{u \in \mathcal{U}_b : \ |u_i(t)| = 1\}.$$

Consider $V = \mathbb{R}^n$, a control set U containing the origin and assume $f(0,0) = 0$. Then the origin is an equilibrium point in the sense that once the origin is reached, it is possible to remain there by switching off the control. With these assumptions, the following results are obtained.

Proposition 2.1.2. *If $x_0 \in \mathcal{C}$ and if y is a point on a trajectory of Σ that joins x_0 to 0, then $y \in \mathcal{C}$.*

Proof. Suppose $\gamma(t)$ is the trajectory joining x_0 and 0 with some control $u(t)$. Say $\gamma(t_1) = 0$ and $\gamma(\tau) = y$. Let v denote the control $v(t) = u(t + \tau)$, then y can be controlled to the origin in time $t_1 - \tau$. Thus, $y \in \mathcal{C}(t_1 - \tau)$. □

Proposition 2.1.3. *$\mathcal{C}$ is arcwise connected.*

Proof. Let $x, y \in \mathcal{C}$. By the previous result, the trajectories from x to the origin and from y to the origin lie entirely in $\mathcal{C}$. Hence, there is an arc contained in $\mathcal{C}$ connecting x and y. □

Proposition 2.1.4. *If $t_1 < t_2$, then $\mathcal{C}(t_1) \subset \mathcal{C}(t_2)$.*

Proof. Suppose $x \in \mathcal{C}(t_1)$, with control $u(t)$. Define

$$v(t) = \begin{cases} u(t), & \text{if } 0 \le t \le t_1 \\ 0, & \text{if } t_1 \le t \le t_2 \end{cases}$$

since $f(0,0) = 0$, the trajectory determined by v stays at the origin when $t_1 \le t \le t_2$. Therefore, $x \in \mathcal{C}(t_2)$. □

Proposition 2.1.5. *$\mathcal{C}$ is open if and only if $0 \in \operatorname{Int} \mathcal{C}$.*

Proof. If $\mathcal{C}$ is open then $0 \in \operatorname{Int} \mathcal{C}$ follows from $\operatorname{Int} \mathcal{C} = \mathcal{C}$ and $0 \in \mathcal{C}$.

Conversely, if $0 \in \operatorname{Int} \mathcal{C}$ there is a ball $B(0, r) \subset \mathcal{C}$ for some $r > 0$. We need to prove that for any $x \in \mathcal{C}$ there is a ball around x completely contained in $\mathcal{C}$. By continuous dependence of solutions, there exists $r_0 > 0$ sufficiently small, such that the ball $B(x, r_0)$ gets mapped into $B(0, r)$. Let $y_1 \in B(x, r_0)$, then y_1 can be steered to some $y \in B(0, r)$. Since $B(0, r) \subset \mathcal{C}$, y can be steered to the origin. Hence, y_1 can be steered to the origin, that is, $y_1 \in \mathcal{C}$. Since this holds for any $y_1 \in B(x, r_0)$ the result is proved. $\square$

More results can be obtained for the controllable set $\mathcal{C}$ if the control system is linear as we shall see in the next section.

2.2 Linear Control Systems

A *linear control system* Σ is defined as

$$\dot{x} = Ax + Bu, \tag{2.2}$$

where $A(n \times n)$ and $B(n \times m)$ are constant matrices, the state space is n-dimensional and the control $u \in \mathcal{U}$, where $\mathcal{U}$ is the class of integrable functions of t.

The solution of the system 2.2 starting at x_0 has the form

$$x(t) = \exp(At) \left(x_0 + \int_0^t \exp(-A\tau) Bu(\tau) \, d\tau \right) \tag{2.3}$$

where the exponential of a matrix is defined by the infinite series $\exp(A) = \Sigma_{k=0}^{\infty} \frac{A^k}{k!}$.

It follows that $x_0 \in \mathcal{C}(t_1)$ if and only if there is an admissible control $u \in \mathcal{U}$ such that $x_0 = -\int_0^{t_1} \exp(-A\tau) Bu(\tau) \, d\tau$.

The following lemma shows some controllability equivalences for a linear system Σ on R^n.

Lemma 2.2.1. *If Σ is a linear control system, then*

(i) *x can be controlled to z if and only if the origin* 0 *can be controlled to* $z - \exp(At)x$.

(ii) Σ *is controllable if and only if the origin* 0 *can be controlled to y for all* $y \in R^n$.

Proof. (i) Note that x can be controlled to $z \Leftrightarrow$ there exists an admissible control $u \in \mathcal{U}$ such that $z = \exp(At)\left(x + \int_0^t \exp(-A\tau)Bu(\tau)\ d\tau\right)$

$\Longleftrightarrow z - \exp(At)x = \exp(At)\left(\int_0^t \exp(-A\tau)Bu(\tau)\ d\tau\right) = \phi(t, 0, 0, u)$

$\Longleftrightarrow$ the origin 0 can be controlled to $z - \exp(At)x$, by definition.

(ii) Let x and z be in R^n and define

$$y := z - \exp(At)x.$$

By part (i) the origin can be controlled to y if and only if x can be controlled to z. $\square$

The following result for the controllable set can be found in [Hoc91].

Proposition 2.2.2. *$\mathcal{C}(t_1)$ and $\mathcal{C}$ are both symmetric and convex.*

Example 2.2.3. *Consider the linear system given by the following state equations*

$$\dot{x_1} = x_1 + u, \quad \dot{x_2} = x_2 + u$$

where $u \in \mathcal{U}_b$ and the matrices A and B are given by $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. *So we have that $x = (x_1, x_2)$ belongs to $\mathcal{C}(t_1)$ if*

$$x_1 = -\int_0^{t_1} \exp(-\tau)u_1\ d\tau = x_2.$$

Since $|u| \leq 1$ then $|x_1| \leq 1 - \exp(-t_1)$.

Therefore, $\mathcal{C}(t_1)$ is the closed diagonal segment $\mathcal{C}(t_1) = \{x_1 = x_2 :\ |x_1| \leq 1 - \exp(-t_1)\}$, *and $\mathcal{C}$ is the open diagonal segment $\mathcal{C} = \{x_1 = x_2 : |x_1| < 1\}$.*

Clearly, $\operatorname{Int}\mathcal{C}$ *is empty and $\mathcal{C}$ is not open (in $\mathbb{R}^2$). In general, it would be impossible to control both components simultaneously with identical controls. To control both components using the same control, the initial deviation of the two components must be equal.*

To get controllability there are two necessary conditions on the controllable set, namely it must have full dimension (not collapse on to a hyperplane as it does in this example) and be unbounded.

2.2.1 The Controllability Matrix

In this section, well-known conditions that ensure controllability for the linear system (2.2) are presented. The results depend on the rank of a $n \times nm$ matrix M, called the *controllability matrix* which is defined as

$$M = [B\ AB\ A^2B\ \ldots\ A^{n-1}B].$$

The property that rank $M = n$ is known in the literature as the ***Kalman condition***.

Proposition 2.2.4. $0 \in \operatorname{Int}\mathcal{C}$ *if and only if* rank $M = n$.

Proof. Suppose that rank $M < n$. Then there is a vector $y \in R^n$ that is orthonormal to every column of M. That is,

$$y^T B = 0 \text{ and } y^T A^k B = 0,\ k = 1, \ldots, n-1.$$

By the Cayley-Hamilton theorem

$$y^T A^k B = 0, \quad \forall k \in \mathbb{N},$$

thus

$$(\forall \tau \geq 0)\ y^T \exp(-A\tau) B = 0.$$

This implies in particular that given any $t_1 > 0$, $y^T x_0 = 0 \ \forall x_0 \in \mathcal{C}(t_1)$. Therefore, $\mathcal{C}(t_1)$ lies in a hyperplane with normal y for all $t_1 > 0$ and $\mathcal{C}$ lies in the same hyperplane. Thus $0 \notin \operatorname{Int} \mathcal{C}$.

Conversely, suppose $0 \notin \operatorname{Int} \mathcal{C}$. Since $\mathcal{C}(t_1) \subset \mathcal{C}$, then $0 \notin \operatorname{Int} \mathcal{C}(t_1)$ for any t_1, but $0 \in \mathcal{C}(t_1)$ and $\mathcal{C}(t_1)$ is convex. Therefore, for each value of t_1 there is a hyperplane through 0 supporting $\mathcal{C}(t_1)$. Let b be the normal to this hyperplane. Then $b^T x \leq 0 \ \forall x \in \mathcal{C}(t_1)$ and

$$\int_0^{t_1} b^T \exp(-A\tau) B u \, d\tau = -b^T x \geq 0,$$

for all $u \in \mathcal{U}$. Since $-u \in \mathcal{U}$ also, the last inequality is actually an equality. Therefore,

$$b^T \exp(-A\tau) B = 0 \text{ for } 0 \leq \tau \leq t_1. \tag{2.4}$$

In particular, $\tau = 0$ gives $b^T B = 0$. Differentiating equation 2.4 k times and setting $\tau = 0$ gives $b^T A^k B = 0$. Then b is orthogonal to all the columns of M; hence, the rank of M is less that n. This contradicts the hypothesis □

When $0 \notin \operatorname{Int} \mathcal{C}$ it is clear that the system cannot be controllable. In the example 2.2.3 of the previous section we determined the points that are not controllable to the origin. The controllability matrix in that example is

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

which has rank equal to one. By Proposition 2.2.4 the system is not controllable at the origin.

If the rank of the controllability matrix M is less than n then $\mathcal{C}$ lies in a hyperplane in $\mathbb{R}^n$ and controllability at the origin will not be satisfied. However, the system may not be controllable at the origin even if the Kalman condition is satisfied. The following result establishes the extra condition needed for controllability at the origin.

Theorem 2.2.5. *If* rank $M = n$*, and* $\mathcal{U} = \mathcal{U}_u$*, then* $\mathcal{C} = \mathbb{R}^n$*.*

For bounded controls the eigenvalues of the matrix A need to be considered as well to get a sufficient conditions for controllability at the origin.

Theorem 2.2.6. *If* rank $M = n$*,* $\mathcal{U} = \mathcal{U}_b$*, and* $Re\lambda_i < 0$ *for each eigenvalue* λ_i *of* A*, then* $\mathcal{C} = \mathbb{R}^n$*.*

The previous hypothesis on the eigenvalues can actually be weakened to $\mathrm{Re}\lambda_i \leq 0$.

2.2.2 The Reachable Set

In this section properties of the reachable set for linear systems on the Euclidean space are presented. These properties will be compared to results of linear systems on Lie groups in Chapter 3.

Define the *reachable set* $\mathcal{R}(x_0, t_1)$ as the set of points that can be reached from the initial state x_0 in time t_1. As pointed out in Section 2.1 there is a reciprocal relationship between reachable sets and the controllable sets, namely if $z \in \mathcal{R}(x_0, t_1)$ then $x_0 \in \mathcal{C}(z, t_1)$.

If the system Σ is linear then $x_1 \in \mathcal{R}(x_0, t_1)$ implies

$$x_1 = \exp(At_1)\left(x_0 + \int_0^{t_1} \exp(-A\tau)Bu(\tau)\ d\tau\right)$$

for some $u \in \mathcal{U}$. The following result is straightforward.

Proposition 2.2.7. *If the system* Σ *is linear then* $\mathcal{R}(x_0, t) = \exp(At)x_0 + \mathcal{R}(0, t)$*.*

The property that any point in $\mathbb{R}^n$ can be reached from the origin is called *reachability.* Lemma 2.2.1 shows that for linear systems reachability and controllability are equivalent.

Local controllability, defined in Section 2.1, can also be linked to a property of the reachable set. First, local controllability at x_0 implies in particular, that

$$x_0 \in \operatorname{Int} \mathcal{R}(x_0).$$

Second, as the proposition below shows, local controllability is equivalent to controllability for linear control systems on $\mathbb{R}^n$.

Proposition 2.2.8. *Let Σ be a linear control system, then*

(a) Σ is locally controllable along some trajectory γ.

(b) Σ is locally controllable along every trajectory γ.

(c) Σ is controllable.

(d) Σ is has the reachability property.

2.3 Geometric Control Theory

In geometric control problems the state variable takes values on an n-dimensional manifold M and the motion of the system at each point of M follows a controlled vector field. The following notations are standard. The tangent space to the manifold M at the point m is denoted by T_mM and TM is the tangent bundle of the manifold M. The set of all smooth vector fields on the manifold M is denoted by $V^\infty(M)$. Given $X, Y \in V^\infty(M)$ their Lie bracket $[X, Y]$ is the smooth vector field defined by $[X, Y] = YX - XY$. Note that $V^\infty(M)$ is a Lie algebra over the reals where the operation is the Lie bracket.

The *integral curve* $\gamma_m(t)$ of a vector field X through $m \in M$ is the differentiable curve of maximal domain that satisfies

$$\frac{d\gamma}{dt} = X(\gamma(t)), \quad \gamma(0) = m.$$

A vector field X is called *complete* if the integral curves through each point $m \in M$ are defined for all $t \in R$. The *flow* of X is the mapping $\Phi : \mathbb{R} \times M \to M$ defined by $\Phi(t, m) = \gamma_m(t)$ and it satisfies:

1. $\Phi(0, x) = x$ for all $x \in M$.
2. $\Phi(t + s, x) = \Phi(t, \Phi(s, x))$ for all $s, t \in \mathbb{R}$ and all $x \in M$.
3. $\dfrac{\partial \Phi(t, x)}{\partial t} = X(\Phi(t, x))$ for all $(t, x) \in M$.
4. The mapping Φ is smooth whenever X is smooth.

For each fixed $t \in \mathbb{R}$, the mapping $\Phi_t(x) : x \mapsto \Phi(t, x)$ is a diffeomorphism in M. The collection $\{\Phi_t : t \in \mathbb{R}\}$ forms a group under the composition of mappings. This group is called the one-parameter group of diffeomorphisms induced by X, and is denoted by $\Phi_t = \exp tX$. The vector field X is called the *infinitesimal generator* of Φ.

A control system Σ on M is defined by

$$\frac{dx}{dt} = F(x, u(t)) \tag{2.5}$$

where the control functions u belong to a class $\mathcal{U}$ of admissible controls with values in a subset U of $\mathbb{R}^n$. The mapping $F : M \times U \to TM$ is such that for each $u \in U$, $F_u = F(\cdot, u) : M \to TM$ is a complete smooth vector field.

When the controls are piecewise-constant functions of t with values in U, a trajectory $x(t)$ of the system (2.5) can be visualized as a continuous curve in M

consisting of pieces of integral curves of vector fields corresponding to different choices of control values.

Definition 2.3.1. *Let $T > 0$, and $x_0 \in M$. The* ***reachable set at time*** *T from x_0, denoted by $\mathcal{R}(x_0, T)$, is the set of terminal points $x(T)$ of solutions of (2.5) that start at x_0.*

The ***reachable set*** *from x_0 is the set $\mathcal{R}(x_0) = \bigcup_{t\geq 0} \mathcal{R}(x_0, t)$.*

Let Ω be the family of vector fields $\{F_u : u \in U\}$, and let $G(\Omega)$ denote the subgroup of the group of diffeomorphisms in M generated by the union of $\{\exp tX : t \in \mathbb{R},\ X \in \Omega\}$. Each element φ of $G(\Omega)$ is a diffeomorphism of M of the form

$$\varphi = (\exp t_k X_k)(\exp t_{k-1} X_{k-1}) \cdots (\exp t_1 X_1), \text{ for } X_i \in \Omega, t_i \in \mathbb{R}, \text{ and } i = 1, \cdots, k.$$

$G(\Omega)$ acts on M and partitions it into orbits. The reachable set through x_0 at time T can be expressed in terms of $G(\Omega)$ as

$$\mathcal{R}(x_0, T) = \{\varphi(x_0) : t_i \geq 0, t_1 + \cdots + t_k = T, X_1, \cdots, X_k \in \Omega\}.$$

Let $\mathcal{S}_f$ denote the semigroup of all elements φ in $G(\Omega)$ of the form

$$\varphi = (\exp t_k X_k)(\exp t_{k-1} X_{k-1}) \cdots (\exp t_1 X_1), \quad \text{for } X_i \in \Omega, t_i \geq 0.$$

Then the reachable set $\mathcal{R}(x_0)$ is equal to the orbit of the semigroup $\mathcal{S}_f$ through x_0, that is,

$$\mathcal{R}(x_0) = \{\varphi(x_0) : \varphi \in \mathcal{S}_f\}.$$

The orbit of the family Ω through each $x \in M$ is $G(\Omega)(x) = \{\varphi(x) : \varphi \in G(\Omega)\}$. A basic property of families of vector fields is that their orbits are manifolds. This result is known as the "orbit theorem", and it is quoted below from [Sus73].

Theorem 2.3.2. *The orbit of Ω through each point $x \in M$ is a connected submanifold of M.*

Definition 2.3.3. *The system Σ is said to have the accessibility property from x_0 if $\mathcal{R}(x_0)$ has nonempty interior. The system Σ is said to have the* ***accessibility property*** *if $\mathcal{R}(x)$ has nonempty interior for every $x \in M$.*

The system Σ is ***locally controllable*** *at x_0 if $x_0 \in \operatorname{Int} \mathcal{R}(x_0)$.*

The system Σ is controllable at x_0 if $\mathcal{R}(x_0) = M$ and Σ is ***controllable*** *if it is controllable at every $x \in M$.*

Let $W(\mathcal{F})$ be the smallest subalgebra of $V^\infty(M)$ containing $\mathcal{F} = \{F_u\}_{u \in U}$. The condition dim $W(\mathcal{F})(x) = n$ is also known as the *rank condition*.

In their work [SJ72], Sussmann and Jurdjevic considered the system Σ as defined above with the condition that F be an analytic function of x. They obtained the following result,

Theorem 2.3.4. *A necessary and sufficient condition for the system Σ to have the accessibility property is that for all x* dim $W(\mathcal{F})(x) = n$.

2.3.1 Lie Groups

Lie groups form an important class of analytic manifolds. Below, the basic definitions and some results from Lie group theory (cf. [War83]) are stated.

Definition 2.3.5. *A* ***Lie group*** *G is an analytic manifold which is also endowed with a group structure such that the map*

$$G \times G \to G, \quad (x, y) \mapsto xy^{-1}$$

is analytic.

For any $g \in G$, ρ_g denotes the right-translation on G by g, that is,

$$\rho_g : G \to G, \quad \rho_g(x) = xg.$$

Let λ_g denote the left-translation by g, that is,

$$\lambda_g : G \to G, \quad \lambda_g(x) = gx.$$

For each $g \in G$, both ρ_g and λ_g are analytic functions and $d\rho_g,\ d\lambda_g$ denote their corresponding differentials.

Definition 2.3.6. *A vector field on G is called right-invariant (respectively, left-invariant) if $d\rho_g \circ X = X \circ \rho_g$ (respectively, $d\lambda_g \circ X = X \circ \lambda_g$)*

Proposition 2.3.7. *Let G be a Lie group and let $\mathfrak{g}$ be its set of right-invariant vector fields. Then*

(i) $\mathfrak{g}$ *is a vector space, and the map $\alpha : \mathfrak{g} \to T_eG$ defined by $X \mapsto X(e)$ is an isomorphism. Consequently,* $\dim\ \mathfrak{g} = \dim\ T_eG = \dim\ G$.

(ii) *Right-invariant vector fields are analytic.*

(iii) *The Lie bracket of right-invariant vector fields is itself right-invariant.*

(iv) $\mathfrak{g}$ *forms a Lie algebra under the Lie bracket operation on vector fields.*

In view of the previous proposition the Lie algebra of a Lie group G can be defined as the Lie algebra of right-invariant vector fields $\mathfrak{g}$, or alternatively as the tangent space at the identity T_eG.

Let Φ be the flow generated by a right-invariant vector field X. For any $g \in G$, the curve $g(t) = \Phi(t, e) \cdot g$, satisfies

$$\frac{dg(t)}{dt} = X(g(t)).$$

So $g(t)$ is the integral curve of X through g, and the following property is satisfied

$$\Phi(t, g) = \Phi(t, e) \cdot g$$

This property is interpreted in terms of the corresponding reachable sets as

$$\mathcal{R}(g) = \mathcal{R}(e) \cdot g$$

Definition 2.3.8. *The **exponential map***

$$\exp : \mathfrak{g} \to G, \quad X \mapsto \exp(X)$$

is a C^∞-map where $\exp(X)$ *denotes the point in G given by* $\Phi(1, e)$.

Theorem 2.3.9. *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively, and $\phi : H \to G$ a homomorphism. Then the following diagram is commutative*

$$\begin{array}{ccc} H & \xrightarrow{\phi} & G \\ {\scriptstyle\exp}\uparrow & & \uparrow{\scriptstyle\exp} \\ \mathfrak{h} & \xrightarrow{d\phi} & \mathfrak{g} \end{array}$$

That is, $\phi(\exp X) = \exp(d\phi(X))$.

A Lie group acts on itself by inner automorphisms

$$a : G \times G \to G, \quad a(g, x) = gxg^{-1} := a_g(x).$$

The identity is a fixed point of this action. Hence, the map

$$g \mapsto da_g|T_eG$$

is a representation of G into $\mathrm{Aut}(\mathfrak{g})$. This is called the *adjoint representation* and it is denoted by

$$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}).$$

The differential of the adjoint representation will be denoted by ad, $d(\mathrm{Ad}) = \mathrm{ad}$, and $\mathrm{Ad}(g)$ will be denoted by Ad_g.

Applying Theorem 2.3.9 to the automorphism a_g of G, we obtain the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{a_g} & G \\ {\scriptstyle\exp}\uparrow & & \uparrow{\scriptstyle\exp} \\ \mathfrak{g} & \xrightarrow{\mathrm{Ad}_g} & \mathfrak{g} \end{array}$$

In other words,

$$\exp t \operatorname{Ad}_g(X) = g \exp(tX) g^{-1}.$$

Proposition 2.3.10. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let $X, Y \in \mathfrak{g}$. Then*

$$\operatorname{ad}(X)Y = [X, Y].$$

2.3.2 Controllability Results

The study of control systems on Lie groups can be traced back to the work by Brockett [Bro72]. In their article [JS72], Jurdjevic and Sussmann study the controllability properties of a right-invariant system Σ on a Lie group G of the form

$$\dot{x} = X_0(x) + \sum_{j=1}^{k} u_j X_j(x),$$

where $X_0, \cdots, X_k$ are right-invariant vector fields on G and $u = (u_1, \cdots, u_k)$ belongs to the class of admissible controls $\mathcal{U}$. In particular, $\mathcal{U}_u$ denotes the class of locally bounded measurable real functions.

A necessary condition for controllability is that the system have the accessibility property. Another necessary condition for controllability for this type of systems is that the reachable set from the identity of the group $\mathcal{R}(e)$ be a subgroup of G. The following results are included in [JS72].

Lemma 2.3.11. *Let Σ be a right-invariant control system on G, then*

(i) $\mathcal{R}(e)$ *is path-connected.*

(ii) $\mathcal{R}(e)$ *is a semigroup.*

Theorem 2.3.12. *If $\mathcal{R}(e)$ is a subgroup of G, then it is the subgroup of G whose Lie algebra L is the subalgebra generated by $X_0, \cdots, X_k$.*

A consequence of this theorem is that the system Σ is controllable if and only if (i) $\mathcal{R}(e)$ is a subgroup, (ii) G is connected, and (iii)L is the Lie algebra $\mathfrak{g}$. This is interpreted in terms of controllability as follows.

Theorem 2.3.13. *A necessary condition for controllability of the right-invariant system Σ is that G be connected and that $L = \mathfrak{g}$. If G is compact, or if the system is homogeneous, the condition is also sufficient.*

Theorem 2.3.14. *If G is connected, $L = \mathfrak{g}$, $\mathcal{U} = \mathcal{U}_u$, and X_0 belongs to the Lie algebra generated by $X_1, \cdots, X_k$, then Σ is controllable.*

Clearly, some of these results resemble the theorems shown in the previous sections for systems on a vector space V.

In many situations the system under consideration is not an invariant system on the Lie group, but induces an invariant system on a Lie group G, which acts on a manifold M.

Definition 2.3.15. *Let G be a Lie group, and M a smooth manifold. The Lie group G is said to act on M if there exists a smooth mapping $\eta : G \times M \to M$ that satisfies*

1. *$\eta(g_1 g_2, m) = \eta(g_1, \eta(g_2, m))$ for any $g_1, g_2 \in G$ and any $m \in M$.*

2. *$\eta(e, m) = m$ for each $m \in M$.*

The action is also denoted by $\eta(g, m) = gm$. For each $g \in G$, define $\eta_g : M \to M$ as $\eta_g(m) = \eta(g, m)$. Then the mapping

$$G \to \mathrm{Diff}(M), \quad g \mapsto \eta_g$$

is a group homomorphism. In particular, if A is an element of $\mathfrak{g}$, then $\{\eta_{\exp(tA)}\}$ is a smooth one-parameter group of diffeomorphisms on M. Let X_A denote its infinitesimal generator. The correspondence

$$\mathfrak{g} \to V^{\infty}, \quad A \mapsto X_A$$

is a Lie algebra homomorphism. Therefore, the family $\Omega = \{X_A : A \in \mathfrak{g}\}$ is a finite-dimensional Lie algebra of complete smooth vector fields on M.

The action is called *transitive* if given any $m, n \in M$, there is a $g \in G$ such that $\eta(g, m) = n$. G is said to act *transitively* on M if each orbit $\{\eta(g, m) : g \in G\}$ is equal to M.

Theorem 2.3.16. *Let $\mathcal{F}$ be an arbitrary subset of $\mathfrak{g}$, and let Ω as above. Denote by H the subgroup of G given by $H = \langle\{\exp(At) : A \in \mathcal{F}, t \in \mathbb{R}\}\rangle$. Then the orbit of Ω through each $m \in M$ is given by the action of H on m, that is,*

$$G(\Omega)(m) = \{\eta_h(m) : h \in H\}.$$

In particular, if $\mathcal{F}$ generates $\mathfrak{g}$ and G acts transitively, then $G(\Omega)(m) = M$ for each $m \in M$.

Manifolds that admit transitive actions of Lie groups are called *homogeneous spaces*. They are precisely equal to the class of manifolds that can be realized as the group of the quotients of Lie groups. In fact, if G acts transitively on M, then the mapping $gH \mapsto \eta_g(m)$ realizes M as G/H. Here H denotes the isotropy group $H = \{g \in G : \eta_g(m) = m\}$ at the point $m \in M$.

To prove the main results of Chapter 4 the idea is to study controllability of the system on the homogeneous space M determined by the Lie group G that defines the state space. Below is a short description of the relationship between controllability at M and transitivity of the action of G on M.

Consider a system Σ on G. Suppose G acts transitively on a manifold M, then $S = \mathcal{R}(e)$ is a subsemigroup of G that also acts on M. In fact, the reachable sets of the original system on M are the orbits defined by S. If S acts transitively on M, then $SM = M$, i.e., the system Σ is controllable on M. In other words, controllability questions can be regarded as questions about transitivity of the action of S on M. This argument is commonly used to treat controllability question using Lie theory of semigroups (see for example [LM02]).

Chapter 3
Linear Control Systems on Lie Groups

This chapter begins with the definition of linear control systems on Lie groups. This definition was introduced by Ayala and Tirao in [AT99]. Section 3.2 presents new results for this type of systems that relate to some of those results discussed in the previous chapter on Section 2.2 for linear control systems on a vector space.

3.1 Definition

In this chapter, G will denote a Lie group with Lie algebra $\mathfrak{g}$. Elements on the Lie algebra are denoted by Y. The value of the vector field at any $g \in G$ will be denoted by $\vec{Y}(g)$. The Lie algebra of all C^∞ vector fields is $V^\infty(G)$. The normalizer of $\mathfrak{g}$ in the Lie algebra $V^\infty(G)$ is denoted $\mathrm{norm}_{V^\infty(G)}(\mathfrak{g})$ and it is defined by

$$\mathrm{norm}_{V^\infty(G)}(\mathfrak{g}) := \{X \in V^\infty(G) : [X, Y] \in \mathfrak{g}, \text{for all } Y \in \mathfrak{g}\}$$

A linear control system Σ on G is defined by

$$\dot{x} = \vec{X}(x) + \sum_{j=1}^{k} u_j \vec{Y}_j(x), \tag{3.1}$$

where the drift vector field $\vec{X}$ has a singularity and is an element of the normalizer of $\mathfrak{g}$, the input functions $u = (u_j)$ belong to a class of admissible controls, and the control vectors $\vec{Y}_j$ are left-invariant vector fields. The Lie algebra $\mathfrak{g}$ will be identified with the left-invariant vector fields.

The above definition generalizes the notion of linear control systems on $\mathbb{R}^n$. This is proved in the example below.

Example 3.1.1. *Linear Systems on* $\mathbb{R}^n$

A linear system on the Euclidean vector space $\mathbb{R}^n$ is given by,

$$\dot{x} = \boldsymbol{A}x + \boldsymbol{B}u,$$

where $\boldsymbol{A}$ is an $n \times n$ real matrix, $\boldsymbol{B}$ is an $n \times k$ real matrix, and u is an k-dimensional column control vector. Let $(b_j)_{j=1,\cdots,k}$ denote the columns of $\boldsymbol{B}$, then the system is equivalent to

$$\dot{x} = \boldsymbol{A}x + \Sigma_{j=1}^{k} u_j b_j \tag{3.2}$$

Lemma 3.1.2. *$norm_{X(G)}(\mathbb{R}^n) = \{$linear vector fields on $\mathbb{R}^n\}$*

By the above lemma, an element in the normalizer with a singularity can be written as $X(x) = \boldsymbol{A}x + d$ for some $\boldsymbol{A} \in M_n(\mathbb{R})$. In particular, X has a singularity if and only if d belongs to the image of A. Since each left-invariant vector field in $\mathbb{R}^n$ is a constant vector field, putting $Y^j(x) \equiv b_j$, the system (3.2) is a linear control system of the form of (3.1) on the Lie group $G = \mathbb{R}^n$.

Next, a proof of the lemma is presented

Proof. Let X be a vector field on $\mathbb{R}^n$ and let U be an element of the Lie algebra $\mathbb{R}^n$ of $\mathbb{R}^n$, i.e., a constant vector field. In terms of local coordinates,

$$\vec{X}(x) = \Sigma_{i=1}^{n} a_i(x)(\frac{\partial}{\partial x_i}),$$

and

$$\vec{U}(x) = \Sigma_{i=1}^{n} b_i(\frac{\partial}{\partial x_i}).$$

The Lie bracket $[X, U] \in \mathfrak{g}$ if $[X, U]$ is a constant vector field. The Lie bracket $[X, U]$, can be written as

$$[X, U] = \Sigma_{i=1}^{n} c_i(x)(\frac{\partial}{\partial x_i}),$$

where $c_i(x) = \Sigma_{j=1}^{n} b_j(\frac{\partial a_j}{\partial x_i})$. Hence, the Lie bracket is a constant vector field if each c_i is constant, but then each a_j is linear, $a_j(x) = \Sigma_{i=1}^{n} a_{ij}x_i + d_j$. Therefore, the vector field $\vec{X}$ is a linear vector field that can be written as

$$\vec{X}(x) = Ax + d$$

for A given by, $A = (a_{ij})_{\{i,j \leq n\}}$ and $d = (d_i)$. $\square$

A linear system on $\mathbb{R}^n$ *is invariant only when it is driftless,* i.e., $A \equiv 0$. *Thus, the known results for right-invariant systems* (2.3) *do not apply to linear control systems in the general form.*

Assume that the drift vector field $\vec{X}$ in (3.1) has the additional restriction of being an infinitesimal automorphism. It is important to note that under this hypothesis $\vec{X} \in \mathrm{norm}_{V^\infty(G)}(\mathfrak{g})$ with the singularity at the identity of the group e. (See Section 2 of [AT99]).

3.2 Infinitesimal Automorphism Properties

Definition 3.2.1. *An* ***infinitesimal automorphism*** X *on* G *is a vector field with flow* $X_t \in \mathrm{Aut}(G)$. *That is,* X_t *is a diffeomorphism that preserves the group operation,* i.e.:

$$X_t(g_1 \cdot g_2) = X_t(g_1) \cdot X_t(g_2).$$

Example 3.2.2. *[Mar81] Let* G *be the matrix Lie group* $GL_n(\mathbb{R})$,i.e., *the group of all linear transformations on* $\mathbb{R}^n$, *and let* $\mathfrak{gl}_n(\mathbb{R})$ *denote its Lie algebra, that is, the vector space of all* $n \times n$ *real matrices. Define the vector field* $\vec{X}$ *by*

$$\vec{X}(P) := AP - PA,$$

for some $A \in \mathfrak{gl}_n(\mathbb{R})$ *and all* $P \in G$. *Then* X *is the infinitesimal generator of the 1-parameter group* X_t *of the form*

$$X_t(g) = e^{tA}\ g\ e^{-tA}.$$

The simple calculation

$$\begin{aligned} X_t(g \cdot h) &= e^{tA}\ g \cdot h\ e^{-tA} \\ &= e^{tA}\ g\ e^{tA} \cdot\ \ e^{-tA} h\ e^{-tA} \\ &= X_t(g) \cdot X_t(h), \end{aligned}$$

shows that X is an infinitesimal automorphism of G.

The following results are generalizations of properties known for linear systems in $\mathbb{R}^n$. The Lemma below shows two useful properties for the flow of an infinitesimal automorphism. The proofs are obtained using basic automorphism properties. The first one states that the inverse of the flow at a point in G is the flow at the inverse of the given point. The second one shows an expression for the vector field at a product of two points with respect to the value at each point. With these properties Proposition 3.2.4 can be proved. This proposition relates the reachable set of Σ from the identity $\mathcal{R}(e, t)$ at time t and the reachable set from any point at that time.

3.2.1 The Flow Property

Lemma 3.2.3. *If $\vec{X}$ is an infinitesimal automorphism and g_0, g, and h are elements in G, then the following statements hold true:*

(i) $(X_t(g_0))^{-1} = X_t(g_0^{-1})$

(ii) $\vec{X}(g.h) = d\rho_h(g)\vec{X}(g) + d\lambda_g(h)\vec{X}(h)$, *where ρ and λ denote right and left multiplication respectively.*

Proof. i) Since $e = X_t(e) = X_t(g_0.g_0^{-1}) = X_t(g_0)X_t(g_0^{-1})$, the desired result is obtained.

ii) Differentiating $X_t(gh) = X_t(g)X_t(h)$ yields

$$\frac{d}{dt}X_t(gh) = d\rho_{X_t(h)}X_t(g)\frac{d}{dt}X_t(g) + d\lambda_{X_t(g)}X_t(h)\frac{d}{dt}X_t(h);$$

evaluating at $t = 0$ gives $\vec{X}(g.h) = d\rho_h(g)\vec{X}(g) + d\lambda_g(h)\vec{X}(h)$. $\square$

For linear systems on $\mathbb{R}^n$ it is well known (see Proposition 2.2.7) that the reachable set of $x_0 \in \mathbb{R}^n$ is the affine subspace $\exp(tA)x_0 + V$ where the subspace V is nothing but the reachable set from the origin 0. A similar result holds true for linear systems on Lie groups in general as is shown in the next section.

3.2.2 The Reachable Set Property

Proposition 3.2.4. *Let Σ be a linear system on the Lie group G. Then*

$$(\forall g_0 \in G) \quad \mathcal{R}(g_0, t) = \mathcal{R}(e, t)X_t(g_0) = X_t(g_0)\mathcal{R}(e, t).$$

Proof. Let $\gamma(t) \in \mathcal{R}(g_0, t)$, that is, γ is a trajectory of (3.1) such that $\gamma(0) = g_0$. Let $\alpha(t) = X_t(g_0^{-1})$ and define

$$\beta(t) := \gamma(t)(X_t(g_0))^{-1}.$$

Then $\beta(0) = e$ and using part (i) of (3.2.3) $\beta(t) = \gamma(t).\alpha(t)$. Now differentiating with respect to t gives

$$\dot{\beta}(t) = d\lambda_{\alpha(t)}(\gamma(t))\dot{\gamma}(t) + d\rho_{\gamma(t)}(\alpha(t))\dot{\alpha}(t),$$

and using that γ is a trajectory of (3.1) and that the control vectors are left-invariant

$$\begin{aligned}
\dot{\beta}(t) &= d\rho_{\gamma(t)}(\alpha(t))\vec{X}(\alpha(t)) + d\lambda_{\alpha(t)}(\gamma(t))[\vec{X}(\gamma(t)) + \sum_{j=1}^{k} u_j d\lambda_{\gamma(t)}(\mathbf{1})Y_j] \\
&= d\rho_{\gamma(t)}(\alpha(t))\vec{X}(\alpha(t)) + d\lambda_{\alpha(t)}(\gamma(t))\vec{X}(\gamma(t)) + \sum_{j=1}^{k} u_j d\lambda_{\alpha(t)}(\gamma(t))d\lambda_{\gamma(t)}(\mathbf{1})Y_j \\
&= \vec{X}(\beta(t)) + \sum_{j=1}^{k} u_j d\lambda_{(\alpha(t)\gamma(t))}(\mathbf{1})Y_j
\end{aligned}$$

where part (ii) of (3.2.3) has also been used.

Therefore,

$$\dot{\beta} = \vec{X}(\beta) + \sum_{j=1}^{k} u_j \vec{Y_j}(\beta),$$

so $\beta(t)$ is a trajectory of (3.1) with $\beta(0) = e$, i.e.: $\beta(t)$ is in $\mathcal{R}(e,t)$ and hence, $\gamma(t) \in \mathcal{R}(e,t)X_t(g_0)$. Thus $\mathcal{R}(g_0,t) \subseteq \mathcal{R}(e,t)X_t(g_0)$.

For the reverse inclusion $\mathcal{R}(g_0,t) \supseteq \mathcal{R}(e,t)X_t(g_0)$ let $\beta(t)$ be the trajectory of (3.1) starting at $\beta(0) = e$ and $\gamma(t) = \beta(t)X_t(g_0)$, then $\gamma(t) \in \mathcal{R}(g_0,t)$.

In order to prove the second equality the same proof works for

$$\tilde{\beta}(t) := (X_t(g_0))^{-1}\,\gamma(t).$$

$\square$

Chapter 4
Local Controllability

This chapter contains the main results of this work. Under certain conditions on the system Σ, Ayala and Tirao [AT99] showed local controllability of the system Σ at the group identity e. In Section 4.3 an alternate proof of the result is given. More important, using the machinery developed here, which is based on a Lie-wedge approach, the result extended in Section 4.4. Briefly, local controllability is obtained as follows. The control system Σ will be lifted to a right-invariant control system $\widehat{\Sigma}$ on an augmented Lie group $\widehat{G}$. A closed subgroup T of $\widehat{G}$ will be defined such that controllability properties of Σ on the homogeneous manifold $M = \widehat{G}/T$ correspond to controllability properties of Σ on G.

4.1 The Augmented System

From this chapter on, the drift vector field $\vec{X}$ will be an infinitesimal automorphism of G (see Section 3.2). Σ will denote the linear control system as defined in Section 3.1, i.e.,

$$\dot{x} = \vec{X}(x) + \sum_{j=1}^{k} u_j \vec{Y_j}(x), \tag{4.1}$$

where the drift vector field $\vec{X}$ is an infinitesimal automorphism (ref. Definition 3.2.1), the input functions $u = (u_j)$ belong to the class of piecewise constant functions, and the control vectors $\vec{Y_j}$ are left-invariant vector fields.

Define $\widehat{G}$ as the semidirect product $G \rtimes_X \mathbb{R}$, i.e., the set of pairs (g, t) with $g \in G$ and $t \in \mathbb{R}$, and with group multiplication given by

$$(g_1, t_1) \cdot (g_2, t_2) = (g_1 X_{t_1}(g_2), t_1 + t_2)$$

It can easily be proved that $(\widehat{G}, \cdot)$ is a group. In particular, the property $X_t(e) = e$ of the flow of the infinitesimal automorphism X, is used to show that the identity of $\widehat{G}$ is $\mathbf{1} = (e, 0)$, and the homomorphism property $X_t(g_1 g_2) = X_t(g_1) X_t(g_2)$ is used to show associativity and to show that an element (g, t) in $\widehat{G}$ has inverse $(X_t(g^{-1}), -t)$. Moreover, the group $\widehat{G}$ is a Lie group (see [Bou98], III.1.4) with Lie algebra $\widehat{\mathfrak{g}} \simeq \mathfrak{g} \times \mathbb{R}$ as a vector space.

Now, let $\vec{\widehat{X}}$, and $\vec{\widehat{Y}}_j$ be the right-invariant vector fields defined at the identity as $\widehat{X} = (0, 1)$ and $\widehat{Y}_j = (Y_j, 0)$. The corresponding flows are $\exp(t\vec{\widehat{X}}) = (e, t)$ and $\exp(t\vec{\widehat{Y}}_j) = (\exp tY^j, 0)$, where $\exp : \widehat{\mathfrak{g}} \to \widehat{G}$ is the exponential map.

Clearly if $p \in G$ then $(p, 0) \in \widehat{G}$. The following calculations show the relationship of the lifted vector fields $\widehat{X}$ and $\widehat{Y}_j$ with the vector fields X and Y^j defining the system (4.1).

$$
\begin{aligned}
\vec{\widehat{X}}(p, 0) &= \left.\frac{d}{dt}\right|_{t=0} ((e, t).(p, 0)) \\
&= \left.\frac{d}{dt}\right|_{t=0} (X_t(p), t) \\
&= (\vec{X}(p), 1)
\end{aligned}
$$

$$
\begin{aligned}
\vec{\widehat{Y}}_j(p, 0) &= \left.\frac{d}{dt}\right|_{t=0} ((\exp tY^j, 0).(p, 0)) \\
&= \left.\frac{d}{dt}\right|_{t=0} (\exp tY^j p, 0) \\
&= (d\rho_p(e) Y^j, 0)
\end{aligned}
$$

The right-invariant augmented system $\widehat{\Sigma}$ on $\widehat{G}$ is defined by

$$
\dot{\gamma} = \vec{\widehat{X}}(\gamma) + \sum_{j=1}^{k} u_j \vec{\widehat{Y}}_j(\gamma), \tag{4.2}
$$

where $\gamma \in \widehat{G}$, $u = (u_j)$ are piecewise constant functions, $\widehat{X}$ and $\widehat{Y}_j$ are in $\widehat{\mathfrak{g}}$.

Example 4.1.1. *In the familiar case of the linear control system Σ on $\mathbb{R}^n$*

$$\dot{x} = Ax + \textstyle\sum_{j=1}^{k} u_j b_j,$$

the Lie group $\widehat{G}$ is the semidirect product $\widehat{G} = \mathbb{R}^n \rtimes_A \mathbb{R}$, with group multiplication

$$(x, s).(y, t) = (x + e^{sA}y, s + t).$$

The Lie algebra $\widehat{\mathfrak{g}}$ is isomorphic, as a vector space, to $\mathbb{R}^n \times \mathbb{R}$. So each $\widehat{X} \in \widehat{\mathfrak{g}}$ can be identified with $(\xi, \tau) \in \mathbb{R}^n \times \mathbb{R}$. Accordingly, $\widehat{V}_i = (e_i, 0)$ with $1 \leq i \leq n$ together with $\widehat{A} = (0, 1)$ form a basis of $\widehat{\mathfrak{g}}$. For $g = (x, s) \in \widehat{G}$ a calculation similar to the general case leads to

$d\rho_g(\mathbf{1})\widehat{V}_i = (e_i, 0)$, *and*

$d\rho_g(\mathbf{1})\widehat{A} = (Ax, 1)$

Now taking $\widehat{Y}_i = (b_i, 0) \in \widehat{\mathfrak{g}}$, the augmented system $\widehat{\Sigma}$ is

$$\dot{g} = d\rho_g(\mathbf{1})(\widehat{A} + \sum_{j=1}^{k} u_j \widehat{Y}_j)$$

which can also be written as

$$\dot{g} = \vec{\widehat{A}}(g) + \sum_{j=1}^{k} u_j \vec{\widehat{Y}}_j(g),$$

furthermore in coordinates this is

$$\begin{pmatrix} \dot{x} \\ \dot{s} \end{pmatrix} = \begin{pmatrix} Ax + \sum_{j=1}^{k} u_j b_j \\ 1 \end{pmatrix}$$

where it is clear that the system Σ is the projection onto the first coordinate of the system $\widehat{\Sigma}$.

4.2 The Fundamental Hypothesis and the Lie Wedge

Let $\mathcal{H} \subseteq \mathfrak{g}$ denote the Lie subalgebra generated by the control vector fields, i.e.,

$$\mathcal{H} = \langle Y_1, \ldots, Y_k \rangle$$

As shown in [AT99] that $\mathfrak{g}$ is $\mathrm{ad}^i(X)$-invariant for each $i \geq 0$

Proof. First note that for $Y \in \mathfrak{g}$

$$[X, Y](e) = -Y(e)X = -\left.\frac{d}{ds}\right|_{s=0} X_{\exp(sY)}.$$

On the other hand,

$$\begin{aligned}
[X, Y](x) &= \left.\frac{d}{dt}\right|_{t=0} (d\exp(-tX))(Y_{X_t(x)}) = \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} X_{-t} \circ Y_s \circ X_t(x) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} X_{-t}(Y_s(X_t(x))) = \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} X_{-t}(X_t(x)\exp(sY)) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} x.X_{-t}(\exp(sY)) = \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} \lambda_x \circ X_{-t}(\exp(sY)) \\
&= -\left.\frac{d}{ds}\right|_{s=0} d\lambda_x X_{\exp(sY)} = d\lambda_x [X, Y](e).
\end{aligned}$$

$\square$

Let $\mathcal{H}^X$ denote the following subspace of $\mathfrak{g}$

$$\mathcal{H}^X = \mathrm{span}\{\mathrm{ad}^i(X)(Y^j) : i \geq 0,\ 1 \leq j \leq k\}$$

Here $\mathrm{ad} = d(\mathrm{Ad})$ is the differential of the adjoint representation; $\mathrm{ad}^0 X(Y) = Y$, $\mathrm{ad}^1 X(Y) = [X, Y]$, $\mathrm{ad}^2 X(Y) = [X, [X, Y]]$, ..., and so on. Thus $\mathcal{H}^X$ is the smallest $\mathrm{ad}(X)$-invariant subspace of $\mathfrak{g}$ containing $\mathcal{H}$. Consequently, only brackets of order $i \leq \dim \mathfrak{g}$ are needed.

Definition 4.2.1. *The system* Σ is said to satisfy the **ad-rank condition** *if*

$$\dim(\mathcal{H}^X) = \dim(G).$$

Remark 4.2.2. *The previous definition is a generalization of the Kalman condition in R^n. For the system $\dot{x} = Ax + ub$, it follows*

$\operatorname{ad}^0 X(Y) = b$, $\operatorname{ad} X(Y) = Ab$, $\operatorname{ad}^2 X(Y) = A^2 b, \ldots$, *and so on. Then,* $\dim(\mathcal{H}^X) = \operatorname{rank} M$*, where M is the controllability matrix (see 2.2.1). Hence, the Kalman condition:* $\operatorname{rank} M = n$ *is equivalent to the ad-rank condition.*

Definition 4.2.3. *A* ***wedge*** *W in a topological vector space V over $\mathbb{R}$ is a subset with the following properties:*

$$\overline{W} = W, \quad W + W = W, \quad \text{and } \mathbb{R}^+ W = W,$$

where $\overline{W}$ denotes the closure of W in V.

The ***edge*** *of the wedge is the largest subspace contained in W denoted by $H(W) := W \cap -W$.*

A ***Lie Wedge*** *is a wedge W in a real Lie algebra $\mathfrak{g}$ which is invariant under the adjoint action of its edge:*

$$e^{adH(W)} W = W.$$

In the augmented setting, define $T = \{e\} \rtimes \mathbb{R}$, which is also equal to $\exp(\mathbb{R}\widehat{X})$. T is a closed subgroup of $\widehat{G}$ since $T = f^{-1}(\{e\})$ where the continuous map f is defined as $f : (g, t) \mapsto X_t(g)$. Therefore, the set of left cosets modulo T yields a homogeneous manifold: $M = \widehat{G}/T$ (see [War83]). There is a natural action of $\widehat{G}$ on M, namely,

$$\widehat{G} \times M \to M, \quad ((g_1, t_1), (g, t)T) \mapsto (g_1 X_{t_1}(g), t)T = (g_1 X_{t_1}(g), 0)T.$$

Clearly, this action is smooth; moreover, it is a transitive action. The map η defined by

$$\eta : \widehat{G} \times G \to G, \quad ((g_1, t_1), g) \mapsto g_1 X_{t_1}(g),$$

is a transitive action of the Lie group $\widehat{G}$ on G. Furthermore, η is a flow as can be seen from the calculation below.

$$\begin{aligned}\eta((g_1,t_1)\cdot(g_2,t_2),g) &= ((g_1,t_1)\cdot(g_2,t_2))g = (g_1X_{t_1}(g_2),t_1+t_2)g\\ &= g_1X_{t_1}(g_2)X_{t_1+t_2}(g) = g_1X_{t_1}(g_2X_{t_2}(g)) = (g_1,t_1)(g_2X_{t_2}(g))\\ &= (g_1,t_1)((g_2,t_2)g) = \eta((g_1,t_1),\eta((g_2,t_2),g)).\end{aligned}$$

Let $\eta_{(g_1,t_1)}$ be defined as in Section 2.3.2, that is,

$$\eta_{(g_1,t_1)} : G \to G, \quad g \mapsto g_1X_{t_1}(g).$$

Note that T is the isotropy group of the identity $e \in G$. Since $M = \widehat{G}/T$, then the mapping

$$M \to G, \quad (g_1,t_1)T \mapsto \eta_{(g_1,t_1)}(e) = g_1$$

is a $\widehat{G}$-flow isomorphism; i.e., an isomorphism of transformation groups.

Let $\widehat{\mathfrak{h}} = \left\langle \widehat{Y}_1,\ldots,\widehat{Y}_k \right\rangle$ denote the subalgebra generated by the control vector fields and let $\widehat{W} \subseteq \widehat{\mathfrak{g}}$ denote the smallest Lie wedge containing $\mathbb{R}^+\widehat{X} + \widehat{\mathfrak{h}}$. Let $\widehat{S} \subseteq \widehat{G}$ denote the semigroup $\widehat{S} = \langle \exp \widehat{W} \rangle$. Moreover, denote with π the natural projection

$$\pi : \widehat{G} \to M$$

which induces a linear surjection

$$d\pi(\mathbf{1}) : T_{\mathbf{1}}\widehat{G} \to T_{x_0}M$$

where $x_0 = \pi(\mathbf{1})$ is the base point in M. $T_{x_0}M$ is isomorphic to $\widehat{\mathfrak{g}}/\mathbb{R}\widehat{X}$ so we can identify $d\pi(\mathbf{1})$ with the quotient map

$$\mathrm{pr} : \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}/\mathbb{R}\widehat{X}.$$

From the above discussion it follows that local controllability at points in M is equivalent to local controllability at the corresponding points on G.

The following theorem will be needed in the following sections.

Theorem 4.2.4. *If $\widehat{W} + \mathbb{R}\widehat{X} = \widehat{\mathfrak{g}}$, then $\widehat{X} \in \mathrm{Int}(\widehat{W})$.*

Proof. Since $\widehat{X} \in \widehat{W}$ then $\mathbb{R}\widehat{X} \subseteq \widehat{W} - \widehat{W}$. By hypothesis $\widehat{W} + \mathbb{R}\widehat{X} = \widehat{\mathfrak{g}}$. So $\widehat{W} - \widehat{W} = \widehat{\mathfrak{g}}$, which implies that $\mathrm{Int}(\widehat{W}) \neq \emptyset$.

Suppose $\widehat{X}$ is in the boundary of $\widehat{W}$. By convexity of $\widehat{W}$ there would be a hyperplane through $\widehat{X}$ supporting $\widehat{W}$. This implies that $\widehat{W}$ is contained in a halfspace. Since also $\mathbb{R}\widehat{X}$ is contained in the hyperplane, $\mathbb{R}\widehat{X}+\widehat{W}$ is contained in the halfspace, a contradiction. Therefore, $\widehat{X} \in \mathrm{Int}(\widehat{W})$. $\square$

4.3 Local Controllability at the Identity

The following lemma, which can be found in Chapter 3 of [HN93], will be used to prove Proposition 4.3.4.

Lemma 4.3.1. *Let S be a subsemigroup of a connected topological group G. Then the following assertions hold:*

(i) $\mathrm{Int}(S)$ *is a semigroup ideal.*

(ii) *If $e \in \overline{\mathrm{Int}(S)}$, then we have that*

$$S \subseteq \overline{\mathrm{Int}(S)} \quad \textit{and} \quad \mathrm{Int}(S) = \mathrm{Int}(\overline{S})$$

Other useful results from Lie theory of semigroups, c.f. [HHL89], are the following:

Lemma 4.3.2. *Let S be a closed subsemigroup of a connected Lie group G. Then the tangent wedge $L(S) = \{X : \exp \mathbb{R}^{+}X \subseteq S\}$ is a Lie wedge. Moreover, $\langle \exp L(S) \rangle \subseteq S$.*

Proposition 4.3.3. *Suppose that a finite dimensional Lie algebra $\mathfrak{g}$ is algebra generated by a subset W. Then the reachable set from the identity $\mathcal{R}_W(e)$ has dense interior.*

The following proposition is an important result linking local controllability of Σ at the identity e and a property of the Lie wedge $\widehat{W}$.

Proposition 4.3.4. *If $\widehat{W} + \mathbb{R}\widehat{X} = \widehat{\mathfrak{g}}$ then Σ is locally controllable at e.*

Proof. If $\widehat{W} + \mathbb{R}\widehat{X} = \widehat{\mathfrak{g}}$ then $d\pi(\mathbf{1})(\widehat{W}) = T_{x_0}M$. Then there exist $\widehat{X}_1, \dots \widehat{X}_k \in \widehat{W} (k = 1 + \dim M)$ such that 0 is in the interior of the convex hull $\operatorname{conv}(X_1, \dots, X_k)$ where $X_j = d\pi(\mathbf{1})(\widehat{X}_j)$. In particular, $X_2 - X_1 \dots X_k - X_1$ form a basis of $T_{x_0}M$.

We also know that $\widehat{X} \in \operatorname{Int} \widehat{W}$ by Theorem 4.2.4 and $\widehat{X} \in \widehat{W} \cap \ker \mathrm{pr}$. This means there is $\varepsilon > 0$ such that

$\widehat{X} + (-\varepsilon, \varepsilon)(\widehat{X}_2 - \widehat{X}_1) + \dots + (-\varepsilon, \varepsilon)(\widehat{X}_k - \widehat{X}_1) \subseteq \widehat{W}$.

The map $F : \mathbb{R}^{k-1} \to M$,

$F(t_2, \dots, t_k) = \exp(\widehat{X} + t_2(\widehat{X}_2 - \widehat{X}_1)) \cdots \exp(\widehat{X} + t_k(\widehat{X}_k - \widehat{X}_1)).x_0$

satisfies that the image of $F'(0)$ contains $X_2 - X_1, \dots, X_k - X_1$, a basis of $T_{x_0}M$; therefore, rank $F'(0) = k - 1 = \dim M$. By the inverse map theorem, F maps any 0-neighborhood in $\mathbb{R}^{k-1}$ onto a neighborhood of $F(0) = x_0$.

The following argument ensures that there is an open neighborhood of x_0 contained in the reachable set of x_0. This is achieved by considering $\Omega = \{\widehat{X} + \mathbb{R}\widehat{Y}_j\}_{j=1,\cdots,k}$ as a set of generators of $\widehat{W}$ (as a Lie wedge). Let $S(\Omega)$ be the semigroup $S(\Omega) = \langle \exp \mathbb{R}^+ \Omega \rangle$, then by Lemma 4.3.1 both semigroups $S(\Omega)$ and $\widehat{S}$ have the same interior, i.e.: $\operatorname{Int} S(\Omega) = \operatorname{Int} \widehat{S}$. Here Lemma 4.3.2 has been used to get that $\widehat{W} = L(\overline{S(\Omega)})$ and therefore, $\widehat{S} = \overline{S(\Omega)}$, and from Proposition 4.3.3 we have that $\mathbf{1} \in \overline{\operatorname{Int} \widehat{S}}$. Hence, it does not matter what set of generators one has chosen. This completes the proof. $\square$

Remark 4.3.5. *The Lie bracket of $\widehat{X}$ and $\widehat{Y}$ is given by*

$$[\widehat{X}, \widehat{Y}] = [(0,1),(Y,0)] = ([X,Y],0) \in \widehat{g}$$

this can be seen by calculation or using known results from Lie theory of semigroups on semidirect products (see Chapter V of [HHL89]*).*

Theorem 4.3.6. *If Σ satisfies the ad-rank condition, then $pr(\widehat{W}) = \mathfrak{g}$. In particular, Σ is locally controllable at e.*

Proof. For $t > 0, \epsilon \in \mathbb{R}$, and $\widehat{Y}_j \in \widehat{W}$

$$\exp(t\widehat{X})\exp(\epsilon\widehat{Y}_j) \in \widehat{S};$$

hence,

$$\pi(\exp(t\widehat{X})\exp(\epsilon\widehat{Y}_j)) = \pi(\exp(t\widehat{X})\exp(\epsilon\widehat{Y}_j)\exp(-t\widehat{X})) \in \pi(\widehat{S})$$

which, in turn, gives

$$(\forall \epsilon \in \mathbb{R}, t > 0) \quad \pi(\exp(\epsilon\ e^{t\,\mathrm{ad}\,\widehat{X}}\widehat{Y}_j)) \in \pi(\widehat{S})$$

Now taking the derivative with respect to ϵ and evaluating $\epsilon = 0$ gives

$d\pi(\mathbf{1})(e^{t\,\mathrm{ad}\,\widehat{X}}\widehat{Y}_j) \in d\pi(\mathbf{1})(\widehat{W}) = \mathrm{pr}(\widehat{W}) = \widehat{W} + \mathbb{R}\widehat{X}$. The same calculation for negative ϵ gives $d\pi(\mathbf{1})(-e^{t\,\mathrm{ad}\,\widehat{X}}\widehat{Y}_j) \in \mathrm{pr}(\widehat{W})$ for all $t > 0$ whence

$d\pi(\mathbf{1})(\pm e^{t\,\mathrm{ad}\,\widehat{X}}\widehat{Y}_j) \in \mathrm{pr}(\widehat{W})$ for all $t > 0$; therefore, using Remark 4.3.5

$\pm e^{t\,\mathrm{ad}(X)}Y_j \in \mathrm{pr}(\widehat{W})$ for all $t > 0$ and hence, differentiating with respect to t l-times and evaluating at $t = 0$

$$(\forall l, j \in \mathbb{N}) \quad \pm\,\mathrm{ad}(X)^l Y_j \in \overline{\mathrm{pr}(\widehat{W})},$$

thus $\mathcal{H}^X \subseteq \overline{\mathrm{pr}(\widehat{W})}$. Since $\dim(\mathcal{H}^X) = \dim(\mathfrak{g})$ then $\mathrm{pr}(\widehat{W})$ is dense in $T_{x_0}M$. But $\mathrm{pr}(\widehat{W})$ is convex, so $\mathrm{pr}(\widehat{W}) = T_{x_0}M$. This implies in particular, that $\widehat{W} + \mathbb{R}\widehat{X} = \widehat{\mathfrak{g}}$. Therefore, local controllability at e follows from Proposition 4.3.4 above. $\square$

Example 4.3.7. *To illustrate the previous result the calculations for a linear control system in R^n are presented. Recall from Example* (4.1.1)

$\widehat{G} = \mathbb{R}^n \rtimes_A \mathbb{R}$ *with Lie algebra* $\widehat{g} = \mathbb{R}^n \times \mathbb{R}$,

$\widehat{A} = (0, 1)$, *and* $\widehat{Y} = (b, 0) \in \widehat{\mathfrak{g}}$.

Here $\widehat{\mathfrak{h}} = \langle (b, 0) \rangle \subseteq \widehat{\mathfrak{g}}$ *and* $\mathcal{H}^X = \operatorname{span}\{\operatorname{ad}^i(A)b : i \geq n\}$. *Proceeding like in the proof of Theorem 4.3.6, the semigroup* $\widehat{S} = \langle \exp \widehat{W} \rangle$ *satisfies*

$\exp(t\widehat{A}). \exp(\epsilon\widehat{Y}). \exp(-t\widehat{A}) \in \widehat{S}$ *for* $t > 0, \epsilon \in \mathbb{R}$, *which implies that its projection satisfies*

$(\forall \epsilon \in \mathbb{R}, t > 0) \quad \pi(\exp(\epsilon\ e^{t \operatorname{ad} \widehat{A}} \widehat{Y})) \in \pi(\widehat{S})$ *from where after taking derivative w.r.t* ϵ *at* $\epsilon = 0$ *gives*

$(\forall l \in \mathbb{N}) \ \pm \operatorname{ad}(A)^l b \in \mathcal{H}^X \subseteq \operatorname{pr}(\widehat{W})$. *Then the ad-rank(Kalman) condition implies* $\operatorname{pr}(\widehat{W})$ *is the whole Lie algebra* $\mathfrak{g}$. *Therefore, the system is locally controllable at the origin* $(0, 0)$.

4.4 Local Controllability: Extended Result

Let $\widehat{H}$ be the subgroup $\widehat{H} = \langle \exp \widehat{\mathfrak{h}} \rangle \subset \widehat{G}$. In this section we will prove that the system Σ is locally controllable at every point on $\pi(\widehat{H})$.

Recall that $\widehat{G}$ acts transitively on the homogeneous manifold M and that for each $p \in M$ the stabilizer of p is the subgroup $G_p = \{g \in \widehat{G} : gp = p\}$. The following result, cf. [SMT99], relates transitivity around a point with a property of its stabilizer.

Proposition 4.4.1. *Let G be a topological group acting on a homogeneous space M. Let $S \subseteq G$ be a subsemigroup with nonempty interior. For $p \in M$, let G_p denote its stabilizer subgroup. If* $\operatorname{Int} S \cap G_p \neq \emptyset$, *then there exists a neighborhood U of p in M, such that S acts transitively on U. Conversely, if $e \in \overline{\operatorname{Int} S}$, and S is* *transitive on a neighborhood of* p*, then* $\operatorname{Int} S \cap G_p \neq \emptyset$*. Finally,* S *acts transitively on* M *if and only if* $(\forall p \in M)$ $\operatorname{Int} S \cap G_p \neq \emptyset$*.*

Definition 4.4.2. *Let* G *be a group with Lie algebra* $\mathfrak{g}$ *and* $S \subseteq G$ *a subsemigroup. The* ***umbrella of S*** *is the set*

$$\operatorname{umb}(S) = \{V \in \mathfrak{g} : \exists T > 0 \text{ such that } \exp(tV) \in S \; \forall t \geq T\}.$$

Theorem 4.4.3. *If* Σ *satisfies the ad-rank condition, then* Σ *is locally controllable at* $p = \pi(h)$ *for all* $h \in \widehat{H}$*.*

Proof. Showing that $\widehat{W} \cap \mathfrak{g}_p \cap \operatorname{umb}(\operatorname{Int} \widehat{S}) \neq \emptyset$ will be enough because this means that there exists $\widehat{X} \in \widehat{W}$ such that $\exp(\widehat{X}) \in G_p \cap \operatorname{Int}(\widehat{S})$. This implies by Proposition 4.4.1 that $\widehat{S}$ acts transitively on a neighborhood of p. So there is a neighborhood V of p such that every q in V can be steered to p, and conversely, p can be steered to any $q \in V$.

Note also that there is a neighborhood U of $\widehat{X}$ such that $\exp(U) \subseteq \operatorname{Int}(\widehat{S})$. Since $\exp(t\widehat{X}) \in \widehat{S}$ for all $t > 0$, there is a neighborhood U' of $\widehat{X}$ contained in U such that $exp(tX).p \subseteq V$ for all t in $[0, 1]$ and X in U'. By construction, the set of points $V' := \{exp(tX).p : t \in [0, 1], X \in U'\}$ is a neighborhood of p. Now for $q = exp(t^*X^*).p$ in V' it is clear that points close to q (in V') can be reached along a trajectory that stays close to the trajectory $t \mapsto exp(tX^*).p$, for t in $[0, t^*]$ from p to q. This is equivalent to the conclusion of the theorem, i.e., Σ is locally controllable at p.

It is left that for $h \in \widehat{H}$

$$\widehat{W} \cap \mathfrak{g}_p \cap \operatorname{umb}(\operatorname{Int}(\widehat{S})) \neq \emptyset$$

Take $p = h.x_0 \in M$ then their stabilizer algebras satisfy $\mathfrak{g}_p = Ad(h)\mathfrak{g}_{x_0}$. Also $Ad(h)\widehat{W} = \widehat{W}$ and $h^{-1}\operatorname{Int}(\widehat{S})h = \operatorname{Int}(\widehat{S})$ because $h \in \widehat{S}$. Theorem 4.3.6 implies, in particular, that $\widehat{W} \cap \mathfrak{g}_{x_0} \cap \operatorname{umb}(\operatorname{Int}(\widehat{S}))$ is not empty.

Therefore, $\widehat{W} \cap \mathfrak{g}_p \cap \operatorname{umb}(\operatorname{Int}(\widehat{S})) \neq \emptyset$. Since $h \in \widehat{H}$ is arbitrary, the system Σ is locally controllable on $\pi(\widehat{H})$. $\square$

Chapter 5
Heisenberg Group Example

The example we study in this chapter shows that the ad-rank condition on Theorem 4.3.6 can not be replaced by a weaker hypothesis. The system considered here has the accessibility property, that is,

$$\dim \operatorname{span}\{\operatorname{ad}^i(X)(Y^j) : \ i \geq 0, \ 1 \leq j \leq k\}_{\mathcal{L}.\mathcal{A}.} = \dim \mathfrak{g},$$

but as we shall see, it is not locally controllable at the identity. This example was originally presented by Ayala and Tirao in [AT99], but a miss-print on their paper led to a simplistic argument for non-local controllability. Here a more elaborated proof of the conclusion is given.

5.1 The Heisenberg Group

In this chapter G will denote the Heisenberg group; i.e., the group of all real 3×3-matrices of the form

$$(a, b, c) = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix},$$

with group multiplication given by $(a_1, b_1, c_1) \cdot (a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2 + b_1 a_2 - a_1 b_2)$ The group identity is the 3×3 identity matrix $e = \mathrm{Id}_{3\times 3}$, and the group inverse is given by

$$(a, b, c)^{-1} = \begin{pmatrix} 1 & -a & -c \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix}.$$

The Lie algebra $\mathfrak{g} \subseteq \mathfrak{gl}_3(\mathbb{R})$ of G is the space of strict upper triangular matrices

$$(\alpha, \beta, \gamma) = \begin{pmatrix} 0 & \alpha & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $\{P, Q, Z\}$ be a basis of $\mathfrak{g}$ satisfying $[P, Q] = Z$ and with all other brackets vanishing. The Lie group G can be identified with $(\mathfrak{g}, *)$ where $*$ denotes the Campbell-Hausdorff-multiplication

$$X * Y = X + Y + \frac{1}{2}[X, Y] \text{ for all } X, Y \in \mathfrak{g},$$

In the $\mathbb{R}^3$ representation the above multiplication is given by

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}(y_1x_2 - x_1y_2)).$$

The left-invariant vector fields on G are of the form

$$\begin{aligned} \vec{X}(x, y, z) &= a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + (c + \frac{ya}{2} - \frac{xb}{2})\frac{\partial}{\partial z} \\ &= a\left(\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial z}\right) + b\left(\frac{\partial}{\partial y} - \frac{x}{2}\frac{\partial}{\partial z}\right) + c\frac{\partial}{\partial z} \end{aligned}$$

for arbitrary constants a, b, c.

5.2 The Accessibility Property

Let Σ be the linear control system on G given by

$$\dot{x} = \vec{D}(x) + u.\vec{Q}(x), \tag{5.1}$$

where the state x belongs to $\mathfrak{g}$ and the control $u \in \mathbb{R}$. Here $\vec{D}$ is the infinitesimal automorphism associated with the derivation $D \in \mathrm{Der}(\mathfrak{g})$ represented with respect to the basis $\{P, Q, Z\}$ by the matrix

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For this example, the augmented Lie group $\widehat{G}$ defined in Chapter 3 is given by the semidirect product $\widehat{G} = \mathfrak{g} \rtimes_D \mathbb{R}$ with group multiplication

$$\begin{aligned}(x,s)\cdot(y,t) &= \big(x * e^{sD}y,\ s+t\big)\\ &= \left(x + e^{sD}y + \frac{1}{2}[x, e^{sD}y],\ s+t\right),\end{aligned}$$

where

$$e^{sD} = \begin{pmatrix} \cosh(s) & \sinh(s) & 0 \\ \sinh(s) & \cosh(s) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\{P, Q, Z, D\}$ is a basis of $\widehat{\mathfrak{g}}$ with $[D, P] = Q$, $[D, Q] = P$, $[P, Q] = Z$, and with all other brackets vanishing. These properties of the Lie brackets imply that the system satisfies the accessibility property. In fact,

$$\begin{aligned}\mathrm{span}\{\mathrm{ad}^i(D)(Q)|\ i \geq 0\}_{\mathcal{L.A.}} &= \mathrm{span}\{Q, [D, Q]\}_{\mathcal{L.A.}}\\ &= \mathrm{span}\{Q, P, [P, Q]\}_{\mathcal{L.A.}}\\ &= \mathrm{span}\{Q, P, Z\}_{\mathcal{L.A.}}\\ &= \mathfrak{g}.\end{aligned}$$

5.3 No Local Controllability

Consider the right invariant augmented system $\widehat{\Sigma}$ on $\widehat{G}$

$$\dot{g}(t) = d\rho_{g(t)}(\mathbf{1})X(t), \qquad X(t) \in U, \tag{5.2}$$

where $U = \mathbb{R}^+ D + \mathbb{R}Q \subseteq \widehat{g}$.

Let $\widehat{W}$ denote the smallest Lie wedge generated by U. The argument used here to show non-local controllability involves two steps. First, the Lie wedge $\widehat{W}$ is identified explicitly. This is followed by a proposition showing that the Lie wedge $\widehat{W}$ is the tangent wedge of the semigroup $\widehat{S} = \langle \exp \widehat{W} \rangle$. Since $d\pi(\mathbf{1})(\widehat{W}) \subsetneq d\pi(\mathbf{1})(\widehat{\mathfrak{g}})$, then $\pi(\widehat{S})$ is not a neighborhood of the base point of x_0. Hence, the system is not locally controllable at x_0. The following proposition identifies $\widehat{W}$ explicitly.

Proposition 5.3.1. *The Lie wedge $\widehat{W} = \mathbb{R}Q + C$, where C is the Lorentzian cone given by $C = \{dD + pP + zZ : 2dz - p^2 \geq 0, d + z \geq 0\}$.*

Proof. First notice that $\mathbb{R}Q+C$ is clearly a wedge on $\mathfrak{g}$ with edge $H(\mathbb{R}Q+C) = \mathbb{R}Q$. To show invariance under the adjoint action of the edge it is enough to prove that $\exp^{tadQ} D \in \mathbb{R}Q + C$.

$$\begin{aligned}
\exp^{tadQ} D &= D + t[Q, D] + \frac{t^2}{2}[Q, [Q, D]] + \frac{t^3}{6}[Q, [Q, [Q, D]] + \cdots \\
&= D + t(-P) + \frac{t^2}{2}[P, Q] + \frac{t^3}{6}[Z, D] + \cdots \\
&= D - tP + \frac{t^2}{2}Z + 0 + \cdots + 0,
\end{aligned}$$

using the Lie bracket relations defined in the previous section. Therefore, $\mathbb{R}Q + C$ is a Lie wedge. Moreover, this is the smallest set satisfying the invariance condition shown above. Let $u \in U$, then $u = dD + pP + qQ + zZ$ with $d > 0$, $q \in \mathbb{R}$, and $p = z = 0$. Thus $2dz - p^2 = 0$ and also $d + z = d > 0$ which implies $u \in \mathbb{R}Q + C$. Therefore, U is contained in the Lie wedge $\mathbb{R}Q + C$. But by definition $\widehat{W}$ is the smallest Lie wedge containing U thus, $\widehat{W} = \mathbb{R}Q + C$. □

The Lie wedge $\widehat{W}$ is a global wedge; i.e., there exists a subsemigroup $\widehat{S} \subseteq \widehat{G}$ such that $L(\widehat{S}) = \widehat{W}$. To prove this some definitions and lemmas from Lie theory of semigroups are needed. These can be found in [HHL89] and in [HN93].

Definition 5.3.2. *If L is a vector space and W is a convex subset, then the* ***algebraic interior*** *of W is the set defined by*

$$\operatorname{algint} W = \{x \in W : \exists t > 0 \textit{ such that } x + ty \in W \ \forall y \in W - W\}.$$

If L is a vector space then L^* will denote its dual. If W is a wedge in L, the *dual wedge* $W^* \subseteq L^*$ is the set of all functionals which are nonnegative on W.

Proposition 5.3.3. *Let L be a topological vector space, and let W be a wedge in L. Then*

(i) *If the vector space $W - W$ is finite dimensional, then*

$$\operatorname{algint} W = \operatorname{Int}_{W-W} W,$$

(ii) *If $\omega \in \operatorname{algint}(W^*)$, then $w(x) > 0$ for all $x \in W \setminus H(W)$, and if $W^* - W^*$ is finite dimensional, then the converse implication is true.*

Definition 5.3.4. *Let $W \subseteq \mathfrak{g}$ be a Lie wedge and $f \in C^\infty(G)$. Then f is said to be strictly W-positive if*

$$f'(g) \in \operatorname{algint}(W^*) \quad \forall g \in G,$$

where $f' : G \to g^*$ is the function defined as $g \mapsto df(g)d\lambda_g(\mathbf{1})$.

Proposition 5.3.5. *Let G be a connected Lie group and let $W \subseteq \mathfrak{g}$ be a Lie wedge. Then the following conditions are equivalent*

1. *W is global in G.*

2. *$H(W)$ is global in G and there exists a strictly W-positive analytic function f on G.*

3. *$H(W)$ is global in G and there is exists a smooth W-positive function f on G such that $f'(\mathbf{1}) \in \operatorname{algint}(W^*)$.*

Proposition 5.3.6. *The Lie wedge $\widehat{W}$ is a global wedge; indeed, $S(\widehat{W}) = \left\langle \exp \widehat{W} \right\rangle$ implies $L(S(\widehat{W})) = \widehat{W}$.*

Proof. The idea is to show that there exists a function $\varphi : \widehat{G} \longrightarrow \mathbb{R}$ such that $\varphi(g(t))$ is increasing along every trajectory $g(t)$ of (5.2), and strictly increasing if $g(t) \notin H(\widehat{W})$. Then $\varphi'(g) := \left(X \mapsto \frac{d}{dt}\big|_{t=0} \varphi(g.\exp tX)\right)$ satisfies that $\varphi'(g) \in \operatorname{algint}(\widehat{W}^*)$ for all $g \in G$. Thus $(\forall X \in W)\ \varphi'(g)X \geq 0$, and it is strictly positive on $\widehat{W} \setminus H(\widehat{W})$. Since the edge $H(\widehat{W}) = \mathbb{R}Q$ is clearly global in G, according to Proposition 5.3.5 the wedge W is global.

For $g = (p, q, z, d) \in \mathfrak{g}$ an expression of $d\lambda_g(\mathbf{1})X = \frac{d}{dt}\big|_{t=0} g.\exp(tX)$ is needed. Since $d\lambda_g(\mathbf{1})$ is linear, it is sufficient to compute $d\lambda_g(\mathbf{1})X$ for X from a basis of $\mathfrak{g}$, say $X \in \{P, Q, Z, D\}$.

$$\begin{aligned}
d\lambda_g(\mathbf{1})Q &= \frac{d}{dt}\bigg|_{t=0} (g.\exp(tQ)) = \frac{d}{dt}\bigg|_{t=0} (p, q, z, d).(0, t, 0, 0) \\
&= \frac{d}{dt}\bigg|_{t=0} \begin{pmatrix} p + t\sinh(d) \\ q + t\cosh(d) \\ z + \frac{t}{2}(p\cosh(d) - q\sinh(d)) \\ d \end{pmatrix} \\
&= \begin{pmatrix} \sinh(d) \\ \cosh(d) \\ \frac{1}{2}(p\cosh(d) - q\sinh(d)) \\ 0 \end{pmatrix} \\
d\lambda_g(\mathbf{1})P &= \frac{d}{dt}\bigg|_{t=0} (p, q, z, d).(t, 0, 0, 0) \\
&= \frac{d}{dt}\bigg|_{t=0} \begin{pmatrix} p + t\cosh(d) \\ q + t\sinh(d) \\ z + \frac{t}{2}(p\sinh(d) - q\cosh(d)) \\ d \end{pmatrix}
\end{aligned}$$

$$d\lambda_g(\mathbf{1})P = \begin{pmatrix} \cosh(d) \\ \sinh(d) \\ \frac{1}{2}(p\sinh(d) - q\cosh(d)) \\ 0 \end{pmatrix}$$

$$d\lambda_g(\mathbf{1})Z = Z$$

$$d\lambda_g(\mathbf{1})D = D.$$

Therefore,

$$d\lambda_g(\mathbf{1}) = \begin{pmatrix} \cosh(d) & \sinh(d) & 0 & 0 \\ \sinh(d) & \cosh(d) & 0 & 0 \\ \frac{1}{2}(p\sinh(d) - q\cosh(d)) & \frac{1}{2}(p\cosh(d) - q\sinh(d)) & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The function φ should satisfy: $\varphi'(g) = d\varphi(g)d\lambda_g(\mathbf{1}) \in \operatorname{algint}(W^*)$ and φ should be constant on the coset $\{g.\exp tQ\}_{t\in\mathbb{R}}$, so $\varphi'(g)d\lambda_g(\mathbf{1})Q = 0$ must hold. Now

$$\begin{aligned} g.\exp(tQ) &= \begin{pmatrix} p + t\sinh(d) \\ q + t\cosh(d) \\ z + \frac{t}{2}(p\cosh(d) - q\sinh(d)) \\ d \end{pmatrix} \\ &= (\tilde{p}, \tilde{q}, \tilde{z}, \tilde{d}). \end{aligned}$$

Taking $t^* = -\frac{q}{\cosh(d)}$, then there are three functions $\varphi_1, \varphi_2, \varphi_3$ which are constant along $g\exp(\mathbb{R}Q)$:

$$\begin{aligned} \varphi_1(p,q,z,d) &= p - q\tanh(d), \\ \varphi_2(p,q,z,d) &= z - \frac{1}{2}pq + \frac{1}{2}q^2\tanh(d), \\ \varphi_3(p,q,z,d) &= d. \end{aligned}$$

Therefore,

$$\begin{aligned}
\varphi_3'(g) &= (0,0,0,1)d\lambda_g(\mathbf{1}) = (0,0,0,1). \\
d\varphi_2(g) &= \left(-\frac{1}{2}q,\ -\frac{1}{2}p + q\tanh(d),\ 1,\ \frac{q^2}{\cosh^2(d)}\right), \\
\implies \varphi_2'(g) &= \left(-\frac{q}{\cosh(d)},\ 0,\ 1,\ \frac{q^2}{\cosh^2(d)}\right).
\end{aligned}$$

Note that the Lorentzian cone C can be identified with

$$C_B := \{v \in \mathbb{R}^3 : v^T B v \geq 0, v^T x_0 \geq 0\},$$

where

$$B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and $x_0 = (0,1,1)$. With this identification the dual cone is

$$C_B^* = \{v \in \mathbb{R}^3 : v^T B^{-1} v \geq 0, v^T \omega_0 \geq 0\}$$

but since $B^{-1} = B$, then

$$C_B^* = \{v \in \mathbb{R}^3 : v^T B v \geq 0, v^T \omega_0 \geq 0\}.$$

So now the dual cone C^* can be characterized by $2\omega_D\omega_Z - \omega_P^2 \geq 0$, $\omega_D + \omega_Z \geq 0$. If we let $\varphi = \varphi_2 + \varphi_3$, then

$$\varphi'(g) = \left(-\frac{q}{\cosh(d)},\ 0,\ 1,\ 1 + \frac{q^2}{\cosh^2(d)}\right) \in \operatorname{algint}(W^*)$$

because of $2\omega_D\omega_Z - \omega_P^2 = 2 + \frac{q^2}{\cosh^2(d)} > 0$ and $\omega_D + \omega_Z = 2 + \frac{q^2}{\cosh^2(d)} > 0$. $\square$

The previous proposition implies that $S(\widehat{W}) \subsetneq G \rtimes_D \mathbb{R}$ is not the whole space. For the base point $x_0 = \pi(\mathbf{1}) \in M$ the projection $\pi(S(\widehat{W})) = S(\widehat{W}).x_0 \subsetneq M$. In particular, since $d\pi(\widehat{W}) \subsetneq d\pi(\mathfrak{g})$ then $\pi(S(\widehat{W}))$ does not contain an open neighborhood of x_0. Hence, the system is not locally controllable at x_0.

5.3.1 The Exponential Map

In the calculation of the exponential map $\exp : \widehat{\mathfrak{g}} \to \widehat{G}$ is given by the formula

$$\exp(\alpha,\beta,\gamma,\delta) = \begin{pmatrix} \alpha\dfrac{\sinh(\delta)}{\delta} + \beta\dfrac{(\cosh(\delta)-1)}{\delta}, \\ \beta\dfrac{\sinh(\delta)}{\delta} + \alpha\dfrac{(\cosh(\delta)-1)}{\delta}, \\ \gamma + (\alpha^2-\beta^2)\dfrac{(\sinh(\delta)-\delta)}{2\delta^2}, \\ \delta \end{pmatrix}$$

which leads to the following proposition.

Proposition 5.3.7. *The exponential function* $\exp : \widehat{\mathfrak{g}} \to \widehat{G}$*, for* $\delta \neq 0$ *is given by*

$$\exp(\alpha,\beta,\gamma,\delta) = \left(f(\delta D)\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \gamma + (\alpha^2-\beta^2)\frac{(\sinh(\delta)-\delta)}{2\delta^2}, \delta \right)$$

where $f(z) = \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} = \frac{e^z-1}{z}$.

Further, $\exp : \widehat{\mathfrak{g}} \to \widehat{G}$ *is a diffeomorphism with global inverse* $\log : \widehat{G} \to \widehat{\mathfrak{g}}$ *given by*

$$\log(p,q,z,d) = \begin{pmatrix} \begin{pmatrix} \frac{d}{2}\left(\dfrac{\sinh(d)}{(\cosh(d)-1)}\right) & -1 \\ -1 & \frac{d}{2}\left(\frac{\sinh(d)}{(\cosh(d)-1)}\right) \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}, \\ z - \dfrac{1}{8}(\sinh(d)-d)(p^2-q^2)(d^2+1), \\ d \end{pmatrix}.$$

Observe that $\mathrm{Spec}(D) \cap 2\pi i\mathbb{Z} = \emptyset$ then $f(\delta D)$ is invertible for all $\delta \in \mathbb{R}$; therefore, using the results below (cf. [Dix57]) there exists a global inverse.

Lemma 5.3.8. $d\exp(X)$ *is invertible if and only if* $\mathrm{Spec}(\mathrm{ad}\, X) \cap 2\pi i(\mathbb{Z}\setminus 0) = \emptyset$.

Proposition 5.3.9. *Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$. Then the exponential map is a diffeomorphism if and only if one of the following holds true*

(i) $(\forall X \in \mathfrak{g})\,\mathrm{Spec}(\mathrm{ad}\,X) \cap 2\pi i(\mathbb{Z} \setminus 0) = \emptyset$.

(ii) $(\forall X \in \mathfrak{g})\,\mathrm{Spec}(\mathrm{ad}\,X) \cap 2\pi i\mathbb{Z} \subseteq 0$.

(iii) $(\forall X \in \mathfrak{g})\,\mathrm{Spec}(\mathrm{ad}\,X) \cap i\mathbb{R} \subseteq 0$.

References

[AT99] V. Ayala and J. Tirao, *Linear control systems on lie groups and local controllability*, Eds. G. Ferreyra et al., Amer. Math. Soc., Providence, RI, 1999.

[Bou98] Nicolas Bourbaki, *Lie groups and Lie algebras. Chapters 1–3*, Springer-Verlag, Berlin, 1998, Translated from the French, Reprint of the 1989 English translation.

[Bro72] R. W. Brockett, *System theory on group manifolds and coset spaces*, SIAM J. Control **10** (1972), 265–284.

[Dix57] J. Dixmier, *L'application exponentielle dans les groupes de Lie résolubles*, Bull. Soc. Math. France **85** (1957), 113–121.

[Dub57] L. E. Dubins, *On curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents*, Amer. J. Math. **79** (1957), 497–516.

[Her63] Robert Hermann, *On the accessibility problem in control theory*, Internat. Sympos. Nonlinear Differential Equations and Nonlinear Mechanics (New York), Academic Press, New York, 1963, pp. 325–332.

[HHL89] Joachim Hilgert, Karl Heinrich Hofmann, and Jimmie D. Lawson, *Lie groups, convex cones, and semigroups*, The Clarendon Press Oxford University Press, New York, 1989, Oxford Science Publications.

[HN93] Joachim Hilgert and Karl-Hermann Neeb, *Lie semigroups and their applications*, Springer-Verlag, Berlin, 1993.

[Hoc91] Leslie M. Hocking, *Optimal control*, The Clarendon Press Oxford University Press, New York, 1991, An introduction to the theory with applications.

[How83] Roger Howe, *Very basic Lie theory*, Amer. Math. Monthly **90** (1983), no. 9, 600–623.

[JK81] V. Jurdjevic and I. Kupka, *Control systems on semisimple Lie groups and their homogeneous spaces*, Ann. Inst. Fourier (Grenoble) **31** (1981), no. 4, vi, 151–179.

[JS72] Velimir Jurdjevic and Héctor J. Sussmann, *Control systems on Lie groups*, J. Differential Equations **12** (1972), 313–329.

[Jur97] Velimir Jurdjevic, *Geometric control theory*, Cambridge University Press, Cambridge, 1997.

[Kre74] Arthur J. Krener, *A generalization of Chow's theorem and the bang-bang theorem to non-linear control problems*, SIAM J. Control **12** (1974), 43–52.

[LaS61] J. P. LaSalle, *The "bang-bang" principle*, Automatic and Remote Control (Proc. First Internat. Congr. Internat. Fed. Automat. Control, Moscow, 1960), Vol. I, Butterworths, London, 1961, pp. 493–497.

[LM67] E. B. Lee and L. Markus, *Foundations of optimal control theory*, John Wiley & Sons Inc., New York, 1967.

[LM02] J. D. Lawson and D. Mittenhuber, *Controllability of Lie systems*, Contemporary trends in nonlinear geometric control theory and its applications (México City, 2000) (River Edge, NJ), World Sci. Publishing, River Edge, NJ, 2002, pp. 53–76.

[Lob70] Claude Lobry, *Contrôlabilité des systèmes non linéaires*, SIAM J. Control **8** (1970), 573–605.

[Mar81] L. Markus, *Controllability of multitrajectories on Lie groups*, Dynamical systems and turbulence, Warwick 1980 (Coventry, 1979/1980) (Berlin), Springer, Berlin, 1981, pp. 250–265.

[Pal57] Richard S. Palais, *A global formulation of the Lie theory of transformation groups*, Mem. Amer. Math. Soc. No. **22** (1957), iii+123.

[PBGM62] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, *The mathematical theory of optimal processes*, Interscience Publishers John Wiley & Sons, Inc. New York-London, 1962.

[SJ72] H. Sussmann and V. Jurdjevic, *Controllability of nonlinear systems*, J. Diff. Equations **12** (1972), 95.

[SMT99] Luiz A. B. San Martin and Pedro A. Tonelli, *Transitive actions of semigroups in semi-simple Lie groups*, Semigroup Forum **58** (1999), no. 1, 142–151.

[Sus73] Héctor J. Sussmann, *Orbits of families of vector fields and integrability of distributions*, Trans. Amer. Math. Soc. **180** (1973), 171–188.

[War83] Frank W. Warner, *Foundations of differentiable manifolds and Lie groups*, Springer-Verlag, New York, 1983, Corrected reprint of the 1971 edition.

Vita

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