Hopf algebras of dimension pq

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Recommended Citation
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Received 12 May 2003
Communicated by Susan Montgomery

Abstract

Let \( H \) be a non-semisimple Hopf algebra with antipode \( S \) of dimension \( pq \) over an algebraically closed field of characteristic 0 where \( p \leq q \) are odd primes. We prove that \( \text{Tr}(S^2p) = p^2d \) where \( d \equiv pq \ (\mod 4) \). As a consequence, if \( p, q \) are twin primes, then any Hopf algebra of dimension \( pq \) is semisimple.

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1. Introduction

Let \( p \) be an odd prime and \( k \) an algebraically closed field of characteristic 0. If \( H \) is a semisimple Hopf algebra of dimension \( p^2 \) over \( k \), then \( H \) is isomorphic to \( k[\mathbb{Z}_p^2] \) or \( k[\mathbb{Z}_p \times \mathbb{Z}_p] \) by [8]. A more general result for semisimple Hopf algebras of dimension \( pq \), where \( p, q \) are primes, was obtained by [4]. In [10], the author proved that non-semisimple Hopf algebras of dimension \( p^2 \) over \( k \) are Taft algebras and hence completed the classification of Hopf algebras of dimension \( p^2 \). However, if \( p, q \) are distinct primes, there is still no example of non-semisimple Hopf algebras of dimension \( pq \). In fact, it was shown in [1] and [3] that there is no non-semisimple Hopf algebras over \( k \) of dimension 14, 15, 21, 35, 55, 77, 65, 91 or 143.

By [10], if \( p \leq q \) are odd primes and \( H \) is a non-semisimple Hopf algebra with antipode \( S \) of dimension \( pq \), then \( S^3p = \text{id}_H \) and \( \text{Tr}(S^2p) = p^2d \) for some odd integer \( d \). In this paper, we prove that \( d \equiv pq \ (\mod 4) \). As a consequence, we prove that if \( p, q \) are twin primes, any Hopf algebra of dimension \( pq \) over \( k \) is semisimple. Recently, Etingof

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and Gelaki also announce an even more general result [5] which covers the cases when $p < q \leq 2p + 1$.

2. Notation and preliminaries

Throughout this paper, $p \leq q$ are odd primes, $k$ denotes an algebraically closed field of characteristic 0, and $H$ denotes a finite-dimensional Hopf algebra over $k$ with antipode $S$. Its comultiplication and counit are, respectively, denoted by $\Delta$ and $\varepsilon$. We will use Sweedler’s notation [16]:

$$\Delta(x) = \sum x_{(1)} \otimes x_{(2)}.$$ 

A non-zero element $a \in H$ is called group-like if $\Delta(a) = a \otimes a$. The set of all group-like elements $G(H)$ of $H$ is a linearly independent set, and it forms a group under the multiplication of $H$. For the details of elementary aspects for finite-dimensional Hopf algebras, readers are referred to [9,16].

Let $\lambda \in H^*$ be a non-zero right integral of $H^*$ and $\Lambda \in H$ a non-zero left integral of $H$. There exists $\alpha \in \text{Alg}(H, k) = G(H^*)$, independent of the choice of $\Lambda$, such that $\Lambda a = \alpha(a) \Lambda$ for $a \in H$. Likewise, there is a group-like element $g \in H$, independent of the choice of $\lambda$, such that $\beta\lambda = \beta(g)\lambda$ for $\beta \in H^*$. We call $g$ the distinguished group-like element of $H$ and $\alpha$ the distinguished group-like element of $H^*$. Then we have a formula for $S^4$ in terms of $\alpha$ and $g$ [11]:

$$S^4(a) = g(\alpha \rightarrow a \leftarrow \alpha^{-1})g^{-1} \quad \text{for } a \in H, \quad (2.1)$$

where $\rightarrow$ and $\leftarrow$ denote the natural actions of the Hopf algebra $H^*$ on $H$ described by

$$\beta \rightarrow a = \sum a_{(1)}\beta(a_{(2)}) \quad \text{and} \quad a \leftarrow \beta = \sum \beta(a_{(1)})a_{(2)}$$

for $\beta \in H^*$ and $a \in H$. There are some useful formulae for the trace of a linear endomorphism on $H$ in terms of $\lambda$ and $\Lambda$.

**Theorem 2.1** [13, Theorem 1]. Let $H$ be a finite-dimensional Hopf algebra with antipode $S$ over the field $k$. Suppose that $\lambda$ is a right integral of $H^*$, and that $\Lambda$ is a left integral of $H$ such that $\lambda(\Lambda) = 1$. Then for any $f \in \text{End}_k(H)$,

$$\text{Tr}(f) = \sum \lambda((S \circ f)(A_{(2)})A_{(1)}) = \sum \lambda((f \circ S)(A_{(2)})A_{(1)}) = \sum \lambda((f \circ S)(A_{(2)})A_{(1)}).$$

Following [10, Section 2], the index of $H$ is the least positive integer $n$ such that

$$S^{4n} = \text{id}_H \quad \text{and} \quad g^n = 1.$$
Suppose that $H$ is a finite-dimensional non-semisimple Hopf algebra of odd index $n$, and that $\omega \in k$ is a primitive $n$th root of unity. Since $g^n = 1$ and $\alpha$ is an algebra map, $\alpha(g)$ is a $n$th root of unity. There exists a unique element $x(\omega, H) \in \mathbb{Z}_n$ such that

$$\alpha(g) = \omega^{x(\omega, H)}.$$ 

Following the notation in [10], we let $H^\omega_{a,i,j} = \{ u \in H \mid S^2(u) = (-1)^{a} \omega^i u \text{ and } ug = \omega^j u \}$ for any $(a, i, j) \in \mathbb{Z}_2 \times \mathbb{Z}_n \times \mathbb{Z}_n$. Since $r(g) \in \text{End}_k(H)$, defined by $r(g)(a) = ag$ for $a \in H$, commutes with $S^2$, we have

$$H = \bigoplus_{a \in K_n} H^\omega_a,$$ 

(2.2)

where $K_n$ denotes the group $\mathbb{Z}_2 \times \mathbb{Z}_n \times \mathbb{Z}_n$.

Using the eigenspace decomposition of $H$ in (2.2), the diagonalization of a left integral $\Lambda$ of $H$ admits the following form (cf. [10]),

$$\Delta(\Lambda) = \sum_{a \in K_n} \left( \sum_{u} u_a \otimes v_{-a+x} \right),$$ 

(2.3)

where $\sum u_a \otimes v_{-a+x} \in H^\omega_a \otimes H^\omega_{-a+x}$ and $x = (0, -x(\omega, H), x(\omega, H))$ in $K_n$.

In the sequel, we will call the expression in Eq. (2.3) the normal form of $\Delta(\Lambda)$ associated with $\omega$. We will simply write $u_a \otimes v_{-a+x}$ for the sum $\sum u_a \otimes v_{-a+x}$ in the normal form of $\Delta(\Lambda)$.

Let $E^\omega_a$, $a \in K_n$, be the set of orthogonal projections associated with the decomposition (2.2). Then

$$\dim(H^\omega_a) = \text{Tr}(E^\omega_a)$$

and we have the following lemma.

**Lemma 2.2.** Let $H$ be a finite-dimensional non-semisimple Hopf algebra with antipode $S$ of odd index $n$ over $k$, and $\omega \in k$ a primitive $n$th root of unity. Then we have

$$\dim(H^\omega_a) = \dim(H^{\omega_a})$$

for all $a \in K_n$ where $x = (0, -x(\omega, H), x(\omega, H))$.

**Proof.** Let $\Lambda$ be a left integral of $H$ and let $\lambda$ be a right integral of $H^*$ such that $\lambda(\Lambda) = 1$. Using the normal form of $\Delta(\Lambda)$ associated with $\omega$ in (2.3) and Theorem 2.1, we have

$$\text{Tr}(E^\omega_a) = \sum_{b \in K_n} \lambda(S(v_{-b+x})E^\omega_a(u_b)) = \lambda(S(v_{-a+x})u_a).$$
By Theorem 2.1 again, we also have

$$\text{Tr}(E_{a+i}^\omega) = \sum_{b \in K} \lambda(S(E_{a+i}^\omega)(v-b+x)u_b) = \lambda(S(v-a+x)u_a).$$

Therefore, $\text{Tr}(E_{a+i}^\omega) = \text{Tr}(E_{a-i}^\omega)$. Since $\dim(H_{a+i}^\omega) = \text{Tr}(E_{a+i}^\omega)$ for any $a \in K_n$, the result follows.

**Theorem 2.3** [10]. Let $H$ be a non-semisimple Hopf algebra of dimension $pq$ over $k$ with antipode $S$, where $p \leq q$ are odd primes. Then the index of $H$ and the order of $S^4$ are equal to $p$, and $\text{Tr}(S^2p) = p^2d$ for some odd integer $d$.

**Lemma 2.4.** Suppose that $H$ is a non-semisimple Hopf algebra of dimension $pq$ over $k$ where $p \leq q$ are odd primes, and that $\omega \in k$ is a primitive $p$th root of unity. Let $g$ and $\alpha$ be the distinguished group-like elements of $H$ and $H^*$, respectively. If $g$ is non-trivial, then the integer $d$ in Theorem 2.3 is given by

$$\dim(H_{a,i,j}^\omega) - \dim(H_{a,i,j}^\sigma) = d$$

for all $i, j \in \mathbb{Z}_p$. Moreover, if both $g$ and $\alpha$ are not trivial, then

$$\dim(H_{a,i,j}^\sigma) = \dim(H_{a,i,j}^\omega)$$

for any $a \in \mathbb{Z}_2$ and $i, j, j' \in \mathbb{Z}_p$.

**Proof.** If $\alpha$ is trivial and $g \neq 1$, then by [10, Lemma 4.3],

$$\dim(H_{0,i,j}^\omega) - \dim(H_{1,i,j}^\sigma) = d.$$

If both $g$ and $\alpha$ are non-trivial, then by the proof of [10, Proposition 5.3], $H$ is isomorphic to the biproduct

$$R \times B$$

as Hopf algebras (cf. [12]) where $B = k[g]$ and $R$ is a left $B$-comodule subalgebra of $H$. It was shown in [2, Section 4] that $R$ is invariant under $S^2$. Moreover, in the identification $H \cong R \otimes B$ given by multiplication, one has

$$S^2 = T \otimes \text{id}_B,$$

where $T$ is the restriction of $S^2$ on $R$. Let

$$R_{a,i} = \{ x \in R \mid S^2(x) = (-1)^a \omega^i x \}$$

for any $(a, i) \in \mathbb{Z}_2 \times \mathbb{Z}_p$. It follows from the proof of [10, Proposition 5.3] that

$$\dim(R_{0,i}) = \dim(R_{1,i}) = d.$$
By (2.4),
\[ H_{a,i,j}^\omega = R_{a,i} \otimes e_j \]
for all \((a, i, j) \in \mathcal{K}_p\) where \(e_j\) is the central idempotent of \(B\) such that \(e_j g = \omega^i e_j\). Thus,
\[ \dim(H_{a,i,j}^\omega) = R_{a,i} \]
for all \((a, i, j) \in \mathcal{K}_p\) and hence
\[ \dim(H_{0,i,j}^\omega) - \dim(H_{1,i,j}^\omega) = d. \]

3. Proofs of main results

**Lemma 3.1.** Let \(H\) be a finite-dimensional non-semisimple Hopf algebra with antipode \(S\) of odd index \(n\) over \(k\), and \(\omega \in k\) a primitive \(n\)th root of unity. Let \(\ell \in \mathbb{Z}\) such that \(2\ell = x(\omega, H)\). Then
\[ \dim(H_{1, -\ell, \ell}^\omega) \]
is even.

**Proof.** Let \(V\) be the space of all \(f \in H^*\) such that \(f(H_{a,i,j}^\omega) = \{0\}\) whenever \((a, i, j) \neq (1, -\ell, \ell)\). Obviously, \(V\) is isomorphic to \((H_{1, -\ell, \ell}^\omega)^*\) and so \(\dim(V) = \dim(H_{1, -\ell, \ell}^\omega)\). Let \(\Lambda\) be a non-zero left integral of \(H\) and
\[ \Delta(\Lambda) = \sum_{a \in \mathcal{K}_n} u_a \otimes v_{-a + x} \]
the normal form of \(\Delta(\Lambda)\) associated with \(\omega\) where \(x = (0, -2\ell, 2\ell)\). Then
\[ (f, h) = (f \otimes h)\Delta(\Lambda) \]
defines a non-degenerate bilinear form on \(H^*\). Let \(f \in V\) such that \((f, h) = 0\) for all \(h \in V\). For any \(h' \in H^*\), there exists \(h \in V\) such that \(h'(u) = h(u)\) for all \(u \in H_{1, -\ell, \ell}^\omega\). Thus
\[ (f, h') = \sum_{a \in \mathcal{K}_n} f(u_a)h'(v_{-a + x}) = f(u_{1, -\ell, \ell})h'(v_{1, -\ell, \ell}) = (f, h) = 0. \]

By the non-degeneracy of \((\cdot, \cdot)\), \(f = 0\). Therefore, \((\cdot, \cdot)\) induces a non-degenerate bilinear form on \(V\). Using [14, Theorem 3(d)], we have
\[ \Delta^{op}(\Lambda) = \sum_{(a, i, j) \in \mathcal{K}_n} (-1)^a \omega^{-i-j} \left( \sum_{a, i, j} u_{a,i,j} \otimes v_{a,-2\ell-i,2\ell-j} \right). \]
Therefore, for any \( f, h \in V \),

\[
(h, f) = (f \otimes h)\Delta^\text{op}(A) = -f(u_1, -\ell, \ell)h(v_1, -\ell, \ell) = -(f, h).
\]

Hence, \( V \) admits a non-degenerate alternating form and so \( \dim(V) \) is even. \( \square \)

If \( H \) is a finite-dimensional Hopf algebra of index \( n \), we define

\[
H^- := \{ u \in H \mid S^2(u) = -u \}, \\
H^+ := \{ u \in H \mid S^2(u) = u \}.
\]

**Corollary 3.2.** Suppose \( H \) is a finite-dimensional non-semisimple Hopf algebra with antipode \( S \) of odd index \( n \) over \( k \). Then, the subspace \( H^- \) is of even dimension.

**Proof.** Let \( \omega \in k \) be an \( n \)th root of unity and \( \ell \in \mathbb{Z} \) such that \( 2\ell = x(\omega, H) \). We then have

\[
H^- = \bigoplus_{i,j \in \mathbb{Z}_n} H^\omega_{1, i, j} = H^\omega_{1, -\ell, \ell} \oplus \left( \bigoplus_{(i,j) \neq (-\ell, \ell)} H^\omega_{1, i, j} \oplus H^\omega_{1, -2\ell-i, 2\ell-j} \right).
\]

It follows from Lemmas 2.2 and 3.1 that \( \dim(H^-) \) is even. \( \square \)

**Theorem 3.3.** Let \( H \) be a non-semisimple Hopf algebra with antipode \( S \) of dimension \( pq \) where \( p \leq q \) are odd primes. Then

\[
\text{Tr}(S^{2p}) = p^2 d \quad \text{and} \quad d \equiv pq \pmod{4}.
\]

**Proof.** By Theorem 2.3, \( H \) is of index \( p \) and \( \text{Tr}(S^{2p}) = p^2 d \) for some odd integer \( d \). Since

\[
\dim(H^+) + \dim(H^-) = pq
\]

and

\[
\text{Tr}(S^{2p}) = \dim(H^+) - \dim(H^-) = p^2 d,
\]

we have

\[
\dim(H^-) = p(q - pd)/2.
\]

By Corollary 3.2, \( p(q - pd) \equiv 0 \pmod{4} \) or \( d \equiv pq \pmod{4} \). \( \square \)

**Theorem 3.4.** For any pair of twin primes \( p < q \), if \( H \) is a Hopf algebra of dimension \( pq \), then \( H \) is semisimple.
Proof. Suppose there is a non-semisimple Hopf algebra $H$ of dimension $pq$. By [6], $H^*$ is also non-semisimple. Since $\dim(H)$ is odd, by [7, Theorem 2.1], $H$ and $H^*$ cannot be both unimodular. By duality, we may simply assume that $H^*$ is not unimodular. It follows from Theorem 2.3 that $|G(H)| = p$ and so

$$\dim(C) \geq p,$$

where $C$ is the coradical of $H$. If $\dim(C) = p$, then $H$ is pointed and hence, by [15, Corollary 4], $H$ is semisimple. Therefore, $\dim(C) > p$ and so we have

$$\text{Tr}(S^p|_{H/C}) \geq -(pq - \dim(C)) > -pq + p = -p^2 - p.$$

It follows from [6, Lemma 3.2] that

$$\text{Tr}(S^p|_C) \geq p.$$

Thus, we have

$$\text{Tr}(S^p) = \text{Tr}(S^p|_C) + \text{Tr}(S^p|_{H/C}) > -p^2.$$  \hspace{1cm} (3.1)

Since $pq \equiv -1 \pmod{4}$, by Theorem 3.3,

$$\text{Tr}(S^p) = -p^2$$

but this contradicts (3.1). □

Acknowledgment

The author thanks P. Etingof for his useful suggestions for Theorem 3.4 and bringing the author’s attention to his recent work [5] with S. Gelaki.

References