Valuations on Rings.

George Vaughn Poynor  
*Louisiana State University and Agricultural & Mechanical College*

Follow this and additional works at: [https://digitalcommons.lsu.edu/gradschool_disstheses](https://digitalcommons.lsu.edu/gradschool_disstheses)

**Recommended Citation**

[https://digitalcommons.lsu.edu/gradschool_disstheses/995](https://digitalcommons.lsu.edu/gradschool_disstheses/995)

This Dissertation is brought to you for free and open access by the Graduate School at LSU Digital Commons. It has been accepted for inclusion in LSU Historical Dissertations and Theses by an authorized administrator of LSU Digital Commons. For more information, please contact [gradetd@lsu.edu](mailto:gradetd@lsu.edu).
This dissertation has been microfilmed exactly as received

POYNOR, George Vaughn, 1938–
VALUATIONS ON RINGS.

Louisiana State University, Ph.D., 1964
Mathematics

University Microfilms, Inc., Ann Arbor, Michigan
VALUATIONS ON RINGS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by

George Vaughn Poynor
B.S., Southeastern Louisiana College, 1959
M.S., Louisiana State University, 1960
August, 1964
ACKNOWLEDGMENT

The author wishes to express his appreciation to Professor Hubert S. Butts for his advice and encouragement.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGMENT</td>
<td>ii</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>iv</td>
</tr>
<tr>
<td>NOTATION</td>
<td>1</td>
</tr>
<tr>
<td>INTRODUCTION</td>
<td>2</td>
</tr>
<tr>
<td>CHAPTER I</td>
<td>5</td>
</tr>
<tr>
<td>CHAPTER II</td>
<td>14</td>
</tr>
<tr>
<td>CHAPTER III</td>
<td>28</td>
</tr>
<tr>
<td>SELECTED BIBLIOGRAPHY</td>
<td>34</td>
</tr>
<tr>
<td>AUTOBIOGRAPHY</td>
<td>35</td>
</tr>
</tbody>
</table>
In this paper $R$ denotes a commutative ring with identity having total quotient ring $Q$. A valuation of $Q$ is a function $v$ from $Q$ onto a partially ordered semigroup such that the following hold:

V1) $v(xy) = v(x) + v(y)$, for all $x, y \in Q$, and

V2) $v(x) \geq v(y)$, $v(z) \geq v(y)$ implies $v(x - z) \geq v(y)$, for all $x, y, z$ in $Q$.

The valuation ring $R_v$ of a valuation $v$ is the set of all elements in $Q$ whose values are $\geq v(1)$. A valuation ideal of a ring $R$ is an $A$ of $R$ such that $A = B \cap R$ for some ideal $B$ of a valuation ring $R_v$ such that $R \subset R_v \subset Q$.

Chapter I consists principally of preparatory material concerning valuations on rings.

In Chapter II we suppose that $R$ is a ring in which the set of zero divisors is an ideal contained in every regular ideal, and let $\mathcal{V}_1(R)$ represent the set of valuation ideals for valuations of $Q$ whose value semigroups are totally ordered. Necessary and sufficient conditions are given for $\mathcal{V}_1(R)$ to be a subset of the collection $\mathcal{Q}(R)$ of primary ideals of $R$. We also find, under suitable assumptions on the ring $R$, necessary and sufficient conditions for $\mathcal{V}_1(R) = \mathcal{Q}(R)$, and for $\mathcal{V}_1(R) \subset \mathcal{P}(R)$, where $\mathcal{P}(R)$ represents the set of prime power ideals in $R$. Finally, if $\mathcal{P}(R)$ represents the set of ideals of $R$
which are prime to the complement of a prime ideal in $R$, necessary and sufficient conditions are presented, again under certain assumptions on $R$, for $\mathcal{V}_1(R)$ to equal $\mathcal{V}(R)$.

In Chapter III we consider rings $R$ with the property that every ideal of $R$ is a product of finitely many quasi-maximal ideals. An ideal $N$ is quasi-maximal if $AN = N$ for every proper divisor $A$ of $N$. It is shown that such rings are multiplication rings, and necessary and sufficient conditions are given in order that such a ring be a ZPI-ring.
NOTATION

In what follows all rings considered will be assumed to be commutative, and to contain an identity distinct from zero. R and Q will denote respectively a ring and its total quotient ring. An element of R which is not a divisor of zero in R will be called a regular element of R. A regular ideal is an ideal which contains at least one regular element. A non-regular ideal consists entirely of zero divisors.

In this dissertation we will be dealing with valuation rings which are not integral domains. If a valuation ring is an integral domain, we will call it a valuation domain.

The symbol ⊂ will be used for set inclusion, with < denoting proper inclusion. In general the notation will be that of Zariski and Samuel [7] and [8].

---

1In the symbol [A; B], A refers to the number of the bibliographical reference; B to the page number in the reference denoted by A.
Various attempts have been made in the literature to generalize the concept of valuation to rings with proper divisors of zero. Among these was that of Maranda [4], whose aim was to provide a basis for an analogue of Krull domains [7; 82] in the ring case, and whose path we follow.

0.1 Definitions. By a partially ordered semigroup we mean a semigroup $\Delta$, together with a binary relation $\leq$ on the elements of $\Delta$, such that $\leq$ is reflexive, anti-symmetric, and transitive, and such that if $x \leq y$ in $\Delta$, then $x + z \leq y + z$ for every $z \in \Delta$. (We will write $\Delta$ as an additive semigroup). A partially ordered group is a partially ordered semigroup which is also a group. A totally ordered semigroup will denote a partially ordered semigroup in which the order is total; i.e., either $x \leq y$ or $y \leq x$ for every pair of elements $x$ and $y$ in the semigroup. A totally ordered group is a totally ordered semigroup which is also a group.

Let $Q$ denote a commutative ring with identity in which every regular element is invertible. A valuation of $Q$ is a function $v$ from $Q$ onto a partially ordered semigroup $\Delta$ satisfying:

1) $v(xy) = v(x) + v(y)$ for all $x, y \in Q$;

2) $v(x) \geq v(z); v(y) \geq v(z)$ implies $v(x - y) \geq v(z)$ for all $x, y, z \in Q$.

We note that the element $v(1)$ is an identity for $\Delta$, and we will therefore denote $v(1) = 0$. Further, $\Delta$ must be commutative, in view of V1. We say that $v$ is non-trivial if there is a regular element $x$ of $Q$
such that \( v(x) \neq 0 \). The set \( R_v = \{ x \in \mathbb{Q} : v(x) > 0 \} \) is a subring of \( \mathbb{Q} \), called the **valuation ring** of \( v \).

If \( \triangle \) is totally ordered, then \( V2 \) can be replaced by the equivalent axiom:

\[ V2^* \quad v(x - y) \geq \min \{v(x), v(y)\} \text{ for all } x, y \in \mathbb{Q}. \]

In the case that \( \mathbb{Q} \) is a field, and \( \triangle = G \cup \{\infty\} \), where \( G \) is a totally ordered group, and the symbol \( \infty \) is subject to the laws \( g < \infty \) for all \( g \in G \), \( g + \infty = \infty + g = \infty \) for all \( g \in \triangle \), then the above yields the customary definition of valuation.

A starting point in this dissertation was the question, can a valuation be associated with a ring, the ideals of which are totally ordered under set inclusion? The answer is yes, as given in 1.13. In the main we restrict our considerations to valuations whose value semigroups are of the form \( \triangle = G \cup \{\infty\} \), as above. In this case, many of the results generally true for valuations of fields have analogues in the ring case.

A further restriction which is made almost from the outset is the confinement of our study to rings in which the set of zero divisors is an ideal contained in every regular ideal. A natural question arises as to the breadth of the class of such rings. A primary ring is of this type, as is the residue class ring of a valuation domain by a proper ideal. Example 3 of Chapter I illustrates a third type of ring with this property which shows that the class is fairly broad.

In [3] Krull considered a class of ideals ("\( b \)-ideals") which were studied utilizing mainly valuation theoretic methods. Zariski [8; 340] defines a similar class of ideals in an integral domain which he calls valuation ideals.
0.2 Definition. An ideal $A$ of ring $R$ is called a \textit{valuation ideal} of $R$ if there is a valuation ring $R_v$ and an ideal $B$ of $R_v$ such that $R \subset R_v \subset Q(R)$, and $A = B \cap R$. Gilmer [1], and Gilmer and Ohm [2], have studied various relationships concerning the valuation ideals of an integral domain $R$, among which are conditions under which every valuation ideal is primary, and vice versa; and conditions under which every valuation ideal is a power of a prime ideal. We have produced analogues to these in the case that $R$ is a commutative ring with identity. As in [2] certain restrictions must be made in order to handle these problems.

In the last chapter of this work, we study a class of rings designated as $qm$-rings; i.e., the class of rings every ideal of which is the product of a finite number of ideals $N$, such that if $A$ is an ideal properly containing $N$, then $AN = N$. Such ideals are called quasi-maximal ideals. It is shown that a $qm$-ring is a multiplication ring, and necessary and sufficient conditions are found for a $qm$-ring to be a ZPI-ring.
1.1 Proposition. Let \( Z \) denote the set of zero divisors of a ring \( R \). If \( Z \) is contained in every regular ideal of \( R \), then either \( Z \) is an ideal of \( R \), or \( R = \mathbb{Q} \).

Proof: Let \( A^* \) denote the intersection of all regular ideals of \( R \). \( Z \subseteq A^* \).

Suppose \( Z \not\subseteq A^* \), and let \( x \in A^* - Z \). Since \( x \) and \( x^2 \) are regular, \( (x) = (x^2) = A^* \), by the minimality of \( A^* \). Hence \( a = ra^2 \) for some \( r \in R \), whence \( 1 = ra \). Thus \( A^* = R \), and hence \( R = \mathbb{Q} \).

1.2 Example. We note that in the ring \( R = \mathbb{I}_2 \oplus \mathbb{I}_2 \), where \( \mathbb{I}_2 \) denotes the field with two elements, \( Z \) consists of every element of \( R \) except the identity, which is the only regular element of \( R \). \( Z \) is not an ideal, but is contained in every regular ideal.

1.3 Lemma. Suppose that the set \( Z \) of zero divisors of \( R \) is contained in every regular ideal of \( R \), and \( R \neq \mathbb{Q} \). Then \( Z \) is an ideal in \( \mathbb{Q} \); i.e., \( Z \mathbb{Q} = Z \).

Furthermore, \( Z \) is the unique maximal ideal of \( \mathbb{Q} \).

Proof: \( Z \) is an ideal of \( R \), by 1.1, and is a prime ideal of \( R \), since the product of two regular elements is regular. Also, \( Z \subseteq Z \mathbb{Q} \), since \( \mathbb{Q} \) contains an identity. Let \( x \in Z \mathbb{Q} \cap R \). Then \( x = \sum a_i (r_i/s_i) \), where \( a_i \in Z \), and \( s_i \) and \( r_i \in R \), \( s_i \) regular. Let \( s = \pi s_i \). Then \( sx \in Z \), \( s \not\in Z \), and \( x \in R \), so that \( x \in Z \). Thus \( Z = Z \mathbb{Q} \cap R \). But \( Z \mathbb{Q} \subseteq R \), for if \( x \in Z \mathbb{Q} \), then \( x = a/s \), where \( a \in Z \), and \( s \) is regular in \( R \). Thus \( a \in sR \), since \( a \in Z \subseteq sR \). Thus
Therefore we have $Z = ZQ \cap R = ZQ$. Since every regular element of $Q$ is a unit, $Z$ is the ideal of non-units of $Q$, and is thus the unique maximal ideal of $Q$.

1.4 Lemma. If the ideals of $R$ are totally ordered under inclusion, then the ideals of $Q$ are totally ordered.

Proof: Let $A$ and $B$ be ideals of $Q$, and set $A^* = \{x \in R: xQ \subset A\}$, $B^* = \{x \in R: xQ \subset B\}$. It is easily shown that $A^*$ and $B^*$ are ideals of $R$.

Suppose $x \in AQ$. Then $x = \sum a_i q_i$, where $a_i \in A$, and $q_i \in Q$. Each $a_i$ is of the form $a_{i1}/s_{i1}$, where $s_{i1}$ is regular in $R$, and $a_{i1} \in R$. Certainly $s_{i1} \in A^*$ since $a_{i1}(1/s_{i1}) \in A$. Thus $s_{i1}Q \subset A$, and hence $x \in \sum a_{i1}Q$, which is contained in $A^*Q$. Thus $AQ \subset A^*Q$, and since $A^*Q \subset A$, we have $A = AQ \subset A^*Q \subset A$, implying that $A = A^*Q$. Similarly $B = B^*Q$. Assuming $A^* \subset B^*$, we have $A = A^*Q \subset B^*Q = B$, proving our lemma.

1.5 Corollary. In order for there to exist a ring $S$, with $R \subset S \subset Q$, having totally ordered ideals, it is both necessary and sufficient that $Q$ have ordered ideals.

Proof: If $Q$ has totally ordered ideals, let $S = Q$. If there exists such an $S$, then $Q(S) = Q$, hence by 1.4, $Q$ has totally ordered ideals.

1.6 Lemma. If $R$ is a ring such that $Q(R)$ has totally ordered ideals, then the set $Z$ of zero divisors of $R$ is an ideal of $R$.

Proof: Let $x$ and $y \in Z$, $r \in R$. There exist non-zero elements $s$ and $t$ in $R$ such that $sx = ty = 0$. We may assume $s \in tQ$, so that $s = (a/b)\cdot t$ for some $a$ and $b \in R$, $b$ regular. Then $bs(x - y) = bsx - bsy = bsx - aty = 0$. However, $bs \neq 0$, since $b$ is regular and $s \neq 0$. Thus $x - y \in Z$. Clearly
rx \in Z$, and hence $Z$ is an ideal.

1.6 Example. Under the hypotheses of 1.6 it does not follow that the ideal of zero divisors of $R$ is contained in every regular ideal. The following is an example: let $R$ denote the field of rational numbers, $x$ and $y$ algebraically independent over $R$. Let $S = R[x, y]_{(x,y)}$ denote the quotient ring of $R[x, y]$ by the maximal ideal $(x, y)$, and let $S_1 = S/(x^2S)$. Then in $S_1$, the ideal $(x)$ is the ideal of zero divisors, but is not contained in the regular ideal $(x^2, y)$ in $S_1$. Further, $Q(S_1) = (S_1)(x)$, a ring whose ideals are totally ordered.

1.7 Lemma. Suppose that the set $Z$ of zero divisors of $R$ is an ideal contained in every regular ideal of $R$. Then if $A$ is a non-regular primary ideal of $R$, $AQ = A$. Furthermore, if $A$ is an ideal of $R$ with radical $Z$, and $AQ = A$, then $A$ is primary.

Proof: If $A$ is a non-regular primary ideal of $R$, then $A^{\text{ec}} = A$ in $R^Z = Q$; but $A^{\text{ec}} = AQ \cap R = AQ$, since $AQ \subseteq ZQ \subseteq R$. Thus $AQ = A$. If $A$ has radical $Z$, and $A = AQ$, then let $xy \in A$, $x \notin Z$, where $x$ and $y$ denote elements of $R$. Then $x$ is a unit in $Q$, and hence $y = xy/y \in QA = A$, whence $A$ is $Z$-primary.

1.8 Lemma. Suppose that $v$ is a valuation of $Q$ onto a totally ordered semigroup $\Delta$. Then if $x$ is a regular element of $Q$, either $x$ or $1/x$ is in $R_v$.

Proof: If $x \notin R_v$, then $v(x) \notin 0$, hence $v(x) < 0$. Thus $0 = v(1) = v(x(1/x)) = v(x) + v(1/x) < 0 + v(1/x) = v(1/x)$. Therefore $1/x \in R_v$.

1.9 Lemma. If $v$ is a valuation of $Q$ onto $\Delta = G \cup \{\infty\}$, where $G$ is a totally ordered semigroup, then $v(x - y) = \min \{v(x), v(y)\}$ whenever $v(x) \neq v(y)$.
Proof: We may assume \( v(x) < v(y) \). Then \( v(x) \geq \min \{v(x - y), v(y)\} \), hence \( v(x) \geq v(x - y) \). Similarly, \( v(x - y) \geq \min \{v(x), v(y)\} = v(x) \). Thus \( v(x - y) = v(x) \).

1.10 Lemma. Let \( v \) be a valuation of \( Q \) onto \( \Delta = G \cup \{\infty\} \) where \( G \) is a totally ordered group. Denote \( P_g = \{x \in Q; v(x) \geq g\} \) for \( g \in \Delta \), and \( P^*_g = \{x \in Q; v(x) > g\} \) for \( g \in G \). Then

i) \( P_\infty = R_v \), the valuation ring of \( v \) in \( Q \);

ii) \( P^*_\infty = P \) is a prime ideal of \( R_v \);

iii) The \( P_g \) and \( P^*_g \) which have radical \( P \) are \( P \)-primary;

iv) \( P_\infty \) (\( P \) for \( g = \infty \)) is a prime ideal in both \( R_v \) and \( Q \);

v) \( \cap P_g = P \) if \( v \) is non-trivial.

Proof: i) and ii) are obvious.

iii) Suppose that \( x \) and \( y \) are elements of \( R_v \) such that \( xy \in P_g \), \( x \notin P \). Then \( v(xy) \geq g \) and \( v(x) = 0 \). Thus \( g \leq v(xy) = v(x) + v(y) = 0 + v(y) = v(y) \), and \( y \in P_g \). A similar argument can be made for \( P^*_g \).

iv) It is clear that \( P \) is an ideal of both \( R_v \) and \( Q \). If \( x, y \notin P \), and \( x, y \in Q \), then \( v(x) \) and \( v(y) \) are finite, therefore \( v(xy) = v(x) + v(y) \) is finite, and \( xy \notin P \).

v) If \( x \in \cap P_g \), then \( v(x) \geq g \) for all \( g \in G \). Thus \( v(x) = \infty \), for if there is an element \( k \in G \) with \( k \geq g \) for all \( g \in G \), then let \( h > 0 \) be an element of \( G \). We have \( k \geq k + h \), contradicting the choice of \( h \). Thus \( \cap P_g \subseteq P_\infty \). Conversely, \( \infty \geq g \) for all \( g \in G \), hence \( P_\infty \subseteq \cap P_g \). This completes the proof of the lemma.

1.11 Corollary. Let \( v \) be a non-trivial valuation of \( Q \) such that the set of zero divisors of \( R_v \) is an ideal of \( R_v \) contained in every regular ideal of \( R_v \), and such that the value semigroup of \( v \) is of the form
G \cup \{\infty\}$, where $G$ is a totally ordered group. Then $P_\infty = Z$.

Proof: Obviously $P_\infty \subset Z$. If $x \in Z$, and $y$ is a regular element of $Q$ such that $v(y) > 0$, then $x \in yR_y$, hence $v(x) \geq v(y) > 0$. If $v(x)$ is finite, say $v(x) = g \in G$, then choose $z \in Q$ so that $v(z) = -g$. Then $0 = v(x) + v(z) = v(xz)$. But this implies $xz$ is regular, since every zero divisor has positive value. This contradiction shows that $x \in P_\infty$.

1.12 Lemma. Let $v$ be a non-trivial valuation of $Q$ onto $G \cup \{\infty\}$, where $G$ is a totally ordered group. Then $R_v \neq Q$. Furthermore, if the set $Z$ of zero divisors of $R_v$ is contained in every regular ideal of $R_v$, then $Z$ is an ideal of $R_v$, equal to the intersection of all regular ideals of $R_v$.

Proof: There is a regular element $x \in Q$ with $v(x) \neq 0$. If $v(x) > 0$, then $v(1/x) < 0$, and $1/x \notin R_v$. Thus $R_v \neq Q$. The remainder of the lemma follows from 1.1.

1.13 Theorem. Suppose that the set $Z$ of zero divisors of $R$ is contained in every regular ideal of $R$, and $R \neq Q$. If the regular ideals of $R$ are ordered, then there exists a non-trivial valuation of $v$ of $Q$ whose value semigroup is of the form $\triangle = G \cup \{\infty\}$, where $G$ is a totally ordered group, such that $R_v = R$. Conversely, if there exists a non-trivial valuation $v$ of $Q$ onto $\triangle = G \cup \{\infty\}$, where $G$ is a totally ordered group, with $R = R_v$, then the regular ideals of $R$ are ordered.

Proof: It follows from 1.3 that $Z$ is a prime ideal of $R$, and the maximal ideal of $Q$. Suppose that the regular ideals of $R$ are linearly ordered, and let $D_v = R/Z$. $D_v$ has quotient field $Q/Z$, and is a valuation domain. Therefore there is a valuation $v'$ of $Q/Z$ with valuation ring $D_v'$. Define
v on Q by \( v(x) = v'(h(x)) \), where \( h \) denotes the natural homomorphism of Q on \( Q/Z \). Then \( v(xy) = v'(h(xy)) = v'(h(x)h(y)) = v'(h(x)) + v'(h(y)) = v(x) + v(y) \), and \( v(x - y) = v'(h(x - y)) = v'(h(x) - h(y)) \geq \min \{ v'(h(x)), v'(h(y)) \} = \min \{ v(x), v(y) \} \), and v is therefore a valuation of Q. Further, \( R = h^{-1}(D_v) \), and hence R is the valuation ring of v. If, conversely, R is the valuation ring of a valuation of Q as described above, then by 1.8, either \( x \) or \( l/x \) is in R for every regular \( x \in Q \). Suppose A and B are regular ideals of Q, and let \( x \in B - A \). Then \( x \) is regular, since \( x \notin A \). Let \( y \in A \). If \( y \) is a zero divisor, then \( y \in B \). If \( y \) is regular, then either \( y/x \) or \( x/y \) is in R. If \( x/y \in R \), then \( x \in Ry \subseteq A \), a contradiction. Thus \( y/x \in R \), or \( y \in Rx \subset B \). Thus \( A \subset B \), and the regular ideals of R are ordered.

1.14 Lemma. Suppose that the set \( Z \) of zero divisors of R is contained in every regular ideal of R, and \( R \neq Q \). Then there is a 1-1 correspondence between the valuation rings \( R_v \) for v a valuation of Q onto \( G \cup \{ \infty \} \), where \( G \) is a totally ordered group, and \( R \subseteq R_v \subseteq Q \), and the valuation domains \( D_v \) such that \( R/Z \subseteq D_v \subseteq Q/Z \).

Proof: If \( R_v \) is a valuation ring of Q as described above, then \( R_v \) has totally ordered regular ideals, by 1.13, and hence \( R_v/Z \) is a valuation domain between \( R/Z \) and \( Q/Z \). We also have \( R_v = h^{-1}(R_v/Z) \), where \( h \) denotes the natural homomorphism of Q on \( Q/Z \). Further, if \( D_v \) is any valuation domain between \( R/Z \) and \( Q/Z \), then \( h^{-1}(D_v) \) is a valuation ring \( R_v \), by 1.13, and also we have \( h(R_v) = D_v \). Thus the correspondence \( R_v \leftrightarrow R_v/Z \) is the required 1-1 correspondence.

1.15 Proposition. In addition to the hypotheses of the preceding lemma, suppose that A is a proper regular prime ideal of R. Then there is a
non-trivial valuation \( v \) of \( \mathbb{Q} \) onto \( \mathbb{G} \cup \{\infty\} \), \( \mathbb{G} \) a totally ordered group, such that \( R \subset R_v \) and \( M_v \cap R = A \), where \( M_v \) denotes the maximal ideal of \( R_v \).

Proof: The theorem is true if \( R \) is an integral domain. Thus, since \( h(A) \) is a proper prime ideal in \( \mathbb{R}/\mathbb{Z} \), there is a non-trivial valuation \( v' \) of \( \mathbb{Q}/\mathbb{Z} \) onto \( \mathbb{G} \cup \{\infty\} \), \( \mathbb{G} \) a totally ordered group, such that \( \mathbb{R}/\mathbb{Z} \subset D_v \subset \mathbb{Q}/\mathbb{Z} \) and \( M_{v'} \cap \mathbb{R}/\mathbb{Z} = h(A) \). The ring \( R_v = h^{-1}(D_v) \) is the valuation ring of the valuation \( v \) defined on \( \mathbb{Q} \) by \( v(x) = v'(h(x)) \) for all \( x \in \mathbb{Q} \), and certainly \( R \subset R_v \subset \mathbb{Q} \). Also \( M_v = h^{-1}(M_{v'}) \), and hence \( M_v \cap R = A \).

1.16 Corollary. Under the same hypotheses as above, except that \( A \) need not be prime, there is a non-trivial valuation \( v \) of \( \mathbb{Q} \) as before, such that \( R \subset R_v \subset \mathbb{Q} \), and \( M_v \cap R \ni A \).

Proof: If \( A \) is a proper regular ideal of \( R \), \( A \) is contained in a prime ideal \( P \) of \( R \). Apply 1.15 for \( P \), we have a non-trivial valuation \( v \) of the required type, such that \( R \subset R_v \subset \mathbb{Q} \), and \( M_v \cap R = P \ni A \).

We conclude this chapter with several examples.

1.17 Example. Let \( D \) be a valuation domain with quotient field \( F \), and let \( R = D \odot F \). Let \( v' \) be a valuation of \( F \) with valuation ring \( D \), and define \( v \) on \( F \odot F \) by \( v(x, y) = V(x) \) for all \((x, y) \in F \odot F \). It is easily shown that \( v \) is a valuation of \( F \odot F \) with valuation ring \( R \). The regular ideals of \( R \) are ordered, but the set of zero divisors of \( R \) is not ideal, and in fact, there are zero divisors in the complement of every proper ideal of \( R \).
1.18 Example. Let $K$ denote the rational number field, and let $x$ and $y$ be algebraically independent over $K$. Define a valuation $v$ on $K(x, y)$ as follows: for $f(x, y) \in K[x, y]$, write $f(x, y) = \sum a_{ij} x^i y^j$, and let $v(f) = (m, n)$, the minimum value of $(i, j)$ for which $a_{ij} \neq 0$ in the lexicographic ordering of $I \oplus I$. (I denotes the set of integers.) Extend $v$ to $K(x, y)$ by $v(f/g) = v(f) - v(g)$. Let $R_v$ be the valuation ring of $v$. $R_v$ has two proper prime ideals, $M = (y)$ and $P = \cap M^n < M$. Let $A = (xy)R_v$, and define $R = R_v/A$. Then the set of zero divisors in $R$ is $M/A$, so that $R = Q(R)$. But the set of nilpotents of $R$ is properly contained in $M/A$, since there are elements $t \in M - P$ in $R_v$, and then $t^n \notin P$ for any $n$, so that in $R$ the image of $t$ is a zero divisor, but not nilpotent.

1.19 Example. Let $K$ be as above, and let $x, y,$ and $z$ be algebraically independent over $K$. Define valuations $v_1$ and $v_2$ on $K(x, y, z)$ as follows: for $f(x, y, z) \in K[x, y, z]$, let $f(x, y, z) = \sum a_{ijk} x^i y^j z^k$, and order $I \oplus I \oplus I$ lexicographically. Let $v_1(f) = (r, s, t)$, where this denotes the minimum of the triples $(i, j, k)$ such that $a_{ijk} \neq 0$, and $v_2(f) = (r, s, t)$ where this denotes the minimum of the triples $(i, k, j)$ such that $a_{ijk} \neq 0$. Both $v_1$ and $v_2$ determine valuations of $K(x, y, z)$ by defining $v_i(f/g) = v_i(f) - v_i(g)$, for $i = 1, 2$. Let $R_1$ and $R_2$ denote the valuation rings of $v_1$ and $v_2$ respectively, and $R = R_1 \cap R_2$. Then $R$ has two maximals, $M_1$ and $M_2 [6; 38]$. Also, the minimal proper primes in $R_1$ and $R_2$ consist of the same set $P$ of elements of $K(x, y, z)$, and hence $P$ is an ideal in $R$. Furthermore, if $t$ is an element of $R$, $t \notin P$, then $P \subset tR$; for $tR = tR_1 \cap tR_2$, since if $w \in tR_1 \cap tR_2$, then $w = tr_1 = tr_2$, hence $r_1 = r_2$ and $w \in tR$. Certainly $tR_1 \supset P$ in $R_1$, for $i = 1$ and $2$, and hence $P \subset tR$. Now let
\( R^* = R/P^2 \). The ideal of zero divisors of \( R^* \) is \( P/P^2 \), for if \( s, t \in R \) with \( st \in P^2 \), and neither \( s \) nor \( t \) in \( P^2 \), then suppose \( s \notin P \). Then 

\[ v_1(s) = (0, a, b) \text{ for some integers } a \text{ and } b; \quad v_1(t) = (1, c, d) \text{ for some integers } c \text{ and } d, \text{ since } t \in P - P^2. \]

Since \( st \in P^2 \), we have 

\[ v_1(st) = (2, e, f), \text{ where } e \text{ and } f \text{ are integers, because } P^2 = \{ z \in R ; v_1(z) = (2, m, n) \text{ for some } m, n \in I \}. \]

But this implies that 

\[ (0, a, b) + (1, c, d) = (2, e, f), \text{ an impossibility. Thus } s \in P, \text{ and } \]

\( P/P^2 \) is the set of zero divisors of \( R^* \). Now if \( t \) is regular in \( R^* \), then 

\( t \notin P/P^2 \), hence no pre-image of \( t \) in \( R \) is in \( P \), whence if \( t_0 \) is a pre-image of \( t \), \( P \subseteq (t_0) \) in \( R \). Therefore \( P/P^2 \subseteq (t_0)/P^2 = (t) \), and every regular ideal of \( R^* \) contains the ideal of zero divisors.

We note here that rings of this type, that is, rings where the set of zero divisors is an ideal contained in every regular ideal, can be constructed with a given (finite) number of maximal ideals, using a technique similar to the above.
2.1 Definition. Let A be an ideal of a ring R. If there is a valuation ring $R_v$ such that $R \subseteq R_v \subseteq Q$, and an ideal $B$ of $R_v$ such that $A = B \cap R$, then A is called a valuation ideal of R. The set of valuation ideals of R will be denoted by $\mathcal{V}(R)$, or simply $\mathcal{V}$ where there is no ambiguity possible. Similarly, the sets of primary ideals and prime power ideals will be denoted by $\mathcal{P}(R)$ and $\mathcal{Q}(R)$ respectively.

2.2 Lemma. The following are equivalent:

1) A is a valuation ideal of $R_v$;

2) $AR_v \cap R = A$ for some valuation ring $R_v$ such that $R \subseteq R_v \subseteq Q$.

Further, if A is a regular ideal of $R_v$, $v$ a non-trivial valuation of $Q$ onto $G \cup \{\infty\}$, where G is a totally ordered group, and the set Z of zero divisors of $R_v$ contained in every regular ideal of $R_v$, then 1) and 2) are equivalent to

3) If $x, y \in R, x \in A$, and $v(y) \geq v(x)$, then $y \in A$.

Proof: If $A \in \mathcal{V}$, then $A \subseteq AR_v \cap R \subseteq B \cap R = A$ for some ideal $B$ of a valuation ring $R_v$ such that $R \subseteq R_v \subseteq Q$. Thus $A = AR_v \cap R$. Conversely, if $A = AR_v \cap R$, let $B = AR_v$, and $A = B \cap R$ follows. Suppose 1) and the additional hypotheses are satisfied, and let A be a regular ideal of R, with $x \in A, y \in R, v(y) \geq v(x)$. There are two cases; if x is regular, then $y/x \in R_v$, and hence $y = (y/x)x \in R_v \cdot x \subseteq A$. If $x \in Z$, then $v(x) = \infty$, by 1.11. Thus $v(y) \geq v(x) = \infty$, implying $y \in Z \subseteq A$. 

14
Thus 3) holds. If 3) holds, let \( x \in \mathbb{A}_v R \cap R \). Then \( x = \sum a_i r_i \), where \( a_i \in A \) and \( r_i \in R_v \). Thus \( v(x) = v(\sum a_i r_i) \geq \min \{ v(a_i r_i) \} = v(a_j) + v(r_j) \geq v(a_j) \) for some \( j \), and since \( a_j \in A \), we have \( x \in A \). Thus \( A = \mathbb{A}_v \mathbb{R} r_j \).

2.3 Remark. From this point on we consider only rings \( R \) with the property that the set \( Z \) of zero divisors of \( R \) is an ideal of \( R \) contained in every regular ideal of \( R \).

2.4 Theorem. If \( P \) is a non-regular prime ideal of \( R \), and \( R_v \) is any valuation ring such that \( R \subset R_v \subset Q \), then \( \mathbb{P}_v \cap R = P \), and there is a prime ideal \( P_1 \) of \( R_v \) such that \( P_1 \cap R = P \). If \( P_1 < P_2 < \ldots < P_n \) are regular prime ideals of \( R \), then there is a valuation ring \( R_v \) such that \( R \subset R_v \subset Q \), and prime ideals \( P_1^* < P_2^* < \ldots < P_n^* \) of \( R_v \) such that \( P_i^* \cap R = P_i \) for \( i = 1, 2, 3, \ldots, n \).

Proof: Let \( P \) be a non-regular prime of \( R \), and \( R_v \) any valuation ring between \( R \) and \( Q \). Let \( x \in \mathbb{P}_v \cap R \). Then \( x = \sum p_i(r_i/s_i) \), where \( p_i \in \mathbb{P} \), and \( r_i, s_i \in R \). Let \( s = ns_i \). Then \( s \) is regular, so that \( sx \in \mathbb{P}, s \notin p \), hence \( x \in P \). Thus \( P = \mathbb{P}_v \cap R \). Now use Zorn's lemma to construct \( P_1 \) in \( R_v \) such that \( P_1 \) is maximal with respect to the property that \( P_1 \cap R = P \). Suppose \( x \notin P_1, y \notin P_1 \), and \( xy \in P_1 \), where \( x \) and \( y \) are elements of \( R_v \). Now \( (P_1, x) \cap R > P \) and \( (P_1, y) \cap R > P \), by the maximality of \( P_1 \). Let \( s \in (P_1, x) \cap R - P \) and \( t \in (P_1, y) \cap R - P \). Then \( s = p_1 + r_1 x \) for some \( p_1 \in \mathbb{P} \) and \( r_1 \in R_v \), and \( t = p_2 + r_2 y \), for some \( p_2 \in \mathbb{P} \) and \( r_2 \in R_v \). Thus \( st = p_1 p_2 + p_1 r_1 x + p_1 r_1 y + r_1 r_2 xy \). Write \( p_1 p_2 + p_2 r_1 x + p_1 r_2 y = a/r \), \( a \) regular \( \in R, b \in R \). Then \( rst = a + r_1 r_2 xy \in R \cap P_1 = P \) since \( st \in P_1 \) and \( rst \in R \). But \( st \notin P \),
hence \( f \in \mathfrak{P} \). But \( f \) is regular, and we thus have a contradiction. Thus \( xy \not\in \mathfrak{P}_1 \), and \( \mathfrak{P}_1 \) is a prime ideal of \( \mathcal{R}_v \). If \( \mathfrak{P}_1 < \mathfrak{P}_2 < \ldots < \mathfrak{P}_n \) are regular primes of \( R \), then \( (\mathfrak{P}_1/\mathbb{Z}) < (\mathfrak{P}_2/\mathbb{Z}) < \ldots < (\mathfrak{P}_n/\mathbb{Z}) \) are proper primes of \( R/\mathbb{Z} \), and hence there are prime ideals \( \mathfrak{P}^*_1 < \mathfrak{P}^*_2 < \ldots < \mathfrak{P}^*_n \) in some valuation domain \( D_v \supset R/\mathbb{Z} \) such that \( \mathfrak{P}^*_1 \cap (R/\mathbb{Z}) = \mathfrak{P}_i/\mathbb{Z} \) for \( i = 1, 2, \ldots, n \).

Let \( R_v = h^{-1}(D_v) \), where \( h \) denotes the natural homomorphism of \( \mathbb{Q} \) on \( \mathbb{Q}/\mathbb{Z} \). Then \( R_v \) is a valuation ring of \( \mathbb{Q} \), and the ideals \( \mathfrak{P}^*_i = h^{-1}(\mathfrak{P}^*_i) \) are the required prime ideals of \( R_v \).

**2.5 Theorem.** Every prime ideal of \( R \) is a valuation ideal. Further, the valuation ring associated with each regular prime ideal can be chosen as the valuation ring of a non-trivial valuation of \( \mathbb{Q} \), whose value semigroup is of the form \( G \cap \{00\} \), where \( G \) is a totally ordered group.

Proof: The theorem follows immediately from 2.4.

**2.6 Lemma.** If the ideals of \( \mathbb{Q} \) are totally ordered, then the non-regular primary ideals of \( R \) are totally ordered.

Proof: \( \mathbb{Q} = R/\mathbb{Z} \) for the prime ideal \( Z \) of \( R \). There is a 1-1 correspondence between the primary ideals of \( R/\mathbb{Z} \) and the primary ideals of \( R \) which are contained in \( Z \).

**2.7 Corollary.** If the ideals of \( \mathbb{Q} \) are totally ordered, then the non-regular prime ideals of \( R \) are totally ordered.

Now let \( A \) be an ideal of a ring \( R \), and define \( \mathcal{J}_A = \{ R' \mid R' \text{ is a ring, } R \subset R' \subset \mathbb{Q}, \text{ and } AR' = A \} \). \( \mathcal{J}_A \) is not empty, since \( R \in \mathcal{J}_A \). By Zorn's lemma, there are maximal elements of \( \mathcal{J}_A \). If \( S_A \) and \( T_A \) are such elements, let \( T = (S_A, T_A) \), the subring of \( \mathbb{Q} \) generated by \( S_A \) and \( T_A \). If
\( x \in AT \), then \( x = \sum a_i t_i \), where \( a_i \in A \), and \( t_i \in T \). But then each \( t_i \) is a finite sum of finite products of elements of \( S_A \) and \( T_A \), and hence 
\[ a_i t_i \in (A S_A, A T_A) = A. \]
Thus \( x \in A \), and hence \( AT = A \), implying that 
\( S_A = T_A = T \). \( S_A \) represents the most inclusive subring of \( Q \) in which \( A \) is an ideal.

2.8 Remark. From this point on we will consider only valuations \( \nu \) of \( Q \) whose value semigroups are totally ordered. We will denote the set of all valuations of this form which are non-negative on a ring \( R \) by \( \nu'(R) \).
We will also denote \( \nu_1(R) = \{ A; A \) is an ideal of \( R, A = AR \cap R \) for some \( \nu \in \nu'(R) \} \).

2.9 Lemma. Every valuation \( \nu \in \nu'(R) \) induces a valuation \( \nu' \in \nu'(R/Z) \) whose value semigroup is of the form \( \Delta = G \cup \{ \infty \} \), where \( G \) is a totally ordered group. Conversely, every valuation \( \nu \in \nu'(R/Z) \) onto \( G \cup \{ \infty \} \) induces a valuation \( \nu' \in \nu'(R) \) with the same value semigroup. Further, in each case, \( \nu \) is non-trivial if and only if \( \nu' \) is non-trivial.

Proof: The set \( G \) of values \( \nu(x) \) for \( x \) regular \( \in Q \) forms a totally ordered subgroup of \( \Delta \). Define \( \nu' \) on \( Q/Z \) by \( \nu'(x + Z) = \nu(x) \), if \( x \) is regular in \( Q \), and \( \nu'(Z) = \infty \). If \( x \) and \( y \) are regular elements of \( Q \), and \( z \in Z \) with \( x = y + z \), then certainly \( \nu(x) \geq \min \{ \nu(y), \nu(z) \} \). But notice that \( \nu(z) \geq \nu(y) \), since \( z \in (y) \), and \( \nu(z) \neq \nu(y) \), for if so \( G \) would have a maximal element. Thus \( \nu(z) > \nu(y) \), so that \( \nu(x) = \nu(y) \). This shows that the function \( \nu' \) is well-defined. It is easily shown that \( \nu' \) is a valuation of \( Q/Z \). If \( \nu \) is trivial, then \( R_\nu = Q \), and \( R_\nu' = Q/Z \), so that \( R_\nu' = R_\nu/Z \). If \( \nu \) is non-trivial, then \( \nu' \) is non-negative for the cosets \( x + Z \) such that \( \nu(x) \) is non-negative. Thus \( R_\nu/Z = R_\nu' \), in this case also.
If \( v \) is a valuation in \( V'(R/Z) \) with value semigroup \( G \cup \{0\} \), \( G \) is a totally ordered group, then define \( v' \) on \( Q \) by \( v'(x) = v(x + Z) \).

Again, it is easily seen that \( v' \in V'(R) \), and has the same value semigroup as \( v \). It is obvious that in both cases above \( v \) is non-trivial if and only if \( v' \) is non-trivial.

2.10 Corollary. \( R \) is the valuation ring of a valuation \( v \in V'(R) \) the regular ideals of \( R \) are totally ordered.

Proof: This follows immediately from 2.9.

2.11 Theorem. Let \( R \) be a ring such that \( Q(R) \) has totally ordered ideals. Then \( \mathcal{V}_1(R) \subseteq \mathcal{Q}(R) \) if and only if

1) regular primes of \( R \) are maximal;
2) the ideal \( Z \) is a minimal prime; and
3) for non-regular ideals \( A \) of \( R \), the only \( S_A \) which contains a valuation ring \( R_v \) of a valuation \( v \in V'(R) \) is \( S_Z = Q \).

Proof: Suppose 1), 2), and 3), and let \( A \) be a proper ideal in \( \mathcal{V}_1(R) \).

If \( R = Q \), then \( A = Z \), a maximal ideal. Thus \( A \) is primary. If \( R \neq Q \), then there are two cases: either \( A \) is regular or non-regular. In the first case \( A \) has prime radical, for \( A = B \cap R \) for some regular ideal \( B \) in the valuation ring \( R_v \) of a valuation \( v \in V'(R) \). If \( v \) is trivial, then \( R_v = Q \), so that \( A = Q \cap R = R \in \mathcal{Q}(R) \). If \( v \) is non-trivial, then the regular ideals of \( R_v \) are totally ordered, so that \( B \) has prime radical. Thus \( A \) has prime radical also. Thus \( A \) is maximal, by 1), and therefore \( A \) is primary. In the second case \( A = AR_v \cap R \) for some valuation ring \( R_v \) of a valuation \( v \in V'(R) \), and since \( A \subseteq Z \), we have \( A = AR_v \cap R = AR_v \).

Thus \( R_v \subseteq S_A \), so that \( S_A = Q \) by 3). Thus \( AQ = A \), so that \( A \) is a primary
by 1.7. Now suppose that one of 1), 2), and 3) fails. If there exist
P and P_1, prime ideals, with Z < P < P_1 < R, then let R_v be the valuation
ring of some valuation v ∈ V'(R), such that P = P* ∩ R and P_1 = P_1* ∩ R for
prime ideals P* and P_1* of R_v. Let x ∈ P_1 - P and let y be a regular
element of P. Define A = (xy)_R_v ∩ R. Then A ∈ U(R), but suppose
A ∈ A(R). Then y ∈ A, since xy ∈ A, and x ∉ P, and hence y = rxy for
some r ∈ R. Thus 1 = rx, a contradiction, since x ∉ P_1 < R_v. Thus
U(R) ∉ A(R). If there is a prime P such that 0 < P < Z, let R_v be
the valuation ring of some v ∈ V'(R). Then PR_v ∩ R = P and ZR_v ∩ R = Z
by 2.4. Let x ∈ Z - P, and y ∈ P, and define A = (xy)_R_v ∩ R.
Certainly A ∈ U(R), but if we suppose A ∈ A(R), then y ∈ A, as before.
Then y = (r/s)xy for some r/s ∈ R_v. Thus y(s - rx) = 0, implying
s - rx ∈ Z. Thus s = (s - rx) + rx ∈ Z, a contradiction. Thus A ∈
U(R) - A(R). Finally, if there is some non-regular ideal A of R
such that S_A ≠ Q, and such that S_A contains the valuation ring R_v of
some v ∈ U(R), then AR_v = A, so that A ∈ U(R). But A_Q ≠ A, so that
A is not primary, by 1.7. Thus A ∈ U(R) - A(R).

2.12 Definition. Let R be a ring with the property that for each prime
ideal P of R, R_P is the valuation ring of some v ∈ V'(R). Then R is
called a Prüfer ring.

2.13 Lemma. Let A be a regular ideal of R. Then A ∈ U(R) if and
only if A/Z ∈ U(R/Z).

The proof is easy, and will be omitted.

2.14 Theorem. If Z is a minimal prime of R, and if the ascending chain
condition for prime ideals is valid in R, then A(R) ⊆ U(R) if and
only if \( R \) is a Prüfer ring.

Proof: If \( R \) is a Prüfer ring, and \( A \) a \( P \)-primary ideal of \( R \), then
\[ A = A_{R_p} \cap R \text{ [7; 225].} \]
Thus \( A \in \mathcal{V}_1(R) \). If \( \mathcal{L}(R) \subset \mathcal{V}_1(R) \), and \( P \) is a prime ideal of \( R \), then we distinguish two cases: when \( P = Z \), then \( R_p Z = Q \), a valuation ring for any trivial valuation in \( V'(R) \). In the case that \( P \) is regular, the ideal \( P/Z \) is a proper prime ideal of \( R/Z \).

Certainly it follows from 2.12 that \( \mathcal{L}(R/Z) \subset \mathcal{V}_1(R/Z) \). Gilmer and Ohm showed in [2] that if the ascending chain condition is valid for prime ideals in \( R/Z \), then \( \mathcal{L}(R/Z) \subset \mathcal{V}_1(R/Z) \) if and only if \( R/Z \) is a Prüfer domain. Thus \( (R/Z)(P/Z) \) has totally ordered ideals, but \( R_p/ZR_p \cong (R/Z)(P/Z), \)
so that the regular ideals of \( R_p \) are ordered. Thus \( R_p \) is the valuation ring of a valuation \( v \in V'(R_p) \subset V'(R) \), by 2.10. Thus \( R \) is a Prüfer ring.

2.15 Definition. A ring is of dimension \( n \) if there is a chain
\[ 0 < P_0 < P_1 < \ldots < P_n < R \]
of prime ideals of \( R \), but no such chain of greater length.

2.16 Theorem. For a ring \( R \), \( \mathcal{V}_1(R) = \mathcal{L}(R) \) if and only if
1) \( R \) is a prüfer ring of dimension \( \leq 1 \); and
2) for non-regular ideals \( A \) of \( R \), the only \( S_A \) to contain the valuation ring \( R_v \) of a valuation \( v \in V'(R) \) is \( R_Z = Q \).

Proof: If \( R = Q \), the theorem is obvious. Suppose \( R \neq Q \). If \( \mathcal{V}_1(R) = \mathcal{L}(R), \) then by 2.11 the dimension of \( R \) is \( \leq 1 \), and thus \( Z \) must be a minimal prime, since \( R \) has a proper regular ideal. Thus \( R \) is a Prüfer ring, by 2.14, and 1) holds. Further, 2) holds by 2.11. If 1) and 2) are valid, then \( Z \) is a minimal prime, since there is a proper regular ideal, and hence \( \mathcal{V}_1(R) \subset \mathcal{L}(R), \) by 2.11. If \( A \in \mathcal{L}(R), \) we distinguish
two cases: if \( A \subset \mathbb{Z} \), then \( \sqrt{A} = \mathbb{Z} \), and \( A = \mathbb{Z} \cap R \) for any valuation \( R \) such that \( v \in V'(R) \); for if \( x \in \mathbb{Z} \cap R \), then write \( x = a/s \), where \( s \) is regular in \( R \), and \( a \in R \). Then \( sx \in A \), \( s \neq z \), so that \( x \in A \). If \( A \) is regular, then \( A \in \mathbb{V} \) by the proof of 2.14. Thus \( \mathbb{V}(R) = \mathcal{L}(R) \).

For the next several lemmas we will only assume that \( R \) is a commutative ring with identity.

2.17 Lemma. If \( \mathcal{L}(R) = \mathcal{P}(R) \), then \( \mathcal{L}(R/A) = \mathcal{P}(R/A) \) for any ideal \( A \) of \( R \), and \( \mathcal{L}(R_p) = \mathcal{P}(R_p) \) for any prime ideal \( P \) of \( R \).

The proof is easy and will be omitted.

2.18 Lemma. If \( \mathcal{L}(R) = \mathcal{P}(R) \), and if \( P_1 \) and \( P_2 \) are prime ideals of \( R \) with \( P_1 > P_2 \), and if there are no primes between \( P_1 \) and \( P_2 \), then \( P_2 = \cap P_1^n \).

Proof: We have \( \mathcal{L} \subset \mathcal{P} \text{ in } R' \) by 2.17, from which it follows that \( \mathcal{L} \subset \mathcal{P} \text{ in } R' = R/P_1P_2 \). Now if \( x \in P_1 - P_2 \), consider the ideal \( P_2 + (x) \). This ideal has radical \( P_1 \), and hence \( \sqrt{(x)} = P_1 \text{ in } R' \), so that \( (x) \) is \( P_1 \)-primary in \( R' \), by the maximality of \( P_1 \) in \( R' \). Thus \( (x) \) is a power of \( P_1 \) in \( R' \), and therefore \( P_1 \) is invertible in \( R' \). Therefore \( R' \) is a Dedekind domain, since \( P_1 \) is its only proper prime ideal. Every proper ideal of \( R' \) is a power of \( P_1 \). Note that \( \sqrt{P_2 + (x)^n} = P_1P_2^1P_2^n \text{ in } R' \), so that \( P_2 + (x)^n \) is primary for \( P_1P_2^1 \text{ in } R \). Thus \( P_2 + (x)^n = P_1P_2^mP_2^n \text{ for } n = 1, 2, 3, \ldots \) Furthermore, \( P_2 + (x)^n \neq P_2 + (x)^k \) for \( n \neq k \), since \( x \) is not a zero divisor in \( R/P_1P_2 \). Thus the sequence \( \{P_1^m\} \text{ is strictly decreasing and each member contains } P_2 \). This means that every power \( P_1 \) contains \( P_2 \), and all of these powers are distinct. Hence \( \cap P_1^n \neq P_2 \text{ in } R \).

But in \( R' \), \( \cap P_1^n = (0) \), so that \( \cap P_1^n \neq P_2 \text{ in } R \). Thus \( \cap P_1^n = P_2 \text{ in } R \). Therefore the same is true in \( R \).
2.19 Lemma. If \( P_1 \) and \( P_2 \) are prime ideals of \( R \), and \( P_1 > P_2 \), then there are prime ideals \( P \) and \( P^* \) such that \( P_1 \supset P > P^* \supset P_2 \), and there are no primes between \( P \) and \( P^* \).

Proof: Let \( x \in P_1 - P_2 \), and let \( P \) be a minimal prime of \( P_2 + (x) \). Suppose that there is no prime \( P^* \) with \( P > P^* \supset P_2 \), such that there are no primes between \( P \) and \( P^* \). Then by Zorn's lemma there is a chain \( \{P_\alpha\} \) of prime ideals of \( R \) such that \( P > P_\alpha \supset P_2 \) for each \( \alpha \), and \( P = \bigcup P_\alpha \). Hence \( x \in P_\alpha \) for some \( \alpha \), so that \( P_2 + (x) \subset P_\alpha < P \). This contradicts the choice of \( P \), and establishes the lemma.

2.20 Lemma. Suppose that \( P_1 \) and \( P_2 \) are prime ideals of \( R \), with \( P_1 > P \). If \( \mathcal{P}(R) \subset \mathcal{O}(R) \), then either \( P = \bigcup P_\alpha \), where \( \{P_\alpha\} \) is a chain of prime ideals such that \( P_2 < P_\alpha < P_1 \) for each \( \alpha \), or there is a prime \( P_\alpha \) such that \( P_1 > P_\alpha \supset P_2 \), with no primes between \( P_1 \) and \( P_\alpha \).

Further, in the first case \( P_1 = P_2 \).

Proof: Suppose the second assertion fails; i.e., for any \( P_\alpha \), with \( P_1 > P_\alpha \supset P_2 \), there are primes between \( P_1 \) and \( P_\alpha \). Then by Zorn's lemma there is a chain \( \{P_\alpha\} \) of prime ideals such that \( P_1 > P_\alpha > P_2 \) for each \( \alpha \), and \( \bigcup P_\alpha = P_1 \). For any \( P_\alpha \), there is a pair of prime ideals \( P(\alpha) \) and \( P^*(\alpha) \) such that \( P_1 \supset P(\alpha) > P^*(\alpha) \supset P_2 \), with no primes between \( P(\alpha) \) and \( P^*(\alpha) \), by 2.19. Thus, by 2.18,
\[
\bigcap_n P^n(\alpha) = P^*(\alpha).
\]
But then \( \bigcap_n P_1 \supset \bigcap_n P^n(\alpha) = P^*(\alpha) \supset P_\alpha \) for every \( \alpha \). Thus \( \bigcap_n P_1 \supset \bigcup P_\alpha = P_1 \), so that \( P_1 = P_2 \). This completes the proof.

2.21 Theorem. Suppose \( \mathcal{P}(R) \subset \mathcal{O}(R) \). If a prime \( P \) properly contains another prime, then \( P^n \) is \( P \)-primary for every \( n \). Further, if \( P \) is a
minimal prime, and \( P^n = \{ x \in R; mx = 0 \text{ for some } m \notin P \} \), then \( P^i \) is primary for \( i = 1, 2, \ldots, n \), and these are the only \( P \)-primary ideals.

**Proof.** Suppose \( P > P_1^2 \). If \( P = P^2 \), the first assertion is true. If \( P \neq P^2 \), then choose \( P^* \) so that \( P > P^* \supseteq P_1 \), and there are no primes between \( P \) and \( P^* \). Then \( \bigcap P^n = P^* \). Now we have that the \( n \)\(^{th} \) symbolic power of \( P \), \( P^{(n)} \), is primary for every \( n \), and hence \( P^{(n)} = P^k \). If \( k_n \neq n \) for some \( n \), then \( P^R_R/P_1^R = P^k/P_1^R \). But these ideals must be different, since \( R^R_R/P_1^R \) is a Dedekind domain. Thus \( k_n = n \), and \( P^n \) is primary. If \( P \) is a minimal prime, then \( R^R_p = (R/N)(P/N) \), where \( N = \{ x \in R; mx = 0 \text{ for some } m \notin P \} \). Note that \( R^R_p \) has only one prime ideal \( PR_p \), and hence every ideal of \( R^R_p \) is primary. Thus \( (0)R^R_p \) is primary and hence \( (0)R^R_p = P^nR_p \) for some \( n \). Choose the smallest possible \( n \) for which this is true, and then \( N = P^n = [(0)R^R_p]^O \), which is \( P \)-primary. But \( \mathcal{Q}(R^R_p) \subset \mathcal{P}(R^R_p) \), thus the only ideals of \( R^R_p \) are powers of \( PR_p \). This means that the only primary ideals of \( R \) that are contained in \( P \) are \( P^i \) for \( i = 1, 2, \ldots, n \).

We now revert to the case when the set \( Z \) of zero divisors of \( R \) is an ideal contained in every regular ideal of \( R \).

2.22 **Theorem.** Let \( R \) be a ring such that \( Q \) has totally ordered ideals. Suppose that for every minimal prime \( P \) of \( R \), the powers of \( P \) are all primary. Then \( V_1(R) \subset \mathcal{P}(R) \) if and only if

1) \( Z \) is a minimal prime of \( R \);
2) \( \mathcal{Q} \subset \mathcal{P} \) for non-regular ideals of \( R \);
3) Every regular prime ideal \( P \) of \( R \) defines a \( P \)-adic valuation in \( V^*(R) \) by \( v_p(x) = n \), where \( x \in P^n - P^{n+1} \), for \( x \in R \);
4) If \( R_v \) is any valuation ring such that \( R \subseteq R_v \subseteq Q \), then \( R_v \) is
the valuation ring of some \( P \)-adic valuation for a regular prime \( P \) of \( R \); and

5) The only non-regular ideals \( A \) of \( R \) for which \( S_A \) contains the valuation ring \( R_v \) of a valuation \( v \in V^*(R) \) are ideals such that \( S_A = Q \).

Proof: If \( V_1(R) \subset \mathcal{O}(R) \), then let \( A \) be a non-regular primary ideal, and \( R_v \) any valuation ring between \( R \) and \( Q \). If \( x \in A R_v \cap R \), then write \( x = a/s \), where \( s \) is regular, \( a \) and \( s \in R \). Thus \( sx \in A \) and \( s \notin \sqrt{A} \) implies \( x \in A \), so that \( A \in V_1(R) \). Thus \( A \subset V_1 \) for the non-regular ideals of \( R \). We assert that \( A = V_1 = \mathcal{O} \) for the non-regular ideals of \( R \).

For if \( P \) is a non-regular prime ideal of \( R \) properly containing another prime of \( R \), then every power of \( P \) is primary, and the powers of a minimal prime are primary by assumption. Thus \( \mathcal{O} \subset A \) for the non-regular ideals of \( R \). This, together with \( A \subset V_1 \subset \mathcal{O} \) for the non-regular ideals of \( R \), proves the assertion. But we have shown in 2.11 that if every non-regular ideal of \( V_1(R) \) is contained in \( A(R) \), then \( Z \) is a minimal prime. The validity of 5) in 2.11 depends only on this fact, also. Thus 1), 2), and 5) are satisfied. Certainly no non-regular prime can define a non-trivial valuation of \( Q \). Thus we need consider only regular primes of \( R \).

It is easily shown that if \( V_1(R) \subset \mathcal{O}(R) \), then \( V_1(R/Z) \subset \mathcal{O}(R/Z) \). Gilmer has shown in [1] that in an integral domain, \( V \subset \mathcal{O} \) if and only if 3) and 4) hold, so that 3) and 4) are valid in \( Q/Z \). But every non-trivial valuation of \( Q/Z \) which is in \( V_1(R/Z) \) induces a non-trivial valuation of \( Q \) which is in \( V_1(R) \), and every such valuation of \( Q \) comes from such a valuation of \( Q/Z \), if we agree to identify all non-trivial valuations with the same valuation ring. Thus 3) and 4) hold in \( Q \).

If 1) - 5) hold, then we note that \( V_1(R/Z) \subset \mathcal{O}(R/Z) \), by the above. Thus every regular ideal in \( V_1(R) \) is a prime power. Gilmer has shown
in [1] that if $\mathcal{V}_1(R/Z) \subseteq \mathcal{P}(R/Z)$, then the proper primes of $R/Z$ are maximal. Thus the regular primes of $R$ are maximal, so that the dimension of $R$ is $\leq 1$. But this together with 5) implies that $\mathcal{V}_1 \subseteq \mathcal{A}$ for the non-regular ideals of $R$, by 2.11. Thus $\mathcal{V}_1 \subseteq \mathcal{A} \subseteq \mathcal{P}$ for the non-regular ideals of $R$ also, by 2), whence $\mathcal{V}_1(R) \subseteq \mathcal{P}(R)$.

2.23 Corollary. Let $R$ be a ring whose total quotient ring $Q$ has totally ordered ideals, and such that for any minimal prime $P$ of $R$, $P^n$ has a primary representation for every $n$. Then $\mathcal{V}_1(R) \subseteq \mathcal{P}(R)$ if and only if $(1) - 5)$ of 2.22 are satisfied.

Proof: If $(0)$ is prime, then all powers of $(0)$ are primary, and hence the corollary follows directly from 2.22. If not, then there is a minimal proper non-regular prime $P$ of $R$. The non-regular primary ideals of $R$ are totally ordered, by 2.6, hence if $P^n = Q_1 \cap Q_2 \cap \ldots \cap Q_n$, there is one $Q_1$ contained in all the rest, so that $P^n = Q_1$. Thus the corollary follows from 2.22.

2.24 Corollary. If $R$ is noetherian, and if the total quotient ring $Q$ of $R$ has totally ordered ideals, then $\mathcal{V}_1(R) \subseteq \mathcal{P}(R)$ if and only if $(1) - 5)$ of 2.22 hold.

Proof: Clearly, if $R$ is noetherian, every prime power has a primary representation. Apply 2.23.

2.25 Theorem. Suppose that $R$ is a ring whose total quotient ring has totally ordered ideals. If $Z$ is a minimal prime, then $\mathcal{V}_1(R) \subseteq \mathcal{P}(R)$ if and only if $(2) - 5)$ of 2.22 are satisfied.

Proof: Suppose $\mathcal{V}_1(R) \subseteq \mathcal{P}(R)$. Now $\mathcal{A} \subseteq \mathcal{V}_1$ for the non-regular ideals
of \( R \), as shown in 2.22. Thus \( \mathcal{L} \subset \mathcal{U}_1 \subset \mathcal{P} \) for the non-regular ideals of \( R \). We note that the only non-regular prime of \( R \) is \( Z \), and that \((z^n)^{\mathcal{R}} = z^n\) in \( R_z = Q \) for every \( n \), since \( ZQ = Z \). Thus the powers of \( Z \) are primary, so that \( \mathcal{L} = \mathcal{U}_1 = \mathcal{P} \) for non-regular ideals of \( R \). Thus 2) has been shown, and 5) follows from 2.11. The validity of 3) and 4) follows exactly as in 2.22. We now suppose that 2) - 5) are satisfied, and note that in the second half of the proof of 2.22 only the assumption 1) - 5) and the fact that the ideals of \( Q \) are totally ordered are used. Thus our assertion here follows as in 2.22.

2.26 Definition. Let \( \psi(\mathcal{R}) \) be the set of ideals \( A \) of \( R \) such that \( A \) is prime to the complement of a prime ideal of \( R \).

We refer the reader to [1] for a discussion of the set \( \psi \). The following generalizes a result from [1].

2.27 Theorem. \( \mathcal{U}_1 \subset \psi \) in \( R \). Further, if \( Z \) is a minimal prime of \( R \), then \( \mathcal{U}_1 = \psi \) if and only if \( R \) is a Prüfer ring.

Proof: Let \( h \) denote the natural homomorphism of \( Q \) on \( Q/Z \). Let \( A \) be a regular ideal in \( \mathcal{U}_1 \). Then \( h(A) = A' \) is a proper ideal of \( R/Z \), and \( A' \in \mathcal{U}_1(\mathcal{R}/Z) \). But in [1] it was shown that \( \mathcal{U} \subset \psi \) in a domain, so that \( A' \in \psi(\mathcal{R}/Z) \). Therefore \( A' \) is prime to \( R/Z - P/Z \), for some prime \( P/Z \) of \( R/Z \). It is obvious that \( A \) is prime to \( R - P \), and hence \( A \in \psi \).

If \( A \) is a non-regular ideal in \( \mathcal{U}_1 \), then \( A = AR_v \) for some \( v \in \mathcal{V}(\mathcal{R}) \), with \( R \subset R_v \subset Q \). Let \( M_v \) denote the maximal ideal of \( R_v \), and let \( P = M_v \cap R \).

If \( x \notin P \), and \( x \in R \), then \( x \) is a unit in \( R_v \), so that for every \( a \in R \) with \( ax \in A \), \( a = (ax)^{-1}/x \in AR_v = A \). Thus \( A \) is prime to \( R - P \), and \( A \in \psi \). The first assertion is thus proved. If \( R \) is a Prüfer ring,
$R_p$ is the valuation ring of a valuation $v \in V'(R)$ for every prime $P$ of $R$.

Thus if $A \in \varphi$, $A$ is prime to $R - P$ for some prime $P$ of $R$. Let $x \in AR_p \cap R$; $x = \sum a_i(r_i/s_i)$, where $a_i \in A$ and $s_i \not\in P$. If $s = \psi s_i$, then $sx \in A$, $s \not\in P$, so that $x \in A$. Therefore $A = AR_p \cap R$, and $a \in \mathcal{U}_1$. If $\mathcal{U}_1 = \varphi$, on the other hand, let $x$ and $y$ be regular elements of $R$, $P$ a prime ideal of $R$, and $A = \{t \in R; t m \in (xy) \text{ for some } m \not\in P\}$. Then $A \in \varphi$, for if $m \not\in P$, $am \in A$ implies $(am)s \in (xy)$ for some $s \not\in P$. Thus $a(ms) \in (xy)$, $ms \not\in P$, so that $a \in A$. Therefore $A \in \mathcal{U}_1$. Now if $A$ is regular, then for $a \in R$, $b \in A$, $v(a) \geq v(b)$, it follows that $a \in A$. It follows easily from this that if $xy \in A$, then either $x^2 \in A$ or $y^2 \in A$. If $x^2 \in A$, then $x^2m \in (xy)$ for some $m \not\in P$, so that $x^2m = dxy$, where $d$ is regular in $R$.

Thus $x/y = d/m \in R_p$. Thus either $x \in yR_p$ or $y \in xR_p$ and hence the regular ideals of $R_p$ are chained. Thus $R_p$ is the valuation ring for some $v \in V'(R)$, and hence $R$ is a Prüfer ring. This completes the theorem.
RINGS WITH QUASI-MAXIMAL IDEAL THEORY

3.1 Definition. An ideal $N$ of a ring $R$ is said to be quasi-maximal if for every proper divisor $A$ of $N$ in $R$, $AN = N$.

3.2 Lemma. A maximal ideal of $R$ is quasi-maximal.

Proof: The only proper divisor of a maximal ideal $N$ is $R$, and $RN = N$.

3.3 Lemma. An idempotent ideal of $R$ is quasi-maximal.

Proof: If $N^2 = N$, and $A > N$, then $N \subseteq AN \subseteq N^2 = N$.

3.4 Lemma. A regular, finitely generated, quasi-maximal ideal of $R$ is maximal.

Proof: Suppose $N$ satisfies the hypotheses, and $A > N$. Then $AN = N$, and hence there is an element $z \in A$ such that $(1 - z)N = (0)$ [7; 215]. But since $N$ is regular, this implies $z = 1$, and hence $A = R$.

3.5 Definition. Let $R$ be a ring in which every ideal can be represented as a product of finitely many quasi-maximal ideals. Then $R$ is said to have a quasi-maximal ideal theory. We will call such rings qm-rings.

In what follows we will assume that $R$ is a qm-ring.

3.6 Definition. A ring $R$ is called a multiplication ring, abbreviated m-ring, if for every pair of ideals $A$ and $B$ of $R$, with $A \subseteq B$, there is an ideal $C$ of $R$ such that $A = BC$. A ring $R$ in which every ideal can
be represented as a product of finitely many prime ideals of $R$ is called a ZPI-ring.

3.7 **Theorem.** If $R$ is a noetherian $qm$-ring, then $R$ is a ZPI-ring.

**Proof:** We use divisor induction. The unit ideal is a prime ideal. Suppose that $A$ is an ideal every proper divisor of which is a product of finitely many prime ideals. Express $A = N_1 \cdot \cdots \cdot N_r$, where $N_i$ is a quasi-maximal ideal of $R$ for $i = 1, 2, \ldots, r$. If $A < N_i$ for all $i$, then each $N_i$ is a product of primes, and thus so is $A$. If $A = N_i$ for some $i$, then suppose $A$ is not prime. Thus there are elements $x, y \not\in A$ such that $xy \in A$. Thus $(A, x) \cdot (A, y) = A$. But both $(A, x)$ and $(A, y)$ are proper divisors of $A$, and hence are products of finitely many primes, by assumption. Thus $A$ is a product of finitely many primes, which proves our assertion.

3.8 **Corollary.** Every non-prime quasi-maximal ideal of $R$ is idempotent.

**Proof:** As above, write $A = BC$, where $B$ and $C$ are proper divisors of $A$. Then $A^2 = A(BC) = (AB)C = AC = A$.

3.9 **Lemma.** If $R$ is any commutative ring with identity, then every regular prime ideal of $R$ contains a non-regular prime ideal of $R$. If $R$ is not an integral domain, this prime may be chosen as a proper ideal of $R$.

**Proof:** If $R$ is an integral domain, then $(0)$ is contained in every regular prime ideal of $R$. If $R$ contains proper divisors of zero, then let $P$ be a regular prime ideal of $R$. $P$ contains proper divisors of zero, for if $x$ is a proper divisor of zero, and $y$ regular $\in P$,
then $xy$ is a proper zero divisor in $P$. We pass to the ring $R_P$. The set of regular elements in $R_P$ is a multiplicatively closed set, and hence there is a prime ideal $\overline{P}_o$ of $R_P$ which misses this system. But there is a 1-1 correspondence between prime ideals of $R$ contained in $P$, and prime ideals of $R_P$. Thus there is a prime ideal $P_0$ of $R$ which is associated with $\overline{P}_o$ in $R_P$. Certainly $P_0$ is non-regular, and contained in $P$.

3.10 Theorem. If every non-zero prime ideal of $R$ is regular, then $R$ is a Dedekind domain.

Proof: $R$ is an integral domain by 3.9. Let $P$ be a proper prime of $R$, and $x \neq 0 \in P$. Then $(x) = N_1 \ldots N_r$, where $N_i$ is quasi-maximal for $i = 1, 2, \ldots r$. Thus $P$ contains some $N_i$. If $P = N_i$, then $(x) = P N'$, where $N' = \prod_{j \neq i} N_j$, and hence $P$ is invertible in $Q(R)$. If $P > N_i$, then $P(x) = (x)$, so that there is an element $z \in P$ such that $(1 - z)(x) = 0$. This implies $z = 1$, and hence $P = R$. Thus every proper prime of $R$ is invertible, so that $R$ is a Dedekind domain.

3.11 Lemma. Every prime ideal of $R$ is quasi-maximal.

Proof: Let $A > P$, where $P$ is a prime ideal of $R$. Write $P = N_1 \ldots N_r$, where $N_i$ is quasi-maximal for $i = 1, 2, \ldots r$. Then $P$ contains some $N_i$. Then $A > P \supset N_i$, so that $AP = A N_1 \ldots N_r = N_1 \ldots (A N_i) \ldots N_r = N_1 \ldots N_r = P$. Thus $P$ is quasi-maximal.

3.12 Theorem. Every regular prime ideal of a qm-ring $R$ is invertible and maximal.

Proof: Invertibility was shown in 3.9. Thus every regular prime ideal of $R$ is finitely generated. Since by 3.11, a regular prime $P$ is
quasi-maximal, we see that $A > P$ implies $AP = P$. But since $P$ is finitely generated, there is an element $z \in A$ such that $(1 - z)P = (0)$. Thus $z = 1$ and $A = R$. Thus $P$ is maximal.

3.13 Theorem. If $A$ is an ideal of a qm-ring $R$, then $R/A$ is a qm-ring. Further, if $M$ is a multiplicative system in $R$, then $RM$ is a qm-ring.

Proof: If $B \supset A$, then let $B = N_1 \cdots N_r$, each $N_i$ quasi-maximal. Certainly $N_i/A$ is quasi-maximal in $R/A$, and $B/A = (N_1/A) \cdots (N_r/A)$. Let $N$ be a quasi-maximal ideal of $R$. If $A' > N^e$ in $R$, let $A = A'C$, the largest ideal of $R$ for which $A^e = A'$. Then $A > N^e \supset N$, so that $AN = N$. Thus $A'N^e = A'C^eN^e = (A'C^eN)^e = (AN)^e = N^e$, whence $N^e$ is quasi-maximal. Thus if $B^e$ is an ideal of $R_m$, and $B = N_1 \cdots N_r$ is a factorization of $B$ into quasi-maximal ideals, we have $B^e = (N_1 \cdots N_r)^e = N_1^e \cdots N_r^e$, a factorization of $B^e$ into quasi-maximal ideals. But every ideal of $R_m$ is an extended ideal, concluding our proof.

3.14 Theorem. If $R$ is a qm-ring, then $R$ is an m-ring.

Proof: We use a theorem of Mott [5], which states that a commutative ring with identity is an m-ring if and only if every prime divisor of an ideal $A$ is a factor of $A$. Let $A \subset P$ in $R$. Write $A = N_1 \cdots N_r$, where each $N_i$ is quasi-maximal. We may assume $P \supset N_1$. If $P = N_1$, then $A = PN_2 \cdots N_r$, and if $P > N_1$, then $PA = PN_1 \cdots N_r = N_1 \cdots N_r = A$. In either case, $P$ is a factor of $A$.

3.15 Corollary. In a qm-ring $R$, every non-maximal prime ideal is idempotent.

Proof: See [5].
3.16 Theorem. Every proper quasi-maximal ideal $N$ of a qm-ring $R$ that is not prime is contained in a minimal prime ideal of $R$.

Proof: There are two cases: if $(N, P) = R$ for all minimal primes $P$ of $R$, then $N = R$, a contradiction. If $(N, P) < R$ for some minimal prime $P$, then let $M$ be a maximal ideal of $R$ such that $(N, P) \subseteq M$. Pass to the Dedekind domain $R/P$. Since $N$ is idempotent, $(N, P)/P$ is idempotent in $R/P$. Thus $(N, P)/P = (0)$ in $R/P$, so that $N \subseteq P$. This establishes the theorem.

3.17 Theorem. Let $R$ be a qm-ring. The following are equivalent:

1) $R$ is a ZPI-ring;
2) Every idempotent ideal of $R$ is finitely generated;
3) Every idempotent ideal of $R$ is a product of prime ideals;
4) Each non-prime quasi-maximal ideal of $R$ is contained in only finitely many minimal prime ideals.

Proof: Obviously 1) implies any of the others. Suppose 4). $R$ is an m-ring, and hence every ideal of $R$ is equal to its kernel. Primary ideals are prime powers, and hence if $N$ is a non-prime quasi-maximal ideal, $N = Q_1^{e_1} \cap \ldots \cap Q_n^{e_n}$, since we can delete all but a finite number of the isolated primary components from the kernel representation of $N$. But this implies $N = Q_1^{e_1} \cdot \ldots \cdot Q_n^{e_n}$, since this product contains $N = N^2$. Thus 1) holds. Suppose 3). Every non-prime quasi-maximal ideal of $R$ is idempotent, and hence as before, 1) holds. Suppose 2). The ascending chain condition is valid for idempotent ideals, since the union of a chain of such ideals is another. Thus we can use divisor induction on the set of idempotent ideals of $R$. Suppose that $N$ is a
non-prime idempotent ideal, and assume that each idempotent proper
divisor of \( N \) is a prime product. We can write \( N = AB \) where \( A \) and \( B \) are
proper divisors of \( N \), as in 3.7. Then we can write \( A = MP \) and \( B = M'Q \),
where \( P \) and \( Q \) denote respectively the products of the primes in a given
factorization of \( A \) and \( B \), and \( M \) and \( M' \) denote the non-prime quasi-maximal
part. \( M \) and \( M' \) are idempotent, and are proper divisors of \( N \), hence are
products of primes. Hence so is \( N = (MP)(M'Q) \), so that 1) holds.
SELECTED BIBLIOGRAPHY

George Vaughn Poynor was born in Amarillo, Texas, on August 28, 1938.
He was educated in the public schools of Baton Rouge and Hammond,
Louisiana. He attended Southeastern Louisiana College from 1955 to 1958,
when he completed the requirements for the B. S. degree in mathematics.
The year 1960 was eventful; Mr. Poynor received the M. S. degree from
Louisiana State University, and the hand of Miss Patricia Anne Hallum
in marriage.
In 1961, after having been a member of the mathematics staff at
Southeastern Louisiana College for one year, he returned to Louisiana
State University for further study. Two years later he became the
proud father of Andree Margaret Poynor. He is currently a candidate
for the Doctor of Philosophy degree in mathematics.
Candidate: GEORGE VAUGHN POYNOR

Major Field: MATHEMATICS

Title of Thesis: VALUATIONS ON RINGS

Approved:

H. S. B关s
Major Professor and Chairman

Max Goodrich
Dean of the Graduate School

EXAMINING COMMITTEE:

L. J. Blake

Peggala Pratelli

R. J. Koch

Hedgell Cohen

Date of Examination: July 29, 1964