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Capturing elements in matroid minors

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CAPTURING ELEMENTS IN MATROID MINORS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
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in partial fulfillment of the
requirements for the degree of
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in

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by

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Abstract

In this dissertation, we begin with an introduction to a matroid as the natural generalization of independence arising in three different fields of mathematics. In the first chapter, we develop graph theory and matroid theory terminology necessary to the topic of this dissertation. In Chapter 2 and Chapter 3, we prove two main results.

A result of Ding, Oporowski, Oxley, and Vertigan reveals that a large 3-connected matroid M has unavoidable structure. For every n exceeding two, there is an integer $f(n)$ so that if $|E(M)|$ exceeds $f(n)$, then M has a minor isomorphic to the rank- n wheel or whirl, a rank- n spike, the cycle or bond matroid of $K_{3,n}$, or $U_{2,n}$ or $U_{n-2,n}$. In Chapter 2, we build on this result to determine what can be said about a large structure using a specified element e of M . In particular, we prove that, for every integer n exceeding two, there is an integer $g(n)$ so that if $|E(M)|$ exceeds $g(n)$, then e is an element of a minor of M isomorphic to the rank- n wheel or whirl, a rank- n spike, the cycle or bond matroid of $K_{1,1,1,n}$, a specific single-element extension of $M(K_{3,n})$ or the dual of this extension, or $U_{2,n}$ or $U_{n-2,n}$.

In Chapter 3, we consider a large 3-connected binary matroid with a specified pair of elements. We extend a corollary of the result of Chapter 2 to show the following result for any pair $\{x, y\}$ of elements of a 3-connected binary matroid M . For every integer n exceeding two, there is an integer $h(n)$ so that if $|E(M)|$ exceeds $h(n)$, then x and y are elements of a minor of M isomorphic to the rank- n wheel, a rank- n binary spike with a tip and a cotip, or the cycle or bond matroid of $K_{1,1,1,n}$.

Chapter 1

Preliminaries

This dissertation concerns matroids, and the matroid theory terminology will follow Oxley [13]. A matroid is a set of elements with some subsets of those elements described as independent. This combinatorial object generalizes the idea of independence arising from three different fields: graph theory, linear algebra, and geometry.

After the following definition of a matroid, taken from Oxley's book [13, p. 7], we will discuss how matroids arise in these different fields and how three ideas of independence converge. A *matroid* M is an ordered pair (E, \mathcal{I}) consisting of the finite set E and a collection \mathcal{I} of subsets of E satisfying the following conditions:

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.

(I3) If I_1 and I_2 are in \mathcal{I} and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ so that $I_1 \cup e \in \mathcal{I}$.

The set E is the *ground set* of M ; the set \mathcal{I} is the set of *independent sets* of M .

Preliminary to discussing matroids in their full generality, we introduce graphs and matroids arising from graphs.

1.1 Definitions - Graphs

The graph terminology used here will follow Diestel [6], with the following exception. The objects that Diestel refers to as graphs and multigraphs will be called simple graphs and graphs, respectively, in this dissertation. Therefore, a *graph* G is a pair (V, E) , where V is a non-empty set of *vertices* and E is a labelled multiset whose elements are unordered pairs

of elements in V . These pairs are called *edges*. We will assume V and E to be finite sets and define $V(G)$ to be V and $E(G)$ to be E .

Let $G = (V, E)$ be a graph. Let e be an edge $\{u, v\}$ for some vertices u and v . If $u = v$, then the edge e is a *loop*. Otherwise, the edge e is *between* u and v , and u and v are *endpoints* of e . Equivalently, the edge e is *incident* with u and v . Any two distinct vertices that are endpoints of an edge are said to be *adjacent*. If u and v are distinct vertices that are both the endpoints of two distinct edges, e and f , then edges e and f are *parallel*. The graph G is a *simple graph* if it has no loops or parallel edges.

Let $P = (V, E)$ be a graph with $V = \{v_0, v_1, \dots, v_k\}$ and $E = \{e_1, e_2, \dots, e_k\}$ with $e_i = \{v_{i-1}, v_i\}$ for all $i \in [k]$. Here and throughout this dissertation, we use $[n]$ to mean $\{1, 2, \dots, n\}$. Then P is a *path* with *endpoints* v_0 and v_k . We will refer to such a path as a *path from* v_0 *to* v_k or a *path between* v_0 *and* v_k . The *length of a path* is the number of edges in the path. For example, P has length k . If we add the edge $e_{k+1} = \{v_k, v_0\}$ to path P , the resulting graph is a *cycle*. The *length of a cycle* is the number of edges or, equivalently, the number of vertices, in the cycle. For example, the cycle we have constructed has length $k + 1$.

Again, let $G = (V, E)$ be a graph with $u, v \in V$ and $e \in E$ with $e = \{u, v\}$. We *delete edge* e from G by removing this edge from the edge set of G . The resulting graph $G' = (V, E - e)$ is denoted $G \setminus e$. We can also *delete vertex* v from the graph G by deleting all edges incident with v and removing v from the vertex set of G .

A graph G is *connected* if there is a path in G between every pair of vertices of G . Further, the graph G is *k -connected* if $|V(G)| \geq k + 1$ and if, for every set $U \subseteq V(G)$ with $|U| < k$, deleting all the vertices of U from G results in a connected graph.

A graph produced by deleting some, possibly empty, sets of edges and vertices from G is a *subgraph of* G . Further, the *induced subgraph* of G by a set $U \subseteq V$ or a set $X \subseteq E$ is the subgraph of G obtained by deleting, in the first case, every vertex of $V - U$ and, in the

second case, every edge of $E - X$ and every vertex that is not incident with at least one edge of X . If a graph G has no subgraph that is a cycle then G is *acyclic*.

Another important graph operation is edge contraction. Consider graph G with an edge $e = \{u, v\}$. This definition appears in Diestel's book [6, p. 29] as contraction in a multigraph. We *contract* e from G by identifying u and v and deleting edge e . This operation yields a vertex incident with every edge, other than e , that was incident with u or v . The resulting graph is denoted G/e .

Any graph obtained from G using edge contraction or vertex or edge deletion is a *minor* of G . In addition, the graph G is viewed as a minor of itself. One important minor of G is the *simplification* of G , denoted $\text{si}(G)$, and is found by deleting a minimal set of edges of G so that the resulting graph is simple.

1.2 Independence in a Graph

Let $G = (V, E)$ be a graph. Using the edge set of G as the elements of a matroid, we associate independence in a matroid with acyclicity in a graph in the following way. Let $M = (E(G), \mathcal{I})$ where $X \in \mathcal{I}$ if and only if $X \subseteq E$ and X does not contain the edge set of any cycle of G . It is easy to see that conditions (I1) and (I2) are satisfied, and a routine exercise shows that (I3) is also satisfied. Therefore, M is a matroid. We refer to M as $M(G)$, the *cycle matroid* of G .

In a matroid (E, \mathcal{I}) , a *circuit* is a minimal set C of elements of E so that $C \notin \mathcal{I}$. If G is a graph, then the circuits of the matroid $M(G)$ are the edge sets of the cycles of G .

1.3 A Matrix from a Graph

Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \dots, v_n\}$ and $E = \{e_1, e_2, \dots, e_m\}$. The *incidence matrix* A of G is $(b_{ij})_{n \times m}$ with

$$b_{ij} = \begin{cases} 1 & \text{if } e_j \text{ is not a loop and } v_i \in e_j; \\ 0 & \text{otherwise.} \end{cases}$$

Consider the column vectors of A , which are labelled by the edges of G , as vectors in a binary vector space. If e_i is a loop, then the associated column vector is the zero vector. It is not difficult to show C is a cycle of G if and only if the vectors labelled by the edges of C are a minimal set of column vectors of A that sum to the zero vector in the binary vector space.

A set S of vectors in a vector space is *linearly independent* if S does not contain a vector of all zeros and if no vector $v \in S$ is a linear combination of vectors of $S - v$. The set is *linearly dependent* if it is not linearly independent. Thus, we test a set X of edges of a graph G for acyclicity by testing the column vectors of the incidence matrix associated with G for linear independence. A set X of elements of $M(G)$ is independent if and only if these column vectors are linearly independent.

It is natural, therefore, to generalize independence in a graphic matroid to linear independence in matrices. We are now ready to define a matroid arising from a matrix.

1.4 A Matroid from a Matrix

Let A be an $n \times m$ matrix over a field \mathbb{F} . Let E be an m -element set whose members label the columns of A . Then the elements of E label elements of the vector space $V(m, \mathbb{F})$. Let \mathcal{I} be the set of subsets of E that are linearly independent in $V(m, \mathbb{F})$. Again, it is easy to see that (I1) and (I2) are satisfied by \mathcal{I} . Given two sets $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$, the dimension of the subspace of $V(m, \mathbb{F})$ spanned by I_1 is less than the dimension of the subspace spanned by I_2 . Using this, it is easy to check that (I3) holds. Thus $M = (E, \mathcal{I})$ is a matroid. In particular, the matroid M is the *vector matroid of A* , denoted $M[A]$.

The matroid $M[A]$ is said to be *representable* over the field \mathbb{F} . A matroid that is representable over the field of two elements is said to be *binary*. Chapter 3 of this dissertation deals specifically with binary matroids.

The *projective geometry associated with $V(m, \mathbb{F})$* is denoted $PG(m - 1, \mathbb{F})$ and consists of a set P of points, a set of lines disjoint from P , and an incidence relation between points

and lines [13, p. 160]. The *points* and *lines* are precisely the 1- and 2- dimensional subspaces of $V(m, \mathbb{F})$, and *incidence* is determined by set inclusion. Thus, if $M[A] = (E, \mathcal{I})$, then we can think of the elements of E that do not label the zero vector as labelling some subset of points in $PG(m-1, \mathbb{F})$. Therefore, it is natural to generalize linear independence to a type of independence, which we will define next, in a set of points.

1.5 A Matroid from a Set of Points

A multiset $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k\}$ of elements from $V(m, \mathbb{F})$ is *affinely dependent* if $k \geq 1$ and there are elements a_1, a_2, \dots, a_k of \mathbb{F} that are not all zero such that $\sum_{i=1}^k a_i \underline{v}_i = 0$ and $\sum_{i=1}^k a_i = 0$ [13, p. 32]. A multiset of elements from $V(m, \mathbb{F})$ that is not affinely dependent is *affinely independent*. A set $X \subseteq V(m, \mathbb{F})$ that is affinely dependent is certainly linearly dependent. Therefore, a set $X \subseteq V(m, \mathbb{F})$ that is linearly independent is certainly affinely independent, and affine independence generalizes linear independence. Let E be a set labelling a multiset of elements from $V(m, \mathbb{F})$. Let \mathcal{I} be the collection of subsets X of E so that X labels an affinely independent subset of $V(m, \mathbb{F})$. It is not difficult to show that (E, \mathcal{I}) is a matroid [13, p. 32].

Affine independence generalizes to a set of points that are not necessarily a subset of a projective geometry. While this generalization can be made for higher dimension, the practice common among matroid theorists is to consider this independence in 2- and 3- dimensions. Here, we will explicitly state the rules for generating a matroid from a set of points in a plane. We will require every element to label a distinct point in a plane, so we will not allow two points to be copunctual. Given a set of distinct points labelled by E in a plane, and a set of (possibly curved) lines, we say a subset I of E is in \mathcal{I} if $|I| \leq 3$ and I does not label 3 points incident with any one line. If every two distinct lines meet in at most one point, then (E, \mathcal{I}) is a matroid [13, p. 35].

Representable matroids arise from matrices over some field and can be studied using tools from linear algebra. Graphic matroids arise from graphs and can be studied using

techniques from graph theory. But these types of matroids comprise a small part of the set of all matroids, and a matroid that is not known to be graphic or representable may be viewed geometrically as dimensions allow. The complexity and variety of general matroids is, in part, due to the fact that there is no one compact representation for a matroid. The most general matroid is, as defined at the start of this chapter, a set E of elements and a collection of subsets of E obeying (I1), (I2), and (I3).

1.6 Definitions - Matroids

In this section, we introduce some more matroid terminology, including element deletion and contraction. Let $M = (E, \mathcal{I})$ be a matroid, and let $e \in E$. If $\{e\} \notin \mathcal{I}$, then e is a *loop* of M . The *deletion* of e from M , denoted $M \setminus e$, results in the matroid $(E - e, \{I \in \mathcal{I} \mid e \notin I\})$. The *restriction* of M to $X \subseteq E$ is $M \setminus (E - X)$ and is denoted $M|X$. The *contraction* of e from M , denoted M/e , is just the deletion of e when e is a loop. Otherwise, $M/e = (E - e, \{I \subseteq E - e \mid I \cup \{e\} \in \mathcal{I}\})$. In both cases, M/e is easily shown to be a matroid. A matroid N is a *minor* of M if and only if N can be produced from M by a possibly empty series of deletions and contractions. It is important to note that if G is a graph with edge e , then $M(G) \setminus e = M(G \setminus e)$ and $M(G)/e = M(G/e)$.

In a matroid (E, \mathcal{I}) , if $e, f \in E$ so that $\{e\} \in \mathcal{I}$ and $\{f\} \in \mathcal{I}$, then these elements are *parallel* if $\{e, f\} \notin \mathcal{I}$. A matroid is *simple* if it has no loops or parallel elements. One important minor of M is the *simplification* of M , denoted $\text{si}(M)$, obtained by deleting a minimal set of elements so that the resulting matroid is simple.

The matroids $M = (E_1, \mathcal{I}_1)$ and $N = (E_2, \mathcal{I}_2)$ are *isomorphic* if there is a bijection ψ from E_1 onto E_2 so that $I \in \mathcal{I}_1$ if and only if $\psi(I) \in \mathcal{I}_2$. If M has a minor isomorphic to N , then M has an *N -minor*.

Let $M = (E, \mathcal{I})$ be a matroid and let B be a largest set of \mathcal{I} . Then $|B|$ is the *rank* of M , denoted $r(M)$. The rank of a set $X \subseteq E$, denoted $r_M(X)$, is $|I|$ where I is a largest subset of X so that $I \in \mathcal{I}$. Let the set of all members of \mathcal{I} with size $r(M)$ be \mathcal{B} . A member

$B \in \mathcal{B}$ is a *basis* of M and \mathcal{B} is the *set of bases* of M . The matroid M can be characterized by the ordered pair (E, \mathcal{B}) . The *dual* of M , denoted M^* , is the matroid with ground set E having $\{E - B \mid B \in \mathcal{B}\}$ as its set of bases. Several properties of a matroid are preserved under duality; that is, if M has that property, then so does M^* . For example, if M can be represented by a matrix over a field \mathbb{F} , then so can M^* .

A matroid $M = (E, \mathcal{I})$ is *connected* or *2-connected* if, for every pair $\{e, f\}$ of distinct elements of M , there is a circuit C of M so that $\{e, f\} \subseteq C$. Equivalently, M is connected if $r_M(X) + r_M(Y) - r(M) \geq 1$ for any partition of the ground set (X, Y) with $|X|, |Y| \geq 1$. The matroid M is *3-connected* if $r_M(X) + r_M(Y) - r(M) \geq 2$ for any partition of the ground set (X, Y) with $|X|, |Y| \geq 2$. While this dissertation concerns 2- and 3-connected matroids, it is easy to see how this definition of connectivity extends to k -connected matroids. One important consequence of 3-connectivity is that, for any element e of a 3-connected matroid M , the matroids M/e and $M \setminus e$ are both 2-connected.

Connectivity in matroid theory is not strictly a generalization of connectivity in graph theory. There are examples of highly connected graphs whose cycle matroids are not even 3-connected. One of the motivations behind the different view of connectivity is that a matroid M is 2- or 3- connected if and only if M^* is 2- or 3- connected, respectively; that is, connectivity is another property that is preserved under duality.

The dual of a graphic matroid is not necessarily graphic. If a matroid M is the cycle matroid of a graph G , then the dual matroid, the *bond matroid* of G , is said to be *cographic*. We denote the bond matroid of G by $M^*(G)$.

Let $M = (E, \mathcal{I})$ be a matroid with $X \subseteq E$. We construct the *closure* of X , denoted $\text{cl}(X)$, by adding to X all the elements of M that are spanned by elements of X , that is, $\text{cl}(X) = X \cup \{e \in E - X : r_M(X \cup e) = r_M(X)\}$. If $X = \text{cl}(X)$, then X is a *closed set* or a *flat*. A *hyperplane* of M is any flat with rank $r(M) - 1$. Some readers may recognize the object we have just defined by the alternate name of copoint.

1.7 Some Important Matroids

The matroids defined in this section feature prominently in this dissertation.

1.7.1 The Rank- n Wheel and Whirl

For an integer $n \geq 3$, let G be the graph with vertex set $\{v_1, v_2, \dots, v_n, h\}$ so that $G[\{v_1, v_2, \dots, v_n\}]$ is a cycle, and h is incident with every other vertex. Then G is a *wheel* or an *n -spoked wheel* and is denoted \mathcal{W}_n . We will refer to the cycle with vertices $\{v_1, v_2, \dots, v_n\}$ as the *rim* of the wheel, and each edge of that cycle as a *rim edge*. Vertex h is the *hub* of the wheel, and every edge incident with h is a *spoke edge*. Figure 1.1 is the graph of a wheel with four spokes.

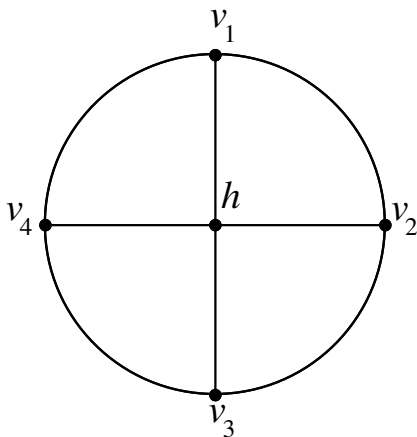


FIGURE 1.1. The graph of \mathcal{W}_4 .

The cycle matroid, $M(\mathcal{W}_n)$, of a wheel has rank n . The matroid $M(\mathcal{W}_4)$ has rank 4 and has a 3-dimensional geometric representation (see Figure 1.2).

This matroid can be represented by a graph and so can be represented by the incidence matrix of the graph. The incidence matrix has $n + 1$ rows, but has rank n . In fact, $M(\mathcal{W}_n)$ is represented by a matrix obtained from the incidence matrix by deleting any row. Arguments in this dissertation rely on the fact that $M(\mathcal{W}_n)$ can be represented by a graph, by an $n \times 2n$ matrix, and by a geometric representation.

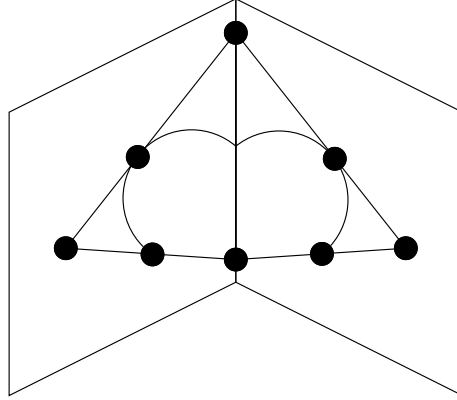


FIGURE 1.2. The geometry of $M(\mathcal{W}_4)$. The straight and curved lines are rank-2 flats, and the four points on the curved lines lie in a plane.

The matroid $M(\mathcal{W}_n)$ is related to the rank- n whirl. In $M(\mathcal{W}_n)$, the set R of rim edges of the graph \mathcal{W}_n is a circuit. This circuit is also a hyperplane of the matroid. For any matroid M_1 with a set \mathcal{B} of bases and a circuit-hyperplane X , there is a matroid M_2 with ground set $E(M_1)$ and $\mathcal{B} \cup \{X\}$ as its set of bases. We say that M_2 is obtained from M_1 by *relaxing* X . In the geometry shown in Figure 1.2, if we move one of the points on one of the curved lines off that curve, leaving it on the straight line containing it, then those four points no longer form a circuit. The resulting matroid is the rank-4 whirl. More generally, relaxing the rim R in $M(\mathcal{W}_n)$ produces the *rank- n whirl*, denoted \mathcal{W}^n .

1.7.2 The Rank- n Spike

There are four types of spikes. Here, we define a spike with a tip (see three examples in Figure 1.3) and define the other three types of spikes in relation to this type of spike.

Let r be an integer so that $r \geq 3$. A matroid M is a *rank- r spike with tip t* if and only if M has the following properties [13, p.41]:

1. $E(M)$ is the union of r lines L_1, L_2, \dots, L_r each of which is a 3-element circuit containing the point t ;
2. for every $k \in [r - 1]$, the union of any k of L_1, L_2, \dots, L_r has rank $k + 1$; and
3. $r(L_1 \cup L_2 \cup \dots \cup L_r) = r$.

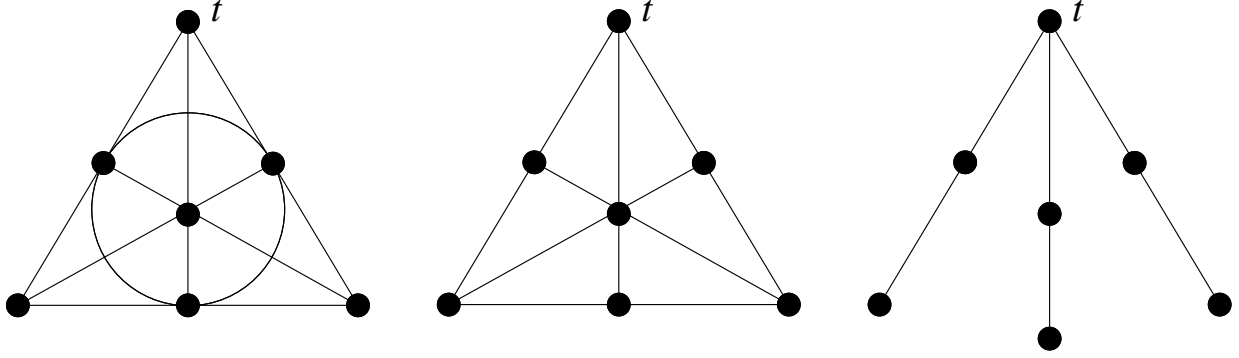


FIGURE 1.3. Three examples of a rank-3 spike with a tip. Straight and curved lines indicate rank-2 flats. These matroids are called the Fano matroid (left), the non-Fano matroid (center), and the rank-3 free spike with tip (right).

Let M be a rank- r spike with a tip t . If we delete the tip from M , then we obtain a *rank- r tipless spike*. If, instead, x is an element of M other than the tip, then $M \setminus x$ is a *rank- r spike with a tip and a cotip*, with t as the tip and the third element on the line of M spanned by $\{t, x\}$ as the *cotip*. We denote by \mathcal{T}_r the set of all rank- r spikes having a tip and a cotip. It is easy to see that if N is a member of \mathcal{T}_r , then N^* is also in \mathcal{T}_r . Moreover, if N has tip t and cotip c , by adding an element y on the line $\{t, c\}$ of N so that y is not copunctual with t or c , we obtain a rank- r spike with tip t . Finally, $M \setminus \{x, t\}$ is a *rank- r tipless spike with a cotip*. In that case, $(M \setminus \{x, t\})^*$ is a rank- $(r - 1)$ spike with a tip.

It is well known that, for all $r \geq 3$, there is a unique rank- r binary spike with a tip, and there is a unique rank- r binary spike with a tip and a cotip. Here, we use I_n as the rank- n identity matrix and J_n to denote the $n \times n$ matrix of all ones. Let $\underline{1}$ be the column vector of all ones. Let A_n be the binary matrix that is obtained from $J_n - I_n$ by replacing the 0 in the bottom right corner with 1. The rank- r binary spike with a tip and the rank- r binary spike with a tip and a cotip are the vector matroids of the binary matrices $[I_r | J_r - I_r | \underline{1}]$ and $[I_r | A_r]$, respectively.

1.7.3 Uniform Matroids

Let M be a matroid with n elements and let $r(M) = r$. If every set of r elements is a basis, then M is the *n -element, rank- r uniform matroid*, denoted $U_{r,n}$. In this matroid, every set

of $r + 1$ elements is a circuit. It is easy to see that if $M \cong U_{r,n}$, then $M^* \cong U_{n-r,n}$. The matroid $U_{2,n}$ is referred to as an n -point line. The dual of this matroid, $U_{n-2,n}$, also appears in this dissertation.

1.7.4 The Cycle and Bond Matroids of $K_{3,n}$ and Two Related Matroids

Let U and V be two disjoint sets of vertices. Then the graph $(U \cup V, \{\{u, v\} : u \in U, v \in V\})$ is a *complete bipartite graph*, denoted $K_{|U|,|V|}$. Let n be a positive integer. The graph $K_{3,n}$ is

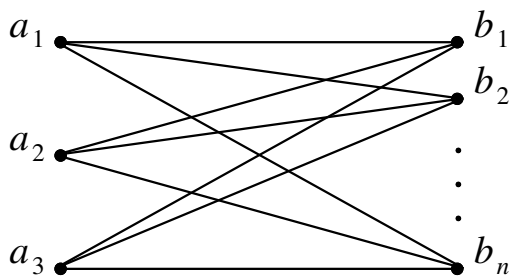


FIGURE 1.4. The graph of $K_{3,n}$.

shown in Figure 1.4. In this graph, without ambiguity, we label the edges by their endpoints, for example, an edge $\{a_1, b_1\}$ would be labelled a_1b_1 . The cycle matroid of $K_{3,n}$ has rank $n + 2$. We can illustrate $M(K_{3,n})$ as a set of rank-3 pages attached across a common line where each added page increases the rank by one (see Figure 1.5).

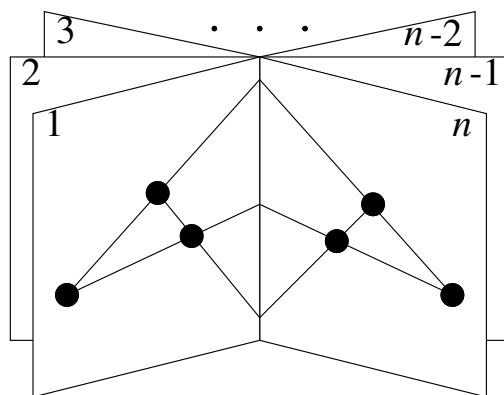


FIGURE 1.5. An illustration of the geometry of $M(K_{3,n})$.

Since $M(K_{3,n})$ is graphic, we can represent this matroid as a geometry, as a graph, or as a binary matrix $[I_n|D_n]$, where D_n is a particular $n \times n$ binary matrix.

The dual matroid, $M^*(K_{3,n})$, is the bond matroid of a graph and can be represented by that graph, by the binary matrix $[D_n^T | I_n]$, or by a geometry. We develop that geometry now. This matroid has rank $2n - 2$. With $K_{3,n}$ labelled as in Figure 1.4, for each i in $[n - 1]$, the set $\{a_1b_i, a_2b_i, a_3b_i\}$ is a 3-point line L_i in $M^*(K_{3,n})$. The set $L_1 \cup L_2 \cup \dots \cup L_{n-1}$ has rank $2n - 2$. View the points of these lines as a subset, S , of $PG(2n - 3, 2)$. The points $a_1b_n, a_2b_n,$ and a_3b_n can be uniquely added to S so that $S \cup \{a_1b_n, a_2b_n, a_3b_n\}$ is a subset of $PG(2n - 3, 2)$ and $\{a_i b_1, a_i b_2, \dots, a_i b_n\}$ is a circuit C_i for all $i \in [3]$. When this is done, it is not hard to show that $\{a_1b_n, a_2b_n, a_3b_n\}$ is also a 3-point line. The matroid of this set of points is $M^*(K_{3,n})$. While this matroid has rank $2n - 2$, we illustrate this geometry as shown in Figure 1.6. In this illustration, straight lines indicate rank-2 flats and the three sets of

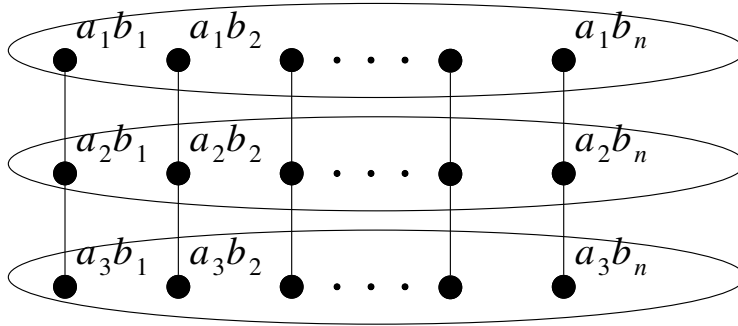


FIGURE 1.6. An illustration of the geometry of $M^*(K_{3,n})$ which has rank $2n - 2$.

circled elements are $C_1, C_2,$ and C_3 . This illustration will be useful in this dissertation even though these objects all appear to lie in a plane. For convenience, we will often replace the three ovals with three straight lines.

The graph of $K_{1,1,1,n}$ is the graph of $K_{3,n}$ shown in Figure 1.4 with the three edges $a_1a_2, a_2a_3,$ and a_1a_3 added. The matroid $M(K_{1,1,1,n})$ is $M(K_{3,n})$ with three elements added as illustrated in Figure 1.7.

The matroid $M(K_{1,1,1,n})$ has n copies of $M(K_4)$ as restrictions, with one 3-point line common to all these restrictions. Using the labelling developed here, this is the line $\{a_1a_2, a_2a_3, a_1a_3\}$. We call this common line the *spine* of $M(K_{1,1,1,n})$. An element p can be added to

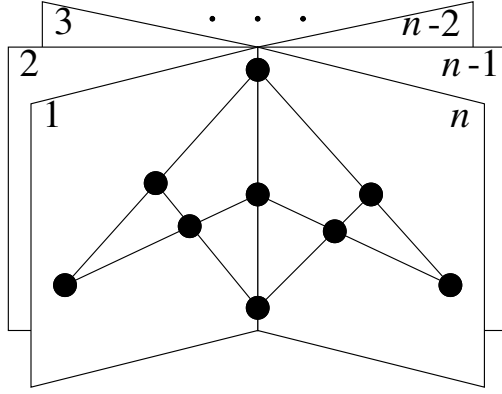


FIGURE 1.7. An illustration of the geometry of $M(K_{1,1,1,n})$.

this spine so that p avoids the three elements already there. Formally, the process of adding p is referred to as *freely adding p* to the flat $\{a_1a_2, a_2a_3, a_1a_3\}$ of $M(K_{1,1,1,n})$ (see [13, p. 270]). Throughout this dissertation, $M(K_{3,n})^+$ will refer to the matroid obtained by adding an element p to $M(K_{1,1,1,n})$ in this manner and then deleting every other element from the spine (see Figure 1.8). Notice that deleting p from $M(K_{3,n})^+$ produces $M(K_{3,n})$.

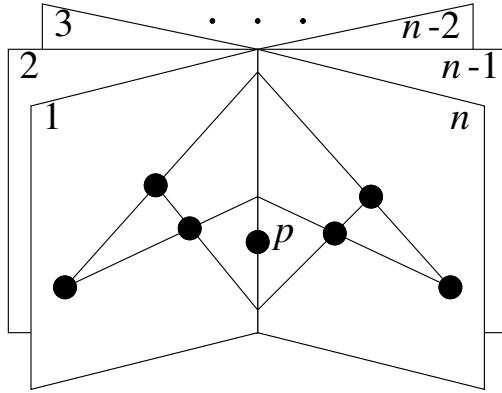


FIGURE 1.8. An illustration of the geometry of $M(K_{3,n})^+$.

1.8 Unavoidable Minors

The topic of this dissertation is capturing elements in matroid minors. The minors we are interested in are known as unavoidable minors. This section presents the results on unavoidable minors that are the foundation of the main proofs in this dissertation. These results make it clear why these minors are called unavoidable.

Graphic matroids are a small subset of matroids, but they are well-understood structures. Often, the behavior of graphs gives intuition for the behavior of binary or even more general matroids. In graph theory, Ramsey properties deal with the appearance of a certain graph as a minor of a very large graph. In this way, Ramsey properties describe structure that arises when a graph is very large. The following theorem is a well-known Ramsey property.

Theorem 1.8.1. *For every positive integer n , there is a number $g_2(n)$ so that every 2-connected graph with at least $g_2(n)$ vertices has a cycle of length n or $K_{2,n}$ as a minor.*

A 1993 result of Oporowski, Oxley, and Thomas [12] shows that every sufficiently large 3-connected graph has one of two large minors.

Theorem 1.8.2. *For any positive integer n , there is a number $g_3(n)$ so that every 3-connected graph with at least $g_3(n)$ vertices has \mathcal{W}_n or $K_{3,n}$ as a minor.*

These Ramsey properties are related to unavoidable-minor results in matroids. Unavoidable-minor results seek structure that arises when a matroid is very large. The following is a result of Lemos and Oxley [11].

Theorem 1.8.3. *For every positive integer n , every 2-connected matroid M with more than $\frac{1}{2}(n-1)^2$ elements has a minor isomorphic to $U_{1,n}$ or $U_{n-1,n}$.*

Ding, Oporowski, Oxley, and Vertigan generalized Theorem 1.8.2 to find unavoidable minors of 3-connected matroids. They identified the unavoidable minors of 3-connected matroids, first in the binary case [7] and later in the general case [8].

Theorem 1.8.4. *For every integer n greater than 2, there is an integer $f_1(n)$ so that every 3-connected binary matroid with more than $f_1(n)$ elements contains a minor isomorphic to $M(\mathcal{W}_n)$, the rank- n binary tipless spike, or the cycle or bond matroid of $K_{3,n}$.*

Theorem 1.8.5. *For every integer n greater than 2, there is an integer $f_2(n)$ so that every 3-connected matroid with more than $f_2(n)$ elements has a minor isomorphic to $M(\mathcal{W}_n)$, \mathcal{W}^n , a rank- n tipless spike, the cycle or bond matroid of $K_{3,n}$, or $U_{2,n}$ or $U_{n-2,n}$.*

In this dissertation, we build on these results to determine what can be said about a large structure using a specified element e of M . This question is motivated by the importance of preserving a specific element through steps of an inductive proof, as induction is one of the most important proof techniques in matroid theory specifically and combinatorics generally. In Chapter 2, we extend Theorem 1.8.5 to show that, by slightly modifying the list of unavoidable minors, we can capture any specified single element of a large 3-connected matroid in a highly structured minor. In Chapter 3, we extend the main result of Chapter 2 in the binary case, showing that we can capture any pair of elements of a large 3-connected binary matroid in a highly structured minor.

Chapter 2

Capturing Matroid Elements in Unavoidable 3-Connected Minors

In this chapter, we extend this theorem to show that, by slightly modifying the list of unavoidable minors in Theorem 1.8.5, we can ensure that we capture any specified single element of a large 3-connected matroid in a large, highly structured minor. The following is the main result of the chapter.

Theorem 2.0.6. *Let M be a 3-connected matroid, and let e be an element of M . For every integer $n > 2$, there is an integer $g(n)$ so that if $|E(M)| \geq g(n)$, then e is an element of a minor of $E(M)$ that is isomorphic to the rank- n wheel or whirl, the cycle or bond matroid of $K_{1,1,1,n}$, $M(K_{3,n})^+$ or its dual, $U_{2,n}$ or $U_{n-2,n}$, or a member of \mathcal{T}_n .*

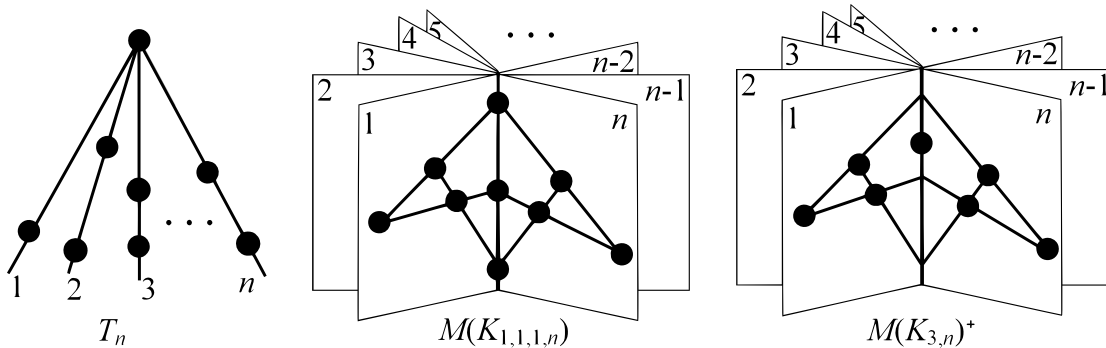


FIGURE 2.1. Geometric illustrations of three matroids: T_n is the rank- n free spike with a tip and a cotip; $M(K_{1,1,1,n})$ and $M(K_{3,n})^+$ have rank $n + 2$.

This theorem shows that not only does every huge 3-connected matroid M contain a large highly structured minor, but a slight modification of such a minor can be chosen to contain any specified element of M . The next two corollaries specialize the main result to the classes of binary and graphic matroids.

Corollary 2.0.7. *Let M be a 3-connected binary matroid, and let e be an element of M . For every integer $n > 2$, there is an integer $h(n)$ so that if $|E(M)| \geq h(n)$, then e is an*

element of a minor of M that is isomorphic to $M(\mathcal{W}_n)$, the rank- n binary spike with a tip and a cotip, $M(K_{1,1,1,n})$, or $M^*(K_{1,1,1,n})$.

Corollary 2.0.8. *Let G be a simple 3-connected graph, and let e be an edge of G . For every integer $n > 2$, there is an integer $k(n)$ so that if $|E(G)| \geq k(n)$, then e is an edge of a minor of G that is isomorphic to \mathcal{W}_n or $K_{1,1,1,n}$.*

Following [7] and [8], we make no attempt to find sharp estimates for the functions $g(n)$, $h(n)$, and $k(n)$ in Theorem 2.0.6, Corollary 2.0.7, and Corollary 2.0.8. By contrast, Lemos and Oxley [11] did find sharp bounds for the functions in the corresponding results for connected matroids.

Theorem 2.0.9. *Let M be a connected matroid having n elements. Then*

- (i) M has a minor isomorphic to $U_{1,m}$ or $U_{m-1,m}$ for some $m \geq \sqrt{2n}$; and
- (ii) for each element e of M , there is a minor of M that uses e and is isomorphic to $U_{1,p}$ or $U_{p-1,p}$ for some $p \geq \sqrt{n-1} + 1$.

In this chapter, we will rely heavily on the following result of Brylawski [2] and Seymour [18] (see also [13, p.129]).

Theorem 2.0.10. *Let N be a connected minor of a connected matroid M , and suppose that $e \in E(M) - E(N)$. Then at least one of $M \setminus e$ and M/e is connected and contains N as a minor.*

By Theorem 1.8.5, a sufficiently large 3-connected matroid has, as a minor, one of the following five matroids:

- (i) an n -element line or its dual;
- (ii) a rank- n spike;
- (iii) a wheel or whirl of rank n ;

(iv) $M(K_{3,n})$; or

(v) $M^*(K_{3,n})$.

In each of the next five sections, we treat one of these cases identifying an unavoidable minor using the special element. That identification is made possible by using Theorem 2.0.10. The main theorem is proved in the last section by combining the results from these five sections.

The reader familiar with the matroid concept of roundedness may be reminded of it by the main theorem of this chapter. Roundedness was introduced by Seymour [20] (see, for example, [13, p.481]) to encompass certain results that were concerned with relating particular minors of a matroid to specific elements of the matroid. For example, Bixby [1] proved that if x is an element of a connected non-binary matroid M , then M has a $U_{2,4}$ -minor using x ; and Seymour [21] extended this showing that if x and y are distinct elements of a non-binary 3-connected matroid M , then M has a $U_{2,4}$ -minor using x and y . The results of this dissertation were motivated in part by the idea of roundedness and by the usefulness of the results that relate to it.

In Section 2.5, we use the following result of Kung [10], which gives an upper bound on the number of elements in a simple matroid that does not contain a long-line minor.

Theorem 2.0.11. *If M is a simple, rank- r matroid with no $U_{2,q+2}$ -minor, then M has at most $\frac{q^r-1}{q-1}$ elements.*

Flowers were introduced by Oxley, Semple, and Whittle [15] to describe crossing 3-separations in 3-connected matroids. We will not require a detailed knowledge of flowers here, but the following definitions will be useful. For a positive integer n , we write $[n]$ for $\{1, 2, \dots, n\}$. Let M be a 3-connected matroid. We will use the connectivity function, $\lambda_M(X)$, and the local connectivity function, $\square_M(X, Y)$ (see, for example, [13, Sections 8.1 and 8.2]). Let (P_1, P_2, \dots, P_n) be an ordered partition Φ of $E(M)$. Consider the following properties.

1. $|P_i| \geq 2$ for all i in $[n]$.
2. $\lambda_M(P_i) = 2$ for all i in $[n]$.
3. $\lambda_M(P_i \cup P_{i+1}) = 2$ for all i in $[n]$ where the indices are considered modulo n .
4. $\lambda_M(\cup_{i \in S} P_i) = 2$ for all proper non-empty subsets S of $[n]$.
5. $\cap(P_i, P_j) = 2$ for all distinct i and j in $[n]$.

If the first three properties hold, then Φ is a *flower* with every set P_i being a *petal*. When the first four properties hold, this flower is an *anemone* [15, 16]. Should all five properties hold, this anemone is a *paddle* [17].

2.1 Long Lines

In this section, we examine the case where a connected matroid with an identified element has a long line or its dual as a minor.

Theorem 2.1.1. *Let M be a connected matroid with a $U_{2,n}$ -minor for some $n \geq 3$. If $e \in E(M)$, then e is an element of a connected minor of M that is isomorphic to $U_{2,m}$ for some $m > \sqrt{n}$.*

Proof. The result is immediate if $n = 3$. Thus we may assume that $n \geq 4$. By Theorem 2.0.10, there is a connected minor N of M so that $N \setminus e \cong U_{2,n}$ or $N/e \cong U_{2,n}$. If $N \setminus e \cong U_{2,n}$, then, as N is connected, $r(N) = r(N \setminus e) = 2$. Thus $N \cong U_{2,n+1}$, or N is obtained from $U_{2,n}$ by adding e parallel to some other element. In either case, we easily identify a $U_{2,n}$ -minor of M using e .

Now assume $N/e \cong U_{2,n}$. Thus $r(N) = 3$ and N is as shown in Figure 2.2 where the possible non-trivial lines through subsets of $\{1, 2, \dots, n\}$ have not been depicted. Let f be an element of N/e . Then N/f has rank 2 and has $\{e\}$ as a rank-1 flat. Simplify N/f without deleting e to produce a minor isomorphic to $U_{2,k}$, for some k . If $k > \sqrt{n}$, then we have identified a desired minor.

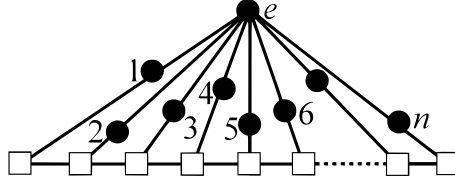


FIGURE 2.2. Geometric illustration of N in rank 3. Possible lines containing subsets of $[n]$ have not been shown.

Assume $k \leq \sqrt{n}$. Since N has $n + 1$ elements, a largest parallel class X of N/f has p elements for some $p \geq \frac{n}{k} \geq \sqrt{n} \geq 2$. As N has no parallel elements, the elements of X are collinear in N , and the matroid $N|(X \cup f) \cong U_{2,p+1}$, so $N|(X \cup f)$ is connected. By Theorem 2.0.10, N has a connected minor N' with ground set $X \cup \{e, f\}$ such that $N|(X \cup f)$ is a minor of N' . Now $r(N')$ is 2 or 3. In the latter case, e is a coloop of N' , a contradiction. Thus $r(N') = 2$, so $N' \cong U_{2,p+2}$, or N is obtained from $U_{2,p+1}$ by adding e parallel to some other element. In either case, we easily identify a $U_{2,p+1}$ -minor of M using e . Since $p + 1 \geq \sqrt{n} + 1$, the lemma holds. \square

2.2 Spike Minors

In this section, we examine the case where a connected matroid with an identified element has a large spike as a minor. It is not hard to show that if a non-tip element is contracted from a rank- r spike with a tip and no cotip, the resulting matroid is a rank- $(r - 1)$ spike with a tip, no cotip, and an extra element parallel to the tip.

It is easy to see that $M(K_{1,1,n})$ is the parallel connection of n 3-point lines, L_1, L_2, \dots, L_n , across a common basepoint (see Figure 2.3). Extending this, we have the following.

Lemma 2.2.1. *Let M be a connected matroid so that $M \setminus x \cong M(K_{1,1,n})$ for some $n \geq 3$. Then $M \setminus x$ is the parallel connection of n 3-point lines, L_1, L_2, \dots, L_n , across a common basepoint, t . If x is not contained in the closure of any proper subset of these lines, then M is a rank- $(n + 1)$ spike with a tip and a cotip.*

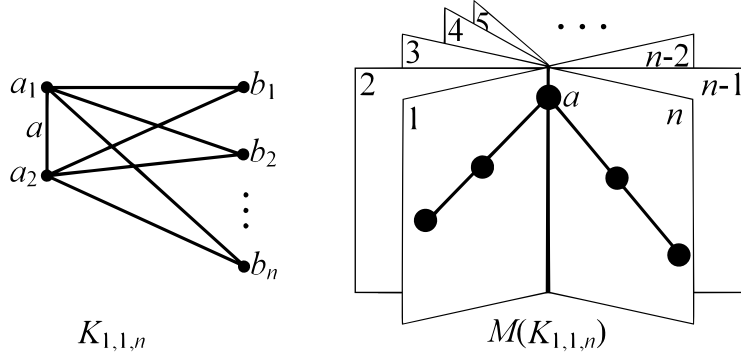


FIGURE 2.3. The graph of $K_{1,1,n}$ and the geometry of $M(K_{1,1,n})$ in rank $n + 1$.

Proof. Freely add a point y on the line of M containing t and x . Let L_{n+1} be the line $\{t, x, y\}$. By the definition of a spike given above, the result is a rank- $(n + 1)$ tipped spike. Hence, without y , the matroid is a rank- $(n + 1)$ spike with a tip and a cotip. \square

Using this characterization, we prove the main theorem of this section.

Theorem 2.2.2. *Let M be a connected matroid with an element e so that $M \setminus e$ is isomorphic to a matroid in \mathcal{T}_n for some $n \geq 6$. Then e is an element of a minor of M that is isomorphic to a matroid in \mathcal{T}_m for some $m \geq \frac{n}{2} \geq 3$.*

Proof. Let t and c be the tip and cotip of $M \setminus e$. If e lies on the line joining c and t , then we can easily find the desired minor. If not, M/c is connected, and $M/c \setminus e$ is a rank- $(n - 1)$ spike with a tip t and no cotip. By definition, this matroid is the union of $n - 1$ lines L_1, L_2, \dots, L_{n-1} , each of which is a 3-element circuit containing the point t so that, for all j in $[n - 2]$, the union of any j of L_1, L_2, \dots, L_{n-1} has rank $j + 1$, and $r(L_1 \cup L_2 \cup \dots \cup L_{n-1}) = n - 1$. Let $\{L_1, L_2, \dots, L_k\}$ be a smallest set of these lines for which $e \in \text{cl}_M(L_1 \cup L_2 \cup \dots \cup L_k)$.

Suppose $k \leq \frac{n}{2}$. Let $\{s_1, s_2, \dots, s_k\}$ be a transversal of $\{L_1, L_2, \dots, L_k\}$ avoiding t . The matroid $M \setminus e / \{c, s_1, s_2, \dots, s_{k-1}\}$ is a spike with a tip, no cotip, and $k - 1$ extra elements parallel to t . In the loopless matroid $M / \{c, s_1, s_2, \dots, s_{k-1}\}$, the element e is in the closure of $(L_1 \cup L_2 \cup \dots \cup L_k) - \{s_1, s_2, \dots, s_{k-1}\}$, so e is in the closure of L_k . Without deleting e , simplify the last matroid. From this simplification, we can remove some set consisting of all

but two elements of the closure of L_k to produce a member of \mathcal{T}_m with e as the tip or cotip and with $m = n - 1 - (k - 1) \geq \frac{n}{2}$.

We may now assume that $k > \frac{n}{2}$. Notice that $k \leq n - 2$, since the union of any $n - 2$ lines has rank $n - 1$, which is the rank of $M/c \setminus e$. Moreover, the restriction $(M/c)|(L_1 \cup L_2 \cup \dots \cup L_k)$ is isomorphic to $M(K_{1,1,k})$. By Lemma 2.2.1, $(M/c)|(L_1 \cup L_2 \cup \dots \cup L_k \cup e)$ is a rank- m spike with a tip t and a cotip x and with $m = k + 1 > \frac{n}{2} + 1$. \square

2.3 Wheels and Whirls

In this section, we consider the case where a connected matroid with an identified element has a large wheel or a large whirl as a minor. First, we define a fan, which can be thought of as a partial wheel or whirl. In a simple, cosimple matroid M , consider a sequence $(s_0, r_1, s_1, \dots, s_{n-1}, r_n, s_n)$ of distinct elements of M so that every set $\{s_{i-1}, r_i, s_i\}$ with $0 < i \leq n$ is a triangle of M and every set $\{r_j, s_j, r_{j+1}\}$ with $0 < j < n$ is a triad of M . Here we call such a sequence a *fan*, noting that this specializes the terminology used in [13], where two other related structures are also called fans. The following result of Seymour shows how closely related fans are to wheels and whirls [19] (see also [13, p. 339]).

Theorem 2.3.1. *Let M be a connected, simple, cosimple matroid having $(s_0, r_1, s_1, \dots, s_{n-1}, r_n, s_n)$ as a fan and having another element r_0 so that $\{r_0, s_0, r_1\}$ and $\{r_n, s_n, r_0\}$ are triads and $\{s_0, r_0, s_n\}$ is a triangle. Then M is a wheel or whirl of rank $n + 1$.*

Viewing a fan as a substructure of a wheel makes it natural to refer to each s_i as a *spoke* element and each r_i as a *rim element*. The following is a technical lemma.

Lemma 2.3.2. *Let M be a 3-connected matroid with an element e so that $M \setminus e$ is 3-connected having a fan $F = (s_0, r_1, s_1, \dots, s_{n-1}, r_n, s_n)$ with $n \geq 3$. Let $E(M) - (F \cup e)$ be a set A having at least two elements. If no triad of F is a triad in M , and M has no $U_{q-2,q}$ -minor, then there is a set X of at least $\frac{n}{q-1}$ elements of $\{r_1, r_2, \dots, r_n\}$ so that $e \in \text{cl}(A \cup (\{r_1, r_2, \dots, r_n\} - X))$.*

Proof. We begin by establishing the following.

2.3.2.1. $\{s_0, s_n\} \subseteq \text{cl}_M(A)$.

Suppose $s_0 \notin \text{cl}(A)$. Then $s_0 \in \text{cl}_{M \setminus e}^*(F - s_0)$. Thus, in $(M \setminus e)^*$, the set F is spanned by $\{r_1, r_2, \dots, r_n, s_n\}$, so $r_{M \setminus e}^*(F) \leq n+1$. But $r_{M \setminus e}(F) \leq n+1$, so $r_{M \setminus e}(F) + r_{M \setminus e}^*(F) - |F| \leq 1$. Hence (F, A) is a 2-separation of $M \setminus e$. This contradiction and symmetry imply that (2.3.2.1) holds.

Let $N = M \setminus \{e, s_1, s_2, \dots, s_{n-1}\}$ and $R = \{r_1, r_2, \dots, r_n\}$. Next we show that

2.3.2.2. $R \cup \{s_0, s_n\}$ is a circuit of M and $r_N^*(R) = 1$.

In $M \setminus e$, every set $\{r_i, s_i, r_{i+1}\}$ with i in $[n-1]$ is a triad. In $M \setminus \{e, s_i\}$, then, $\{r_i, r_{i+1}\}$ is a series pair, and so has corank 1. In N , it follows that every set $\{r_i, r_{i+1}\}$ has corank at most 1. A straightforward induction argument establishes that the set $R \cup \{s_0, s_n\}$ is a circuit of M and so is a circuit of $M \setminus \{e, s_1, s_2, \dots, s_{n-1}\}$. Hence, by orthogonality, no element of $R \cup \{s_0, s_n\}$ is a coloop. Thus $r_N^*(\{r_1\}) = 1$ and $r_N^*(\{r_i, r_{i+1}\}) = 1$ for all i in $[n-1]$. Hence $\text{cl}_{N^*}(\{r_1\}) = R$ and (2.3.2.2) follows.

Suppose $i \in [n-1]$. By hypothesis, $\{r_i, s_i, r_{i+1}\}$ is not a triad of M , so M has $\{e, r_i, s_i, r_{i+1}\}$ as a cocircuit. Let $M_1 = M \setminus \{s_1, s_2, \dots, s_{n-1}\}$. We will show next that

2.3.2.3. $\{e, r_i, r_{i+1}\}$ is a triad of M_1 .

Every set $\{e, r_i, r_{i+1}\}$ with i in $[n-1]$ is a union of cocircuits of M_1 . By orthogonality with the circuit $R \cup \{s_0, s_n\}$, it follows that a cocircuit contained in $\{e, r_i, r_{i+1}\}$ contains r_i if and only if it contains r_{i+1} . Thus $\{e, r_i, r_{i+1}\}$ is a triad of M_1 ; otherwise e would be a coloop of M_1 , so $\{e, s_1, s_2, \dots, s_{n-1}\}$ would contain a cocircuit of M containing e ; a contradiction to orthogonality with the triangles of F .

In M_1 , the set $R \cup e$ has corank at least 2, since it contains a triad. Now $N = M_1 \setminus e$ so, by (2.3.2.2), $r_{M_1 \setminus e}^*(R) = 1$. Therefore, $r_{M_1}^*(R) \leq 2$, so

2.3.2.4. $r_{M_1}^*(R \cup e) = 2$.

Consider M_1^* . In this matroid, $R \cup e$ has rank 2 and, by (2.3.2.3), every set $\{e, r_i, r_{i+1}\}$ with i in $[n-1]$ is a triangle. Since M_1^* has no $U_{2,q}$ -minor, M_1^* has fewer than q parallel classes. As one of these classes is e , and since the n elements of R are contained in the other parallel classes, there is a parallel class X with at least $\frac{n}{q-1}$ elements. In $M_1^*(R \cup e)$, the set X is a hyperplane, and its complement, $(R \cup e) - X$, is a cocircuit. Moreover, $R - X \neq \emptyset$. Thus, in M_1 , the element e is contained in the closure of $E(M_1) - e - X$. Hence $e \in \text{cl}(E(M) - \{s_1, s_2, \dots, s_{n-1}\} - e - X)$. Recall that $A = E(M) - (F \cup e) = E(M) - \{s_0, s_1, s_2, \dots, s_{n-1}, s_n\} - e - R$. By (2.3.2.1), $\{s_0, s_n\} \subseteq \text{cl}(A)$, so $e \in \text{cl}(E(M) - \{s_0, s_1, s_2, \dots, s_{n-1}, s_n\} - e - X) = \text{cl}(A \cup (R - X))$. \square

We are now ready to prove the main theorem of this section.

Theorem 2.3.3. *Let M be a connected matroid with no $U_{q-2,q}$ -minor and no $U_{2,q}$ -minor for some $q \geq 4$. If M has an element e and a minor isomorphic to $M(\mathcal{W}_{q^3})$ or \mathcal{W}^{q^3} , then M has a minor containing e isomorphic to $M(\mathcal{W}_q)$ or \mathcal{W}^q .*

Proof. Let $n = q^3$. By Theorem 2.0.10 and by switching to the dual if necessary, we may assume $M \setminus e = N$ where N is a rank- n wheel or whirl. Let $\{s_0, s_1, \dots, s_{n-1}\}$ be the set S of spokes of N , and $\{r_1, r_2, \dots, r_n\}$ be the set R of rim elements of N , where every set $T_i = \{s_{i-1}, r_i, s_i\}$ is a triangle of N and every set $T_j^* = \{r_j, s_j, r_{j+1}\}$ is a triad of N where all subscripts are interpreted modulo n .

The matroid M is connected, and $M \setminus e$ is 3-connected. If e is parallel to another element x in M , then $M \setminus x$ is a minor of the desired type. Hence we may assume that e is not parallel to another element. Thus M is 3-connected.

The set S is a basis of M , so $S \cup e$ contains a unique circuit C containing e . Let X be a largest set of consecutive spokes avoiding C . Without loss of generality, when X is non-empty, we may assume that $X = \{s_1, s_2, \dots, s_k\}$. Every set $\{s_{i-1}, r_i, s_i\}$ with i in $[k+1]$ is a

triangle in $M \setminus e$ and so is a triangle in M . Now suppose $i \in [k]$. Then either $\{r_i, s_i, r_{i+1}\}$ or $\{e, r_i, s_i, r_{i+1}\}$ is a cocircuit of M . Since the circuit C is a subset of $(S - X) \cup e$, the spoke $s_i \notin C$. By orthogonality, $\{e, r_i, s_i, r_{i+1}\}$ is not a cocircuit of M . Thus $\{r_i, s_i, r_{i+1}\}$ is a triad of M , and M contains the fan $F = (s_0, r_1, s_1, r_2, s_2, \dots, r_k, s_k)$.

Assume that $k \geq q$. The circuit C is contained in the complement of F , so we will remove this complement to produce a wheel or whirl minor. It is clear that we may contract rim elements of a wheel or whirl and simplify to produce a smaller wheel or whirl. Let $R' = \{r_{k+1}, r_{k+2}, \dots, r_n\}$. The matroid N/R' has $\{s_k, s_{k+1}, \dots, s_{n-1}, s_0\}$ as a parallel class, and the elements of $C - e$ are contained in this parallel class. The matroid M/R' has e in the closure of the set $C - e$ and so $e \in \text{cl}(\{s_k, s_{k+1}, \dots, s_{n-1}, s_0\})$. Either M/R' has $\{e, s_k, s_{k+1}, \dots, s_{n-1}, s_0\}$ as a parallel class or has e as a loop. In the latter case, contract the elements of R' from M one at a time until e becomes parallel to some remaining element of R' . In both cases, by simplifying the resulting matroid without removing e , we obtain a wheel or whirl minor with at least q spokes.

Now assume that $k \leq q - 1$, noting that this includes the case when X is empty. Then every set of q consecutive spokes of N contains an element of C . Let $|C - e| = m$. Then $m \geq \frac{q^3}{q} = q^2$. Suppose $s_i \in S - C$. The set $\{r_i, s_i, r_{i+1}\}$ is a triad of $M \setminus e$. By orthogonality with the circuit C of M , it follows that $\{r_i, s_i, r_{i+1}\}$ is a cocircuit of M . In M , the complement

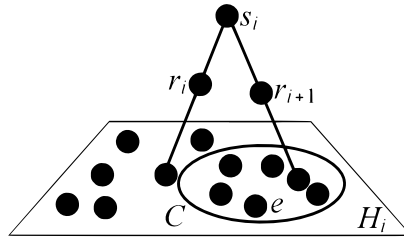


FIGURE 2.4. The triad $T_i^* = \{r_i, s_i, r_{i+1}\}$ and the complementary hyperplane H_i .

of $\{r_i, s_i, r_{i+1}\}$ is a hyperplane H_i containing C (see Figure 2.4). The element $s_i \notin \text{cl}(C)$, so $M \setminus s_i / r_i$ has C as a circuit. Notice that $(M \setminus s_i / r_i) \setminus e$ is a wheel or whirl of rank $n - 1$. In

this way, we may remove s_i and r_i for all s_i in $S - C$ to produce a matroid M_1 in which C is a circuit and $M_1 \setminus e$ is a wheel or whirl of rank m having $C - e$ as its set of spokes.

Reindex both the set of spokes and the set of rim elements of $M_1 \setminus e$ so that $M_1 \setminus e$ has $\{s_{i-1}, r_i, s_i\}$ as a triangle and $\{r_i, s_i, r_{i+1}\}$ as a triad for all i in $[m]$, where $s_m = s_0$ and $r_{m+1} = r_1$. By orthogonality with C , it follows that each $\{e, r_i, s_i, r_{i+1}\}$ is a cocircuit of M_1 . Now $(s_0, r_1, s_1, \dots, r_{m-2}, s_{m-2})$ is a fan F_1 of $M_1 \setminus e$ and no triad of F_1 is a triad of M_1 . Let $A_1 = E(M_1 \setminus e) - F_1$. Then $A_1 = \{r_{m-1}, s_{m-1}, r_m\}$. Thus $|A_1| \geq 2$ and we may apply Lemma 2.3.2 to get a subset Y of $\{r_1, r_2, \dots, r_{m-2}\}$ having at least $\frac{m-2}{q-1}$ elements so that $e \in \text{cl}(A_1 \cup (\{r_1, r_2, \dots, r_{m-2}\} - Y))$. Observe that

$$|Y| \geq \frac{q^2 - 2}{q - 1} = \frac{q^2 - 1}{q - 1} - \frac{1}{q - 1} = q + 1 - \frac{1}{q - 1}.$$

As $q \geq 4$, it follows that $|Y| \geq q + 1$.

Let $M_2 = M_1 / (\{r_1, r_2, \dots, r_{m-2}\} - Y)$. In this matroid, e is in the closure of A_1 . Suppose e is a loop of M_2 . Then contracting some proper subset of $\{r_1, r_2, \dots, r_{m-2}\} - Y$ from M_1 makes e parallel to some other element of the set. On the other hand, when e is not a loop of M_2 , it is spanned by the set A_1 , which equals $\{r_{m-1}, s_{m-1}, r_m\}$. In this case, contracting some subset of $\{r_{m-1}, r_m\}$ from M_2 makes e parallel to some element of $\{r_{m-1}, s_{m-1}, r_m\}$. In either case, by simplifying the resulting matroid without deleting e , we obtain a wheel or whirl that uses e and has rank at least $|Y|$. We conclude that the theorem holds. \square

2.4 $M(K_{3,n})$

In this section, we consider the case where a connected matroid with an identified element has a minor isomorphic to a large $M(K_{3,n})$. We give a lower bound on the rank of a similar minor containing the identified element. It is easy to show that the elements of the matroid $M(K_{3,n})$ can be partitioned into a rank- $(n+2)$ paddle with n petals of rank 3 (see Figure 2.5). Two related matroids are $M(K_{1,1,1,n})$ and $M(K_{3,n})^+$ (see Figure 2.6). If a matroid has an $M(K_{3,n})$ -minor, then, by contracting a set of two elements from this minor and simplifying,

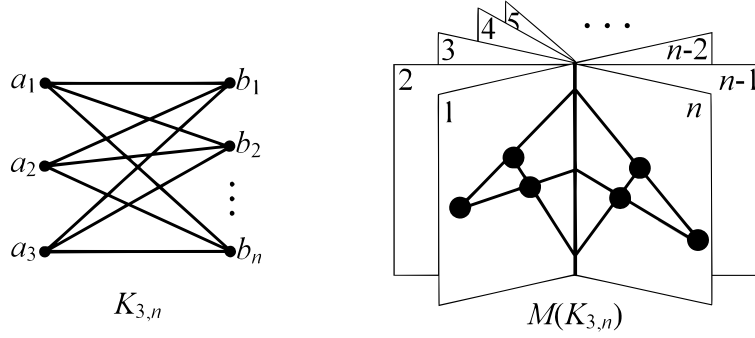


FIGURE 2.5. The graph $K_{3,n}$ and the geometry of $M(K_{3,n})$.

we obtain a minor isomorphic to $M(K_{1,1,1,n-2})$. The matroid $M(K_{3,n})^+$ was formally defined in the introduction. Recall that \mathcal{T}_n denotes the set of rank- n spikes with a tip and a cotip.

Theorem 2.4.1. *Let M be a connected matroid with $M \setminus e = N \cong M(K_{1,1,1,n})$ for some $n \geq 3$. Then e is an element of a minor of M isomorphic to $M(K_{1,1,1,m})$, $M(K_{3,m})^+$, or some member of \mathcal{T}_m for some $m \geq \frac{n}{2}$.*

Proof. The matroid N has exactly n triads $\{P_1, P_2, P_3, \dots, P_n\}$. These sets are disjoint and their union is $E(N) - S$, where S is the spine of N . Moreover, $(P_1, P_2, P_3, \dots, P_n)$ is a rank- $(n+2)$ paddle Φ in $N \setminus S$. Let $P = \{P_1, P_2, P_3, \dots, P_k\}$ be a minimal set P of petals of Φ whose closure contains e in M .

Suppose $k \leq \frac{n}{2}$. Let X be a transversal of P . Contract elements of X from M one at a time until the first time that either e becomes parallel to an element of N , or $e \in \text{cl}(S)$. Simplify the resulting matroid without deleting e to form M' . Either $M' \cong M(K_{1,1,1,m})$

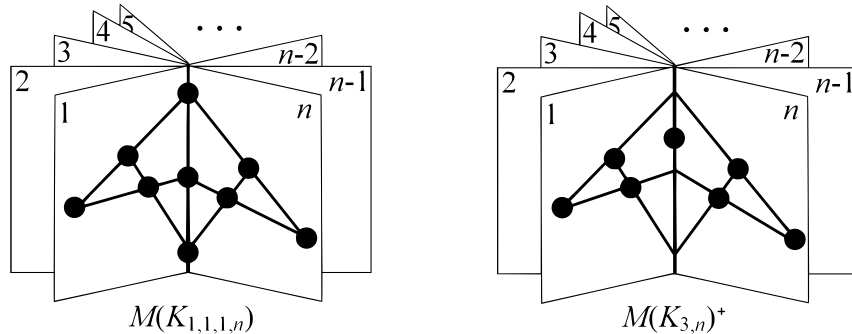


FIGURE 2.6. Geometric representations of $M(K_{1,1,1,n})$ and $M(K_{3,n})^+$.

for some $m \geq n - k \geq \frac{n}{2}$, or $e \in \text{cl}(S)$ and this line has four elements. In the latter case, delete the elements other than e on this line to form an $M(K_{3,m})^+$ -minor of M for some $m \geq n - k \geq \frac{n}{2}$.

Now suppose $k > \frac{n}{2}$ and consider $M|(P_1 \cup P_2 \cup \dots \cup P_k \cup S \cup e)$. Let Y be a transversal of P so that $M \setminus (Y \cup e)$ is a set of $k + 1$ 3-point lines all intersecting at some point a of S . Then $M \setminus (Y \cup e) \cong M(K_{1,1,k+1})$ (see Figure 2.7). For each j in $[k]$, let $L_j = P_j - Y$. By the choice of P , the element e is in the closure of $L_1 \cup L_2 \cup \dots \cup L_k \cup S$, but it is not

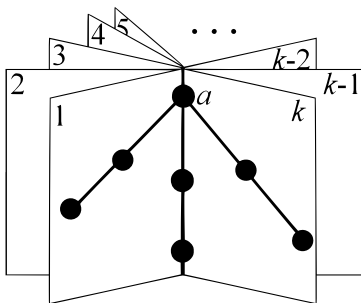


FIGURE 2.7. Geometric representation of $M(K_{1,1,k+1})$ in rank $k + 1$.

in $\text{cl}((L_1 \cup L_2 \cup \dots \cup L_k \cup S) - L_i)$ for any i in $[k]$. This leaves the possibility that $e \in \text{cl}(L_1 \cup L_2 \cup \dots \cup L_k)$. Thus, for some m in $\{k, k + 1\}$, there are m lines in $\{L_1, L_2, \dots, L_k, S\}$ whose union spans e , and no proper subset of these m lines spans e . By Lemma 2.2.1, the restriction of M to the union of e with the elements of these m lines is a member of \mathcal{T}_{m+1} with a as the tip and e as the cotip. \square

2.5 $M^*(K_{3,n})$

In this section, we consider the case where a connected matroid with an identified element has a minor isomorphic to a large $M^*(K_{3,n})$. We give a lower bound on the rank of a similar minor, $M^*(K_{1,1,1,m})$, containing the identified element. This result relies on work of Geelen and Whittle. In particular, we extend arguments in [9, Theorem 9.43 and 9.44] to prove the following theorem. The dual of a paddle is a *copaddle*.

Theorem 2.5.1. *Let M be a connected matroid with $M \setminus e = N \cong M^*(K_{3,m^5})$ for some $m \geq 4$. If M has no $U_{m,m+2}$ -minor, then e is an element of a minor of M that is isomorphic to $M^*(K_{1,1,1,m-1})$.*

Proof. The minor N has a copaddle $\Phi = \{T_1, T_2, \dots, T_{m^5}\}$, with each petal T_i being a triangle $\{\alpha_i, \beta_i, \gamma_i\}$. The set $\{\alpha_1, \alpha_2, \dots, \alpha_{m^5}\}$ is a transversal A of the petals. Let i be an element of $[m^5]$. The matroid N has rank $2m^5 - 2$ and is spanned by $E(N) - T_i$. Since $\{\beta_j, \gamma_j\}$ spans T_j for all j in $[m^5]$, the set $E(N \setminus A) - \{\beta_i, \gamma_i\}$ is a basis B_i of N and is therefore a basis of M . Let C_i be the fundamental circuit $C(e, B_i)$ and let \mathcal{Q}_i be the set of petals of Φ that meet C_i .

We show next that

2.5.1.1. *\mathcal{Q}_i is a minimal set of petals of Φ whose closure contains e .*

Suppose that $T_j \in \mathcal{Q}_i$ but that $e \in \text{cl}(\cup_{T_l \in \mathcal{Q}_i} T_l - T_j)$. Then $e \in \text{cl}(\cup_{T_l \in \mathcal{Q}_i} T_l - (T_j \cup A))$, so M has a circuit that is contained in $B_i \cup e$ but differs from C_i ; a contradiction. Thus (2.5.1.1) holds.

2.5.1.2. *If $T_j \cap C_i = \emptyset$, then $C_j = C_i$, so $\mathcal{Q}_j = \mathcal{Q}_i$.*

To see this, observe that, since $T_j \cap C_i = \emptyset$, we have $C_i \subseteq B_j \cup e$. But C_j is the unique circuit contained in $B_j \cup e$. Hence $C_j = C_i$ and (2.5.1.2) holds.

By relabelling if necessary, we may assume that the set \mathbf{Q} of distinct sets \mathcal{Q}_i with $i \in [m^5]$ is $\{\mathcal{Q}_1, \mathcal{Q}_2, \dots, \mathcal{Q}_k\}$. Then \mathbf{Q} is the set of all distinct minimal sets of petals whose closure contains e .

2.5.1.3. *If i and j are distinct elements of $[k]$, then every petal of Φ is in \mathcal{Q}_i or \mathcal{Q}_j .*

Suppose that some petal T_s is in neither \mathcal{Q}_i nor \mathcal{Q}_j . Then $T_s \cap C_i = \emptyset = T_s \cap C_j$ so, by (2.5.1.2), $\mathcal{Q}_i = \mathcal{Q}_j$; a contradiction. Thus (2.5.1.3) holds.

For each \mathcal{Q}_i in \mathbf{Q} , let \mathcal{X}_i be the set of petals of Φ that are not in \mathcal{Q}_i . By (2.5.1.3), if $i \neq j$, then $\mathcal{X}_i \cap \mathcal{X}_j = \emptyset$. By construction, for all s in $[k]$, the petal T_s is not in \mathcal{Q}_s , so $T_s \in \mathcal{X}_s$. It follows from (2.5.1.2) that $\{\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_k\}$ is a partition \mathbf{X} of the petals of Φ .

Since Φ has m^5 petals, it follows by the pigeonhole principle that either

(i) $k \geq m^4$, or

(ii) $|\mathcal{X}_i| \geq m$ for some i .

First, assume that (ii) holds. Then, without loss of generality, $|\mathcal{X}_1| \geq m$. Thus \mathcal{Q}_1 avoids at least m petals of Φ . Now \mathcal{Q}_1 is a minimal set of petals of Φ whose closure contains e . Choose one petal T_j in \mathcal{Q}_1 . For each T_i in $\mathcal{Q}_1 - T_j$, delete α_i and contract $\{\beta_i, \gamma_i\}$. In the resulting matroid N' , the element e is in the closure of T_j . We observe that $N' \setminus e \cong M^*(K_{3,m'})$ for some $m' \geq m + 1$.

Suppose that e is parallel to another element f of N' . Then $N' \setminus f$ contains e and is isomorphic to $M^*(K_{3,m'})$. The last matroid contains e in a minor isomorphic to $M^*(K_{1,1,1,m'-2})$ and $m' - 2 \geq m - 1$, so the theorem holds in this case. Thus we may assume that e is not parallel to any other element of N' . Then $\text{cl}_{N'}(T_j)$ is a 4-point line containing e . As $N' \setminus e$ is cographic, $N' \setminus e / \alpha_i$ is cographic. Since N' / α_i has $\{e, \beta_i, \gamma_i\}$ as a parallel class, N' / α_i is cographic. Without deleting e , take the simplification of N' / α_i . This matroid is isomorphic to $M^*(K'_{3,m'-1})$, where the graph $K'_{3,m'-1}$ is shown in Figure 2.8. Contracting the edge $a_3 b_1$

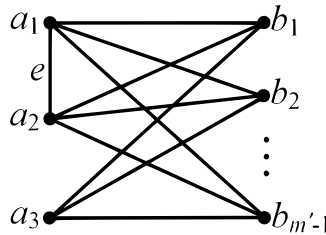


FIGURE 2.8. The graph $K'_{3,m'-1}$.

in this graph produces a $K_{1,1,1,m'-2}$ -minor using e . Hence N' has an $M^*(K_{1,1,1,m'-2})$ -minor using e . As $m' - 2 \geq m - 1$, we conclude that the theorem holds in case (ii).

We may now assume that (i) holds, that is, $k \geq m^4$. In addition, we assume that some $|\mathcal{X}_i| > 1$. In particular, we suppose that \mathcal{X}_1 contains T_i and at least one other petal. Consider the matroid $M/T_i = M'$. Observe that $M' = M/\{\beta_i, \gamma_i\} \setminus \alpha_i$ as α_i is a loop of $M/\{\beta_i, \gamma_i\}$. Since $e \notin \text{cl}(T_i)$, the matroid M' is connected. Notice that $M' \setminus e \cong M^*(K_{3, m^5-1})$. Let $\mathbf{X}' = \{\mathcal{X}_1 - T_i, \mathcal{X}_2, \dots, \mathcal{X}_k\}$. It is clear that \mathbf{X}' partitions the set of triangles of $M^*(K_{3, m^5-1})$, and these triangles are the petals of a copaddle Φ' in $M' \setminus e$.

In the partition \mathbf{X} , the set \mathcal{X}_1 contains T_i . Hence $T_i \in \mathcal{Q}_j$ for all $j \neq 1$. Now let $\mathcal{Q}'_t = \mathcal{Q}_t - \{T_i\}$ for all t in $[k]$. We will show that the set $\mathbf{Q}' = \{\mathcal{Q}'_1, \mathcal{Q}'_2, \dots, \mathcal{Q}'_k\}$ has the same properties in M' as the set \mathbf{Q} has in M .

Suppose $1 < j \leq m^5 - 1$. The set $E(N \setminus A) - T_j$ is a basis B_j of M . Since $T_i \in \mathcal{Q}_j$, it follows from (2.5.1.1) that the fundamental circuit C_j of e with respect to B_j contains β_i or γ_i . Let $C'_j = C_j - \{\beta_i, \gamma_i\}$. We show next that

2.5.1.4. C'_j is a circuit of $M/\{\beta_i, \gamma_i\} \setminus \alpha_i$.

This is certainly true if $\{\beta_i, \gamma_i\} \subseteq C_j$. Thus, we may assume, by symmetry, that C_j contains β_i but not γ_i . Then M/β_i has C'_j as a circuit. Clearly C'_j is contained in $(B_j \cup e) - \{\beta_i, \gamma_i\}$. In M/β_i , the set $B_j - \beta_i$ is a basis and $(B_j \cup e) - \beta_i$ contains a unique circuit, namely C'_j . Thus $\gamma_i \notin \text{cl}_{M/\beta_i}(C'_j)$, so $M/\{\beta_i, \gamma_i\}$ has C'_j as a circuit. Hence so does $M/\{\beta_i, \gamma_i\} \setminus \alpha_i$. Thus (2.5.1.4) holds.

2.5.1.5. In M' , the set \mathbf{Q}' is precisely the set of minimal sets of petals of Φ whose closure contains e .

To see this, observe that, in M' , the set $E(N \setminus A) - T_j$ is a basis B'_j , and $B'_j \cup e$ contains a unique circuit. As $B'_j \cup e$ contains C'_j , that circuit is C'_j . The set of triangles intersecting this circuit is exactly \mathcal{Q}'_j , and (2.5.1.5) follows without difficulty.

2.5.1.6. The members of \mathbf{Q}' are distinct.

Suppose $1 < s < t \leq k$. As $\mathcal{Q}_s \neq \mathcal{Q}_t$, clearly $\mathcal{Q}_s - \{T_i\} \neq \mathcal{Q}_t - \{T_i\}$, that is, $\mathcal{Q}'_s \neq \mathcal{Q}'_t$. Now suppose $\mathcal{Q}'_1 = \mathcal{Q}'_s$. Then \mathcal{Q}_1 is a proper subset of \mathcal{Q}_s , contradicting (2.5.1.1).

We have now shown that when $T_i \in \mathcal{X}_1$ and $|\mathcal{X}_1| > 1$, we can construct a new matroid M' from M so that $M' \setminus e$ has a copaddle Φ' whose petals are all the petals of the copaddle Φ of $M \setminus e$ except for T_i . In particular, $M' = M/T_i$. Moreover, $(\mathcal{X}_1 - \{T_i\}, \mathcal{X}_2, \dots, \mathcal{X}_k)$ partitions the set of petals of $M' \setminus e$ and $(\mathcal{Q}_1 - \{T_i\}, \mathcal{Q}_2 - \{T_i\}, \dots, \mathcal{Q}_k - \{T_i\})$ is a collection of distinct sets coinciding with the set of minimal sets of petals of Φ' whose union spans e . We may repeat this process of shrinking the size of the matroid we are dealing with until each \mathcal{X}_i is reduced to containing a single petal, that is, $|\mathcal{X}_i| = 1$ for all i in $[k]$. We now consider this case, letting the matroid in which it occurs be M_0 . Then $M_0 \setminus e = N_0 \cong M^*(K_{3,k})$. Thus N_0 has a copaddle Φ_0 whose petals are triangles. By relabelling if necessary, we may assume that these triangles are T_1, T_2, \dots, T_k where $\mathcal{X}_i = \{T_i\}$ for all i in $[k]$.

Let \mathbf{P}_0 be the set of petals of Φ_0 . By construction, the minimal sets of petals of Φ_0 whose closure contains e are precisely the k sets $\mathbf{P}_0 - \{T_i\}$ for all i in $[k]$. As before, let $T_i = \{\alpha_i, \beta_i, \gamma_i\}$ for all i in $[k]$, and let $A = \{\alpha_1, \alpha_2, \dots, \alpha_k\}$.

Suppose i and j are distinct elements of $[k]$. We show next that

$$\mathbf{2.5.1.7.} \quad r_{M_0 \setminus A}^*(\{\beta_i, \gamma_i, \beta_j, \gamma_j\}) = 3.$$

Let $Y = \{\beta_i, \gamma_i, \beta_j, \gamma_j\}$. The set $E(M_0 \setminus A) - (Y \cup e)$ is independent and does not span e . Hence $r_{M_0 \setminus A}(E(M_0 \setminus A) - Y) = 2(k - 2) + 1$, so

$$\begin{aligned} r_{M_0 \setminus A}^*(Y) &= |Y| + r_{M_0 \setminus A}(E(M_0 \setminus A) - Y) - r(M_0 \setminus A) \\ &= 4 + 2(k - 2) + 1 - 2(k - 1) \\ &= 3. \end{aligned}$$

Thus (2.5.1.7) holds.

Consider M_0^* . It has rank $k + 3$ since $M_0^*/e = (M_0 \setminus e)^* \cong M(K_{3,k})$. Now $M(K_{3,k})$ has T_1, T_2, \dots, T_k as triads and, by orthogonality, has A as an independent set. Thus $r_{M_0^*}(A) = k$, so $r(M_0^*/A) = r(M_0^*) - k = 3$.

By (2.5.1.7), the lines of M_0^*/A spanned by $\{\beta_i, \gamma_i\}$ and $\{\beta_j, \gamma_j\}$ are distinct for all distinct i and j in $[k]$. Thus the k lines spanned by $\{\beta_1, \gamma_1\}, \{\beta_2, \gamma_2\}, \dots, \{\beta_k, \gamma_k\}$ are all distinct.

Next observe that

2.5.1.8. M_0^*/A has at least $(2k)^{1/2} + \frac{1}{2}$ distinct parallel classes.

Suppose that there are exactly p points in the simplification of the rank-3 matroid M_0^*/A . The number of distinct lines determined by these points is at most $\binom{p}{2}$, that is, $\frac{1}{2}(p^2 - p)$. But M_0^*/A has at least k distinct lines, so $k \leq \frac{1}{2}(p^2 - p)$. Thus $2k + \frac{1}{4} \leq p^2 - p + \frac{1}{4}$. Hence $2k + \frac{1}{4} \leq (p - \frac{1}{2})^2$, and (2.5.1.8) holds.

By Theorem 2.0.11, as $r(M_0^*/A) = 3$ and M_0^*/A has no $U_{2,m+2}$ -minor, M_0^*/A has at most $\frac{m^3-1}{m-1}$ distinct parallel classes. Thus

$$(2k)^{1/2} + \frac{1}{2} \leq \frac{m^3 - 1}{m - 1} = m^2 + m + 1,$$

so

$$(2k)^{1/2} \leq m^2 + m + \frac{1}{2} < (m + 2^{-1/2})^2.$$

Hence

$$(2k)^{1/4} - 2^{-1/2} < m.$$

But $k \geq m^4$, so $2^{1/4}m - 2^{-1/2} < m$ and it follows that $m < 2^{-1/2}(2^{1/4} - 1)^{-1} < 4$. This contradiction completes the proof of the theorem. \square

2.6 The Main Theorem

In this section, we prove the main result, which is restated below. Recall that \mathcal{T}_n denotes the set of rank- n spikes having a tip and a cotip. We let \mathcal{S}_n denote the set of rank- n spikes having neither a tip nor a cotip.

Theorem 2.0.6. *Let M be a 3-connected matroid, and let e be an element of M . For every integer $n > 2$, there is an integer $g(n)$ so that if $|E(M)| \geq g(n)$, then e is an element of a minor of $E(M)$ that is isomorphic to the rank- n wheel or whirl, the cycle or bond matroid of $K_{1,1,1,n}$, $M(K_{3,n})^+$ or its dual, $U_{2,n}$ or $U_{n-2,n}$, or a member of \mathcal{T}_n .*

Proof. By Theorem 1.8.5, there is a function f_2 so that if $|E(M)| \geq f_2(n^{10})$, then M has a minor isomorphic to a member of the set $\mathcal{M} = \mathcal{S}_{n^{10}} \cup \{M(\mathcal{W}_{n^{10}}), \mathcal{W}^{n^{10}}, M(K_{3,n^{10}}), M^*(K_{3,n^{10}}), U_{2,n^{10}}, U_{n^{10}-2,n^{10}}\}$. By Theorem 2.0.10, M has a connected minor N using e so that N/e or $N \setminus e$ is a member of \mathcal{M} .

If N has a U_{2,n^2} -minor, then, by Theorem 2.1.1, as $n^2 \geq 3$ by assumption, there is an $m \geq n$ so that N has a $U_{2,m}$ -minor containing e . Dually, if N has a U_{n^2-2,n^2} -minor, then N has a $U_{n-2,n}$ -minor containing e . Therefore, we will assume that M has no minor isomorphic to U_{2,n^2} or U_{n^2-2,n^2} .

Consider the case when $N/e \in \mathcal{M}$. Then, since \mathcal{M} is closed under duality, $N^* \setminus e \in \mathcal{M}$. In the theorem statement, the list of potential minors of M containing e is also closed under duality, so we may assume that $N \setminus e \in \mathcal{M}$. As N has no minor isomorphic to U_{2,n^2} or U_{n^2-2,n^2} , we deduce that $N \setminus e$ is a member of $\mathcal{S}_{n^{10}} \cup \{M(\mathcal{W}_{n^{10}}), \mathcal{W}^{n^{10}}, M(K_{3,n^{10}}), M^*(K_{3,n^{10}})\}$.

Suppose first that $N \setminus e \in \mathcal{S}_{n^{10}}$. Choose an element x of $E(N) - e$ that is not parallel to e in N . Then $N/x \setminus e$ is a rank- $(n^{10} - 1)$ spike with a tip and no cotip. Hence N/x is connected. Let y be an element of $N/x \setminus e$ other than the tip. Then $N/x \setminus y \setminus e \in \mathcal{T}_{n^{10}-1}$, and $N/x \setminus y$ is connected. By Theorem 2.2.2, as $n^{10} - 1 \geq 6$ by assumption, there is an $m \geq \frac{n^{10}-1}{2}$ so that $N/x \setminus y$ has a \mathcal{T}_m -minor containing e .

Next suppose that $N \setminus e \in \{M(\mathcal{W}_{n^{10}}), \mathcal{W}^{n^{10}}\}$. Then, by Theorem 2.0.10 and duality, we may assume that N has a connected minor N' containing e so that $N' \setminus e \in \{M(\mathcal{W}_{n^6}), \mathcal{W}^{n^6}\}$. We have assumed that N has no minor isomorphic to U_{2,n^2} or U_{n^2-2,n^2} . Thus, by Theorem 2.3.3, there is an $m \geq n^2$ so that N' and hence N has a minor that contains e and is isomorphic to $M(\mathcal{W}_m)$ or \mathcal{W}^m .

Now let $N \setminus e = M(K_{3,n^{10}})$. Then $N \setminus e$ is a paddle whose petals are triads. As $n^{10} \geq 4$, we can find petals P_1, P_2 , and P_3 of $N \setminus e$ none of whose elements is parallel to e in N . Moreover, there are elements e_1, e_2 , and e_3 of P_1, P_2 , and P_3 , respectively, such that $\text{si}(N \setminus e / e_1, e_2) \cong M(K_{1,1,1,n^{10}-2}) \cong \text{si}(N \setminus e / e_1, e_3)$. Thus N has a connected minor N' containing e so that $N' \setminus e = M(K_{1,1,1,n^{10}-2})$ unless both $\{e_1, e_2, e\}$ and $\{e_1, e_3, e\}$ are circuits of N . The exceptional case cannot arise since it implies the contradiction that $\{e_1, e_2, e_3\}$ is a circuit of $N \setminus e$. Therefore we can apply Theorem 2.4.1 to N' to get that there is an $m \geq \frac{n^{10}}{2}$ so that N' and hence N has a minor that contains e and is isomorphic to $M(K_{1,1,1,m})$, $M(K_{3,m})^+$, or some member of \mathcal{T}_m .

Finally, suppose that $N \setminus e = M^*(K_{3,n^{10}})$. Since N has no minor isomorphic to U_{2,n^2} or U_{n^2-2,n^2} and $n^2 \geq 4$, it follows by Theorem 2.5.1 that there is an $m \geq n^2$ so that N has a minor that contains e and is isomorphic to $M^*(K_{1,1,1,m-1})$. \square

Chapter 3

Pairs of Elements in Unavoidable Minors of 3-Connected Binary Matroids

In this chapter, we will extend Corollary 2.0.7 by examining the case where a 3-connected binary matroid M contains two special elements, x and y , that we would like to include in a highly structured minor. From Corollary 2.0.7, x is an element of a minor of M that is isomorphic to one of four highly structured minors. By examining each of these cases individually, we find that x and y are elements of a minor of M that is isomorphic to one of these four matroids. The following is the main result of this chapter.

Theorem 3.0.1. *Let M be a 3-connected binary matroid, and let x and y be elements of M . For every integer n greater than 2, there is an integer $h(n)$ so that if $|E(M)| > h(n)$, then x and y are elements of a minor of M that is isomorphic to $M(\mathcal{W}_n)$, the vector matroid of the binary matrix $[I_n|A_n]$, or the cycle or bond matroid of $K_{1,1,1,n}$.*

If we restrict our interest to graphic matroids, the next corollary follows from this theorem.

Corollary 3.0.2. *Let G be a 3-connected graph with edges e and f . For every integer n greater than 2, there is an integer $j(n)$ so that if $|E(G)| > j(n)$, then e and f are edges of a minor of G that is isomorphic to \mathcal{W}_n or $K_{1,1,1,n}$.*

3.1 Background

First, we recall the definition of a fan used in this dissertation. In a simple, cosimple matroid M , consider a sequence $(s_0, r_1, s_1, \dots, s_{n-1}, r_n, s_n)$ of distinct elements of M so that every set $\{s_{i-1}, r_i, s_i\}$ with $0 < i \leq n$ is a triangle of M and every set $\{r_j, s_j, r_{j+1}\}$ with $0 < j < n$ is a triad of M . Here we call such a sequence a *fan*, noting that this specializes the terminology used in [13], where two other related structures are also called fans. In this chapter, we

will rely heavily on a modification of the next theorem, which is a result of Bixby and Coullard [4] (see also [13, p. 479]).

Theorem 3.1.1. *Let N be a 3-connected minor of a 3-connected matroid M . Suppose that $|E(N)| \geq 4$, $x \in E(M) - E(N)$, and that M has no 3-connected proper minor that both contains x and has N as a minor. Then, for some (N_1, M_1) in $\{(N, M), (N^*, M^*)\}$, one of the following holds:*

(i) $N_1 = M_1 \setminus x$.

(ii) $N_1 = M_1 \setminus x/e$, and N_1 has an element t so that $\{e, x, t\}$ is a circuit of M_1 .

(iii) $N_1 = M_1 \setminus x, e/f$ where M_1 has a fan (x, f, t, e) . Moreover, $M_1 \setminus x$ is 3-connected.

(iv) $N_1 = M_1 \setminus x, e, f$ where M_1 has a fan (t, e, x, f, s) .

(v) $N_1 = M_1 \setminus x, e/f, g$ where M_1 has a fan (x, f, t, e, g) . Moreover, $M_1 \setminus x$ and $M_1 \setminus x/f$ are 3-connected.

The next basic connectivity result is known as Bixby's Lemma [3] (see also [13, p.333]).

Lemma 3.1.2. *Let M be a 3-connected matroid and suppose $e \in E(M)$. Then either $M \setminus e$ or M/e has no non-minimal 2-separations, so either $\text{si}(M/e)$ or $\text{si}(M^*/e)$ is 3-connected.*

This chapter will employ grafts, which are discussed in [13, Section 10.3]. A *graft* is a pair (G, γ) where G is a graph and γ is a subset of vertices of G . We will say the graft element is incident with the vertices in γ . The *incidence matrix*, $A_{(G, \gamma)}$, of (G, γ) is the matrix that is obtained from the mod-2 vertex-edge incidence matrix of G by adjoining a new column, e_γ corresponding to γ . Specifically, e_γ is the incidence vector of the set γ , that is, e_γ has a 1 in each row corresponding to a vertex of γ and a 0 in every other row. The matroid $M(G, \gamma)$ associated with the graft (G, γ) is the vector matroid $M[A_{(G, \gamma)}]$ where $A_{(G, \gamma)}$ is viewed as a matrix over $GF(2)$. Thus the *graft matroid*, $M(G, \gamma)$ has ground set $E(G) \cup e_\gamma$. If the graft

element is incident with an odd number of vertices, this element is a coloop in M . In this chapter, we will require any graft element to contain an even number of elements.

Let (G, γ) be a graft, and let $e \in E(G)$. To obtain the *deletion* $(G, \gamma) \setminus e$ and the *contraction* $(G, \gamma) / e$ of e from (G, γ) , we delete or contract e from G leaving the set of vertices of γ unchanged except when e is contracted and has distinct ends u and v . In the exceptional case, $(G, \gamma) / e = (G/e, \gamma')$ where the vertex w that results from identifying u and v is in γ' if and only if exactly one of u and v is. Equivalently, $A_{(G/e, \gamma')}$ is obtained from $A_{(G, \gamma)}$ by deleting column e and replacing rows u and v by a single row equal to their sum modulo 2. Notice that if $|\gamma|$ is even, then so is $|\gamma'|$. The *minors* of (G, γ) are those grafts that can be produced by a sequence of single-edge deletions and contractions. It is routine to check that $M((G, \gamma) \setminus e) = M(G, \gamma) \setminus e$ and that $M((G, \gamma) / e) = M(G, \gamma) / e$.

As in the previous chapter, the reader familiar with the matroid concept of roundedness may be reminded of it by the results of this chapter. Roundedness was introduced by Seymour [20] to encompass certain results that were concerned with relating particular minors of a matroid to specific elements of the matroid. The next lemma contains two examples of such results. The first part follows by combining results of Seymour [21] and Oxley and Reid [14] (see also [13, p.481]). The second part follows from the first.

Lemma 3.1.3. *Let $t \in \{3, 4\}$ and let M be a binary matroid with an $M(\mathcal{W}_t)$ -minor.*

- (i) *If M is 3-connected and $e, f \in E(M)$, then M has an $M(\mathcal{W}_t)$ -minor using $\{e, f\}$.*
- (ii) *If M is 2-connected and $e \in E(M)$, then M has an $M(\mathcal{W}_t)$ -minor using $\{e\}$.*

3.2 A Modification of Bixby and Coullard's Theorem

Theorem 3.1.1 reveals that if M is a 3-connected matroid with a 3-connected minor N and a fixed element x , then M has a 3-connected minor M' that uses x , has N as a minor, and has at most four elements more than N . As noted in [4], it is easy to see that M'' , a smallest minor of M that uses x and has a minor isomorphic to N , has at most $|E(N)| + 1$ elements.

In this section, we consider the case where M'' must use a specified element of N . We will show that, in this case, M'' has at most $|E(N)| + 2$ elements and prove the following result.

Theorem 3.2.1. *Let N be a 3-connected minor of a 3-connected matroid M with $|E(N)| \geq 4$. Let $x \in E(M) - E(N)$ and $y \in E(N)$. Suppose M has no 3-connected proper minor that both contains x and y and has a minor isomorphic to N . Then, for some (N_1, M_1) such that either $N_1 \cong N$ and $M_1 \cong M$ or $N_1 \cong N^*$ and $M_1 \cong M^*$, one of the following holds:*

- (i) $N_1 = M_1 \setminus x$, and y is contained in this minor.
- (ii) $N_1 = M_1 \setminus x/z$, and this minor contains y and an element s so that $\{x, z, s\}$ is a circuit of M_1 .

Proof. As M has N as a minor, but no proper minor contains $\{x, y\}$ and a minor isomorphic to N , it follows that M has no proper minor that uses x and has N as a minor. Thus, for some (N_1, M_1) in $\{(N, M), (N^*, M^*)\}$, one of five cases identified in Theorem 3.1.1 holds.

In case (v), $N_1 = M_1 \setminus x, e/f, g$ where M_1 has (x, f, t, e, g) as a fan (see Figure 3.1). Now M_1/f has $\{t, x\}$ as a circuit, $M_1 \setminus t$ has $\{e, f\}$ as a cocircuit, and M_1/e has $\{t, g\}$ as

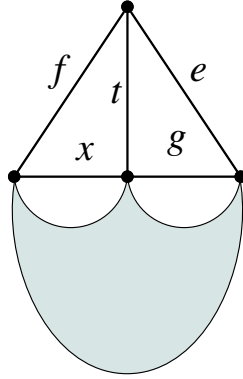


FIGURE 3.1. Part of M_1 in case (v) that behaves like a graph (see [13, p. 480]).

a circuit. Thus $M_1 \setminus x/f = M_1/f \setminus x \cong M_1/f \setminus t \cong M_1/e \setminus t \cong M_1/e \setminus g$. Hence $M_1/e \setminus g$ is a 3-connected proper minor of M_1 containing x and every element of N_1 , and $M_1/e \setminus g$ has a minor isomorphic to N_1 ; a contradiction.

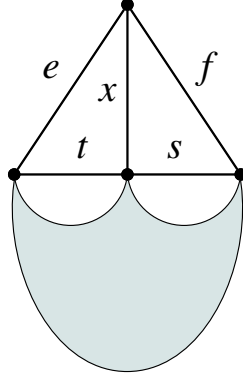


FIGURE 3.2. Part of M_1 in case (iv) that behaves like a graph (see [13, p. 480]).

In case (iv), $N_1 = M_1 \setminus x, e, f$ where M_1 has (t, e, x, f, s) as a fan (see Figure 3.2). By symmetry, we may assume $t \neq y$. Now $N_1 = M_1 \setminus x, f/e$. As $\{t, x\}$ is a circuit of M_1/e , it follows that $N_1 \cong M_1 \setminus t, f/e$. But $\{x, y\}$ is a subset of $E(M_1 \setminus t, f/e)$, so we have a contradiction.

In case (iii), $N_1 = M_1 \setminus x, e/f$ and (x, f, t, e) is a fan of M_1 (see Figure 3.3). As M_1/f has

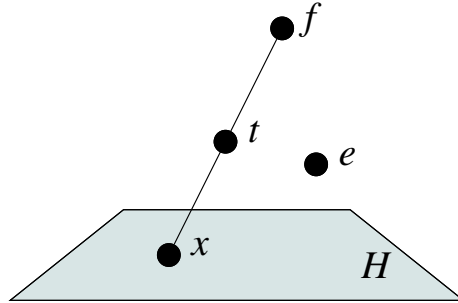


FIGURE 3.3. The matroid M_1 in case (iii) with hyperplane H complementing triad $\{f, t, e\}$.

$\{x, t\}$ as a circuit, $N_1 \cong M_1 \setminus t, e/f$. Thus $M_1 \setminus t, e/f$ contradicts the minimality of M_1 unless $t = y$. Consider the exceptional case. Since $M_1 \setminus e$ has $\{x, f, y\}$ as a circuit and $\{f, y\}$ as a cocircuit, $(\{x, f, y\}, E(M_1 \setminus e) - \{x, f, y\})$ is a non-minimal 2-separation of $M_1 \setminus e$. Hence, by Bixby's Lemma, M_1/e has no non-minimal 2-separations, so $\text{si}(M_1/e)$ is 3-connected. Recall that $N_1 \cong M_1 \setminus y, e/f$. Since $\{t, e, f\}$ is a cocircuit of M_1 , it follows that $N_1 \cong M_1 \setminus y, e, f \cong M_1 \setminus y, f/e$. Hence N_1 is a minor of M_1/e , and hence of $\text{si}(M_1/e)$. By the minimality of M_1 , it follows that $\text{si}(M_1/e)$ does not contain $\{x, y\}$. Hence M_1/e has $\{x, y\}$ as a circuit, so M_1 has $\{e, x, y\}$ as a circuit. Thus $\{e, f, x, y\}$ is 2-separating in M_1 ; a contradiction. \square

3.3 A Wheel-Minor Containing x and y

In this section, we consider the case where a 3-connected matroid with two identified elements, x and y , has a large wheel-minor. As this minor is a graphic matroid, it will be useful to refer to the graph shown in Figure 3.4.

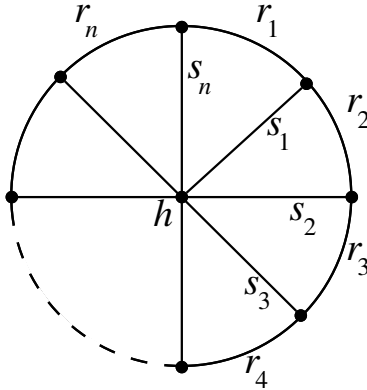


FIGURE 3.4. Graph of a rank- n wheel with some spoke and rim edges labelled and with the hub vertex labelled h .

Now, we develop two lemmas. The first lemma relates to case (i) identified in Theorem 3.2.1, and we assume the removal of one element of a 3-connected binary matroid produces a wheel-minor.

Lemma 3.3.1. *Let M be a 3-connected binary matroid with distinct elements x and y . Suppose M has a minor $N \cong M(\mathcal{W}_k)$ for some integer k greater than 2 and that $|E(M) - E(N)| = 1$. Then there is an integer m with $m \geq \frac{k}{4}$ so that x and y are elements of a minor of M that is isomorphic to $M(\mathcal{W}_m)$.*

Proof. By Lemma 3.1.3, this theorem holds for $k \leq 16$. We will assume $k \geq 17$.

If $x, y \in E(N)$, then N is a large wheel-minor of M containing x and y . We will assume that $x \in E(M) - E(N)$. In addition, by duality, we may assume $M \setminus x = N$. As M is binary and a single-element extension of a graph, M is the matroid of a graft (G, γ_x) with $G \cong \mathcal{W}_k$ where x corresponds to the graft element incident with the set $\gamma_x \subseteq V(G)$. The hub vertex of G is labelled h as shown in Figure 3.4.

This proof is divided into four main cases depending on whether h is in γ_x and whether y corresponds to a spoke or rim edge of G . We will operate on the matroid M by operating exclusively on the graft (G, γ_x) , and we operate on this graft as described in the Section 3.1. Label the edge of (G, γ_x) corresponding to y as edge e_y .

First, assume that $h \in \gamma_x$, and e_y is a spoke of G . One endpoint of e_y is h , label the other v . As established in Section 3.1, $|\gamma_x|$ is even. Since x is not parallel to any element of M , the set γ_x contains h and at least three other vertices. Now, assume $v \notin \gamma_x$. Let P be the shortest path along the rim of G that both contains v and has endpoints in γ_x . Label the end points of this path u and w . For the sake of notation, we also relabel graph G as G' and set γ_x as γ'_x . If, however, $v \in \gamma_x$, we will operate on the graft to remove v from this set.

Label the vertex of γ_x that is the shortest distance along the rim from v as vertex v' . Contract the edges of the shortest path from v to v' along the rim of G , noting that at most $\frac{k-1}{2}$ edges are removed this way. Label the vertex resulting from these contractions as vertex v . Simplify the underlying graph without removing e_y to produce the graft (G', γ'_x) with $G' \cong \mathcal{W}_n$ for some $n \geq \frac{k+1}{2}$ and $\gamma'_x = \gamma_x - \{v, v'\}$. If $|\gamma'_x| = 2$, then the graft element is an edge. Here, the incidences of the graft element are the endpoints of an edge f . Therefore $(G', \gamma'_x) \setminus f \cong \mathcal{W}_n$, and we have identified a wheel-minor containing edges corresponding to elements x and y . We may assume, instead, that γ'_x contains h and at least three other vertices. Let P be the shortest path along the rim of G' that both contains v and has endpoints in γ'_x . Label the end points of this path u and w .

The graft (G', γ'_x) now appears as illustrated in Figure 3.5. We now partition the rim edges of G' into $E(P)$, and two other sets. For convenience, we color one set red and the other set blue in the following way. Consider the $|\gamma'_x - h|$ paths of $G' \setminus h$ with both endpoints in $\gamma'_x - h$ and with no two distinct paths sharing an edge. Let \mathcal{P} be the edge sets of these paths. Clearly, $E(P) \in \mathcal{P}$. As $|\gamma'_x - h|$ is odd, $|\mathcal{P} - E(P)|$ is even. Color the edges of

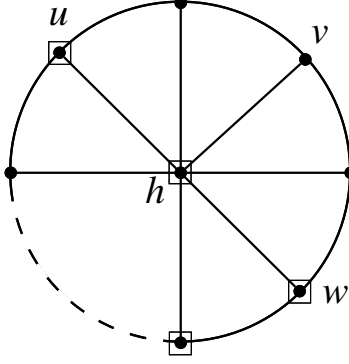


FIGURE 3.5. Graft (G', γ'_x) with boxes around vertices of γ'_x .

$E(G' \setminus h) - E(P)$ so that every vertex of $\gamma'_x - \{u, w\}$ meets one red edge and one blue edge and for every $P_i \in \mathcal{P} - E(P)$, the edges in P_i are monochromatic.

Without loss of generality, there are at least as many blue edges as red edges. Contract the red edges and simplify the underlying graph without deleting e_y . The resulting graft, (G'', γ''_x) , has $G'' \cong \mathcal{W}_m$ and $\gamma''_x = \{h, u\}$ or $\gamma''_x = \{h, w\}$. Thus the graft element is an edge parallel to a spoke f of G'' . Recall e_y is incident with h and v , and $v \notin \{u, w\}$. Therefore, $(G'', \gamma''_x) \setminus f \cong \mathcal{W}_m$ with $m \geq \frac{1}{2}(\frac{k+1}{2}) = \frac{k+1}{4}$, and this minor contains edges corresponding to x and y .

Next, we assume that $h \in \gamma_x$ and that e_y is a rim element. Let P be the shortest path of $G \setminus h$ that both contains e_y and has endpoints in γ_x . Label these endpoints u and w . Again, it will be convenient to partition the rim edges of G into $E(P)$, a red set, and a blue set. Consider the $|\gamma_x - h|$ paths of $G \setminus h$ with both endpoints in $\gamma_x \cup h$ and with no two distinct paths sharing an edge. Let \mathcal{P} be the edge sets of these paths. Clearly $E(P) \in \mathcal{P}$. Since $|\gamma_x - h|$ is odd, $\mathcal{P} - E(P)$ is even. Color the edges of $E(G \setminus h) - E(P)$ so that every vertex of $\gamma_x - \{u, w\}$ meets one red edge and one blue edge and for every $P_i \in \mathcal{P} - E(P)$, the edges in P_i are monochromatic.

Without loss of generality, there are no more than $\frac{k - |E(P)|}{2}$ red edges. Contract the red edges and simplify the underlying graph without deleting e_y . The resulting graft, (G', γ'_x) , has $G' \cong \mathcal{W}_m$ and $\gamma'_x = \{h, u\}$ or $\gamma'_x = \{h, w\}$. The edge e_y lies on the rim of G' . Therefore,

in $M(G', \gamma'_x)$, the element x is parallel to the element corresponding to a spoke edge of G' , and this matroid can be simplified without deleting x or y to produce a \mathcal{W}_m -minor with $m \geq k - \frac{k-|E(P)|}{2} \geq \frac{k+1}{2}$.

Finally, we may assume $h \notin \gamma_x$. Partition the edges of $G \setminus h$ into a red set and a blue set in the following way. Consider the $|\gamma_x|$ paths of $G \setminus h$ with both endpoints in γ_x and with no two distinct paths having a common edge. Let \mathcal{P} be the edge sets of these paths. As $|\gamma_x|$ is even, $|\mathcal{P}|$ is even. Color the edges of $G \setminus h$ so that every vertex of γ_x meets one red edge and one blue edge and for every $P_i \in \mathcal{P}$, the edges in P_i are monochromatic.

Without loss of generality, there are at most $\frac{k}{2}$ red edges. If e_y is not red, then contract all but one of the red edges. Simplify the underlying graph without deleting e_y to produce the graft (G', γ'_x) with $G' \cong \mathcal{W}_m$ and $\gamma'_x = \{u, w\}$, where u and w are endpoints of the remaining red edge of G' . Thus, we delete the remaining red edge to produce a \mathcal{W}_m -minor using the edge corresponding to x and e_y with $m \geq k - (\frac{k}{2} - 1) \geq \frac{k}{2} + 1$.

We may assume, then, that e_y is red in G . As M is 3-connected, G has another red edge, e_z . Contract the red edges other than e_y and e_z from (G, γ_x) and simplify the underlying graph to produce (G', γ'_x) , one of the grafts shown in Figure 3.6 with either $\gamma'_x = \{u, v\}$

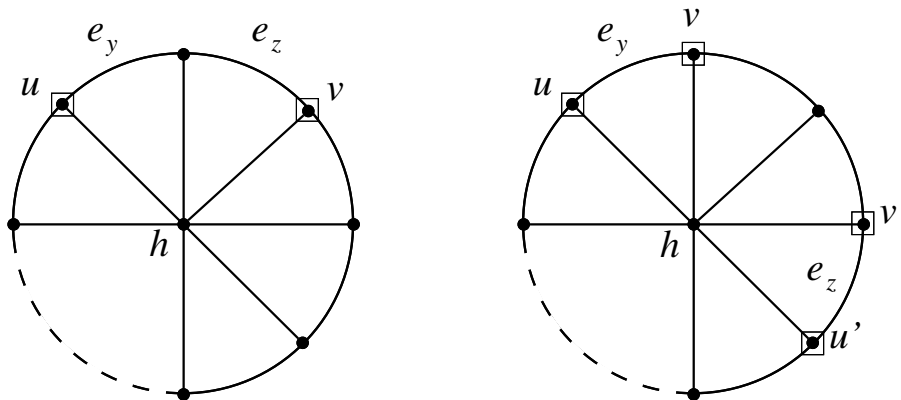


FIGURE 3.6. The two possible configurations of (G', γ'_x) with e_y and e_z as shown and with a box around every vertex of γ'_x .

or $\gamma'_x = \{u, u', v, v'\}$. If e_y and e_z are not adjacent in G' (see the graft shown on the right of Figure 3.6), then we will contract edges to make them adjacent. Without loss of

generality, the path of blue edges of G' between v and v' has at most half of the blue edges. Contract these edges and simplify the underlying graph. Now, in either case, the graft element has two incidences and so is an edge e_x in a graph H shown in Figure 3.7. We

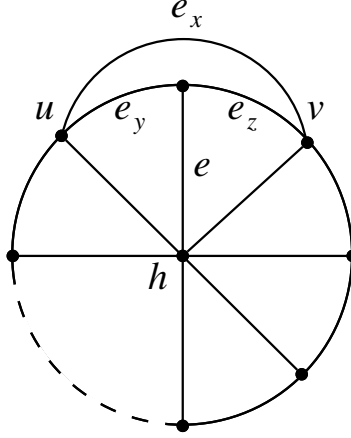


FIGURE 3.7. The graph H .

can simplify the graph H/e without deleting e_x or e_y to produce a graph isomorphic to \mathcal{W}_m with $m \geq \frac{1}{2}(k - (\frac{k}{2} - 2)) - 1 = \frac{k}{4}$. \square

We have considered the case where the removal of one element from a 3-connected binary matroid results in a wheel. Before considering the next case, we require a technical lemma.

For an integer $k \geq 3$, let $[I_k|D_k]$ be the following binary matrix.

$$\left[\begin{array}{cccc|ccccc} b_1 & b_2 & b_3 & \dots & b_k & a_1 & a_2 & a_3 & \dots & a_k \\ \hline & & & & & 1 & 0 & 0 & \dots & 1 \\ & & & & & 1 & 1 & 0 & \dots & 0 \\ & & & & & 0 & 1 & 1 & \dots & 0 \\ & & & & & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & & & 0 & 0 & 0 & \dots & 1 \end{array} \right]$$

Then $M[I_k|D_k] \cong M(\mathcal{W}_k)$. The spoke and rim edges of \mathcal{W}_k correspond to the column vectors labelled b_i and a_i , respectively, for $i \in [k]$. Let $V(k, 2)$ be the k -dimensional vector space over $\text{GF}(2)$ and view it's elements as column vectors.

Lemma 3.3.2. *Every vector of $V(k, 2)$ that has an even number of ones is spanned by $\{a_1, a_2, \dots, a_k\}$.*

Proof. The set of vectors (x_1, x_2, \dots, x_k) so that $\sum_{i=1}^k x_i \equiv_2 0$ forms a hyperplane H of $V(k, 2)$. This hyperplane contains $\{a_1, a_2, \dots, a_k\}$. As the last set is a circuit of $M[I_k|D_k]$, it has rank k , and so spans H . \square

We now consider the case where two elements must be removed to form the wheel-minor, which corresponds with case (ii) in Theorem 3.2.1. It will be convenient to represent the rank- n wheel geometrically as shown in Figure 3.8.

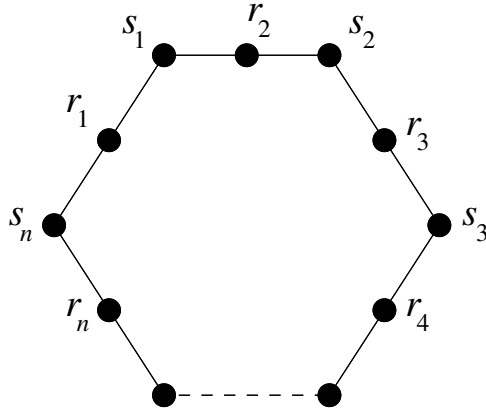


FIGURE 3.8. A geometric illustration of the rank- n wheel with labels corresponding to those in Figure 3.4.

Lemma 3.3.3. *Let M be a 3-connected binary matroid with $M \setminus x / f = N \cong M(\mathcal{W}_k)$ for some integer k greater than 2, and let $y \in E(N)$. Suppose N has an element s so that $\{x, f, s\}$ is a circuit of M . Then there is an integer m with $m \geq \frac{k}{4}$ so that x and y are elements of a minor of M that is isomorphic to $M(\mathcal{W}_m)$.*

Proof. By Lemma 3.1.3, this theorem holds for $k \leq 16$. We will assume $k \geq 17$. Since $\{x, f, s\}$ is a circuit, M/f has x parallel to s . Therefore, the matroid $M/f \setminus s \cong M/f \setminus x$, which is isomorphic to $M(\mathcal{W}_k)$. This large wheel-minor contains x and y as long as $y \neq s$. We may assume, then, that $y = s$.

We consider the following cases:

(I) y is a spoke element of N ; and

(II) y is a rim element of N .

In M^* , the set $\{x, f, y\}$ is a cocircuit. Label the complementing hyperplane H . As $M^* \setminus f/x = N^*$, the matroid $M^*|H = N^* \setminus y$. As we are working in the dual, the element y will be a rim element of N^* in case I and a spoke element of N^* in case II. The matroid M^* is represented in Figure 3.9. There is a unique binary matroid, M_1 obtained by adding z to H to form a

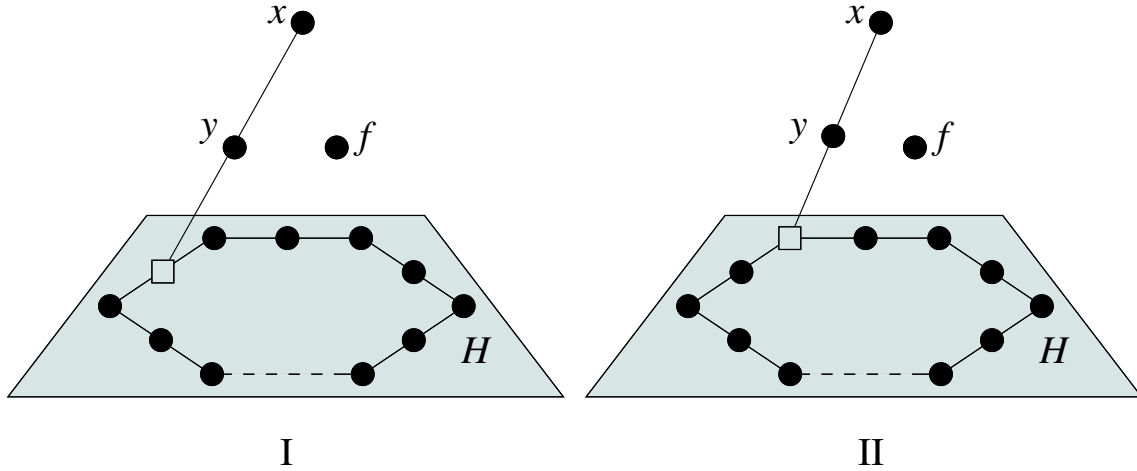


FIGURE 3.9. Geometric illustration of M^* for case I and case II.

triangle with x and y . The matroid $M_1 \setminus f/x$ has z parallel to y , and it is easy to see that $M_1|(H \cup z) \cong N^*$. While M_1 is not and M_1/z may not be a minor of M^* , the matroid $M_1 \setminus z$ is a minor of M^* . We take care throughout this proof to avoid contracting z and to ensure z is deleted to produce a minor of M^* .

The matroid M_1/f is $M_1|(H \cup z)$ with two extra elements, x and y (see Figure 3.10). First we consider case II. The following matrix represents M_1/f .

$$\begin{array}{c}
z \quad e_2 \quad e_3 \quad \dots \quad e_k \\
\left[\begin{array}{c|cccccccc}
e_{k+1} & e_{k+2} & e_{k+3} & \dots & e_{2k} & x & y \\
\hline
1 & 0 & 0 & \dots & 1 & 1 & 0 \\
1 & 1 & 0 & \dots & 0 & a_2 & a_2 \\
0 & 1 & 1 & \dots & 0 & a_3 & a_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 1 & a_k & a_k
\end{array} \right]
\end{array}$$

The sum $\sum_{i=2}^k a_i$ is odd or even, so, by Lemma 3.3.2, x or y , respectively, is spanned by the circuit of rim elements $C = \{e_{k+1}, e_{k+2}, \dots, e_{2k}\}$. Without loss of generality, $x \in \text{cl}(C)$, and $y \notin \text{cl}(C)$. The smallest circuit C_x containing x in $\{x, e_{k+1}, e_{k+2}, \dots, e_{2k}\}$ has at most $\frac{k}{2} + 1$ elements, otherwise a smaller circuit can be found in the symmetric difference of C_x and C . There is an $i \in [k]$ so that e_{k+i} is an element of $C_x - x$.

The matroid $(M_1/f)/(C_x - \{x, e_{k+i}\})$ has x parallel to e_{k+i} . Notice that y is not a loop of this matroid, as $y \notin \text{cl}_{M_1/f}(C)$. Simplify $(M_1/f)/(C_x - \{x, e_{k+i}\})$ without deleting x or y to produce M_2 (see Figure 3.11). It may be that triangle $\{x, y, z\}$ is a triangle of the rank- $r(M_2)$ wheel of M_2 . In this case, contract one rim element other than y to make z parallel to another element and delete z to produce a wheel-minor of M^* that uses x and y and has appropriate rank.

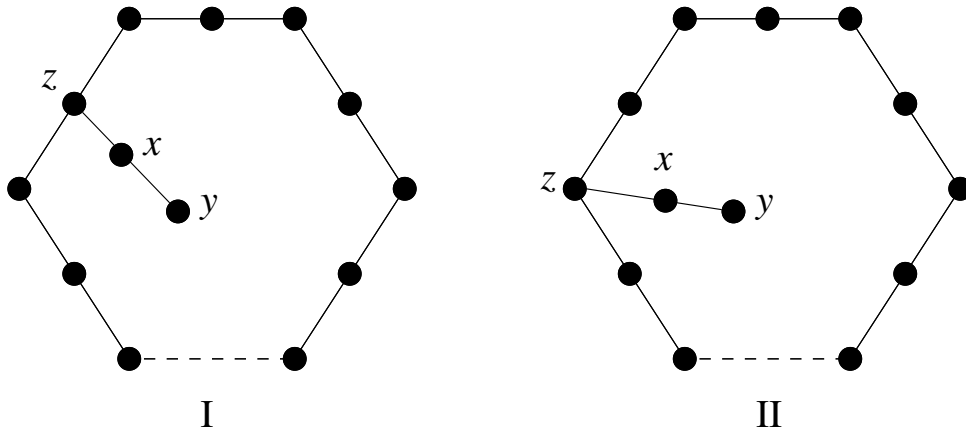


FIGURE 3.10. Geometric illustration of M_1/f for case I and case II.

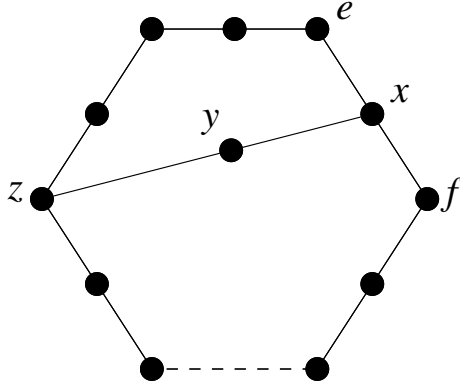


FIGURE 3.11. Geometric illustration of M_2 .

Otherwise, Figure 3.11 shows that triangle $\{x, y, z\}$ is a triangle of two different wheel-minors that are restrictions of M_2 and that the elements of these wheel-minors are the elements of M_2 . These wheel-minors share elements $\{e, f, x, y, z\}$ and contain x as a spoke element. Restrict M_2 to the larger of these wheel-minors. Contract one rim element to make z parallel to an element of M^* and delete z to identify a minor of M^* that uses x and y and is isomorphic to $M(\mathcal{W}_m)$ for some integer m with $m \geq \frac{r(M_2)+2}{2} - 1 \geq \frac{1}{2}(k - (|C_x| - 2)) \geq \frac{k+2}{4}$.

Now consider case I shown in Figure 3.10. The following matrix represents the matroid M_1/f .

$$\begin{array}{c}
 \begin{array}{cccccc}
 e_1 & e_2 & e_3 & \dots & e_k & \\
 \end{array} \\
 \left[\begin{array}{cccccc|cccc}
 & & & & & z & e_{k+2} & e_{k+3} & \dots & e_{2k} & x & y \\
 & & & & & 1 & 0 & 0 & \dots & 1 & a_1 & a_1 + 1 \\
 & & & & & 1 & 1 & 0 & \dots & 0 & a_2 & a_2 + 1 \\
 & & & & & 0 & 1 & 1 & \dots & 0 & a_3 & a_3 \\
 & & & & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
 & & & & & 0 & 0 & 0 & \dots & 1 & a_k & a_k
 \end{array} \right] \\
 \begin{array}{c}
 I_k
 \end{array}
 \end{array}$$

The vectors corresponding to x and y both have an even number of ones or they both have an odd number of ones.

First, assume the parity is even. Let $I = \{e_{k+2}, e_{k+3}, \dots, e_{2k}\}$. Since I spans circuit $I \cup z$, by Lemma 3.3.2, the vectors representing the elements of I span the hyperplane of column

vectors of $V(k, 2)$ with an even number of ones. Hence, the independent set I spans x and y . Therefore, sets $I \cup x$ and $I \cup y$ contain unique circuits, C_x and C_y , respectively.

As M_1/f is binary, the symmetric difference of circuits of this matroid is the disjoint union of circuits. Thus $\{x, y, z\} \Delta (I \cup z) = \{x, y\} \cup I$ is the union of disjoint circuits. The set I is independent, hence these disjoint circuits are precisely C_x and C_y , and $C_x \dot{\cup} C_y = \{x, y\} \cup I$. Without loss of generality, C_x is smaller than C_y , thus $|C_x| \leq \frac{|I \cup \{x, y\}|}{2} = \frac{(k-1)+2}{2}$.

There is an $i \in [k]$ so that e_{k+i} is an element of $C_x - x$. The matroid $M_1/f/(C_x - \{x, e_{k+i}\})$ has x parallel to e_{k+i} . Notice that y is not a loop of this matroid, as $y \notin \text{cl}_{M_1/f}(C_x)$. Simplify $(M_1/f)/(C_x - \{x, e_{k+i}\})$ without deleting x or y to produce M_2 .

It may be that x is in a triangle with e_1 or e_2 , as shown in Figure 3.12. As shown in the

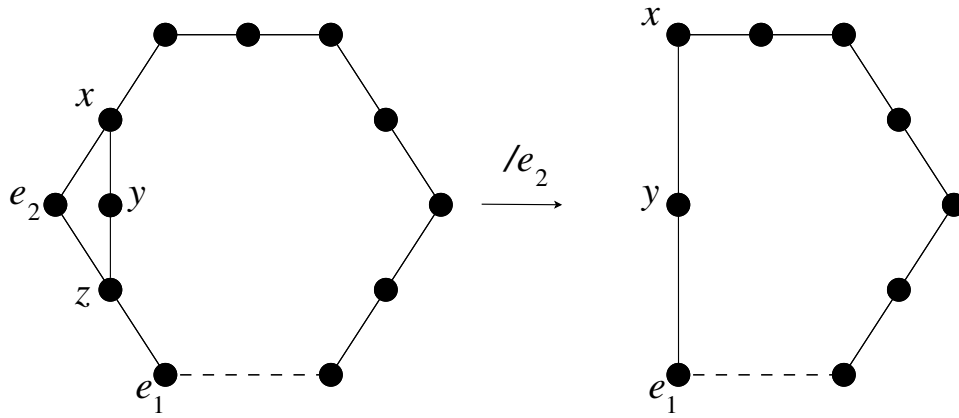


FIGURE 3.12. Geometric illustration of one possible configuration of M_2 .

figure, we contract a spoke element and delete z and another spoke element to produce a wheel-minor of M^* that uses x and y and has appropriate rank.

Otherwise, x is not in a triangle with e_1 or e_2 (see Figure 3.13). Figure 3.13 shows that triangle $\{x, y, z\}$ is a triangle of two wheel-minors of M_2 , and these wheel-minors share elements $\{e, f, g, h, x, y, z\}$ and contain all the elements of M_2 . In each of these wheel-minors, x and z are spoke elements and y is a rim element. Restrict to the larger of these minors, which is isomorphic to $M(\mathcal{W}_r)$ for some integer $r \geq \frac{r(M_2)}{2} + 1 \geq \frac{1}{2}(k - (|C_x| - 2)) + 1 \geq \frac{k+6}{4}$. Contract one rim element to make z parallel to an element other than x or y , then delete z

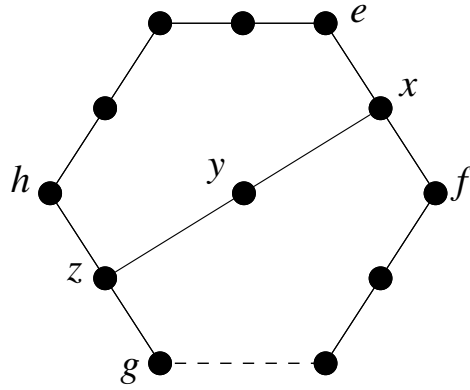


FIGURE 3.13. Geometric illustration of M_2 .

to produce a minor of M^* . This minor has x and y and is isomorphic to $M(\mathcal{W}_m)$ for some integer m with $m \geq r - 1 \geq \frac{k+2}{4}$.

We may now assume in case I, that $\sum_{i=1}^k a_i$ is odd. Recall that this case came from case I depicted in Figure 3.9. Because M^* is binary, there is a unique binary matroid, M_1 , obtained by adding elements z , x' and y' so that $\{x, y, z\}$, $\{x, f, x'\}$, and $\{y, f, y'\}$ are triangles (see Figure 3.14). The following matrix represents M_1 .

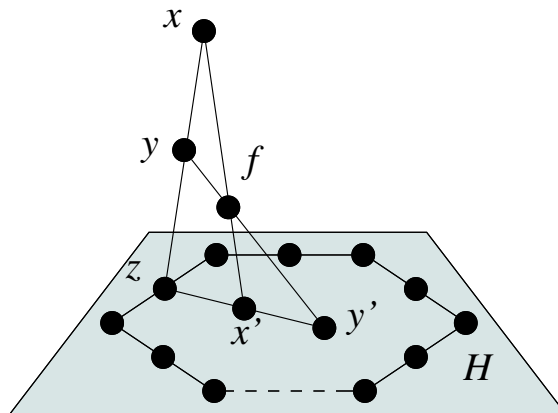


FIGURE 3.14. Geometric illustration of M^* with x' , y' , and z added to produce matroid M_1 . Here H is the complement of triad $\{x, f, y\}$.

$$\begin{array}{c}
\begin{array}{cccccccc}
e_1 & e_2 & e_3 & \dots & e_k & f & z & e_{k+2} & e_{k+3} & \dots & e_{2k} & x & y & x' & y'
\end{array} \\
\left[\begin{array}{c}
I_{k+1} \\
\begin{array}{cccccccc}
1 & 0 & 0 & \dots & 1 & a_1 & a_1 + 1 & a_1 & a_1 + 1 \\
1 & 1 & 0 & \dots & 0 & a_2 & a_2 + 1 & a_2 & a_2 + 1 \\
0 & 1 & 1 & \dots & 0 & a_3 & a_3 & a_3 & a_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 1 & a_k & a_k & a_k & a_k \\
0 & 0 & 0 & \dots & 0 & 1 & 1 & 0 & 0
\end{array}
\end{array} \right]
\end{array}$$

Since $\sum_{i=1}^k a_i$ is odd, neither x' nor y' is spanned by $\{e_{k+2}, e_{k+3}, \dots, e_{2k}\}$. Let H be the hyperplane complementing triad $\{x, f, y\}$. The independent sets $I_x = \{x', e_{k+2}, e_{k+3}, \dots, e_{2k}\}$ and $I_y = \{y', e_{k+2}, e_{k+3}, \dots, e_{2k}\}$ span H . The set $I_x \cup \{y'\}$ contains a circuit, C . As C is not contained in I_x or I_y , it must be that $\{x', y'\} \subseteq C$. Since $\{z, e_{k+2}, e_{k+3}, \dots, e_{2k}\}$ and $\{x', y', z\}$ are circuits of M_1 , their symmetric difference, $\{x', y', e_{k+2}, e_{k+3}, \dots, e_{2k}\}$ is the union of disjoint circuits. But any circuit of this set has x' and y' as elements, therefore $C = \{x', y', e_{k+2}, e_{k+3}, \dots, e_{2k}\}$.

The hyperplane H also has the set $\{e_i, e_{k+2}, e_{k+3}, \dots, e_{2k}\}$ as a basis B_i for every $i \in [k]$. Choose any $1 \leq i \leq k$. The set $B_i \cup x' = I_x \cup e_i$ contains a circuit C_x that contains $\{x', e_i\}$. Similarly, $B_i \cup y' = I_y \cup e_i$ contains a circuit C_y that contains $\{y', e_i\}$. The set $C_x \Delta C_y$ is the disjoint union of circuits, and $C_x \Delta C_y \subset I_x \cup I_y = C$. It follows that $C_x \Delta C_y = C$. Without loss of generality, $|C_x| \leq |C_y|$ and C_x has size at most $\frac{|C|}{2} + 1 = \frac{k+3}{2}$.

Contract the elements $C_x - \{x', e_i\}$ to make x' parallel to e_i . Since y' is not contained in the closure of $C_x - \{x, e_i\}$, the element y' has not become a loop in this process. Simplify the matroid without deleting any element of $\{x, y, f, x', y', z\}$ to produce M_2 shown on the left in Figure 3.15. If e_1 has been replaced by x' , then, since $\{x', y', z\}$ is a triangle of M_2 , the element e_2 has been replace by y' . In this case, there are two elements whose contraction produces, after relabelling two elements as e_1 and e_2 , the matroid shown on the right in

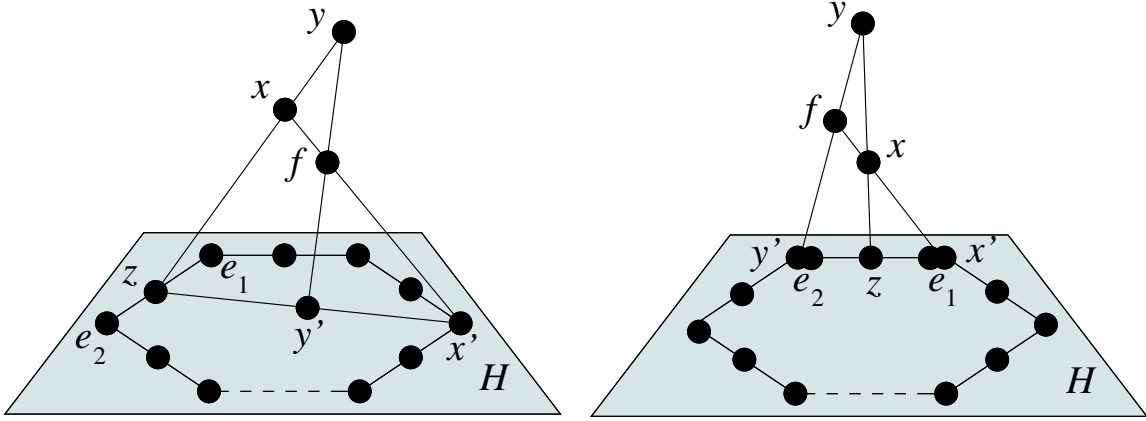


FIGURE 3.15. Geometric illustration of M_2 (left) and M_2 with some modification (right).

Figure 3.15. Otherwise, $M_2 \setminus \{x, y, f\}$ has two wheel-minor restrictions using $\{x', y', z\}$ as a triangle of the wheel. These two minors overlap in elements $\{x', y', z, e_1, e_2\}$ and each element of $M_2 \setminus \{x, y, f\}$ is an element of at least one of these minors. Let R be the set of rim elements of the smaller wheel-minor. In M_2 , contract $R - \{e_1, e_2, y'\}$ to produce the matroid shown on the right in Figure 3.15.

In both cases, remove the added elements, x' , y' , and z and simplify to produce an $M(\mathcal{W}_m)$ -minor, for some m with $m \geq \frac{r(M_2)}{2} + 1 \geq \frac{1}{2}(k - (|C_x| - 2)) + 1 \geq \frac{1}{2}(k - (\frac{k-1}{2})) + 1 = \frac{k+1}{4} + 1$. \square

3.4 A Spike-Minor Containing x and y

In this section, we examine the case where a 3-connected binary matroid with two identified elements has a large spike-minor. A rank- n , binary spike with no tip or cotip has a representation $[I_n | J_n]$ and will be denoted S_n . If the spike has a tip and no cotip, it has a representation $[I_n | J_n | \mathbf{1}]$ where $\mathbf{1}$ is the column of n ones and this column represents the tip of the spike.

For any $i \in [n]$, the matroid elements represented by the i th column and the $(i + n)$ th column form a triangle with the tip element. Delete any column of this matrix other than the column corresponding to the tip to produce a rank- n binary spike with a tip and a cotip. If the deleted element was in a triangle with c and the tip, then c is the cotip of this spike.

We denote a rank- n binary spike with a tip and cotip T_n . Deleting the tip from this matroid results in a rank- n binary spike with a cotip and no tip.

First, we prove a lemma.

Lemma 3.4.1. *Let N be a rank- n binary spike with a tip, t , and no cotip for some $n \geq 4$. Let M be a 3-connected matroid so that $M \setminus x = N$. If T is the set of elements of a minimal set of triangles of N spanning x , then $M|(T \cup x)$ and $M \setminus (T - t)$ are spikes with tip t and cotip x .*

Proof. The following matrix is a representation for M .

$$\left[\begin{array}{cccccc|cccc} e_1 & e_2 & e_3 & \dots & e_n & f_1 & f_2 & f_3 & \dots & f_n & t & x \\ \hline & & & & & 0 & 1 & 1 & \dots & 1 & 1 & x_1 \\ & & & & & 1 & 0 & 1 & \dots & 1 & 1 & x_2 \\ & & & & & 1 & 1 & 0 & \dots & 1 & 1 & x_3 \\ & & & & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ & & & & & 1 & 1 & 1 & \dots & 0 & 1 & x_n \end{array} \right]$$

Without loss of generality, T is comprised of the elements of triangles $\{e_i, f_i, t\}$ for $1 \leq i \leq k$. As this is a minimal set of triangles spanning x , and M is 3-connected, notice that $2 \leq k \leq n - 2$. The set $\{e_1, e_2, \dots, e_k, t\}$ is a basis of $M|T$, and so spans x .

By the minimality of k , no set $\{e_1, e_2, \dots, e_k, t\} - e_i$ for $1 \leq i \leq k$ spans x . Therefore, either

$$x_i = \begin{cases} 1 & \text{if } 1 \leq i \leq k, \\ 0 & \text{otherwise;} \end{cases}$$

or

$$x_i = \begin{cases} 0 & \text{if } 1 \leq i \leq k, \\ 1 & \text{otherwise.} \end{cases}$$

The first k rows of this matrix are identical. If we remove the redundant rows and rearrange the matrix we produce the following representation of our matroid.

$$\begin{array}{c}
 \\
 \\
 k+1 \\
 k+2 \\
 k+3 \\
 \vdots \\
 n-1 \\
 n \\
 k
 \end{array}
 \left[\begin{array}{cccccccc}
 e_{k+1} & e_{k+2} & e_{k+3} & \dots & e_{n-1} & e_n & x & \\
 & & & & & & & f_{k+1} & f_{k+2} & f_{k+3} & \dots & f_{n-1} & f_n & t \\
 & & & & & & & 0 & 1 & 1 & \dots & 1 & 1 & 1 \\
 & & & & & & & 1 & 0 & 1 & \dots & 1 & 1 & 1 \\
 & & & & & & & 1 & 1 & 0 & \dots & 1 & 1 & 1 \\
 & & & & & & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
 & & & & & & & 1 & 1 & 1 & \dots & 0 & 1 & 1 \\
 & & & & & & & 1 & 1 & 1 & \dots & 1 & 0 & 1 \\
 & & & & & & & 1 & 1 & 1 & \dots & 1 & 1 & 1
 \end{array} \right]$$

Therefore, our matroid is isomorphic to T_{n-k+1} with tip t and cotip x . □

Using this lemma, we consider the case that a 3-connected, binary matroid M is the single-element extension of T_n .

Lemma 3.4.2. *Let $N \cong T_n$ for some integer n greater than 2. Let M be a 3-connected binary matroid with elements x and y so that $M \setminus x = N$. Then there is an integer m with $m \geq \frac{n}{2}$ so that x and y are elements of a minor of M isomorphic to T_m .*

Proof. By Lemma 3.1.3, as T_3 is isomorphic to $M(\mathcal{W}_3)$, the theorem holds for $n \leq 6$. So we may assume $n \geq 7$.

The matroid N is n copunctual lines, L_1, L_2, \dots, L'_n so that each line, $L_i = \{t, e_i, f_i\}$ for $i \in [n-1]$, with $L'_n = \{t, e_n\}$. There is a unique binary matroid obtained by adding z to M to form M_1 so that $\{t, e_n, z\}$ is a triangle of M_1 . Let $L_n = \{t, e_n, z\}$. The following is a representation of M_1 .

$$\begin{array}{c}
1 \\
2 \\
3 \\
\vdots \\
n-1 \\
n
\end{array}
\left[\begin{array}{cccccc|cccc}
e_1 & e_2 & e_3 & \dots & e_{n-1} & e_n & f_1 & f_2 & f_3 & \dots & f_{n-1} & z & t & x \\
0 & 1 & 1 & \dots & 1 & 1 & 1 & 1 & 1 & & & & & 1 \\
1 & 0 & 1 & \dots & 1 & 1 & 1 & & & & & & & x_1 \\
1 & 1 & 0 & \dots & 1 & 1 & 1 & & & & & & & x_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & & & & & & \vdots \\
1 & 1 & 1 & \dots & 0 & 1 & 1 & & & & & & & x_{n-1} \\
1 & 1 & 1 & \dots & 1 & 0 & 1 & & & & & & & x_n
\end{array} \right]$$

If x is spanned by L_i for some $i \in [n]$, then $x \in \text{cl}(L_n)$. Otherwise x would be parallel to an element of M , and M would not be 3-connected. In this case, x is parallel to z in M_1 . Delete z and any element other than x , y , or t from M_1 to produce a T_n -minor of M containing x and y .

Now, we assume that x is not spanned by L_i for any i . The element z has not been placed parallel to any element of M , thus M_1 is 3-connected. Let A be the set of elements of a minimal set of these lines whose closure spans x . Let k be the number of lines L_i that are subsets of A . We also consider the $n - k$ lines that are not subsets of A . Let $B = E(M_1) - (A - t)$. By Lemma 3.4.1, then $x \in \text{cl}_{M_1}(B)$, and $M_1|(A \cup x)$ and $M_1|(B \cup x)$ are spikes with tip t and cotip x . Without loss of generality, $k \leq n - k$. We may assume that $A = \{t, e_1, f_1, e_2, f_2, \dots, e_k, f_k\}$ or $A = \{t, e_1, f_1, e_2, f_2, \dots, e_{k-1}, f_{k-1}, e_n, z\}$. Thus for some $c \in \{0, 1\}$,

$$x_i = \begin{cases} c & \text{if } 1 \leq i \leq k, \\ c - 1 & \text{otherwise;} \end{cases}$$

or

$$x_i = \begin{cases} c - 1 & \text{if } k \leq i \leq n - 1, \\ c & \text{otherwise.} \end{cases}$$

The element y may be contained in A . By the symmetry of the matroid M_1 , we may assume that, if it is, $y \in \{t, e_1, f_1, e_n\}$. Let $M_2 = M_1 / \{e_2, e_3, \dots, e_{k-1}\} \setminus \{f_2, f_3, \dots, f_{k-1}\}$.

The matroid M_2 has the following representation.

$$\begin{array}{cccccc|cccccc}
 & e_1 & e_k & e_{k+1} & \dots & e_{n-1} & e_n & f_1 & f_k & f_{k+1} & \dots & f_{n-1} & z & t & x \\
 1 & & & & & & & 0 & 1 & 1 & \dots & 1 & 1 & 1 & c \\
 k & & & & & & & 1 & 0 & 1 & \dots & 1 & 1 & 1 & x_k \\
 k+1 & & & & & & & 1 & 1 & 0 & \dots & 1 & 1 & 1 & c-1 \\
 \vdots & & & & & & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
 n-1 & & & & & & & 1 & 1 & 1 & \dots & 0 & 1 & 1 & c-1 \\
 n & & & & & & & 1 & 1 & 1 & \dots & 1 & 0 & 1 & x_k-1
 \end{array}$$

Depending on c and x_k one of four cases holds. First, if $c = x_k = 0$, then $\{e_1, f_k, x\}$ and $\{e_k, f_1, x\}$ are the only triangles of M_2 containing x . Secondly, if $c = x_k = 1$, then $\{e_1, e_k, x\}$ and $\{f_1, f_k, x\}$ are the only triangles of M_2 containing x . In either of these cases, contract e_1 if $e_1 \neq y$ otherwise, contract f_1 . In the resulting matroid, x is parallel to e_k or f_k . Delete this parallel element and z to produce a T_m -minor of M using x and y for some integer m with $m = n - (k - 2) - 1 \geq \frac{n}{2} + 1$.

We may assume $c \neq x$. In the third case, $c = 1$ and $x = 0$, thus $\{e_1, e_n, x\}$ and $\{f_1, z, x\}$ are the only triangles of M_2 containing x . And finally, if $c = 0$ and $x = 1$, then $\{e_1, z, x\}$ and $\{e_n, f_1, x\}$ are the only triangles of M_2 containing x . In these two cases, if the triangle containing $\{x, e_1\}$ avoids y , contract e_1 , otherwise, contract f_1 . Now x is parallel to an element of M_2 other than y . Delete z and simplify without deleting x or y to produce a spike-minor of M using x and y . If this minor has no cotip, delete an element other than t , x , or y to produce a T_m -minor for some integer m with $m \geq \frac{n}{2} + 1$ \square

We now consider the case where two elements must be removed to form the minor.

Lemma 3.4.3. *Let M be a 3-connected binary matroid with $M \setminus x / f = N \cong T_n$ for some integer n with $n \geq 4$, and let $y \in E(N)$. Suppose N has an element s so that $\{x, f, s\}$ is a circuit of M . Then there is an integer m with $m \geq \frac{n}{2}$ so that x and y are elements of a minor of M isomorphic to T_m .*

Proof. By Lemma 3.1.3, since $T_3 \cong M(\mathcal{W}_3)$, the theorem holds for $n \leq 6$. So we may assume $n \geq 7$.

Since $\{x, f, s\}$ is a triangle, the matroid M/f contains the parallel pair $\{x, s\}$. Thus $M/f \setminus x \cong M/f \setminus s \cong N$. If y is not s , then this minor is a T_n -minor containing x and y . We will assume $y = s$.

In M^* , the set $\{x, f, y\}$ is a cocircuit complementing a hyperplane, H . We consider three cases based on three possible locations of y in $M^* \setminus f/x$. These cases are illustrated in Figure 3.16. There is a unique binary matroid, M_0 , obtained by adding z to M^* so that $\{x, y, z\}$

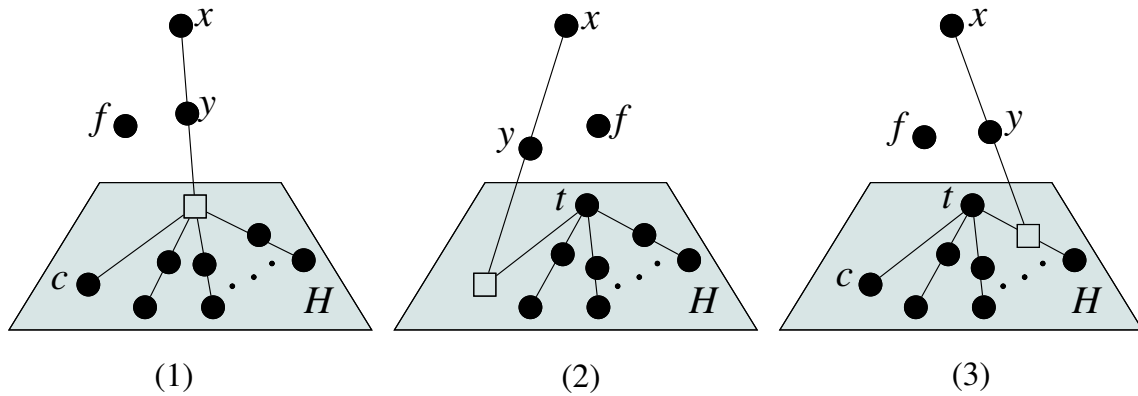


FIGURE 3.16. Geometric illustrations for the three possible configurations of matroid M^* .

is a triangle. In $M_0 \setminus \{x, f, y\}$, the element z is either (1) the tip, (2) the cotip, or (3) neither the tip nor the cotip. The matroid M_0 is not a minor of M^* . Throughout this proof, we avoid contracting z . In each case, we will produce a minor of M^* from a minor of M_0 by deleting z .

In the first case, contract f from M_0 to produce the matroid shown in Figure 3.17. This contraction projects elements from the complement of H into the hyperplane, but will not affect the matroid in any other way. The matroid M_0/f has the following representation.

$$\begin{array}{c}
\begin{array}{cccccccc}
e_1 & e_2 & e_3 & \dots & e_{n-1} & e_n & f_1 & f_2 & f_3 & \dots & f_{n-1} & z & x & y
\end{array} \\
\left[\begin{array}{c}
I_n \\
\begin{array}{cccccccc}
0 & 1 & 1 & \dots & 1 & 1 & x_1 & x_1 + 1 \\
1 & 0 & 1 & \dots & 1 & 1 & x_2 & x_2 + 1 \\
1 & 1 & 0 & \dots & 1 & 1 & x_3 & x_3 + 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & \dots & 0 & 1 & x_{n-1} & x_{n-1} + 1 \\
1 & 1 & 1 & \dots & 1 & 1 & x_n & x_n + 1
\end{array}
\end{array} \right]
\end{array}$$

Let the number of non-zero members of $\{x_1, x_2, \dots, x_n\}$ be k . By interchanging x and y if necessary, we may assume that $k \leq \frac{n}{2}$, so that $n - k \geq \frac{n}{2}$. Suppose first that $k = 1$. If $x_n = 1$, then deleting e_{n-1} and e_n from M_0/f produces a T_n -minor using x and y . If $x_j = 1$ for some $j \neq n$, then deleting e_j and f_j produces a T_n -minor using x and y . Thus we may assume that $k > 1$. Without loss of generality, $x_1 = x_2 = \dots = x_{k-1} = 1$ and either (i) $x_k = 1$ or (ii) $x_n = 1$. In case (i), contract $\{e_2, e_3, \dots, e_k\}$ and delete $\{f_3, f_4, \dots, f_k\}$. Then deleting $\{e_1, f_1, z\}$ gives a T_{n-k+1} -minor of M^* with tip f_2 and cotip e_n using x and y . In case (ii), first contract $\{e_2, e_n\}$ then contract $\{e_3, e_4, \dots, e_{k-1}\}$ and delete $\{f_3, f_4, \dots, f_{k-1}\}$. Then deleting $\{e_1, f_1, z, f_{n-1}\}$ gives a T_{n-k+1} -minor of M^* with tip f_2 and cotip e_{n-1} using x and y .

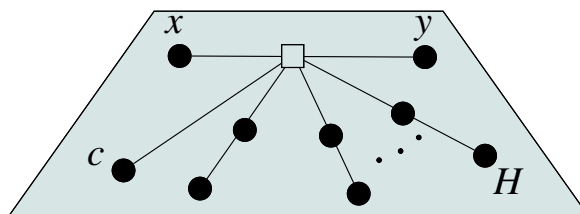


FIGURE 3.17. Geometric illustration of M_0/f in case (1) with a square representing z .

We may now assume that $k \leq \frac{n-2}{2}$. As x is not a loop, $x_j = 1$ for some $j \neq n$. Without loss of generality, we may assume that $x_1 = x_2 = \dots = x_k = 1$. Contract $\{e_2, e_3, \dots, e_k\}$ and delete $\{f_2, f_3, \dots, f_k\}$ to produce a matroid with the following representation.

$$\left[\begin{array}{cccccc|cccc} e_1 & e_{k+1} & e_{k+2} & \dots & e_{n-1} & z & f_1 & f_{k+1} & f_{k+2} & \dots & f_{n-1} & t & x & y \\ \hline & & & & & & 0 & 1 & 1 & \dots & 1 & 1 & 1 & 1 \\ & & & & & & 1 & 0 & 1 & \dots & 1 & 1 & 0 & 0 \\ & & & & & & 1 & 1 & 0 & \dots & 1 & 1 & 0 & 0 \\ & & & & & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ & & & & & & 1 & 1 & 1 & \dots & 0 & 1 & 0 & 0 \\ & & & & & & 1 & 1 & 1 & \dots & 1 & 1 & 0 & 1 \end{array} \right]$$

I_{n-k+1}

The element x is now parallel to e_1 . Delete e_1 and contract f_1 to obtain a matroid with the following representation. (To obtain this matrix explicitly, add the last row to the first, then pivot on first entry of f_1 and delete the first row as well as the column f_1 from the resulting matrix.)

$$\left[\begin{array}{cccccc|cccc} e_1 & e_{k+1} & e_{k+2} & \dots & e_{n-1} & z & f_1 & f_{k+1} & f_{k+2} & \dots & f_{n-1} & t & x & y \\ \hline & & & & & & 1 & 0 & 1 & 1 & \dots & 1 & 1 & 1 & 0 \\ & & & & & & 1 & 1 & 0 & 1 & \dots & 1 & 1 & 1 & 0 \\ & & & & & & 1 & 1 & 1 & 0 & \dots & 1 & 1 & 1 & 0 \\ & & & & & & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ & & & & & & 1 & 1 & 1 & 1 & \dots & 0 & 1 & 1 & 0 \\ & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 1 & 1 & \dots & 1 & 1 & 1 & 1 \end{array} \right]$$

I_{n-k}

Delete z and t to produce a minor of M^* isomorphic to T_m for some $m = n - k \geq n - \frac{n-2}{2} \geq \frac{n}{2} + 1$ with tip x and cotip y .

We may now assume we are in case (3) from Figure 3.16. Here, z forms a triangle with t and e . Without loss of generality, the matroid M_1/f has the following matrix representation.

$$\begin{array}{c}
 \begin{array}{cccccc}
 e & e_2 & e_3 & \dots & e_{n-1} & e_n
 \end{array} \\
 \left[\begin{array}{cccccc|cccc}
 & & & & & & z & f_2 & f_3 & \dots & f_{n-1} & t & x & y \\
 & & & & & & 0 & 1 & 1 & \dots & 1 & 1 & x_1 & x_1 \\
 & & & & & & 1 & 0 & 1 & \dots & 1 & 1 & x_2 & x_2 + 1 \\
 & & & & & & 1 & 1 & 0 & \dots & 1 & 1 & x_3 & x_3 + 1 \\
 & & & & & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
 & & & & & & 1 & 1 & 1 & \dots & 0 & 1 & x_{n-1} & x_{n-1} + 1 \\
 & & & & & & 1 & 1 & 1 & \dots & 1 & 1 & x_n & x_n + 1
 \end{array} \right] \\
 \begin{array}{c}
 I_n
 \end{array}
 \end{array}$$

As long as e is not parallel to x or y , we can contract e , delete z , and relabel t as z to give this matrix the same form as the matrix representation of M_0/f in case (1). In this case, we reduce case (3) to case (1) and find a T_m -minor of M^* using x and y for some integer $m \geq n - 1 - \frac{n-1}{2} = \frac{n+1}{2}$. If, however, e is parallel to x or y , then z is parallel to y or x , respectively. Here, delete e and z to produce a T_n -minor of M^* that uses x and y . \square

3.5 A Minor Isomorphic to the Cycle or Bond Matroid of $K_{1,1,1,n}$ Containing x and y .

In this section, we examine the case when M has a minor isomorphic to the cycle or bond matroid of $K_{1,1,1,n}$. We will refer to Figure 3.19, which shows the graph of $K_{1,1,1,n}$ and illustrates the geometry of this rank- $(n + 2)$ matroid.

First, we consider the case where the deletion of one element of M results in a $M(K_{1,1,1,n})$ -minor.

Lemma 3.5.1. *Let M be a 3-connected binary matroid so that $M \setminus x = N \cong M(K_{1,1,1,n})$ for some positive integer n . Suppose $y \in E(N)$. Then there is an integer m with $m \geq \frac{n-1}{2}$ so that x and y are elements of a minor of M isomorphic to T_m or $M(K_{1,1,1,m})$.*

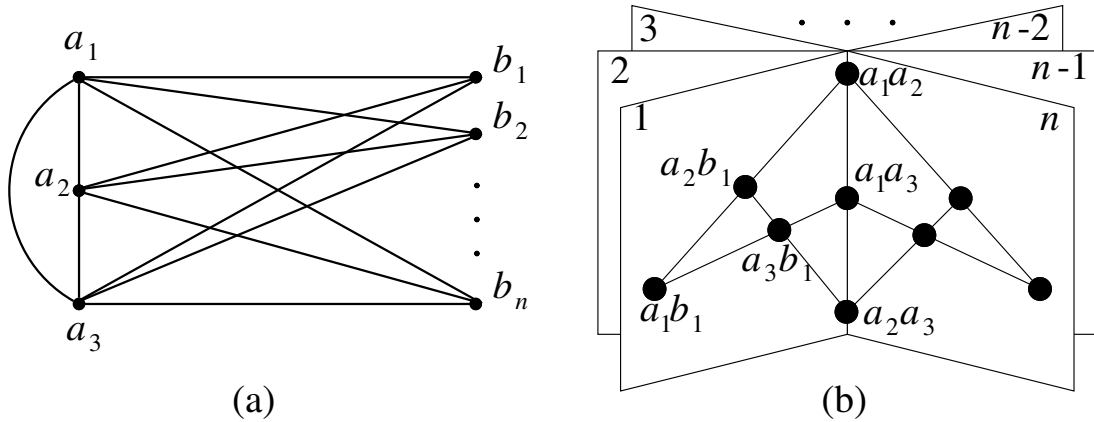


FIGURE 3.19. (a) The graph of $K_{1,1,1,n}$ and (b) a geometric illustration of the graph's cycle matroid.

Proof. As $T_3 \cong M(\mathcal{W}_3)$, by Lemma 3.1.3, x and y are elements of a minor of M isomorphic to T_3 . Hence we may assume that $n \geq 7$.

The matroid $M = M(G, \gamma_x)$ for the graft (G, γ_x) where $G = K_{1,1,1,n}$ and γ_x is the set of vertices of G incident with the graft element x . Label G as shown in Figure 3.19. If $|\gamma_x| = 2$, then, as M is simple, we may assume that $\gamma_x = \{b_1, b_2\}$, and then $M/a_3 b_2 \setminus \{a_1 a_3, a_2 a_3, a_3 b_1\}$ is an $M(K_{1,1,1,m})$ -minor that contains x and y and has $m \geq n - 1$.

In this proof, we operate on M by operating on the graft (G, γ_x) . Recall that a graft element is incident with an even number of vertices, and contracting and deleting edges of G will not change this. We may assume that $|\gamma_x| > 2$. Therefore $|\gamma_x| \geq 4$. Let A_x and B_x be the vertex sets $\{a_1, a_2, a_3\} \cap \gamma_x$ and $\{b_1, b_2, \dots, b_n\} \cap \gamma_x$, respectively. By the symmetry of $K_{1,1,1,n}$, we may assume that y is $a_1 a_2$ or $a_1 b_1$.

First, let $|B_x| \leq \frac{n}{2} + 1$. Assume a_1 or a_2 is not in γ_x . Then, the set B_x has at least two vertices. Thus $B_x - b_1$ has at least one vertex; without loss of generality, $b_2 \in B_x$. Contract the edges from vertices of $B_x - b_2$ to a_3 and label the resulting composite vertex a_3 . Simplify the underlying graph without deleting y . The resulting graft (G', γ'_x) has $G \cong K_{1,1,1,m}$ for some integer m with $m = n - |B_x - b_2| \geq \frac{n}{2}$. In G' , the edge y has a_1 as one endpoint, and the other endpoint is in $\{a_2, a_3, b_1\}$. The set γ'_x contains b_2 and some subset of $\{a_1, a_2, a_3\}$. Since $|\gamma'_x|$ is even, and either a_1 or a_2 is not in γ'_x , the set $\gamma'_x = \{a_i, b_2\}$ for some $i \in [3]$. In

$M(G', \gamma'_x)$, then, x is parallel to the element $a_i b_2$, and since y is not incident with b_2 in G' , the graft $(G' \setminus a_i b_2, \gamma'_x)$ is a $K_{1,1,1,m}$ -minor of (G, γ_x) using x and y . Therefore $M(G' \setminus a_i b_2, \gamma'_x)$ is the minor we seek.

Now, assume that both a_1 and a_2 are in γ_x . Since $|\gamma_x| \geq 4$, there is a vertex b_k in B_x . If $b_1 \in B_x$, let $k = 1$, otherwise, let b_k be any vertex of B_x . Contract $a_2 b_k$ from the graft, labelling the resulting vertex a_2 . Simplify the underlying graph without deleting y to produce the graft (G', γ'_x) , with $G' \cong K_{1,1,1,n-1}$ and $\gamma'_x = \gamma_x - \{a_2, b_k\}$. If $|\gamma'_x| = 2$, then $\gamma'_x = \{a_1, a_3\}$ or $\gamma'_x = \{a_1, b_i\}$ for some $i \neq 1$. In either case, $M(G', \gamma'_x)$ has x parallel to some element other than y , so we may simplify to produce an $M(K_{1,1,1,n-1})$ -minor containing x and y . Thus, we may assume that $|\gamma_x - \{a_2, b_k\}| \geq 4$. Since $a_2 \notin \gamma'_x$, this case is reduced to the case considered in the previous paragraph. Thus, there is a $K_{1,1,1,m}$ -minor containing x and y with $m \geq \frac{n-1}{2}$.

Finally, we may assume $|B_x| \geq \frac{n+1}{2} + 1 \geq \frac{8}{2}$. Since B_x has at least four vertices, $B_x - b_1$ has at least three. Without loss of generality, $\{b_2, b_3, b_4\} \subseteq B_x$. For every $a_i \in A_x$, contract the edge $a_i b_{i+1}$ and label the resulting vertex a_i . Also contract the set of edges from $\{b_1, b_2, \dots, b_n\} - B_x$ to a_3 and label the composite vertex a_3 . The resulting graft has graft element $\gamma'_x = B_x - \{b_{i+1} : a_i \in A_x\}$ and has the vertex set $\{a_1, a_2, a_3\} \cup \gamma'_x$. Simplify the underlying graph without deleting y to produce the graft (G', γ'_x) with $G' \cong K_{1,1,1,m}$ for some integer m with $m = |B_x| - |A_x| \geq \frac{n-1}{2} - 1$.

At this point, $y \in \{a_1 a_2, a_1 b_1, a_1 a_3\}$. Without loss of generality, $y \neq a_1 a_3$. Delete all edges of G' incident with a_3 to produce a graft (G'', γ'_x) with y an edge of G'' . After relabelling the remaining vertices of B_x as $\{b_1, b_2, \dots, b_m\}$, the graft (G'', γ'_x) has the following incidence matrix.

$$\begin{array}{c}
a_1 \\
a_2 \\
b_1 \\
b_2 \\
b_3 \\
\vdots \\
b_m
\end{array}
\left[\begin{array}{c|cccccc|cccc|c}
a_1 a_2 & a_1 b_1 & a_1 b_2 & a_1 b_3 & \dots & a_1 b_m & a_2 b_1 & a_2 b_2 & a_2 b_3 & \dots & a_2 b_m & x \\
1 & 1 & 1 & 1 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\
1 & 0 & 0 & 0 & \dots & 0 & 1 & 1 & 1 & \dots & 1 & 0 \\
0 & & & & & & & & & & & 1 \\
0 & & & & & & & & & & & 1 \\
0 & & & & & & & & & & & 1 \\
\vdots & \vdots & & & & & & & & & & \vdots \\
0 & & & & & & & & & & & 1
\end{array} \right]$$

The matroid $M(G'', \gamma'_x)$ equals the vector matroid of the matrix below, which can be obtained from the incidence matrix by deleting the row labelled by a_1 and then adding the row labelled by a_2 to every other row.

$$\begin{array}{c}
a_2 \\
b_1 \\
b_2 \\
b_3 \\
\vdots \\
b_m
\end{array}
\left[\begin{array}{c|cccccc|cccc|c}
a_1 a_2 & a_1 b_1 & a_1 b_2 & a_1 b_3 & \dots & a_1 b_m & a_2 b_1 & a_2 b_2 & a_2 b_3 & \dots & a_2 b_m & x \\
1 & 0 & 0 & 0 & \dots & 0 & 1 & 1 & 1 & \dots & 1 & 0 \\
1 & & & & & & 0 & 1 & 1 & \dots & 1 & 1 \\
1 & & & & & & 1 & 0 & 1 & \dots & 1 & 1 \\
1 & & & & & & 1 & 1 & 0 & \dots & 1 & 1 \\
\vdots & \vdots & & & & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & & & & & & 1 & 1 & 1 & \dots & 0 & 1
\end{array} \right]$$

Thus, $M(G'', \gamma'_x)$ contains x and y and is isomorphic to T_{m+1} with $m+1 \geq \frac{n-1}{2}$. \square

Now, we consider the matroid $M^*(K_{1,1,1,n})$. While the $M(K_{1,1,1,n})$ is depicted in Figure 3.19, it will still be useful to develop a geometric illustration for $M^*(K_{1,1,1,n})$ itself.

In $K_{3,n+1}$, let the vertex classes be labelled $\{a_1, a_2, a_3\}$ and $\{b_0, b_1, \dots, b_n\}$. Perform a Y- Δ exchange on the triad $\{b_0 a_1, b_0 a_2, b_0 a_3\}$. The resulting triangle is $\{a_1 a_2, a_2 a_3, a_1 a_3\}$ and the resulting graph is $K_{1,1,1,n}$. Thus, in $M^*(K_{1,1,1,n})$, if we perform a Y- Δ exchange on the

triad $\{a_1a_2, a_2a_3, a_1a_3\}$, we get $M^*(K_{3,n+1})$. Geometrically, $M^*(K_{3,n+1})$ can be formed in the following way. Take the direct sum of n triangles $Z_i = \{a_1b_i, a_2b_i, a_3b_i\}$ for all $i \in [n]$. There is a unique binary matroid M_0 that can be obtained by adding elements $z_1, z_2,$ and z_3 so that $\{a_jb_1, a_jb_2, \dots, a_jb_n, z_j\}$ is a circuit of M_0 for each $j \in [3]$. By taking the symmetric difference of the union of these three $(n+1)$ -element circuits with the union of the n triangles Z_i , we find that $\{z_1, z_2, z_3\}$ is a triangle of M_0 . From above, we see that performing a Δ -Y exchange on the triangle $\{z_1, z_2, z_3\}$ gives the triad $\{a_1a_2, a_2a_3, a_1a_3\}$ in the matroid $M^*(K_{1,1,1,n})$ where $A_1 = \{a_1a_2, a_1a_3, a_1b_1, a_1b_2, \dots, a_1b_n\}$, $A_2 = \{a_1a_2, a_2a_3, a_2b_1, a_2b_2, \dots, a_2b_n\}$, and $A_3 = \{a_1a_3, a_2a_3, a_3b_1, a_3b_2, \dots, a_3b_n\}$ are circuits.

While $M^*(K_{1,1,1,n})$ has rank $2n + 1$, an illustration is useful. Figure 3.20 shows triad $\{a_1a_2, a_1a_3, a_2a_3\}$ complementing a hyperplane labelled H . The white squares indicate the position of triangle $\{z_1, z_2, z_3\}$ which was removed. The triangles are shown as vertical, 3-point lines and each circuit A_i is indicated by a horizontal line that bends at a white square so that each such line includes $n + 2$ points.

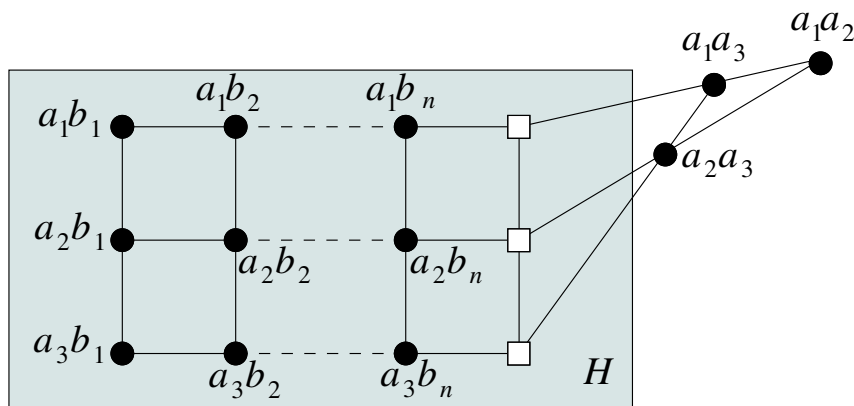


FIGURE 3.20. A geometric illustration of $M^*(K_{1,1,1,n})$.

Now we consider the case where the deletion of one element of M produces an $M^*(K_{1,1,1,n})$ -minor.

Lemma 3.5.2. *Let M be a 3-connected binary matroid so that $M \setminus x = N \cong M^*(K_{1,1,1,n})$ for a positive integer n . Suppose $y \in E(N)$. Then there is an integer m with $m \geq \frac{n}{4} - 2$ so that x and y are elements of a minor of M isomorphic to $M^*(K_{1,1,1,m})$.*

Proof. As $M^*(K_{1,1,1,1}) \cong M(\mathcal{W}_3)$, by Lemma 3.1.3, the theorem holds for $n \leq 12$. Thus we may assume that $n \geq 13$. We will also assume N is labelled as depicted in Figure 3.20, with triangles $Z_i = \{a_1b_i, a_2b_i, a_3b_i\}$ for all $i \in [n]$ and a triad $Z_0^* = \{a_1a_2, a_1a_3, a_2a_3\}$.

Let C_x be a circuit of M containing x meeting a minimum-sized subset \mathcal{Z} of $\{Z_1, Z_2, \dots, Z_n\}$. Choose C_x to also have minimal size. It follows that, $|C_x \cap Z_i| \leq 1$ for all $i \in [n]$; otherwise, for some i , a circuit contained in $C_x \Delta Z_i$ containing x contradicts the minimality of C_x . Let $k = |\mathcal{Z}|$. Without loss of generality, $\mathcal{Z} = \{Z_1, Z_2, \dots, Z_k\}$ and $y \in \{a_1a_2, a_1b_1, a_1b_{k+1}\}$.

First, we assume $k > \frac{3}{4}n$. By the pigeonhole principle, for some $j \in [3]$, say $j = 1$, the set C_x meets $\{a_jb_1, a_jb_2, \dots, a_jb_n\}$ in at least $\frac{1}{3}|\mathcal{Z}|$ elements. Thus $C_x \Delta \{a_1b_1, a_1b_2, \dots, a_1b_n, a_1a_2, a_1a_3\}$ contains a circuit C'_x containing x that avoids at least $\frac{|\mathcal{Z}|}{3}$ triangles of N . Then C'_x meets at most $n - \frac{|\mathcal{Z}|}{3} < \frac{3}{4}n$ triangles of N , contradicting the minimality of $|\mathcal{Z}|$. Therefore, $k \leq \frac{3}{4}n$.

Now, we consider the case where $k = 0$. Then $x \in \text{cl}(Z_0^*)$. Since M is 3-connected, x is not parallel to any element. As M is binary, $M \setminus x$ is illustrated in Figure 3.21 with four

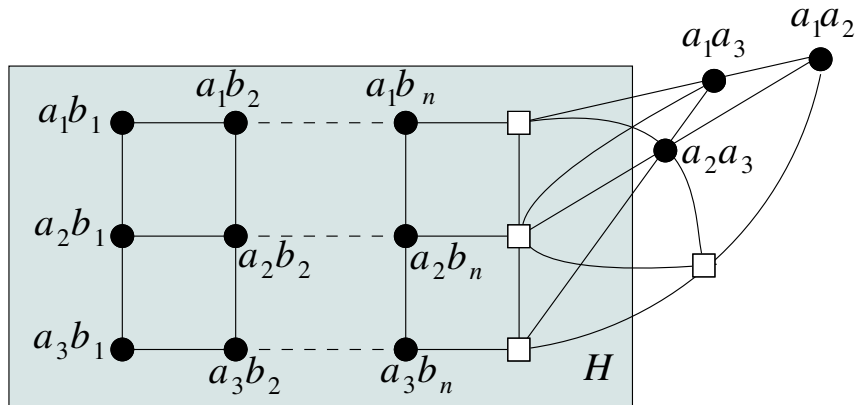


FIGURE 3.21. The matroid $M \setminus x$ with four boxes representing the possible locations for x in $\text{cl}(Z_0^*)$.

possible locations for x in M represented by squares. If x is not in $\text{cl}(H)$, then delete a_1a_3 to produce an $M^*(K_{1,1,1,n})$ -minor using x and y . If x is not in a triangle with a_1a_3 and a_2a_3 , then we can contract one of these elements to produce an $M^*(K_{3,n+1})$ -minor using x and y . In this case, we can easily find an $M^*(K_{1,1,1,n-1})$ -minor using x and y . Thus, we may assume $\{x, a_1a_3, a_2a_3\}$ is a triangle (see Figure 3.21). The matroid M is the vector matroid of the following binary matrix.

$$\begin{array}{c}
 \begin{array}{cccccc|cccc|ccc}
 a_1b_1 & \dots & a_1b_n & a_2b_1 & \dots & a_2b_n & a_1a_2 & a_3b_1 & a_3b_2 & \dots & a_3b_n & a_1a_3 & a_2a_3 & x \\
 \hline
 & & & & & & & 1 & 0 & \dots & 0 & 1 & 0 & 1 \\
 & & & & & & & 0 & 1 & \dots & 0 & 1 & 0 & 1 \\
 & & & & & & & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
 & & & & & & & 0 & 0 & \dots & 1 & 1 & 0 & 1 \\
 & & & & & & & 1 & 0 & \dots & 0 & 0 & 1 & 1 \\
 & & & & & & & 0 & 1 & \dots & 0 & 0 & 1 & 1 \\
 & & & & & & & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
 & & & & & & & 0 & 0 & \dots & 1 & 0 & 1 & 1 \\
 & & & & & & & 0 & 0 & \dots & 0 & 1 & 1 & 0
 \end{array} \\
 \\
 I_{2n+1}
 \end{array}$$

Thus, the following matrix represents M^* . This matrix is obtained by taking the representation $[I_{2n+1}|D]$ for M and first constructing $[D^T|I_{n+3}]$. In the resulting matrix, we add rows $n+1$ and $n+2$ to row $n+3$. Finally, we adjoin a new row that is the sum of all the current rows.

$$\left[\begin{array}{ccc|ccc|c|ccc|cc} a_1b_1 & \dots & a_1b_n & a_2b_1 & \dots & a_2b_n & a_1a_2 & a_3b_1 & \dots & a_3b_n & a_1a_3 & a_2a_3 & x \\ & & & & & & 0 & & & & 0 & 0 & 0 \\ & & I_n & & & I_n & \vdots & & & I_n & \vdots & \vdots & \vdots \\ & & & & & & 0 & & & & 0 & 0 & 0 \\ 1 & \dots & 1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & \dots & 0 & 1 & \dots & 1 & 1 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 & 1 & 1 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 1 & 0 & 0 & 1 \end{array} \right]$$

Therefore, M is cographic, with its dual represented by the graph G shown in Figure 3.22. It is easy to see that $G/a_3b_n \cong K_{1,1,1,n}$, and this graph contains x and y . Therefore, $M \setminus a_3b_n \cong M^*(K_{1,1,1,n})$ is a minor of M that uses elements x and y .

Now, we may assume that $k \geq 1$. Just as we may delete a triad from $K_{1,1,1,n}$ to produce $K_{1,1,1,n-1}$, we may contract a triangle of $M^*(K_{1,1,1,n})$ to produce $M^*(K_{1,1,1,n-1})$. One by one, contract triangles in $\mathcal{Z} - Z_1$ until one of the following holds:

- (1) x is in $\text{cl}(Z_j)$ for some $j \in [n]$, or
- (2) x is in $\text{cl}(Z_j \cup Z_0^*)$ for some $j \in [n]$.

The resulting matroid, M_1 , is a single-element extension of $M^*(K_{1,1,1,m})$ for some $m \geq n - k \geq \frac{n}{4}$.

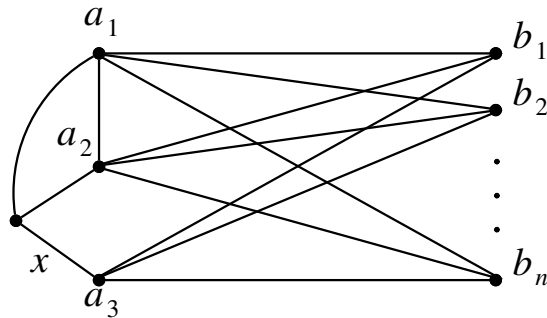


FIGURE 3.22. A graph G representing M^* .

In case (1), M_1 has $x \in \text{cl}(Z_j)$ for some $j \in [n]$, and $k \geq 2$. By the minimality of $|\mathcal{Z}|$, it follows that $j = 1$. If x is not parallel to y , we may simplify to find the desired minor, so assume they are parallel. In this case, $y = a_1b_1$. Let M_0 be the matroid obtained from M by contracting the triangles of \mathcal{Z} other than Z_1 and Z_2 . By the minimality of $|\mathcal{Z}|$, contracting Z_2 from M_0 creates the parallel class $\{x, y\}$. Hence $\{x, a_1b_1, a_i b_2\}$ is a circuit for some $i \in [3]$. Since $M_0 \setminus x$ has $(Z_1 \cup Z_2)$ as a 3-separating set, and $x \in \text{cl}(Z_1 \cup Z_2)$, the matroid M_0 can be represented as a 3-sum of the type shown in Figure 3.23. Here we take advantage of the fact that a 3-connected binary matroid with a given 3-separation has a unique representation as a 3-sum.

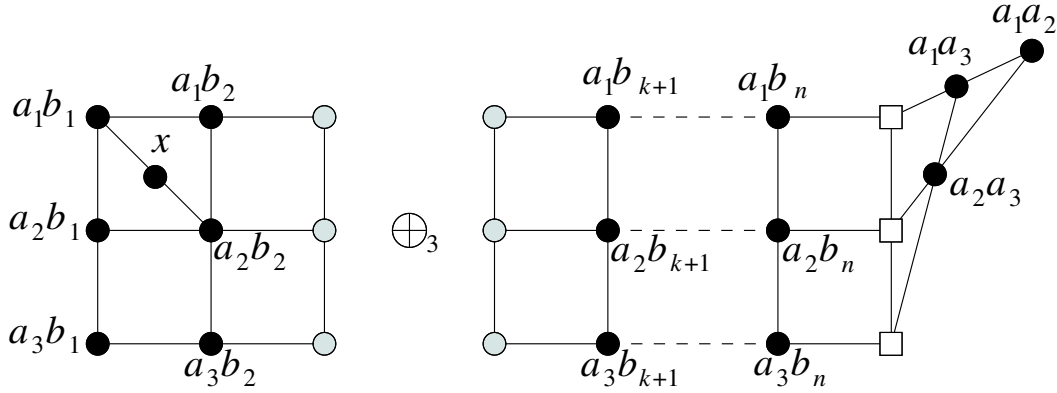


FIGURE 3.23. An illustration of M_0 as a 3-sum when $i = 2$. The gray elements are the elements that are deleted when the 3-sum is taken.

If $i \neq 1$, then, without loss of generality, $i = 2$. Contract a_1b_2 and a_2b_1 to produce a matroid with the 3-sum illustrated in Figure 3.24. Delete a_3b_1 and a_3b_2 to produce an $M^*(K_{1,1,1,n-k+1})$ -minor using x and y . If $i = 1$, then x is parallel to a gray element in Figure 3.23, and x and y are elements of $M_0 / \{a_2b_2, a_3b_1\} \setminus \{a_2b_1, a_3b_2\} \cong M^*(K_{1,1,1,n-k+1})$.

Finally, we consider case (2). In M_1 , the element x is in $\text{cl}(Z_0^* \cup Z_j)$. By the minimality of $|\mathcal{Z}|$, it follows that $j = 1$. Since $(Z_1 \cup Z_0^*)$ is a 3-separating set in $M_1 \setminus x$ and $x \in \text{cl}_{M_1}(Z_1 \cup Z_0^*)$, we can view M_1 as the 3-sum shown in Figure 3.25.

The set $\{x, a_1a_2, a_1a_3, a_2a_3, a_1b_1, a_2b_1, a_3b_1\}$ contains a minimum-sized subset C'_x that is a circuit of M_1 containing x . By orthogonality, C'_x intersects the cocircuit $\{x, a_1a_2, a_1a_3, a_2a_3\}$

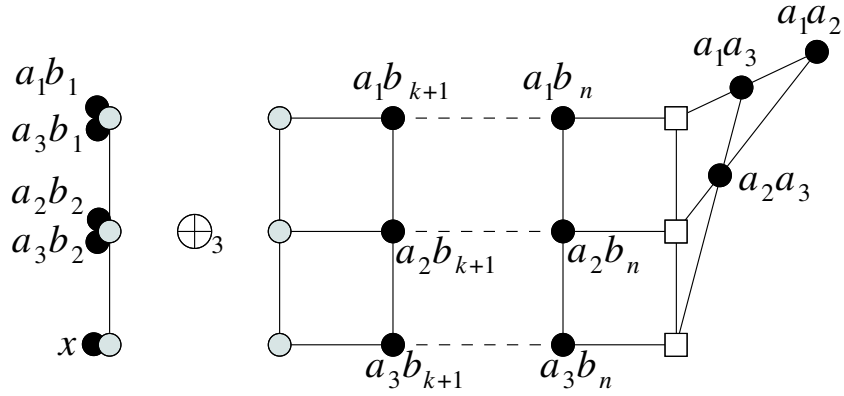


FIGURE 3.24. The matroid $M_0/\{a_1b_2, a_2b_1\}$ illustrated as a 3-sum.

in an even set. As $k \geq 1$, the circuit C'_x meets Z_1 . We can assume $C'_x \cap Z_1 = a_i b_1$ otherwise $C'_x \Delta Z_1$ contains a circuit containing x that contradicts the minimality of C'_x . Therefore, either $\{x, a_1 a_2, a_1 a_3, a_2 a_3, a_i b_1\}$ or $\{x, a_j a_k, a_i b_1\}$ for some $i \in [3]$ and some $a_j a_k \in \{a_1 a_2, a_1 a_3, a_2 a_3\}$ is a circuit. By choosing the basis $\{a_1 a_2, a_1 a_3, a_2 a_3, a_1 b_1, a_2 b_1\}$, we obtain the following binary representation for the left side, M_2 , of the 3-sum displayed in Figure 3.25.

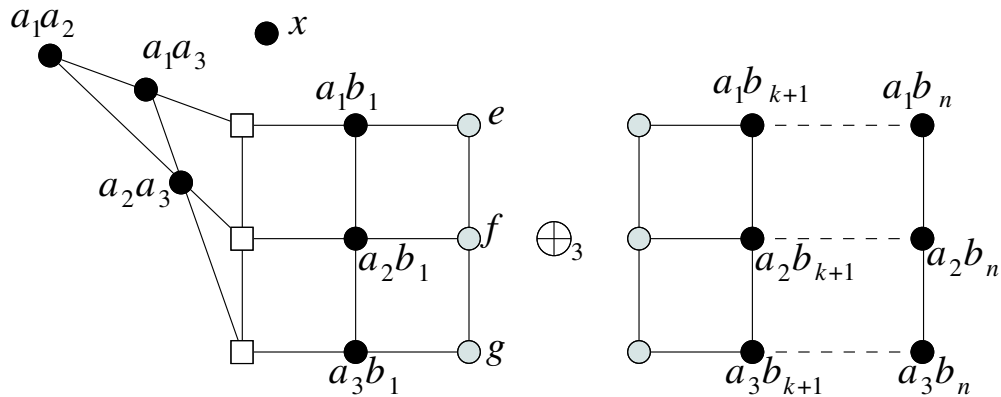


FIGURE 3.25. The matroid M_1 with cocircuit $\{x, a_1 a_2, a_1 a_3, a_2 a_3\}$ illustrated as a 3-sum.

$$\left[\begin{array}{ccccc|ccccc}
a_1a_2 & a_1a_3 & a_2a_3 & a_1b_1 & a_2b_1 & a_3b_1 & e & f & g & x \\
& & & & & 0 & 1 & 1 & 0 & x_1 \\
& & & & & 0 & 1 & 0 & 1 & x_2 \\
& & & & & 0 & 0 & 1 & 1 & x_3 \\
& & & & & 1 & 1 & 0 & 1 & x_4 \\
& & & & & 1 & 0 & 1 & 1 & x_5
\end{array} \right]$$

If $a_1a_2 \in C'_x$, then either $\{x, a_1a_2, a_1a_3, a_2a_3, a_i b_1\}$ or $\{x, a_1a_2, a_i b_1\}$ for some $i \in [3]$ is a circuit. In this case, $M_2/\{a_1a_3, a_2a_3\}$ has the following representation with (x_4, x_5) being either $(1, 0)$, $(0, 1)$, or $(1, 1)$.

$$\left[\begin{array}{ccc|ccccc}
a_1a_2 & a_1b_1 & a_2b_1 & a_3b_1 & e & f & g & x \\
& & & 0 & 1 & 1 & 0 & 1 \\
& & & 1 & 1 & 0 & 1 & x_4 \\
& & & 1 & 0 & 1 & 1 & x_5
\end{array} \right]$$

If (x_4, x_5) is $(1, 0)$ or $(1, 1)$, then contracting a_2b_1 from this rank-3 matroid produces a rank-2 line with every gray element parallel to another element and with x not parallel to y . Thus, we may simplify $M_1/\{a_1a_3, a_2a_3\}/a_2b_1$ to find an $M^*(K_{3, n-k+1})$ -minor using x and y . If, instead, $(x_4, x_5) = (0, 1)$, then contract whichever of a_1a_2 and a_1b_1 is not y from $M_1/\{a_1a_3, a_2a_3\}$ to find an $M^*(K_{3, n-k+1})$ -minor using x and y . In either case, we can easily find an $M^*(K_{1,1,1, n-k-1})$ -minor using x and y , and $n - k - 1 \geq \frac{n}{4} - 1$.

Now, we may assume that $a_1a_2 \notin C'_x$. Thus $C'_x = \{x, a_j a_k, a_i b_1\}$ for some $i \in [3]$ and some $a_j a_k \in \{a_1a_3, a_2a_3\}$. Thus M_2 has the following binary representation with (x_2, x_3) being either $(1, 0)$ or $(0, 1)$ and with (x_4, x_5) being either $(1, 0)$, $(0, 1)$, or $(1, 1)$. By symmetry, we

may assume $(x_2, x_3) = (1, 0)$.

$$I_5 \left[\begin{array}{ccccc|ccccc} a_1a_2 & a_1a_3 & a_2a_3 & a_1b_1 & a_2b_1 & a_3b_1 & e & f & g & x \\ & & & & & 0 & 1 & 1 & 0 & 0 \\ & & & & & 0 & 1 & 0 & 1 & x_2 \\ & & & & & 0 & 0 & 1 & 1 & x_3 \\ & & & & & 1 & 1 & 0 & 1 & x_4 \\ & & & & & 1 & 0 & 1 & 1 & x_5 \end{array} \right]$$

If $y \neq a_1b_1$, then $M_2/\{a_1b_1, a_2b_1\} \setminus a_3b_1$ has x parallel to a_1a_3 . In this case, $M_1/\{a_1b_1, a_2b_1\} \setminus \{a_3b_1, a_1a_3\} \cong M^*(K_{1,1,1,n-k})$, and this matroid contains x and y . Thus, we may assume that $y = a_1b_1$.

If $(x_4, x_5) \neq (1, 0)$, then $M_2/\{a_1a_2, a_1a_3, a_2a_3\}$ has y , a_2b_1 , and a_3b_1 parallel to e , f , and g , respectively, with x parallel to f or g . Therefore, we may simplify $M_1/\{a_1a_2, a_1a_3, a_2a_3\}$ to find an $M^*(K_{3,n-k+1})$ -matroid containing x and y . From this matroid, we can easily find an $M^*(K_{1,1,1,n-k-1})$ -minor using x and y . Instead, we assume that $(x_4, x_5) = (1, 0)$, so (x_2, x_3, x_4, x_5) is $(1, 0, 1, 0)$. Then, $M_1/\{a_1a_2, a_2a_3, a_3b_1\} \setminus a_2b_1 \cong M^*(K_{3,n-k+1})$. This minor contains x and y , and we can easily find an $M^*(K_{1,1,1,n-k-1})$ -minor using x and y . Recall that $n - k - 1 \geq \frac{n}{4} - 1$. \square

In the next lemma, we consider the case where removing two elements of M produces an $M(K_{1,1,1,n})$ -minor.

Lemma 3.5.3. *Let M be a 3-connected binary matroid so that $M \setminus x/f = N \cong M(K_{1,1,1,n})$ with $n \geq 1$. Let N have an element s so that $\{x, f, s\}$ is a circuit of M . Suppose $y \in E(N)$. There is an integer m with $m \geq \frac{n}{16} - 5$ so that x and y are elements of a minor of M that is isomorphic to $M(K_{1,1,1,m})$.*

Proof. As $M(K_{1,1,1,1}) \cong M(\mathcal{W}_3)$, by Lemma 3.1.3, the theorem holds for $n \geq 96$. We will assume $n \geq 97$.

Since $\{x, f, s\}$ is a triangle, the matroid M/f contains the parallel pair $\{x, s\}$. Thus $M/f \setminus s \cong M/f \setminus x$. If $y \neq s$, then $M/f \setminus s$ is an $M(K_{1,1,1,n})$ -minor containing x and y . Thus, we may assume that $y = s$.

In M^* , the set $\{x, f, y\}$ is a triad, and $M^*/x \setminus f \cong M^*(K_{1,1,1,n})$. There is a unique binary matroid, M_0 , obtained from M^* by adding an element z so that $\{x, y, z\}$ is a circuit of M_0 . Since $M_0 \setminus z$ is 3-connected, and z is not added as a loop, a coloop, or parallel to another element, M_0 is 3-connected. Let H be the hyperplane of M^* that is the complement of $\{x, f, y\}$.

Now, $z \in \text{cl}_{M_0}(H)$ and M_0/x has the parallel pair $\{y, z\}$. Thus $M_0|(H \cup z) = M_0/x \setminus \{f, y\} \cong M_0/x \setminus \{f, z\} = M^*/x \setminus f \cong M^*(K_{1,1,1,n})$. Hence, M_0 contains z in an $M^*(K_{1,1,1,n})$ -restriction. We will assume this restriction is labelled as in Figure 3.20. Without loss of generality, $z \in \{a_1b_1, a_1a_2\}$. One of these cases is depicted in Figure 3.26.

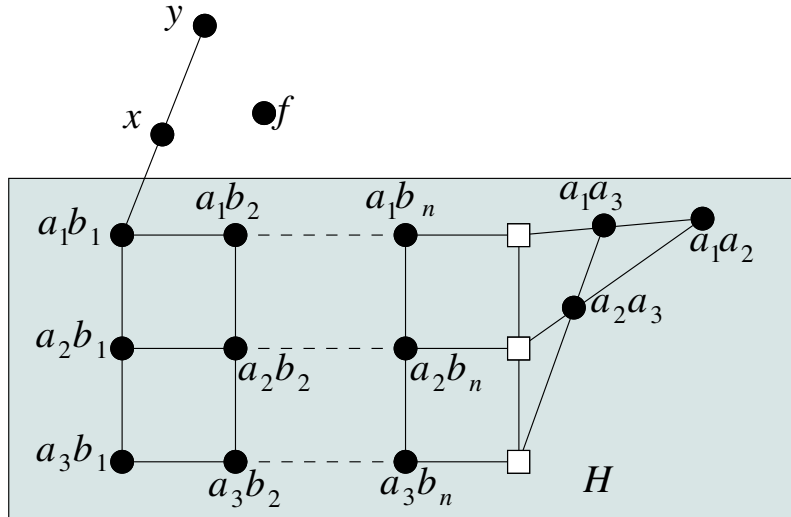


FIGURE 3.26. A geometric illustration of M_0 with hyperplane H complementing cocircuit $\{x, f, y\}$ if $z = a_1b_1$.

Consider M_0/f . Since $M_0/f \setminus \{x, y\} \cong M^*(K_{1,1,1,n})$, the matroids $M_0/f \setminus y$ and $M_0/f \setminus x$ are single-element extensions of $M^*(K_{1,1,1,n})$. If one of these is 3-connected, then without loss of generality, $M_0/f \setminus y$ is 3-connected. By Lemma 3.5.2, $M_0/f \setminus y$ has x and z in a minor,

$(M_0/f\setminus y)/C\setminus D \cong M^*(K_{1,1,1,k})$ for $k \geq \frac{n}{4} - 2$. The matroid $M_0/(C \cup f)\setminus D$ is the single-element extension of $M^*(K_{1,1,1,k})$ by an element y added so that $\{x, y, z\}$ is a circuit.

Suppose $M_0/(C \cup f)\setminus D$ is not 3-connected. Then y is parallel to an element c . In this case, $M_0/(C \cup f)\setminus(D \cup c) \cong M^*(K_{1,1,1,k})$, and $M_0/(C \cup f)\setminus(D \cup c)$ has $\{x, y, z\}$ as a triangle. Then, without loss of generality, $x = a_1b_1$, $y = a_2b_1$, and $z = a_3b_1$ (see Figure 3.27). Contract the cocircuit $\{a_1a_2, a_1a_3, a_2a_3\}$ from this matroid to produce an $M^*(K_{3,k})$ -minor.

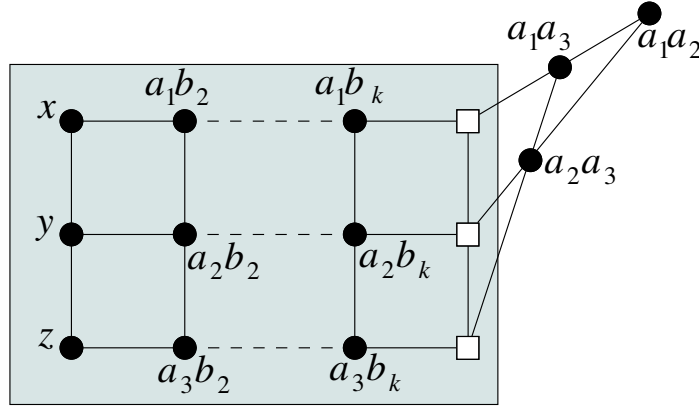


FIGURE 3.27. A geometric illustration of $M^*(K_{1,1,1,k})$ with triangle $\{x, y, z\}$.

Delete $\{z, a_2b_2\}$ and contract a_3b_2 to produce an $M^*(K_{1,1,1,k-2})$ -minor using x and y . As we have deleted z , this minor is also a minor of M^* .

We may now assume that $M_0/(C \cup f)\setminus D$ is a 3-connected, single-element extension of $M^*(K_{1,1,1,k})$ with elements x and y . By Lemma 3.5.2, this matroid has x and y in a minor, N_1 , which is isomorphic to $M^*(K_{1,1,1,j})$ for some $j \geq \frac{k}{4} - 2 \geq \frac{n}{16} - 3$. Since x and y are not parallel in N_1 , the element z has not been contracted to produce N_1 . Therefore, either z has been deleted to produce N_1 so N_1 is a minor of M^* , or z is an element of the triangle $\{x, y, z\}$ in N_1 . In the latter case, using the argument above, we can delete z and identify an $M^*(K_{1,1,1,j-2})$ -minor of M^* that contains x and y . Recall $j - 2 \geq \frac{n}{16} - 5$.

Finally, we consider the case that both $M_0/f\setminus y$ and $M_0/f\setminus x$ fail to be 3-connected. Recall that M_0 is 3-connected. Since $M_0/f\setminus\{x, y\}$ is 3-connected, in M_0/f , the elements x and y are parallel to e and d , respectively. Thus $M_0/f\setminus\{e, d\} \cong M^*(K_{1,1,1,n})$, and $\{x, y, z\}$ is a

triangle of this matroid. Again, by the argument above, we may delete z and produce an $M^*(K_{1,1,1,n-2})$ minor of M^* using x and y . \square

Finally, we consider the case where the removal of two elements produces an $M^*(K_{1,1,1,n})$ -minor. The reader will note in the next lemma that one of the outcomes involves getting a spike-minor but does not mention the elements x and y .

Lemma 3.5.4. *Let M be a 3-connected binary matroid so that $M \setminus x / f = N \cong M^*(K_{1,1,1,n})$ for some positive integer n . Let N have an element s so that $\{x, f, s\}$ is a circuit of M . Suppose $y \in E(N)$. Then there is an integer m with $m \geq \frac{n}{4} - 3$ so that either M has a minor isomorphic to T_m , or M has a minor that uses $\{x, y\}$ and is isomorphic to $M^*(K_{1,1,1,m})$.*

Proof. As $M^*(K_{1,1,1,1}) \cong M(\mathcal{W}_3)$, by Lemma 3.1.3, the theorem holds for $n \leq 16$. We will assume $n \geq 17$. If M has a minor isomorphic to T_r for $r \geq \frac{n}{4} - 3$, then we are done, so we assume otherwise.

Since $\{x, f, s\}$ is a triangle, M/f has $\{x, s\}$ as a parallel pair. Thus $M/f \setminus s \cong M/f \setminus x$. If $s \neq y$, then $M \setminus f / s$ is an $M^*(K_{1,1,1,n})$ -minor using x and y . Thus we may assume $s = y$.

In M^* , the set $\{x, f, y\}$ is a triad complementing a hyperplane H . The matroid $M^* / x \setminus f \cong M(K_{1,1,1,n})$. There is a unique binary matroid M_0 obtained from M^* by adding an element z so that $\{x, y, z\}$ is a triangle in M_0 . Since $M_0 \setminus z$ is 3-connected, and z is not a loop, a coloop, or parallel to another element, M_0 is 3-connected.

Now, $z \in \text{cl}_{M_0}(H)$, and M_0/x has the parallel pair $\{y, z\}$. Thus, $M_0|(H \cup z) = M_0/x \setminus \{f, y\} \cong M_0/x \setminus \{f, z\} = M^* / x \setminus f \cong M(K_{1,1,1,n})$. Hence, M_0 contains z in an $M(K_{1,1,1,n})$ -restriction. We will assume this restriction is labelled as depicted in Figure 3.19. Without loss of generality, $z \in \{a_1 b_1, a_1 a_2\}$. One of these case is depicted in Figure 3.28.

Consider M_0/f . Since $M_0/f \setminus \{x, y\} \cong M(K_{1,1,1,n})$, the matroids $M_0/f \setminus y$ and $M_0/f \setminus x$ are single-element extensions of $M(K_{1,1,1,n})$. If one of these matroids is 3-connected, then without loss of generality, $M_0/f \setminus y$ is 3-connected. By Lemma 3.5.1, $M_0/f \setminus y$ has x and

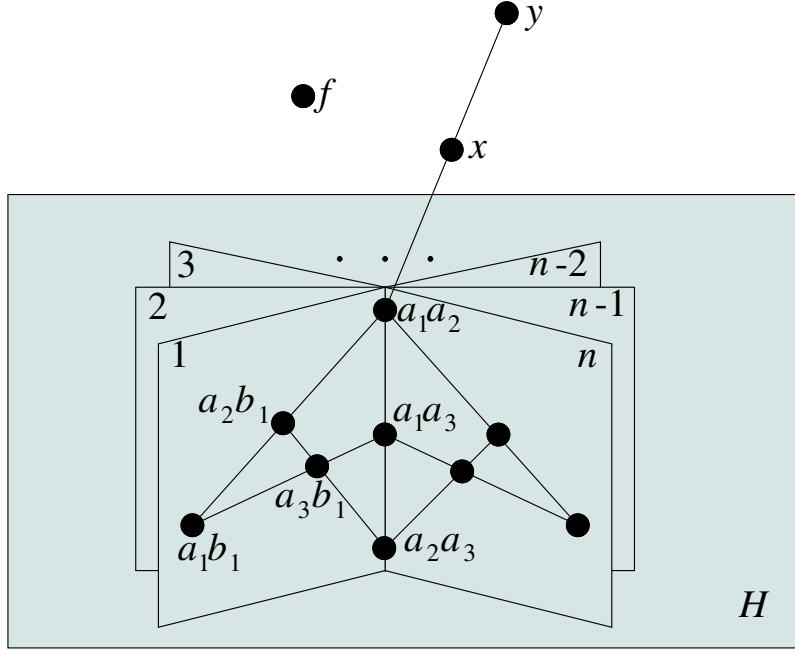


FIGURE 3.28. A geometric illustration of M_0 with hyperplane H complementing bond $\{x, f, y\}$ if $z = a_1 a_2$.

z in a minor, $(M_0/f \setminus y)/C \setminus D$, that is isomorphic to T_k or $M(K_{1,1,1,k})$ for some $k \geq \frac{n-1}{2}$. If $(M_0/f \setminus y)/C \setminus D \cong T_k$, then $(M_0/f)/C \setminus D$ is a spike T_k with an extra element, y , added in the closure of two elements. It is routine to check that $((M_0/f)/C \setminus D)/y \setminus z$ or $((M_0/f)/C \setminus D)/x \setminus z$ contains a T_{k-1} -minor. Since z has been deleted, T_{k-1} is also a minor of M^* , contradicting the assumption that M has no T_r -minor for $r \geq \frac{n}{4} - 3$. Thus $(M_0/f \setminus y)/C \setminus D \cong M(K_{1,1,1,k})$. The matroid $M_0/(C \cup f) \setminus D$ is the single-element extension of $M(K_{1,1,1,k})$ by the element y added so that $\{x, y, z\}$ is a circuit.

If $M_0/(C \cup f) \setminus D$ is not 3-connected, then y is parallel to an element c . In this case, $M_0/(C \cup f) \setminus (D \cup c) = M_1 \cong M(K_{1,1,1,k})$, and M_1 has $\{x, y, z\}$ as a triangle. If $\{x, y, z\}$ is $\{a_1 a_2, a_1 a_3, a_2 a_3\}$, then $M_1 \setminus z$ has an $M(K_{1,1,1,k-1})$ -minor using x and y . Otherwise, without loss of generality, $\{x, y, z\} = \{a_1 a_2, a_1 b_1, a_2 b_1\}$ (see Figure 3.29). In $M_1/a_3 b_1$, the elements $\{x, y, z, a_1 a_3, a_2 a_3\}$ are collinear. Delete z and any elements parallel to x and y to produce a minor isomorphic to $M(K_{1,1,1,k-1})$ or $M(K_{1,2,k-1})$. In the latter case, we can easily find

a minor isomorphic to $M(K_{1,1,1,k-2})$ that contains x and y . In either case, since we have deleted z , this minor is also a minor of M^* .

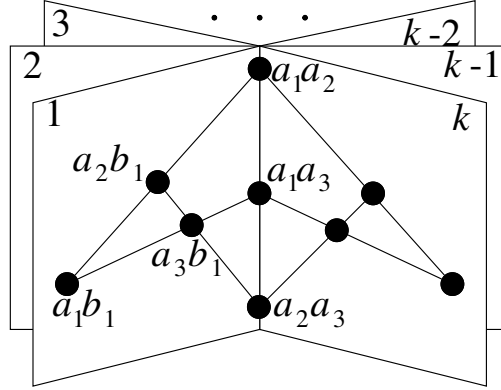


FIGURE 3.29. A geometric illustration of $M_1 \cong M(K_{1,1,1,k})$.

We may now assume that $M_0/(C \cup f) \setminus D$ is a 3-connected, single-element extension of $M(K_{1,1,1,k})$ that uses x and y . By Lemma 3.5.1, this matroid has x and y in a minor, N_1 , that is isomorphic to $M(K_{1,1,1,j})$ or T_j for some $j \geq \frac{k-1}{2} \geq \frac{n-3}{4}$. Since x and y are not parallel in N_1 , the element z has not been contracted to produce N_1 . Therefore, either z has been deleted to produce N_1 so N_1 is a minor of M^* and the lemma holds; or z is an element of the triangle $\{x, y, z\}$ in N_1 . In the latter case, suppose first that $N_1 \cong T_j$. The spike T_j is not a minor of M^* by assumption. Therefore, $z \in E(N_1)$ and this T_j -minor has triangle $\{x, y, z\}$. As the only triangles of T_j are those including the tip, and without loss of generality, x is not the tip of N_1 , it is routine to check that $N_1/x \setminus z \cong T_{j-1}$. Since z has been deleted, this matroid is a minor of M^* , a contradiction.

We may now assume that $N_1 \cong M(K_{1,1,1,j})$ and $\{x, y, z\}$ is a triangle of N_1 . In this case, using the argument in the second to last paragraph, we can identify an $M(K_{1,1,1,j-2})$ -minor of M_0 that contains x and y and avoids z . Therefore this matroid is also a minor of M^* . Recall $j - 2 \geq \frac{n-3}{4} - 2$.

Finally, we consider the case that both $M_0/f \setminus y$ and $M_0/f \setminus x$ fail to be 3-connected. As M_0 and $M_0/f \setminus \{x, y\}$ are both 3-connected, in M_0/f , the elements x and y are parallel to e and

d , respectively. Thus $M_0/f \setminus \{e, d\} \cong M(K_{1,1,1,n})$, and $\{x, y, z\}$ is a triangle of this matroid. Again, by the argument above, we can delete z and produce an $M(K_{1,1,1,n-2})$ minor of M_0 using x and y that is also a minor of M^* . \square

3.6 The Proof of the Main Result

We are now ready to prove the main result, which is restated below.

Theorem 3.0.1. *Let M be a 3-connected, binary matroid, and let x and y be elements of M . For every $n > 2$, there is an integer $h(n)$ so that if $|E(M)| > h(n)$, then x and y are elements of a minor of M that is isomorphic to the rank- n wheel, T_n , $M(K_{1,1,1,n})$, or $M^*(K_{1,1,1,n})$.*

Proof. By Theorem 2.0.7, there is a function g so that if $|E(M)| \geq g(100n)$, then y is an element of a minor N of M that is isomorphic to $M(\mathcal{W}_{100n})$, T_{100n} , $M(K_{1,1,1,100n})$, or $M^*(K_{1,1,1,100n})$. If x is an element of N , then the theorem holds, so we assume $x \in E(M) - E(N)$. Let M' be a minimum sized minor of M so that M' has elements x and y and a minor isomorphic to N . By Theorem 3.2.1, for some (N_1, M_1) such that either $N_1 \cong N$ and $M_1 \cong M'$ or $N_1 \cong N^*$ and $M_1 \cong (M')^*$, one of the following holds:

- (i) $N_1 = M_1 \setminus x$, and y is contained in this minor.
- (ii) $N_1 = M_1 \setminus x/z$, and this minor contains y and an element s so that $\{x, z, s\}$ is a circuit of M_1 .

As $\{M(\mathcal{W}_{100n}), T_{100n}, M(K_{1,1,1,100n}), M^*(K_{1,1,1,100n})\}$ is closed under duality, may assume that $N_1 \in \{M(\mathcal{W}_{100n}), T_{100n}, M(K_{1,1,1,100n}), M^*(K_{1,1,1,100n})\}$. We consider these cases.

First, we consider the case that $N_1 \cong M(\mathcal{W}_{100n})$. In case (i), by Lemma 3.3.1, x and y are elements of a minor of M_1 that is isomorphic to $M(\mathcal{W}_m)$ for some $m \geq 25n$. In case (ii), by Lemma 3.3.3, x and y are elements of a minor of M_1 that is isomorphic to $M(\mathcal{W}_m)$ for some $m \geq 25n$.

Now, consider the case that N_1 is isomorphic to the cycle or bond matroid of $K_{1,1,1,100n}$. In case (i), by Lemma 3.5.1 and Lemma 3.5.2, either M_1 has a T_k -minor or x and y are elements of a minor of M_1 isomorphic to the cycle or bond matroid of $K_{1,1,1,k}$ for some $k \geq 25n - 2$. In case (ii), by Lemma 3.5.3 and Lemma 3.5.4, either M_1 has a T_k -minor for some $k \geq 25n - 3$, or x and y are elements of a minor of M_1 isomorphic to the cycle or bond matroid of $K_{1,1,1,m}$ for some $m \geq \frac{25n}{4} - 5 \geq 4n$.

Finally, we may assume M_1 has a minor isomorphic to T_k for some $k \geq 25n - 3$. By Theorem 2.2.2, x is an element of a minor of M_1 that is isomorphic to T_j for some $j \geq \frac{25n-3}{2}$. Let M'' be a minimum sized minor of M_1 so that M'' has elements x and y and a minor isomorphic to T_j . By Theorem 3.2.1, for some M_2 such that either $M_2 \cong M''$ or $M_2 \cong (M'')^*$, one of the following holds:

- (i) $T_j \cong M_2 \setminus x$, and y is contained in this minor.
- (ii) $T_j \cong M_2 \setminus x/z$, and this minor contains y and an element s so that $\{x, z, s\}$ is a circuit of M_2 .

In case (i) and case (ii), by Lemma 3.4.2 and Lemma 3.4.3 respectively, x and y are elements of a minor of M_2 that is isomorphic to T_i for some $i \geq \frac{j}{2} \geq 6n - 1$. □

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Vita

Deborah Ann Chun was born in November 1980, in Pennsylvania. She double majored in mathematics and engineering at Harvey Mudd College, earning a bachelor of science degree in May of 2002. While working for three years as a signals analyst with the National Security Agency, she completed a master of science degree in applied and computational mathematics at the Johns Hopkins University Whiting School of Engineering. In August 2005, she came to Louisiana State University to study mathematics, particularly graph and matroid theory, as a graduate student. She earned a master of science degree in mathematics from Louisiana State University in May 2010. She is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August 2011.