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The effect of initial selections in estimating the missing comparisons in an incomplete AHP matrix

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THE EFFECT OF INITIAL SELECTIONS IN ESTIMATING THE MISSING COMPARISONS IN AN INCOMPLETE AHP MATRIX

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ABSTRACT

One of the most widely used methods in decision-making is the Analytic Hierarchy Process (AHP). With its technique of comparing the alternatives by means of a sequence of pairwise comparison matrices, the AHP is both easy to understand and very versatile. This research aims at contributing some insights on this method, in particular, regarding to what is known as the incomplete AHP. The core of this research is to investigate whether the initial comparisons used to extract the data for a multi-criteria decision making problem, will play a role in producing a relatively accurate estimation of the ranking of the alternatives. Three problems are investigated in this work. The first problem is to determine the optimal number of the initial comparisons. As the number of initial comparisons increases, a complete pairwise comparison matrix will more likely be estimated accurately. Consequently, the time required to calculate these initial comparisons will also increase. These conflicting goals will be investigated further in this thesis.

The second problem of this research is to determine which initial comparisons should be asked as the starting point. Using the minimal number of initial comparisons (i.e., $n-1$ comparisons), five different strategies will be investigated. Lastly, the final problem is to determine if the method that we use to estimate the missing comparisons will also affect the accuracy of the weight vector. Two methods will be compared in this thesis, namely the Least Squares, and the Geometric Mean methods.

In order to determine whether a matrix is accurately estimated, two methods are used to compare the estimated and the original weight vectors. One method is to compare the ranking order of the alternatives, while the other is to compute the average difference between the two vectors. The smaller the average difference, the better the corresponding selection strategy is.

Furthermore, the two methodologies will be compared based on their computation requirements. The methodology with less computational time and better accuracy will be considered better than the other. The final results of this thesis will provide more insight into the incomplete AHP in general, thus hopefully providing the decision maker a reliable tool to optimally use this method.

CHAPTER 1

INTRODUCTION

1.1. Background Information

The Analytic Hierarchy Process (AHP) was first introduced by Saaty (1980). The simple representation of comparing multiple alternatives in the form of a pairwise comparison matrix has made it one of the most widely used decision analysis methods. Recently, using the Yahoo! search engine, over 6,500 sites were found containing the keyword “analytic hierarchy process”, ranging from explanation of the general methodology to specific applications of the AHP. Some of the applications mentioned are in health care, agriculture, operations research, economics, and transportation models. Moreover, an international symposium (called the ISAHP) is held approximately every two years to discuss developments related to the AHP.

However, this method is not without its drawbacks. In the early days, a new finding has proven that the order of selecting the best alternative can vary depending in a way that is counter-intuitive (Belton and Gear, 1983). In that paper, the ranking of the best alternative can differ when a copy of a non-optimal alternative is added. This phenomenon might be intuitively unacceptable, since adding a new non-optimal alternative should not affect the ranking of the best alternative.

Shen *et. al.* (1992) also discovered another irregularity in the methodology. They proposed to divide the incomplete comparison matrix into smaller subsets such that each subset is a complete matrix. Since the subsets are complete matrices, we can use the eigenvector approach to calculate the weight vector for each subset. The final weight vector is then calculated by combining the weight vectors from each subset using their common parts. However, using this method of calculating the final weight vector, the ranking of the alternatives can be different from the weight vector that was calculated using the original AHP methodology.

Another case of ranking irregularities can be found in Triantaphyllou (2001). In that paper, two methods for comparing the alternatives are investigated. The first is to compare all of them simultaneously, while the other compares two alternatives at a time. The ranking of the alternatives (given that we are looking for the best) can be different between the two methods. Nevertheless, these weaknesses along with many other criticisms of the AHP cannot change the fact that it is one of the most widely used methods in decision-making today. More related discussions on the AHP can be found in ((Dyer, 1990), (Harker and Vargas, 1990), (Holder, 1990), (Millet and Harker, 1990), (Perez, 1995), and (Wedley, 1993)).

With all the enthusiasm surrounding AHP, this research will try to contribute some new analysis of the method, specifically regarding the incomplete AHP. The details of the general method of the AHP will be discussed in the next section.

1.2. The AHP Methodology

The AHP was designed to structure and effectively solve multi-criteria decision-making problems. There are two phases involved with this methodology. The first phase involves structuring the problem in a tree-like hierarchy. The root of the tree consists of the goal (i.e., the main objective). The branches out of the root correspond to the major decision criteria, which may be further branched upon by sub-criteria. Following the criteria, the set of alternatives is represented as leaves. The second phase involves numerical evaluations of the nodes in the tree in a bottom-up manner. At the root, we can draw a conclusion as to the best alternative satisfying the goal given a set of criteria.

In order to illustrate this methodology, we will consider a simplified problem of choosing a car. In selecting the most suitable car, several criteria are used to quantify suitability. Some of the criteria might be the interior comfort, the style, and the reliability of the cars. All of these factors are placed at the same level of the tree. These criteria can further be divided into sub-criteria. For example, the reliability criterion can be divided into the price of parts that need to be maintained regularly, the efficiency of the dealership in handling the maintenance, how easy it is to get the spare parts in regular stores, etc. However, these sub-criteria are omitted to simplify the illustration. Finally, the bottom level of the hierarchical tree will be the alternatives. Suppose that we are interested in three different types of cars, say Car A, Car B, and Car C. Then, the hierarchical structure for our example can be represented as follows:

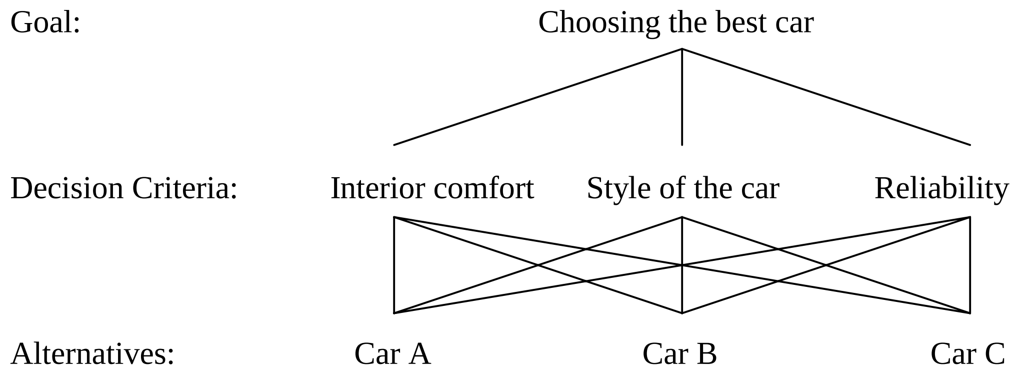

Figure 1-1: An illustrated example for the AHP.

This hierarchical design can help the decision maker understand the problem better as to the importance of each level of the decision problem. The most important aspect, known as the goal (e.g., choosing the best car), is on the root, followed by the primary decision criteria (e.g., interior comfort, style of the car, and reliability). More sub-criteria can be branched out of each criterion to improve the model representation of a real life situation. After these criteria, the leaves represent the alternatives, which are compared to determine the best solution.

The second phase of the AHP methodology is to perform numerical computations at each level of the hierarchy. Each alternative is compared relative to a single criterion, and each criterion to the goal. In our example, the main focus of this step is to assign a value of how "good" each car is when it is compared to another car under each one of the criteria, and how "important" each criterion is, relative to the goal.

Assigning the values to these comparisons can be done in two different ways. One approach is to assign an absolute value to each decision criterion and alternative. For example, we can give Car A a "good" rating on the interior comfort criterion without comparing it with other cars. This type of comparison is highly unusual, since as humans, we always need some kind of relational references for the comparison. To say that something is "good" without any reference as what is "bad" is unreliable, thus this method is not recommended.

The second method for assigning the comparison value is through a sequence of relative judgments. In this method, the alternatives (or the criteria) are compared with each other using a fixed scale. Saaty has proposed a numerical scale to represent the degree of “importance” of one alternative (or criterion) compared with another. The scale consists of the discrete numbers in the set of {1/9, 1/8, 1/7, 1/6, 1/5, 1/4, 1/3, 1/2, 1, 2, 3, 4, 5, 6, 7, 8, 9}. The explanation of these values is given in Table 1.1 (Saaty, 1980).

Table 1.1: Explanation of the Saaty scale.

Numerical Value	Linguistic Definition	Explanation
1	Equal Importance	Two activities contribute equally to the objective
3	Moderate Importance	Experience and judgment slightly favor one activity over another
5	Strong Importance	Experience and judgment strongly favor one activity over another
7	Very Strong or Demonstrated Importance	An activity is favored very strongly over another; its dominance demonstrated in practice
9	Extreme Importance	The evidence favoring one activity over another is of the highest possible order of affirmation
2, 4, 6, 8	Intermediate values	To reflect the compromise between two adjacent judgments

This scale is widely accepted in the decision-making community, mostly in the AHP context, due to its support by a psychological study which stated that a person cannot compare more than seven entities (plus or minus two) at the same time (Miller, 1956).

Using this scale, the decision maker can express his/her opinion on the importance of an alternative or criterion compared to another. In order to ensure consistency, the comparisons of the alternatives must be based on a common criterion at a time.

While comparing the alternatives, the assigned value of a single comparison can be based on a quantity-based judgment, or on a quality-based judgment. For example, we can compare the exact fuel consumption (in miles per gallon) of each car in order to determine the value of the comparisons for the fuel consumption criterion. For example, the mileage ratio of Car A relative to Car B can be used to determine the comparison value of Car A to Car B. This type of judgment is called quantity-based judgment.

For this type of judgment, the Saaty scale is no longer useful, since it is most commonly used for quality-based judgments. Thus, in this illustrative example, we would like to consider only quality-based judgments for our comparison matrices.

The other type of judgment (i.e., quality-based judgment) is more of a personal preference of the decision maker to a particular alternative (or criterion) relative to another. For example, we cannot quantify the style of Car A is better than Car B with high precision. Personal preferences play a major factor in determining this type of judgment.

To illustrate the comparisons more effectively, a comparison matrix is introduced to represent all the possible pairs of comparisons. Consider a matrix A comparing n alternatives with respect to a particular criterion. The entry $a_{i,j}$ $(a_{i,j} > 0)$ reflects the value of how "important" alternative i is when it is compared to alternative j. Obviously, we can assume that the following statements are true: $a_{i,j} = 1/ a_{j,i}$, for all $i,j = 1, \ldots, n$ and the diagonal entries are equal to 1. That is, $a_{i,i} = 1$, for all $i = 1, \ldots, n$.

In our example, we can derive the comparison matrix for each criterion. First we should compare the alternatives in terms of the interior comfort criterion. The following matrix is assumed to be the comparisons of the three cars:

Table 1.2: Comparisons of alternatives based on the interior comfort criterion.

Interior comfort	Car A	Car B	Car C
Car A	1	7	6
Car B	1/7	1	1/3
Car C	1/6	3	1

The entries in this matrix are the relative judgments of the decision maker about the preference of one car to another among all three alternatives based on the comfort criterion. For example, entry $a_{1,2}$ in table 1.2 is equal to 7, which indicate that Car A is *"demonstrably more important (more comfortable) than"* Car B. This number is determined by the decision maker's opinion on this matter. Similarly, the entry $a_{2,1}$ is the comfort comparison of Car B to Car A. Obviously, the entry of $a_{2,1}$ should be 1/7, which is the reciprocal of entry $a_{1,2}$. The comparison of the alternative with itself should always be equally important, thus all the diagonal values are equal to 1.

An important observation that can be added regarding this criterion is that the entries in the comparison matrix are derived using quality-based judgments. In the real world, some problems require quantity-based judgments since exact values for the comparisons can be available. An example of these problems is the comparison of two cars based on their fuel consumption. If we compare the cars based on their actual fuel consumptions, the entries of the matrix are likely to include decimal numbers. For example, Car A can travel as far as 32 miles per gallon, while Car B can only travel 23 miles per gallon. With this information, the ratio of Car A and Car B is equal to 1.39.

From the above discussion, we can see that the Saaty scale is not very useful in representing the quantity-based judgments. However, in this M.S. thesis, the Saaty scale will be utilized as the basis to determine the comparison values between the alternatives. With this assumption in mind, some values in the matrix may not be consistent with the Saaty scale, thus an approximation is needed. This situation will be investigated further in the later chapters.

Another observation that we can derive from the previous matrix is that not all entries (which are $3 \times 3 = 9$) need to be decided by the decision maker. The comparison values in the upper triangular portion of the matrix are exactly the reciprocals of the ones in the lower triangular portion of the matrix and the diagonal values are always equal to 1. Thus the number of pairwise comparisons needed to complete the whole matrix is equal to 3. In general, the decision maker needs to make $n(n-1)/2$ pairwise comparisons in order to complete the comparison matrix, where n is the number of alternatives (or criteria) to be compared.

In a perfectly consistent matrix, we can easily conclude that the following formula is true:

$$a_{i,j} = a_{i,k} \times a_{k,j}, \qquad for \quad i, j, k = 1, ..., n. \qquad (1.1)$$

As stated earlier, the entry $a_{i,j}$ is the comparison of alternative *i* with alternative *j*, more appropriately, the ratio of the relative weights of alternative *i* to alternative *j* in terms of a single criterion. The following formula is derived for entry $a_{i,j}$ in a perfectly consistent pairwise matrix *A*:

$$a_{i,j} = \frac{w_i}{w_j}, \qquad for \ \ i, j = 1, ..., n. \qquad (1.2)$$

where w_i is the relative weight (an unknown) of alternative *i* (for $i = 1, 2, ..., n$). From the above formulas, we can easily prove that:

$$a_{i,j} = a_{i,k} \times a_{k,j} \left(= \frac{w_i}{w_k} \times \frac{w_k}{w_j} = \frac{w_i}{w_j} = a_{i,j} \right)$$

and $$a_{i,j} = \frac{w_i}{w_j} = \frac{1}{w_j / w_i} = \frac{1}{a_{j,i}}$$

The following formula is a derivation of the above formulas.

$$a_{i,j} \times \frac{w_j}{w_i} = 1, \qquad for \ \ i, j = 1, ..., n. \qquad (1.3)$$

With respect to a specific row *i*, formula (1.3) can be summed for the total of *n* columns in matrix *A*, which can be expressed as follows:

$$\sum_{j=1}^{n} a_{i,j} w_j \frac{1}{w_i} = n, \qquad for \ \ i = 1, ..., n.$$

Furthermore, it can be rewritten as:

$$\sum_{j=1}^{n} a_{i,j} w_j = n w_i, \qquad for \ \ i = 1, ..., n. \qquad (1.4)$$

In general, formula (1.4) can be written as:

$$Aw = nw \qquad (1.5)$$

where *A* represents the pairwise matrix *A* and *w* is the relative weight vector.

The above formula can be depicted in more detail as follows:

$$\begin{bmatrix} \frac{w_1}{w_1} & \frac{w_1}{w_2} & \cdots & \frac{w_1}{w_{n-1}} & \frac{w_1}{w_n} \\ \frac{w_2}{w_1} & \frac{w_2}{w_2} & \cdots & \frac{w_2}{w_{n-1}} & \frac{w_2}{w_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{w_{n-1}}{w_1} & \frac{w_{n-1}}{w_2} & \cdots & \frac{w_{n-1}}{w_{n-1}} & \frac{w_{n-1}}{w_n} \\ \frac{w_n}{w_1} & \frac{w_n}{w_2} & \cdots & \frac{w_n}{w_{n-1}} & \frac{w_n}{w_n} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{n-1} \\ w_n \end{bmatrix} = n \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{n-1} \\ w_n \end{bmatrix}$$

From formula (1.5), we can determine that *n* is the principal right eigenvalue of the comparison matrix *A*, thus the formula can be rewritten as:

$$Aw = \lambda_{\max} w$$

where $\lambda_{\max}$ is the principal right eigenvalue of the matrix *A*, and the weight vector *w* is a right principle eigenvector of matrix *A*. Note that the above assumption of $\lambda_{\max} = n$ is only true for a perfectly consistent pairwise matrix. In an inconsistent matrix, $\lambda_{\max} > n$ (Saaty, 1980).

The formulation of the eigenvector is the most widely used method to calculate the relative weights of the alternatives in decision-making problems. However, since this method is not easily done, an approximation method was introduced in (Saaty, 1980). This approximation method calculates the geometric mean of each row in the matrix *A* as the corresponding element of the eigenvector. The calculation involves the multiplication of the elements of each row together, then to take the *n*-th root (where *n* is the dimension of matrix *A*). An important point that needs to be clarified is that this method will only approximate the values of the weight vectors in an inconsistent matrix. In a perfectly consistent matrix, the approximated values derived by using the geometric mean method will be identical to those by the eigenvector approach.

Since we usually want to have $\sum_{i=1}^{n} w_i = 1$, the approximated values should be normalized. The procedure of the geometric mean calculation of the relative weights can be expressed as follows:

$$w_i = \frac{\sqrt[n]{\prod_{j=1}^{n} a_{i,j}}}{\sum_{i=1}^{n}\sqrt[n]{\prod_{j=1}^{n} a_{i,j}}}, \qquad \text{where } a_{i,j} \text{ is the } (i,j) \text{ element in matrix } A. \quad (1.6)$$

After the calculation of the weight vector, we can determine (or estimate it for the inconsistent case) the eigenvalue using formula (1.5). The method is as follows:

$$\sum_{j=1}^{n} a_{i,j} w_j = \lambda_{\max} w_i, \qquad for \;\; i = 1,\ldots,n.$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j} w_j = \sum_{i=1}^{n} \lambda_{\max} w_i = \lambda_{\max} \qquad (1.7)$$

Using this approximation method we can calculate the relative weight of each alternative in our example. The principal eigenvalue $\lambda_{\max}$ for our car selection matrix for the interior comfort criterion is equal to 3.096 and the corresponding approximated eigenvector (i.e., the vector with the relative weight of each alternative) is equal to $(0.750, 0.078, 0.172)^T$ (where "T" stands for transpose). From this calculation, we can conclude that Car A is the most favorable one in terms of the interior comfort criterion, followed by Car C, thus making Car B the least favorable one.

The above method can be applied to calculate the best alternative under each one of the criteria. For example, the following matrices could be formed to compare the cars in terms of the other two criteria, namely the style of the car and the reliability.

Table 1.3: Comparisons of the alternatives based on the style criterion.

Style of the car	Car A	Car B	Car C
Car A	1	1/4	3
Car B	4	1	7
Car C	1/3	1/7	1

Table 1.4: Comparisons of the alternatives based on the reliability criterion.

Reliability	Car A	Car B	Car C
Car A	1	1/3	1/6
Car B	3	1	1/4
Car C	6	4	1

The geometric mean method is applied to estimate the right principal eigenvalue and corresponding eigenvector (i.e., the weight vector) of each matrix. For the style of the car criterion, the eigenvalue is 3.030 and the corresponding weight vector is $(0.211, 0.705, 0.084)^T$. In terms of the reliability criterion, the eigenvalue is 3.05 and the weight vector is $(0.091, 0.218, 0.691)^T$.

The results from the above calculations are derived using the assumption of having a perfectly consistent matrix. However, the case of having a perfect consistency is highly unlikely. As the eigenvalues of the above matrices are not equal to 3 (i.e., $\lambda_{\max} \neq n$), we can conclude that those matrices are not perfectly consistent.

Furthermore, in real world applications, inconsistent matrices are more common, due to inconsistencies in human judgments. With this assumption, a standard must be established to determine if a matrix can be accepted as being "adequately" consistent. A matrix is considered to be adequately consistent if the corresponding Consistency Ratio (*CR*) is less than 10% (Vargas, 1982). In order to determine the value of *CR*, we need to calculate the Consistency Index (*CI*) of the matrix compared to the Random Consistency Index (*RCI*). The formula to calculate *CI* is expressed as follows ((Saaty, 1980) and (Vargas, 1982)):

$$CI = \frac{(\lambda_{\max} - n)}{(n-1)} \tag{1.8}$$

where $\lambda_{\max}$ is the principal right eigenvalue of the corresponding matrix, and n is the dimension of the comparison matrix. *RCI* is a parameter, which is used to determine an upper limit on how much inconsistency can be tolerated in a specific comparison matrix. The *RCI* values for different matrices of n dimension have been determined by Saaty (1980) to be as follows:

Table 1.5: The corresponding *RCI* values for random matrices of dimension *n*.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
RCI	0	0	0.52	0.89	1.11	1.25	1.35	1.40	1.45	1.49	1.51	1.54	1.56	1.57	1.58

Using the above values, the *CR* can be obtained by using the following formula: *CR* = *CI/RCI*. If the *CR* value is greater than 10%, we need to re-calculate some of the comparisons in the respective matrix in order to achieve an acceptable consistency (Saaty, 1980).

In our example, the *CI* value for the interior comfort criterion matrix is 0.048 and the *RCI* value is 0.52. Thus, the *CR* value is 0.092, which is less than 10%, so the interior comfort criterion matrix is acceptably consistent. The *CR* values for the two remaining matrices are 0.028 and 0.048 for the style of the car and reliability criteria, respectively. Since none of the *CR* values is greater than 10%, we can conclude that the matrices are adequately consistent.

After calculating the relative weight of each alternative with respect to each criterion, we also need to calculate the relative weight of each criterion in terms of the overall goal. In general, we need to calculate the relative weights of the alternatives on each level of the hierarchy tree, starting from the bottom level and continue up the tree until the root (i.e., the goal).

The following matrix is assumed to illustrate the comparisons between each criterion with respect to the main objective (i.e., choosing the best car).

Table 1.6: Comparisons of decision criteria based on the main objective.

Choose the best car	Interior comfort	Style of the car	Reliability
Interior comfort	1	8	5
Style of the car	1/8	1	2
Reliability	1/5	1/2	1

Upon calculating the relative weights of the above matrix, we next combine the results from the other matrices to form the decision matrix for this illustrative problem as follows:

Table 1.7: The decision matrix for choosing the best car.

Choose the best car	Interior comfort 0.714	Style of the car 0.132	Reliability 0.154
Car A	0.750	0.211	0.091
Car B	0.078	0.705	0.218
Car C	0.171	0.084	0.691

Entry $\alpha_{i,j}$ in the above decision matrix is the relative weight of alternative *i* in terms of criterion *j*. Thus, $\alpha_{1,1}$ indicates the relative weight of Car A in terms of the interior comfort criterion, which is 0.75. Next, we need to determine the final weight (performance value) of alternative *i* in terms of the overall goal. The formula to calculate the final weight is as follows:

$$P_i = \sum_{j=1}^{m} \alpha_{i,j} w_j, \qquad for \;\; i = 1, \ldots, n, \qquad (1.9)$$

where $\alpha_{i,j}$ is the relative weight of alternative *i* in terms of criterion *j*, and w_j is the weight of criterion *j* to the overall goal.

Using formula (1.9), we can calculate the following final preference values:

For Car A: P_1 = (0.714×0.750) + (0.132×0.211) + (0.154×0.091) = 0.577.

Similarly, the relative weights of the other two alternatives are:

For Car B: P_2 = 0.182.

For Car C: P_3 = 0.241.

From the above results, we can observe that Car A has the highest relative value when it is compared with the other alternatives, thus making it the best choice. As a conclusion, for the goal of choosing the best car, the final ranking is Car A, then Car C followed by Car B (or $A > C > B$, where > means "better than").

1.3. The Case of Conflicting Criteria

The above example only considers that the higher the weight vector, the better an alternative (or criterion) is. In a real-world application, this may not be the case, since conflicting criteria might be included. An illustrative example is the price criterion. If we use this criterion in our example, we will consider the lower the relative weight of a particular car, the more appealing that car will be. A study of this situation is reported in (Triantaphyllou and Baig, 2000).

CHAPTER 2

PROBLEM DESCRIPTION

2.1. The Incomplete AHP

As described in the previous chapter, the AHP requires to form a series of comparison matrices in order to calculate the relative weight vectors, which in turn form the decision matrix. In order to complete a single comparison matrix, the decision maker needs to make $n(n-1)/2$ comparisons. These comparisons are enough to complete the whole matrix of size *n*, since all diagonal comparison values are equal to 1, and the other part of the matrix can be calculated using the reciprocal formula $\left(a_{i,j} = \frac{1}{a_{j,i}} \right)$, where $a_{i,j}$ is the comparison value of alternative *i* with alternative *j*.

For a more general case with multiple criteria, the number of comparisons that the decision maker needs to make is $n(n-1)/2 + n(m(m-1))/2)$, where *m* is the number of alternatives, and *n* is the number of criteria. Note that this formula only applies for a single hierarchy level of criteria. Multiple hierarchy levels of criteria are not considered in the formula.

In a consistent matrix formula (1.1) can be used to calculate the missing comparisons, thus only $n-1$ initial comparisons are needed. However, in a real-life situation inconsistency in the matrix is highly expected due to human errors. In order to reduce the errors of the estimated missing comparisons, additional comparisons may be necessary. Naturally, as the number of comparisons increases, we can be more confident of the correctness of the estimated values of the missing comparisons.

On the other hand, the more comparisons being conducted by the decision maker, the more likely he/she will induce some errors in the judgments. As the size of the matrix (i.e., the *n* value) increases, the number of the pairwise comparisons necessary to complete the matrix will increase dramatically, along with the time required to complete the comparisons. After some time, the fatigue factor will start to influence the decision maker, making his/her judgments more likely to be grossly inaccurate. With this aspect in mind, our original intention of making more initial comparisons for improved accuracy may be impractical.

A solution to this problem can be seen in the incomplete AHP. In an incomplete comparison matrix, the decision maker has to make between $n-1$ and $n(n-1)/2$ pairwise comparisons to partially fill the matrix. The rest of the missing comparisons can be calculated using some of the methods that will be discussed later.

2.2. Determining the Initial Comparisons

In the previous discussion, we can see that the time and effort to complete a judgment matrix may be too large. The main idea of the incomplete AHP is to fill in a judgment matrix with some initial comparisons, followed by a calculation method to estimate the remaining missing comparisons. These initial comparisons are obtained directly from the decision maker, thus the values will be more accurate (i.e., they are not estimated from other comparisons). Although these values are more accurate, consistency in the matrix may not be the case. In a perfectly consistent matrix, we only need $n-1$ initial comparisons, and then the entire matrix can be filled in with perfect accuracy.

However, in general, a perfectly consistent matrix is highly unlikely, thus we will probably need more than just $n-1$ initial comparison values. The main question that arises first is **how many initial comparisons do we need in order to ensure a relatively accurate estimation of the whole matrix?** Intuitively, the more comparisons one makes (up to some number), the more likely is the matrix to be estimated accurately. On the other hand, the more comparisons one makes, the longer it will take, thus increasing the likelihood of errors in the judgments.

Another question that we will address here is **which pairwise comparisons do we need to ask as the starting point?** Some researchers (e.g., (Harker, 1987) and (Chen, 1997)) have chosen to use random selection to determine these initial values. Weiss and Rao (1987) have proposed a methodology called Balanced Incomplete Block Designs (BIBD) to represent the initial comparisons in a large-scale AHP matrix. Using this method, a large matrix was divided into smaller subsets. A decision maker would evaluate each subset, thus making the calculations much easier. However, since the rule of dividing these subsets was not established, the above question remains unanswered.

The focus of this research is to determine whether some type of initial selection will produce a better estimation of the missing comparisons.

2.3. Estimating the Missing Values in an Incomplete Matrix

After somehow determining some initial comparisons, a method is necessary to estimate the remaining missing comparisons in the matrix. Harker (1987) proposed the Geometric Mean, and the Revised Geometric Mean method to calculate the missing comparisons. Using one of the above methods, we will be able to estimate the missing comparison values in our incomplete matrix. After estimating the missing comparisons, the corresponding weight vector of the estimated matrix can be calculated using the AHP method discussed in chapter 1.

Chen (1997) has also proposed to use the Revised Geometric Mean method along with the Least Squares method to estimate the missing comparisons. She used the same stopping criteria as Harker (1987) to terminate the estimation of the missing comparison values. Chen has concluded that the two methods will produce almost identical results in terms of estimating the weight vector. With this finding, this research expects no major differences will be found in the previously introduced methods. Instead, the main focus is to determine whether the selection of initial values will play an important role in estimating the correct weight vector.

2.4. The Geometric Mean Method

One of the methods in estimating the missing comparisons is the Geometric Mean method (Harker, 1987). We will introduce $X_{i,j}$ to denote the missing comparison value in the i^{th} row and j^{th} column. From formula (1.1), for a perfectly consistent case we can conclude that:

$$X_{i,j} = a_{i,k} \times a_{k,j}, \qquad \text{for } i,j,k = 1, \ldots, n \qquad (2.1)$$

where $a_{i,k}$ and $a_{k,j}$ are known initial comparisons.

Formula (2.1) can only be true if the matrix is perfectly consistent. In the case of an inconsistent matrix, the formula will only estimate the comparison value. The combination of $a_{i,k}$ and $a_{k,j}$ is called an elementary path (of length 2) connecting the missing comparison of items *i* and *j* (Harker, 1987). It is important that such connecting paths are comprised of a pair of known comparison values. If one of the elements in the pair is missing, the Geometric Mean method cannot be applied.

As stated earlier, formula (2.1) will accurately calculate the comparison value in a perfectly consistent matrix. In an inconsistent matrix, we should consider calculating $X_{i,j}$ using more than one elementary path. The formulation involves multiplying all the possible elementary paths between *i* and *j*, then taking the q^{th} root (where *q* is the number of all possible paths). Note that an elementary path does not always consist of two elements. In a matrix of size *n* the number of elements in an elementary path can be from 2 up to $n-1$. Thus, formula (2.1) can be extended to include these additional elements. We will use CP_r to represent a connecting path with $r+1$ elements. The parameter *r* (called the connecting path index) will define the number of elements in the connecting path (Harker, 1987).

$$CP_r : X_{i,j} = a_{i,k1} \times a_{k1,k2} \times ... \times a_{kr,j}, \quad \text{for } i, j, k1, ..., kr = 1, ..., n \quad \text{and } 1 \le r \le n-2 \qquad (2.2)$$

The following formula provides the general geometric mean estimation:

$$X_{i,j} = \sqrt[q]{\prod_{r=1}^{q} CP_r} \qquad (2.3)$$

where CP_r is a connecting path with $r+1$ elements, *r* is the connecting path index, and *q* is the number of all possible connecting paths for $1 \le r \le n-2$.

To illustrate the Geometric Mean method, we will use the following incomplete matrix as an example:

$$A = \begin{bmatrix} 1.000 & 4.000 & 0.500 & - & 0.250 \\ 0.250 & 1.000 & 2.000 & 8.000 & 0.200 \\ 2.000 & 0.500 & 1.000 & 0.333 & 6.000 \\ - & 0.125 & 3.000 & 1.000 & - \\ 4.000 & 5.000 & 0.167 & - & 1.000 \end{bmatrix}$$

From the previous matrix *A*, the missing comparisons are $X_{1,4}, X_{4,1}, X_{4,5}$ and $X_{4,5}$.

Using formulas (2.1) and (2.2), we can calculate a missing comparison value. The following elementary paths can be determined for $X_{1,4}$.

$$X_{1,4} \approx a_{1,2}a_{2,4}$$
$$X_{1,4} \approx a_{1,3}a_{3,4}$$
$$X_{1,4} \approx a_{1,2}a_{2,3}a_{3,4}$$
$$X_{1,4} \approx a_{1,3}a_{3,2}a_{2,4}$$
$$X_{1,4} \approx a_{1,5}a_{5,2}a_{2,4}$$
$$X_{1,4} \approx a_{1,5}a_{5,3}a_{3,4} \quad \text{... and so on.}$$

In order to simplify the calculations in this example, we will only consider these six elementary paths. Applying formula (2.3), we can estimate the value of $X_{1,4}$ as follows:

$$X_{1,4} = \sqrt[6]{a_{1,2}a_{2,4} \times a_{1,3}a_{3,4} \times a_{1,2}a_{2,3}a_{3,4} \times a_{1,3}a_{3,2}a_{2,4} \times a_{1,5}a_{5,2}a_{2,4} \times a_{1,5}a_{5,3}a_{3,4}}$$

The same procedure can be done to estimate the remaining missing comparisons. One issue that we should observe is that all the elements in an elementary path should be of known values, and not part of the missing comparisons.

In the above calculation, we omit the connecting paths with four elements to simplify the illustration. For a more accurate estimation of the missing value, all possible connecting paths should be included in the multiplication. After all the missing comparisons are estimated, we can replace them with our calculated values, thus completing the whole matrix.

2.5. The Revised Geometric Mean Method

In the previous section, we can see that determining all the possible connecting paths can be computationally difficult. As the dimension of the matrix becomes very large, the number of such connecting paths can be astronomically large. To deal with this problem Harker (1987) has proposed another method called the Revised Geometric Mean method. In this method, instead of estimating the value of $X_{i,j}$, we will simply put that value to be equal to w_i / w_j. These values will be used to transform the matrix into another matrix by multiplying it with the weight vector.

One possible drawback of this methodology is that it is only applicable in an irreducible matrix. An irreducible matrix is a matrix that cannot be decomposed into the form of:

$$\begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix}$$

where A_1 and A_3 are square matrices and 0 is the zero matrix. For an irreducible matrix, there exists a path between all paired entities. This restriction will ensure that a missing comparison can be calculated using the Geometric Mean Method discussed in the previous section.

An example of this methodology is taken from (Harker, 1987) and is presented next. Consider an illustrative matrix A as follows:

$$A = \begin{bmatrix} 1 & 2 & - \\ 1/2 & 1 & 2 \\ - & 1/2 & 1 \end{bmatrix}$$

From formula (1.2), we can replace the missing comparisons with their respective ratios of weights as follows:

$$A = \begin{bmatrix} 1 & 2 & w_1 / w_3 \\ 1/2 & 1 & 2 \\ w_3 / w_1 & 1/2 & 1 \end{bmatrix}$$

Multiplying this revised matrix with its weight vector, we can obtain the following matrix:

$$AW = \begin{bmatrix} 1 & 2 & w_1 / w_3 \\ 1/2 & 1 & 2 \\ w_3 / w_1 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 2w_1 + 2w_2 \\ 1/2w_1 + w_2 + 2w_3 \\ 1/2w_2 + 2w_3 \end{bmatrix}$$

Using the formula of $AW = CW$, a new matrix C can be calculated as follows:

$$C = \begin{bmatrix} 2 & 2 & 0 \\ 1/2 & 1 & 2 \\ 0 & 1/2 & 2 \end{bmatrix}$$

This new matrix C can be used to calculate the right principle eigenvector and also determine the weight vector of matrix A.

In order to determine the transformed matrix (i.e., matrix C in the above example), Harker (1987) provided the following rules:

$$\begin{aligned} c_{i,j} &= a_{i,j}, && \text{if } a_{i,j} \text{ is a real number} > 0 \text{ and } i \neq j \\ c_{i,j} &= 0, && \text{otherwise (and } i \neq j) \\ c_{i,i} &= 1 + m_i, && \text{where } m_i \text{ is the number of unanswered questions in row } i = 1,\dots,n, \end{aligned}$$

where $c_{i,j}$ and $a_{i,j}$ are the elements of matrix C and incomplete matrix A of the above example, respectively.

Using this methodology, we will require the calculation of w_i / w_j to fill in the missing comparisons. This value can be a continuous value, which is not consistent with the Saaty scale. In order to estimate this value to the closest Saaty scale value, we use an approximation rule as discussed in the following section.

- **An Approximation Rule (Chen, 1997)**

This rule will determine the closest value to one of the discrete comparison values in the Saaty scale. If the estimation of w_i / w_j is exactly equal to one of the values in the Saaty scale, we will use this number as the final estimated comparison values. However, more likely, the estimated value will be between two neighboring numbers of the Saaty scale, thus we need to determine which one is closest to the estimated value. Since we are dealing with the Geometric Mean method, we want to use the square root of the product of these neighboring numbers as the criterion to determine the closest scale number.

As an example, let the estimated $X_{i,j}$ value be equal to 3.324. This number is between the two neighboring numbers of 3 and 4 from the Saaty scale. To maintain consistency in the matrix, this value must be approximated either by 3.000 or 4.000. The criterion to determine the closest number is calculated as $(3 \times 4)^{1/2}$, which equals to 3.4641. Since the value of $X_{i,j}$ (= 3.324) is less than 3.4641, we should approximate it to 3.000 as the closest value. This approximated number is then used as the value of the missing comparison.

2.6. The Least Squares Method

This method investigates the error factor in estimating the comparison value (see also Triantaphyllou, *et. al.,* 1990). In a perfectly consistent matrix, formula (1.2) can be used to estimate any comparison value without any error. However, for an inconsistent matrix, formula (1.2) can be modified as follows:

$$a_{i,j} = \frac{w_i}{w_j} d_{i,j}, \quad \text{for } i, j = 1, \ldots, n. \tag{2.4}$$

where $d_{i,j}$ is the deviation of $a_{i,j}$ due to inconsistency.

From formula (2.4), we can determine the error of the comparison $a_{i,j}$ as follows:

$$\varepsilon_{i,j} = d_{i,j} - 1$$

where $\varepsilon_{i,j}$ is the error value in the comparison. Furthermore, the above formula can be written as follows:

$$\varepsilon_{i,j} = \frac{w_j}{w_i} a_{i,j} - 1, \quad \text{for } i, j = 1, \ldots, n. \tag{2.5}$$

The values of $\varepsilon_{i,j}$ will be equal to zero in a perfectly consistent matrix. Moreover, the relative weights must be normalized, which means that the sum of all the weights must be equal to 1.

With these assumptions, a new matrix form can be derived as follows: $BW = b$, where the vector b has zero entries everywhere, except the last one which is equal to 1. W is the weight vector, which we would like to determine, and matrix B is of the form:

$$B = \begin{bmatrix}
-1 & a_{1,2} & & & & & & \\
-1 & & a_{1,3} & & & & & \\
-1 & & & a_{1,4} & & & & \\
\vdots & & & & \ddots & & & \\
\vdots & & & & & \ddots & & \\
-1 & & & & & & a_{1,n-1} & \\
-1 & & & & & & & a_{1,n} \\
 & -1 & a_{2,3} & & & & & \\
 & -1 & & a_{2,4} & & & & \\
 & -1 & & & a_{2,5} & & & \\
 & \vdots & & & & \ddots & & \\
 & -1 & & & & & a_{2,n-1} & \\
 & -1 & & & & & & a_{2,n} \\
 & & & & & & \vdots & \vdots \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}$$

In the previous matrix *B*, empty cells correspond to 0 value.

In an inconsistent matrix, the relation of $BW = b$ will not be true. We can assign a residual vector $r = b - BW$, which we would like to minimize. Since this value can be positive or negative, the idea is to minimize the sum of squares of the vector *r*, which can be written as follows ((Triantaphyllou, *et. al*, 1990) and (Chen, 1997)):

$$\min f^2(x) = \|b - BW\|_2^2 \tag{2.6}$$

By solving the above minimization problem, we can derive the weight vector *W*. Another way to calculate the weight vector *W* is described as follows:

$$W = (B^T B)^{-1} B^T b\,, \tag{2.7}$$

where B^T is the transpose of matrix *B*.

This weight vector can be used to determine the missing comparison values by using formula (1.2). If the calculated value is not one of the scale values, an approximation rule can be used as discussed in section 2.5.

To illustrate this method, we will use the incomplete matrix *A* given in section 2.4. A central task in applying this method is to determine the matrix *B*. The elements on the top row in matrix *A* can be used to determine the top four rows in matrix *B*. The first column will be filled with –1, and each element of the first row in matrix *A* will be placed in matrix *B* in sequence, where the column number remains the same. The first part of matrix *B* is as follows:

$$\begin{bmatrix} -1.000 & 4.000 & 0 & 0 & 0 \\ -1.000 & 0 & 0.500 & 0 & 0 \\ -1.000 & 0 & 0 & 0 & 0.250 \end{bmatrix}$$

We can use this method to determine the second part of matrix *B*, which corresponds to the second row in the incomplete matrix *A:*

$$\begin{bmatrix} 0 & -1.000 & 2.000 & 0 & 0 \\ 0 & -1.000 & 0 & 8.000 & 0 \\ 0 & -1.000 & 0 & 0 & 0.2000 \end{bmatrix}$$

In this method, we will only consider the upper triangular part of the incomplete matrix, and the values of –1 are placed in the same column number as the row where we use its elements. For example, for all the elements in row 1, we will assign the value of –1 in column 1, and for elements from row 2, the value of –1 will be in column 2 and so on.

Another important observation that can be derived is that the number of columns will be the same as the size of the incomplete matrix, while the number of rows will depend on the number of missing comparisons. Since the number of elements on the upper triangular of the matrix is equal to $n(n-1)/2$, the number of rows will be equal to $(n(n-1)/2)-m+1$, where *m* is the number of missing comparisons.

The final matrix *B* from the incomplete matrix *A* is as follows:

$$B = \begin{bmatrix} -1.000 & 4.000 & 0 & 0 & 0 \\ -1.000 & 0 & 0.500 & 0 & 0 \\ -1.000 & 0 & 0 & 0 & 0.250 \\ 0 & -1.000 & 2.000 & 0 & 0 \\ 0 & -1.000 & 0 & 8.000 & 0 \\ 0 & -1.000 & 0 & 0 & 0.200 \\ 0 & 0 & -1.000 & 0.333 & 0 \\ 0 & 0 & -1.000 & 0 & 6.000 \\ 1.000 & 1.000 & 1.000 & 1.000 & 1.000 \end{bmatrix}$$

Using the matrix *B*, we can compute the weight vector *W* using formula (2.7), thus the missing comparisons can be estimated.

2.7. The Goals of This Research

As with the previous discussions, this section summarizes the goals of this work. In this thesis, we will investigate three different questions that arise from the previous sections.

Goal #1: If we start with the minimum number of comparisons (i.e., $n-1$ comparisons), which ones could produce the best estimation of the weight vector?

Five different selection rules of the initial comparisons will be investigated to see their effect on estimating the weight vector. These selection criteria will be discussed further in the next chapter.

Goal #2: If we add additional comparisons one at a time, after how many (or what percentage of all the $n(n-1)/2$ comparisons) can we stop?

Our intuition has suggested that the more initial comparisons one uses, the more likely the weight vector can be estimated correctly. With this assumption in mind, we will try to investigate the effect of adding additional comparisons above the minimum (i.e., $n-1$ comparisons) on the correctness of estimating the weight vector. Another sub-goal that we can investigate is how to select the next comparison values. Chen (1997) has conducted a study on this inquiry, and concluded that it does not make any difference of how we select the next comparisons. A random selection will produce almost identical results as the guided method proposed by Harker (1987).

Goal #3: Does the estimation method (i.e., the Geometric Mean or the Least Squares) have any effect on estimating the missing comparison values?

The effect of these two different methods will be investigated to determine if they play an important role in correctly estimating the missing comparisons. Previous work by Chen (1997) has also concluded that the two methods will produce similar results, thus no significant differences are expected to be found.

As a summary, this research is intended as an addition to Chen's work regarding the incomplete AHP. However, a new question of which selection rule will produce the best estimate will be the main focus of this research. Furthermore, this study will help us gain more understanding into this fascinating methodology.

CHAPTER 3

PROPOSED METHODOLOGY

3.1. Determining the Number of Initial Comparisons

In our discussion in the previous chapter, we concluded that the number of initial comparisons must be between $n-1$ and $n(n-1)/2$, where n is the size of the matrix. Furthermore, in determining these initial comparisons, we should not consider any value in the diagonal entries, since these values are always equal to 1.

Intuitively, the more initial comparisons we have, the more likely we will correctly estimate the remaining missing comparisons. We will start with $n-1$ initial comparisons. An increment of additional initial comparisons will then be added to determine its effect on the estimated weight vector.

3.2. Determining which will be the Initial Comparisons

Previous work on the incomplete AHP failed to address this subject. A random selection of the initial values is accepted to be the easiest and best solution. Carmone *et. al*. (1997) have conducted a study on several deletion rules of a complete matrix to determine if some patterns of the initial comparisons play a role in determining the accuracy of the estimated matrix. In their paper, they concluded that a matrix with high comparison values initially will be more likely to be estimated correctly. The question still remains, as how to determine which comparisons will produce the highest comparison values. Nevertheless, two of their selection rules, namely the best and worst will be included in this paper.

Ra (1999) has also proposed a selection rule for the initial comparisons. In that paper, n initial comparisons were chosen instead of the minimum $n-1$. These n comparisons were positioned in the $n-1$ cells, which were to the right of the diagonal entries, and the one cell in the upper right corner. This extra comparison would enable the consistency of the matrix to be easily checked. Since this selection rule was proposed solely to support his method of Chainwise Paired Comparisons to estimate the missing comparisons, fair evaluation with different selection rules cannot be performed.

Considering the above discussions, this paper proposes five selection rules of the initial comparisons. Initially, the decision maker needs to make at least $n-1$ comparisons. The selection of the comparisons can be done using any one of the following rules:

Rule # 1: All comparisons are made based on one alternative, thus the $a_{i,j}$ value will be determined on a common row (i) (e.g., $a_{i,2}, a_{i,3}, \cdots, a_{i,n}$ excluding $a_{i,i}$), or a common column (j) (e.g., $a_{1,j}, a_{2,j}, \cdots, a_{n,j}$ excluding $a_{j,j}$).

Rule # 2: The alternatives are arranged in decreasing order of their weights. Note that these values are unknown, but the decision maker is assumed to be able to rank them correctly by using his/her previous knowledge. Then, an alternative is compared with the next alternative in the order that they are ranked. This rule will produce the initial comparisons in a diagonal manner (e.g., $a_{1,2}, a_{2,3}, \cdots, a_{n-1,n}$), assuming that the alternatives are ranked as $A_1 \geq A_2 \geq A_3 \geq \cdots \geq A_n$ (where $\geq$ means "better than").

Rule # 3: The selection of the questions regarding the $a_{i,j}$ values is random.

Rule # 4: The comparisons with the highest $a_{i,j}$ values are selected. If the selected comparison lies in the diagonal section of the matrix (i.e., $a_{1,1}, a_{2,2}, \cdots, a_{n,n}$), then the next highest comparison is selected to replace the original comparison to complete the $n-1$ initial comparisons (see also Carmone *et. al.* (1997)).

Rule # 5: The comparisons are ranked starting from the highest value. Then, the comparison in the median is selected as the first initial comparison. To complete the minimum initial comparisons, the comparisons whose ranking neighboring this median comparisons are selected (see also Carmone *et. al.* (1997)).

The effect of these different selection rules will be the main focus of this research.

3.3. Illustration of the Proposed Methodology for the Computational Study

The purpose of this study is to determine whether a selection rule of how to select the initial comparisons will affect the accuracy of estimating the missing comparisons, and ultimately, the accuracy of the weight vector and the ranking of the alternatives. In order to estimate the missing comparisons, we can use any of the above methods, namely the Revised Geometric Mean, or the Least Squares method.

To test the accuracy of our estimated missing values from the incomplete matrix, a complete matrix is needed as the reference. This complete matrix is unknown to the decision maker, thus he/she will estimate the missing comparisons from the incomplete matrix. Obviously, if a complete comparison matrix and its weight vector are already known, there will be no reason trying to estimate the missing comparisons. Thus, the effect of the selection rule of the initial comparisons cannot be determined.

With this assumption, an experimental study will then be conducted in a reverse sequence of the normal decision-making process. First, a complete matrix will be generated as the basis of the experiment. Using this matrix, we will then create an incomplete matrix by deleting some of the comparison values, thus leaving $n-1$ comparisons. Next, the missing comparisons from the incomplete matrix are estimated. After completely estimating the missing comparisons, the weight vector of this estimated matrix is next calculated and compared to the original weight vector from our initial complete matrix.

This study is performed repeatedly for the different selection rules of initial comparisons, along with different size matrices. The details of this methodology are explained in the following steps.

Step 1: Generating random relative weights of the alternatives.

In this step, we randomly generate n relative weights for the alternatives (or criteria) that are to be compared. These numbers are generated from the uniform distribution between 0 and 1 with each random number corresponding to the relative weight of a particular entity. A weight vector W is then formed from these weights. This vector is then normalized into $W^{'}$ with each weight divided by the sum of all the weights in the vector. The formula of the normalization method is as follows:

$$W_i^{'} = \frac{W_i}{\sum_{i=1}^{n} W_i} \quad for \quad i = 1,2,\cdots,n \tag{3.1}$$

where W_i is the relative weight for a particular alternative (or criterion).

Note that these relative weights of the weight vector are unknown to the decision maker. The decision maker is expected to form a similar weight vector using his/her previous knowledge. The value of the relative weights may not be the same, but the ranking order is expected to be identical.

Step 2: Forming the comparison matrix.

After the weight vector W is normalized (i.e., the W' vector is derived), we will calculate the element $a_{i,j}$ using formula (1.2) in the form of a comparison matrix. This comparison matrix is known as the Real Continuous Pairwise (RCP) matrix ((Triantaphyllou, 1995) and (Chen, 1997)) as the entities in the matrix are continuous numbers. Since the above matrix is calculated directly from the weight vector, this matrix will be perfectly consistent.

Step 3: Approximate the continuous comparison values with the ones from the Saaty scale.

In the previous chapter, the decision maker is expected to make the comparisons using the scale proposed by Saaty. For our methodology, the elements in an RCP matrix are likely to be different from the values in the Saaty scale, thus an approximation of these values is required. The approximation method discussed in section 2.5 can be used to transform the continuous comparison values into their nearest Saaty values. The new modified matrix is known as the Closest Discrete Pairwise (CDP) matrix ((Triantaphyllou, 1995) and (Chen, 1997)).

The CDP matrix can be assumed is the matrix to be provided by the decision maker as described in chapter 1. Due to the approximation nature, a CDP matrix may no longer be perfectly consistent. This condition is acceptable since the decision maker may also make some minor errors in the comparisons. If the matrix is reasonably consistent (i.e., its *CR* value is less than 10%), the relative weights will then be calculated using formula (1.5).

The above CDP matrix is considered as the best-case scenario in our study. If and only if the decision maker can be as accurate as possible in every single pairwise comparison, he/she will produce the above CDP matrix. In general, the decision maker will produce a comparison matrix similar to the above CDP matrix due to human errors. The differences between the actual estimated matrix and our best-case scenario matrix (i.e., the CDP matrix) may not be important, as long as the weight vectors are identical in terms of the ranking order of the alternatives (or criteria).

Step 4: Forming an incomplete comparison matrix.

In order to determine if a specific selection rule for the initial pairwise comparisons will lead to a better estimation of the missing comparisons, we will delete most of the comparisons from the above CDP matrix, leaving $n-1$ comparisons according to each one of the rules described in section 3.2. The deleted comparison values are assumed to be undetermined by the decision maker, either by his/her hesitation to make the comparisons, or due to the time constraint of determining the comparisons. Using the Least Squares or the Revised Geometric Mean methods, we will try to estimate the deleted comparison values.

Note that the incomplete matrix is formed with the actual relative weight vector and the ranking of all the weights is known. This original weight vector will then be compared with our estimated one from the incomplete matrix.

Step 5: Estimating the missing comparisons.

Using one of the above methods (i.e., the Least Squares or the Geometric Mean method), the missing comparisons in the incomplete matrix will be estimated. This estimation method will not guarantee an identical result as our original CDP matrix. Next, the weight vector will be calculated and compared with the actual values from the original CDP matrix. The difference between them will determine the accuracy of the estimated matrix.

Step 6: Updating the incomplete matrix.

The accuracy of the estimated weight vector may be unsatisfactory, thus a new missing comparison needs to be estimated. The selection of the next missing comparison can be done either randomly, or using a guided rule as in (Harker, 1987). Intuitively, the increased number of comparisons will improve the accuracy of the weight vector. Thus, the decision maker will go back to step 5 to estimate additional comparisons.

As an illustration to the last two steps, the decision maker first determines the minimum number of $n-1$ initial comparisons. These values are selected using one of the selection rules discussed in section 3.2. After the weight vector is calculated, the next comparison is added to the incomplete matrix (thus, increasing the number of the comparisons by one). A new weight vector is calculated, then it is compared with the first weight vector to see the differences between them.

The last two steps will be performed in a loop until the decision maker is satisfied with the accuracy of the weight vector, or all the missing comparisons are calculated. Harker (1987) has suggested stopping the loop if the maximum absolute difference in the attribute weights from one question to the next is $\leq \alpha$, where α is a given constant (e.g., α = 10%). Of course, the decision maker can stop the loop at anytime, either due to a time constraint or other factors, without using Harker’s rule.

CHAPTER 4

AN EXTENSIVE ILLUSTRATIVE EXAMPLE

4.1. Determining the Original Weight Vector

To illustrate the steps described in the previous chapter, the following weight vector W is generated containing five entities:

$$W = \begin{bmatrix} 0.66757076 \\ 0.34038541 \\ 0.45607005 \\ 0.60076497 \\ 0.23282216 \end{bmatrix}$$

Note that the relative weights on the above vector W are generated randomly from the uniform distribution between 0 and 1. Using formula (3.1), this weight vector is then normalized as follows:

$$W' = \begin{bmatrix} 0.29054965 \\ 0.14814738 \\ 0.19849730 \\ 0.26147348 \\ 0.10133217 \end{bmatrix}$$

Next, an RCP matrix is formed by using the above weight vector W'. An illustration of the calculation can be seen as we take as example the $a_{1,2}$ value. This value is calculated from the ratio of values from the first row of our normalized weight vector (= 0.29054965) with the value from the second row of the same vector (= 0.14814738). Thus, the value of $a_{1,2}$ is equal to $\frac{0.29054965}{0.14814738} = 1.96122031$. The complete RCP matrix A is as follows:

$$A = \begin{bmatrix} 1 & 1.96122031 & 1.46374611 & 1.11120122 & 2.86729920 \\ 0.50988662 & 1 & 0.74634456 & 0.56658664 & 1.46199751 \\ 0.68317859 & 1.33986372 & 1 & 0.75914881 & 1.95887742 \\ 0.89992702 & 1.76495514 & 1.31726479 & 1 & 2.58036002 \\ 0.34876025 & 0.68399569 & 0.51049647 & 0.38754282 & 1 \end{bmatrix}$$

From the above RCP matrix *A*, we can form the CDP matrix by approximating the comparisons with continuous numbers in the above matrix into their closest Saaty values. The criterion to determine the closest scale number is calculated using the square root of the product of the two neighboring scale numbers.

As an example, observe that the actual value of $a_{1,3}$ is 1.463746106. This continuous value is different from the ones in the Saaty scale, which only consists of the discrete numbers (i.e., 1, 2, 3, …, 9) and their reciprocals. Using the approximation rule discussed in section 2.5, we can estimate this value to its closest discrete value. Since this value is between the two neighboring numbers of 1 and 2, it must be approximated to either 1 or 2. The criterion to determine the closest number is calculated as $(1 \times 2)^{1/2}$, which equals to 1.41421. As the value of $a_{1,3}$ (= 1.463746106) is greater than 1.41421, it will be approximated to 2 as the closest value.

This process is repeated until all the continuous values in our comparison matrix are estimated to their closest discrete numbers according to the Saaty scale. The modified comparison matrix A' is as follows:

$$A' = \begin{bmatrix} 1 & 2 & 2 & 1 & 3 \\ 1/2 & 1 & 1 & 1/2 & 2 \\ 1/2 & 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 & 3 \\ 1/3 & 1/2 & 1/2 & 1/3 & 1 \end{bmatrix}$$

Using matrix A' (which is the CDP matrix for this example), the weight vector can be determined using formula (1.5). However, the method of using the eigenvector calculation will consume extensive time, thus the approximation method using the geometric means method will be used. When formula (1.6) is used, the weight vector of matrix $A^{'}$ is as follows:

$$W'' = \begin{bmatrix} 1.64375183 \\ 0.87055056 \\ 1 \\ 1.43096908 \\ 0.48835934 \end{bmatrix}$$

After we normalize the above weight vector, our reference weight vector is:

$$W^{*} = \begin{bmatrix} 0.30251445 \\ 0.16021526 \\ 0.18403900 \\ 0.26335412 \\ 0.08987717 \end{bmatrix}$$

The actual values of the weight vector may not be as important as the order ranking of each individual alternative (or criterion). The decision maker is more likely to appreciate which alternative is the best than knowing the actual value of its weight vector. With this assumption, the accuracy of the estimated weight vector is determined based on how accurate the ranking of the alternatives is compared to the ranking from the original weight vector. Using the assumption that the higher the weight, the better it is, the original ranking of the alternatives is 1-4-3-2-5. One may compare different rankings by employing the methods discussed in (Ray and Triantaphyllou, 1998 and 1999).

Another method that one can use to determine the accuracy of the estimated weight vector is to calculate the average difference between two vectors. The smaller the difference, the better a methodology is. As an illustration, let W be the new estimated weight vector. We should consider the difference between W and W^{*} (i.e., $\frac{\sum_{i=1}^{n} \left| w_i^{*} - w_i \right|}{n}$)

to determine the accuracy of the estimated weight vector. This method can be very useful to determine which alternative is better if two or more values are very close to each other (e.g., the values of second and third alternatives in our original weight vector).

4.2. Calculation of the Weight Vector by Using the Least Squares Method

In this section, we would like to compare different selection rules for the initial comparisons discussed in section 3.2. Using the Least Squares method, the missing comparisons can be calculated, and consequently the weight vector can be estimated. This calculated weight vector, specifically the order of the ranking, is then compared with our initial reference weight vector. Recall that the reference weight vector for the current illustrative example is:

$$W^* = \begin{bmatrix} 0.30251445 \\ 0.16021526 \\ 0.18403900 \\ 0.26335412 \\ 0.08987717 \end{bmatrix}$$

Then the steps to take are described in the following subsections.

4.2.1. Initial Comparisons Based on a Common Row or Column

From the previous matrix A', we will delete most of the comparisons leaving $n-1$ comparisons on a common row or column. As an example, we will use the first row (and consequently the first column) as our initial comparison values. The modified matrix (which is assumed to be the initial incomplete comparison matrix for this example) is as follows:

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 3 \\ 1/2 & 1 & - & - & - \\ 1/2 & - & 1 & - & - \\ 1 & - & - & 1 & - \\ 1/3 & - & - & - & 1 \end{bmatrix}$$

In order to determine its weight vector, formula (2.7) is used, thus we need to determine matrix *B* as described in section 2.5. From the above incomplete matrix *A*, the matrix *B* is formed as follows:

$$B = \begin{bmatrix} -1.000 & 2.000 & 0 & 0 & 0 \\ -1.000 & 0 & 2.000 & 0 & 0 \\ -1.000 & 0 & 0 & 1.000 & 0 \\ -1.000 & 0 & 0 & 0 & 3.000 \\ 1.000 & 1.000 & 1.000 & 1.000 & 1.000 \end{bmatrix}$$

Using formula (2.7), the weight vector is calculated as follows:

$$W = \begin{bmatrix} 0.30 \\ 0.15 \\ 0.15 \\ 0.30 \\ 0.10 \end{bmatrix}$$

This calculated weight vector would be used to fill in the missing comparisons in our initial matrix. According to Carmone *et. al.* (1997), an incomplete matrix with high $a_{i,j}$ values as their initial values will be more likely to be estimated correctly. With this information, $a_{4,5}$ (= 3.000) is chosen to be included as the next entry in our new incomplete matrix. Thus, the revised matrix *A* is as follows:

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 3 \\ 1/2 & 1 & - & - & - \\ 1/2 & - & 1 & - & - \\ 1 & - & - & - & 3 \\ 1/3 & - & - & 1/3 & 1 \end{bmatrix}$$

Again, using formula (2.7), the new weight vector is calculated as follows:

$$W = \begin{bmatrix} 0.30 \\ 0.15 \\ 0.15 \\ 0.30 \\ 0.10 \end{bmatrix}$$

This process is repeated until all the missing comparisons are calculated. The complete matrix *A* and its weight vector *W* can be shown to be as follows:

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 3 \\ 1/2 & 1 & 1 & 1/2 & 3/2 \\ 1/2 & 1 & 1 & 1/2 & 3/2 \\ 1 & 2 & 2 & 1 & 3 \\ 1/3 & 2/3 & 2/3 & 1/3 & 1 \end{bmatrix} \qquad W = \begin{bmatrix} 0.30 \\ 0.15 \\ 0.15 \\ 0.30 \\ 0.10 \end{bmatrix}$$

In this illustrative example, we cannot distinguish the ranking between alternatives 1 and 4, and the same holds true with alternatives 2 and 3. Thus, the ranking of the alternatives can be any of the following: 1-3-4-2-5, 1-4-3-2-5, 2-3-4-1-5, or 2-4-3-1-5. Of all the selections, one of them matches our initial ranking perfectly, while the other three are very close.

Another method that we can use to determine whether the estimated weight vector is as good as the reference one will be to calculate $\frac{\sum_{i=1}^{n} \left| w_i^* - w_i \right|}{n}$. The differences between the two vectors are as follows:

$$\left| W^* - W \right| = \left| \begin{bmatrix} 0.30251445 \\ 0.16021526 \\ 0.18403900 \\ 0.26335412 \\ 0.08987717 \end{bmatrix} - \begin{bmatrix} 0.30 \\ 0.15 \\ 0.15 \\ 0.30 \\ 0.10 \end{bmatrix} \right| = \begin{bmatrix} 0.00251445 \\ 0.01021526 \\ 0.03403900 \\ 0.03664588 \\ 0.01012283 \end{bmatrix}$$

Using the above vector, we can calculate the average difference of the two vectors. Thus, $\frac{\sum_{i=1}^{n}\left|w_i^* - w_i\right|}{n}$ is equal to 0.018707484. This value is then compared with the others from different selection strategies of the initial comparisons. The selection strategy with the smallest average difference value is considered to be the best.

4.2.2. Initial Comparisons Based on Pre-Ranking of the Alternatives

In this section, we will re-arrange the alternatives in decreasing order of their values in the weight vector. At this point, we assume that the decision maker does not have the exact value of this weight vector, but using his/her previous knowledge, he/she is able to re-arrange the alternatives according to their importance in decreasing order. In our example, the complete matrix $A^{'}$ is re-arranged as follows:

$$A^{''} = \begin{bmatrix} 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 \\ 1/2 & 1 & 1 & 1 & 2 \\ 1/2 & 1/2 & 1 & 1 & 2 \\ 1/3 & 1/3 & 1/2 & 1/2 & 1 \end{bmatrix}$$

With this modification, the ranking of the alternatives will be: 1-2-3-4-5. The above matrix A is transformed into an incomplete matrix using the second rule mentioned in section 3.2 as follows:

$$A = \begin{bmatrix} 1 & 1 & - & - & - \\ 1 & 1 & 1 & - & - \\ - & 1 & 1 & 1 & - \\ - & - & 1 & 1 & 2 \\ - & - & - & 1/2 & 1 \end{bmatrix}$$

By repeating the same steps as in section 4.2.1, the initial matrix *B* (as described in section 2.5) and its weight vector are as follows:

$$B = \begin{bmatrix} -1.000 & 1.000 & 0 & 0 & 0 \\ 0 & -1.000 & 1.000 & 0 & 0 \\ 0 & 0 & -1.000 & 1.000 & 0 \\ 0 & 0 & 0 & -1.000 & 2.000 \\ 1.000 & 1.000 & 1.000 & 1.000 & 1.000 \end{bmatrix}$$

$$W = \begin{bmatrix} 0.222 \\ 0.222 \\ 0.222 \\ 0.222 \\ 0.111 \end{bmatrix}$$

With this rule of selecting the initial comparisons, the ranking of the alternatives may become more confusing. The first four alternatives have the same weight, thus the order of their ranks is practically unknown.

Using the differences between the estimated and the reference weight vector, we can form the following vector:

$$\left| W^* - W \right| = \begin{bmatrix} 0.08051445 \\ 0.06178474 \\ 0.03796100 \\ 0.04135412 \\ 0.02112283 \end{bmatrix}$$

From the above vector, the average difference of the two vectors is 0.08091238.

4.2.3. Random Selection of the Initial Comparisons

Another rule that we will investigate is random selection. In this section, the incomplete matrix *A* is selected as follows:

$$A = \begin{bmatrix} 1 & - & 2 & - & 3 \\ - & 1 & - & 1/2 & - \\ 1/2 & - & 1 & 1 & - \\ - & 2 & 1 & 1 & - \\ 1/3 & - & - & - & 1 \end{bmatrix}$$

That is, the initial set of known comparisons is at random places.

One important issue that we need to discuss regarding this selection rule is the presence of a part that connects all the alternatives. It means that there is a connection between an alternative and every other alternative either directly or indirectly. Since the selection of these initial comparisons is done randomly, a connecting path is not guarantee. When this problem occurs, we reselect the comparisons again.

Formula (2.7) is used to determine the matrix *B* and eventually its weight vector. The matrix *B* and its calculated weight vector are as follows:

$$B = \begin{bmatrix} -1.000 & 0 & 2.000 & 0 & 0 \\ -1.000 & 0 & 0 & 0 & 3.000 \\ 0 & -1.000 & 0 & 0.500 & 0 \\ 0 & 0 & -1.000 & 1.000 & 0 \\ 1.000 & 1.000 & 1.000 & 1.000 & 1.000 \end{bmatrix} \qquad W = \begin{bmatrix} 0.387 \\ 0.097 \\ 0.194 \\ 0.194 \\ 0.129 \end{bmatrix}$$

Using this random selection, the corresponding ranking of the weight vector can be any of the following: 1-5-2-3-4 or 1-5-3-2-4. Neither one of these results matches with our original ranking, thus this methodology might be unpredictable.

If we use the differences of $\left|W^* - W\right|$, the following vector is formed:

$$\left|W^* - W\right| = \begin{bmatrix} 0.08448555 \\ 0.06321526 \\ 0.00996100 \\ 0.06935412 \\ 0.03912283 \end{bmatrix}$$

Thus, the average difference of the two vectors is 0.08871292.

4.2.4. Initial Comparisons Based on the Highest Comparison Values

In this selection rule, each comparison in our reference matrix is ranked starting from the highest (i.e., the highest value be the number one ranking). When two or more comparisons are of equal values, the rankings will be chosen arbitrarily. Thus, the rankings of the comparisons in the reference matrix are as follows:

$$R = \begin{bmatrix} 8 & 3 & 4 & 9 & 1 \\ 19 & 10 & 11 & 20 & 5 \\ 21 & 12 & 13 & 14 & 6 \\ 15 & 7 & 16 & 17 & 2 \\ 24 & 22 & 23 & 25 & 18 \end{bmatrix}$$

Using this ranking matrix, we select the $n-1$ comparisons with the highest ranks. Thus, the resulting incomplete matrix is as follows:

$$A = \begin{bmatrix} 1 & 2 & 2 & - & 3 \\ 1/2 & 1 & - & - & - \\ 1/2 & - & 1 & - & - \\ - & - & - & 1 & 3 \\ 1/3 & - & - & 1/3 & 1 \end{bmatrix}$$

Note that from the above example, the matrix has a path that connects all of alternatives. However, because the reference matrix is generated randomly, sometimes a connecting path does not exist. If such case arises, we need to add more comparisons beyond the minimum $n-1$ comparisons to solve this inherit problem.

Using formula (2.7), we determine the B matrix, along with the weight vector. The B matrix and its weight vector are as follows:

$$B = \begin{bmatrix} -1.000 & 2.000 & 0 & 0 & 0 \\ -1.000 & 0 & 2.000 & 0 & 0 \\ -1.000 & 0 & 0 & 0 & 3.000 \\ 0 & 0 & 0 & -1.000 & 3.000 \\ 1.000 & 1.000 & 1.000 & 1.000 & 1.000 \end{bmatrix} \qquad W = \begin{bmatrix} 0.3 \\ 0.15 \\ 0.15 \\ 0.3 \\ 0.1 \end{bmatrix}$$

In this selection rule, the first and forth entries have the same weights, thus we cannot distinguish which alternative is more important. The same is also true for the second and third alternatives. Thus the ranking of this weight vector can be: 1-3-4-2-5, 1-4-3-2-5, 2-3-4-1-5, or 2-4-3-1-5. To distinguish the impact of this selection rule, we can use the difference between the estimated and the reference weight vector. Thus, the following vector is formed:

$$\left|W^* - W\right| = \left\| \begin{bmatrix} 0.30251445 \\ 0.16021526 \\ 0.18403900 \\ 0.26335412 \\ 0.08987717 \end{bmatrix} - \begin{bmatrix} 0.30 \\ 0.15 \\ 0.15 \\ 0.30 \\ 0.10 \end{bmatrix} \right\| = \begin{bmatrix} 0.00251445 \\ 0.01021526 \\ 0.03403900 \\ 0.03664588 \\ 0.01012283 \end{bmatrix}$$

The average difference between these two vectors is 0.018707484.

4.2.5. Initial Comparisons Based on the Median Comparison Values

With this selection rule, we will choose the comparisons in the median range. Recall that the rankings of our reference matrix is as follows:

$$R = \begin{bmatrix} 8 & 3 & 4 & 9 & 1 \\ 19 & 10 & 11 & 20 & 5 \\ 21 & 12 & 13 & 14 & 6 \\ 15 & 7 & 16 & 17 & 2 \\ 24 & 22 & 23 & 25 & 18 \end{bmatrix}$$

In this example, the median is the 13th ranked comparison. In order to easily select the initial comparisons, the above ranking matrix can be modified. The 13th ranked comparison can be seen as the highest ranked comparison, then the 14th ranked comparison as the second highest ranked, the 12th ranked comparison as the third highest ranked, the 15th ranked comparison as the forth highest ranked, and so on. This procedure is done until all the reference rankings are modified into the following matrix:

$$R' = \begin{bmatrix} 10 & 20 & 18 & 8 & 24 \\ 13 & 6 & 4 & 15 & 16 \\ 17 & 2 & 1 & 3 & 14 \\ 5 & 12 & 7 & 9 & 22 \\ 13 & 19 & 21 & 25 & 11 \end{bmatrix}$$

After modifying the ranking matrix, the initial comparisons can be selected with the same method by using the highest ranked comparisons. In this example, we selected the top four comparisons.

Note that the first ranked comparison lies on the diagonal entries. Since the diagonal entries are already included in our incomplete matrix, we need to replace this comparison with the next ranked comparison. For our example, the next ranked comparison is no 5.

Furthermore, we also need to check if all these comparisons are unique. This means that they are not the reciprocal-pair of an existing comparison. For example, the 2nd and 4th ranked comparisons are the reciprocal-pair of one another, thus we need to select another additional comparison to fill the initial incomplete matrix.

After incorporating all the information, our initial incomplete matrix is as follows:

$$A = \begin{bmatrix} 1 & - & - & 1 & - \\ - & 1 & 1 & 1/2 & - \\ - & 1 & 1 & 1 & - \\ 1 & 2 & 1 & 1 & - \\ - & - & - & - & 1 \end{bmatrix}$$

Note that in this example, the fifth alternative has no connecting path with any other alternative. Therefore, we need to add more comparisons in the same way until such connecting path is achieved. The modified initial incomplete matrix is as follows:

$$A = \begin{bmatrix} 1 & - & - & 1 & - \\ - & 1 & 1 & 1/2 & - \\ - & 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 & - \\ - & - & 1/2 & - & 1 \end{bmatrix}$$

Again, by using formula (2.7) we can determine the *B* matrix and finally the weight vector to be as follows:

$$B = \begin{bmatrix} -1.000 & 0 & 0 & 1.000 & 0 \\ 0 & -1.000 & 1.000 & 0 & 0 \\ 0 & -1.000 & 0 & 0.500 & 0 \\ 0 & 0 & -1.000 & 1.000 & 0 \\ 0 & 0 & -1.000 & 0 & 2.000 \\ 1.000 & 1.000 & 1.000 & 1.000 & 1.000 \end{bmatrix} \qquad W = \begin{bmatrix} 0.253 \\ 0.172 \\ 0.214 \\ 0.248 \\ 0.108 \end{bmatrix}$$

The ranking of the weight vector is 1-4-3-2-5, which is exactly the same as our reference weight vector. The difference of the incomplete and reference weight vector is as follows:

$$\left| W^* - W \right| = \left\| \begin{bmatrix} 0.30251445 \\ 0.16021526 \\ 0.18403900 \\ 0.26335412 \\ 0.08987717 \end{bmatrix} - \begin{bmatrix} 0.253 \\ 0.172 \\ 0.214 \\ 0.248 \\ 0.108 \end{bmatrix} \right\| = \begin{bmatrix} 0.04951445 \\ 0.01178474 \\ 0.02996100 \\ 0.01535412 \\ 0.01812283 \end{bmatrix}$$

Thus, the average difference is 0.024947428.

From the previous five rules of selecting the initial comparisons, the first rule produced the best result for this illustrative example. Of course, we cannot make this generalization based on our small example. More computational experiments are needed using extensive test matrices.

4.3. Calculation of the Weight Vector by Using the Revised Geometric Mean Method

As in the previous section, the calculation of the missing comparisons, and their weight vector, can also be done by using the Geometric Mean method. Again, different selection rules of the initial comparisons will be investigated in this section. Using the same initial weight vector as in section 4.1, the same matrix $A^{'}$ can be calculated as follows:

$$A^{'} = \begin{bmatrix} 1 & 2 & 2 & 1 & 3 \\ 1/2 & 1 & 1 & 1/2 & 2 \\ 1/2 & 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 & 3 \\ 1/3 & 1/2 & 1/2 & 1/3 & 1 \end{bmatrix}$$

4.3.1. Initial Comparisons Based on a Common Row or Column

From the complete matrix $A^{'}$, we will select the first row (and consequently its corresponding first column) as our initial comparisons. Note that we chosen the first row arbitrarily. A similar result is expected when other row is chosen. The incomplete matrix A is as follows:

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 3 \\ 1/2 & 1 & - & - & - \\ 1/2 & - & 1 & - & - \\ 1 & - & - & 1 & - \\ 1/3 & - & - & - & 1 \end{bmatrix}$$

Using the formula of $AW = CW$, we can derived the matrix C as follows:

$$C = \begin{bmatrix} 1 & 2 & 2 & 1 & 3 \\ 1/2 & 4 & 0 & 0 & 0 \\ 1/2 & 0 & 4 & 0 & 0 \\ 1 & 0 & 0 & 4 & 0 \\ 1/3 & 0 & 0 & 0 & 4 \end{bmatrix}$$

From matrix *C*, we can determine the corresponding weight vector using the right principle eigenvector method. The weight vector is calculated as follows:

$$W = \begin{bmatrix} 0.300 \\ 0.150 \\ 0.150 \\ 0.300 \\ 0.100 \end{bmatrix}$$

In this example, we cannot distinguish the ranking between alternatives 1 and 4, and the same with alternatives 2 and 3. Thus, the ranking of the alternatives can be any of the following: 1-3-4-2-5, 1-4-3-2-5, 2-4-3-1-5, or 2-4-3-1-5. One of them matches our original ranking perfectly, while the rest of them are very close.

As before, another method that one can use to determine the accuracy of the estimated weight vector is to calculate the average difference between the estimated and the reference weight vector. The differences between the two vectors are as follows:

$$\left| W^{*} - W \right| = \begin{bmatrix} 0.00251445 \\ 0.01021526 \\ 0.03403900 \\ 0.03664588 \\ 0.01012283 \end{bmatrix}$$

Using the above vector, the average difference $\left(\text{i.e., } \frac{\sum_{i=1}^{n} \left| w_i^* - w_i \right|}{n} \right)$ is 0.018707484. Note that this value is identical to the one produced by the Least Squares method.

4.3.2. Initial Comparisons Based on Pre-Ranking of the Alternatives

As described in section 4.2.2, we will re-arrange the alternatives in decreasing order of their values in the weight vector. In our example, the complete matrix $A^{'}$ is re-arranged as follows:

$$A^{''} = \begin{bmatrix} 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 \\ 1/2 & 1 & 1 & 1 & 2 \\ 1/2 & 1/2 & 1 & 1 & 2 \\ 1/3 & 1/3 & 1/2 & 1/2 & 1 \end{bmatrix}$$

With this modification, the ranking of the alternatives will also change to 1-2-3-4-5. The above matrix is then transformed into an incomplete matrix by using the second rule in section 3.2 as follows:

$$A = \begin{bmatrix} 1 & 1 & - & - & - \\ 1 & 1 & 1 & - & - \\ - & 1 & 1 & 1 & - \\ - & - & 1 & 1 & 2 \\ - & - & - & 1/2 & 1 \end{bmatrix}$$

Using the formula $AW = CW$, we can determine the new matrix C as follows:

$$C = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1/2 & 4 \end{bmatrix}$$

Again, by using the right principle eigenvector method, the following weight vector can be determined:

$$W = \begin{bmatrix} 0.222 \\ 0.222 \\ 0.222 \\ 0.222 \\ 0.111 \end{bmatrix}$$

The result of this second selection rule is identical to the one using the Least Squares method, where the first four alternatives have the same values. Thus, the order of their ranks is unknown.

Another method using the differences between the estimated and the reference weight vector produces the following vector:

$$\left| W^* - W \right| = \begin{bmatrix} 0.08051445 \\ 0.06178474 \\ 0.03796100 \\ 0.04135412 \\ 0.02112283 \end{bmatrix}$$

The average difference of the two vectors is 0.08091238. Again, this value of the average difference of the two vectors is identical to the one produced by the Least Squares method.

4.3.3. Random Selection of the Initial Comparisons

In this section, we will randomly select $n-1$ comparisons as our initial values in the incomplete matrix *A* while making sure they form a connecting path. The matrix *A* is formulated as follows:

$$A = \begin{bmatrix} 1 & - & 2 & - & 3 \\ - & 1 & - & 1/2 & - \\ 1/2 & - & 1 & 1 & - \\ - & 2 & 1 & 1 & - \\ 1/3 & - & - & - & 1 \end{bmatrix}$$

Similar to the calculation from the previous section, a new matrix *C* is formed using the formula $AW = CW$. The new matrix *C* is as follows:

$$C = \begin{bmatrix} 3 & 0 & 2 & 0 & 3 \\ 0 & 4 & 0 & 1/2 & 0 \\ 1/2 & 0 & 3 & 1 & 0 \\ 0 & 2 & 1 & 3 & 0 \\ 1/3 & 0 & 0 & 0 & 4 \end{bmatrix}$$

From the matrix *C*, the weight vector can be estimated using the right principle eigenvector. The weight vector *W* can be expressed as follows:

$$W = \begin{bmatrix} 0.387 \\ 0.097 \\ 0.194 \\ 0.194 \\ 0.129 \end{bmatrix}$$

Since we cannot distinguish the ranking between alternatives 3 and 4, the ranking of the alternatives can be either of the following: 1-5-2-3-4, or 1-5-3-2-4. One of these results is very close to the original ranking (i.e., 1-4-3-2-5).

Another approach that we can use to determine the difference in the two rankings of the alternatives is to calculate the average difference of the two weight vectors. The difference of the two vectors is as follows:

$$\left|W^{*}-W\right| = \begin{bmatrix} 0.08448555 \\ 0.06321526 \\ 0.00996100 \\ 0.06935412 \\ 0.03912283 \end{bmatrix}$$

Thus, the average difference of the estimated weight and the reference vectors is 0.08871292.

4.3.4. Initial Comparisons Based on the Highest Comparison Values

As with the Least Squares Method, this selection rule will have the same initial incomplete matrix *A* as follows:

$$A = \begin{bmatrix} 1 & 2 & 2 & - & 3 \\ 1/2 & 1 & - & - & - \\ 1/2 & - & 1 & - & - \\ - & - & - & 1 & 3 \\ 1/3 & - & - & 1/3 & 1 \end{bmatrix}$$

From this matrix, we can calculate the *C* matrix using the formula $AW = CW$:

$$C = \begin{bmatrix} 2 & 2 & 2 & 0 & 3 \\ 1/2 & 4 & 0 & 0 & 0 \\ 1/2 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 3 \\ 1/3 & 0 & 0 & 1/3 & 3 \end{bmatrix}$$

Using this matrix, we can estimate the weight vector by using right principal eigenvector. The weight vector is calculated as follows:

$$W = \begin{bmatrix} 0.3 \\ 0.15 \\ 0.15 \\ 0.3 \\ 0.1 \end{bmatrix}$$

The resulting weight vector is exactly the same using the Least Squares Method, thus the resulting rankings along with the average difference of the incomplete and reference vector are identical.

4.3.5. Initial Comparisons Based on the Median Comparison Values

Again, this selection rule is identical to the one used for the Least Squares Method. The initial incomplete matrix is as follows:

$$A = \begin{bmatrix} 1 & - & - & 1 & - \\ - & 1 & 1 & 1/2 & - \\ - & 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 & - \\ - & - & 1/2 & - & 1 \end{bmatrix}$$

Thus, using the formula $AW = CW$, we can calculate the C matrix as follows:

$$C = \begin{bmatrix} 4 & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 1/2 & 0 \\ 0 & 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 & 0 \\ 0 & 0 & 1/2 & 0 & 4 \end{bmatrix}$$

By means of right eigenvector method, we can estimate the weight vector as:

$$W = \begin{bmatrix} 0.253 \\ 0.172 \\ 0.214 \\ 0.248 \\ 0.108 \end{bmatrix}$$

Again, this weight vector is identical to the one calculated by using the Least Squares Method. Therefore, the resulting rankings and the mean difference will be the same.

CHAPTER 5

COMPUTATIONAL RESULTS

As stated earlier, the main focus of this thesis research is to determine which selection rule will produce the best-estimated weight vector. In order to simulate the real world application, a computer program is built to produce the necessary calculations. The procedure for this experiment is described in section 3.3. Matrices of dimension 10, 15 and 20 are chosen to determine which selection rule will perform the best. These dimensions are chosen because of their "medium" size, thus similar results can be expected for smaller or larger dimensions.

For the forth and fifth selection rules, the average number of initial comparisons is expected to be greater than $n-1$. Thus, additional matrices with different dimensions are formed to determine if there is any correlation between the size of the matrix and the average number of initial comparisons.

In order to preserve accuracy in the methodology, 100 random matrices are generated to determine the effect of each selection rule in estimating the weight vector. On each repetition, a reference matrix is generated from random numbers, ensuring the unique randomness of each repetition.

The computational experiment was performed using IBM SP cluster, Power 3 – II architecture machines on the LSU mainframe system (also known as Casper), and the program is written in the FORTRAN language with the use of some subroutines from IMSL 4.01 Library.

5.1. Results of the Best Selection Rule (Research Goal #1)

In order to determine which selection rule most accurately will estimate the weight vector, four comparison criteria are used to compare the estimated and the reference weight vectors. They are as follows:

- Mean absolute difference.

The difference of each weight from the estimated and reference weight vectors is calculated in terms of the absolute value. Using absolute values, the average difference can then be calculated.

- Mean sum-square-error difference.

  Similar to the previous method, each weight from the estimated weight vector is compared to each weight from the reference vector in terms of sum-square-error method. After all the sum-square-error differences are known, the average can be calculated.

- Percentage of the sum-square-error in ranking difference.

  The weights in the estimated and reference weight vectors are first ranked starting from the highest. Then, these ranking vectors are compared in terms of the sum-square-error method, and its percentage is calculated.

- Percentage of weighted-error in ranking difference.

  As in the previous method, the ranking vectors are compared. However, the differences in the ranking are weighted according to the importance of each weight. For the weight with the best rank (#1), we assign the highest penalty should the ranking of the two vectors are different. The percentage of the weighted errors compared to the possible maximum error is then calculated.

In order to illustrate the comparison criteria in more detail, the following weight vectors can be used:

$$W_1 = \begin{bmatrix} 0.50 \\ 0.25 \\ 0.15 \\ 0.10 \end{bmatrix} \qquad W_2 = \begin{bmatrix} 0.25 \\ 0.35 \\ 0.10 \\ 0.30 \end{bmatrix}$$

From the two weight vectors, we can calculate the absolute difference as follows:

$$W^* = \left| \begin{bmatrix} 0.50 \\ 0.25 \\ 0.15 \\ 0.10 \end{bmatrix} - \begin{bmatrix} 0.25 \\ 0.35 \\ 0.10 \\ 0.30 \end{bmatrix} \right| = \begin{bmatrix} 0.25 \\ 0.10 \\ 0.05 \\ 0.20 \end{bmatrix}$$

Thus the mean absolute difference is 0.15.

Another comparison criterion is to compare the two weight vectors using the sum-square-error method. The difference can be calculated as follows:

$$W^* = \left(\begin{bmatrix} 0.50 \\ 0.25 \\ 0.15 \\ 0.10 \end{bmatrix} - \begin{bmatrix} 0.25 \\ 0.35 \\ 0.10 \\ 0.30 \end{bmatrix} \right)^2 = \begin{bmatrix} 0.0625 \\ 0.0100 \\ 0.0025 \\ 0.0400 \end{bmatrix}$$

Thus the average sum-square-error difference is 0.02875.

The third and forth methods are based on comparing the estimated and reference weight vector in respect of their ranking. Using W_1 and W_2, the ranking vectors of their weights can be shown to be as follows:

$$R_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad R_2 = \begin{bmatrix} 3 \\ 1 \\ 4 \\ 2 \end{bmatrix}$$

For the third comparison criterion, the two ranking vectors are compared using the sum-square-error method. The resulting vector is as follows:

$$R^* = \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 4 \\ 2 \end{bmatrix} \right)^2 = \begin{bmatrix} 4 \\ 1 \\ 1 \\ 4 \end{bmatrix}$$

In order to see the impact of this criterion, the sum of the sum-square-error ranking difference is compared to the sum of the maximum difference. The maximum difference for the sum-square-error method can be achieved if the two ranking vectors are the reciprocal of one another.

$$R_{max} = \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} \right)^2 = \begin{bmatrix} 9 \\ 1 \\ 1 \\ 9 \end{bmatrix}$$

The sum of $R*$ is then compared to the sum of R_{max} to determine the percentage sum-square-error difference in the ranking. The result is 50 %.

The forth comparison criterion is similar to the third one. Instead of using the sum square error, each weight on the reference vector is given a penalty, with the highest ranked weight having the highest penalty. For our example, say that R_1 is the reference vector. Thus, the penalty can be assigned as follows:

$$P = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

For our example, if the first weights of the reference and estimated weight vector are different in ranking, a penalty of 4 is assigned; if the difference lies in the second weight, a penalty of 3 is assigned, and so on. The sum of all penalties is then compared to the maximum possible penalty to determine its percentage. For our example, since all the weight rankings are different, the percentage will be 100 %.

5.1.1. Comparisons of the Selection Rules Using the Mean Absolute Difference

In this section, the selection rules are compared using the mean absolute difference method. Random matrices of dimension 10, 15 and 20 were used and the number of replication was equal to 100. The results for dimension 10 are presented in Figure 5-1 (parts *a* and *b*), for dimension 15 are presented in Figure 5-1 (parts *c* and *d*), and for dimension 20 are presented in Figure 5-1 (parts *e* and *f*). Based on these results, the following observations can be drawn:

Observations:

- The second selection rule performs better than the other selection rules, specially using the Least Squares Method. The difference is not apparent when using the Geometric Mean Method.
- When comparing only the minimum initial comparisons (i.e., $n-1$ comparisons), the forth selection rule seems to be the best choice. However, as discovered earlier, the average minimum initial comparisons for the forth and fifth rules are more than $n-1$ comparisons. Because of this finding, the forth selection rule cannot be compared fairly based only on the minimum initial comparisons (i.e., $n-1$ comparisons). Thus, the first selection rule can be seen as the best when estimating the minimum initial comparisons.
- Between the two estimating methods (i.e., the Least Squares and the Geometric Mean), the Geometric Mean method performs consistently better regardless of the selection rule.

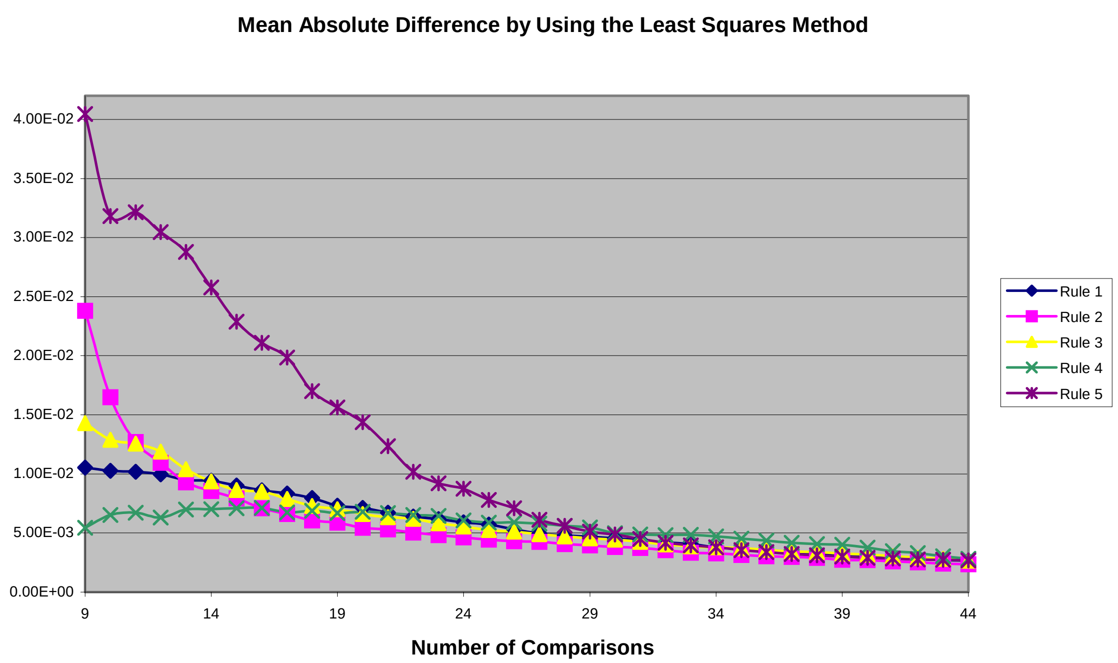

Figure 5-1 a: Comparison of the selection rules by mean absolute difference using the Least Squares Method (n = 10)

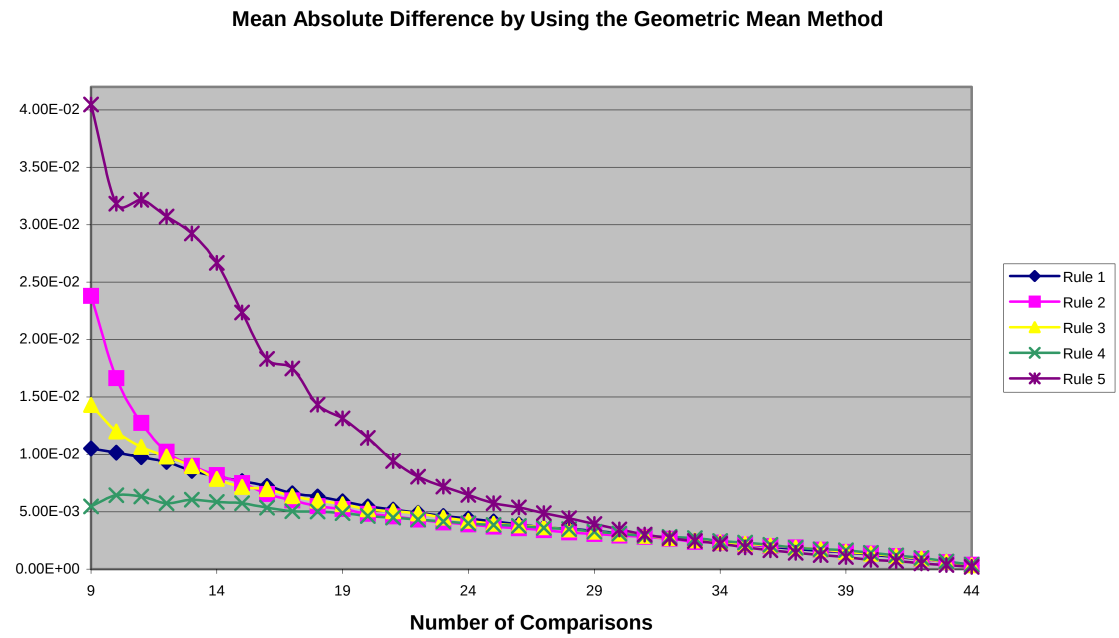

Figure 5-1 b: Comparison of the selection rules by mean absolute difference using the Geometric Mean Method (n = 10)

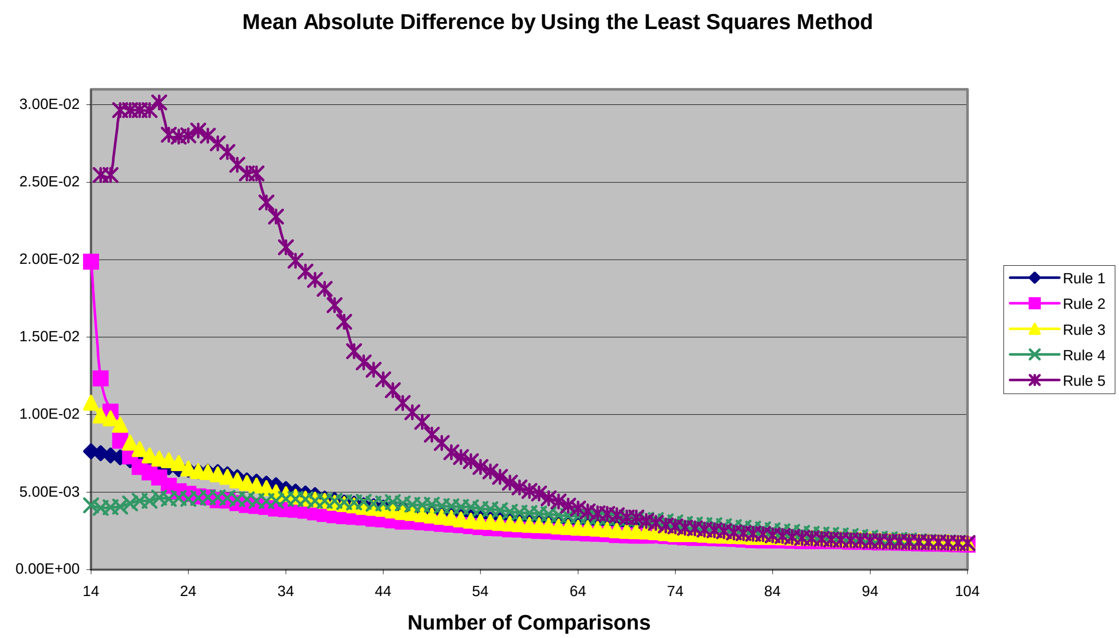

Figure 5-1 c: Comparison of the selection rules by mean absolute difference using the Least Squares Method (n = 15)

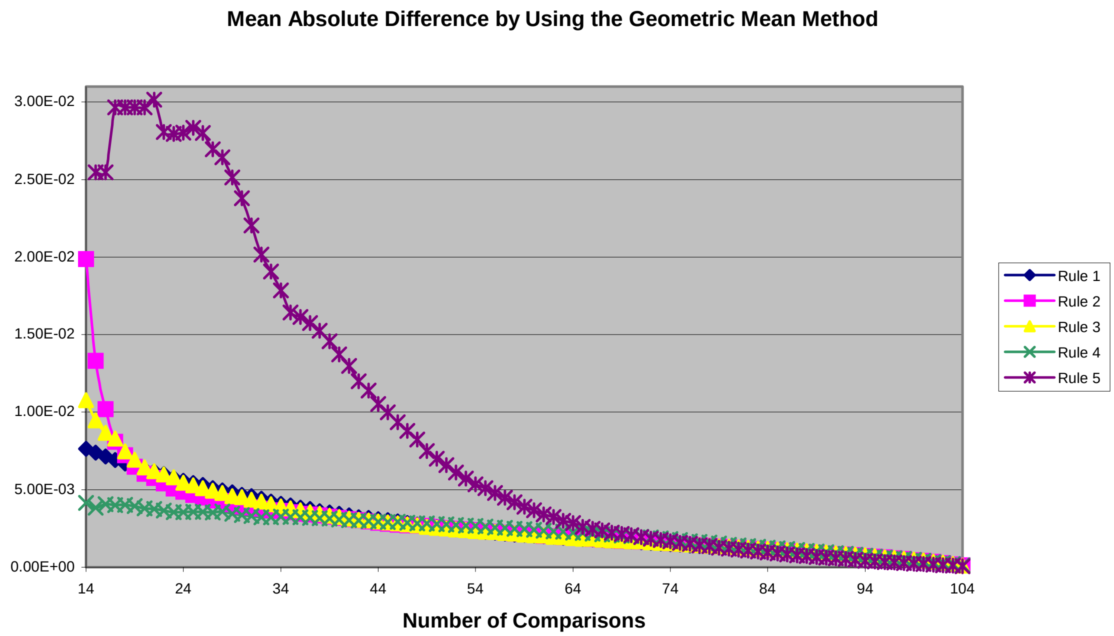

Figure 5-1 d: Comparison of the selection rules by mean absolute difference using the Geometric Mean Method (*n* = 15)

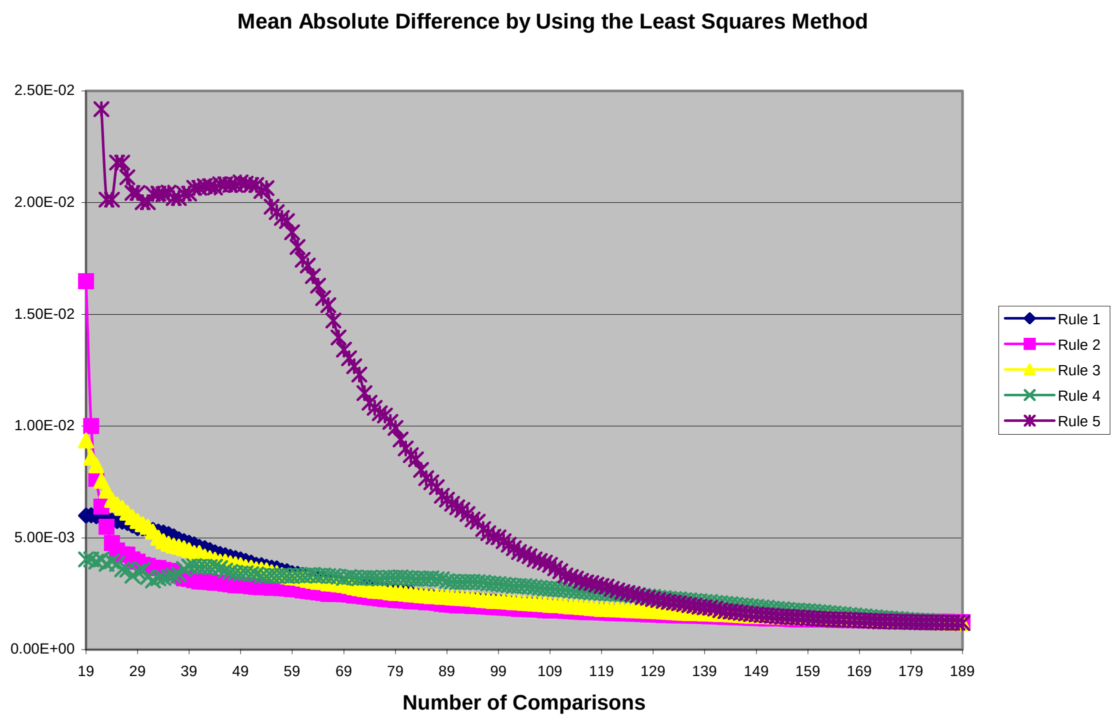

Figure 5-1 e: Comparison of the selection rules by mean absolute difference using the Least Squares Method (*n* = 20)

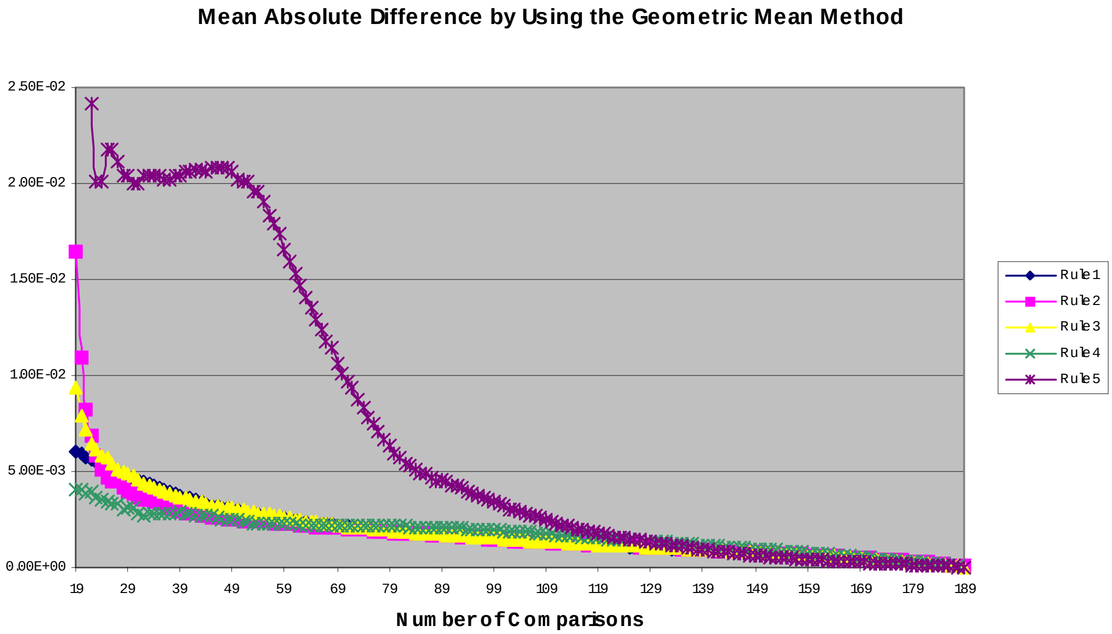

Figure 5-1 f: Comparison of the selection rules by mean absolute difference using the Geometric Mean Method (n = 20)

Note: The points on each series in the graph represent steps of iterations, starting from minimum initial comparisons (i.e., $n-1$ comparisons), until the incomplete matrix is left with only one missing comparison (i.e., $\frac{n(n-1)}{2}-1$ comparisons). Additionally, although the number of initial comparisons for the forth and fifth selection rules seems to start from $n-1$ or n, it only happens on a few replications. The average number for those rules is much higher than the minimum initial comparisons.

5.1.2. Comparisons of Selection Rules Using the Mean Sum-Square-Error Difference

Another method to evaluate the differences of the estimated and reference weight vectors is by the sum square error. As with the previous section, a similar procedure is used and the following observations can be derived:

Observations:

- The best selection rule is the second one. This fact is apparent when using the Least Squares Method. On the other hand, when using the Geometric Mean Method, all the selection rules seem to perform in a similar manner.
- Again, while comparing the minimum initial comparisons (i.e., $n-1$ comparisons), the forth selection rule appears to be the best. However, because its average number of initial comparisons is beyond $n-1$ comparisons, this fact remains untrue. With this finding, the first selection rule can be seen as the better choice.
- The Geometric Mean performs better than the Least Squares Method.

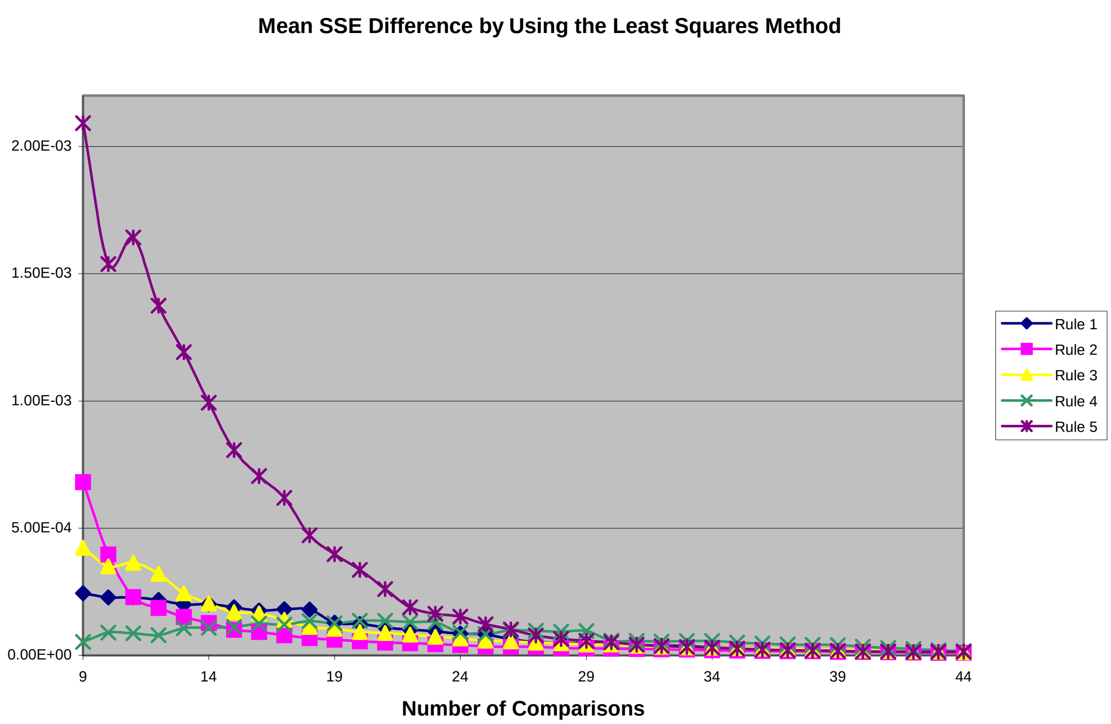

Figure 5-2 a: Comparison of the selection rules by mean sum-square-error difference using the Least Squares Method (n = 10)

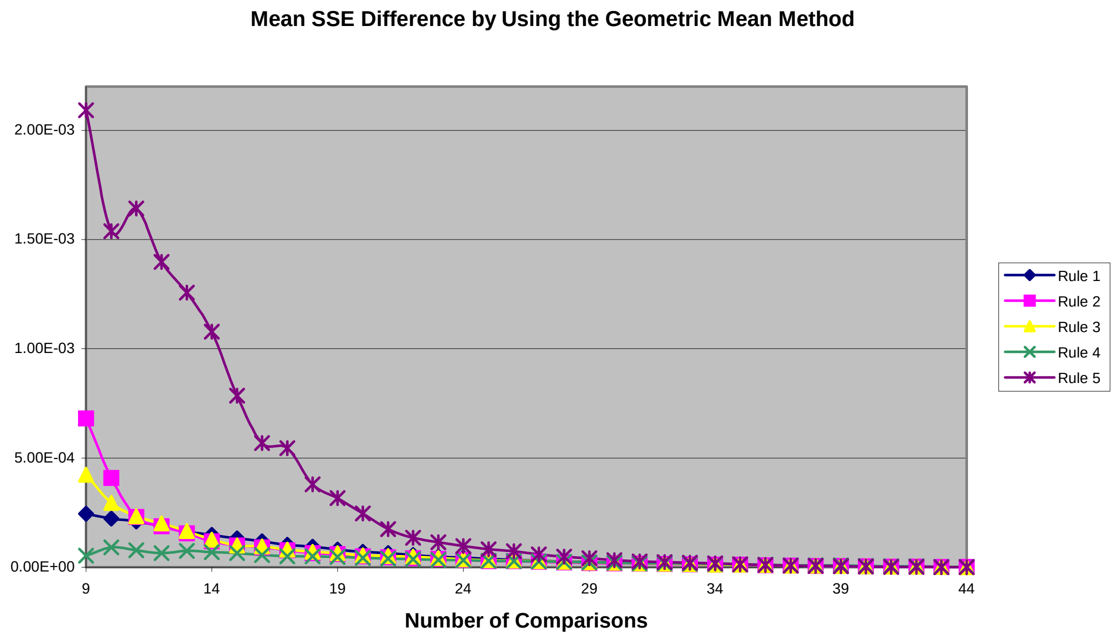

Figure 5-2 b: Comparison of the selection rules by mean sum-square-error difference using the Geometric Mean Method (n = 10)

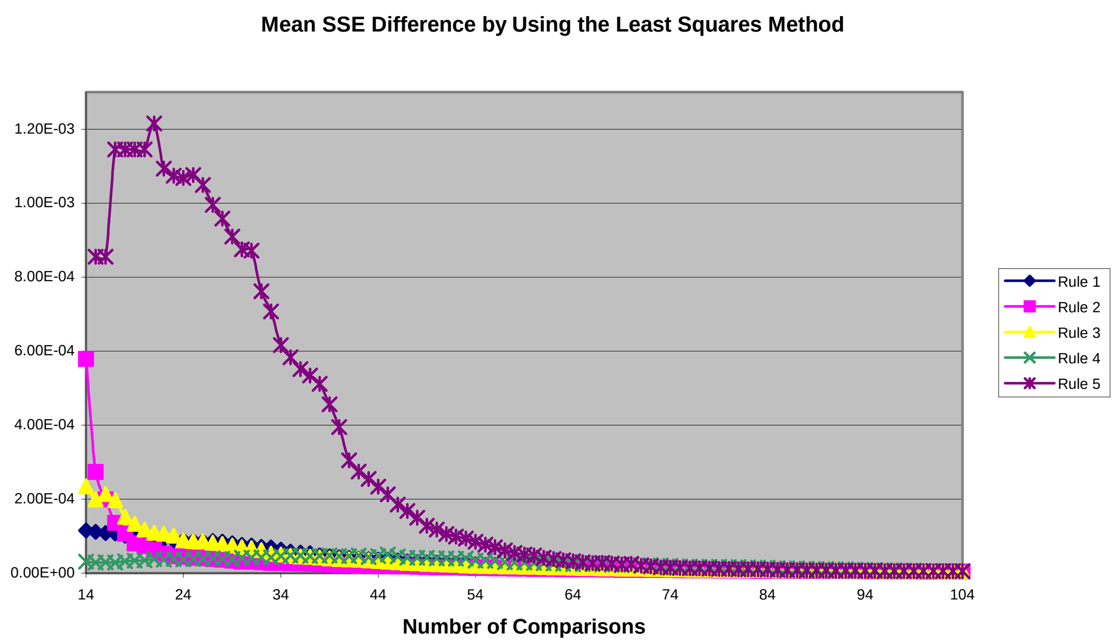

Figure 5-2 c: Comparison of the selection rules by mean sum-square-error difference using the Least Squares Method (n = 15)

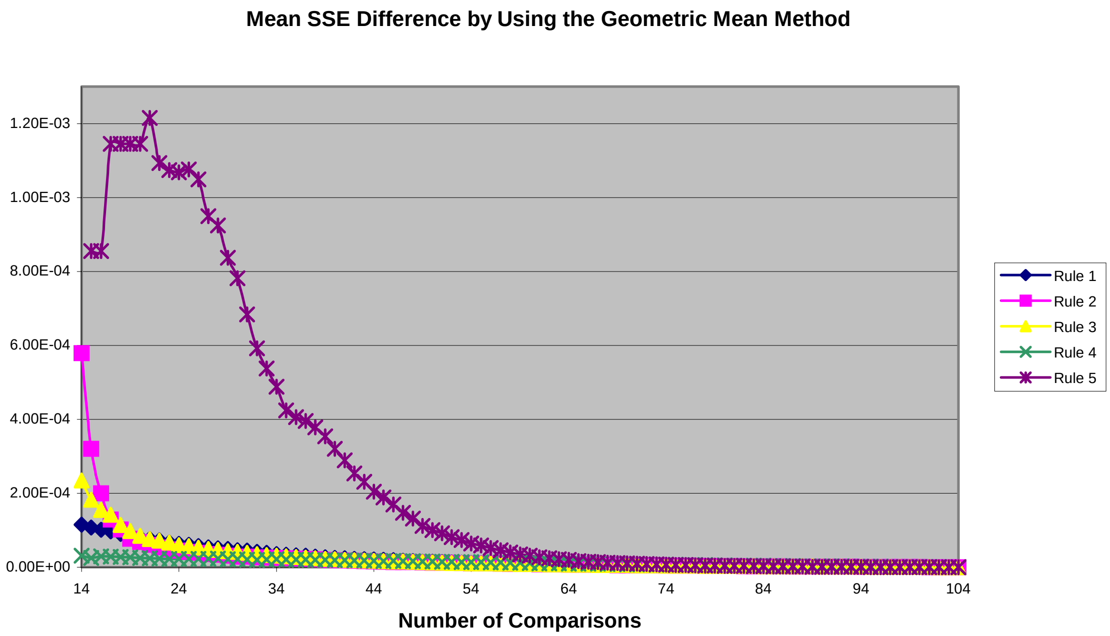

Figure 5-2 d: Comparison of the selection rules by mean sum-square-error difference using the Geometric Mean Method (n = 15)

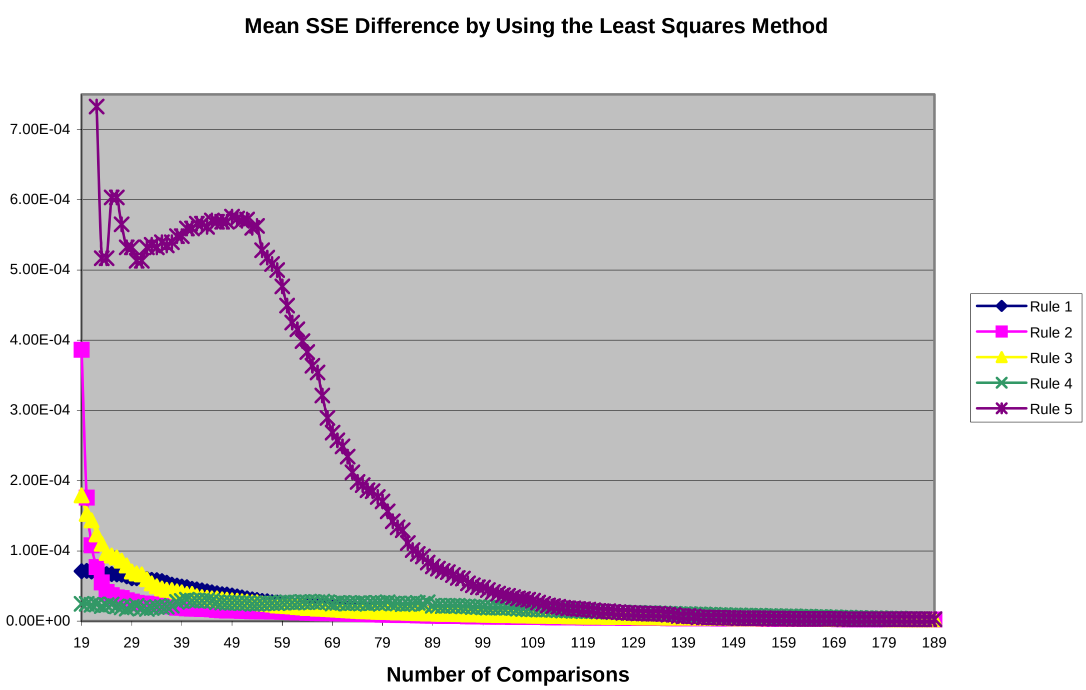

Figure 5-2 e: Comparison of the selection rules by mean sum-square-error difference using the Least Squares Method (n = 20)

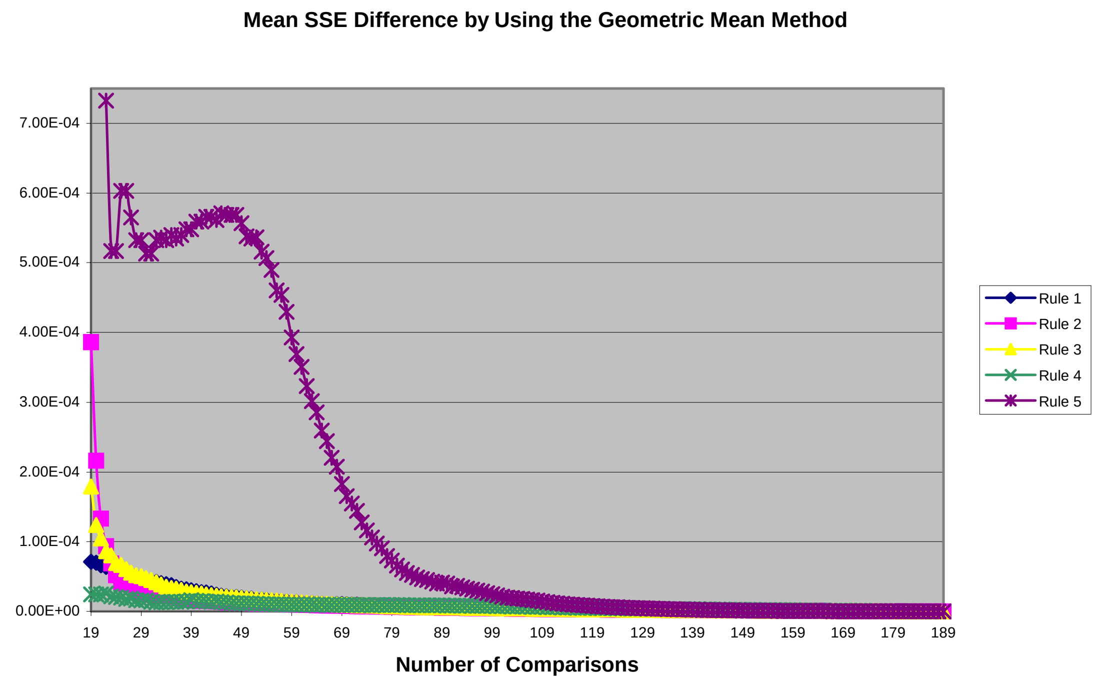

Figure 5-2 f: Comparison of the selection rules by mean sum-square-error difference using the Geometric Mean Method (n = 20)

5.1.3. Comparisons of Selection Rules Using the Sum-Square-Error Ranking Difference

Initially, the weights in the estimated and reference weight vectors are ranked according to their values starting from the highest. Then, the rankings from each weight vector are compared using the sum square error. The error is then divided by the maximum error possible to determine its percentage.

From the results, the following observations can be drawn:

Observations:

- Similar with the previous method, the second selection rule performs better than any other selection rule under the Least Squares Method. However, the forth selection rule seems to be better when the Geometric Mean Method is used, although the difference is very small.
- As for the minimum initial comparisons, the forth rule also appears to be the best. Nevertheless, since to its average number of minimum initial comparisons is greater than $n-1$, the first rule is believed to be the best choice in this manner.
- The Geometric Mean Method performs better than the Least Squares Method.

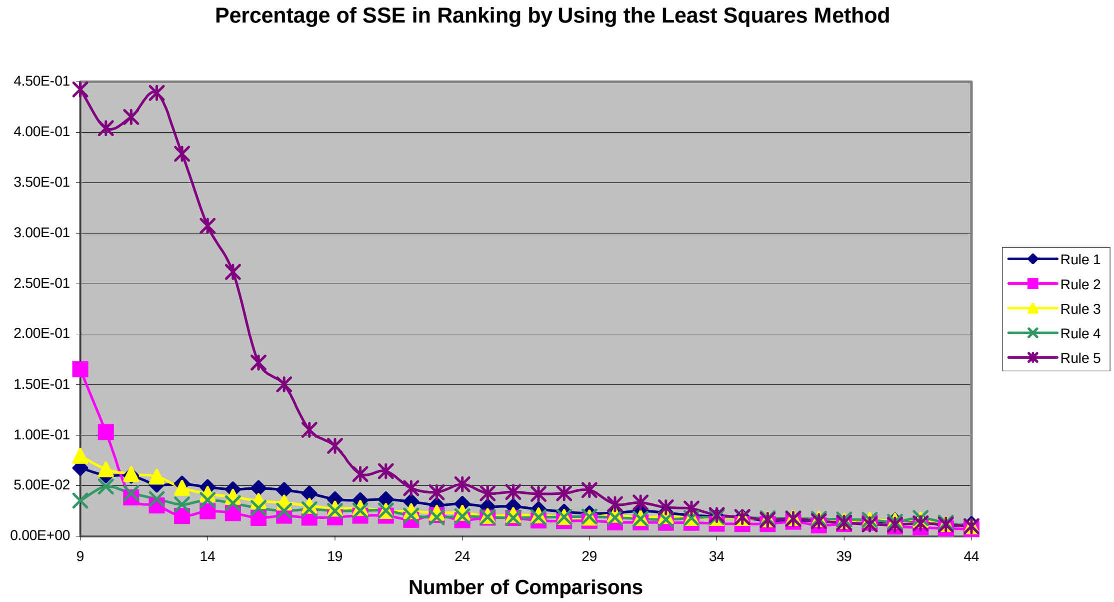

Figure 5-3 a: Comparison of the selection rules by percentage of sum-square-error ranking difference using the Least Squares Method (n = 10)

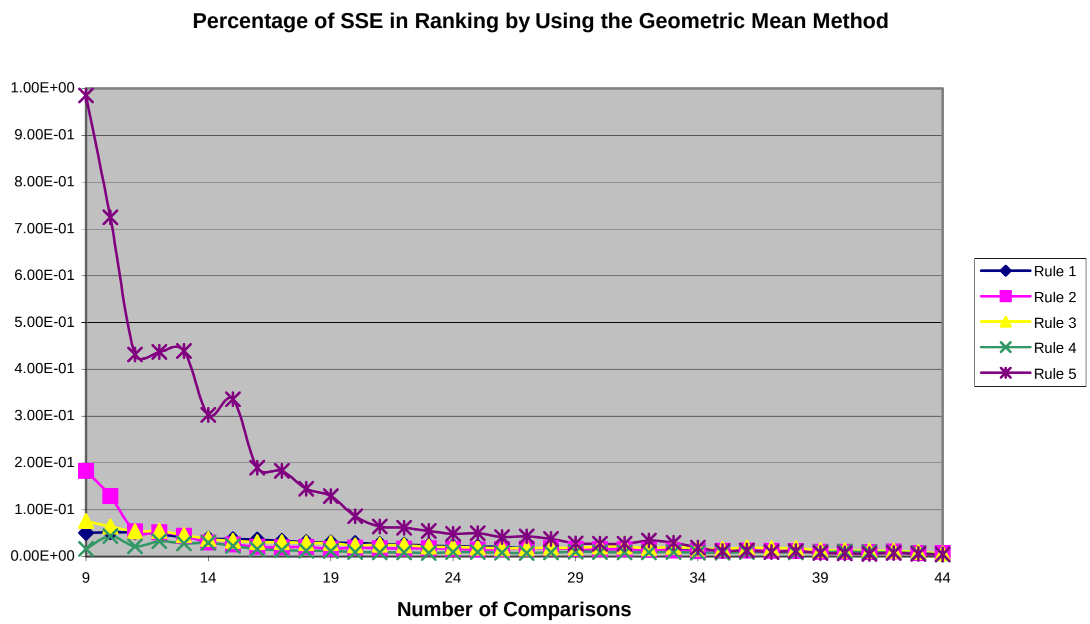

Figure 5-3 b: Comparison of the selection rules by percentage of sum-square-error ranking difference using the Geometric Mean Method (n = 10)

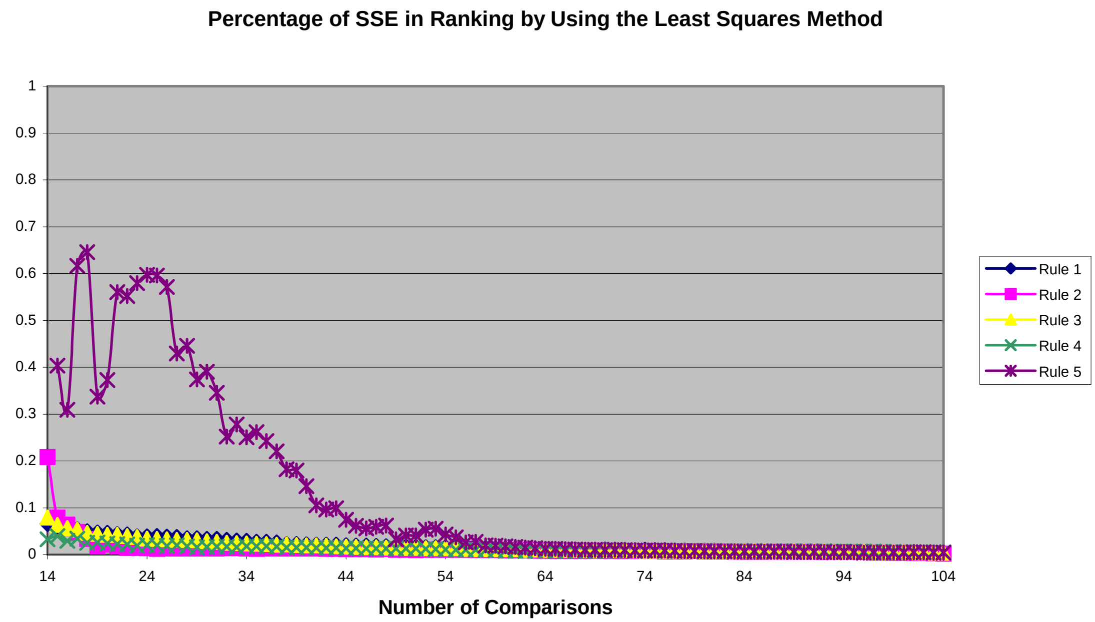

Figure 5-3 c: Comparison of the selection rules by percentage of sum-square-error ranking difference using the Least Squares Method (n = 15)

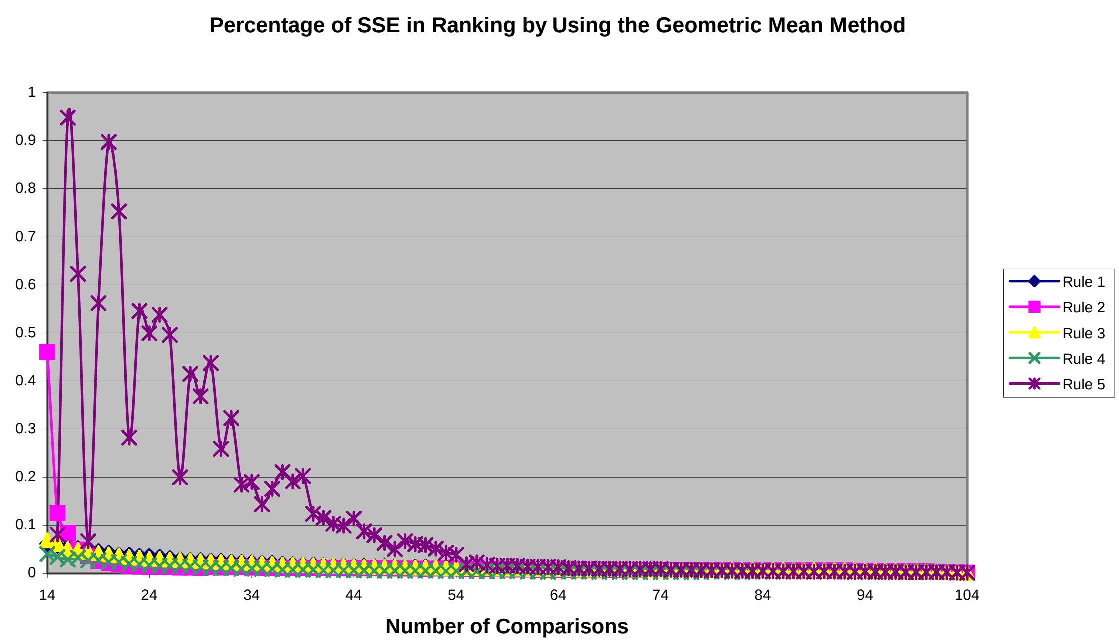

Figure 5-3 d: Comparison of the selection rules by percentage of sum-square-error ranking difference using the Geometric Mean Method (n = 15)

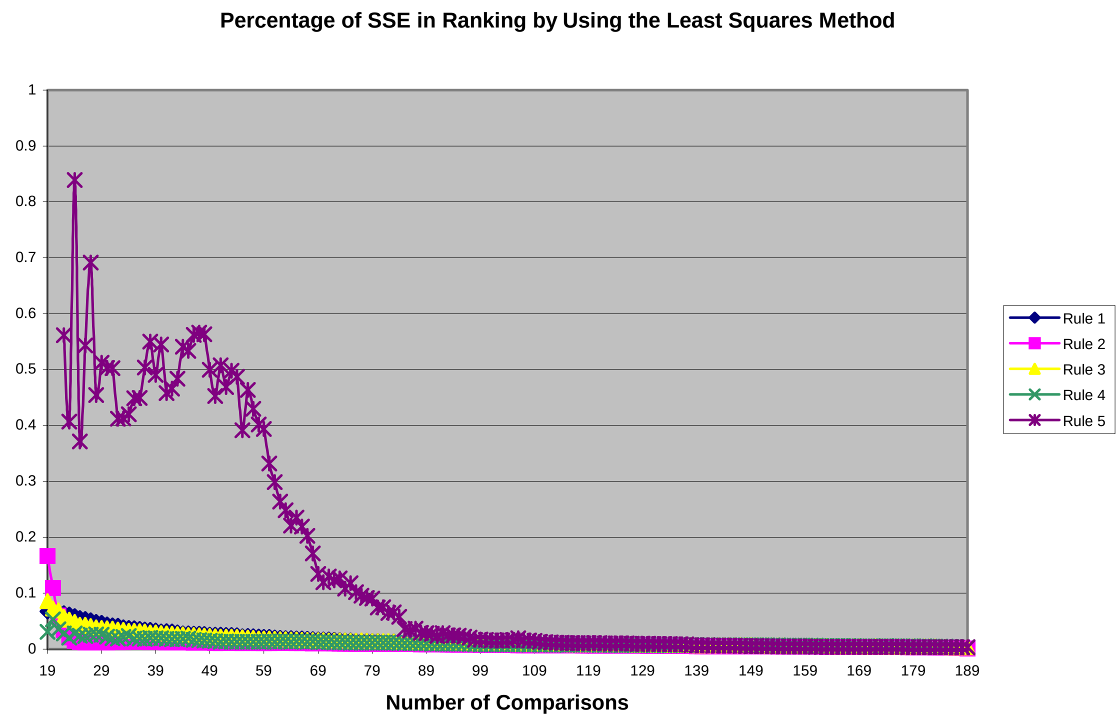

Figure 5-3 e: Comparison of the selection rules by percentage of sum-square-error ranking difference using the Least Squares Method (n = 20)

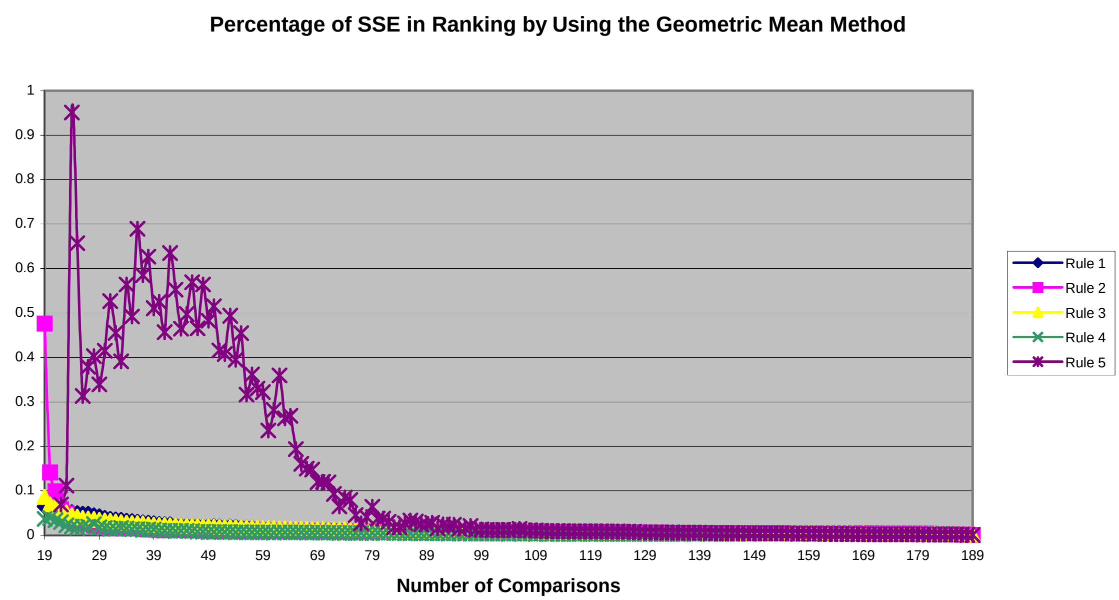

Figure 5-3 f: Comparison of the selection rules by percentage of sum-square-error ranking difference using the Geometric Mean Method (n = 20)

5.1.4. Comparisons of Selection Rules Using the Weighted-Error Ranking Difference

In this section, each ranking of the weights from the reference vector are given a certain penalty. The first-ranking weight will be given the highest penalty, while the last-ranking weight will have the smallest penalty. For our computational experiment, the weight vector will be of dimension 10. Thus, the highest weight (i.e., rank #1) will have a penalty of 10, then consequently reduces by 1 until the smallest weight (i.e., rank #10) will only have a penalty of 1.

The rankings from the estimated and the reference weight vectors are then compared. If the ranking of the weights are different, the consequent penalty is calculated. The total penalty is then divided with the maximum possible penalty to calculate its percentage of error.

The following observations can be derived from these results:

Observations:

- The result is inconclusive with the Least Squares Method, however, when using the Geometric Mean Method, the forth selection rule clearly the best choice.
- For evaluation of minimum initial comparisons, the forth rule seems to be the best. Nevertheless, as stated earlier, the first rule still being considered a better choice, since the average minimum initial comparisons for the forth selection rule are above $n-1$ comparisons.
- The Geometric Mean Method performs better than the Least Squares Method.

An important observation is found during the analysis of the results regarding the comparisons using the ranking differences. The weights of the estimated weight vectors sometimes may not be distinguishable because their differences are very small. On the other hand, these small differences have a major impact when it comes to their rankings. Often, the difference between the first and second ranked weights is smaller than 10^{-7}.

With this finding, the analysis of the selection rules by using any of the ranking difference methods maybe misleading. Therefore, the analysis by using the mean difference of the estimated and reference weight vectors is more reliable.

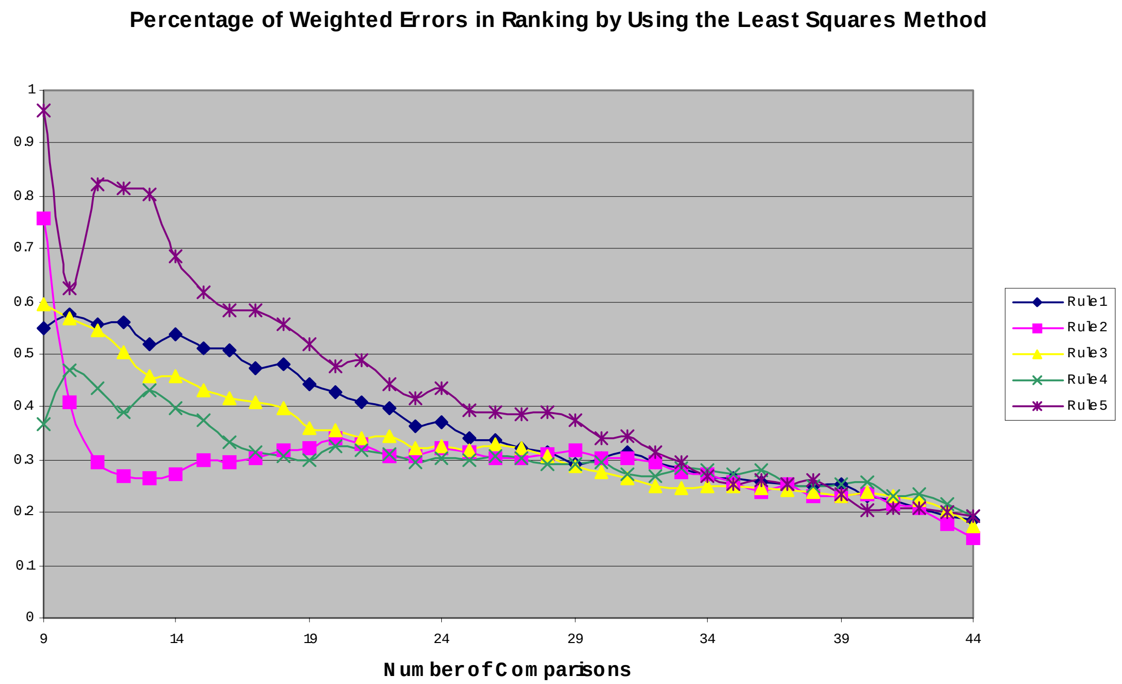

Figure 5-4 a: Comparison of the selection rules by percentage of weighted-error ranking difference using the Least Squares Method (n = 10)

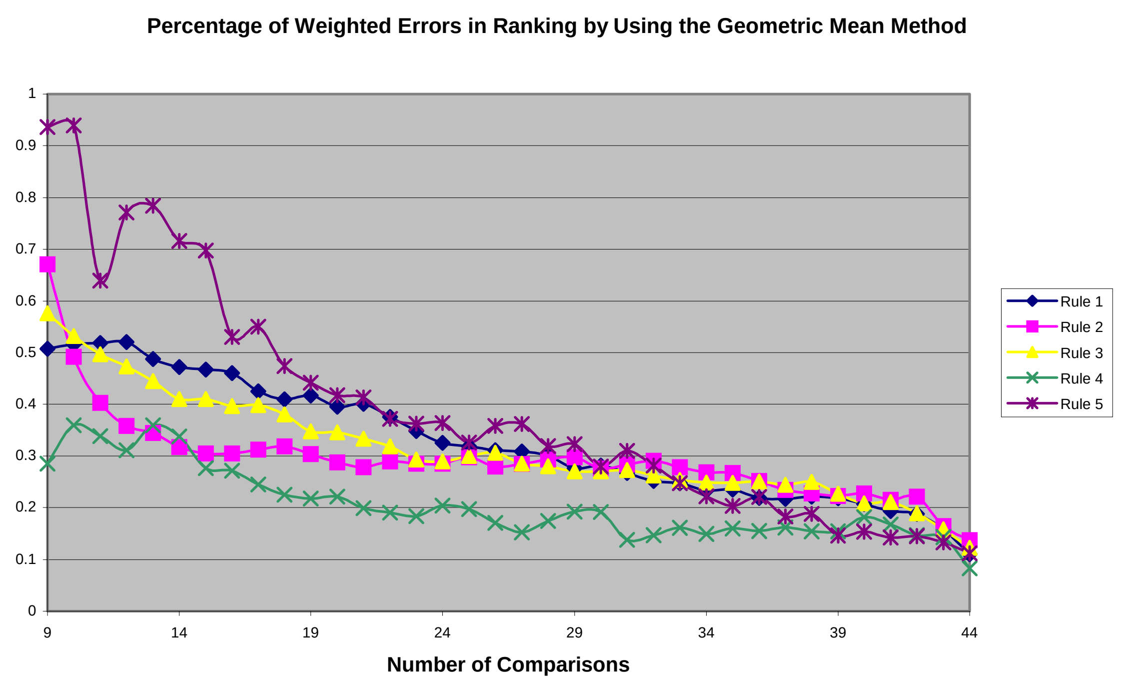

Figure 5-4 b: Comparison of the selection rules by percentage of weighted-error ranking difference using the Geometric Mean Method (n = 10)

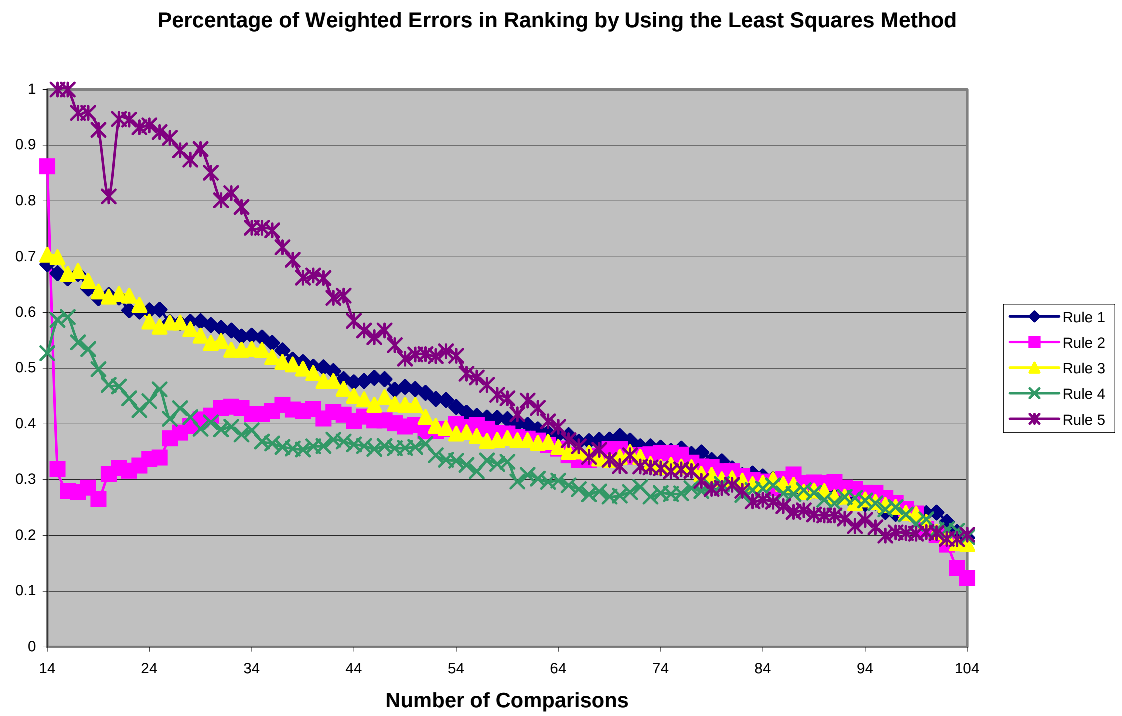

Figure 5-4 c: Comparison of the selection rules by percentage of weighted-error ranking difference using the Least Squares Method (n = 15)

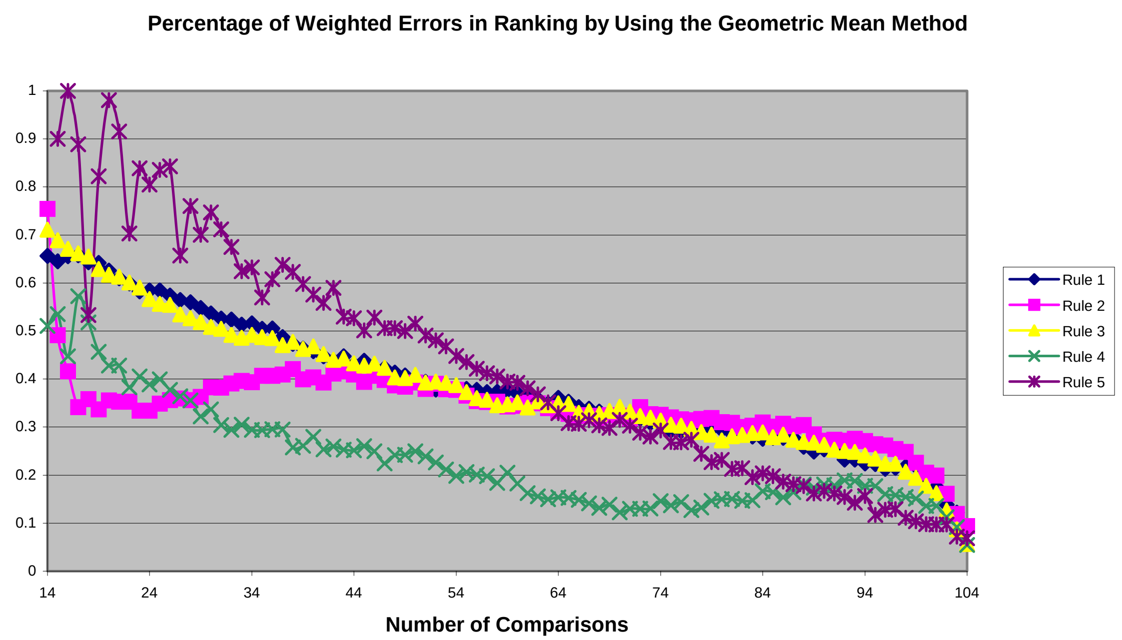

Figure 5-4 d: Comparison of the selection rules by percentage of weighted-error ranking difference using the Geometric Mean Method (n = 15)

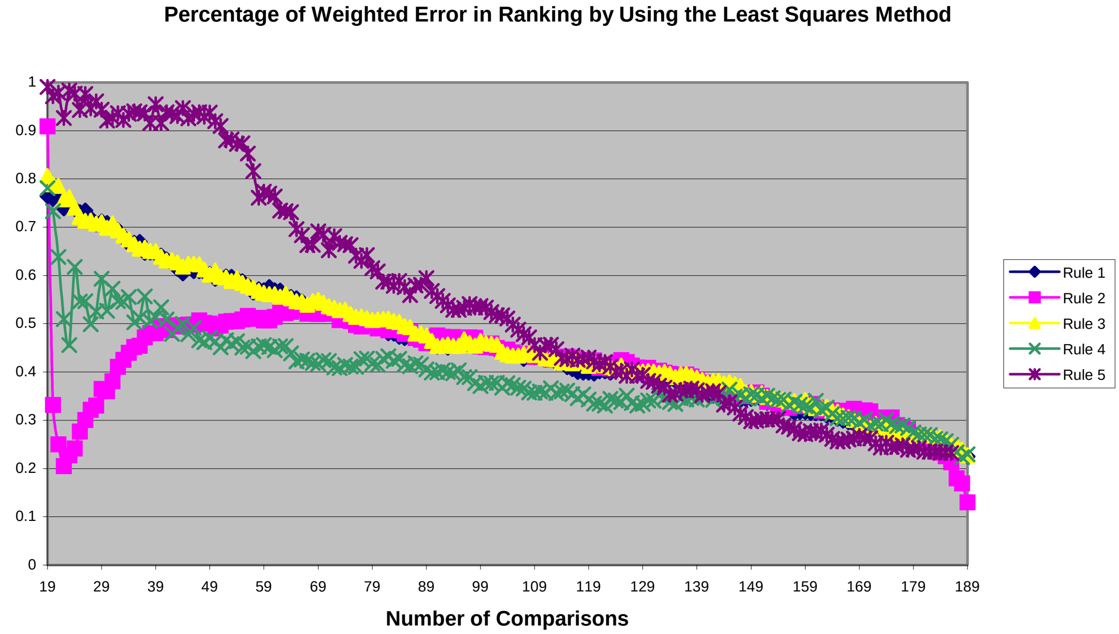

Figure 5-4 e: Comparison of the selection rules by percentage of weighted-error ranking difference using the Least Squares Method (n = 20)

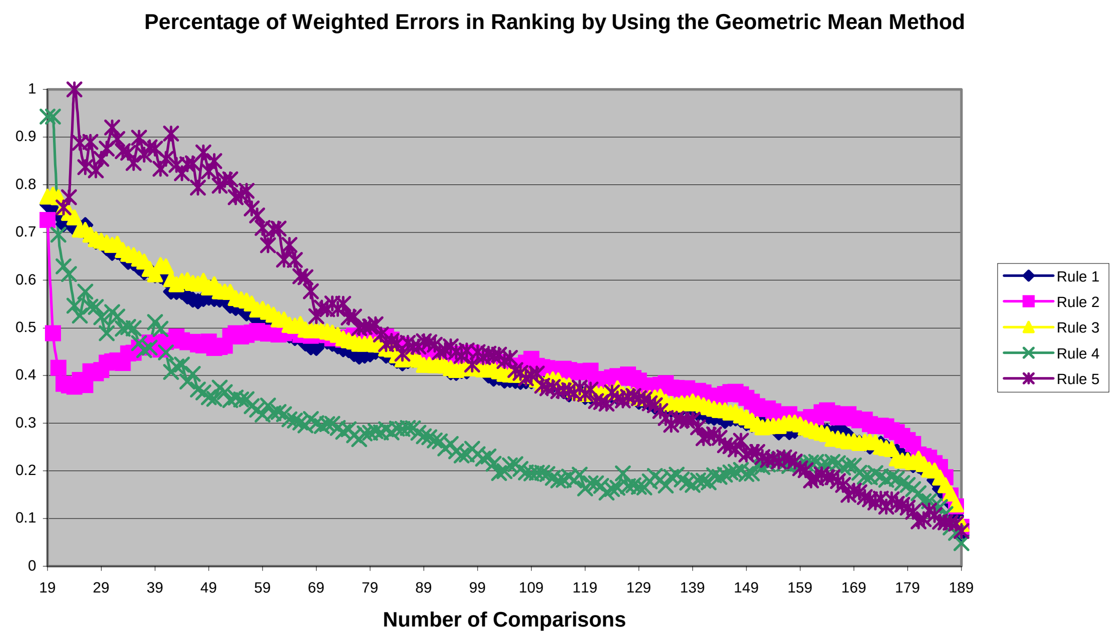

Figure 5-4 f: Comparison of the selection rules by percentage of weighted-error ranking difference using the Geometric Mean Method (n = 20)

5.1.5. The Average Minimum Number of Initial Comparisons for the Forth and Fifth Selection Rule

In the earlier section, we discovered that additional comparisons may be needed on top of the minimum $n-1$ initial comparisons, to ensure that there will be a connecting path between all the alternatives, either directly or indirectly. To investigate the effect of the matrix dimension with the average minimum number of initial comparisons, a computational experiment was conducted with 10,000 replications with different matrix dimensions. The following graphs show the correlation between the matrix dimension, and the additional number of initial comparisons above the $n-1$ comparisons:

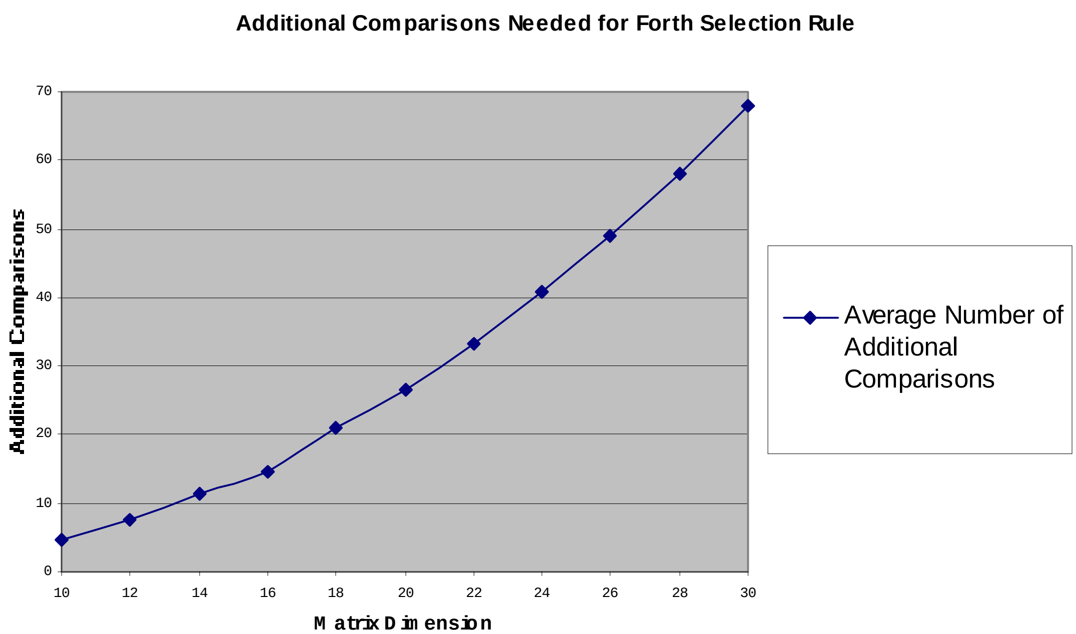

Figure 5-5 a: Additional comparisons needed to ensure a connecting path for all alternatives using forth selection rule

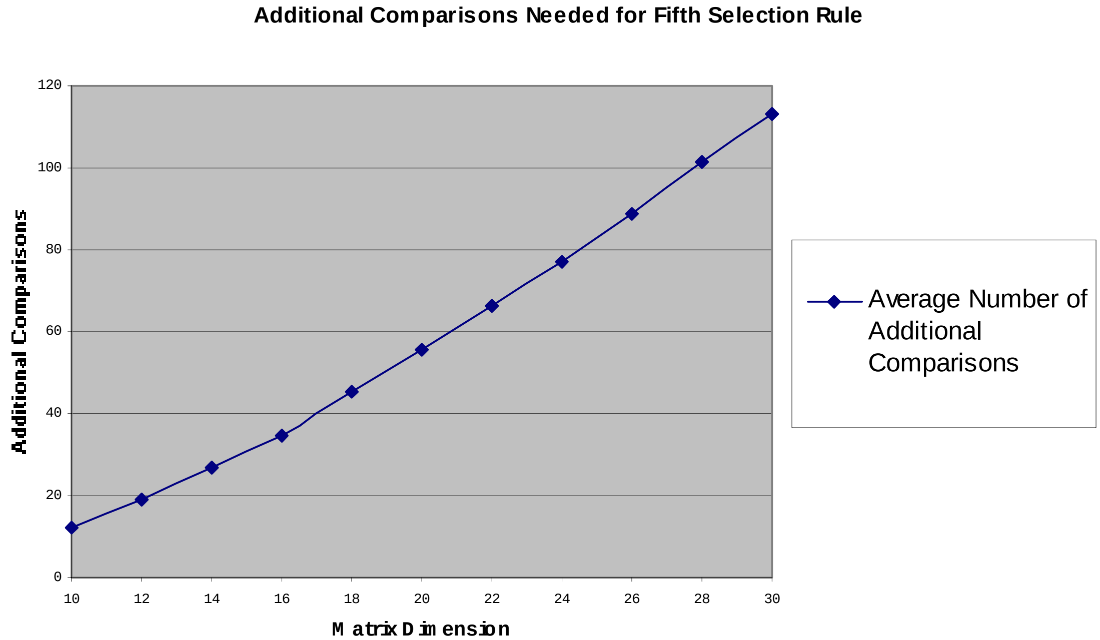

Figure 5-5 b: Additional comparisons needed to ensure a connecting path for all alternatives using fifth selection rule

Observations:

- The number of required additional initial comparisons appears to be proportional to the matrix dimension.
- The average essential additional comparisons is about 15.23 % of the remaining missing comparisons for the forth selection rule, and about 31.8 % for the fifth selection rule.
- Although the correlation seems to be proportional to the matrix dimension, the percentage of the necessary additional comparisons is increasing with the matrix dimension when using the forth selection rule, and decreasing when the fifth selection rule is used.

5.2. Percentage of Necessary Comparisons to Estimate the Weight Vector Correctly (Research Goal #2)

Another goal of this thesis is to determine the necessary number of comparisons in order to estimate the weight vector accurately. Although the error in the estimation cannot be eliminated completely, a certain confidence level that our estimation is relatively accurate can be achieved. From the above figures, the differences between the estimated and reference weight vectors are small, even with $n-1$ number of comparisons.

5.2.1. Results for *n* = 10

For the minimum number of comparisons (i.e., $n-1$ comparisons), the weight vector can be estimated with 97.6 % through 98.9 % accuracy, depending on the selection rule used. The calculation of the above accuracy is based on the mean absolute difference method without the forth and fifth selection rules. Since the average number of minimum comparisons for the forth and fifth selection rules is greater than $n-1$, the inclusion of these selection rules will be misleading.

If a higher accuracy is required, with $n+11$ comparisons we can get an accuracy of 99.5 % regardless of the selection rule, except for the fifth rule. Therefore, for 99.5 % accuracy, the percentage of necessary comparisons is around 34.29 % above the minimum $n-1$ comparisons.

5.2.2. Results for *n* = 15

Using the mean absolute difference method on the minimum number of comparisons, the weight vector in this matrix dimension can be estimated with 98.0 % through 99.2 % accuracy. This accuracy range does not include the forth and fifth selection rule, because the average number of minimum comparisons for these selection rules are more than $n-1$ comparisons.

When $n+21$ comparisons are used, a higher accuracy of 99.5 % can be achieved regardless of the selection rule, except the fifth rule. Hence, to achieve 99.5 % accuracy, the percentage of necessary comparisons is around 23.33 % above the required $n-1$ comparisons.

5.2.3. Results for *n* = 20

The weight vector can be estimated with 98.3 % through 99.4 % by using only the minimum number of comparisons (i.e., $n-1$ comparisons). Again, the mean absolute difference method is used to determine the accuracy of the estimated weight vector, without considering the forth and fifth selection rules.

For a higher accuracy of 99.8 %, $n + 50$ comparisons are needed regardless of the selection rule, except for the fifth rule. Thus, the percentage of necessary comparisons is around 30 % above the minimum $n - 1$ comparisons in order to achieve 99.8 % accuracy.

5.3. Comparisons of the Least Squares Method and Geometric Mean Method (Research Goal #3)

The last goal of this thesis research was to compare the Least Squares Method with the Geometric Mean Method. Chen (1997) has concluded that the two methods perform in a similar matter. In order to confirm this finding, the two methods are again compared using different selection rules. The following graphs will show the comparisons:

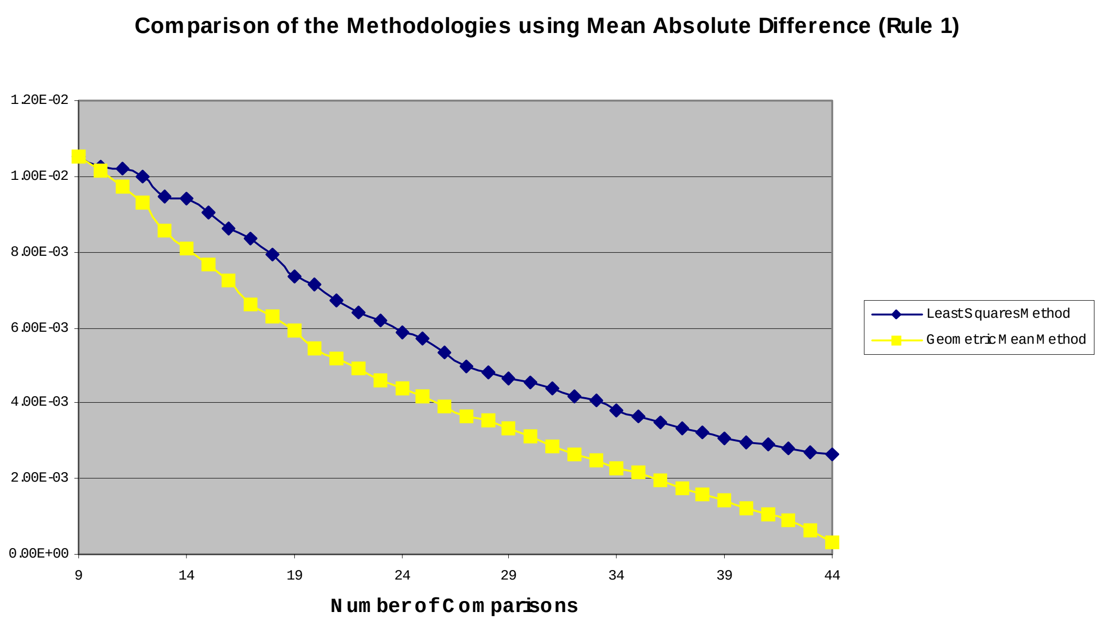

Figure 5-6-1 a: Comparison of the methodologies using the mean absolute difference for the first selection rule (n = 10)

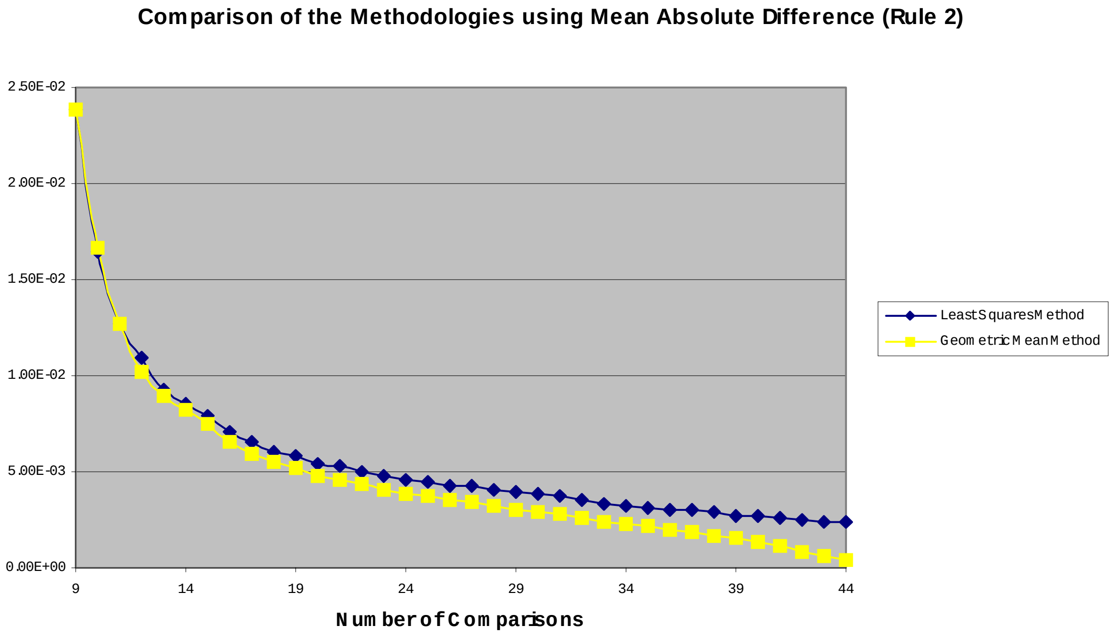

Figure 5-6-1 b: Comparison of the methodologies using the mean absolute difference for the second selection rule (n = 10)

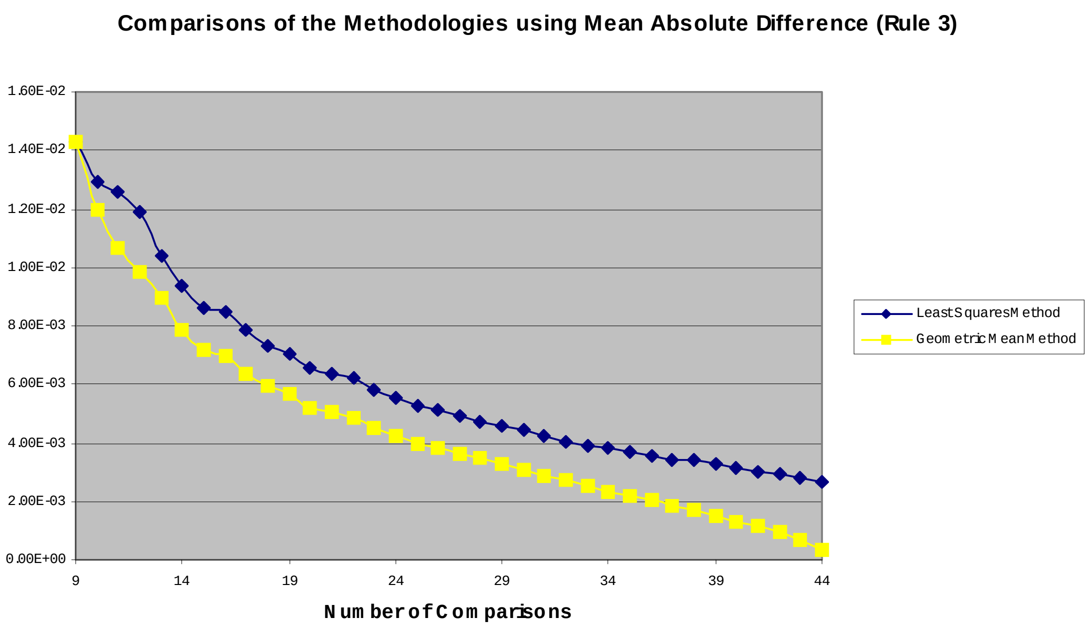

Figure 5-6-1 c: Comparison of the methodologies using the mean absolute difference for the third selection rule (n = 10)

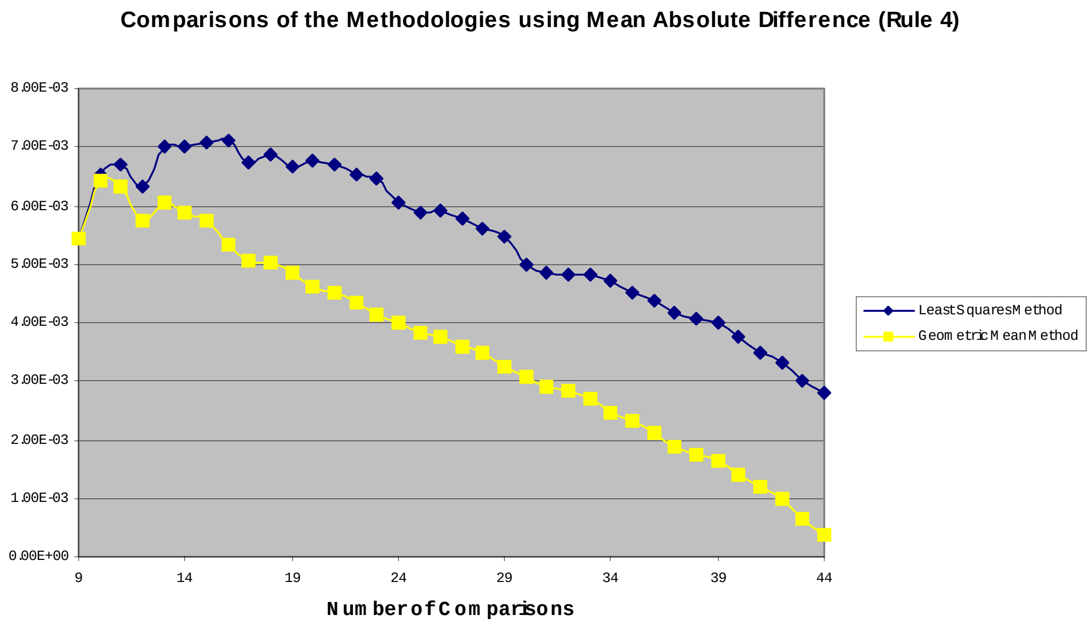

Figure 5-6-1 d: Comparison of the methodologies using the mean absolute difference for the forth selection rule (n = 10)

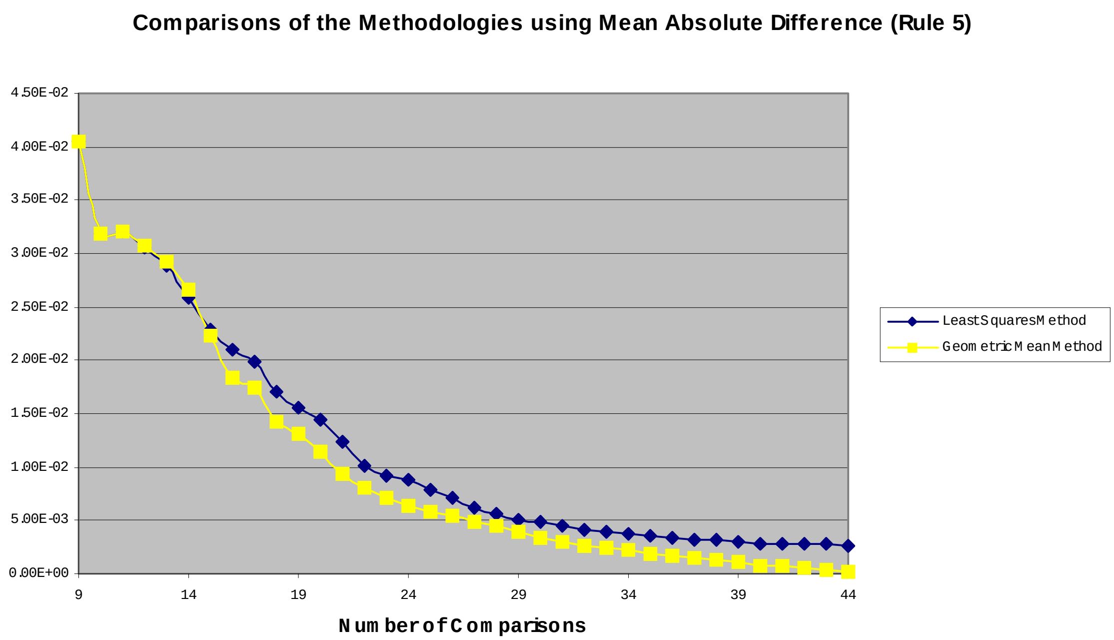

Figure 5-6-1 e: Comparison of the methodologies using the mean absolute difference for the fifth selection rule (n = 10)

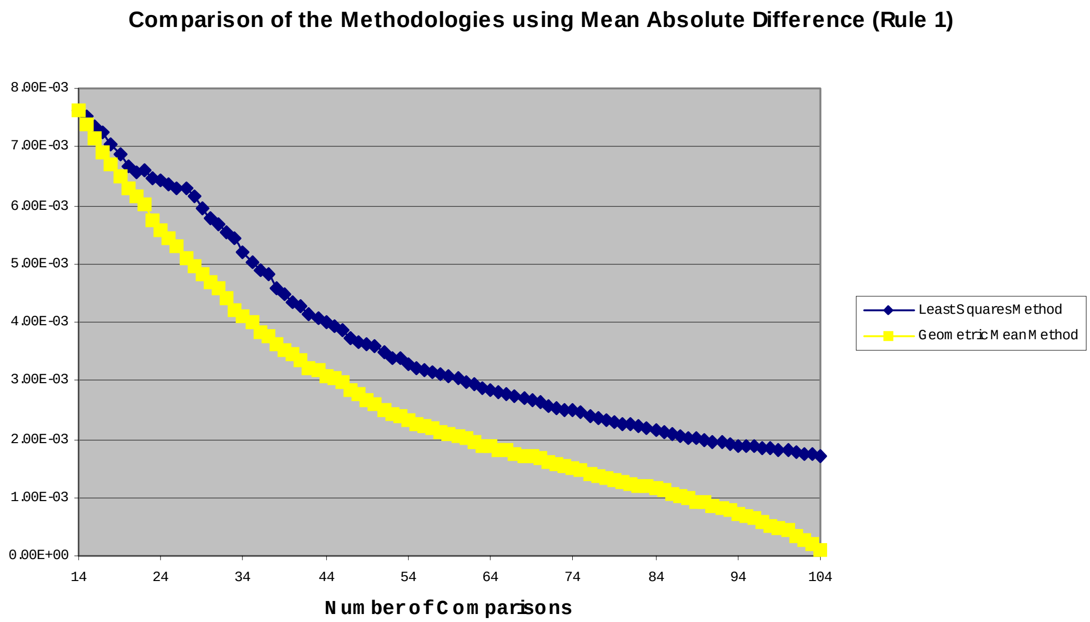

Figure 5-6-2 a: Comparison of the methodologies using the mean absolute difference for the first selection rule (n = 15)

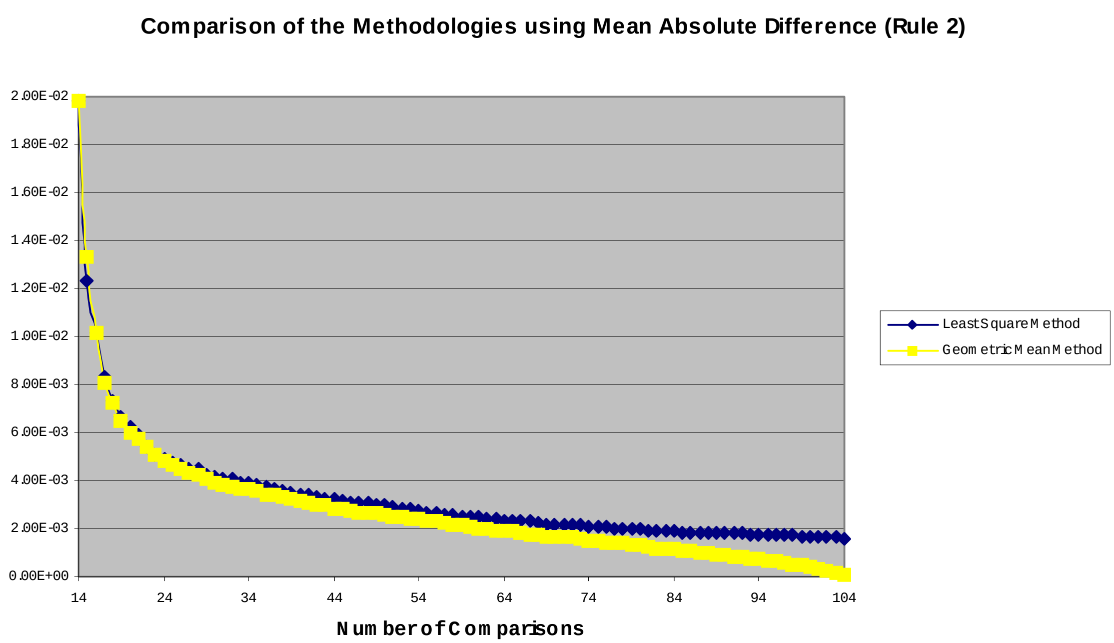

Figure 5-6-2 b: Comparison of the methodologies using the mean absolute difference for the second selection rule (n = 15)

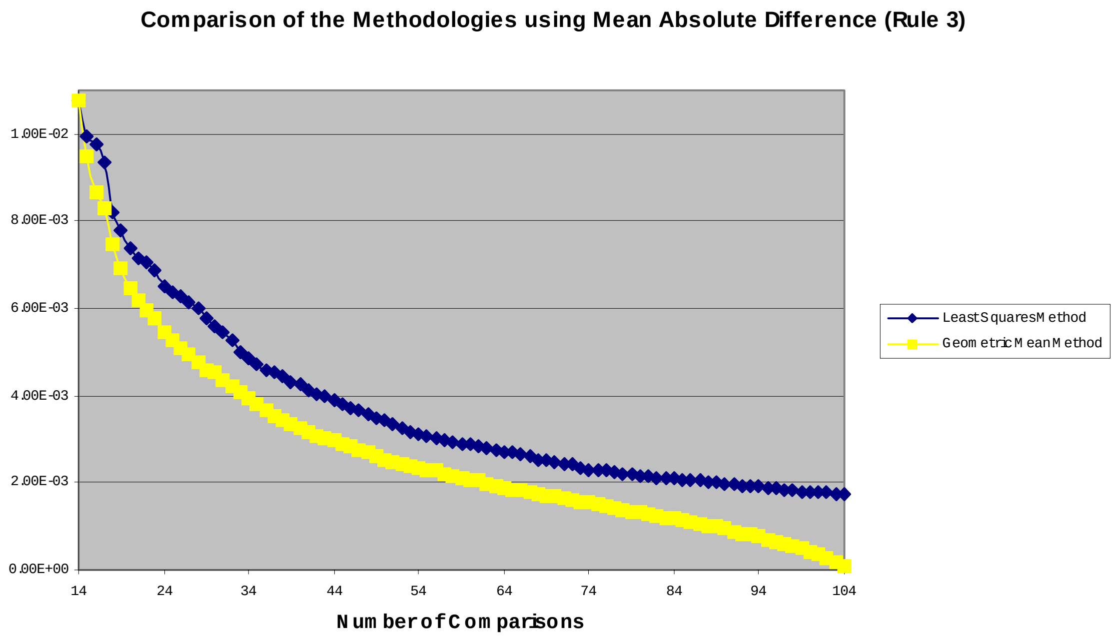

Figure 5-6-2 c: Comparison of the methodologies using the mean absolute difference for the third selection rule (n = 15)

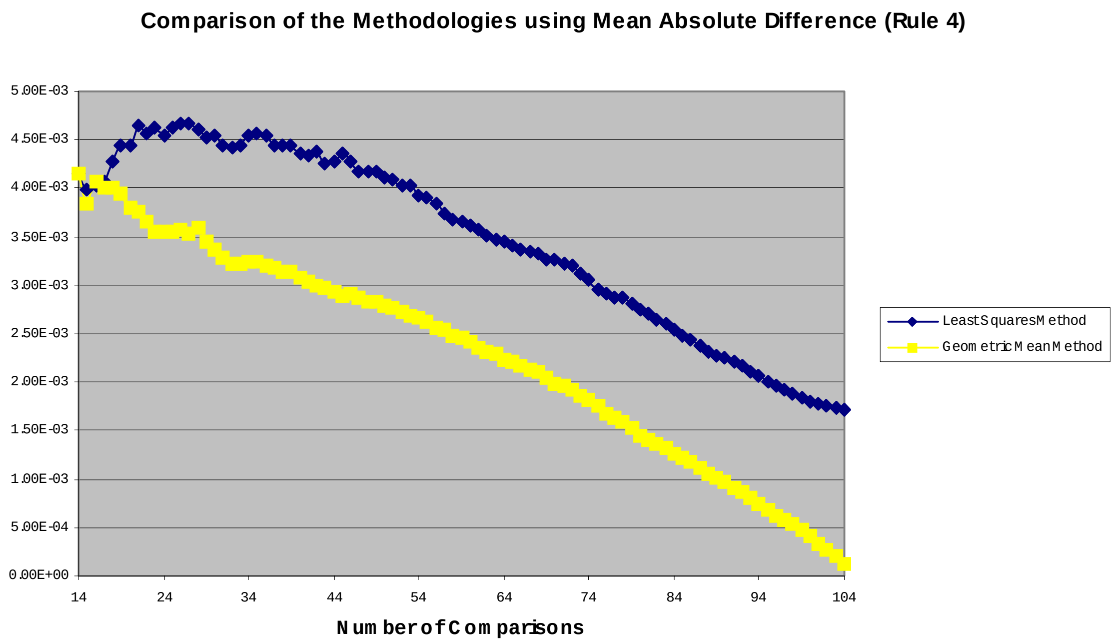

Figure 5-6-2 d: Comparison of the methodologies using the mean absolute difference for the forth selection rule (n = 15)

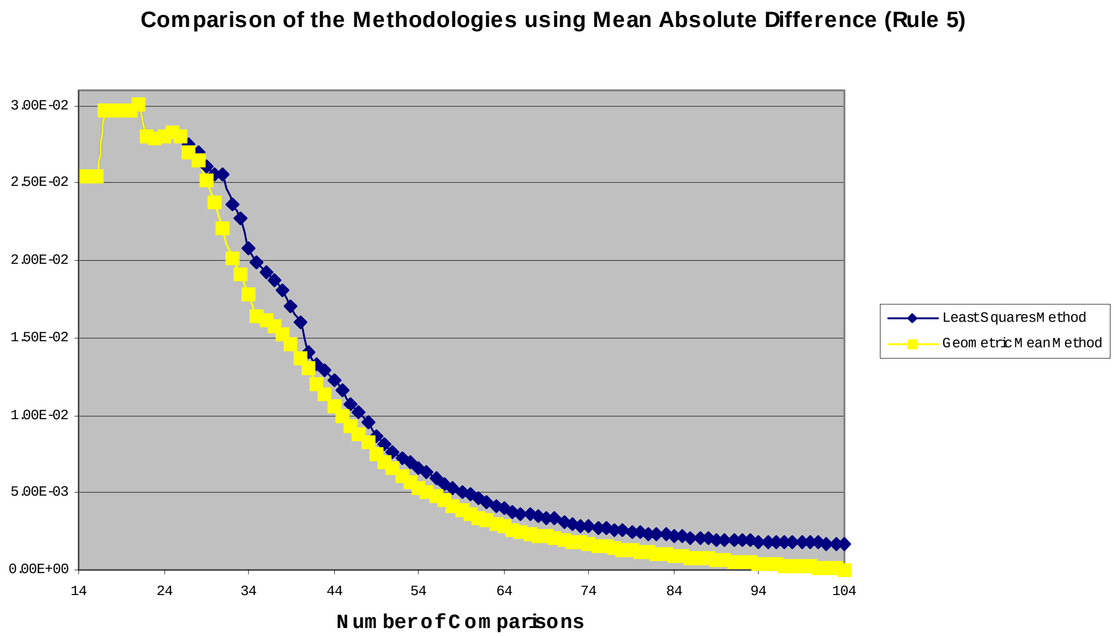

Figure 5-6-2 e: Comparison of the methodologies using the mean absolute difference for the fifth selection rule (n = 15)

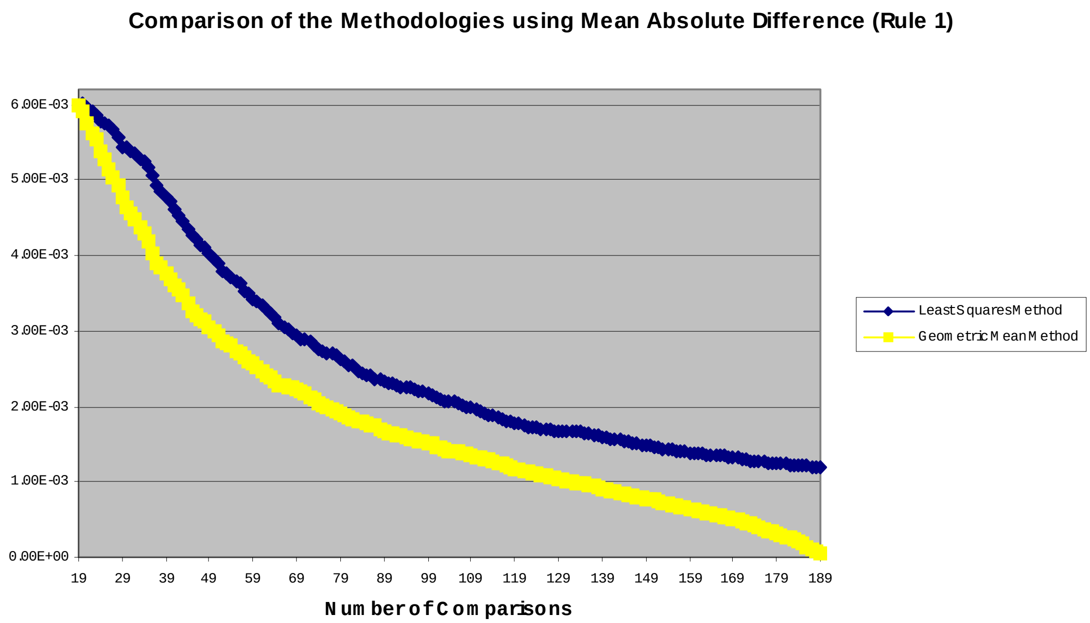

Figure 5-6-3 a: Comparison of the methodologies using the mean absolute difference for the first selection rule (n = 20)

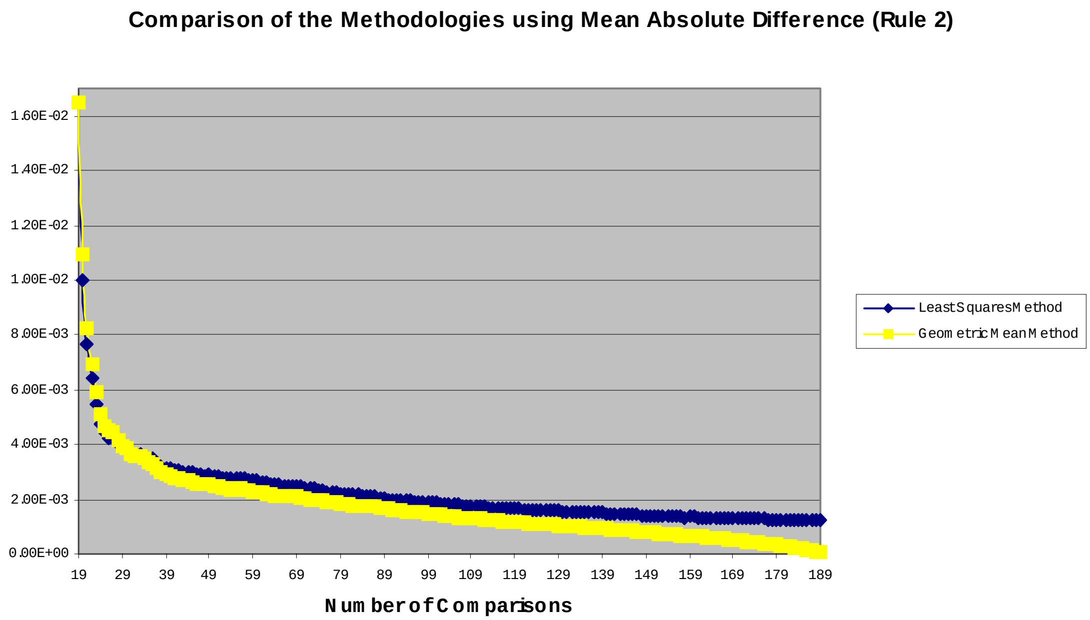

Figure 5-6-3 b: Comparison of the methodologies using the mean absolute difference for the second selection rule (n = 20)

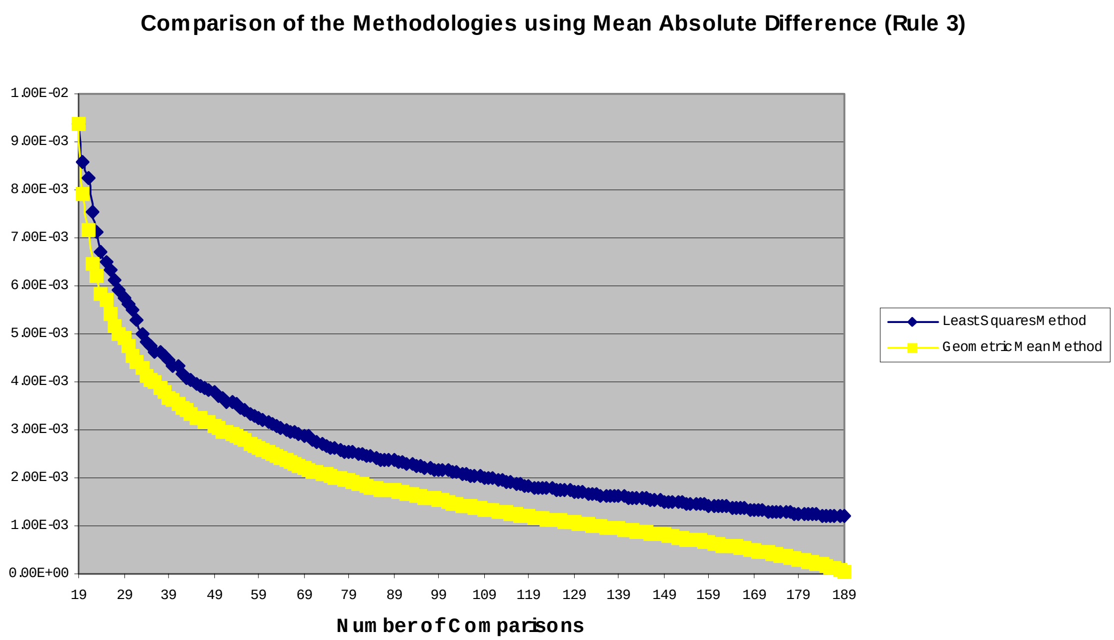

Figure 5-6-3 c: Comparison of the methodologies using the mean absolute difference for the third selection rule (n = 20)

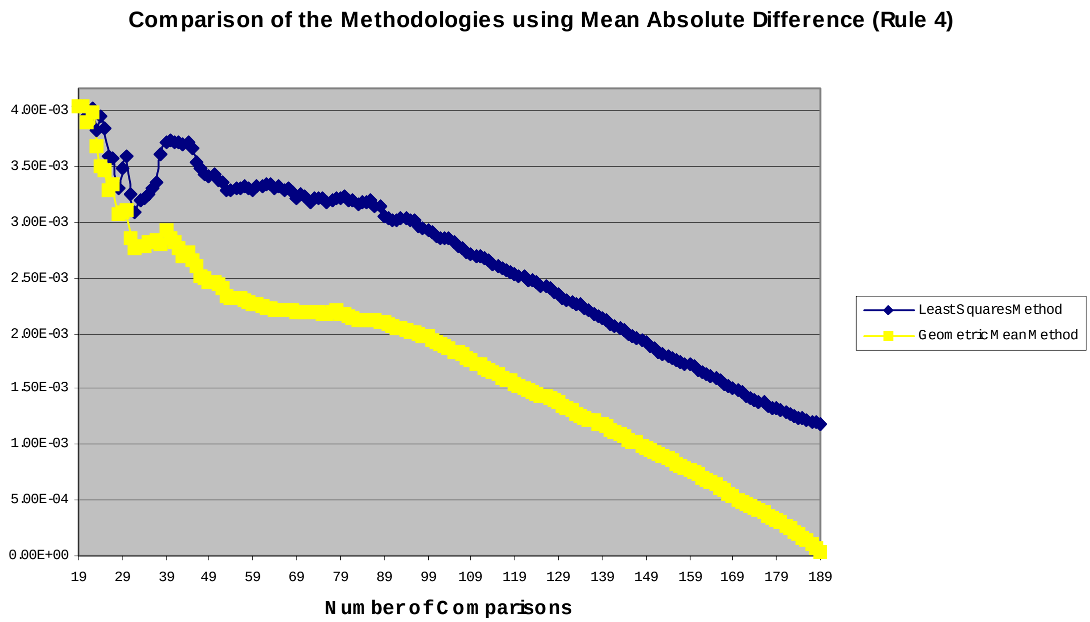

Figure 5-6-3 d: Comparison of the methodologies using the mean absolute difference for the forth selection rule (n = 20)

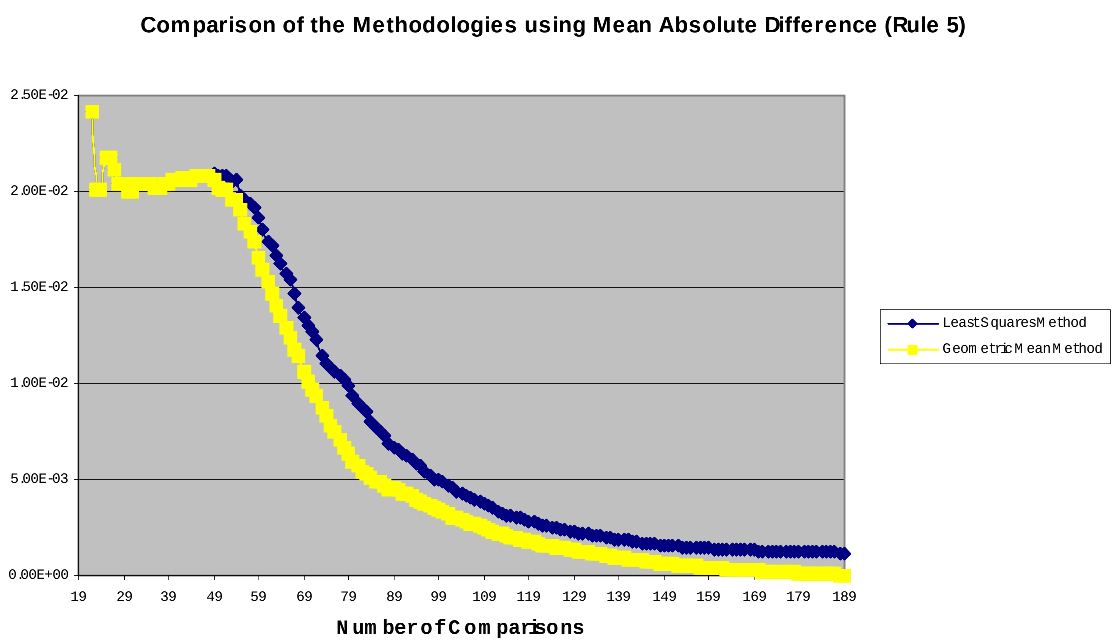

Figure 5-6-3 e: Comparison of the methodologies using the mean absolute difference for the fifth selection rule (n = 20)

Observations:

- The Geometric Mean Method performs consistently better than the Least Squares Method, regardless which selection rule is used.
- The difference is not so significant under the second and fifth selection rules. However, this small difference will have a major impact to the ranking of the alternatives.

To support these observations, more comparisons by using mean sum-square-error difference are conducted. The following graphs will show the comparisons of the two methodologies:

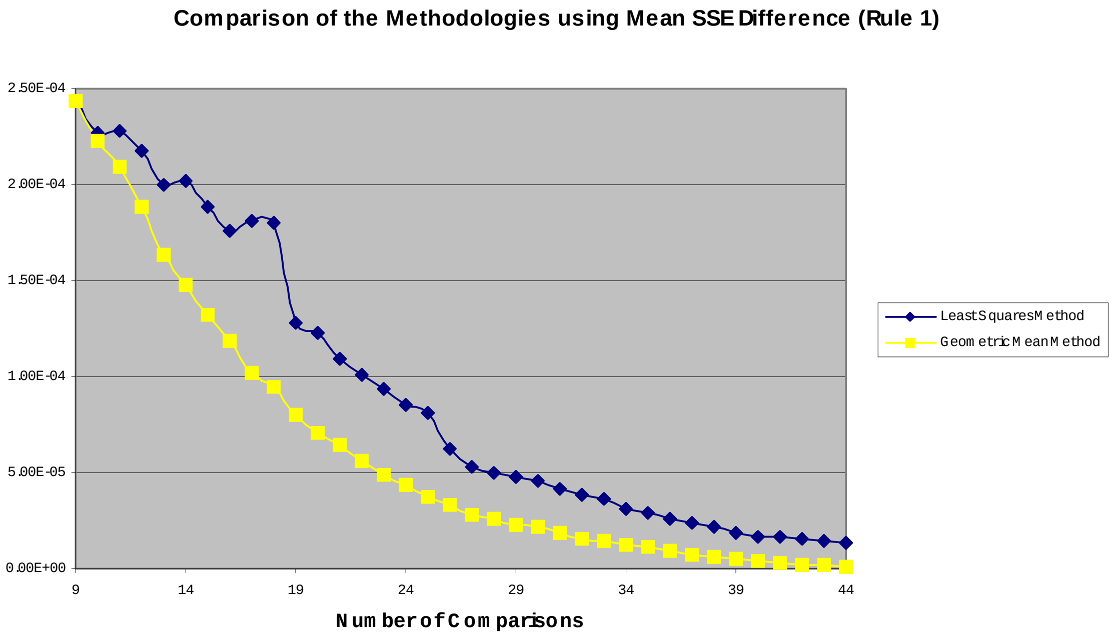

Figure 5-7-1 a: Comparison of the methodologies using the mean sum-square-error difference for the first selection rule (n = 10)

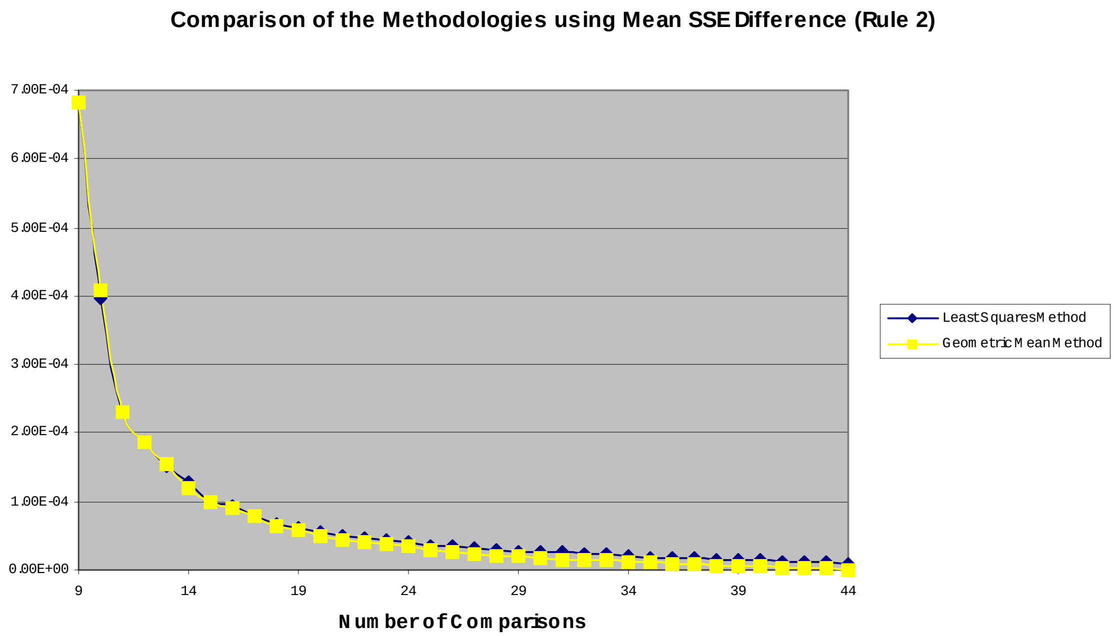

Figure 5-7-1 b: Comparison of the methodologies using the mean sum-square-error difference for the second selection rule (n = 10)

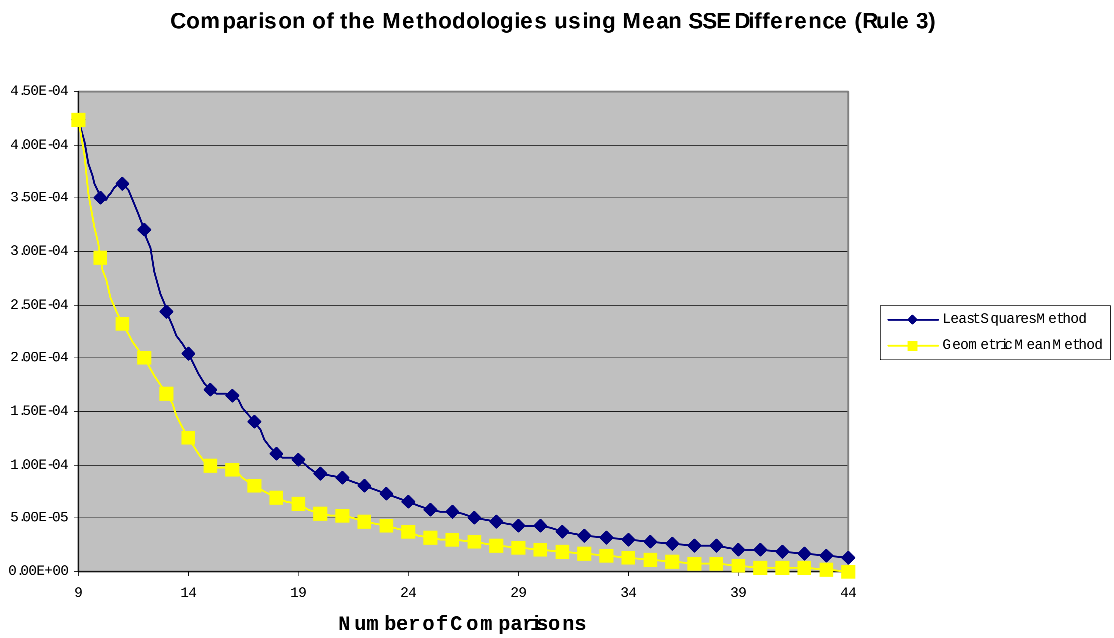

Figure 5-7-1 c: Comparison of the methodologies using the mean sum-square-error difference for the third selection rule (n = 10)

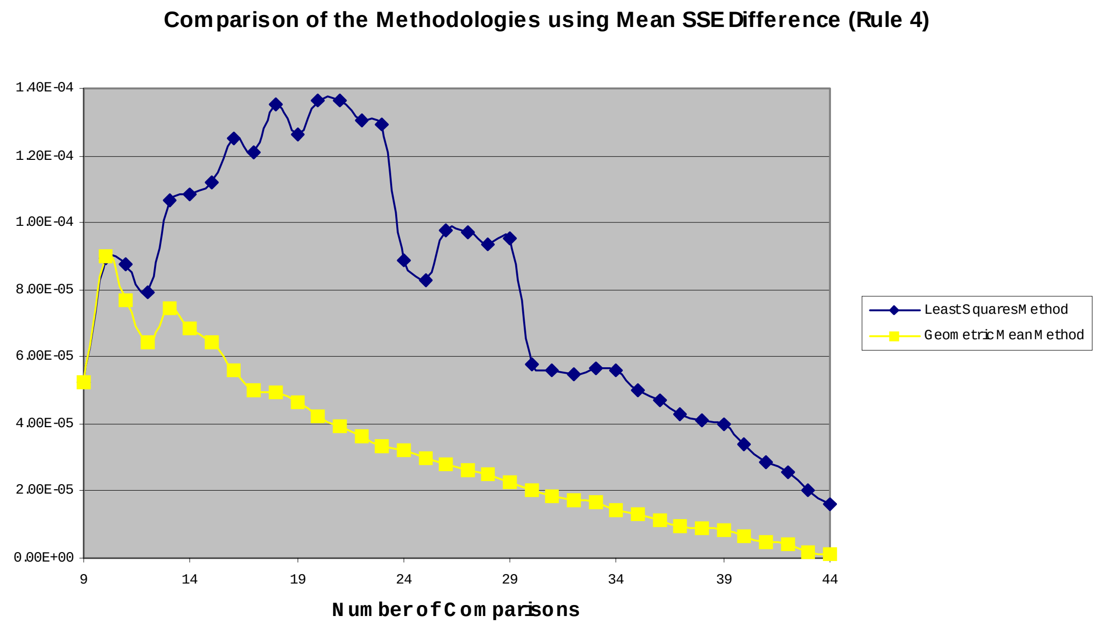

Figure 5-7-1 d: Comparison of the methodologies using the mean sum-square-error difference for the forth selection rule (n = 10)

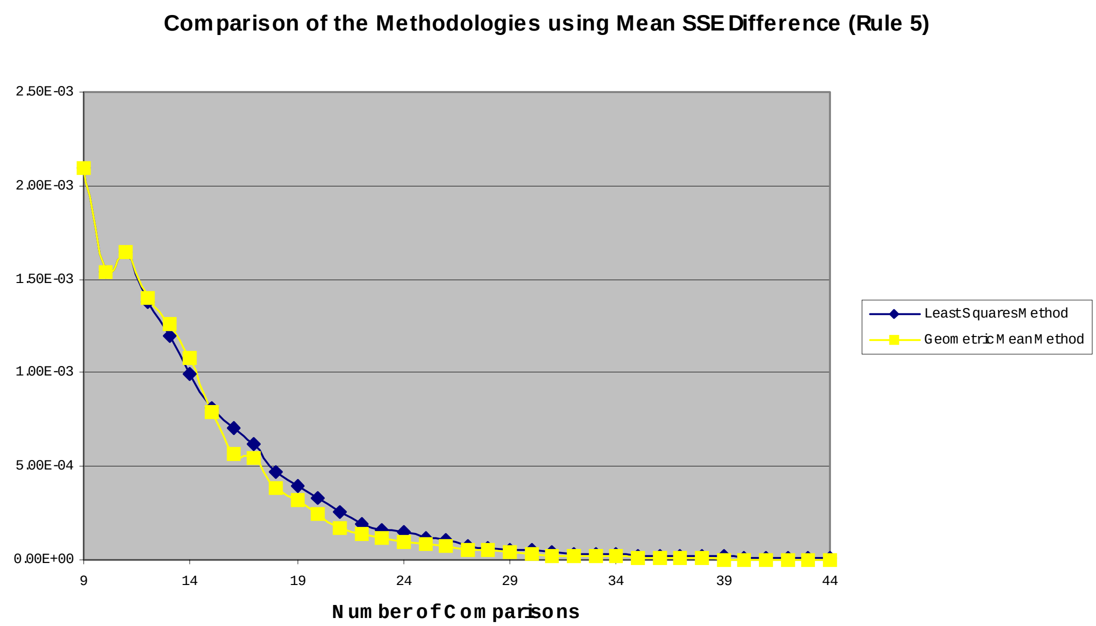

Figure 5-7-1 e: Comparison of the methodologies using the mean sum-square-error difference for the fifth selection rule (n = 10)

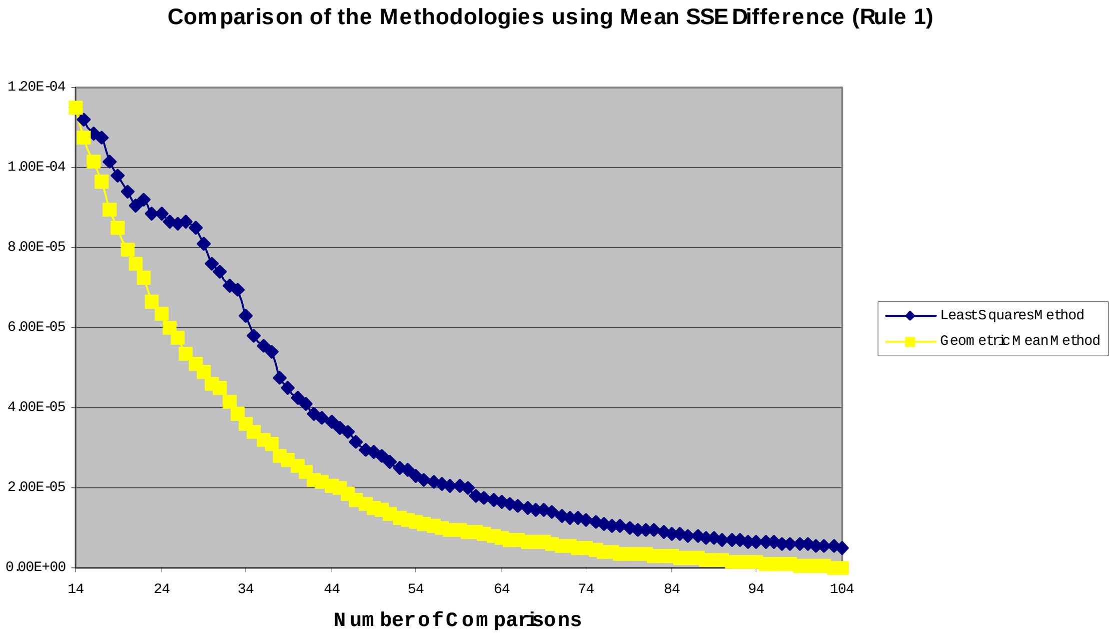

Figure 5-7-2 a: Comparison of the methodologies using the mean sum-square-error difference for the first selection rule (n = 15)

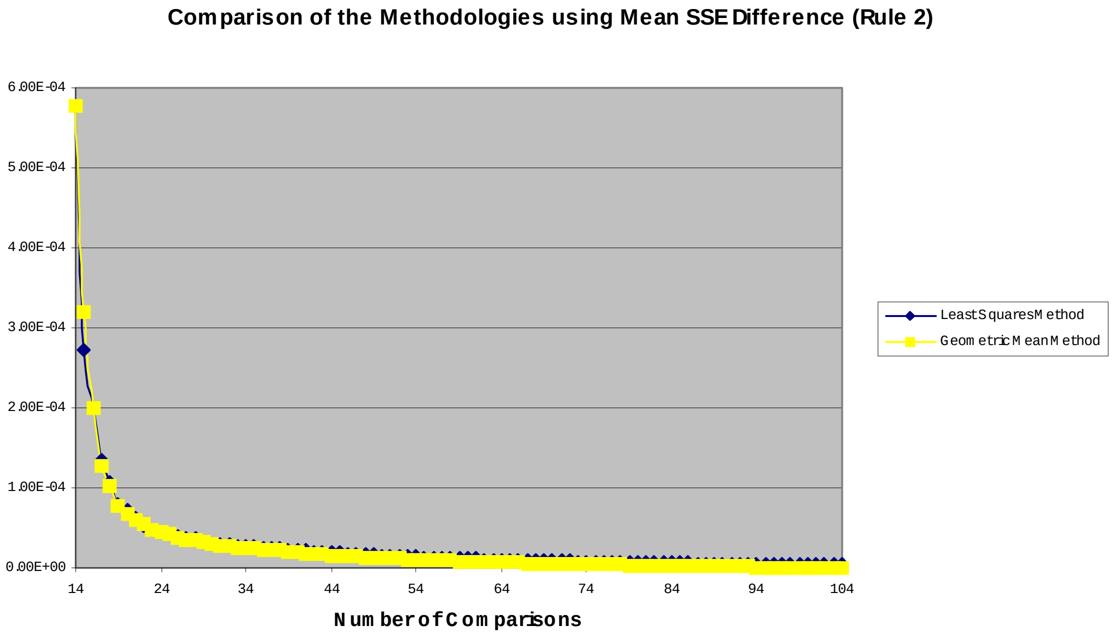

Figure 5-7-2 b: Comparison of the methodologies using the mean sum-square-error difference for the second selection rule (n = 15)

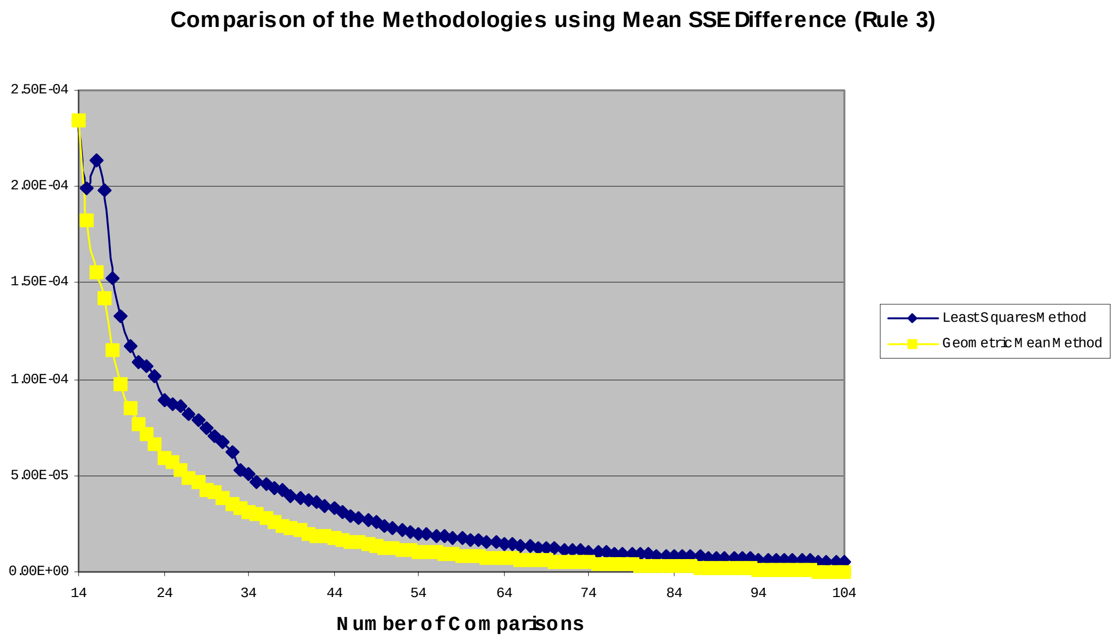

Figure 5-7-2 c: Comparison of the methodologies using the mean sum-square-error difference for the third selection rule (n = 15)

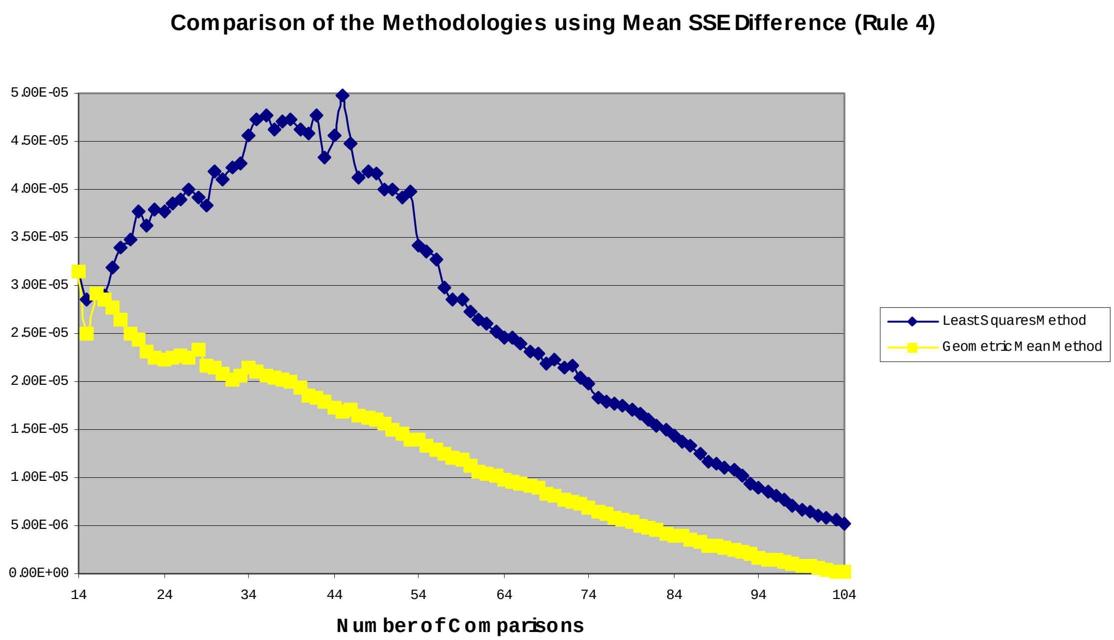

Figure 5-7-2 d: Comparison of the methodologies using the mean sum-square-error difference for the forth selection rule (n = 15)

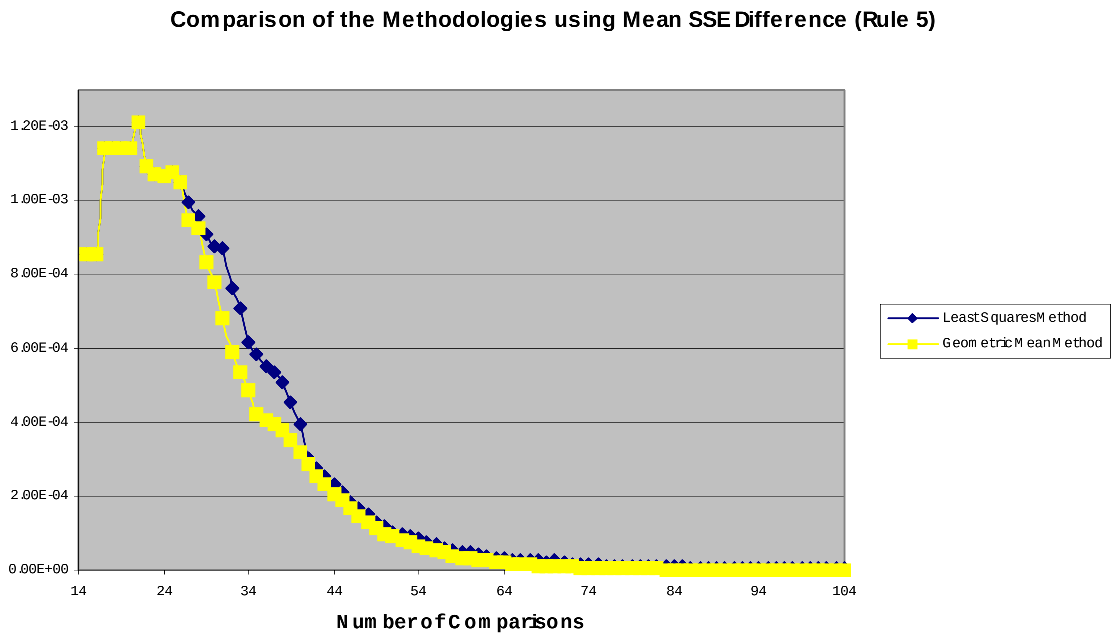

Figure 5-7-2 e: Comparison of the methodologies using the mean sum-square-error difference for the fifth selection rule (n = 15)

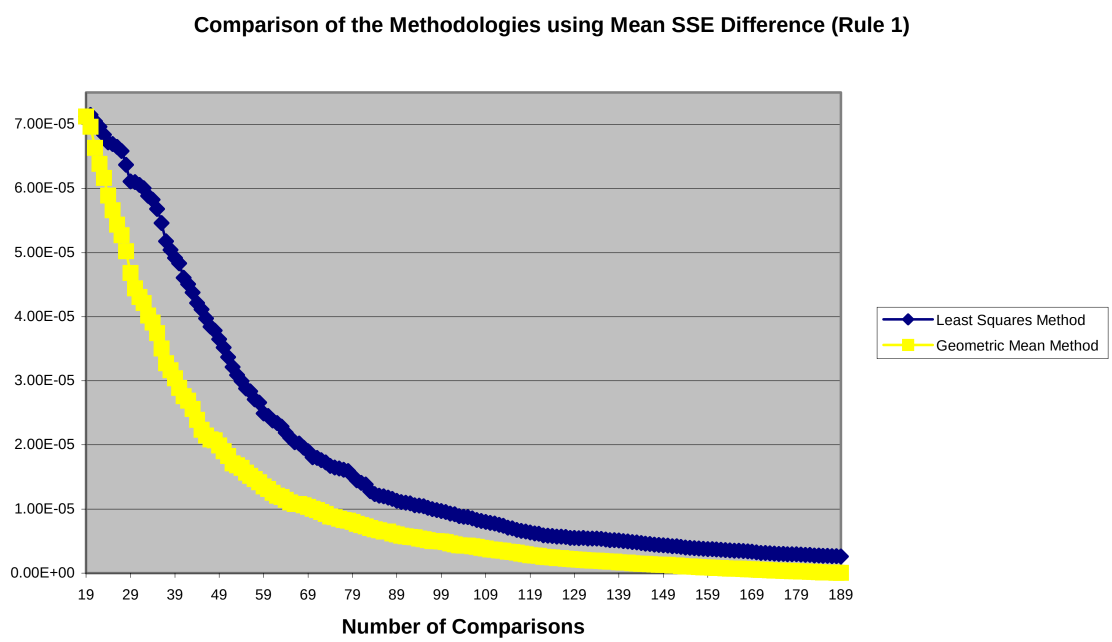

Figure 5-7-3 a: Comparison of the methodologies using the mean sum-square-error difference for the first selection rule (n = 20)

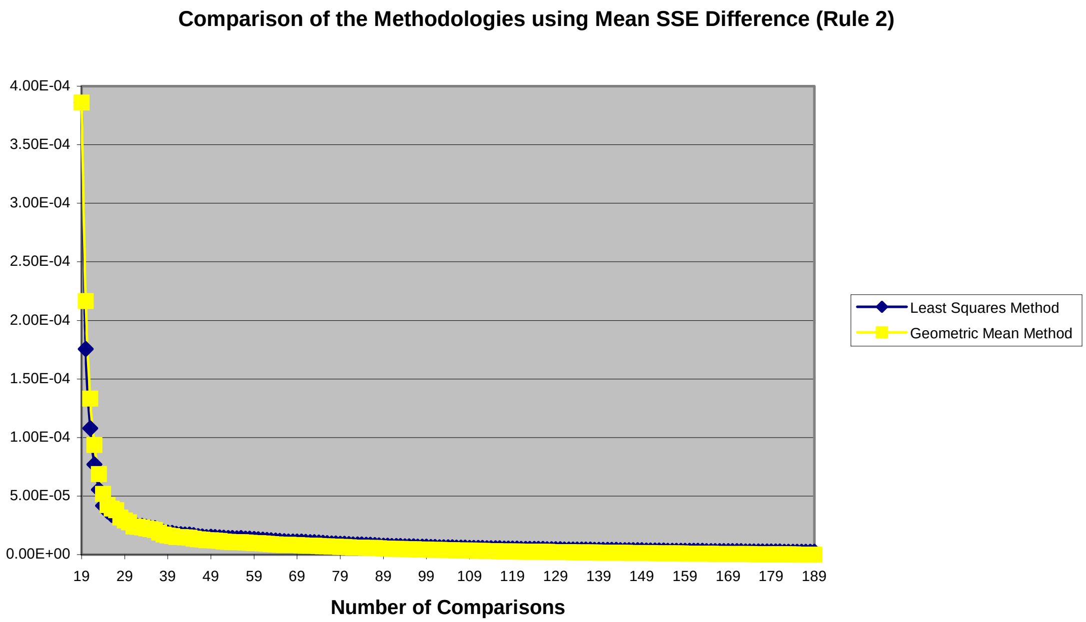

Figure 5-7-3 b: Comparison of the methodologies using the mean sum-square-error difference for the second selection rule (n = 20)

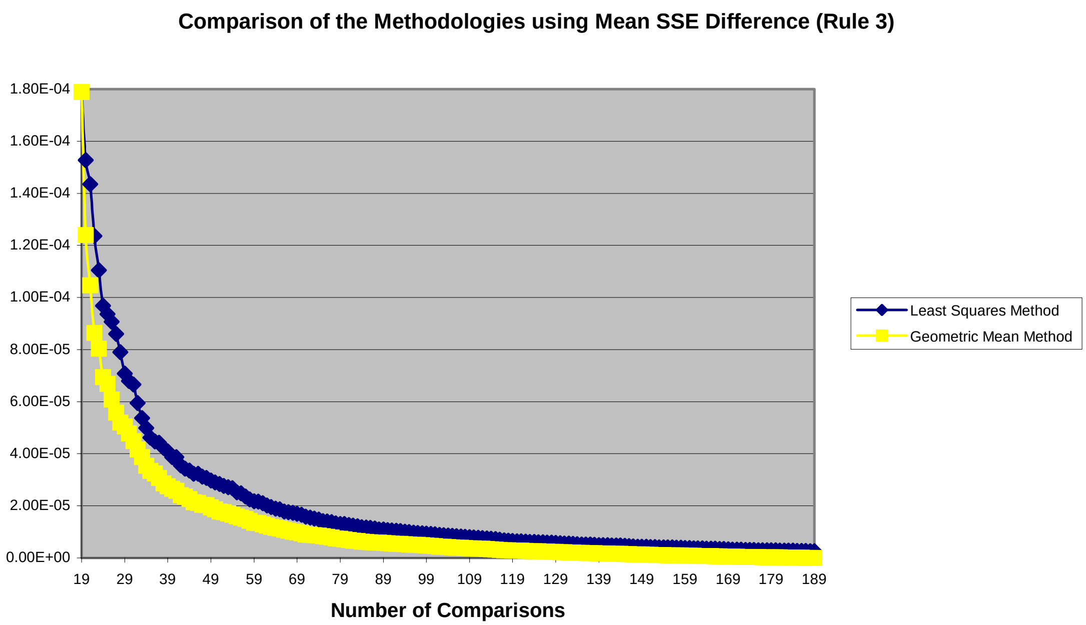

Figure 5-7-3 c: Comparison of the methodologies using the mean sum-square-error difference for the third selection rule (n = 20)

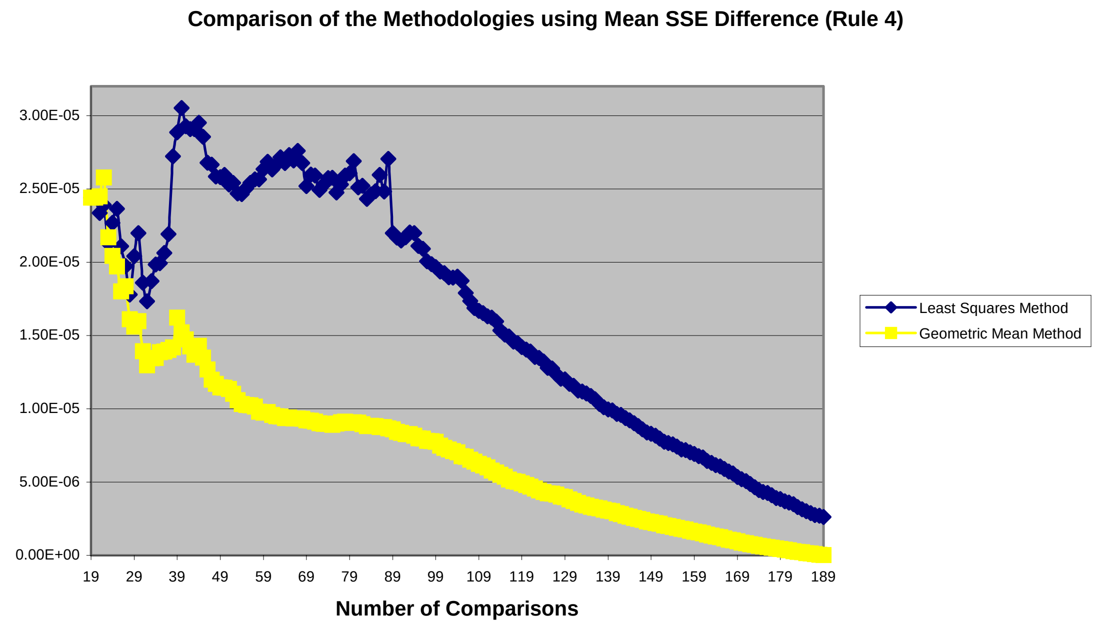

Figure 5-7-3 d: Comparison of the methodologies using the mean sum-square-error difference for the forth selection rule (n = 20)

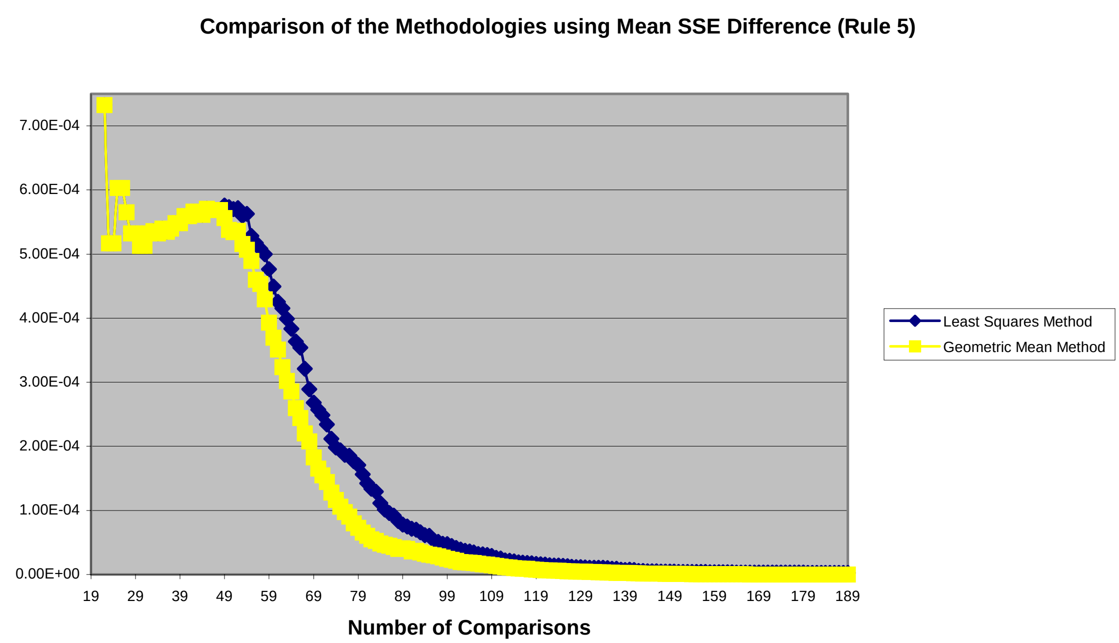

Figure 5-7-3 e: Comparison of the methodologies using the mean sum-square-error difference for the fifth selection rule (n = 20)

Observations:

- The Geometric Mean Method still performs better than the Least Squares Method.
- Again, although the difference under the second and fifth selection rules might be small, it may have a major impact in determining the ranking of the alternatives.

CHAPTER 6

CONCLUSIONS

As stated in earlier chapters, the main focus of this thesis research was to determine if the way that we select our initial comparisons would have an effect in accurately estimating the weight vector from the incomplete AHP matrices. In order to answer this question, we have selected five selection rules, which are simple and easy to understand. These selection rules are discussed in chapter 3 with some numerical examples in chapter 4.

6.1. Conclusions Regarding the First Research Goal

- Overall, the second selection rule performs better than any other selection rules. Please recall that this rule requires the decision maker to first arrange the alternatives in decreasing order of their weights using his/her previous knowledge on the subject. Then, an alternative is compared with the next alternative in order that they are ranked.
- This finding is expected, since the alternatives are ordered prior to be being estimated. Thus, more information is used.
- Due to the nature of the second selection rule, the decision maker is expected to be able to rank the alternatives correctly. If this step is not possible, this selection rule cannot be used.
- The third selection rule (i.e., random selection) is the next best rule that one can use to estimate the weight vector.
- When we only consider the minimum initial comparisons (i.e., the $n-1$ comparisons), the first selection rule (i.e., comparisons based on a single alternative) performs better than the other selection rules.
- These results are inherited, regardless of the matrix dimension used.

6.2. Conclusions Regarding the Second Research Goal

- For n = 10, the minimum number of initial comparisons results in accuracies of 97.6 % to 98.9 % depending on the selection rule used.
- For n = 15, the accuracies of 98.0 % to 99.2 % can be achieved by only using the minimum number of initial comparisons (i.e., $n-1$ comparisons) depending on the selection rule used.
- For n = 20, the $n-1$ initial comparisons results in accuracies of 98.3 % to 99.4 % depending on the selection rule used.
- The above accuracy is based on the mean absolute difference method, without considering the forth and fifth selection rules (i.e., the highest and median comparison values). Since the average numbers of initial comparisons for those rules are above $n-1$, their accuracy cannot be considered.
- If a more accurate result is needed, we have to add additional comparisons until the expected accuracy is achieved.
- For 99.5 % accuracy, most of the selection rules will be indistinguishable. The number of comparisons is approximately equal to $n+11$ (i.e., 34.29% above the minimum $n-1$ comparisons) for n = 10.
- For n = 15, using $n+21$ comparisons (i.e., 23.33 % above the minimum $n-1$ comparisons), an accuracy of 99.5 % can be obtained.
- For n = 20, with $n+50$ comparisons (i.e., 30 % above the minimum $n-1$ comparisons), a slightly higher accuracy of 99.8 % can be obtained.

6.3. Conclusions Regarding the Third Research Goal

- The two methodologies are compared namely the Least Squares Method and the Geometric Mean Method. The Geometric Mean Method performs consistently better than the Least Squares Method, regardless of the selection rule used.
- This finding is due to the way the Least Squares Method is formulated to calculate the weight vector. The calculation of $W = (B^T B)^{-1} B^T b$ may not be very accurate because of the approximation on the matrix inverse step.

6.4. Other Findings

- The comparisons of the selection rules using the ranking differences are not very accurate, since the differences in the weights are very close.
- A connecting path, which connects all the alternatives either directly or indirectly is sometimes hard to be formed, thus additional comparisons are required until such path is established.
- The forth and fifth selection rules have more than $n-1$ comparisons as their minimum number of initial comparisons in order to ensure a connecting path between all the alternatives.
- The number of required additional comparisons increases, as the matrix dimension increases.

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VITA

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