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The new stochastic integral and anticipating stochastic differential equations

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Abstract

In this work, we develop further the theory of stochastic integration of adapted and instantly independent stochastic processes started by Wided Ayed and Hui-Hsiung Kuo in [1, 2]. We provide a first counterpart to the Itô isometry that accounts for both adapted and instantly independent processes. We also present several Itô formulas for the new stochastic integral. Finally, we apply the new Itô formula to solve a linear stochastic differential equations with anticipating initial conditions.

Chapter 1
Introduction

1.1 Historical background

In discussing the history of the theory behind this work, one could start some 2200 years ago, with Archimedes' approximation of the number π — the first documented work involving limits. But it was not until 1821 that Augustin-Louis Cauchy [7] defined the limit precisely. Before that, Sir Isaac Newton [24] and Gottfried Leibniz introduced the notions of derivatives and integrals laying grounds for classical calculus.

In 1854, Bernhard Riemann, in his Habilitationsschrift (published later in [30]), defined the concept of the integral, now called the Riemann integral. It involves computation of a limit of, what we now call, Riemann sums and turns out to be, in a certain sense, an operation opposite to that of differentiation. However Riemann integral has its limitations, most of which were overcame by the introduction of the Lebesgue integral. Introduced in 1902 by Henri Lebesgue [22], the Lebesgue integral is defined using the step functions and approximation of measurable functions by step functions.

Before we discuss the stochastic integral, we need to introduce the notions of randomness, random variables and stochastic processes. And of course the most well-known stochastic process of them all — the Brownian motion. This process is named after a Scottish botanist Robert Brown [5], who in 1827 observed that pollen grains suspended in water and observed through a microscope continuously move in a random fashion. Although Brown did not explain the source of the motion he discovered, the mathematical object, a stochastic process, describing the motion is named after him. In 1900, in his doctoral thesis, Louis Bachelier used Brownian motion to model a stochastic differential equation to evaluate stock options, pioneering the field of mathematical finance.

In 1905 Albert Einstein [8], and independently, in 1906 a Polish physicist Marian Smoluchowski [34] described the Brownian motion using terminology closer to contemporary mathematics. Einstein used Brown's observations to establish the existence of atoms (a fact that was not commonly agreed upon in the early twentieth century).

Even though Brownian motion was already studied and used in applications for many years, it was not until 1923 when Norbert Wiener [36] provided a precise mathematical definition of a stochastic process now known as the Brownian motion or the Wiener process. Wiener also proved, through construction, the existence of the object he defined.

Some twenty years later, in 1944, Kiyosi Itô [12] defined a new kind of an integral — the stochastic integral with respect to Brownian motion. It was a starting point for a whole new area of both pure and applied mathematics, known nowadays as stochastic calculus. The definition proposed by Itô lays somewhat between the definitions of Riemann and Lebesgue integrals, in the sense that, it is built upon Riemann–like sums with Brownian increments instead of deterministic increments, but the construction is carried out through step stochastic processes, like in the construction of Lebesgue integral.

Probably the most well-known application of the theory of stochastic integration is the famous Black–Scholes formula for the price of the European option derived by Fisher Black and Myron Scholes [4] in 1973. For their work in mathematical finance, Robert Merton [23] (who deepened the mathematical understanding of the Black–Scholes model) and Scholes received a 1997 Nobel prize in Economics.

Even though Itô's theory has many applications, it has certain restrictions, the main one being the fact that it can only be applied to processes that do not see into the future. We call the processes that do not depend on the future adapted or non-anticipating, while those that do depend on the future, non-adapted or anticipating. The ability to integrate the anticipating processes becomes important once we want to model certain events like "insider trading" where the knowledge of the future events is the basis for trade decision.

There were several attempts to overcome this limitation of the Itô stochastic integral. Even Itô studied the possible extension of the theory onto anticipating processes. During the 1976 Kyoto symposium on stochastic differential equations, he proposed an approach, based on the enlargement of the filtration, to evaluate the integral $\int_0^1 B_1 \, dB_t$, mathematical meaning of which will be discussed in forthcoming chapters. The main drawback of his approach is the fact that the Brownian motion loses its good properties in the new setting.

Another approach was proposed independently by Masuyuki Hitsuda [10, 11] and Anatoliy Skorokhod [33]. Their work results in, what is nowadays called, the Hitsuda–Skorokhod integral that can be defined as either the adjoint operator to Malliavin derivative (see [25] for more details), or the white noise integral (see [14] for more details). However, both of the definitions require substantial background theory and result in an object that might be a generalized function rather than a regular random variable.

Recently, Wided Ayed and Hui-Hsiung Kuo [1, 2] proposed a different approach, one that is anchored in the basic probability theory and does not require the use of generalized functions or enlargement of filtration. Their idea will be discussed in Chapter 3 and developed further in the following chapters of this dissertation. The main idea of Ayed and Kuo is to identify a counterpart to the adapted processes that are used in the Itô theory, and exploit their properties to define the new stochastic integral. This theory is currently being developed further by Kuo, Anuwat Sae-Tang and the author of this dissertation (see [18, 19, 20, 21].)

1.2 Mathematical background

We will begin the theoretical part of this dissertation with the definitions of several basic, but key, concepts from probability theory that are used throughout this work. The first definition is that of a probability space.

Definition 1.1 (Probability space)**.** The *probability space* is a triple $(\Omega, \mathcal{F}, P)$ consisting of:

1. sample space Ω,

2. σ-field of events $\mathcal{F}$,

3. probability measure P.

Henceforth, we fix an arbitrary complete probability space $(\Omega, \mathcal{F}, P)$. That is a space for which the following condition holds: if a set B is such that $P(B) = 0$ and $B \subset A$, where $A \in \mathcal{F}$, then $B \in \mathcal{F}$.

Definition 1.2 (Random variable)**.** A real valued *random variable* on the probability space $(\Omega, \mathcal{F}, P)$ is a Borel measurable function $X : \Omega \to \mathbb{R}$, that is for each Borel set $B \subseteq \mathbb{R}$ the set $X^{-1}(B) = \{\omega : X^{-1}(\omega) \in B\}$ is $\mathcal{F}$-measurable.

It is not possible to talk about stochastic analysis without the object we wish to analyze — stochastic process. Below we give the definition of a stochastic process which we illustrate with the most famous stochastic process, namely Brownian motion.

Definition 1.3 (Stochastic process). A collection $\{X_t\}_{t\in T}$ of random variables on $(\Omega, \mathcal{F}, P)$ indexed by a set T is called a *stochastic process.*

Whenever the index set T is one of $[a, b]$, $[a, T]$, $[0, \infty)$, we will suppress the index set and write $\{X_t\}$, X_t or even $X(t)$ instead of $\{X_t\}_{t\in T}$. Notice that the stochastic process $\{X_t\}$ is a function

$$T \times \Omega \ni (t, \omega) \longmapsto X(t, \omega) \in \mathbb{R}$$

such that

1. $X(t, \cdot) : \Omega \to \mathbb{R}$ is a random variable for each $t \in T$;

2. $X(\cdot, \omega) : T \to \mathbb{R}$ is a measurable function for each $\omega \in \Omega$;

Definition 1.4 (Brownian motion). *Brownian motion* B_t is a stochastic process that satisfies the following properties:

1. The process B_t starts at 0 almost surely, that is

$$P(B_0 = 0) = 1;$$

2. For any $0 \le s < t$, the random variable $B_t - B_s$ has normal distribution with mean 0 and variance $t - s$;

3. The process B_t has independent increments, that is, for any $n \in \mathbb{N}$ and any $0 \le t_1 < t_2 < \ldots < t_{n-1} < t_n$, the random variables

$$B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$$

are independent;

4. P-almost all sample paths of B_t are continuous, that is,

$$P(\omega : B.(\omega) \text{ is continuous}) = 1.$$

Since it is a well-known fact, and there are several constructions available, we will omit the proof of existence of Brownian motion.

Definition 1.5 (Filtration)**.** A *filtration* $\{\mathcal{F}_t\}_{t\in T}$ on a probability space $(\Omega, \mathcal{F}, P)$ is an increasing sequence of sub-σ-fields of $\mathcal{F}$.

We can think of the filtration as the information available to us at a given time. As the time progresses, we gain more and more information.

Example 1.6 (Natural Brownian filtration)**.** The natural Brownian filtration is given by

$$\mathcal{F}_t = \sigma\left(B_s | s \in [0,t]\right), \qquad \text{for } t \in [0,\infty).$$

Henceforth, we fix an arbitrary filtration $\{\mathcal{F}_t\}$ such that

1. B_t is measurable with respect to $\mathcal{F}_t$ for each $t \geq 0$;

2. $B_t - B_s$ is independent of $\mathcal{F}_s$ for each $0 \leq s < t$.

An example of such a filtration is the natural Brownian filtration defined in Example 1.6.

Definition 1.7 (Adapted stochastic process)**.** We say that a stochastic process $\{X_t\}_{t\in T}$ is *adapted* to the filtration $\{\mathcal{F}_t\}_{t\in T}$ if for all $t \in T$, the random variable X_t is $\mathcal{F}_t$-measurable.

The notion of adaptedness is quite natural, as it can be thought of as inability to see into the future. That is, an adapted process depends only on the past and the present and has no knowledge about future events.

Example 1.8. Trivially, Brownian motion B_t is adapted to its natural filtration.

Definition 1.9 (Conditional Expectation)**.** A *conditional expectation* of an integrable random variable X on $(\Omega, \mathcal{F}, P)$ with respect to a σ-field $\mathcal{G} \subset \mathcal{F}$ is a random variable denoted by $\mathbb{E}[X|\mathcal{G}]$ that satisfies the following conditions

1. $\mathbb{E}[X|\mathcal{G}]$ is measurable with respect to $\mathcal{G}$;

2. $\int_A \mathbb{E}[X|\mathcal{G}]dP = \int_A XdP$ for all $A \in \mathcal{G}$.

We can think of the conditional expectation $\mathbb{E}[X|\mathcal{G}]$ as a "best guess" for the value of the random variable X given information from $\mathcal{G}$. This will become more natural once we introduce the notion of martingales.

Remark 1.10 (Properties of conditional expectation)**.** Suppose that X and Y are integrable random variables on $(\Omega, \mathcal{F}, P)$. Suppose also that $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$ are σ-fields and $a \in \mathbb{R}$ is a constant. The following are simple properties of the conditional expectation:

$$\mathbb{E}[aX + Y|\mathcal{G}] = a\mathbb{E}[X|\mathcal{G}] + \mathbb{E}[Y|\mathcal{G}], \tag{CE-1}$$

$$\mathbb{E}\left(\mathbb{E}[X|\mathcal{G}]\right) = \mathbb{E}[X], \tag{CE-2}$$

$$\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X], \qquad \text{if } X \perp \mathcal{G} \tag{CE-3}$$

$$E[XY|\mathcal{G}] = Y\mathbb{E}[X|\mathcal{G}], \qquad \text{if } Y \in \mathcal{G} \text{ and } \mathbb{E}|XY| < \infty \tag{CE-4}$$

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]. \tag{CE-5}$$

Above and henceforth, we use the notation $X \perp \mathcal{G}$ to mean that X and $\mathcal{G}$ are independent, and $Y \in \mathcal{G}$ to mean that Y is $\mathcal{G}$-measurable.

Definition 1.11 (Martingale)**.** A stochastic process $\{X_t\}$ is said to be a *martingale* with respect to a filtration $\{\mathcal{F}_t\}$ if the following conditions hold:

1. $\mathbb{E}\,|X_t| < \infty$ for all $t \geq 0$;

2. $\mathbb{E}\left[X_t|\mathcal{F}_s\right] = X_s$ for all $0 < s < t$.

So, a martingale is a stochastic process whose "best estimation" for the value at a future time t given the history up until the current time s is the current value of the process itself. In other words, if I had to bet on the future value of X_t knowing it's value now, the most reasonable bet I can make is the current value of X_s.

Remark 1.12. Note that condition 2 of Definition 1.11 implies that if a stochastic process $\{X_t\}$ is a martingale with respect to a filtration $\{\mathcal{F}_t\}$, then it is adapted to that filtration.

Example 1.13. Brownian motion is a martingale with respect to its natural filtration. To see this, we will use the Definition 1.4 of the Brownian motion:

$$\begin{aligned}
\mathbb{E}\left[B_t|\mathcal{F}_s\right] &= \mathbb{E}\left[B_t - B_s + B_s|\mathcal{F}_s\right] \\
&= \mathbb{E}\left[B_t - B_s|\mathcal{F}_s\right] + \mathbb{E}\left[B_s|\mathcal{F}_s\right] && \text{by (CE-1)} \\
&= \mathbb{E}\left[B_t - B_s\right] + B_s && \text{by (CE-3) and (CE-4)} \\
&= B_s
\end{aligned}$$

Above we have also used the fact that $\mathbb{E}[B_t - B_s] = 0$ as in Definition 1.4.

Definition 1.14 (Quadratic variation)**.** The *quadratic variation* of a function $f(t)$ on the interval $[a, b]$ is given by

$$\mathrm{QV}_{[a,b]}(f) = \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^{n} \left(f\left(t_i\right) - f\left(t_{i-1}\right)\right)^2, \tag{1.1}$$

whenever the limit exists.

Definition 1.15. If $f(t)$ in Definition 1.14 is a stochastic process, then the quadratic variation is given by the same limit evaluated in probability, whenever it exists.

Example 1.16. If $f(t)$ is a C^1 function, then $QV(f) = 0$. This follows from the mean value theorem for derivatives:

$$\begin{aligned}
\mathrm{QV}_{[a,b]}(f) &= \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} \left(f\left(t_i\right) - f\left(t_{i-1}\right)\right)^2 \\
&= \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} \left(f'\left(t_i^*\right)\left(t_i - t_{i-1}\right)\right)^2 \\
&\leq \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} \|f'\|_\infty^2 \left(t_i - t_{i-1}\right)^2 \\
&\leq \|f'\|_\infty^2 \lim_{\|\Delta_n\|\to 0} \|\Delta_n\| \sum_{i=1}^{n} \left(t_i - t_{i-1}\right) \\
&= \|f'\|_\infty^2 \lim_{\|\Delta_n\|\to 0} \|\Delta_n\| \left(b - a\right) \\
&= 0.
\end{aligned}$$

Fact 1.17 (Quadratic variation of Brownian motion)**.** It is a well-known fact that the quadratic variation of Brownian motion on the interval $[a, b]$ is given by $\mathrm{QV}_{[a,b]}(B) = b - a$.

Chapter 2
Itô Integral

In this chapter, we introduce the concept of the Itô integral. It is not our intention to provide a comprehensive review of the Itô theory of stochastic integration. We merely provide well-known results pertinent to the results presented in the chapters following this one. For a more complete presentation of this topic see [15, 28].

2.1 Definition

Although the stochastic integral may be defined for very general classes of stochastic processes (see [15, Chapters 4–6]), for the purpose of this dissertation we will restrict ourselves to the definition of the Itô integral for square integrable adapted stochastic processes which we define below. Also, the integrand in the stochastic integral may be much more general than Brownian motion, but again we will only consider the stochastic integral with respect to Brownian motion.

Definition 2.1. We denote by $L^2_{\text{ad}}(\Omega \times [a,b])$ the space of all adapted stochastic processes $f(t)$ such that

$$\mathbb{E}\left[\int_a^b f^2(t)\,dt\right] < \infty. \tag{2.1}$$

We can now state the definition of the Itô integral. It is worth noting that $I(f)$ as defined below is in fact a well-defined mathematical object (see [15, Chapter 4]), however we will omit the proof of this well-known fact.

Definition 2.2 (Itô integral)**.** Suppose that $f \in L^2_{\text{ad}}(\Omega \times [a,b])$. Let $\Delta_n = \{a = t_0 < t_1 < \cdots < t_n = b\}$ and $\Delta B_i = B_{t_i} - B_{t_{i-1}}$. We define the *Itô integral* of f with respect to Brownian motion as

$$I(f) = \int_a^b f(t)\, dB_t = \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^{n} f(t_{i-1}) \Delta B_i, \tag{2.2}$$

whenever the limit in probability exists.

Note that in Equation (2.2), the stochastic process $f(t)$ is always evaluated at the left endpoint of the sub-interval of the partition of $[a, b]$. This is a crucial observation made by Itô, that leads to many important properties of the stochastic integral. For example, as we will show later, the stochastic integral is a martingale. It also ensures that the expectation of the stochastic integral is zero and is the source of the so-called Itô isometry.

Now, let us illustrate the computational complexity of the stochastic integral on a very basic example. In order to present the next example in a concise form, we will need the following simple identity

$$b^2 = (b - a + a)^2 = (b - a)^2 + 2a(b - a) + a^2. \tag{2.3}$$

Solving equation (2.3) for $a(b - a)$ yields

$$a(b - a) = \frac{1}{2} \left(b^2 - a^2 - (b - a)^2 \right). \tag{2.4}$$

Example 2.3. We compute (using the definition) the following integral

$$\begin{aligned}
\int_a^b B_t dB_t &= \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^{n} B_{t_{i-1}} \left(B_{t_i} - B_{t_{i-1}} \right) \\
&= \frac{1}{2} \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^{n} \left(B_{t_i}^2 - B_{t_{i-1}}^2 - \left(B_{t_{i-1}} - B_{t_i} \right)^2 \right) \\
&= \frac{1}{2} \left(B_b^2 - B_a^2 - (b - a) \right). \qquad (2.5)
\end{aligned}$$

The second equality in the above equation follows from Equation (2.4). In the third equality, we have used the fact that the quadratic variation of Brownian motion on the interval $[a, b]$ is equal to $b - a$ (see Fact 1.17) as well as the fact that the first two factors under the sum form a telescoping sum, that is

$$\sum_{i=1}^{n} \left(B_{t_i}^2 - B_{t_{i-1}}^2 \right) = B_{t_n}^2 - B_{t_0}^2 = B_b^2 - B_a^2.$$

2.2 The Itô Formula

As we have seen in the last example of the previous section, the evaluation of even the simplest stochastic integral may be quite a complex task. This is also the case with the Newton–Leibniz calculus, where the Fundamental Theorem of Calculus allows us to use antiderivative rules in order to evaluate definite integrals. This approach simplifies computations in most cases and it is only natural to wonder, if we have a similar tool in the theory of stochastic integration. We will present such a tool in this section.

The famous Itô formula is a generalization of the well-known chain rule in the classical calculus. The chain rule states that if f and g are differentiable functions, then

$$\frac{d}{dt} f\left(g(t)\right) = f'\left(g(t)\right) g'(t).$$

On the other hand, using the integral notation, the above can be restated as

$$f\left(g(b)\right) - f\left(g(a)\right) = \int_a^b f'\left(g(t)\right) g'(t) dt = \int_a^b f'\left(g(t)\right) dg(t), \tag{2.6}$$

where the last integral is of the Riemann–Stieltjes type.

It is only natural to ask if there is a connection between the integral of a derivative of a function of Brownian motion and the function itself. One hopes that the formula describing this connection is as simple and elegant as the one presented in Equation (2.6). Below we give the simplest case of the Itô formula — a counterpart of the chain rule in the Itô calculus setting.

Theorem 2.4 (Itô formula)**.** *For any twice continuously differentiable function f we have*

$$f(B_b) - f(B_a) = \int_a^b f'(B_t)\, dB_t + \frac{1}{2}\int_a^b f''(B_t)\, dt, \tag{2.7}$$

where the above are the Itô integral and the Riemann integral for each sample path of B_t, respectively.

Comparing formulas in Equations (2.6) and (2.7), we notice the extra term in the latter. It comes from the fact that Brownian motion has non-zero quadratic variation. In fact, in Equation (2.6) we assume that g is a differentiable function, hence its quadratic variation is zero (see Example 1.16).

To illustrate the power of the Itô formula, we will find the value of the same integral as in Example 2.3. This will show that, whenever possible to apply, Itô formula provides the answer in fewer mental steps then integration using the definition of the stochastic integral.

Example 2.5. As in Example 2.3, we would like to integrate B_t on the interval $[a, b]$ with respect to Brownian motion. This means, that we need to apply the Itô formula from Theorem 2.4 with $f'(x) = x$ in Equation (2.7). Hence, $f(x) = \frac{x^2}{2}$ and $f''(x) = 1$. Therefore, we obtain

$$\frac{B_b^2}{2} - \frac{B_a^2}{2} = \int_a^b B_t\, dB_t + \frac{1}{2}\int_a^b 1\, dt.$$

Thus,

$$\int_a^b B_t\, dB_t = \frac{1}{2}\left(B_b^2 - B_a^2\right) - \frac{1}{2}(b - a).$$

Theorem 2.4, although very useful, has its limitations. One of them is the fact that the only process that we can apply this formula to is Brownian motion. There are many more general stochastic processes, and to use the Itô formula in a different setting, we need to introduce a class of stochastic processes called Itô processes.

The definition below is restricted to the processes from the $L^2_{\text{ad}}(\Omega \times [a,b])$ class, however, it can be generalized onto a larger class of stochastic processes that go beyond the scope of this work (for more details, see [15, Chapter 7].

Definition 2.6 (Itô process)**.** We say that X_t is an *Itô process* if

$$X_t = X_a + \int_a^t f(s)\, dB_s + \int_a^t g(s)\, ds, \qquad a \le t \le b, \tag{2.8}$$

where X_a is an $\mathcal{F}_a$-measurable random variable and $f, g \in L^2_{\text{ad}}(\Omega \times [a,b])$.

The next theorem extends Theorem 2.4, generalizing the Itô formula onto the class of the Itô processes. It also allows for the function θ to explicitly depend on t.

Theorem 2.7. *Suppose that X_t is an Itô process given by Equation (2.8) and $\theta(t,x)$ is a continuous function with continuous partial derivatives $\frac{\partial\theta}{\partial t}$, $\frac{\partial\theta}{\partial x}$ and $\frac{\partial^2\theta}{\partial x^2}$. Then*

$$\begin{aligned}\theta(t, X_t) = \theta(a, X_a) &+ \int_a^t \frac{\partial\theta}{\partial x}(s, X_s)\, f(s)\, dB_s \\ &+ \int_a^t \left[\frac{\partial\theta}{\partial t}(s, X_s) + \frac{\partial\theta}{\partial x}(s, X_s)\, g(s) + \frac{1}{2}\frac{\partial^2\theta}{\partial x^2}(s, X_s)\, f(s)^2\right] ds.\end{aligned} \tag{2.9}$$

It is often useful to use the differential notation and the so-called Itô table (see Table 2.1) to state and use the Itô formula for the Itô processes. Using the

$\cdot$	dB_t	dt
dB_t	dt	0
dt	0	0

TABLE 2.1. The Itô table.

differential notation, the Itô process from Equation (2.8) can be written as

$$dX_t = f(t)\, dB_t + g(t)\, dt. \tag{2.10}$$

Now, we can restate the conclusion of Theorem 2.7, that is Equation (2.9) as

$$d\theta(t, X_t) = \frac{\partial\theta}{\partial x}(t, X_t)\, dX_t + \frac{1}{2}\frac{\partial^2\theta}{\partial x^2}(t, X_t)(dX_t)^2 + \frac{\partial\theta}{\partial t}(t, X_t)\, dt. \tag{2.11}$$

Note that the meaning of the $(dX_t)^2$ term can be understood through the Itô table presented in Table 2.1. Using Equation (2.10) and the Itô table, we have

$$\begin{aligned}(dX_t)^2 &= \big(f(t)\,dB_t + g(t)\,dt\big)^2 \\ &= f^2(t)(dB_t)^2 + 2f(t)g(t)(dB_t)(dt) + g(t)(dt)^2 \\ &= f^2(t)\,dt \end{aligned} \tag{2.12}$$

Putting together Equations (2.11) and (2.12), we obtain

$$\begin{aligned} d\theta(t, X_t) &= \frac{\partial\theta}{\partial x}(t, X_t)\,dX_t + \frac{1}{2}\frac{\partial^2\theta}{\partial x^2}(t, X_t)(dX_t)^2 + \frac{\partial\theta}{\partial t}(t, X_t)\,dt \\ &= \frac{\partial\theta}{\partial x}(t, X_t)\Big(f(t)\,dB_t + g(t)\,dt\Big) + \frac{1}{2}\frac{\partial^2\theta}{\partial x^2}(t, X_t)\left(f^2(t)\,dt\right) + \frac{\partial\theta}{\partial t}(t, X_t)\,dt \\ &= \frac{\partial\theta}{\partial x}(t, X_t)f(t)\,dB_t \\ &\quad + \left[\frac{\partial\theta}{\partial x}(t, X_t)g(t) + \frac{1}{2}\frac{\partial^2\theta}{\partial x^2}(t, X_t)f^2(t) + \frac{\partial\theta}{\partial t}(t, X_t)\right]\,dt. \end{aligned} \tag{2.13}$$

Since $B(t)$ is nowhere differentiable, the meaning of Equation (2.13) is that of Equation (2.9). We will be using the Itô table often, especially in Chapters 5 and 6 to establish the Itô formula for the new stochastic integral and to find the solutions of stochastic differential equations.

2.3 Itô Isometry

As we have mentioned previously, the fact that the Itô integral is well-defined requires a proof that is rather technical and tedious. The key idea in this proof is the Itô isometry, a theorem that enables us to compute the variance of the Itô integral.

First, let us compute the expectation of the Itô integral.

Theorem 2.8. *For any $f \in L^2_{ad}(\Omega \times [a, b])$ the expectation of the Itô integral of $f(t)$ is zero, that is*

$$\mathbb{E}\left[\int_a^b f(t)\,dB_t\right] = 0.$$

Proof. By Definition 2.2 we have

$$\begin{aligned}
\mathbb{E}\left[\int_a^b f(t)\,dB_t\right] &= \mathbb{E}\left[\lim_{\|\Delta_n\|\to 0}\sum_{i=1}^{n} f(t_{i-1})\Delta B_i\right] \\
&= \lim_{\|\Delta_n\|\to 0}\sum_{i=1}^{n}\mathbb{E}\left[f(t_{i-1})\Delta B_i\right].
\end{aligned}$$

Therefore, to establish the zero-expectation property of the Itô integral, it is enough to show that $\mathbb{E}\left[f(t_{i-1})\Delta B_i\right] = 0$ for all $i \in \{1, 2, 3, \ldots, n\}$ and all $n \in \mathbb{N}$. We have

$$\begin{aligned}
\mathbb{E}\left[f(t_{i-1})\Delta B_i\right] &= \mathbb{E}\left[\mathbb{E}\left[f(t_{i-1})\Delta B_i \middle| \mathcal{F}_{t_{i-1}}\right]\right] && \text{by (CE-2)} \\
&= \mathbb{E}\left[f(t_{i-1})\mathbb{E}\left[\Delta B_i \middle| \mathcal{F}_{t_{i-1}}\right]\right] && \text{by (CE-4)} \\
&= \mathbb{E}\left[f(t_{i-1})\mathbb{E}\left[\Delta B_i\right]\right] && \text{by (CE-3)} \\
&= \mathbb{E}\left[f(t_{i-1})\right]\mathbb{E}\left[\Delta B_i\right] \\
&= 0 && \text{by Def. 1.4 pt. 2.}
\end{aligned}$$

Hence the proof is complete. □

By the above theorem, it is enough to compute the second moment of the Itô integral in order to find its variance.

Theorem 2.9 (Itô isometry)**.** *For any $f \in L^2_{ad}\left(\Omega \times [a,b]\right)$ the following isometry holds*

$$\mathbb{E}\left[\int_a^b f(t)dB_t\right]^2 = \mathbb{E}\left[\int_a^b f^2(t)dt\right].$$

As stated previously, the above theorem plays a crucial role in the construction of the Itô integral. In Definition 2.2 we see the Riemann-like sums, but it is the Lebesgue's approach that makes it possible to show that the Itô integral is a well-defined object for large class of stochastic processes. One has to define the Itô integral for step stochastic processes first and then use an approximation lemma based on the Itô isometry in order to extend the definition onto the $L^2_{\text{ad}}\left(\Omega \times [a,b]\right)$

class of stochastic processes. This, and the fact that Itô isometry gives the variance of the stochastic integral are the two main reasons why Itô isometry is such an important result.

2.4 Itô integral and the martingales

As stated previously, one of the consequences of the choice of the left endpoint as the evaluation point in the Riemann-like sum in the Itô integral is the fact that the Itô integral is a martingale. It might not be entirely clear what this means until we introduce the stochastic process associated with the stochastic integral.

Definition 2.10. Suppose that $f \in L^2_{\text{ad}}(\Omega \times [a,b])$. We define the *stochastic process associated* with $f(t)$ by

$$X_t = \int_a^t f(s)\, dB_s. \tag{2.14}$$

Naturally, we would like for the associated stochastic process to have useful properties. One of such properties is the fact that it is a martingale with respect to $\{\mathcal{F}_t\}$. This fact is stated in the following theorem.

Theorem 2.11. *Suppose that $f \in L^2_{ad}(\Omega \times [a,b])$. Then the associated stochastic process is a martingale with respect to $\{\mathcal{F}_t\}$. That is, for all $a \le s < t \le b$ we have*

$$\mathbb{E}\left[\int_a^t f(u)\, dB_u \middle| \mathcal{F}_s\right] = \int_a^s f(u)\, dB_u. \tag{2.15}$$

Proof. Note that due to the fact that $\int_a^s f(u) dB_u$ is $\mathcal{F}_s$-measurable, by the property from Equation (CE-4), Equation (2.15) can be rewritten as

$$\mathbb{E}\left[\int_a^t f(u)\, dB_u - \int_a^s f(u)\, dB_u \middle| \mathcal{F}_s\right] = 0. \tag{2.16}$$

Now, by the linearity of the Itô integral with respect to the region of integration, we can write Equation (2.16) as

$$\mathbb{E}\left[\int_s^t f(u)\, dB_u \middle| \mathcal{F}_s\right] = 0. \tag{2.17}$$

Let $\Delta_n = \{s = u_0 < u_1 < \ldots < u_{n-1} < u_n = t\}$ be any partition of the interval $[s, t]$ and let $\Delta B_i = u_i - u_{i-1}$. From the Definition 2.2 of the stochastic integral, the left-hand side of Equation (2.17) can be written as

$$
\begin{aligned}
&\mathbb{E}\left[\int_s^t f(u)\, dB_u \middle| \mathcal{F}_s\right] && \text{by Def. 2.2} \\
&= \mathbb{E}\left[\lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} f(u_{i-1})\Delta B_i \middle| \mathcal{F}_s\right] \\
&= \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} \mathbb{E}\left[f(u_{i-1})\Delta B_i | \mathcal{F}_s\right] && \text{by (CE-5)} \\
&= \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} \mathbb{E}\left[\mathbb{E}\left[f(u_{i-1})\Delta B_i \middle| \mathcal{F}_{u_{i-1}}\right] \middle| \mathcal{F}_s\right] && \text{by (CE-4)} \\
&= \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} \mathbb{E}\left[f(u_{i-1})\mathbb{E}\left[\Delta B_i \middle| \mathcal{F}_{u_{i-1}}\right] \middle| \mathcal{F}_s\right] && \text{by (CE-3)} \\
&= \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} \mathbb{E}\left[f(u_{i-1})\mathbb{E}\left[\Delta B_i\right] | \mathcal{F}_s\right] \\
&= \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} \mathbb{E}\left[f(u_{i-1}) | \mathcal{F}_s\right] \mathbb{E}\left[\Delta B_i\right] && \text{by Def. 1.4 pt. 2} \\
&= 0.
\end{aligned}
$$

Therefore the proof is complete. □

Remark 2.12. In the proofs of Theorems 2.8 and 2.11, we have used a technique very common in stochastic analysis, namely evaluation by conditioning. By the so-called tower property (CE-5) of the conditional expectation, we can condition what is already a conditional expectation, provided that the σ -field we use is larger than the σ -field already in place. By property (CE-2) of the conditional expectation, we can apply the conditional expectation with respect to any σ -field under the regular expectation. As we will see in the forthcoming chapters, clever choices of the σ -field lead to useful results.

Chapter 3
The new stochastic integral

One of the limitations of the Itô integral is the fact that the integrands have to be adapted. Recently, more and more applied problems require us to integrate non-adapted (or anticipating) stochastic processes. In this chapter, we present very recent developments in the theory of stochastic integration. In 2008, Wided Ayed and Hui-Hsiung Kuo introduced an extension of the Itô integral by decomposing an anticipating stochastic process into an adapted part and an instantly independent part. In this chapter, we present the main ideas of [1, 2] that will be used in further development of this theory in the forthcoming chapters.

3.1 Instantly independent processes

In order to define the extension of the Itô integral, Ayed and Kuo identified a property of stochastic processes that could be exploited in order to uniformly treat both adapted and anticipating stochastic processes. For any t, one can regard the σ-field $\mathcal{F}_t$ as the information available at time t. Therefore, adapted stochastic processes are thought of as processes that do not look into the future, and depend only on the present and past, that is, for each t, the random variable X_t is $\mathcal{F}_t$-measurable. Among all the anticipating (non-adapted) processes, a special class of instantly independent processes can be regarded as a counterpart to the adapted processes. Those are instantly independent processes. While adapted processes depend only on the past and present, instantly independent processes are independent of the past and present.

Definition 3.1 (Instantly independent process)**.** We say that a stochastic process X_t is *instantly independent* of the filtration (or instantly independent with respect to the filtration) $\{\mathcal{F}_t\}$, if for each t, the random variable X_t is independent of the σ-field $\mathcal{F}_t$.

Example 3.2. Let $X_t = B_1 - B_t$. Observe that

1. for $t \in [0, 1]$, the process X_t is instantly independent of $\{\mathcal{F}_t\}$;
2. For $t > 1$, the process X_t is adapted to $\{\mathcal{F}_t\}$.

Example 3.3. Let $Y_t = (B_1 - B_t)(B_2 - B_t)$. It can be easily verified that

1. For $t \in [0, 1]$, the process Y_t is instantly independent of $\{\mathcal{F}_t\}$;
2. For $t \in [1, 2]$, the process Y_t is neither instantly independent nor adapted to $\{\mathcal{F}_t\}$;
3. For $t > 2$, the process Y_t is adapted with respect to $\{\mathcal{F}_t\}$.

As we have seen in the previous two examples, there are processes that are instantly independent for some time and then adapted. We have also seen a process that is neither adapted nor instantly independent. The natural question to ask now is if there are processes that are both at the same time. The answer to this question (presented in the next theorem) gives us the reason to call the instantly independent processes a counterpart to the adapted processes.

Theorem 3.4. *If X_t is instantly independent and adapted to $\{\mathcal{F}_t\}$, then X_t is deterministic.*

Proof. Let us examine the conditional expectation of X_t with respect to $\mathcal{F}_t$. On one hand, X_t is adapted to $\{\mathcal{F}_t\}$, so, by the property (CE-4) of the conditional expectation,

$$\mathbb{E}\left[X_t|\mathcal{F}_t\right] = X_t, \tag{3.1}$$

but at the same time it is independent of $\mathcal{F}_t$, hence, by the property (CE-3) of the conditional expectation,

$$\mathbb{E}\left[X_t|\mathcal{F}_t\right] = \mathbb{E}\left[X_t\right]. \tag{3.2}$$

Equations (3.1) and (3.2) yield, $X_t = \mathbb{E}\left[X_t\right]$ and so X_t is a deterministic function.

□

3.2 Definition

The crucial idea of Itô in defining the stochastic integral was to choose the evaluation point for the stochastic process in the Riemann-like sums as the left endpoint of the sub-interval. This can be intuitively understood through an example. Suppose that in the integral $\int_a^b f(t)\,dB_t$ process $f(t)$ is the amount of stock we have at time t and that B_t is the value of the stock at time t. Each component of the Riemann-like sum has the form $f(t_{i-1})\Delta B_i$, where ΔB_i is the change of B_t over the interval $[t_{i-1}, t_i]$. So, in this setting, we can regard the $f(t_{i-1})\Delta B_i$ term as the change of our wealth over the time interval $[t_{i-1}, t_i]$, but we cannot predict the future, so we need to use the $f(t_{i-1})$ to estimate the change of our wealth. If, for example, we chose to use $f(t_i)$ instead of $f(t_{i-1})$, it would mean that we know how the stock price will change and based on this we anticipate the best amount of stock to have. This illustrates why the choice of t_{i-1} as the evaluation point is reasonable.

Now, if we really had an insight into the future, it would be best for us to use it, and choose the right endpoint of the sub-interval $[t_{i-1}, t_i]$ to evaluate the wealth of our stock portfolio. Thus it seems reasonable to choose t_i as the evaluation point in the Riemann-like sum for all the processes that are independent of the past and the present. As it turns out, this idea gives rise to the Ayed–Kuo integral that retains crucial properties of the Itô integral, for example, the zero expectation property, while at the same time allows for the integrands to be anticipating.

Definition 3.5. Suppose that $f(t)$ is adapted to $\{\mathcal{F}_t\}$ and $\varphi(t)$ is instantly independent with respect to $\{\mathcal{F}_t\}$. Let $\Delta_n = \{a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\}$ be the partition of the interval $[a, b]$. We define the *stochastic integral* of $f(t)\varphi(t)$ over the interval $[a, b]$ to be

$$I(f\varphi) = \int_a^b f(t)\varphi(t)\, dB_t = \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} f(t_{i-1})\varphi(t_i)\Delta B_i, \tag{3.3}$$

whenever the limit exists in probability.

Example 3.6. Let us compute the following integral

$$\int_0^1 B_1 - B_t\, dB_t.$$

We use the same notation as in the Definition 3.5 to write

$$\begin{aligned}
\int_0^1 B_1 - B_t\, dB_t &= \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} \left[(B_1 - B_{t_i})\, \Delta B_i\right] \\
&= \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} \left[\left(B_1 - (B_{t_i} - B_{t_{i-1}}) - B_{t_{i-1}}\right) \Delta B_i\right] \\
&= \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} \left[\left(B_1 - (B_{t_i} - B_{t_{i-1}}) - B_{t_{i-1}}\right) \Delta B_i\right] \\
&= \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} \left[B_1\Delta B_i - (\Delta B_i)^2 - B_{t_{i-1}}\Delta B_i\right] \\
&= \lim_{\|\Delta_n\|\to 0} \left[B_1 \sum_{i=1}^{n} \Delta B_i - \sum_{i=1}^{n} (\Delta B_i)^2 - \sum_{i=1}^{n} B_{t_{i-1}}\Delta B_i\right]. \qquad (3.4)
\end{aligned}$$

Note that the first sum in Equation (3.4) is a telescoping one, that is

$$\sum_{i=1}^{n} \Delta B_i = \sum_{i=1}^{n} \left(B_{t_i} - B_{t_{i-1}}\right) = B_1 - B_0 = B_1.$$

From Fact 1.17 we know that the second sum in Equation (3.4) is equal to 1 and in the last sum of Equation (3.4) we recognize the classical Itô integral from Definition

2.2. Therefore, we obtain

$$\int_0^1 B_1 - B_t \, dB_t = B_1^2 - 1 - \int_0^1 B_t \, dB_t. \tag{3.5}$$

Upon evaluating the last integral in Equation (3.5), we obtain

$$\int_0^1 B_1 - B_t \, dB_t = \frac{B_1^2}{2} - \frac{1}{2}.$$

Example 3.7. Note that the new integral can also be used to evaluate

$$\int_0^1 B_1 \, dB_t. \tag{3.6}$$

In this case, the integrand is not a product of an adapted and instantly independent processes, however we can decompose it in such a way, that the new integral is applicable. That is

$$B_1 = (B_1 - B_t) + B_t.$$

The first part, $B_1 - B_t$ is independent of $\mathcal{F}_t$, because it is an increment of Brownian motion and intervals (a, t) and (t, b) are disjoint. On the other hand, the second part, namely B_t, is adapted to $\mathcal{F}_t$. Hence, using the linearity of the stochastic integral, we have

$$\begin{aligned} \int_0^1 B_1 \, dB_t &= \int_0^1 (B_1 - B_t) + B_t \, dB_t \\ &= \int_0^1 (B_1 - B_t) \, dB_t + \int_0^1 B_t \, dB_t. \end{aligned} \tag{3.7}$$

Now, Equations (3.7) and (3.5) yield

$$\int_0^1 B_1 \, dB_t = B_1^2 - 1.$$

As previously mentioned, the new integral has zero expectation as shown in the next theorem.

Theorem 3.8. *Suppose that $f(t)$ is adapted to $\{\mathcal{F}_t\}$ and $\varphi(t)$ is instantly independent with respect to $\{\mathcal{F}_t\}$. Suppose also that $\int_a^b f(t)\varphi(t)\,dB_t$ exists, and $\mathbb{E}\,|f(t)| < \infty$ and $\mathbb{E}\,|\varphi(t)| < \infty$ for all $t \in [a,b]$. Then*

$$\mathbb{E}\left[\int_a^b f(t)\varphi(t)\,dB_t\right] = 0. \tag{3.8}$$

Proof. As in the proof of the Theorem 2.8, it is enough to examine one of the terms in the Riemann-like sums.

$$\begin{aligned}
\mathbb{E}\left[f(t_{i-1})\varphi(t_i)\Delta B_i\right] &= \mathbb{E}\left[\mathbb{E}\left[f(t_{i-1})\varphi(t_i)\Delta B_i|\mathcal{F}_{t_i}\right]\right] && \text{by (CE-2)}\\
&= \mathbb{E}\left[f(t_{i-1})\Delta B_i \mathbb{E}\left[\varphi(t_i)|\mathcal{F}_{t_i}\right]\right] && \text{by (CE-4)}\\
&= \mathbb{E}\left[f(t_{i-1})\Delta B_i \mathbb{E}\left[\varphi(t_i)\right]\right] && \text{by (CE-3)}\\
&= \mathbb{E}\left[\mathbb{E}\left[\, f(t_{i-1})\Delta B_i \middle| \mathcal{F}_{t_{i-1}}\right]\right]\mathbb{E}\left[\varphi(t_i)\right] && \text{by (CE-2)}\\
&= \mathbb{E}\left[f(t_{i-1})\mathbb{E}\left[\Delta B_i \middle| \mathcal{F}_{t_{i-1}}\right]\right]\mathbb{E}\left[\varphi(t_i)\right] && \text{by (CE-4)}\\
&= \mathbb{E}\left[f(t_{i-1})\mathbb{E}\left[\Delta B_i\right]\right]\mathbb{E}\left[\varphi(t_i)\right] && \text{by (CE-3)}\\
&= 0 && \text{by Def. 1.4 pt. 2.}
\end{aligned}$$

Therefore, using the above in the Definition 3.5, we have

$$\begin{aligned}
\mathbb{E}\left[\int_a^b f(t)\varphi(t)\,dB_t\right] &= \mathbb{E}\left[\lim_{\|\Delta_n\|\to 0}\sum_{i=1}^n f(t_{i-1})\varphi(t_i)\Delta B_i\right]\\
&= \lim_{\|\Delta_n\|\to 0}\sum_{i=1}^n \mathbb{E}\left[f(t_{i-1})\varphi(t_i)\Delta B_i\right]\\
&= 0.
\end{aligned}$$

Thus the proof is complete. □

Remark 3.9. Notice, that in the proof above, we have used evaluation by conditioning, a technique already mentioned in Remark 2.12 and used in the proofs of Theorems 2.8 and 2.11. Theorem 3.8 is an analogue of Theorem 2.8 for the new stochastic integral, but here we had to use conditioning twice. This is a consequence of having two processes f and φ that are evaluated at different points. We will see this kind of approach to proofs often in the forthcoming chapters.

Example 3.10. From Example 3.7 we know that

$$\int_0^1 B_1\, dB_t = B_1^2 - 1.$$

Using property 2 of the Definition 1.4 of Brownian motion, we see that

$$\mathbb{E}\left[\int_0^1 B_1\, dB_t\right] = \mathbb{E}\left[B_1^2 - 1\right] = \mathbb{E}\left[B_1^2\right] - 1 = 1 - 1 = 0.$$

3.3 The first Itô formula in the new theory

In 2008, Ayed and Kuo presented a new version of the Itô formula (see [1, Theorem 3.2]) for the new stochastic integral they defined. We recall this theorem below, as it is first of its kind and will serve as a basis for further development of this type of formulas in Chapter 5.

Theorem 3.11 ([1, Theorem 3.2])**.** *Let $f(x)$ and $\varphi(x)$ be C^2-functions and let $\theta(x,y) = f(x)\varphi(y-x)$. Then the following equality holds for $a \leq t \leq b$,*

$$\begin{aligned}\theta\left(B_t, B_b\right) = \theta\left(B_a, B_b\right) &+ \int_a^t \frac{\partial \theta}{\partial x}\left(B_s, B_b\right) dB_s \\ &+ \int_a^t \left[\frac{1}{2}\frac{\partial^2 \theta}{\partial x^2}\left(B_s, B_b\right) + \frac{\partial^2 \theta}{\partial x \partial y}\left(B_s, B_b\right)\right] ds. \end{aligned} \tag{3.9}$$

Example 3.12. To illustrate the application of Theorem 3.11, we will compute again the integral from Example 3.6, namely

$$\int_0^1 B_1 - B_t\, dB_t. \tag{3.10}$$

We will apply the Itô formula from Theorem 3.11 to a function $\theta(x,y) = x(y-x)$. We have

$$\frac{\partial \theta}{\partial x}(x,y) = y - 2x, \qquad \frac{\partial^2 \theta}{\partial x^2}(x,y) = -2, \qquad \frac{\partial^2 \theta}{\partial x \partial y}(x,y) = 1,$$

Now, Theorem 3.11 gives us

$$B_1\left(B_1 - B_1\right) = B_0\left(B_1 - B_0\right) + \int_0^1 B_1 - 2B_t\, dB_t + \int_0^1 0\, dt.$$

Hence

$$0 = \int_0^1 B_1 - 2B_t\, dB_t = \int_0^1 B_1 - B_t\, dB_t - \int_0^1 B_t\, dB_t,$$

or equivalently

$$\int_0^1 B_1 - B_t\, dB_t = \int_0^1 B_t\, dB_t.$$

Using Equation (2.5), we have

$$\int_0^1 B_1 - B_t\, dB_t = \frac{B_1^2}{2} - \frac{1}{2}.$$

3.4 The new integral and near-martingales

As we have seen in Theorem 2.11, the stochastic process associated with the Itô integral is a martingale. By Remark 1.12, we know that the stochastic process associated with the new stochastic integral is not a martingale with respect to $\{\mathcal{F}_t\}$ because it is not adapted to $\{\mathcal{F}_t\}$. However, if we take a closer look at Condition 2 in Definition 1.11, we see that it can be restated as

$$\mathbb{E}\left[X_t - X_s | \mathcal{F}_s\right] = 0. \tag{3.11}$$

Note that if X_t is adapted to $\{\mathcal{F}_t\}$, then Equation (3.11) is equivalent to Condition 2 of Definition 1.11. The main difference between Equation (3.11) and Condition 2 of Definition 1.11 is the fact that the latter implies that X_t is adapted to $\{\mathcal{F}_t\}$, while the former may be valid even for non-adapted processes X_t. This is the motivation behind the next definition.

Definition 3.13 (Near-martingale)**.** A process X_t is said to be a *near-martingale* with respect to filtration $\{\mathcal{F}_t\}$ if the following conditions are satisfied:

1. $\mathbb{E}|X_t| < \infty$ for all $t \geq 0$,

2. $\mathbb{E}[X_t - X_s|\mathcal{F}_s] = 0$ for all $0 \leq s \leq t$.

The next theorem states that the stochastic process associated to the new stochastic integral is a near-martingale. It is an analogue of Theorem 2.11 in the new integral setting and a generalization of [18, Theorem 1.5], where the processes f and φ depend explicitly on B_t and $B_b - B_t$, respectively.

Theorem 3.14. *Suppose that $f(t)$ is adapted to $\{\mathcal{F}_t\}$ and $\varphi(t)$ is instantly independent of $\{\mathcal{F}_t\}$. Suppose also that $f(t)$ and $\varphi(t)$ are continuous with probability* 1. *If $\mathbb{E}\left|\int_a^b f(t)\varphi(t)\,dB_t\right| < \infty$, then*

$$X_t = \int_a^t f(s)\varphi(s)\,dB_s \tag{3.12}$$

is a near-martingale with respect to $\{\mathcal{F}_t\}$.

Proof. By the definition of X_t, the Definition 3.13 of the near-martingale, and the Definition 3.5 of the new stochastic integral, we have

$$\begin{aligned}\mathbb{E}[X_t - X_s|\mathcal{F}_s] &= \mathbb{E}\left[\int_s^t f(u)\varphi(u)\,dB_u\middle|\mathcal{F}_s\right] \\ &= \mathbb{E}\left[\lim_{\|\Delta_n\|\to 0}\sum_{i=1}^n f(u_{i-1})\varphi(u_i)\Delta B_i\middle|\mathcal{F}_s\right],\end{aligned}$$

where $\Delta_n = \{s = u_0 < u_1 < \cdots < u_{n-1} < u_n = t\}$ is a partition of the interval $[s,t]$. Now, using the tower property (Equation (CE-5)) of the conditional expectation, the above becomes

$$\mathbb{E}[X_t - X_s|\mathcal{F}_s] = \lim_{\|\Delta_n\|\to 0}\sum_{i=1}^n \mathbb{E}\left[\mathbb{E}\left[f(u_{i-1})\varphi(u_i)\Delta B_i\middle|\mathcal{F}_{t_{i-1}}\right]\middle|\mathcal{F}_s\right].$$

Since $f(t_{i-1})$ is $\mathcal{F}_{t_{i-1}}$-measurable, property (CE-4) of the conditional expectation transforms the above into

$$\mathbb{E}[X_t - X_s|\mathcal{F}_s] = \lim_{\|\Delta_n\|\to 0}\sum_{i=1}^n \mathbb{E}\left[f(u_{i-1})\mathbb{E}\left[\varphi(u_i)\Delta B_i\middle|\mathcal{F}_{t_{i-1}}\right]\middle|\mathcal{F}_s\right].$$

Applying the tower property (CE-5) again, and using the fact that ΔB_i is $\mathcal{F}_{t_i}$-measurable, we obtain

$$\begin{aligned}\mathbb{E}\left[X_t - X_s|\mathcal{F}_s\right] &= \lim_{\|\Delta_n\|\to 0}\sum_{i=1}^{n}\mathbb{E}\left[f(u_{i-1})\mathbb{E}\left[\mathbb{E}\left[\varphi(u_i)\Delta B_i|\mathcal{F}_{t_i}\right]\big|\mathcal{F}_{t_{i-1}}\right]\big|\mathcal{F}_s\right] \\ &= \lim_{\|\Delta_n\|\to 0}\sum_{i=1}^{n}\mathbb{E}\left[f(u_{i-1})\mathbb{E}\left[\Delta B_i\mathbb{E}\left[\varphi(u_i)|\mathcal{F}_{t_i}\right]\big|\mathcal{F}_{t_{i-1}}\right]\big|\mathcal{F}_s\right]\end{aligned}$$

Since $\varphi(t_i)$ is independent of $\mathcal{F}_{t_i}$, the above is

$$\begin{aligned}\mathbb{E}\left[X_t - X_s|\mathcal{F}_s\right] &= \lim_{\|\Delta_n\|\to 0}\sum_{i=1}^{n}\mathbb{E}\left[f(u_{i-1})\mathbb{E}\left[\Delta B_i\mathbb{E}\left[\varphi(u_i)\right]\big|\mathcal{F}_{t_{i-1}}\right]\big|\mathcal{F}_s\right] \\ &= \lim_{\|\Delta_n\|\to 0}\sum_{i=1}^{n}\mathbb{E}\left[\varphi(u_i)\right]\mathbb{E}\left[f(u_{i-1})\mathbb{E}\left[\Delta B_i\big|\mathcal{F}_{t_{i-1}}\right]\big|\mathcal{F}_s\right]\end{aligned}$$

The independence of ΔB_i and $\mathcal{F}_{t_i}$ yields

$$\begin{aligned}\mathbb{E}\left[X_t - X_s|\mathcal{F}_s\right] &= \lim_{\|\Delta_n\|\to 0}\sum_{i=1}^{n}\mathbb{E}\left[\varphi(u_i)\right]\mathbb{E}\left[\Delta B_i\right]\mathbb{E}\left[f(u_{i-1})|\mathcal{F}_s\right] \\ &= 0.\end{aligned}$$

The last equality is due to the fact that Brownian increments have mean equal to zero (see Definition 1.4 part 2.) This completes the proof. □

Remark 3.15. Note that a weaker form of Theorem 3.14 has been first published in [18] and in [32]. The advantage of this version over the one in [18, 32] is the fact that Theorem 3.14 does not require the integrands to be functions of Brownian motion, but general stochastic processes.

Chapter 4
Itô isometry

Throughout this and following chapters, we will work with stochastic processes built upon a special class of real functions, namely, functions whose Maclaurin expansion has infinite radius of convergence. We will denote this space by $\mathcal{M}^\infty$, that is

$$\mathcal{M}^\infty = \left\{ f \in C^\infty(\mathbb{R}) \;\middle|\; f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n, \text{ for all } x \in \mathbb{R} \right\}.$$

We will be working with processes of the form $f(B_b - B_t)$, where $f \in \mathcal{M}^\infty$. This construction of stochastic processes allows us to manipulate the argument using the power series expansion and binomial theorem, so that we can exploit the properties of the Brownian motion and its increments. While this choice of processes may seem restrictive, processes built upon functions $f \in \mathcal{M}^\infty$ form a class wide enough to be useful in applications, as we will see in Chapter 6.

4.1 Simple case

In this section, we present a simple Itô isometry result for stochastic processes of the form $X_t = \varphi(B_b - B_t)$. Note that this are purely instantly independent stochastic processes and in this particular case, Itô isometry has the same form as in the classical adapted case (see Theorem 2.9).

The results of this section can be found in a paper by Kuo–Sae-Tang–Szozda [18] and in the doctoral dissertation of Sae-Tang [32].

The first result presented below is a formula for the covariance between stochastic integrals of powers of $B_b - B_s$. It is a technical result that is used later to establish the main result of this section — Theorem 4.3.

Lemma 4.1 ([18, Theorem 1.10])**.** *For $a \le t \le b$, we have*

$$\mathbb{E}\left[\left(\int_a^t (B_b - B_s)^m \, dB_s\right)\left(\int_a^t (B_b - B_s)^n \, dB_s\right)\right]$$
$$= \begin{cases} \dfrac{(m+n-1)!!}{\frac{m+n}{2}+1}\left((b-a)^{\frac{m+n}{2}+1} - (b-t)^{\frac{m+n}{2}+1}\right), & \textit{if } m+n \textit{ is even,} \\ 0, & \textit{if } m+n \textit{ is odd.} \end{cases} \tag{4.1}$$

Proof. We begin with the Taylor expansion of e^x with $x = \tau(B_b - B_t)$, to obtain

$$e^{\tau(B_b - B_t)} = \sum_{m=0}^{\infty} \frac{(B_b - B_t)^m}{m!} \tau^m. \tag{4.2}$$

Applying the Itô integral over the interval $[a, b]$ on both sides of Equation (4.2) yields

$$\int_a^t e^{\tau(B_b - B_s)} \, dB_s = \sum_{m=0}^{\infty} \frac{\tau^m}{m!} \int_a^t (B_b - B_s)^m \, dB_s. \tag{4.3}$$

By taking the expectation of both sides of the Equation (4.3), we get

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_a^t e^{\tau(B_b - B_s)} \, dB_s\right)\left(\int_a^t e^{\sigma(B_b - B_s)} \, dB_s\right)\right] \\
&= \mathbb{E}\left[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\tau^m \sigma^n}{n!m!}\left(\int_a^t (B_b - B_s)^m dB_s\right)\left(\int_a^t (B_b - B_s)^n \, dB_s\right)\right] \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\tau^m \sigma^n}{n!m!} \mathbb{E}\left[\left(\int_a^t (B_b - B_s)^m \, dB_s\right)\left(\int_a^t (B_b - B_s)^n \, dB_s\right)\right].
\end{aligned} \tag{4.4}$$

On the other hand, by the Itô formula in Theorem 3.11, we have

$$\int_a^t e^{\tau(B_b - B_s)} \, dB_s = \frac{1}{\tau} e^{\tau(B_b - B_a)} - \frac{1}{\tau} e^{\tau(B_b - B_t)} - \frac{\tau}{2} \int_a^t e^{\tau(B_b - B_s)} \, ds,$$

and this leads to the following expectation that, to simplify the notation, we will denote by $\mathcal{E}_{\tau\sigma}$

$$\begin{aligned}
\mathcal{E}_{\tau\sigma} &= \mathbb{E}\left[\left(\int_a^t e^{\tau(B_b - B_s)} \, dB_s\right)\left(\int_a^t e^{\sigma(B_b - B_s)} \, dB_s\right)\right] \\
&= \mathbb{E}\left[\left(\frac{1}{\tau} e^{\tau(B_b - B_a)} - \frac{1}{\tau} e^{\tau(B_b - B_t)} - \frac{\tau}{2} \int_a^t e^{\tau(B_b - B_s)} \, ds\right)\right.
\end{aligned}$$

$$\times \left(\frac{1}{\sigma} e^{\sigma(B_b - B_a)} - \frac{1}{\sigma} e^{\sigma(B_b - B_t)} - \frac{\sigma}{2} \int_a^t e^{\sigma(B_b - B_s)}\, ds \right) \Bigg] . \tag{4.5}$$

It is straightforward but very tedious to carry out the computations to show that the right-hand side of Equation (4.5) is equal to

$$\begin{aligned}
\mathcal{E}_{\tau\sigma} &= \frac{2}{(\tau+\sigma)^2} \left(e^{\frac{1}{2}(b-a)(\tau+\sigma)^2} - e^{\frac{1}{2}(b-t)(\tau+\sigma)^2} \right) \\
&= \frac{2}{(\tau+\sigma)^2} \left[\sum_{n=0}^{\infty} \frac{(b-a)^n}{2^n n!} (\tau+\sigma)^{2n} - \sum_{n=0}^{\infty} \frac{(b-t)^n}{2^n n!} (\tau+\sigma)^{2n} \right] \\
&= \sum_{n=1}^{\infty} \frac{(b-a)^n - (b-t)^n}{2^{n-1} n!} (\tau+\sigma)^{2(n-1)} \\
&= \sum_{n=0}^{\infty} \frac{(b-a)^{n+1} - (b-t)^{n+1}}{2^n (n+1)!} (\tau+\sigma)^{2n}. \\
&= \sum_{n=0}^{\infty} \beta_n (\tau+\sigma)^{2n},
\end{aligned} \tag{4.6}$$

where, for simplicity of the notation, we let $\beta_n = \frac{(b-a)^{n+1} - (b-t)^{n+1}}{2^n (n+1)!}$. We use the binomial theorem and change the order of summation to show that (4.6) equals

$$\mathcal{E}_{\tau\sigma} = \sum_{k=0}^{\infty} \sum_{n=\lceil \frac{k}{2} \rceil}^{\infty} \beta_n \binom{2n}{k} \sigma^k \tau^{2n-k}.$$

We split the summation into two parts: for $k = 2j+1$ and $k = 2j$ with $j = 0, 1, 2, \ldots$. Hence the last summation becomes

$$\mathcal{E}_{\tau\sigma} = \sum_{j=0}^{\infty} \left[\sum_{n=j+1}^{\infty} \beta_n \binom{2n}{2j+1} \sigma^{2j+1} \tau^{2n-2j-1} + \sum_{n=j}^{\infty} \beta_n \binom{2n}{2j} \sigma^{2j} \tau^{2n-2j} \right].$$

We substitute $m = 2n - 2j - 1$ for the first summation and $m = 2n - 2j$ for the second summation. Note that m in the first summation will cover all the positive odd numbers, while m in the second summation will cover all the positive even numbers and zero. Therefore,

$$\mathcal{E}_{\tau\sigma} = \sum_{j=0}^{\infty} \left[\sum_{m:\ \text{odd}} \beta_{\lceil \frac{m+2j+1}{2} \rceil} \binom{m+2j+1}{2j+1} \sigma^{2j+1} \tau^m \right.$$

$$+ \sum_{m:\text{ even}} \beta_{\lceil \frac{m+2j}{2} \rceil} \binom{m+2j}{2j} \sigma^{2j}\tau^m \Bigg]$$

$$= \sum_{n,m:\text{ odd}} \beta_{\frac{m+n}{2}} \binom{m+n}{n} \sigma^n \tau^m + \sum_{n,m:\text{ even}} \beta_{\frac{m+n}{2}} \binom{m+n}{n} \sigma^n \tau^m.$$

Therefore,

$$\begin{aligned}\mathcal{E}_{\tau\sigma} &= \mathbb{E}\left[\left(\int_a^t e^{\tau(B_b-B_s)}\, dB_s\right)\left(\int_a^t e^{\sigma(B_b-B_s)}\, dB_s\right)\right] \\ &= \sum_{n,m:\text{ odd}} \beta_{\frac{m+n}{2}} \binom{m+n}{n} \tau^m \sigma^n + \sum_{n,m:\text{ even}} \beta_{\frac{m+n}{2}} \binom{m+n}{n} \tau^m \sigma^n. \end{aligned} \tag{4.7}$$

By comparing the coefficients of $\tau^m\sigma^n$ in Equations (4.7) and (4.4), we obtain the desired result. □

The following theorem establishes the covariance of the new stochastic integrals of powers of Brownian increments. It is the next step in the proof of Theorem 4.3.

Theorem 4.2 ([18, Theorem 1.11])**.** *For any $a \le t \le b$ and $m, n \in \{0, 1, 2, \ldots\}$ we have*

$$\begin{aligned} &\mathbb{E}\left[\left(\int_a^t (B_b - B_s)^n\, dB_s\right)\left(\int_a^t (B_b - B_s)^m\, dB_s\right)\right] \\ &= \int_a^t \mathbb{E}\left[(B_b - B_s)^{n+m}\right] ds. \end{aligned} \tag{4.8}$$

Proof. First, observe that by Definition 1.4 pt. 2, we know that

$$B_b - B_s \sim \mathcal{N}(0, b-s).$$

Hence, if $n+m$ is odd, then $\mathbb{E}\left[(B_b - B_s)^{n+m}\right] = 0$ as in Equation (4.1). On the other hand, when $n+m$ is even, we have

$$\begin{aligned}\int_a^t \mathbb{E}\left[(B_b - B_s)^{n+m}\right] ds &= \int_a^t (b-s)^{\frac{m+n}{2}} \frac{(m+n)!}{2^{\frac{m+n}{2}}\left(\frac{m+n}{2}\right)!}\, ds \\ &= \frac{(m+n)!}{2^{\frac{m+n}{2}}\left(\frac{m+n}{2}+1\right)!}\left((b-a)^{\frac{m+n}{2}+1} - (b-t)^{\frac{m+n}{2}+1}\right)\end{aligned}$$

$$= \frac{(m+n-1)!!}{\frac{m+n}{2}+1}\left((b-a)^{\frac{m+n}{2}+1} - (b-t)^{\frac{m+n}{2}+1}\right).$$

Hence Equation (4.8) holds. $\square$

Finally, we are ready to prove the main result of this section, namely the Itô isometry for processes of the form $\varphi(B_b - B_t)$.

Theorem 4.3 ([18, Theorem 1.12]). *Suppose that $\varphi \in \mathcal{M}^\infty$ is such that $\int_a^b \mathbb{E}\left[\varphi(B_b - B_t)\right]^2 dt < \infty$ and $\int_a^b \varphi(B_b - B_t)\, dB_t$ exists. Then for all $a \le t \le b$,*

$$\mathbb{E}\left[\left(\int_a^t \varphi(B_b - B_s)\, dB_s\right)^2\right] = \int_a^t \mathbb{E}\left[\varphi(B_b - B_s)^2\right] ds. \tag{4.9}$$

Proof. We give an informal derivation of the formula in the Equation (4.9). Since $\varphi \in \mathcal{M}^\infty$, we can write it as

$$\varphi(x) = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} x^n, \qquad x \in \mathbb{R},$$

where $\varphi^{(n)}(0)$ is the n-th derivative of $\varphi(x)$ at zero. Writing out the left-hand side of Equation (4.9), we get

$$\begin{aligned}
&\mathbb{E}\left[\int_a^t \varphi(B_b - B_s)\, dB_s\right]^2 \\
&= \mathbb{E}\left[\int_a^t \left(\sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!}(B_b - B_s)^n\right) dB_s\right]^2 \\
&= \mathbb{E}\left[\sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \int_a^t (B_b - B_s)^n\, dB_s\right]^2 \\
&= \mathbb{E}\left[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left\{\frac{\varphi^{(n)}(0)\varphi^{(m)}(0)}{n!m!}\left(\int_a^t (B_b - B_s)^n\, dB_s\right)\left(\int_a^t (B_b - B_s)^m\, dB_s\right)\right\}\right] \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left\{\frac{\varphi^{(n)}(0)\varphi^{(m)}(0)}{n!m!}\mathbb{E}\left[\left(\int_a^t (B_b - B_s)^n\, dB_s\right)\left(\int_a^t (B_b - B_s)^m dB_s\right)\right]\right\}.
\end{aligned} \tag{4.10}$$

On the other hand, we can use similar arguments to those above to show that

$$\int_a^t \mathbb{E}\left[\varphi(B_b - B_s)\right]^2 ds = \int_a^t \mathbb{E}\left[\sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!}(B_b - B_s)^n\right]^2 ds$$

$$= \int_a^t \mathbb{E}\left[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\varphi^{(n)}(0)\varphi^{(m)}(0)}{n!m!}(B_b - B_s)^{n+m}\right] ds$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\varphi^{(n)}(0)\varphi^{(m)}(0)}{n!m!}\int_a^t \mathbb{E}\left[(B_b - B_s)^{n+m}\right] ds. \quad (4.11)$$

Therefore the assertion of the theorem follows from Equations (4.10), (4.11) and Theorem 4.2. □

4.2 More general case

In this section, we derive an isometry formula for processes of the form $X_t = f(B_t)\varphi(B_b - B_t)$, where $f, \varphi \in \mathcal{M}^\infty$. This is the first step towards a general Itô isometry for the new stochastic integral. We will show how to obtain this result in two ways, namely using the white noise theory and basic probability theory. We begin with the former approach.

Theorem 4.4 (in White Noise Theory)**.** *Suppose $\varphi(x) \in L^2\left([a,b]; \mathcal{W}^{1/2}\right)$. Then*

$$\left\|\int_a^b \partial_t^*\varphi(t)\, dt\right\|^2 = \int_a^b \|\varphi(t)\|^2\, dt + \int_a^b\int_a^b \big((\partial_t\varphi(s), \partial_s\varphi(t))\big)\, dsdt.$$

Here $\mathcal{W}^{1/2} = \left\{\varphi \in (L^2) : \|(N+1)^{1/2}\varphi\|_0 < \infty\right\}$, where N is the number operator, that is $N\varphi = \sum_{n=1}^{\infty} n\langle : \cdot^{\otimes n} :, f_n\rangle$ for any $\varphi = \sum_{n=0}^{\infty}\langle : \cdot^{\otimes n} :, f_n\rangle$, $: \cdot^{\otimes n} :$ is the Wick tensor, $: \cdot :$ is the renormalization, ∂_t is the white noise derivative, and ∂_t^* is the operator adjoint to ∂_t. For details, see [14, Theorem 13.16].

Example 4.5. This example illustrates, how we can informally derive the formula in Theorem 4.7 from Theorem 4.4. Consider function $f(B_t)\varphi(B_b - B_t)$ on the interval $[a,b]$, where $f(x)$ and $\varphi(x)$ are differentiable functions. To simplify the notation, we will write $\theta(t) = f(B_t)\varphi(B_b - B_t)$. Using theorem 4.4, we get

$$\left\|\int_a^b \partial_t^*\theta(t)\, dt\right\|^2 = \int_a^b \|\theta(t)\|^2\, dt + \int_a^b\int_a^b \big((\partial_s\theta(t), \partial_t\theta(s))\big)\, dsdt \quad (4.12)$$

Which can be rewritten as

$$\mathbb{E}\left[\int_a^b \partial_t^*\theta(t)\, dt\right]^2 = \int_a^b \mathbb{E}\left[\theta(t)\right]^2\, dt + \int_a^b\int_a^b \big((\partial_s\theta(t), \partial_t\theta(s))\big)\, dsdt \quad (4.13)$$

Of course, the first integral on the right-hand side of the above equation is equal to the first integral on the right-hand side of Equation (4.14). So we need to show that the double integral above is equal to the double integral in Equation (4.14). Let us consider the expression under the double integral in the Equation (4.12). To evaluate it, we need to compute

$$\begin{aligned}\partial_s\theta(t) &= \partial_s\big(f(B_t)\varphi(B_b - B_t)\big)\\ &= f'(B_t)\partial_s(B_t)\varphi(B_b - B_t) + f(B_t)\varphi'(B_b - B_t)\partial_s(B_b - B_t)\end{aligned}$$

Using the methods of white noise theory, we can compute (for $a \le t \le b$)

$$\begin{aligned}\partial_s B_t &= \partial_s \int_a^t \dot{B}_u du\\ &= \int_a^t \partial_s \dot{B}_u du\\ &= \int_a^t \delta_s(u) du\\ &= \begin{cases} 1, & \text{if } s < t,\\ 0, & \text{if } s > t.\end{cases}\end{aligned}$$

Using similar approach, we can compute $\partial_s(B_b - B_t)$, and so we obtain

$$\partial_s(B_t) = \begin{cases} 1, & \text{if } s < t,\\ 0, & \text{if } s > t,\end{cases} \qquad \text{and} \qquad \partial_s(B_b - B_t) = \begin{cases} 0, & \text{if } s < t,\\ 1, & \text{if } s > t.\end{cases}$$

Therefore

$$\partial_s\theta(t) = \begin{cases} f(B_t)\varphi'(B_b - B_t), & \text{if } s > t,\\ f'(B_t)\varphi(B_b - B_t), & \text{if } s < t,\end{cases}$$

and exchanging the roles of t and s, we obtain similar formula for $\partial_t\theta(s)$. Hence

$$\big(\!\big(\partial_s\theta(t), \partial_t\theta(s)\big)\!\big) = \mathbb{E}\left[f(B_{t\wedge s})f'(B_{t\vee s})\varphi(B_b - B_{t\vee s})\varphi'(B_b - B_{t\wedge s})\right],$$

and therefore

$$\int_a^b |\partial_t^*(\theta(t))\,dt|^2 = \int_a^b \mathbb{E}\left[\theta(t)\right]^2\,dt \\ + \int_a^b \int_a^b \mathbb{E}\left[f(B_{t\wedge s})f'(B_{t\vee s})\varphi(B_b - B_{t\vee s})\varphi'(B_b - B_{t\wedge s})\right]\,dsdt.$$

This result can be also obtained using Malliavin calculus. More precisely, one can use [27, Proposition 3.1], which we state below.

Theorem 4.6 (in Malliavin Calculus)**.** *Let $u \in L^2(T\times\Omega)$ be such that*

$$\int_T \mathbb{E}\left[\|Du_t\|_H^2\right]\mu(dt) < \infty.$$

Then the Hitsuda–Skorokhod integral δu of u exists and

$$\mathbb{E}\left[(\delta u)^2\right] = \int_T \mathbb{E}\left[u_t^2\right]\mu(dt) + \int_{T^2} \mathbb{E}\left[Du_t(s)Du_s(t)\right]\mu(dt)\mu(ds).$$

Here D stands for the Malliavin derivative, $\|\cdot\|_H$ is a norm on a separable Hilbert space H and μ is a finite atomless measure on a measurable space $(T,\mathcal{B})$. For details, see [27].

Finally, the main result of this section is presented in Theorem 4.7 which is proved with the use of only basic probability theory.

Theorem 4.7. *If $f,\varphi \in \mathcal{M}^\infty$, then*

$$\mathbb{E}\left[\left(\int_a^b f(B_t)\varphi(B_b - B_t)\,dB_t\right)^2\right] = \int_a^b \mathbb{E}\left[f^2(B_t)\varphi^2(B_t)\right]\,dt \\ + 2\int_a^b\int_a^t \mathbb{E}\left[f(B_s)\varphi'(B_b - B_s)f'(B_t)\varphi(B_b - B_t)\right]\,dsdt. \tag{4.14}$$

Remark 4.8. Note that if $\varphi(x) = 1$ or $f(x) = 1$, then the formula takes the form of original Itô isometry. Thus, it is in fact an extension of the Itô isometry. Moreover, Itô isometry for purely instantly independent stochastic processes lacks the extra term, just as the formula for the adapted processes.

Lemma 4.9. *If $f(x)$ and $\varphi(x)$ are continuous, $\Delta_n = \{a = t_0 < t_1 < \ldots < t_n = b\}$ and $\Delta B_i = B_{t_i} - B_{t_{i-1}}$ then*

$$\lim_{\|\Delta_n\|\to 0}\left(\sum_{i=0}^{n}\mathbb{E}\left[f^2(B_{t_{i-1}})\varphi^2(B_b - B_{t_i})(\Delta B_i)^2\right]\right) = \int_a^b \mathbb{E}\left[f^2(B_t)\varphi^2(B_b - B_t)\right]\, dt. \tag{4.15}$$

Proof. To simplify the notation, we will write $f_i = f(B_{t_i})$ and $\varphi_i = \varphi(B_b - B_{t_i})$. Let us consider the expectation in the left-hand side of Equation (4.15). Using the properties of the conditional expectation and the fact that f_{i-1} and ΔB_i are $\mathcal{F}_{t_i}$-measurable, we have

$$\begin{aligned}\mathbb{E}\left[f_{i-1}^2\varphi_i^2(\Delta B_i)^2\right] &= \mathbb{E}\left[\mathbb{E}\left[f_{i-1}^2\varphi_i^2(\Delta B_i)^2\middle|\mathcal{F}_{t_i}\right]\right]\\ &= \mathbb{E}\left[f_{i-1}^2(\Delta B_i)^2\mathbb{E}\left[\varphi_i^2\middle|\mathcal{F}_{t_i}\right]\right].\end{aligned}$$

Since φ_i is independent of $\mathcal{F}_{t_i}$, the previous expression is equal to

$$\mathbb{E}\left[f_{i-1}^2(\Delta B_i)^2\mathbb{E}\left[\varphi_i^2\right]\right] = \mathbb{E}\left[\varphi_i^2\right]\mathbb{E}\left[f_{i-1}^2(\Delta B_i)^2\right].$$

Now, using the same property of conditional expectation as before, and the fact that f_{i-1} is $\mathcal{F}_{t_{i-1}}$-measurable and ΔB_i is independent of $\mathcal{F}_{t_{i-1}}$, the above is equal to

$$\begin{aligned}\mathbb{E}\left[\varphi_i^2\right]\mathbb{E}\left[f_{i-1}^2\mathbb{E}\left[(\Delta B_i)^2\middle|\mathcal{F}_{t_{i-1}}\right]\right] &= \mathbb{E}\left[\varphi_i^2\right]\mathbb{E}\left[f_{i-1}^2\mathbb{E}\left[(\Delta B_i)^2\right]\right]\\ &= \mathbb{E}\left[\varphi_i^2\right]\mathbb{E}\left[f_{i-1}^2\right]\mathbb{E}\left[(\Delta B_i)^2\right]\\ &= \mathbb{E}\left[\varphi_i^2\right]\mathbb{E}\left[f_{i-1}^2\right]\Delta t_i.\end{aligned}$$

Since φ_i^2, and f_{i-1}^2 are functions of Brownian increments over disjoint intervals, they are independent, and so we get

$$\mathbb{E}\left[f_{i-1}^2\varphi_i^2(\Delta B_i)^2\right] = \mathbb{E}\left[\varphi_i^2 f_{i-1}^2\right]\Delta t_i.$$

Applying the limit as $\|\Delta_n\| \to 0$ and summation over $1 \leq i \leq n$, we get

$$\lim_{\|\Delta_n\|\to 0}\sum_{i=1}^{n}\mathbb{E}\left[f_{i-1}^2\varphi_i^2(\Delta B_i)^2\right] = \lim_{\|\Delta_n\|\to 0}\sum_{i=1}^{n}\mathbb{E}\left[\varphi_i^2 f_{i-1}^2\right]\Delta t_i$$

$$= \int_a^b \mathbb{E}\left[\varphi^2(B_b - B_t) f^2(B_t)\right] \, dt.$$

And this completes the proof. □

Proof of Theorem 4.7. We will split the proof of this theorem into several steps.

Step 1. As in the proof of Lemma 4.9, we will use the following $f_i = f(B_{t_i}),\ \varphi_i = \varphi(B_b - B_{t_i})$ to simplify the notation. Writing out the left-hand side of the equation (4.14), using the definition of the integral and distributing the square, we get

$$\mathbb{E}\left[\left(\int_a^b f(B_t)\varphi(B_b - B_t)\, dB_t\right)^2\right]$$
$$= \lim_{\|\Delta_n\|\to 0} \left(\sum_{i=0}^{N} \mathbb{E}\left[f_{i-1}^2 \varphi_i^2 (\Delta B_i)^2\right] + 2\sum_{i=2}^{N}\sum_{j=1}^{i-1} \mathbb{E}\left[f_{i-1}\varphi_i f_{j-1}\varphi_j \Delta B_i \Delta B_j\right]\right). \tag{4.16}$$

By Lemma 4.9, we have

$$\lim_{\|\Delta_n\|\to 0} \left(\sum_{i=0}^{N} \mathbb{E}\left[f_{i-1}^2 \varphi_i^2 (\Delta B_i)^2\right]\right) = \int_a^b \mathbb{E}\left[f^2(B_t)\varphi^2(B_b - B_t)\right] \, dt, \tag{4.17}$$

so it remains to show that

$$\lim_{\|\Delta_n\|\to 0} \sum_{i=2}^{N}\sum_{j=1}^{i-1} \mathbb{E}\left[f_{i-1}\varphi_i f_{j-1}\varphi_j \Delta B_i \Delta B_j\right]$$
$$= \int_a^b \int_a^t \mathbb{E}\left[f(B_s)\varphi'(B_b - B_s) f'(B_t)\varphi(B_b - B_t)\right] dsdt.$$

From now on, we will focus on the expectation under the double sum in equation (4.16), which we will denote by

$$\mathcal{E}_{ij} = \mathbb{E}\left[f_{i-1}\varphi_i f_{j-1}\varphi_j \Delta B_i \Delta B_j\right].$$

Step 2. Using properties (CE-2), (CE-3) and (CE-4) of the conditional expectation, we obtain

$$\mathcal{E}_{ij} = \mathbb{E}\left[\mathbb{E}\left[f_{i-1}\varphi_i f_{j-1}\varphi_j \Delta B_i \Delta B_j | \mathcal{F}_{t_i}\right]\right]$$
$$= \mathbb{E}\left[f_{i-1} f_{j-1} \Delta B_i \Delta B_j \mathbb{E}\left[\varphi_i \varphi_j | \mathcal{F}_{t_i}\right]\right] \tag{4.18}$$

Since $\varphi \in \mathcal{M}^\infty$, we can use its power series expansion at 0 to obtain

$$\begin{aligned}\mathbb{E}\left[\varphi_i\varphi_j|\mathcal{F}_{t_i}\right] &= \mathbb{E}\left[\varphi_i\varphi(B_b - B_{t_j})\big|\mathcal{F}_{t_i}\right] \\ &= \mathbb{E}\left[\varphi_i \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!}(B_b - B_{t_j})^n \middle| \mathcal{F}_{t_i}\right] \\ &= \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!}\mathbb{E}\left[\varphi_i(B_b - B_{t_j})^n\big|\mathcal{F}_{t_i}\right]. \qquad (4.19)\end{aligned}$$

We can apply the binomial theorem to express $(B_b - B_{t_j})^n$ as follows

$$\begin{aligned}(B_b - B_{t_j})^n &= (B_b - B_{t_i} + B_{t_i} - B_{t_j})^n \\ &= \sum_{k=0}^{n} \binom{n}{k}(B_b - B_{t_i})^k(B_{t_i} - B_{t_j})^{n-k}\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{E}\left[\varphi_i\varphi_j|\mathcal{F}_{t_i}\right] &= \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!}\mathbb{E}\left[\varphi_i \sum_{k=0}^{n}\binom{n}{k}(B_b - B_{t_i})^k(B_{t_i} - B_{t_j})^{n-k}\middle|\mathcal{F}_{t_i}\right] \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{\varphi^{(n)}(0)}{n!}\mathbb{E}\left[\varphi_i(B_b - B_{t_i})^k(B_{t_i} - B_{t_j})^{n-k}\big|\mathcal{F}_{t_i}\right]. \qquad (4.20)\end{aligned}$$

Step 3. Since $(B_{t_i} - B_{t_j})$ is $\mathcal{F}_{t_i}$-measurable and $(B_b - B_{t_i})$ is independent of $\mathcal{F}_{t_i}$,, properties (CE-3) and (CE-4) of the conditional expectation and Equations (4.18), (4.19) and (4.20) give

$$\begin{aligned}\mathcal{E}_{ij} = &\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{\varphi^{(n)}(0)}{n!} \\ &\times \mathbb{E}\left[\varphi_i(B_b - B_{t_i})^k\right]\mathbb{E}\left[f_{i-1}f_{j-1}\Delta B_i\Delta B_j(B_{t_i} - B_{t_j})^{n-k}\right]. \qquad (4.21)\end{aligned}$$

Since $f_{i-1},\ f_{j-1}$ and ΔB_j are $\mathcal{F}_{t_{i-1}}$-measurable, properties (CE-2) and (CE-4) of the conditional expectation, yield

$$\begin{aligned}&\mathbb{E}\left[f_{i-1}f_{j-1}\Delta B_i\Delta B_j(B_{t_i} - B_{t_j})^{n-k}\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[f_{i-1}f_{j-1}\Delta B_i\Delta B_j(B_{t_i} - B_{t_j})^{n-k}\big|\mathcal{F}_{t_{i-1}}\right]\right] \\ &= \mathbb{E}\left[f_{i-1}f_{j-1}\Delta B_j\,\mathbb{E}\left[\Delta B_i(B_{t_i} - B_{t_j})^{n-k}\big|\mathcal{F}_{t_{i-1}}\right]\right]. \qquad (4.22)\end{aligned}$$

Applying the binomial theorem to $(B_{t_i} - B_{t_j})^{n-k}$, we obtain

$$\begin{aligned}(B_{t_i} - B_{t_j})^{n-k} &= (B_{t_i} - B_{t_{i-1}} + B_{t_{i-1}} - B_{t_j})^{n-k} \\ &= (\Delta B_i + (B_{t_{i-1}} - B_{t_j}))^{n-k} \\ &= \sum_{l=0}^{n-k} \binom{n-k}{l} (\Delta B_i)^l (B_{t_{i-1}} - B_{t_j})^{n-k-l}. \end{aligned} \tag{4.23}$$

Since ΔB_i is independent of $\mathcal{F}_{t_{i-1}}$ and $(B_{t_i} - B_{t_j})$ is $\mathcal{F}_{t_{i-1}}$-measurable, properties (CE-3) and (CE-4) of the conditional expectation, and Equation (4.23) give

$$\begin{aligned}&\mathbb{E}\left[\Delta B_i (B_{t_i} - B_{t_j})^{n-k} \middle| \mathcal{F}_{t_{i-1}}\right] \\ &= \mathbb{E}\left[\Delta B_i \sum_{l=0}^{n-k} \binom{n-k}{l} (\Delta B_i)^l (B_{t_{i-1}} - B_{t_j})^{n-k-l} \middle| \mathcal{F}_{t_{i-1}}\right] \\ &= \sum_{l=0}^{n-k} \binom{n-k}{l} \mathbb{E}\left[(\Delta B_i)^{l+1} (B_{t_{i-1}} - B_{t_j})^{n-k-l} \middle| \mathcal{F}_{t_{i-1}}\right] \\ &= \sum_{l=0}^{n-k} \binom{n-k}{l} (B_{t_{i-1}} - B_{t_j})^{n-k-l} \mathbb{E}\left[(\Delta B_i)^{l+1}\right]. \end{aligned} \tag{4.24}$$

Now, Equation (4.24) together with the fact that $\mathbb{E}\left[(\Delta B_i)^{l+1}\right] = 0$ for $l = 0$, $\mathbb{E}\left[(\Delta B_i)^{l+1}\right] = \Delta t_i$ for $l = 1$ and $\mathbb{E}\left[(\Delta B_i)^{l+1}\right] = o(\Delta t_i)$ for $l > 2$, yield

$$\mathbb{E}\left[\Delta B_i (B_{t_i} - B_{t_j})^{n-k} \middle| \mathcal{F}_{t_{i-1}}\right] \approx (n-k)(B_{t_{i-1}} - B_{t_j})^{n-k-1} \Delta t_i. \tag{4.25}$$

Putting together Equations (4.21), (4.22) and (4.25), we obtain

$$\begin{aligned}\mathcal{E}_{ij} \approx & \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{\varphi^{(n)}(0)}{n!} (n-k) \Delta t_i \\ & \times \mathbb{E}\left[\varphi_i (B_b - B_{t_i})^k\right] \mathbb{E}\left[f_{i-1} f_{j-1} \Delta B_j (B_{t_{i-1}} - B_{t_j})^{n-k-1}\right] \end{aligned} \tag{4.26}$$

Observe that due to the $(n-k)$ term in the Equation (4.26), the term with $n = k$ does not contribute to the sum, hence we can exclude those terms from the summation by letting $n \in \{1, 2, 3, \ldots\}$ and $k \in \{0, 1, 2, \ldots, n-1\}$. Hence

$$\begin{aligned}\mathcal{E}_{ij} \approx & \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \binom{n}{k} \frac{\varphi^{(n)}(0)}{n!} (n-k) \Delta t_i \\ & \times \mathbb{E}\left[\varphi_i (B_b - B_{t_i})^k\right] \mathbb{E}\left[f_{i-1} f_{j-1} \Delta B_j (B_{t_{i-1}} - B_{t_j})^{n-k-1}\right] \end{aligned} \tag{4.27}$$

Step 4. Since f_{j-1} and ΔB_j are $\mathcal{F}_{t_j}$-measurable, properties (CE-2) and (CE-4) of the conditional expectation give

$$\begin{aligned}
&\mathbb{E}\left[f_{i-1}f_{j-1}\Delta B_j(B_{t_{i-1}}-B_{t_j})^{n-k-1}\right] \\
&=\mathbb{E}\left[\mathbb{E}\left[f_{j-1}\Delta B_j f_{i-1}(B_{t_{i-1}}-B_{t_j})^{n-k-1}\middle|\mathcal{F}_{t_j}\right]\right] \\
&=\mathbb{E}\left[f_{j-1}\Delta B_j\mathbb{E}\left[f_{i-1}(B_{t_{i-1}}-B_{t_j})^{n-k-1}\middle|\mathcal{F}_{t_j}\right]\right]. \qquad (4.28)
\end{aligned}$$

Since $f \in \mathcal{M}^\infty$, we can use its Maclaurin expansion to obtain

$$\begin{aligned}
&\mathbb{E}\left[f_{i-1}(B_{t_{i-1}}-B_{t_j})^{n-k-1}\middle|\mathcal{F}_{t_j}\right] \\
&=\mathbb{E}\left[f(B_{t_{i-1}})(B_{t_{i-1}}-B_{t_j})^{n-k-1}\middle|\mathcal{F}_{t_j}\right] \\
&=\mathbb{E}\left[(B_{t_{i-1}}-B_{t_j})^{n-k-1}\sum_{m=0}^{\infty}\frac{f^{(m)}(0)}{m!}(B_{t_{i-1}})^m\middle|\mathcal{F}_{t_j}\right] \\
&=\sum_{m=0}^{\infty}\frac{f^{(m)}(0)}{m!}\mathbb{E}\left[(B_{t_{i-1}}-B_{t_j})^{n-k-1}(B_{t_{i-1}})^m\middle|\mathcal{F}_{t_j}\right] \qquad (4.29)
\end{aligned}$$

We can apply the binomial theorem again to obtain

$$\begin{aligned}
(B_{t_{i-1}})^m &= (B_{t_{i-1}}-B_{t_j}+B_{t_j})^m \\
&= \sum_{l=0}^{m}\binom{m}{l}(B_{t_{i-1}}-B_{t_j})^l(B_{t_j})^{m-l}. \qquad (4.30)
\end{aligned}$$

Since B_{t_j} is $\mathcal{F}_{t_j}$-measurable and $B_{t_i}-B_{t_j}$ is independent of $\mathcal{F}_{t_j}$, properties (CE-3) and (CE-4), together with Equations (4.29) and (4.30), yield

$$\begin{aligned}
&\mathbb{E}\left[f_{i-1}(B_{t_{i-1}}-B_{t_j})^{n-k-1}\middle|\mathcal{F}_{t_j}\right] \\
&=\sum_{m=0}^{\infty}\sum_{l=0}^{m}\binom{m}{l}\frac{f^{(m)}(0)}{m!}\mathbb{E}\left[(B_{t_{i-1}}-B_{t_j})^{n-k-1}(B_{t_{i-1}})^m\middle|\mathcal{F}_{t_j}\right] \\
&=\sum_{m=0}^{\infty}\sum_{l=0}^{m}\binom{m}{l}\frac{f^{(m)}(0)}{m!}(B_{t_{i-1}})^m\mathbb{E}\left[(B_{t_{i-1}}-B_{t_j})^{n-k-1}\right] \qquad (4.31)
\end{aligned}$$

Putting together Equations (4.27), (4.28) and (4.31), we obtain

$$\begin{aligned}
\mathcal{E}_{ij} \approx &\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\sum_{m=0}^{\infty}\sum_{l=0}^{m}\binom{n}{k}\frac{\varphi^{(n)}(0)}{n!}(n-k)\binom{m}{l}\frac{f^{(m)}(0)}{m!}\Delta t_i \\
&\times\mathbb{E}\left[\varphi_i(B_b-B_{t_i})^k\right]\mathbb{E}\left[(B_{t_{i-1}}-B_{t_j})^{n-k-1+l}\right]\mathbb{E}\left[f_{j-1}(B_{t_j})^{m-l}\Delta B_j\right] \qquad (4.32)
\end{aligned}$$

Step 5. Since f_{j-1} is $\mathcal{F}_{t_{j-1}}$-measurable, applying again properties (CE-2) and (CE-4) of the conditional expectation, gives

$$\begin{aligned}\mathbb{E}\left[f_{j-1}(B_{t_j})^{m-l}\Delta B_j\right] &= \mathbb{E}\left[\mathbb{E}\left[f_{j-1}(B_{t_j})^{m-l}\Delta B_j\middle|\mathcal{F}_{t_{j-1}}\right]\right]\\ &= \mathbb{E}\left[f_{j-1}\mathbb{E}\left[(B_{t_j})^{m-l}\Delta B_j\middle|\mathcal{F}_{t_{j-1}}\right]\right] \qquad (4.33)\end{aligned}$$

Applying the binomial theorem to obtain

$$\begin{aligned}(B_{t_j})^{m-l} &= (B_{t_j} - B_{t_{j-1}} + B_{t_{j-1}})^{m-l}\\ &= (\Delta B_j + B_{t_{j-1}})^{m-l}\\ &= \sum_{q=0}^{m-l}\binom{m-l}{q}(\Delta B_j)^q(B_{t_{j-1}})^{m-l-q},\end{aligned}$$

together with properties (CE-3) and (CE-4) and the fact that ΔB_j is independent of $\mathcal{F}_{t_{j-1}}$ and $B_{t_{j-1}}$ is $\mathcal{F}_{t_{j-1}}$-measurable, gives us

$$\begin{aligned}\mathbb{E}\left[(B_{t_j})^{m-l}\Delta B_j\middle|\mathcal{F}_{t_{j-1}}\right] &= \mathbb{E}\left[\sum_{q=0}^{m-l}\binom{m-l}{q}(\Delta B_j)^q(B_{t_{j-1}})^{m-l-q}\Delta B_j\middle|\mathcal{F}_{t_{j-1}}\right]\\ &= \sum_{q=0}^{m-l}\binom{m-l}{q}\mathbb{E}\left[(\Delta B_j)^{q+1}(B_{t_{j-1}})^{m-l-q}\middle|\mathcal{F}_{t_{j-1}}\right]\\ &= \sum_{q=0}^{m-l}\binom{m-l}{q}(B_{t_{j-1}})^{m-l-q}\mathbb{E}\left[(\Delta B_j)^{q+1}\right]. \qquad (4.34)\end{aligned}$$

Observe that

$$\mathbb{E}\left[(\Delta B_j)^{q+1}\right] = \begin{cases} 0, & \text{if } q = 0,\\ \Delta t_j, & \text{if } q = 1,\\ o(\Delta t_i), & \text{if } q > 1.\end{cases}$$

Hence the only term of the sum in Equation (4.34) that contributes to the limit as $\|\Delta_n\| \to 0$ is the one with $q = 1$. So Equation (4.34) becomes

$$\mathbb{E}\left[(B_{t_j})^{m-l}\Delta B_j\middle|\mathcal{F}_{t_{j-1}}\right] = (m-l)(B_{t_{j-1}})^{m-l-1}\mathbb{E}\left[(\Delta B_j)^2\right]$$

$$= (m-l)(B_{t_{j-1}})^{m-l-1}\Delta t_j \tag{4.35}$$

Putting together Equations (4.32), (4.33) and (4.35) yields

$$\begin{aligned}\mathcal{E}_{ij} \approx \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\sum_{m=0}^{\infty}\sum_{l=0}^{m}\binom{n}{k}\frac{\varphi^{(n)}(0)}{n!}(n-k)\binom{m}{l}\frac{f^{(m)}(0)}{m!}(m-l)\Delta t_i\Delta t_j \\ \times\mathbb{E}\left[\varphi_i(B_b-B_{t_i})^k\right]\mathbb{E}\left[(B_{t_{i-1}}-B_{t_j})^{n-k-1+l}\right]\mathbb{E}\left[f(B_{t_{j-1}})(B_{t_{j-1}})^{m-l-1}\right].\end{aligned} \tag{4.36}$$

Due to the $(m-l)$ term in Equation (4.36), we can disregard those components of the sum, for which $m = l$. Hence we have

$$\begin{aligned}\mathcal{E}_{ij} \approx \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\sum_{m=1}^{\infty}\sum_{l=0}^{m-1}\binom{n}{k}\frac{\varphi^{(n)}(0)}{n!}(n-k)\binom{m}{l}\frac{f^{(m)}(0)}{m!}(m-l)\Delta t_i\Delta t_j \\ \times\mathbb{E}\left[\varphi(B_b-B_{t_i})(B_b-B_{t_i})^k\right]\mathbb{E}\left[(B_{t_{i-1}}-B_{t_j})^{n-k-1+l}\right] \\ \times\mathbb{E}\left[f(B_{t_{j-1}})(B_{t_{j-1}})^{m-l-1}\right].\end{aligned} \tag{4.37}$$

Since B_t is a Brownian motion and $(a, t_{j-1}), (t_j, t_{i-1})$ and (t_i, b) are disjoint intervals, we can rewrite Equation (4.37) as

$$\begin{aligned}\mathcal{E}_{ij} \approx \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\sum_{m=1}^{\infty}\sum_{l=0}^{m-1}\binom{n}{k}\frac{\varphi^{(n)}(0)}{n!}(n-k)\binom{m}{l}\frac{f^{(m)}(0)}{m!}(m-l)\Delta t_i\Delta t_j \\ \times\mathbb{E}\left[f_{j-1}\varphi_i(B_b-B_{t_i})^k(B_{t_{i-1}}-B_{t_j})^{n-k-1+l}(B_{t_{j-1}})^{m-l-1}\right]\end{aligned}$$

Step 6. Using the definitions of the binomial symbol and the factorial, we obtain

$$\binom{n}{k}\frac{n-k}{n!} = \frac{n!(n-k)}{(n-k)!k!n!} = \frac{(n-1)!}{(n-k-1)!k!}\frac{1}{(n-1)!} = \binom{n-1}{k}\frac{1}{(n-1)!}, \tag{4.38}$$

and

$$\binom{m}{l}\frac{m-l}{m!} = \frac{m!(m-l)}{(m-l)!l!m!} = \frac{(m-1)!}{(m-l-1)!l!}\frac{1}{(m-1)!} = \binom{m-1}{l}\frac{1}{(m-1)!}. \tag{4.39}$$

Putting Equations (4.38) and (4.39) into Equation (4.37), yields

$$\mathcal{E}_{ij} \approx \mathbb{E}\Bigg[\Delta t_i \Delta t_j f_{j-1}\varphi_i \times \sum_{m=1}^{\infty}\sum_{l=0}^{m-1}\binom{m-1}{l}\frac{f^{(m)}(0)}{(m-1)!}(B_{t_{i-1}} - B_{t_j})^l (B_{t_{j-1}})^{m-l-1} \times \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{\varphi^{(n)}(0)}{(n-1)!}(B_b - B_{t_i})^k (B_{t_{i-1}} - B_{t_j})^{n-k-1}\Bigg]. \tag{4.40}$$

Using the binomial theorem again, Equation (4.40) can be written as

$$\begin{aligned}\mathcal{E}_{ij} &\approx \mathbb{E}\Bigg[f_{j-1}\varphi_i \sum_{m=1}^{\infty}\frac{f^{(m)}(0)}{(m-1)!}\sum_{l=0}^{m-1}\binom{m-1}{l}(B_{t_{i-1}} - B_{t_j})^l (B_{t_{j-1}})^{m-l-1} \\ &\quad \times \sum_{n=1}^{\infty}\frac{\varphi^{(n)}(0)}{(n-1)!}\sum_{k=0}^{n-1}\binom{n-1}{k}(B_b - B_{t_i})^k (B_{t_{i-1}} - B_{t_j})^{n-k-1}\Bigg]\Delta t_i \Delta t_j \\ &= \mathbb{E}\Bigg[f_{j-1}\varphi_i \sum_{m=1}^{\infty}\frac{f^{(m)}(0)}{(m-1)!}(B_{t_{i-1}} - B_{t_j} + B_{t_{j-1}})^{m-1} \\ &\quad \times \sum_{n=1}^{\infty}\frac{\varphi^{(n)}(0)}{(n-1)!}(B_b - B_{t_i} + B_{t_{i-1}} - B_{t_j})^{n-1}\Bigg]\Delta t_i \Delta t_j. \end{aligned} \tag{4.41}$$

It is a well known fact that for any function $f \in \mathcal{M}^{\infty}$, we have

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n \quad \Longleftrightarrow \quad f'(x) = \sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{(n-1)!}x^{n-1}. \tag{4.42}$$

Applying Equation (4.42) to functions f and φ in Equation (4.41), we have

$$\begin{aligned}\mathcal{E}_{ij} &\approx \mathbb{E}\Bigg[f_{j-1}\varphi_i \sum_{m=1}^{\infty}\frac{f^{(m)}(0)}{(m-1)!}(B_{t_{i-1}} - \Delta B_j)^{m-1} \\ &\quad \times \sum_{n=1}^{\infty}\frac{\varphi^{(n)}(0)}{(n-1)!}(B_b - B_{t_j} - \Delta B_i)^{n-1}\Bigg]\Delta t_i \Delta t_j \\ &= \mathbb{E}\left[f_{j-1}\varphi_i f'(B_{t_{i-1}} - \Delta B_j)\varphi'(B_b - B_{t_j} - \Delta B_i)\right]\Delta t_i \Delta t_j. \end{aligned} \tag{4.43}$$

Step 7. Finally, taking the limit as $\|\Delta_N\| \to 0$, we obtain

$$\lim_{\|\Delta_N\|\to 0}\sum_{i=1}^{N}\sum_{j=0}^{i-1}\mathcal{E}_{ij}$$

$$= \lim_{\|\Delta_N\| \to 0} \sum_{i=1}^{N} \sum_{j=0}^{i-1} \mathbb{E}\left[f_{j-1}\varphi_i \varphi'(B_b - B_{t_j} - \Delta B_i) f'(B_{t_{i-1}} - \Delta B_j) \right] \Delta t_i \Delta t_j$$

$$= \int_a^b \int_a^t \mathbb{E}\left[f(B_s) f'(B_t) \varphi'(B_b - B_s) \varphi(B_b - B_t) \right] dsdt. \tag{4.44}$$

Combining Equations (4.16), (4.17) and (4.44) we obtain the desired result. □

Example 4.10. Consider $f(x) = \varphi(x) = x$. A stochastic process arising from these functions on the interval $[0, 1]$ is $\theta(t) = f(B_t)\varphi(B_1 - B_t)$. Of course $f'(x) = \varphi'(x) = 1$, so applying Theorem 4.7 gives

$$\begin{aligned} \mathbb{E}\left[\int_0^1 \theta(t)\, dB_t \right]^2 &= \int_0^1 \mathbb{E}\left[B_t^2 B_1 - B_t^2 \right] dt \\ &\quad + 2 \int_0^1 \int_0^t \mathbb{E}\left[B_s (B_1 - B_t) \right] dsdt. \end{aligned} \tag{4.45}$$

Note that the double integral is equal to 0, because B_s and $B_1 - B_t$ are independent, and so $\mathbb{E}\left[B_s(B_1 - B_t) \right] = 0$. Therefore

$$\begin{aligned} \mathbb{E}\left[\int_0^1 \theta(t)\, dB_t \right]^2 &= \int_0^1 \mathbb{E}\left[B_t^2 B_1 - B_t^2 \right] dt \\ &= \int_0^1 \mathbb{E}\left[B_t^2 \right] \mathbb{E}\left[B_1 - B_t^2 \right] dt \\ &= \int_0^1 t(1 - t)\, dt \\ &= \frac{1}{6}. \end{aligned}$$

We can also compute the expectation in the left-hand side of (4.45) explicitly. To do this, we first compute the Itô integral under the expectation using the Itô formula applied to $\theta(x, y) = \frac{x^2 y}{2} - \frac{x^3}{3}$. This yields

$$\begin{aligned} \frac{B_1^2 B_1}{2} - \frac{B_1^3}{3} = \frac{B_0^2 B_1}{2} - \frac{B_0^3}{3} &+ \int_0^1 B_t (B_1 - B_t)\, dB_t \\ &+ \int_0^1 \left\{ \frac{1}{2}(B_1 - 2B_t) + B_t \right\} dt. \end{aligned}$$

Therefore

$$\begin{aligned}\int_0^1 B_t(B_1 - B_t)\,dB_t &= \frac{B_1^2 B_1}{2} - \frac{B_1^3}{3} - \frac{1}{2}\int_0^1 B_1\,dt \\ &= \frac{B_1^3}{6} - \frac{B_1}{2}.\end{aligned}$$

And so the expectation in the left-hand side of (4.45) is equal to

$$\begin{aligned}\mathbb{E}\left[\frac{B_1^3}{6} - \frac{B_1}{2}\right]^2 &= \frac{1}{36}\mathbb{E}\left[B_1^6\right] - \frac{1}{6}\mathbb{E}\left[B_1^4\right] + \frac{1}{4}\mathbb{E}\left[B_1^2\right] \\ &= \frac{15}{36} - \frac{3}{6} + \frac{1}{4} \\ &= \frac{1}{6}\end{aligned}$$

Example 4.11. In [18, Example 2], Kuo–Sae-Tang–Szozda give an example of a process for which the Itô isometry from Theorem 4.3 is not applicable. It was shown that for

$$\int_0^1 B_t^2(B_1 - B_t)^2\,dB_t,$$

we have

$$\mathbb{E}\left[\int_0^1 B_t^2(B_1 - B_t)^2\,dB_t\right]^2 = \frac{11}{30}, \tag{4.46}$$

but

$$\mathbb{E}\left[\int_0^1 \left(B_t^2(B_1 - B_t)^2\right)^2\,dt\right] = \frac{3}{10}. \tag{4.47}$$

Hence, it follows that the last term in the right-hand side of Equation (4.14) is equal to $\frac{1}{15}$. In fact, with $f(x) = \varphi(x) = x^2$, we have

$$\begin{aligned}&2\int_0^1\int_0^t \mathbb{E}\left[2B_t B_s^2(B_1 - B_t)^2 2(B_1 - B_s)\right]\,dsdt \\ &= 8\int_0^1\int_0^t s(s-t)(1-t)\,dsdt \\ &= \frac{1}{15}.\end{aligned}$$

Example 4.12. We will compute the second moment of the integral

$$\int_0^1 B_t e^{B_1 - B_t}\, dB_t. \tag{4.48}$$

In view of Equation (4.14), with $f(x) = x$, $f'(x) = 1$ and $\varphi(x) = \varphi'(x) = e^x$, we have

$$\begin{aligned}
\mathbb{E}\left[\int_0^1 B_t e^{B_1 - B_t}\, dB_t\right]^2 &= \int_0^1 \mathbb{E}\left[B_t^2 e^{2(B_1 - B_t)}\right] dt \\
&\quad + 2\int_0^1 \int_0^t \mathbb{E}\left[B_t e^{B_1 - B_s} e^{B_1 - B_t}\right] dsdt \\
&= \int_0^1 t e^{2(1-t)}\, dt + 2\int_0^1 \int_0^t (t-s) e^{2 - \frac{s}{2} - \frac{3t}{2}}\, dsdt \\
&= \frac{e^2}{4} - \frac{3}{4} + 2\left(\frac{2}{9}e^2 + \frac{4}{9}\sqrt{e} - 2\right) \\
&= \frac{25}{36}e^2 + \frac{8}{9}\sqrt{e} - \frac{19}{4}.
\end{aligned}$$

Chapter 5
The new Itô formula

As we have already seen in Section 2.2, the Itô formula is one of the key tools in the stochastic analysis. We will see one of its many applications in Chapter 6, where we will use it to obtain the solution to a certain class of linear stochastic differential equations. In this chapter, present several versions of the Itô formula.

5.1 The Itô formula for more general processes

As we have seen in Section 2.2, the elementary Itô formula for Brownian motion can be generalized to be applicable to Itô processes. Our goal is to generalize Theorem 3.11 in a similar way to how Theorem 2.7 generalizes Theorem 2.4 in classical Itô calculus. That is, we would like to change $f(B_t)$ and $\varphi(B_b - B_t)$ into $f(X_t)$ and $\varphi(Y_t)$ where X_t and Y_t are appropriate stochastic processes. The natural candidate for X_t is an Itô process. Our intuition suggests that a process of a similar form should be a counterpart to the Itô process and that is how we will choose process Y_t.

In order to simplify notation, we introduce the notion of a backward associated stochastic process in the following way

$$Y^{(t)} = \int_t^b h(s)\, dB_s + \int_t^b \chi(s)\, ds. \tag{5.1}$$

This is an instantly independent counterpart to the Itô process. The reason for this is the fact that for $f(s) = 1$ and $\chi(s) = 0$, we have

$$Y^{(t)} = \int_t^b 1\, dB_s = B_b - B_t,$$

which is exactly the process appearing in the first Itô formula introduced by Ayed and Kuo in [1] and recalled here in Theorem 3.11. For the purpose of this dissertation, we will assume that $h, \chi \in L^2([a,b])$ are deterministic functions. As it turns out, the process $Y^{(t)}$ is an instantly independent stochastic process.

Theorem 5.1. *Suppose that $h, \chi \in L^2([a,b])$. Then $Y^{(t)}$ as defined in Equation* (5.1) *is an instantly independent stochastic process.*

Proof. For $Y^{(t)}$ as defined in Equation (5.1) to be an instantly independent stochastic process, we need to show, that for each $t \in [a,b]$ the random variable $Y^{(t)}$ is independent of $\mathcal{F}_t$. Recall that $\mathcal{F}_t$ is a σ-field generated by Brownian motion up to time t. Now, by Definition 3.5, we have that

$$Y^{(t)} = \int_t^b h(s)\, dB_s + \int_t^b \chi(s)\, ds = \lim_{\|\Delta_n\| \to 0} \left[\sum_{i=1}^n h(s_i) \Delta B_i + \sum_{i=0}^n \chi(s_i) \Delta s_i \right],$$

where $\Delta_n = \{t = s_0 < s_1 < \cdots < s_{n-1} < s_n = b\}$ is a partition of the interval $[t,b]$. First note, that the latter sum in the above expression is deterministic, and hence does not influence the independence of $Y^{(t)}$ and $\mathcal{F}_t$. Note also, that for all $i \in \{1, 2, \ldots, n\}$ value $h(s_i)$ is deterministic and ΔB_i is independent of the σ-field $\mathcal{F}_t$ as a Brownian increment following t. Therefore, $\sum_{i=1}^n h(s_i)\Delta B_i$ is independent of $\mathcal{F}_t$ for each $n \in \mathbb{N}$. Thus $Y^{(t)}$ as a limit is also independent of $\mathcal{F}_t$. □

Theorem 5.2. *Suppose that $\theta(x,y) = f(x)\varphi(y)$, where $f, \varphi \in C^2(\mathbb{R})$. Let*

$$X_t = \int_a^t g(s)\, dB_s + \int_a^t \gamma(s)\, ds, \qquad \text{and} \qquad Y^{(t)} = \int_t^b h(s)\, dB_s + \int_t^b \chi(s)\, ds,$$

where, $g, \gamma \in L^2_{ad}(\Omega \times [a,b])$ and $h, \chi \in L^2[a,b]$ are deterministic functions. Then

$$\begin{aligned}\theta(X_t, Y^{(t)}) = \theta(X_a, Y^{(a)}) & \\ &+ \int_a^t \frac{\partial \theta}{\partial x}(X_s, Y^{(s)}) dX_s + \frac{1}{2} \int_a^t \frac{\partial^2 \theta}{\partial x^2}(X_s, Y^{(s)}) \left(dX_s\right)^2 \\ &+ \int_a^t \frac{\partial \theta}{\partial y}(X_s, Y^{(s)}) dY^{(s)} - \frac{1}{2} \int_a^t \frac{\partial^2 \theta}{\partial y^2}(X_s, Y^{(s)}) \left(dY^{(s)}\right)^2. \qquad (5.2)\end{aligned}$$

Proof. Throughout this proof, we will use the standard notation introduced earlier, namely $\Delta_n = \{a = t_0 < t_1, \cdots < t_{n-1} < t_n = t\}$. To establish the formula in Equation (5.2), we begin by writing the difference $\theta\left(X_t, Y^{(t)}\right) - \theta\left(X_a, Y^{(a)}\right)$ in the form of a telescoping sum.

$$\begin{aligned}
\theta\left(X_t, Y^{(t)}\right) - \theta\left(X_a, Y^{(a)}\right) &= \sum_{i=1}^{n} \left[\theta\left(X_{t_i}, Y^{(t_i)}\right) - \theta\left(X_{t_{i-1}}, Y^{(t_{i-1})}\right)\right] \\
&= \sum_{i=1}^{n} \left[f\left(X_{t_i}\right)\varphi\left(Y^{(t_i)}\right) - f\left(X_{t_{i-1}}\right)\varphi\left(Y^{(t_{i-1})}\right)\right]
\end{aligned}$$

Note that in order to obtain the new stochastic integral in the sum above, we have to use left endpoint of the interval $[t_{i-1}, t_i]$ as the evaluation point in every occurrence of function f and the right endpoint of the same interval to evaluate every occurrence of φ. To get to this point, we will use second order Taylor expansion of functions f and φ. We can restrict the Taylor expansion to the second order, because for $k > 2$, we have

$$(\Delta X_i)^k = o(\Delta t_i) \qquad \text{and} \qquad (\Delta Y_i)^k = o(\Delta t_i),$$

hence higher order terms will converge to zero as $\|\Delta_n\| \to 0$. Therefore

$$\begin{aligned}
&\theta\left(X_t, Y^{(t)}\right) - \theta\left(X_a, Y^{(a)}\right) \\
&\approx \sum_{i=1}^{n} \Bigg[\left(f\left(X_{t_{i-1}}\right) + f'\left(X_{t_{i-1}}\right)(\Delta X_i) + \frac{1}{2}f''\left(X_{t_{i-1}}\right)(\Delta X_i)^2\right)\varphi\left(Y^{(t_i)}\right) \\
&\quad - f\left(X_{t_{i-1}}\right)\left(\varphi\left(Y^{(t_i)}\right) + \varphi'\left(Y^{(t_i)}\right)(-\Delta Y_i) + \frac{1}{2}\varphi''\left(Y^{(t_i)}\right)(-\Delta Y_i)^2\right)\Bigg] \\
&= \sum_{i=1}^{n} \Bigg[\left(f'\left(X_{t_{i-1}}\right)\varphi\left(Y^{(t_i)}\right)(\Delta X_i) + \frac{1}{2}f''\left(X_{t_{i-1}}\right)\varphi\left(Y^{(t_i)}\right)(\Delta X_i)^2\right) \\
&\quad - \left(f\left(X_{t_{i-1}}\right)\varphi'\left(Y^{(t_i)}\right)(-\Delta Y_i) + \frac{1}{2}f\left(X_{t_{i-1}}\right)\varphi''\left(Y^{(t_i)}\right)(-\Delta Y_i)^2\right)\Bigg] \\
&= \sum_{i=1}^{n} \Bigg[\frac{\partial\theta}{\partial x}\left(X_{t_{i-1}}, Y^{(t_i)}\right)\Delta X_i + \frac{1}{2}\frac{\partial^2\theta}{\partial x^2}\left(X_{t_{i-1}}, Y^{(t_i)}\right)(\Delta X_i)^2 \\
&\quad + \frac{\partial\theta}{\partial y}\left(X_{t_{i-1}}, Y^{(t_i)}\right)\Delta Y_i - \frac{1}{2}\frac{\partial^2\theta}{\partial y^2}\left(X_{t_{i-1}}, Y^{(t_i)}\right)(\Delta Y_i)^2\Bigg]
\end{aligned}$$

$$\xrightarrow{\|\Delta_n\|\to 0} \int_a^t \frac{\partial\theta}{\partial x}(X_s, Y^{(s)})dX_s + \frac{1}{2}\int_a^t \frac{\partial^2\theta}{\partial x^2}(X_s, Y^{(s)})\,(dX_s)^2$$
$$+ \int_a^t \frac{\partial\theta}{\partial y}(X_s, Y^{(s)})dY^{(s)} - \frac{1}{2}\int_a^t \frac{\partial^2\theta}{\partial y^2}(X_s, Y^{(s)})\left(dY^{(s)}\right)^2.$$

Hence the formula in Equation (5.2) holds. □

Arguments similar to the ones in the proof of Theorem 5.2 can be used to prove the following corollary.

Corollary 5.3. *Suppose that* $\theta(t,x,y) = \tau(t)f(x)\varphi(y)$*, where* $f, \varphi \in C^2(\mathbb{R})$ *and* $\tau \in C^1([a,b])$*. Let*

$$X_t = \int_a^t g(s)\,dB_s + \int_a^t \gamma(s)\,ds, \qquad \text{and} \qquad Y^{(t)} = \int_t^b h(s)\,dB_s + \int_t^b \chi(s)\,ds,$$

where, $g, \gamma \in L^2(\Omega \times [a,b])$ *and* $h, \chi \in L^2[a,b]$ *are deterministic functions. Then*

$$\begin{aligned}\theta(t, X_t, Y^{(t)}) = \theta(a, X_a, Y^{(a)}) & \\ &+ \int_a^t \frac{\partial\theta}{\partial x}(s, X_s, Y^{(s)})dX_s + \frac{1}{2}\int_a^t \frac{\partial^2\theta}{\partial x^2}(s, X_s, Y^{(s)})\,(dX_s)^2 \\ &+ \int_a^t \frac{\partial\theta}{\partial y}(s, X_s, Y^{(s)})dY^{(s)} - \frac{1}{2}\int_a^t \frac{\partial^2\theta}{\partial y^2}(s, X_s, Y^{(s)})\left(dY^{(s)}\right)^2 \\ &+ \int_a^t \frac{\partial\theta}{\partial s}(s, X_s, Y^{(s)})\,ds.\end{aligned}$$

Now, in preparation for the next Chapter, we introduce the differential notation for stochastic integrals. In deterministic calculus it is customary to write $df(t) = g(t)\,dt$ instead of $f(t) = f(a) + \int_a^t g(s)\,ds$, or even $df(t) = g(t)\,dh(t)$ instead of $f(t) = f(a) + \int_a^t g(s)\,dh(s)$, where the last integral is understood as Riemann–Stieltjes integral. This is not only a notation, but the differentials $df(t)$ are well-defined for a wide class of functions. It is not the case with the Brownian differential dB_t, because Brownian motion is not differentiable. Hence, we introduce the following conventional notation

$$dX_t = g(t)\,dB_t + \gamma(t)\,dt$$

and understand this as a shorthand for

$$X_t = X_a + \int_a^t g(s)\,dB_s + \int_a^t \gamma(s)\,ds.$$

We will also write

$$dY^{(t)} = -h(t)dB_t - \chi(t)\,dt$$

for

$$Y^{(t)} = \int_t^b h(s)\,dB_s + \int_t^b \chi\,ds.$$

This allows us to rewrite the conclusion of Theorem 5.2 in Equation (5.2) as

$$\begin{aligned} d\theta(X_t, Y^{(t)}) = {} & \frac{\partial\theta}{\partial x}(X_t, Y^{(t)})dX_t + \frac{1}{2}\frac{\partial^2\theta}{\partial x^2}(X_t, Y^{(t)})\,(dX_t)^2 \\ & + \frac{\partial\theta}{\partial y}(X_t, Y^{(t)})dY^{(t)} - \frac{1}{2}\frac{\partial^2\theta}{\partial y^2}(X_t, Y^{(t)})\left(dY^{(t)}\right)^2. \end{aligned}$$

As we have seen in Example 3.7, upon appropriate decomposition of the integrand, it is possible to apply the new integral to processes that are not instantly independent. Our next goal is to establish an Itô formula for such processes. Note that using the notation of Equation (5.1), with $h(s) = 1$ and $\chi(s) = 0$, on the interval $[0, 1]$, we have

$$Y^{(a)} = \int_0^1 1\,dB_t = B_1 - B_0 = B_1.$$

Thus, $Y^{(a)}$ is to $Y^{(t)}$ what B_1 is to $B_1 - B_t$. Hence, we want to establish an Itô formula for $\theta(X_t, Y^{(a)})$, with X_t and $Y^{(t)}$ as defined in Theorem 5.2. Keeping in mind that the definition of the new integral does not allow processes that are anticipating and not instantly independent, we have to impose an additional structure on function θ in order to move freely between $Y^{(a)}$ and $Y^{(t)}$. As we have seen in previous chapters, such structure can be introduced by the use of functions whose Maclaurin series expansion has infinite radius of convergence, namely functions from the class $\mathcal{M}^\infty$.

Theorem 5.4. *Suppose that $\theta(x,y) = f(x)\varphi(y)$, where $f \in C^2(\mathbb{R})$, and $\varphi \in \mathcal{M}^\infty$. Let*

$$X_t = \int_a^t g(s)\,dB_s + \int_a^t \gamma(s)\,ds, \qquad \text{and} \qquad Y^{(t)} = \int_t^b h(s)\,dB_s + \int_t^b \chi(s)\,ds,$$

where, $g, \gamma \in L^2(\Omega \times [a,b])$ and $h, \chi \in L^2[a,b]$ are deterministic functions. Then

$$\begin{aligned} \theta(X_t, Y^{(a)}) &= \theta(X_a, Y^{(a)}) \\ &\quad + \int_a^t \frac{\partial \theta}{\partial x}\left(X_s, Y^{(a)}\right) dX_s + \frac{1}{2}\int_a^t \frac{\partial^2 \theta}{\partial x^2}\left(X_s, Y^{(a)}\right)(dX_s)^2 \\ &\quad - \int_a^t \frac{\partial^2 \theta}{\partial x \partial y}\left(X_s, Y^{(a)}\right)(dX_s)\left(dY^{(s)}\right). \end{aligned} \tag{5.3}$$

Proof. We will derive the formula in Equation (5.3) symbolically using the differential notation introduced earlier. That is, we need to establish that

$$\begin{aligned} d\theta(X_t, Y^{(a)}) = \frac{\partial \theta}{\partial x}\left(X_t, Y^{(a)}\right) dX_t &+ \frac{1}{2}\frac{\partial^2 \theta}{\partial x^2}\left(X_t, Y^{(a)}\right)(dX_t)^2 \\ &- \frac{\partial^2 \theta}{\partial x \partial y}\left(X_t, Y^{(a)}\right)(dX_t)\left(dY^{(t)}\right). \end{aligned} \tag{5.4}$$

To simplify the notation, we will write $\mathcal{D} = d\left(\theta\left(X_t, Y^{(a)}\right)\right)$. Now, let us consider

$$\begin{aligned} \mathcal{D} &= d\left(\theta\left(X_t, Y^{(a)}\right)\right) \\ &= d\left(f(X_t)\varphi\left(Y^{(a)}\right)\right) \\ &= d\left(f(X_t)\sum_{n=0}^{\infty}\frac{\varphi^{(n)}(0)}{n!}\left(Y^{(a)}\right)^n\right) \\ &= d\left(f(X_t)\sum_{n=0}^{\infty}\frac{\varphi^{(n)}(0)}{n!}\left(Y^{(a)} - Y_t + Y_t\right)^n\right) \end{aligned} \tag{5.5}$$

Applying the binomial theorem and the fact that $Y^{(a)} - Y_t = Y^{(t)}$ yields

$$\mathcal{D} = d\left(f(X_t)\sum_{n=0}^{\infty}\frac{\varphi^{(n)}(0)}{n!}\sum_{k=0}^{n}\binom{n}{k}\left(Y^{(t)}\right)^k (Y_t)^{n-k}\right)$$

$$= \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{k=0}^{n} \binom{n}{k} d\left(f(X_t)\,(Y_t)^{n-k}\left(Y^{(t)}\right)^k\right). \tag{5.6}$$

Note that $Z_t = f(X_t)Y_t^{n-k}$ as a product of Itô processes is an Itô process itself, hence we can use the Itô formula from Theorem 5.2 to evaluate the differential under the sum in Equation (5.6). We take $\eta(z,y) = zy^k$ to obtain $\eta_z(z,y) = y^k, \eta_{zz}(z,y) = 0, \eta_y(z,y) = kzy^{k-1}$ and $\eta(z,y)_{yy} = k(k-1)zy^{k-2}$ which yields

$$\begin{aligned} d\left(\eta\left(Z_t, Y^{(t)}\right)\right) = {} & \left(Y^{(t)}\right)^k d(Z_t) + kZ_t\left(Y^{(t)}\right)^{k-1} dY^{(t)} \\ & - \frac{1}{2}k(k-1)Z_t\left(Y^{(t)}\right)^{k-2}\left(dY^{(t)}\right)^2 \end{aligned} \tag{5.7}$$

Using the Itô product rule for Itô processes, we easily see that

$$\begin{aligned} dZ_t &= f(X_t)\,d\left(Y_t^{n-k}\right) + Y_t^{n-k}\,df(X_t) + (df(X_t))\left(dY_t^{n-k}\right) \\ &= f(X_t)\left[(n-k)Y_t^{n-k-1}\,dY_t + \frac{1}{2}(n-k)(n-k-1)Y_t^{n-k-2}\,(dY_t)^2\right] \\ &\quad + Y_t^{n-k}\left[f'(X_t)\,dX_t + \frac{1}{2}f''(X_t)\,(dX_t)^2\right] \\ &\quad + (n-k)f'(X_t)Y_t^{n-k-1}\,(dX_t)\,(dY_t) \\ &= f(X_t)(n-k)Y_t^{n-k-1}\,dY_t \\ &\quad + \frac{1}{2}(n-k)(n-k-1)f(X_t)Y_t^{n-k-2}\,(dY_t)^2 \\ &\quad + f'(X_t)Y_t^{n-k}\,dX_t \\ &\quad + \frac{1}{2}f''(X_t)Y_t^{n-k}\,(dX_t)^2 \\ &\quad + (n-k)f'(X_t)Y_t^{n-k-1}\,(dX_t)\,(dY_t) \end{aligned} \tag{5.8}$$

Putting together Equations (5.6), (5.7) and (5.8), we see that in order to complete this proof, we have to evaluate

$$\begin{aligned} \mathcal{D} = {} & f(X_t)\sum_{n=0}^{\infty}\frac{\varphi^{(n)}(0)}{n!}\sum_{k=0}^{n}\binom{n}{k}(n-k)Y_t^{n-k-1}\left(Y^{(t)}\right)^k\,dY_t \\ & + \frac{1}{2}f(X_t)\sum_{n=0}^{\infty}\frac{\varphi^{(n)}(0)}{n!}\sum_{k=0}^{n}\binom{n}{k}(n-k)(n-k-1)Y_t^{n-k-2}\left(Y^{(t)}\right)^k(dY_t)^2 \end{aligned}$$

$$
\begin{aligned}
&+ f'(X_t) \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{k=0}^{n} \binom{n}{k} Y_t^{n-k} \left(Y^{(t)}\right)^k \, dX_t \\
&+ \frac{1}{2} f''(X_t) \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{k=0}^{n} \binom{n}{k} Y_t^{n-k} \left(Y^{(t)}\right)^k (dX_t)^2 \\
&+ f'(X_t) \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{k=0}^{n} \binom{n}{k} (n-k) Y_t^{n-k-1} \left(Y^{(t)}\right)^k (dX_t)\,(dY_t) \\
&+ f(X_t) \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{k=0}^{n} \binom{n}{k} k Y_t^{n-k} \left(Y^{(t)}\right)^{k-1} dY^{(t)} \\
&- f(X_t) \frac{1}{2} \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{k=0}^{n} \binom{n}{k} k(k-1) Y_t^{n-k} \left(Y^{(t)}\right)^{k-2} \left(dY^{(t)}\right)^2 \qquad (5.9)
\end{aligned}
$$

In order to simplify $\mathcal{D}$ in Equation (5.9), we need to evaluate the following 5 sums:

1. The first sum is given by

$$
S_1 = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{k=0}^{n} \binom{n}{k} (n-k) Y_t^{n-k-1} \left(Y^{(t)}\right)^k . \qquad (5.10)
$$

Note that for $n = k$ the expression under the sum is equal to zero, so we have

$$
S_1 = \sum_{n=1}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{k=0}^{n-1} \binom{n}{k} (n-k) Y_t^{n-k-1} \left(Y^{(t)}\right)^k .
$$

Now, since $\frac{1}{n!}\binom{n}{k}(n-k) = \frac{1}{(n-1)!}\binom{n-1}{k}$, we get

$$
S_1 = \sum_{n=1}^{\infty} \frac{\varphi^{(n)}(0)}{(n-1)!} \sum_{k=0}^{n-1} \binom{n-1}{k} Y_t^{(n-1)-k} \left(Y^{(t)}\right)^k ,
$$

and application of the binomial theorem yields

$$
S_1 = \sum_{n=1}^{\infty} \frac{\varphi^{(n)}(0)}{(n-1)!} \left(Y_t + Y^{(t)}\right)^{n-1} .
$$

Since, by definition, $Y_t + Y^{(t)} = Y^{(a)}$ and $\sum_{n=1}^{\infty} \frac{\varphi^{(n)}(0)}{(n-1)!} x^{n-1} = \varphi'(x)$, we obtain

$$
S_1 = \varphi'\left(Y^{(a)}\right) . \qquad (5.11)
$$

2. The second sum we have to evaluate is

$$
S_2 = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{k=0}^{n} \binom{n}{k} (n-k)(n-k-1) Y_t^{n-k-2} \left(Y^{(t)}\right)^k . \qquad (5.12)
$$

Due to the $n-k$ and $n-k-1$ factors, the terms with $k=n$ and $k=n-1$ do not contribute to the sum, hence

$$S_2 = \sum_{n=2}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{k=0}^{n-2} \binom{n}{k} (n-k)(n-k-1) Y_t^{n-k-2} \left(Y^{(t)}\right)^k .$$

Since $\frac{1}{n!}\binom{n}{k}(n-k)(n-k-1) = \frac{1}{(n-2)!}\binom{n-2}{k}$, we have

$$S_2 = \sum_{n=2}^{\infty} \frac{\varphi^{(n)}(0)}{(n-2)!} \sum_{k=0}^{n-2} \binom{n-2}{k} Y_t^{n-k-2} \left(Y^{(t)}\right)^k .$$

Using the binomial theorem, we obtain

$$S_2 = \sum_{n=2}^{\infty} \frac{\varphi^{(n)}(0)}{(n-2)!} \left(Y_t + Y^{(t)}\right)^{n-2} .$$

Using the facts that $\varphi''(x) = \sum_{n=2}^{\infty} \frac{\varphi^{(n)}(0)}{(n-2)} x^{n-2}$ and $Y_t + Y^{(t)} = Y^{(a)}$ we get

$$S_2 = \varphi''\left(Y^{(a)}\right) . \tag{5.13}$$

3. Using the same reasoning as previously, we can write the next sum appearing in Equation (5.9) as

$$\begin{aligned} S_3 &= \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{k=0}^{\infty} \binom{n}{k} Y_t^{n-k} \left(Y^{(t)}\right)^k \\ &= \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \left(Y_t + Y^{(t)}\right)^n \\ &= \varphi\left(Y^{(a)}\right) . \end{aligned} \tag{5.14}$$

4. Now, we evaluate the following sum

$$S_4 = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{k=0}^{n} \binom{n}{k} k Y_t^{n-k} \left(Y^{(t)}\right)^{k-1} .$$

Notice that substitution $j = n-k$ together with the fact that $\binom{n}{n-j} = \binom{n}{j}$ yields

$$S_4 = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{j=0}^{n} \binom{n}{n-j} (n-j) Y_t^j \left(Y^{(t)}\right)^{n-j-1}$$

$$= \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{j=0}^{n} \binom{n}{j} (n-j) Y_t^j \left(Y^{(t)}\right)^{n-j-1},$$

and this is the same sum we have evaluated in Equation (5.10), hence

$$S_4 = \varphi\left(Y^{(a)}\right). \tag{5.15}$$

5. The last sum needed is

$$S_5 = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{k=0}^{n} \binom{n}{k} k(k-1) Y_t^{n-k} \left(Y^{(t)}\right)^{k-2}.$$

Using the substitution $j = n - k$ and the fact that $\binom{n}{n-j} = \binom{n}{j}$ again, we obtain

$$\begin{aligned} S_5 &= \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{k=0}^{n} \binom{n}{n-j} (n-j)(n-j-1) Y_t^j \left(Y^{(t)}\right)^{n-j-2} \\ &= \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{k=0}^{n} \binom{n}{j} (n-j)(n-j-1) Y_t^j \left(Y^{(t)}\right)^{n-j-2}. \end{aligned}$$

And this sum appears in Equation (5.12), thus

$$S_5 = \varphi''\left(Y^{(a)}\right). \tag{5.16}$$

Now, putting together Equations (5.9), (5.11), (5.13), (5.14), (5.15) and (5.16) we obtain

$$\begin{aligned} \mathcal{D} &= f(X_t) S_1 \, dY_t + \frac{1}{2} f(X_t) S_2 \left(dY_t\right)^2 + f'(X_t) S_3 \, dX_t + \frac{1}{2} f''(X_t) S_3 \left(dX_t\right)^2 \\ &\quad + f'(X_t) S_1 \left(dX_t\right)\left(dY_t\right) + f(X_t) S_4 \, dY^{(t)} - \frac{1}{2} f(X_t) S_5 \left(dY^{(t)}\right)^2 \\ &= f(X_t) \varphi'\left(Y^{(a)}\right) dY_t + \frac{1}{2} f(X_t) \varphi''\left(Y^{(a)}\right) \left(dY_t\right)^2 + f'(X_t) \varphi\left(Y^{(a)}\right) dX_t \\ &\quad + \frac{1}{2} f''(X_t) \varphi\left(Y^{(a)}\right) \left(dX_t\right)^2 + f'(X_t) \varphi'\left(Y^{(a)}\right) \left(dX_t\right)\left(dY_t\right) \\ &\quad + f(X_t) \varphi'\left(Y^{(a)}\right) dY^{(t)} - \frac{1}{2} f(X_t) \varphi''\left(Y^{(a)}\right) \left(dY^{(t)}\right)^2 \end{aligned}$$

Since $dY_t = -dY^{(t)}$, we obtain

$$\begin{aligned}\mathcal{D} = &-f(X_t)\varphi'\left(Y^{(a)}\right) dY^{(t)} + \frac{1}{2}f(X_t)\varphi''\left(Y^{(a)}\right)\left(dY^{(t)}\right)^2 + f'(X_t)\varphi\left(Y^{(a)}\right) dX_t \\ &+ \frac{1}{2}f''(X_t)\varphi\left(Y^{(a)}\right)(dX_t)^2 - f'(X_t)\varphi'\left(Y^{(a)}\right)(dX_t)\left(dY^{(t)}\right) \\ &+ f(X_t)\varphi'\left(Y^{(a)}\right) dY^{(t)} - \frac{1}{2}f(X_t)\varphi''\left(Y^{(a)}\right)\left(dY^{(t)}\right)^2 \\ = &\frac{\partial\theta}{\partial x}\left(X_t, Y^{(a)}\right) dX_t + \frac{1}{2}\frac{\partial^2\theta}{\partial x^2}\left(X_t, Y^{(a)}\right)(dX_t)^2 - \frac{\partial^2\theta}{\partial x\partial y}\left(X_t, Y^{(a)}\right)(dX_t)\left(dY^{(t)}\right)\end{aligned}$$

And this completes the proof. □

As with Corollary 5.3, we can easily deduce, that if we had a component of θ that is deterministic and depends only on t, the following Corollary will hold.

Corollary 5.5. *Suppose that $\theta(t, x, y) = \tau(t)f(x)\varphi(y)$, where $\tau \in C^1(\mathbb{R})$, $f \in C^2(\mathbb{R})$, and $\varphi \in \mathcal{M}^\infty$. Let*

$$X_t = \int_a^t g(s)\, dB_s + \int_a^t \gamma(s)\, ds, \qquad \text{and} \qquad Y^{(t)} = \int_t^b h(s)\, dB_s + \int_t^b \chi(s)\, ds,$$

where, $g, \gamma \in L^2_{ad}(\Omega \times [a, b])$ and $h, \chi \in L^2[a, b]$ are deterministic functions. Then

$$\begin{aligned}\theta(t, X_t, Y^{(a)}) = \theta(a, X_a, Y^{(a)}) \\ &+ \int_a^t \frac{\partial\theta}{\partial x}\left(s, X_s, Y^{(a)}\right) dX_s + \frac{1}{2}\int_a^t \frac{\partial^2\theta}{\partial x^2}\left(s, X_s, Y^{(a)}\right)(dX_s)^2 \\ &- \int_a^t \frac{\partial^2\theta}{\partial x\partial y}\left(s, X_s, Y^{(a)}\right)(dX_s)\left(dY^{(s)}\right) \\ &+ \int_a^t \frac{\partial\theta}{\partial t}\left(s, X_s, Y^{(a)}\right) ds. \qquad (5.17)\end{aligned}$$

The next corollary is a simple consequence of the previous one. It is a version that we will use often in the next chapter, when we solve the stochastic differential equations with anticipating initial conditions.

Corollary 5.6. *Suppose that $\theta(t, x, y) = \tau(t)f(x)\varphi(y)$, where $\tau \in C^1(\mathbb{R})$, $f \in C^2(\mathbb{R})$, and $\varphi \in \mathcal{M}^\infty$. Let*

$$X_t = \int_a^t g(s)\, dB_s + \int_a^t \gamma(s)\, ds,$$

where $g, \gamma \in L^2_{ad}(\Omega \times [a,b])$. *Then*

$$\begin{aligned}\theta(t, X_t, B_b - B_a) = \theta(a, X_a, B_b - B_a) &+ \int_a^t \frac{\partial \theta}{\partial x}(s, X_s, B_b - B_a)\ dX_s \\ &+ \frac{1}{2}\int_a^t \frac{\partial^2 \theta}{\partial x^2}(s, X_s, B_b - B_a)\ (dX_s)^2 \\ &+ \int_a^t \frac{\partial^2 \theta}{\partial x \partial y}(s, X_s, B_b - B_a)\ (dX_s)(dB_s) \\ &+ \int_a^t \frac{\partial \theta}{\partial t}(s, X_s, B_b - B_a)\ ds. \end{aligned} \tag{5.18}$$

Equivalently, we can write the Equation (5.18) *in a differential form as*

$$\begin{aligned} d\theta(t, X_t, B_b - B_a) = &\frac{\partial \theta}{\partial x}(t, X_t, B_b - B_a)\ dX_t + \frac{1}{2}\frac{\partial^2 \theta}{\partial x^2}(s, X_t, B_b - B_a)\ (dX_t)^2 \\ &+ \frac{\partial^2 \theta}{\partial x \partial y}(t, X_t, B_b - B_a)\ (dX_t)(dB_t) \\ &+ \frac{\partial \theta}{\partial t}(t, X_t, B_b - B_a)\ ds. \end{aligned} \tag{5.19}$$

The last corollary we present in this chapter will also be used in the proof of the main result of the next chapter.

Corollary 5.7. *Suppose that* $\theta(t,x,y) = x\sum_{n=0}^{\infty}\theta_n(t)y^n$*, where for each* $t \geq 0$*, the radius of convergence of* $\sum_{n=0}^{\infty}\theta_n(t)y^n$ *is infinite. Suppose also that* X_t *is given by*

$$X_t = \int_a^t g(s)\, dB_s + \int_a^t \gamma(s)\, ds,$$

where $g, \gamma \in L^2_{ad}(\Omega \times [a,b])$. *Then*

$$\begin{aligned}\theta(t, X_t, B_b - B_a) = \theta(t, X_a, B_b - B_a)& \\ &+ \int_a^t \frac{\partial \theta}{\partial x}(s, X_s, B_b)\, dX_s + \int_a^t \frac{\partial \theta}{\partial t}(s, X_s, B_b)\, ds \\ &+ \int_a^t \frac{\partial \theta}{\partial y}(s, X_s, B_b)\, (dX_s)\, (dB_s)\,. \end{aligned}$$

The above equation can be written in the differential form as follows

$$d\theta(t, X_t, B_b) = \frac{\partial \theta}{\partial x}(t, X_t, B_b)\, dX_t + \frac{\partial \theta}{\partial t}(t, X_t, B_b)\, dt + \frac{\partial \theta}{\partial y}(t, X_t, B_b)\, (dX_t)\, (dB_t)\,. \tag{5.20}$$

Proof. We will prove symbolically, that Equation (5.20) holds. Consider

$$\begin{aligned} d\left(\theta(t, X_t, B_b - B_a)\right) &= d\left(X_t \sum_{n=0}^{\infty} \theta_n(t)(B_b - B_a)^n\right) \\ &= \sum_{n=0}^{\infty} d\left(X_t \theta_n(t)(B_b - B_a)^n\right). \end{aligned}$$

Note that the function $x\theta_n(t)y^n$ satisfies the assumptions of Corollary 5.6, hence we have

$$\begin{aligned} d\left(\theta(t, X_t, B_b - B_a\right) &= \sum_{n=0}^{\infty} \Big[\theta_n(t)(B_b - B_a)^n \, dX_t + X_t \theta_n'(t)(B_b - B_a)^n \, dt \\ &\quad + X_t \theta_n(t) n (B_b - B_a)^{n-1}(dX_t)(dB_t)\Big] \\ &= \frac{\partial \theta}{\partial x}(t, X_t, B_b) \, dX_t + \frac{\partial \theta}{\partial t}(t, X_t, B_b) \, dt \\ &\quad + \frac{\partial \theta}{\partial y}(t, X_t, B_b) \left(dX_t\right) \left(dB_t\right). \end{aligned}$$

Therefore the corollary holds. □

Chapter 6
Stochastic Differential Equations

In this chapter, we will discuss linear stochastic differential equations (SDEs). Our goal is to generalize the following theorem known from the classical Itô theory (see, for example, [15, Section 11.1]).

Theorem 6.1. *Suppose that $\alpha, \beta \in L^2_{\mathrm{ad}}(\Omega \times [a,b])$. Then, the solution of the linear stochastic differential equation*

$$\begin{cases} dZ_t = \alpha(t) Z_t dB_t + \beta(t) Z_t dt \\ Z_a = z, \end{cases} \tag{6.1}$$

is given by

$$Z_t = z \exp\left\{ \int_a^t \alpha(s) dB_s + \int_a^t \left(\beta(s) - \frac{1}{2}\alpha(s)^2 \right) ds \right\}. \tag{6.2}$$

It is worth noting that notation in Equation (6.1) is only a symbolic notation and the real meaning of this equation is the following

$$Z_t = z + \int_a^t \alpha(s) Z_s dB_s + \int_a^t \beta(s) Z_s ds. \tag{6.3}$$

This is due to the fact that stochastic differentials, such as dB_t, have no meaning other than that expressed in Equation (6.3).

Equation (6.1) is a simplified version of a Theorem 11.1.1 from [15], that is, we do not include the drifts in Equation (6.1). Since processes α and β are adapted and $z \in \mathbb{R}$, the solution Z_t is an adapted stochastic process. We will extend this result for case with $z = p(B_b)$, which is clearly not adapted to the filtration $\{\mathcal{F}_t\}$.

Hence, we wish to find a solution to the following SDE

$$\begin{cases} dZ_t = \alpha(t) Z_t dB_t + \beta(t) Z_t dt \\ Z_a = p\left(B_b\right), \end{cases} \tag{6.4}$$

where $p(x)$ is a function with infinite radius of convergence of its Maclaurin series.

6.1 Motivational example

In this section we present an example that illustrates the method for obtaining a solution of Equation (6.4). Let us examine the above mentioned equation with $\alpha \equiv 1, \beta \equiv 0$ and $p(x) = x$. We will consider this problem on the interval $[0, 1]$. That is, we wish find a solution to

$$\begin{cases} dZ_t = Z_t dB_t, & t \in [0,1] \\ Z_0 = B_1. \end{cases} \tag{6.5}$$

The natural guess for the solution of Equation (6.5) is obtained by putting B_1 for z in Equation (6.2) to obtain

$$Z_t = B_1 \exp\left\{B_t - \frac{1}{2}t\right\}. \tag{6.6}$$

Note that since Z_t can be written as $Z_t = e^{-\frac{1}{2}t} e^{B_t} B(1)$, we can apply the Itô formula from Corollary 5.6 to check, if our guess in fact solves Equation (6.5). With $\theta(t, x, y) = y e^{x - \frac{1}{2}t}$, we have

$$\begin{aligned} d\theta\left(t, B_t, B_1\right) = B_1 e^{B_t - \frac{1}{2}t}\, dB_t + \frac{1}{2} B_1 e^{B_t - \frac{1}{2}t} \left(dB_t\right)^2 \\ + e^{B_t - \frac{1}{2}t} \left(dB_t\right) + \left(-\frac{1}{2} B_1 e^{B_t - \frac{1}{2}t}\right) dt. \end{aligned}$$

Now, since $(dB_t)^2 = dt$, we obtain

$$dZ_t = Z_t\, dB_t + e^{B_t - \frac{1}{2}t} dt. \tag{6.7}$$

Comparing Equations (6.5) and (6.7), we see that process Z_t proposed in Equation (6.6) is not a solution of Equation (6.5).

The reason for our failure in this approach is the fact that we do not account for the new factor in the equation, namely B_1. As we have seen, this introduces an extra term in the expression for dZ_t. To counteract this effect, we introduce a correction term as follows.

We will look for the solution of Equation (6.5) in the following form

$$Z_t = (B_1 - \xi(t)) \exp\left\{B_t - \frac{1}{2}t\right\}, \tag{6.8}$$

where $\xi(t)$ is a deterministic function.

The reason for this particular choice is simple. We see that the difference between Equations (6.7) and (6.5) is the term $\exp\{B_t - \frac{1}{2}t\}\,dt$, and to counteract this, we need to introduce another dt-term with the opposite sign. Looking at the Itô formula in Corollary 5.6, we see that we have to introduce a correction factor that depends only on t.

We use the Itô formula from Corollary 5.6 with $\theta(t,x,y) = (y-\xi(t))e^{x-\frac{1}{2}t}$, and

$$\begin{aligned}
\theta_t &= -\xi'(t)e^{x-\frac{1}{2}t} - \frac{1}{2}(y-\xi(t))e^{x-\frac{1}{2}t},\\
\theta_x &= (y-\xi(t))e^{x-\frac{1}{2}t},\\
\theta_{xx} &= (y-\xi(t))e^{x-\frac{1}{2}t},\\
\theta_{xy} &= e^{x-\frac{1}{2}t},
\end{aligned}$$

to obtain

$$\begin{aligned}
d\theta(t,B_t,B_1) =&\ (B_1-\xi(t))\,e^{B_t-\frac{1}{2}}\,dB_t + \frac{1}{2}\,(B_1-\xi(t))\,e^{B_t-\frac{1}{2}}\,dt\\
&+ e^{B_t-\frac{1}{2}t}\,dt - \left(\xi'(t)e^{B_t-\frac{1}{2}t} + \frac{1}{2}(B_1-\xi(t))e^{B_t-\frac{1}{2}t}\right)\,dt\\
=&\ (B_1-\xi(t))\,e^{B_t-\frac{1}{2}}\,dB_t + \left(e^{B_t-\frac{1}{2}t} - \xi'(t)e^{B_t-\frac{1}{2}t}\right)\,dt
\end{aligned}$$

So for $Z_t = \theta(t, B_t, B_1)$ to be the solution of Equation (6.5), function $\xi(t)$ has to satisfy the following differential equation

$$\begin{cases} \xi'(t) = 1, \quad t \in [0,1] \\ \xi(0) = 0. \end{cases} \tag{6.9}$$

Thus, if $\xi(t) = t$, process Z_t given in Equation (6.8) is a solution to stochastic differential equation (6.5), that is

$$Z_t = (B_1 - t) \exp\left\{ B_t - \frac{1}{2} t \right\}$$

solves Equation (6.5).

6.2 Linear SDEs with anticipating initial conditions

Before we present the main result of this Chapter, let us recall a key analytical tool that will be used in the construction of the solution of the linear stochastic differential equation with anticipating conditions.

Definition 6.2. We define the Schwartz class of rapidly decreasing functions to be

$$\mathcal{S}(\mathbb{R}) = \left\{ f \in C^\infty(\mathbb{R}) \,\middle|\, \sup_{x \in \mathbb{R}} \left| x^n \frac{d^m f(x)}{dx^m} \right| < \infty, \text{ for all } m, n \in \mathbb{N} \right\}. \tag{6.10}$$

Definition 6.3. For $f \in \mathcal{S}(\mathbb{R})$, we define the *Fourier transform* of the function $f(x)$ to be

$$\hat{f}(\zeta) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \zeta} \, dx. \tag{6.11}$$

Remark 6.4. It is a well-known fact, that the inverse to the Fourier transform is given by

$$f(x) = \int_{\mathbb{R}} \hat{f}(\zeta) e^{2\pi i x \zeta} \, d\zeta. \tag{6.12}$$

Also,

$$\left(\widehat{\frac{d}{dx} f(x)} \right)(\zeta) = 2\pi i \zeta \hat{f}(x). \tag{6.13}$$

Now we are ready to state and prove the main result.

Theorem 6.5. *Suppose that* $\alpha \in L^2([a,b])$ *and* $\beta \in L^2_{ad}(\Omega \times [a,b])$*. Suppose also that* $p \in \mathcal{M}^\infty \cap \mathcal{S}(\mathbb{R})$*. Then the stochastic differential equation*

$$\begin{cases} dZ_t = \alpha(t) Z_t \, dB_t + \beta(t) Z_t \, dt, \quad t \in [a,b] \\ Z_a = p(B_b - B_a), \end{cases} \tag{6.14}$$

has a solution given by

$$Z_t = \left(p(B_b - B_a) - \xi(t, B_b - B_a)\right) X_t, \tag{6.15}$$

where

$$\xi(t,y) = \int_a^t \alpha(s) p'\left(y - \int_s^t \alpha(u)\, du\right) ds,$$

and

$$X_t = \exp\left\{ \int_a^t \alpha(s)\, dB_s + \int_a^t \left(\beta(s) - \frac{1}{2}\alpha^2(s)\right) ds \right\}.$$

Proof. First, observe that X_t is a solution of a stochastic differential equation given by

$$\begin{cases} dX_t = \alpha(t) X_t \, dB_t + \beta(t) X_t \, dt, \quad t \in [a,b] \\ X_a = 1. \end{cases}$$

Consider

$$\begin{aligned} dZ_t &= d\Big[\big(p(B_b - B_a) - \xi(t, B_b - B_a)\big) X_t\Big] \\ &= d\big[p(B_b - B_a) X_t\big] - d\big[\xi(t, B_b - B_a) X_t\big], \end{aligned}$$

where

$$\xi(t,x) = \sum_{n=0}^\infty \xi_n(t) x^n, \qquad \text{for all } t \geq 0,\ x \in \mathbb{R}.$$

Using Corollaries 5.6 and 5.7 we obtain

$$\begin{aligned} dZ_t = {} & p(B_b - B_a) dX_t + p'(B_b - B_a)(dX_t)(dB_t) \\ & - \left[\frac{\partial \xi}{\partial t}(t, B_b - B_a) X_t \, dt + \xi(t, B_b - B_a)\, dX_t + \frac{\partial \xi}{\partial y}(t, B_b - B_a)(dX_t)(dB_t)\right] \end{aligned}$$

$$= \big[p(B_b - B_a) - \xi(t, B_b - B_a)\big]\, dX_t$$
$$+ \Big[p'(B_b - B_a)(dX_t)(dB_t) - \frac{\partial \xi}{\partial t}(t, B_b - B_a) X_t\, dt$$
$$- \frac{\partial \xi}{\partial y}(t, B_b - B_a)(dX_t)(dB_t)\Big].$$

So for Z_t to be a solution of Equation (6.14), we need

$$p'(B_b - B_a)(dX_t)(dB_t) - \frac{\partial \xi}{\partial t}(t, B_b - B_a) X_t\, dt - \frac{\partial \xi}{\partial y}(t, B_b - B_a)(dX_t)(dB_t) = 0 \quad (6.16)$$

for all $t \in [a, b]$.. Note that

$$(dX_t)(dB_t) = \big(\alpha(t) X_t\, dB_t + \beta(t) X_t\, dt\big)(dB_t)$$
$$= \alpha(t) X_t\, dt. \quad (6.17)$$

Thus, putting together Equations (6.16) and (6.17) we get

$$p'(B_b - B_a)\alpha(t) X_t\, dt - \frac{\partial \xi}{\partial t}(t, B_b - B_a) X_t\, dt - \frac{\partial \xi}{\partial y}(t, B_b - B_a)\alpha(t) X_t\, dt = 0.$$

Or, equivalently

$$\big[p'(B_b - B_a)\alpha(t) - \frac{\partial \xi}{\partial t}(t, B_b - B_a) - \frac{\partial \xi}{\partial y}(t, B_b - B_a)\alpha(t)\big] X_t\, dt = 0.$$

Hence it is enough to find $\xi(t, y)$ such that

$$\begin{cases} p'(y)\alpha(t) - \frac{\partial \xi}{\partial t}(t, y) - \frac{\partial \xi}{\partial y}(t, y)\alpha(t) = 0, \quad t \in [a, b] \\ \xi(0, y) = 0. \end{cases} \quad (6.18)$$

Thus the problem of finding a solution to the stochastic differential equation (6.14) has been reduced to that of finding a solution to the deterministic partial differential equation (6.18). In order to solve Equation (6.18), we will apply the Fourier transform to obtain

$$\widehat{p'}(\zeta)\alpha(t) - \frac{\partial}{\partial t}\widehat{\xi}(t, \zeta) - 2\pi i \zeta \widehat{\xi}(t, \zeta)\alpha(t) = 0. \quad (6.19)$$

Note that Equation (6.19) is an ordinary differential equation in t, with an integrating factor

$$\exp\left\{2\pi i\zeta \int_a^t \alpha(s)\,ds\right\}.$$

Hence Equation (6.19) is equivalent to

$$\frac{\partial}{\partial t}\left(\widehat{\xi}(t,\zeta)\exp\left\{2\pi i\zeta\int_a^t \alpha(s)\,ds\right\}\right) = \widehat{p'}(\zeta)\alpha(t)\exp\left\{2\pi i\zeta\int_a^t \alpha(s)\,ds\right\}. \quad (6.20)$$

Integration with respect to t of both sides of Equation (6.20) yields

$$\widehat{\xi}(t,\zeta)\exp\left\{2\pi i\zeta\int_a^t \alpha(s)\,ds\right\} = \widehat{p'}(\zeta)\int_a^t \alpha(s)\exp\left\{2\pi i\zeta\int_a^s \alpha(u)\,du\right\}\,ds + \widehat{C}(\zeta). \quad (6.21)$$

Thus, the Fourier transform of function $\xi(t,y)$, that is a solution of Equation (6.18), is given by

$$\begin{aligned}\widehat{\xi}(t,\zeta) = \widehat{p'}(\zeta)\int_a^t \alpha(s)\exp\left\{-2\pi i\zeta\int_s^t \alpha(u)\,du\right\}\,ds \\ + \widehat{C}(\zeta)\exp\left\{-2\pi i\zeta\int_a^t \alpha(s)\,ds\right\}.\end{aligned} \quad (6.22)$$

Now, we apply the inverse Fourier transform to get

$$\begin{aligned}\xi(t,y) &= \int_{\mathbb{R}} \widehat{p'}(\zeta)\int_a^t \alpha(s)\exp\left\{-2\pi i\zeta\int_s^t \alpha(u)\,du\right\}\,ds\exp\{-2\pi iy\zeta\}\,d\zeta \\ &\quad + \int_{\mathbb{R}} \widehat{C}(\zeta)\exp\left\{-2\pi i\zeta\int_a^t \alpha(s)\,ds\right\}\exp\{2\pi iy\zeta\}\,d\zeta \\ &= \int_a^t \alpha(s)\int_{\mathbb{R}} \widehat{p'}(\zeta)\exp\left\{a\pi i\zeta\left(y-\int_s^t \alpha(u)\,du\right)\right\}\,d\zeta\,ds \\ &\quad + \int_{\mathbb{R}} \widehat{C}(\zeta)\exp\left\{a\pi i\zeta\left(y-\int_s^t \alpha(u)\,du\right)\right\}\,d\zeta \\ &= \int_a^t \alpha(s)p'\left(y-\int_s^t \alpha(u)\,du\right)\,ds + C\left(y-\int_a^t \alpha(s)\,ds\right).\end{aligned}$$

Using the initial condition from Equation (6.18), we see that $C(y)\equiv 0$. Hence the proof is complete. □

6.3 Examples

Below we present several examples of stochastic differential equations and their solutions. Each of the examples is given with two different initial conditions, namely deterministic and anticipating.

Example 6.6 (Adapted)**.** Equation

$$\begin{cases} dZ_t = Z_t dB_t + Z_t dt \\ Z_0 = z \end{cases}$$

has solution given by

$$Z_t = z \exp\left\{B_t + \frac{1}{2}t\right\}.$$

Example 6.7 (Anticipating, compare with Example 6.6)**.** Equation

$$\begin{cases} dZ_t = Z_t dB_t + Z_t dt \\ Z_0 = B_1 \end{cases}$$

has solution given by

$$Z_t = (B_1 - t)\exp\left\{B_t + \frac{1}{2}t\right\}.$$

Example 6.8 (Anticipating, compare with Example 6.6)**.** Equation

$$\begin{cases} dZ_t = Z_t dB_t + Z_t dt \\ Z_0 = e^{B_1} \end{cases}$$

has a solution given by

$$Z_t = e^{B_1}(1 - e^{-t})\exp\left\{B_t - \frac{1}{2}t\right\}$$

Example 6.9 (Adapted)**.** Equation

$$\begin{cases} dZ_t = \alpha(t) Z_t dB_t + \beta(t) Z_t dt \\ Z_0 = z \end{cases}$$

has solution given by

$$Z_t = z \exp \left\{ \int_0^t \alpha(s) dB_s + \int_0^t \left(\beta(s) - \frac{1}{2}\alpha(s)^2 \right) ds \right\}.$$

Example 6.10 (Anticipating, compare with Example 6.9)**.** Equation

$$\begin{cases} dZ_t = \alpha(t) Z_t dB_t + \beta(t) Z_t dt \\ Z_0 = B_1 \end{cases}$$

has solution given by

$$Z_t = \left(B_1 - \int_0^t \alpha(s) ds \right) \exp \left\{ \int_0^t \alpha(s) dB_s + \int_0^t \left(\beta(s) - \frac{1}{2}\alpha(s)^2 \right) ds \right\}.$$

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Vita

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