On Certain Algebras of Centralizers.

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ABSTRACT

This paper is devoted primarily to a study of the algebras of centralizers of certain Banach algebras. The algebra of centralizers of a Banach algebra $X$, denoted by $\mathcal{C}(X)$, is the set of all mappings $T$ from $X$ into $X$ which satisfy the identity $x(Ty) = (Tx)y$. By $R(X)$ [$L(X)$] we will denote the algebra of right [left] centralizers which is by definition, the set of bounded linear operators $T$ from $X$ into $X$ which satisfy $T(xy) = (Tx)y$ [$T(xy) = x(Ty)$]. If $X$ is commutative, a function $F$ will be called a multiplier provided $F \hat{\mathcal{C}} \subseteq \hat{X}$ where $\hat{\cdot}$ is the Gelfand transform. We will use $M(X)$ to denote the set of all multipliers.

In Chapter I, we show that $\mathcal{C}(X)$ is a commutative Banach subalgebra of the bounded linear operators on $X$ and is the intersection of $R(X)$ and $L(X)$. For $X$ commutative, we give a proof of the known fact that $\mathcal{C}(X)$ and $M(X)$ are isomorphic.

In the first part of Chapter II, it is shown that $R(H)$ and $L(H)$ are $W^*$-algebras for $H$ an $H^*$-algebra. We prove that the commutant of $R(H)$ [$L(H)$] is $L(H)$ [$R(H)$]. In order to do this, we show that an $H^*$-algebra has both left and right approximate identities.

The second part of Chapter II is concerned with commutative $H^*$-algebras (in which case $R(H) = L(H) =$
We show that $\mathcal{L}(H)$ is isometric $*$-algebra isomorphic to the set of all bounded (continuous) functions on a certain discrete space. Characterizations are given for compact centralizers and projection centralizers. We prove that two $H^*$-algebras are topologically and algebraically equivalent if and only if they have the same dimension. As a corollary, we have that if $G$ and $H$ are compact abelian topological groups, then $L^2(G)$ is isometric $*$-algebra isomorphic to $L^2(H)$ if and only if their respective dual groups are in one-to-one correspondence. Chapter II is concluded by giving a characterization of the Banach algebra of bounded complex-valued functions on a set as an algebra of centralizers.

In Chapter III, we study the centralizers of the Banach algebras $L^p(G)$ for $1 \leq p < \infty$ and $G$ a compact abelian topological group. A result of Chapter II gives us that $M(L^2(G))$ is the space of bounded continuous functions on the discrete dual group of $G$ and it is known that $M(L^1(G))$ is the set of all Fourier-Stieltjes transforms of complex-valued regular Borel measures on $G$. We show that if $1 \leq p < \infty$, $1 \leq q < \infty$ and $1/p + 1/q = 1$, then $M(L^p(G))$ and $M(L^q(G))$ are identical. For $G$ a compact abelian topological group with an element of infinite order in its dual, we prove that the union of all the sets $M(L^p(G))$, for $p \neq 2$, is properly contained in $M(L^2(G))$. For the same class of groups, we also have that $M(L^1(G))$ is properly contained in the intersection
of all the sets $M(L^p(G))$, for $p > 1$. These results are obtained first for the circle group and then extended to the group $G$.

In Chapter IV, this paper is concluded by listing several unsolved problems which have arisen in connection with this work.
INTRODUCTION

The terminology and notation used throughout this paper will, for the most part, be that of Loomis [10]\(^1\) although we will depart from this in several places and use the terminology of Rickart [13]. These departures are found mostly in discussions of operators in Chapter II, where the appropriate definitions and notation will be given.

The terms right centralizer and left centralizer seem to appear first in a paper of J. G. Wendel [16], where they are defined as the set of all operators, \(T\), on \(L^1(G)\), for \(G\) a locally compact topological group, for which the identities \(T(fg) = (Tf)g\) and \(T(fg) = f(Tg)\), respectively hold. Operators similar to these, namely those satisfying the identity \(f(Tg) = (Tf)g\), have been studied by Wang in [15] over a general commutative Banach algebra, where they are called 'multipliers'. Wang shows that this algebra of operators is isomorphic to an algebra of functions on the space of regular maximal ideals of the underlying Banach algebra. In this paper, we will use the term centralizer to denote the operators and reserve the term

\(^1\)Pairs of numbers in brackets refer to correspondingly numbered references in the Selected Bibliography and page numbers, respectively. A single number in a bracket refers to the correspondingly numbered reference in the Selected Bibliography.
multiplier for the corresponding functions. It should be noted that multiplier will have meaning only when the Banach algebra is commutative.

This paper is divided into four chapters, the first of which is concerned with the general properties of centralizers on a Banach algebra. By Banach algebra, we will always mean a complex Banach algebra. Included in Chapter I are extensions of several of the results of [15] to non-commutative algebras, and the concepts of right centralizer, left centralizer and centralizer are related. A proof of the theorem relating centralizers and multipliers, due to Wang [15], is also included.

Chapter II is devoted to a detailed study of \( H^* \)-algebras and centralizers of \( H^* \)-algebras. The majority of the material is motivated by \( L^2(G) \), where \( G \) is a compact topological group. It is shown that the right centralizers and the left centralizers are each \( W^* \)-algebras and are, in fact, the \( W^* \)-algebras generated by the sets of left and right multiplication operators, respectively. We prove that the commutant of the right [left] centralizers is the left [right] centralizers. In order to show the above commutant property, we need a fact which is interesting in its own right, namely that any \( H^* \)-algebra has a left approximate identity which is unbounded if the algebra is not finite dimensional. By involution we obtain a right approximate identity with the same property.

The latter part of Chapter II involves only commutative \( H^* \)-algebras. In this context, each right
centralizer is a left centralizer and vice versa. Hence we can here talk about centralizers without ambiguity. A characterization is given for the $W^*$-algebra of centralizers of a commutative $H^*$-algebra as the set of all bounded, a priori continuous, functions on a discrete space. The compact operators which are centralizers and the projection operators which are centralizers are characterized. In order to obtain the above results, we have had to rely on several deep and complicated theorems from the general theory of operators on a Hilbert space.

In discussing an abelian topological group $G$, $\hat{G}$ will always denote the dual group of $G$ (the group of continuous homomorphisms of $G$ into the circle group). We show, in the case of the $H^*$-algebra $L^2(G)$, where $G$ is a compact abelian topological group with an element of infinite order in $\hat{G}$, that the operator norm closure of the left multiplication operators defined by regular Borel measures must be properly contained in the algebra of centralizers. An example is included to show that non-isomorphic groups can have $L^2$-spaces which are isomorphic and isometric, a result which is certainly not the case for $L^1$-spaces. This example and the method of proof used in the characterization of the algebra of centralizers lead to a theorem which states that two $H^*$-algebras are topologically and algebraically equivalent if and only if they have the same dimension as Hilbert spaces. Chapter II is concluded by showing that every commutative $H^*$-algebra is $*$-algebra isomorphic and topologically equivalent to $L^2(G)$ for some compact
abelian topological group $G$, and a characterization of the Banach algebra of bounded functions on a set as an algebra of centralizers is given.

Chapter III of this paper deals with centralizers of the Banach algebras $L^p(G)$ for $1 \leq p < \infty$ and $G$ a compact abelian topological group. Due to the nature of the $L^p$-norm for $1 < p < \infty$ and $p \neq 2$, the algebras of multipliers are somewhat easier to work with than the corresponding algebras of centralizers. The multipliers have the added asset that they are all contained in $C(\hat{G})$, the bounded complex-valued functions on the discrete space $\hat{G}$. In view of a result of Chapter II, we have that the algebra of multipliers of $L^2(G)$ is all of $C(\hat{G})$. Helson shows in [7] that the algebra of multipliers of $L^1(G)$ is the set of Fourier-Stieltjes transforms of regular Borel measures. Wendel's results in [16] give a non-commutative analogue which yields the same result in the case that $G$ is commutative. It should be noted that the set of Fourier-Stieltjes transforms of regular Borel measures must be properly contained in $C(\hat{G})$ if $G$ is not finite (cf.[8]).

We show that if $1/p + 1/q = 1$, $1 < p < \infty$ and $1 < q < \infty$, then the multiplier algebra of $L^p(G)$ is the same as the multiplier algebra of $L^q(G)$. It then follows that the algebra of centralizers of $L^p(G)$ is isometric and isomorphic to the algebra of centralizers of $L^q(G)$, where $p$ and $q$ are as above. We next obtain a version of a theorem of Rudin [14,55] which states that if $H$ is a closed subgroup of $G$,
then the Fourier transforms of elements of $L^1(G/H)$ are precisely the restrictions to the annihilator of $H$ of transforms of elements of $L^1(G)$. Our theorem depends on the fact that $G$ is compact and we obtain the same conclusion where $1$ is replaced by $p$, for $1 \leq p < \infty$. It is shown that if $G$ is a compact abelian topological group with an element of infinite order in $G$, then the union of all the multiplier algebras for $p \neq 2$ is properly contained in $C(G)$. This is accomplished by first proving it for the circle group and then extending to the general case by means of the analogue to the theorem from Rudin. For the same class of groups, it is also shown that the intersection of all the multiplier algebras for $p > 1$ properly contains the multiplier algebra of $L^1(G)$.

In Chapter IV we conclude this paper by listing several interesting unsolved problems and conjectures which have arisen in connection with this work.
CHAPTER I

The purpose of this chapter is to give the necessary notation and some of the properties of centralizers which we will need throughout this paper. The theorems for non-commutative algebras are straightforward generalizations of results found in [15]. Finally, for the sake of completeness, we give in detail the proof of the theorem due to Wang [15] which relates the centralizers and the multipliers.

The definition of Banach algebra and the properties used herein can be found in [10] and [13]. The symbol $B(X)$ will be used to denote the collection of all bounded linear operators from the Banach space $X$ into $X$. With usual sum, multiplication defined by composition and operator norm, $B(X)$ is a Banach algebra with identity.

Definition 1.1. If $X$ is a Banach algebra, then a mapping $T$ from $X$ into $X$ will be called a centralizer of $X$ if $T$ satisfies the identity $x(Ty) = (Tx)y$, and we will use the symbol $C(X)$ to denote the set of all centralizers of $X$.

Definition 1.2. If $X$ is a Banach algebra, then following Wendel [16], a bounded linear operator $T$ from $X$ into $X$ will be called a right [left] centralizer of $X$ if $T$ satisfies the identity $T(xy) = (Tx)y$ [$T(xy) = x(Ty)$]. We will use $R(X)$ and $L(X)$ to denote the right and left centralizers, respectively.
Definition 1.3. A Banach algebra $X$ is said to be without order if $yX = (0)$ or $Xy = (0)$ implies that $y = 0$.

Lemma 1.1.1. If $X$ is a Banach algebra which is without order, then $\mathcal{C}(X) \subset B(X)$.

Proof. Let $T$ be a fixed element of $\mathcal{C}(X)$. For $x, y, z \in X$ and $a$ and $b$ complex numbers, we have that

$$x[T(ay + bz)] = (Tx)(ay + bz) = a(Tx)y + b(Tx)z = ax(Ty) + bx(Tz) = x[aTy + bTz].$$

Since $x$ was arbitrary and $X$ is without order, $T(ay + bz) = aTy + bTz$ and thus $T$ is linear. Further, suppose that $y, z \in X$ and that $(y_n)_{n=1}^{\infty}$ is a sequence in $X$ such that $\|y_n - y\| \to 0$ and $\|Ty_n - z\| \to 0$.

For $x \in X$, we have $\|xz - x(Ty)\| \leq \|xz - x(Ty_n)\| + \|(Tx)y_n - (Tx)y\| \leq \|x\|\|z - Ty_n\| + \|Tx\|\|y_n - y\|$ which converges to zero with $n$. Therefore $xz = x(Ty)$ for each $x \in X$ and since $X$ is without order, we get $z = Ty$. We now apply the Closed Graph Theorem [1;41] to conclude that $T$ is bounded and hence $\mathcal{C}(X) \subset B(X)$.

Lemma 1.1.2. If $X$ is a Banach algebra, then $R(X) \cap L(X) \subset \mathcal{C}(X)$ and equality holds if $X$ is without order.

Proof. Let $T \in R(X) \cap L(X)$ and $x, y \in X$. Then $T(xy) = (Tx)y$ and $T(xy) = x(Ty)$ since $T$ is a member of $R(X)$ and $L(X)$, respectively. Combining these two equalities we see that $x(Ty) = (Tx)y$, for $x, y \in X$, and therefore $T \in \mathcal{C}(X)$. Now suppose that $X$ is without order and let $T \in \mathcal{C}(X)$. For $x, y, z \in X$, we have $x[T(yz)] = (Tx)(yz) = [(Tx)y]z = [x(Ty)]z = x[(Ty)z]$. Thus, since $x$
was arbitrary and \( X \) is without order, \( T(yz) = (Ty)z \), which implies that \( T \in R(X) \). Also, \( x[T(yz)] = (Tx)(yz) = [(Tx)y]z = [x(Ty)]z = x[(Ty)z] = x[y(Tz)] \). Then, as above, \( T(yz) = y(Tz) \) and hence \( T \in R(X) \cap L(X) \).

**Theorem 1.1.** If \( X \) is a Banach algebra which is without order, then \( \mathcal{C}(X) \) is a commutative Banach subalgebra of \( B(X) \) which contains the identity operator.

**Proof.** As previously mentioned, we will take our operations to be usual sum and composition, namely \((S + T)x = Sx + Tx\) and \((ST)x = S(Tx)\). By Lemma 1.1.1, we already have that \( \mathcal{C}(X) \subseteq B(X) \). If \( T, S \in \mathcal{C}(X) \) and \( x, y \in X \), then \( x[(S + T)y] = x[Sy + Ty] = x(Sy) + x(Ty) = (Sx)y + (Tx)y = [Sx + Tx]y = [(S + T)x]y \) and therefore \( S + T \in \mathcal{C}(X) \). By Lemma 1.1.2, both \( T \) and \( S \) are in \( R(X) \) and hence \( [(TS)x]y = [T(Sx)]y = T[(Sx)y] = T[S(xy)] = (TS)(xy) \) which implies that \( TS \in R(X) \). Similarly, \( x[(TS)y] = x[T(Sy)] = T[x(Sy)] = T[S(xy)] = (TS)(xy) \), since both \( T \) and \( S \) are in \( L(X) \), and hence \( TS \in L(X) \). Thus \( TS \in R(X) \cap L(X) \) and application of Lemma 1.1.2 gives us that \( TS \in \mathcal{C}(X) \). Furthermore, if \( a \) is a complex number, then \( x(aT)y = x[aTy] = ax(Ty) = a(Tx)y = [(aT)x]y \) and \( aT \in \mathcal{C}(X) \). We have now shown that \( \mathcal{C}(X) \) is a subalgebra \( B(X) \). If \( T, S \in \mathcal{C}(X) \) and \( x, y \in X \), then \( x[(TS)y] = x[T(Sy)] = (Tx)(Sy) = [S(Tx)]y = [(ST)x]y = x[(ST)y] \). Since \( X \) is without order, we conclude that \( (TS)y = (ST)y \) and that \( \mathcal{C}(X) \) is commutative. If we denote by \( I \) the identity operator, then \( x(Iy) = xy = (Ix)y \) and hence \( I \in \mathcal{C}(X) \). We will use \( || \cdot ||_6 \) to represent the operator
norm for $B(X)$, namely $\|T\|_0 = \sup\{\|Tx\|: x \in X, \|x\| = 1\}$ for $T \in B(X)$. Suppose that $\{T_n\}_{n=1}^\infty$ is a sequence in $\mathcal{L}(X)$ which is Cauchy with respect to the operator norm defined above. Since $\mathcal{L}(X) \subseteq B(X)$ and $B(X)$ is complete, there exists $T \in B(X)$ such that $\|T_n - T\|_0 \to 0$. If $x, y \in X$, then
\[
\|x(Ty) - (Tx)y\| \leq \|x(Ty) - x(T_ny)\| + \|(T_nx)y - (Tx)y\| \\
\leq \|x\|\|Ty - T_ny\| + \|T_nx - Tx\|\|y\| \leq 2\|x\|\|y\|\|T_n - T\|_0
\]
which converges to zero with $n$. Hence $x(Ty) = (Tx)y$ and $T \in \mathcal{L}(X)$. Therefore $\mathcal{L}(X)$ is complete with respect to the operator norm and the proof of the theorem is complete.

It should be noted that the theorem just proved is a very straightforward generalization of several of the results found in Section 2 of [15]. The next theorem is also in [15] and is included (with a more lucid proof) for the sake of completeness.

We will first need a few facts which can be found in [10]. We use the symbol $\mathcal{M}_X$ to denote the space of regular maximal ideals of the commutative Banach algebra $X$. An ideal $I$ in $X$ is said to be regular $X/I$ has a unit. If $\mathcal{M}_X$ is given the weakest topology for which all the functions $\hat{x}$, for $x \in X$, are continuous, then it becomes a locally compact Hausdorff space. The function $\hat{x}$ is defined on $\mathcal{M}_X$ by $\hat{x}(M) = h_M(x)$ where $M \in \mathcal{M}_X$ and $h_M$ is the multiplicative linear functional onto the complex numbers associated with $M$. In this paper, we will consider $\mathcal{M}_X$ as the corresponding multiplicative linear functionals onto the complexes which are imbedded in the adjoint space.
of $X$ and given the relative weak-$*$ topology. We also need two definitions before stating the theorem relating centralizers and multipliers.

**Definition 1.4.** [10;62] A commutative Banach algebra is said to be semi-simple if the intersection of all of its regular maximal ideals is zero. Note that this condition is equivalent to the condition that the correspondence between $x$ and $\hat{x}$ be one-to-one.

**Definition 1.5.** If $X$ is a commutative Banach algebra, then a function $f$ from $\mathcal{M}_X$ into the complex numbers will be called a multiplier of $X$ provided $f \hat{x} \subseteq \hat{x}$, where $\hat{x} = \{x : x \in X\}$ and the set of all multipliers of $X$ will be denoted by $M(X)$.

**Theorem 1.2.** [15] Let $X$ be a semi-simple commutative Banach algebra. Then $M(X) \subseteq C(\mathcal{M}_X)$, the bounded continuous complex-valued functions on $\mathcal{M}_X$, and there is a natural mapping from $\mathcal{C}(X)$ onto $M(X)$ which is a norm-decreasing algebra isomorphism.

**Remark.** We will often refer to this mapping as the Wang mapping.

**Proof.** We will first show how to obtain the multiplier which corresponds to a given centralizer. Fix $T \in \mathcal{C}(X)$ and $h \in \mathcal{M}_X$. Suppose that $x$ and $y$ are two elements of $X$ such that $\hat{x}(h) \neq 0 \neq \hat{y}(h)$. Then

$$\hat{T_x(h)} = \hat{T_x(h)}\hat{y}(h) = (\hat{T_x}\hat{y})(h) = \hat{x(T_y)(h)} = \hat{x(h)}\hat{y}(h) = \hat{T_y(h)} .$$

Now, for $h \in \mathcal{M}_X$, choose $x_h \in X$ such that $\hat{x_h}(h) \neq 0$ and define $F$ from $\mathcal{M}_X$ into the complex numbers by $F(h) =$
of $F$ is independent of choice of the $x_h$ such that $\hat{x}_h(h) \neq 0$. Since $\hat{x}_h$ and $\hat{F}_h$ are continuous at $h$ and $\hat{x}_h(h) \neq 0$, it follows that $F$ is continuous at $h$. Suppose that $x \in X$ such that $\hat{x}(h) = 0$. Then $0 = \hat{x}(h) \hat{F}_h(h) = \hat{x}(\hat{T}_h)(h) = (\hat{T}_h)\hat{x}_h(h) = \hat{T}_h(\hat{x}_h(h))$ and since $\hat{x}_h(h) \neq 0$, it is true that $\hat{T}_h(h) = 0$. Hence, $\hat{x}(h)F(h) = \hat{T}_h(h)$ holds for all $x \in X$ and $h \in \mathcal{M}_X$: if $\hat{x}(h) \neq 0$, then the definition of $F$ implies the validity of the above equation and if $\hat{x}(h) = 0$, the previous argument shows that both sides of the equation are zero. Since $F \hat{x} = \hat{T}_h$ for all $x \in X$, it is clear that $F$ is a multiplier. We have already noted that $F$ is continuous and now we will show that it is bounded. In order to do this, for each $h \in \mathcal{M}_X$, let $K_h = \sup\{|\hat{x}_h(h)| : x \in X, ||x|| = 1\}$. Since $h \in \mathcal{M}_X$, there is an element $x_h \in X$ such that $\hat{x}_h(h) \neq 0$. Also $\frac{x_h}{||x_h||} \in X$ and has norm unity. Therefore $K_h \geq \left| \frac{\hat{x}_h(h)}{||x_h||} \right| = \frac{||\hat{x}_h(h)||}{||x_h||} > 0$.

Further, $x \in X$ and $||x|| = 1$ implies that $||\hat{x}_h||_\infty \leq 1$, where $||\cdot||_\infty$ is the supremum norm for $C(\mathcal{M}_X)$, and hence $0 < K_h \leq 1$.

Now, fix $x \in X$. If $T_x = 0$, then $\hat{F}(h) = 0$ and $|F(h)\hat{x}(h)| = |\hat{T}_h(h)| = 0 \leq K_h ||T||_o ||x||$, where $||T||_o$ is the operator norm of $T$. If $T_x \neq 0$, then $\frac{\hat{T}_h}{||\hat{T}_h||} \in X$ and has norm one.

Thus, $K_h \geq \left| \frac{\hat{T}_h(h)}{||\hat{T}_h||} \right|$ and hence $|\hat{T}_h(h)| \leq K_h ||T||_o ||x||$. Therefore, $|F(h)\hat{x}(h)| = |\hat{T}_h(h)| \leq K_h ||T||_o ||x||$ for all $x \in X$. Hence, for $x \in X$ such that $||x|| = 1$, we have that
\[ |F(h)| \leq K_h \|T\|_o \text{ and therefore } \sup \{|F(h)| : h \in \mathcal{H}, |x| = 1 \} \leq K_h \|T\|_o. \text{ Therefore } |F(h)| \leq |T\|_o, \text{ } F \text{ is bounded and } |F|_\infty \leq |T|_o. \text{ Now, define } \theta \text{ from } \mathcal{L}(X) \text{ into } \mathcal{M}(X) \text{ by } \theta(T) = F, \text{ where } \hat{T}x = FX \text{ for all } x \in X.

Since, for every } h \in \mathcal{M}_X, \text{ there is } x_h \in X \text{ such that } \hat{x}_h(h) \neq 0, \text{ it follows that } \theta \text{ is well defined. We have already noted that } \theta \text{ is into } \mathcal{M}(X) \text{ and is norm-decreasing.}

For } T, S \in \mathcal{L}(X), h \in \mathcal{M}_X \text{ and } x, y \in X, \text{ we have that}

\[ \theta(T + S)(\hat{x}(h)) = \theta(T)(\hat{x}(h)) + \theta(S)(\hat{x}(h)) = \hat{T}x(h) + \hat{S}x(h) = \hat{(T + S)x}(h) = \theta(T + S)(\hat{x}(h)) \text{ and hence}

\[ \theta(T + S) = \theta(T) + \theta(S). \] Further, \[ \theta(ST)(\hat{x}(h)) = (ST)(\hat{x}(h)) = S(T(xh))(h) = \hat{Sx}(h)\hat{Tx}(h) = \theta(S)(\hat{x}(h))\theta(T)(\hat{x}(h)) \text{ and thus}

\[ \theta(ST) = \theta(S)\theta(T). \] Since \[ \theta(aT)(\hat{x}(h)) = a\hat{T}x(h) = a\hat{T}x(h) = a\theta(T)(\hat{x}(h)), \text{ we have } \theta(aT) = a\theta(T) \text{ for all } a \in \mathbb{C}.

The mapping } \theta \text{ is one-to-one since if } \theta(T) = 0, \text{ then } \hat{T}x(h) = 0 \text{ for all } x \in X \text{ and } h \in \mathcal{M}_X. \text{ The semi-simplicity of } X \text{ implies that } Tx = 0 \text{ for all } x \in X \text{ and hence } T = 0.

We will now show that } \theta \text{ is actually onto. In order to do this, let } F \in \mathcal{M}(X). \text{ Since } F \mathcal{X} \subseteq \hat{\mathcal{X}}, \text{ for each } x \in X, \text{ there is an element } y \in X \text{ such that } F\hat{x} = \hat{y}. \text{ Since the mapping } \wedge \text{ is one-to-one, there can be only one such } y \in X. \text{ Now define } T \text{ from } X \text{ into } X \text{ by } Tx = y \text{ where } y \text{ is related to } x \text{ by } F\hat{x} = \hat{y}. \text{ Since } y \text{ is uniquely determined by } x, \text{ the mapping } T \text{ is well-defined. It is clear that } \theta(T) = F \text{ since}

\[ \hat{T}x(h) = \hat{y}(h) = F(h)x(h). \] For } x, y \in X, \text{ \[ \hat{x}(h)\hat{T}y(h) = \hat{x}(h)F(h)\hat{y}(h) = F(h)\hat{x}(h)\hat{y}(h) = \hat{T}x(h)\hat{y}(h) =}
\((T_x)y(h)\) for all \(h \in \mathcal{M}_X^\perp\). Therefore \(x(Ty) = (Tx)y\) and the fact that \(\sim\) is one-to-one gives us that \(x(Ty) = (Tx)y\).

Thus \(T \in \mathcal{C}(X)\) and \(\theta(T) = F\) implies that \(\theta\) is onto \(M(X)\).

It now follows that \(M(X)\) is a subset of \(C(\mathcal{M}_X^\perp)\) since the image of \(\theta\) is contained in \(C(\mathcal{M}_X^\perp)\).
CHAPTER II

This chapter will be devoted to a detailed study of H*-algebras and centralizers of H*-algebras.

Definition 2.1. [10;100][[11;330]] An H*-algebra (Hilbert Ring) is a Banach algebra H which is also a Hilbert space under the same norm and has an involution * satisfying:

a). \((ax + by)^* = \bar{a}x^* + \bar{b}y^*\)
b). \((xy)^* = y^*x^*\)
c). \(x^{**} = x\)
d). \((xy, z) = (y, x^*z)\)
e). \(||x^*|| = ||x||\)
f). \(x^*x \neq 0 \text{ if } x \neq 0\)

for all \(x, y, z \in H\) and \(a\) and \(b\) complex numbers.

Definition 2.2. [13;180][[11;188]] A Banach algebra X will be called a Banach *-algebra (symmetric Banach ring) if there is an involution on X satisfying a), b), c) and e) of Definition 2.1.

Definition 2.3. [13;180] [[11;228]] A Banach *-algebra, X, is called a B*-algebra (completely regular Banach ring) if \(||x^*x|| = ||x||^2\) for each \(x \in X\).

Definition 2.4. [13;181] A closed self-adjoint subalgebra of B(H), where H is a Hilbert space, is called a C*-algebra. By self-adjoint, we mean that the natural involution of B(H) maps the subalgebra into itself.
Remark. It is clear that a C*-algebra is a B*-algebra and every complex B*-algebra is isometrically *-algebra isomorphic to a C*-algebra, (cf. [13;244]).

Definition 2.5. [13;289] ([11;444]) A W*-algebra (weakly closed ring) is a self-adjoint subalgebra of B(H), where H is a Hilbert space, which is closed relative to the weak operator topology of B(H).

Now, assume that H is an H*-algebra. We have already shown in Theorem 1.1 that C(H) is a commutative Banach subalgebra of B(H). We will now study the spaces R(H) and L(H) in considerably more detail.

Theorem 2.1. If H is an H*-algebra, then R(H) and L(H) are C*-algebras (hence B*-algebras) each of which contains the identity operator I.

Proof. The proof will be given for R(H) since the arguments for L(H) are quite similar. Choose A,B \in R(H) and x,y \in H. First, \((A + B)(xy) = A(xy) + B(xy) = A(xy) + B(xy) = (Ax)y + (Bx)y = [(A + B)x]y\) and thus \(A + B \in R(H)\). Second, \((AB)(xy) = A[B(xy)] = A[(Bx)y] = [A(Bx)]y = [(AB)x]y\), AB \in B(H) and hence AB \in R(H). If A \in R(H) and x,y,z \in H, then \((A^*(xy),z) = (xy,Az) = (x,(Az)y^*) = (x,A(zy^*)) = (A^*x,zy^*) = ((A^*x)y,z)\). Hence \(A^*(xy) = (A^*x)y\) and \(A^* \in R(H)\). It is clear that aA \in R(H) whenever a is complex and A \in R(H). Furthermore, if A \in B(H) and if \(\{A_n\}_{n=1}^{\infty}\) is a sequence in R(H) such that \(\|A_n - A\|_o \rightarrow 0\), then \(\|A(xy) - (Ax)y\| \leq \|A(xy) - A_n(xy)\| + \|(A_nx)y - (Ax)y\| \leq 2\|x\|\|y\|\|A_n - A\|_o\) which converges with
n to zero. Hence $A(xy) = (Ax)y$ and $A \in R(H)$. Therefore $R(H)$ is a closed self-adjoint subalgebra of $B(H)$. Since $R(H)$ clearly contains the identity operator, we have the conclusion of the theorem.

**Definition 2.6.** [11;445] Suppose $S$ is a set in $B(H)$. Then the commutant of $S$, denoted by $S'$, is the set of all operators in $B(H)$ that commute with all the operators in $S \cup S^*$. 

**Definition 2.7.** [11;445] The principal identity of a set $S$ in $B(H)$ is the projection onto $H - N$ where $N$ is the set of all elements in $H$ on which all the operators in $S \cup S^*$ vanish and $H - N$ is the orthogonal complement of $N$ in $H$.

The following propositions, ending with Corollary 2.2.2, will be used in the next few theorems and are included, without proof, for the sake of completeness.

**Lemma 2.2.1.** [11;445] If $S$ is an arbitrary set in $B(H)$, then the principal identity of $S$ belongs to $S'$ and $S'' = (S')'$.

**Theorem 2.2.** [11;446] Suppose $S$ is an arbitrary subset of $B(H)$ and that $E_0$ is the principal identity of the set $S$. The $W^*$-algebra generated by $S$ consists of those and only those elements $A \in S''$ which satisfy the condition $E_0 A = AE_0 = A$.

**Corollary 2.2.1.** [11;448] If $X$ is a self-adjoint subalgebra of $B(H)$, then its weak and strong operator closures coincide.
Corollary 2.2.2. [11;448] If $X$ is a $W^*$-algebra containing the identity, then $X^{''} = X$.

Theorem 2.3. Let $H$ be an $H^*$-algebra. Then $R(H)$ and $L(H)$ are $W^*$-algebras.

Proof. In view of Theorem 2.1, $R(H)$ is a self-adjoint subalgebra of $B(H)$ and hence we must only show that it is weak operator closed. To do this, let $A \in B(H)$ and $\{A_a\}_{a \in D}$ be a net in $R(H)$ with the property that $\{A_a\}_{a \in D}$ converges to $A$ in the strong operator topology. By a net, we mean a function from a directed set into some space, in this case from $D$ into $R(H)$ (e.g., see [9;65]). Then, for $x,y \in H$, we have

$$
||A(xy) - (Ax)y|| \leq ||A(xy) - A_a(xy)|| + ||A_a(xy) - (Ax)y|| \leq ||A(xy) - A_a(xy)|| + ||A_a x - Ax|| ||y||
$$

which converges to zero with $a$. Hence $A(xy) = (Ax)y$, $A \in R(H)$ and $R(H)$ is closed relative to the strong operator topology. Since $R(H)$ is a self-adjoint subalgebra of $B(H)$, the strong and weak operator closures of $R(H)$ coincide by Corollary 2.2.1. Hence $R(H)$ is weak operator closed and thus is a $W^*$-algebra.

The proof for $L(H)$ is similar and will be omitted.

Theorem 2.4. If $H$ is an $H^*$-algebra, then there is a net $\{e_a\}_{a \in D}$ contained in $H$ with the property that $\{e_a x\}_{a \in D}$ converges to $x$ and $\{x e_a^*\}_{a \in D}$ converges to $x$ for every $x \in H$.

Proof. For $x \in H$, define the mapping, $L_x$, from $H$ into $H$ by $L_x(y) = xy$. Let $\mathcal{L}(H) = \{L_x : x \in H\}$ and it
is easily seen that \( \mathfrak{L}(H) \subseteq R(H) \). For \( S \) an arbitrary subset of \( B(H) \), denote by \( W(S) \) the smallest \( W^* \)-algebra containing \( S \). Since operator sum and composition in \( \mathfrak{L}(H) \) correspond to the sum and multiplication in \( H \) and 
\[
(L_x)^* = L_{x^*},
\]
we have that \( \mathfrak{L}(H) \) is a self-adjoint subalgebra of \( B(H) \). It now follows that the weak operator closure of \( \mathfrak{L}(H) \) is a \( W^* \)-algebra and hence \( W(\mathfrak{L}(H)) \) is the weak operator closure of \( \mathfrak{L}(H) \). Now, let \( N = \{ x \in H : Ax = 0 \text{ for all } A \in \mathfrak{L}(H) \} \). Since \( x \in N \) implies that \( L_y(x) = 0 \) for all \( y \in H \), we have in particular, 
\[
L_{x^*}(x) = x^*x = 0
\]
and thus \( x = 0 \). Hence \( N = \{0\} \) and the principal identity of \( \mathfrak{L}(H) \) is \( I \), the identity operator, and by Lemma 2.2.1, \( I \in \mathfrak{L}(H)' \cap \mathfrak{L}(H)'' \). By Theorem 2.2, we have that \( W(\mathfrak{L}(H)) = \{ A \in \mathfrak{L}(H)'' : AI = IA = A \} = \mathfrak{L}(H)'' \). Combining these two statements, we conclude that \( I \in W(\mathfrak{L}(H)) \) and hence \( I \) is an element of the weak operator closure of \( \mathfrak{L}(H) \). Since \( \mathfrak{L}(H) \) is a self-adjoint subalgebra of \( B(H) \), we have, via Corollary 2.2.1, that \( I \) is an element of the strong operator closure of \( \mathfrak{L}(H) \). Hence, there exists a net of operators, \( \{ A_a \}_{a \in D} \), contained in \( \mathfrak{L}(H) \) such that \( \{ A_a \}_{a \in D} \) converges to \( I \) in the strong operator topology. Since \( A_a \in \mathfrak{L}(H) \), there exists an \( e_a \in H \) such that \( A_a = L_{e_a} \) for all \( a \in D \). For \( x \in H \), we have that 
\[
\lim_{a \to D} \| L_{e_a}(x) - I(x) \| = 0,
\]
since \( \{ L_{e_a} \}_{a \in A} \) converges to \( I \) in the strong operator topology. Hence 
\[
\lim_{a \to D} \| e_a x - x \| = 0
\]
and \( \{ e_a x \}_{a \in D} \) converges to \( x \). By applying the same
argument to $x^*$, we can conclude that $\|e_a x^* - x^*\| \to 0$.
However, this implies that $\|xe_a^* - x\| \to 0$ and thus $\{xe_a^*\}_{a \in D}$ converges to $x$.

**Corollary 2.4.1.** If $H$ is an $H^*$-algebra, then $R(H)$ is the $W^*$-algebra generated by the left multiplication operators, or equivalently, $R(H) = W(L(H))$. If we define right multiplication operators in a similar manner, then we get the analogous result for $L(H)$.

**Proof.** Let $\{e_a\}_{a \in D}$ be the net of Theorem 2.4 in $H$.

For $A \in R(H)$, $L_{Ae_a}(x) = (Ae_a)x = A(e_a x)$ and since $\{e_a x\}_{a \in D}$ converges to $x$, we have that $\{L_{Ae_a}(x)\}_{a \in D}$ converges to $Ax$ for each $x \in H$. Therefore, the net of operators $\{L_{Ae_a}\}_{a \in D}$ converges to $A$ in the strong operator topology and hence $L(H)$ is strong operator dense in $R(H)$. As seen in the proof of Theorem 2.4, $L(H)$ is a self-adjoint subalgebra of $B(H)$, so its weak and strong operator closures coincide by Corollary 2.2.1. Since $R(H)$ is a $W^*$-algebra, we now have that the weak operator closure of $L(H)$, which is the $W^*$-algebra generated by $L(H)$, is precisely $R(H)$. If we define $R_x$ from $H$ into $H$ by $R_x(y) = yx$, for each $x \in H$, then $\mathcal{G}(H) = \{R_x : x \in H\}$ is a self-adjoint subalgebra of $B(H)$ and $\mathcal{G}(H) \subseteq L(H)$. For $A \in L(H)$, the net $\{R_{Ae_a^*}\}_{a \in D}$ converges to $A$ in the strong operator topology and the previous arguments applied to $\mathcal{G}(H)$ and $L(H)$ give that $W(\mathcal{G}(H)) = L(H)$. 

Theorem 2.5. If $H$ is an $H^*$-algebra, then the commutant of $R(H)$, $R(H)'$, is $L(H)$ and $L(H)' = R(H)$.

Proof. It is known, see [11;445], that if $S$ is any set in $B(H)$, then $S' = W(S)'$. This fact, together with Corollary 2.4.1, allows us to conclude that $R(H)' = W(L(H))' = L(H)'$. Now, for $A \in R(H)'$, we have that $A \in L(H)'$ and hence $AL_x = L_x A$ for all $x \in H$. Thus $A(xy) = A(L_x y) = (AL_x)y = (L_x A)y = L_x(Ay) = x(Ay)$, $A \in L(H)$ and $R(H)' \subseteq L(H)$. For $A \in L(H)$, it will now be shown that $A$ commutes with $L_x$ for each $x \in H$. If $x, y \in H$, then $(AL_x)y = A(L_x y) = A(xy) = x(Ay) = L_x(Ay) = (L_x A)y$. Hence $A \in L(H)'$ and thus $A \in R(H)'$. Therefore we have shown that $L(H) \subseteq R(H)'$ and hence $L(H) = R(H)'$. Also, we have that $R(H)' = L(H)'$ and since $R(H)$ is a $W^*$-algebra containing the identity, we can conclude, by Corollary 2.2.2., that $L(H)' = R(H)$.

Definition 2.8. [13;233] ([11;217]) A Banach $^*$-algebra $X$ with identity, $e$, will be called symmetric (completely symmetric) if $e + x^*x$ has an inverse in $X$ for each $x \in X$.

Definition 2.9. [11;259] A Banach $^*$-algebra $X$, with identity, $e$, will be called reduced if 

\[ \{ x \in X : f(x^*x) = 0 \text{ for every positive functional, } f, \text{ on } X \} = (0), \]

where $f$ is called a positive functional if $f$ is a linear functional and $f(y^*y) \geq 0$ for all $y \in X$.

Theorem 2.6. $R(H)$ and $L(H)$ are both symmetric and reduced (hence semi-simple) algebras if $H$ is an $H^*$-algebra.
Proof. It is known that every closed self-adjoint subalgebra of $B(H)$, which contains $I$, is symmetric (e.g., see [11;299]), hence each of $R(H)$ and $L(H)$ is symmetric. For $x \in H$, define $f_x$ from $R(H)$ into the complex numbers by $f_x(A) = (Ax,x)$. It is easily verified that $f_x$ is a linear functional on $R(H)$ and $f_x(A^*A) = (A^*Ax,x) = \|Ax\|^2 \geq 0$. If $A \neq 0$, then there exists an $x \in H$ such that $Ax \neq 0$ and hence $\|Ax\|^2 > 0$. Therefore, the $f_x$ corresponding to $x$ has the property that $f_x(A^*A) > 0$ and thus $R(H)$ is reduced. It then follows, see [11;260], that $R(H)$ is semi-simple.

**Theorem 2.7.** If $H$ is an $H^*$-algebra, then $\mathcal{C}(H)$ is a commutative $W^*$-algebra with the identity operator as unit.

**Proof.** By Theorem 1.1, $\mathcal{C}(H)$ is a closed, commutative subalgebra of $B(H)$. Since each of $R(H)$ and $L(H)$ is a $W^*$-algebra and $\mathcal{C}(H) = R(H) \cap L(H)$, $\mathcal{C}(H)$ is a $W^*$-algebra with unit. Note that $\mathcal{C}(H)$ is also symmetric, reduced and semi-simple since it is a closed subalgebra of $B(H)$ containing $I$ and is also isometric $*$-algebra isomorphic to the bounded continuous functions on its compact maximal ideal space (e.g., see [11;232]). This concludes the proof of the theorem.

We will now focus our attention on commutative $H^*$-algebras and first give a characterization of the centralizers of a commutative $H^*$-algebra as the set of all bounded, (continuous), complex-valued functions on a discrete space. For easy reference, the following discussion from [10;106] is included. If $H$ is a
commutative $H^*$-algebra, then $H$ is the direct sum of its minimal ideals, $N_a$, each of which is a one-dimensional ideal consisting of all scalar multiples of its identity $e_a$. The minimal ideals, $N_a$, are the orthogonal comple-
ments of the maximal ideals, $M_a$, and the homomorphism which maps $x$ to $\hat{x}$ is given by $\hat{x}(M_a) = (x, e_a)/\|e_a\|^2$.

Since $\hat{\delta}_b$ is a continuous function which has value one at $M_b$ and zero otherwise, we know that $\mathcal{M}_H$ is discrete.

Now, let $E = \{e_a/\|e_0\|: e_a \in N_a\}$ where $e_0$ is any fixed irreducible self-adjoint idempotent. An idempotent is said to be irreducible if it is not the sum of two non-zero idempotents, $x$ and $y$, such that $xy = 0$. Note that all the irreducible self-adjoint idempotents, which are the identities of the minimal ideals, have the same norm. We can now identify $E$ with $\mathcal{M}_H$ and if we give $E$ the discrete topology, then $E$ and $\mathcal{M}_H$ are also topologically equivalent. Note also that $E$ is a complete orthonormal basis for $H$.

**Theorem 2.8.** Let $H$ be a commutative $H^*$-algebra. Then there exists a $^*$-algebra isomorphism which is an isometry from $\mathcal{C}(H)$ onto $C(E)$, the set of all bounded complex-valued functions on the discrete space $E$, where $E$ is defined as in the previous discussion.

**Proof.** We will construct two mappings which are inverses of each other and such that each is norm-decreasing. If $A \in \mathcal{C}(H)$, then define $f_A$ from $E$ into the complex numbers by $f_A(e) = (Ae, e)$, for each $e \in E$, and define $\emptyset$ from $\mathcal{C}(H)$ into the set of functions on $E$
by $\mathcal{O}(A) = f_A$. Since $|f_A(e)| = |(Ae,e)| \leq ||A||_0 ||e||^2 = ||A||_0$, $f_A$ is a bounded function for each $A \in \mathcal{C}(H)$ and $||f_A||_\infty = ||\mathcal{O}(A)||_\infty \leq ||A||_0$. Now, fix $g \in C(E)$. Define the mapping $B^g$ on $H$ by $B^g(x) = \sum_{E} (x,e)g(e)e$ for each $x \in H$. We will show that $B^g$ is a mapping from $H$ to $H$ and that $B^g \in \mathcal{C}(H)$. For $x \in H$ (since $E$ is a complete orthonormal basis for $H$) we have that $x = \sum_{E} (x,e)e$ and $||x||^2 = \sum_{E} |(x,e)|^2$. In particular, there exists a countable subset of $E$ such that $(x,e) = 0$ for all $e$ not in the subset. Denote this subset of $E$ by $\{e_i\}_1$. Let $y = \sum_{i=1}^{n} (x,e_i)g(e_i)e_i$. If $n > m$, then $y_n - y_m = \sum_{i=m+1}^{n} (x,e_i)g(e_i)e_i$ and each coefficient of $y_n - y_m$, relative to $E$, is $(x,e_i)g(e_i)$. Hence $||y_n - y_m||^2 = \sum_{i=m+1}^{n} |(x,e_i)g(e_i)|^2 \leq ||g||_\infty \sum_{i=m+1}^{n} |(x,e_i)|^2$ which converges to zero with $m$ and $n$. Therefore $\{y_n\}_{n=1}^\infty$ is Cauchy and since $H$ is complete there exists $z \in H$ such that $z = \sum_{E} (x,e)g(e)e$. Thus, $B^g(x) = z$ and $B^g$ is a mapping from $H$ into $H$. Note that if $e,f \in E$, then $e = e_a/||e_o||$ and $f = e_b/||e_o||$, where $e_a$ and $e_b$ are irreducible self-adjoint idempotents. Therefore $ef = 0$ if $e \neq f$ and $ee = (e_a/||e_o||)(e_a/||e_o||) = e_a/||e_o||^2 = e/||e_o||$. We also have that $xe = \sum_{feE} (x,f)e = \sum_{feE} (x,f)e = (x,e)e/||e_o||$.

Now, if $x,y \in H$, then $x(B^g y) = x[\sum_{E} (y,e)g(e)e] = \sum_{E} (y,e)g(e)xe = \sum_{E} (y,e)g(e)(x,e)e/||e_o||$ and $(B^g x) y = \sum_{E} (y,e)g(e)xe = \sum_{E} (y,e)g(e)(x,e)e/||e_o||$.
\[
\sum_{E} (x,e)g(e)e\ y = \sum_{E} (x,e)g(e)e\ y = \sum_{E} (x,e)g(e)(y,e)e/\|e\|_0. \text{ Therefore } x(Bg\ y) = (Bg\ x)y \text{ and } B_g \in \mathcal{L}(H). \text{ Define } \theta \text{ from } C(E) \text{ into } \mathcal{L}(H) \text{ by } \theta(g) = B_g.
\]

Note that \(\|\theta(g)\|_0 = \|B_g\|_0 = \sup\{||B_g x|: x \in H, \|x\| = 1\}\) and \(\|B_g x\|^2 = \sum_{E} |(x,e)g(e)|^2 \leq \|g\|_\infty^2 \|x\|^2\), so that \(\|\theta(g)\|_0 \leq \|g\|_\infty\). We will now show that \(\emptyset\) and \(\theta\) are inverses. For a fixed \(g \in C(E)\), \((\emptyset\theta)(g) = \emptyset(\theta(g)) = \emptyset(B_g) = f_{B_g}\). If \(e \in E\), then \(f_{B_g}(e) = (B_g e,e) = (\sum_{f \in E} (e,f)g(f)f,e) = ((e,e)g(e)e,e) = g(e)\). Hence \((\emptyset\theta)(g) = g\). Also, for a fixed \(A \in \mathcal{L}(H)\), \((\emptyset\theta)(A) = \theta(\emptyset(A)) = \theta(f_A) = B_{f_A}\). Note that \((Ae,f) = (A||e_0||e)e, f) = ||e_0||((Ae)e, f) = ||e_0||(Ae, fe) = 0\) for \(e \neq f\). Therefore \(B_{f_A}(x) = \sum_{E} (x,e)f_A(e)e = \sum_{E} (x,e)(Ae,e)e\) and \(A(x) = \sum_{E} (Ae,e)e = \sum_{e \in E} (A[\sum_{f \in E} (x,f)f], e)e = \sum_{e \in E} (\sum_{f \in E} (x,f)f)e = \sum_{e \in E} (x,f)(Af, e)e = \sum_{E} (x,e)(Af, e)e\). Hence \(B_{f_A} = A\) and \((\emptyset\theta)(A) = A\). It now follows that each of \(\theta\) and \(\emptyset\) is one-to-one onto their respective spaces. Since both are norm-decreasing, we have that \(\|A\|_0 = \|\emptyset(A)\|_0 \leq \|\theta(A)\|_\infty \leq \|A\|_0\) and thus \(\emptyset\) (and similarly \(\theta\)) is an isometry. We will now show that \(\emptyset\) is an algebra isomorphism and that \(\emptyset(A^*) = \emptyset(A)\) (the natural involution for continuous functions). First,
for $A, B \in \mathcal{F}(H)$, $e \in E$ and $a$ and $b$ complex numbers, we have $\varnothing(aA + bB)(e) = ((aA + bB)(e), e) = a(Ae, e) + b(Be, e) = [a\varnothing(A) + b\varnothing(B)](e)$ and hence $\varnothing$ is linear. Second,

$\varnothing(AB)(e) = ((AB)e, e) = (A(Be), e) = (A||Be||e, e) = ||e||((Be)(Ae), e)$. But $Be = \sum_{E} (Be, f)f = (Be, e)e$, since $(Be, f) = 0$ for $e \neq f$.

Therefore $\varnothing(AB)(e) = ||e||((Be, e)eAe, e) = ||e||((Be, e)(A(ee), e) = (A||e||ee, e)(Be, e) = (Ae, e)(Be, e) = [\varnothing(A)\varnothing(B)](e)$ and hence $\varnothing$ is an algebra isomorphism. Further, $\varnothing(M)(e) = (A^*e, e) = (e, Ae) = (\text{Ae}, e) = \varnothing(A)(e)$, which completes the proof.

Notice that a somewhat shorter proof can be given for Theorem 2.8 by using Theorem 1.2. The preceding proof was chronologically our first and was made without knowledge of Theorem 1.2. We will include here the alternate proof using Theorem 1.2.

**Alternate proof of Theorem 2.8.** By Theorem 1.2, there is a mapping from $\mathcal{F}(H)$ onto $M(H)$, a subset of $C(\mathcal{M}_{H})$. By identifying $\mathcal{M}_{H}$ with $E$, we have $M(H)$ as a subset of $C(E)$. The correspondence between $\mathcal{M}_{H}$ and $E$ gives the Gelfand transform the form $\hat{x}(e) = (x, e)/||e||$, since $e \in E$ has previously been normalized. Notice that the defining equation for the Wang mapping was $\varnothing(A)(h)\hat{x}(h) = \hat{A}x(h)$, for all $x \in H$ and $h \in \mathcal{M}_{H}$. Since $E \subset H$, we have that $\varnothing(A)(e)\hat{x}(e) = \hat{A}e(e)$ which implies that $\varnothing(A)(e)(e, e)/||e|| = (Ae, e)/||e||$ and hence $\varnothing(A)(e) = (Ae, e)$. Therefore the Wang mapping is the same as the mapping constructed in the previous proof. Now, let
Then, just as before, for \( x \in H \), \( z = \sum_{E} (x,e)g(e)e \) is an element of \( H \). We will now show that \( g \in M(H) \). If \( x \in H \), then \( g(e)\hat{x}(e) = g(e)(x,e)/||e_0|| \) and \( \hat{x}(e) = (x,e)g(e)/||e_0|| \). Therefore \( g(e)\hat{x}(e) = \hat{x}(e) \) and thus \( \hat{g} \subset \hat{H} \), so that \( g \in M(H) \). The mapping clearly takes \( A^* \) into the conjugate of the image of \( A \) and thus the only thing remaining is to prove that the Wang mapping \( \varnothing \) is an isometry. Theorem 1.2 gives us that \( ||\varnothing(A)||_\infty \leq ||A||_o \). For \( A \in \mathcal{L}(H) \), we have that \((Ae,f) = 0 \) for \( e \neq f \) and \((Ax,e) = (A[x,f],e) = \sum_{f \in E} (x,f)(Af,e) = (x,e)(Ae,e) \). Therefore \( ||Ax||^2 = ||\sum_{E} (Ax,e)e||^2 = \sum_{E} |(Ax,e)|^2 = \sum_{E} |(x,e)|^2 |(Ae,e)|^2 \leq ||\varnothing(A)||^2_\infty ||x||^2 \) and hence \( ||A||_o \leq ||\varnothing(A)||_\infty \), so that \( \varnothing \) is an isometry. This completes the alternate proof of the theorem.

We will now use the mapping of Theorem 2.8 to characterize the compact operators in \( \mathcal{L}(H) \), where \( H \) is again a commutative \( H^* \)-algebra. The proof will use the following important lemma which gives a necessary and sufficient condition for a projection operator to be in \( \mathcal{L}(H) \).

**Lemma 2.9.1.** Let \( H \) be a commutative \( H^* \)-algebra and \( P \) a projection operator on \( H \). Then \( P \in \mathcal{L}(H) \) if and only if the following three conditions are satisfied:

a). \( H = I_1 \oplus I_2 \), where \( I_1 \) is an ideal and \( I_2 \) is a subalgebra of \( H \) with \( P = P_{I_2} \) (here \( P_{I_2} \) is
the projection onto $I_2$ and $\oplus$ denotes direct sum in the Hilbert space sense).

b). $I_1$ is orthogonal to $I_2$.

c). $I_1I_2 = (0)$.

**Proof.** First, assume that $P \in \mathcal{L}(H)$, let $I_2 = P(H)$, and let $I_1$ be the orthogonal complement of $I_2$ in $H$. By the definition of $I_1$ and $I_2$, b) is satisfied. Let $x \in I_1$ and $y \in H$. Then $P(xy) = (Px)y$ and since $x$ is in the orthogonal complement of the range of $P$, $Px = 0$. Hence, $P(xy) = 0$, $xy \in I_1$ and $I_1$ is an ideal of $H$. Furthermore, if $x, y \in I_2$, then $Px = x$ and $P(xy) = (Px)y = xy$ which implies that $xy \in I_2$. Hence $I_2$ is a subalgebra of $H$ and we have a) satisfied. If $x \in I_1$ and $y \in I_2$, then $xy = x(Py) = P(xy) = 0$ since $x \in I_1$ (an ideal) and $P$ is the projection onto its orthogonal complement. Thus $I_1I_2 = (0)$ and c) is satisfied. Conversely, suppose that the conditions a), b), and c) are satisfied for the projection operator $P$. Let $x, y \in H$. Since $H = I_1 \oplus I_2$, $x = x_1 + x_2$ and $y = y_1 + y_2$ where $x_1, y_1 \in I_1$ and $x_2, y_2 \in I_2$. Since $P = P_{I_2}$, $Px = P(x_1 + x_2) = x_2$ and $Py = P(y_1 + y_2) = y_2$. Condition c) implies that $x_2y_1 = 0 = y_2x_1$. Hence $x(Py) = (x_1 + x_2)y_2 = x_2y_2$, $(Px)y = x_2(y_1 + y_2) = x_2y_2$ and $x(Py) = (Px)y$. Therefore we have $P \in \mathcal{L}(H)$, concluding the proof.

It now seems appropriate to introduce some notation to be used in the following work. By $I_o(H)$, we will denote the set of all compact operators on $H$. We will denote by $C_o(X)$ and $C_\infty(X)$, respectively, the subspaces of
C(X) which are the functions with compact support and the functions which vanish at \( \infty \). Let \( \mathcal{C}_0(H) = \mathcal{C}_0(E) \) and \( \mathcal{C}_0(H) = \mathcal{C}_0(E) \), where \( \mathcal{C}_0(E) \) is the mapping from \( \mathcal{C}(H) \) onto \( \mathcal{M}(H) \) of Theorem 2.8.

**Theorem 2.9.** Let \( H \) be a commutative \( H^* \)-algebra. Then \( \mathcal{C}_0(H) = \mathcal{I}_0(H) \cap \mathcal{C}(H) \), the space of all compact centralizers on \( H \).

**Proof.** If \( A \in \mathcal{C}_0(H) \), then \( \mathcal{C}(A) \in \mathcal{C}_0(E) \) and since \( E \) is discrete we have that \( \mathcal{C}(A) \) is finitely non-zero on \( E \). Let \( \{e_i\}_{i=1}^n \) be the set of points \( e \) in \( E \) such that \( \mathcal{C}(A)(e) \neq 0 \). Then, for \( x \in H \), \( Ax = \sum_{E} (Ax,e)e = \sum_{E} (x,e)(Ae,e)e \) (see for example alternate proof of Theorem 2.8). Hence \( Ax = \sum_{i=1}^n (x,e_i)(Ae_i,e_i)e_i \) and therefore \( A(H) \subset \sum_{i=1}^n \oplus N_i \), where \( e_i \in N_i \), a minimal ideal of \( H \). Since each \( N_i \) is one-dimensional, we have that the range of \( A \) is finite dimensional and hence \( A \in \mathcal{I}_0(H) \). Therefore \( \mathcal{C}_0(H) \subset \mathcal{I}_0(H) \cap \mathcal{C}(H) \) and since each of \( \mathcal{I}_0(H) \) and \( \mathcal{C}(H) \) is closed relative to the operator norm, we have that the operator norm closure of \( \mathcal{C}_0(H) \) is contained in \( \mathcal{I}_0(H) \cap \mathcal{C}(H) \). Since \( \mathcal{C}_0(E) \) is supremum norm dense in \( \mathcal{C}_0(E) \) and \( \mathcal{C}_0(E) \) is an isometry we see that \( \mathcal{C}_0(H) \) is operator norm dense in \( \mathcal{C}_0(E) \), and \( \mathcal{C}_0(H) \subset \mathcal{I}_0(H) \cap \mathcal{C}(H) \). It is known, (cf. [11;250]), that a bounded self-adjoint operator \( A \) is compact if and only if it has the form \( A = \sum_{k=1}^{\infty} a_k P_k \) where the \( P_k \) are...
mutually orthogonal projection operators onto finite dimensional subspaces and \( a_k \) is a sequence of real numbers which tend to zero as \( k \) approaches \( \infty \). It should also be noted that \( P_k = P(a_k) \) where \( P(a) \) is the operator-valued function obtained from the Spectral Theorem. It is also known, [11;448], that if \( A \) is a bounded self-adjoint operator with spectral function \( P(a) \), then \( A \) is in a \( W^* \)-algebra \( M \) if and only if all the operators \( P(a) \) for \( a < 0 \) and \( I - P(a) \) for \( a > 0 \) belong to \( M \). Using these facts, we will now prove the other containment of the theorem. Let \( A \in I_o(H) \cap \mathcal{L}(H) \), \( B = (A + A^*)/2 \) and \( C = (A - A^*)/2i \). Then, since \( A \in I_o(H) \cap \mathcal{L}(H) \), it follows that \( B, C \in I_o(H) \cap \mathcal{L}(H) \), \( B = B^* \) and \( C = C^* \). Hence \( B \) is a bounded self-adjoint operator which belongs to the \( W^* \)-algebra \( \mathcal{L}(H) \). By the remark above, \( P(a) \in \mathcal{L}(H) \) for \( a < 0 \) and \( I - P(a) \in \mathcal{L}(H) \) for \( a > 0 \) where \( P(a) \) is the spectral function of \( B \). Also, since \( I \in \mathcal{L}(H) \), we have that \( I - (I - P(a)) = P(a) \in \mathcal{L}(H) \) for \( a \geq 0 \) and thus \( P(a) \in \mathcal{L}(H) \) for all \( a \) real. Further, \( B \in I_o(H) \) implies that \( B = \sum_{k=1}^{\infty} a_k P_k \) where \( P_k = P(a_k) \), each \( P_k \) is a projection onto a finite dimensional subspace, and \( a_k \to 0 \) as \( k \to \infty \). Hence, we have that \( B = \sum_{k=1}^{\infty} a_k P_k \) where \( P_k \in \mathcal{L}(H) \). We will now show that the function on \( E \), which maps \( e \) to \((P_k e, e)\), is finitely non-zero for each \( k \). Let \( e \in E \) such that \( P_k e \neq 0 \). By Lemma 2.9.1, we know that \( H = I_1 \oplus I_2 \) where \( P_k = P_{I_2} \), \( I_1 \) and \( I_2 \) are orthogonal, \( I_1 \) is an ideal and \( I_2 \) is a subalgebra.
Since $e \in E \subset H$, there exists $e_1 \in I_1$ and $e_2 \in I_2$ such that $e = e_1 + e_2$. Since $e \in E$, there is an irreducible self-adjoint idempotent, $f$, such that $e = f/||e_0||$.

Therefore, $||e_0||(e_1 + e_2) = ||e_0||e = f = ff = ||e_0||^2(e_1 + e_2)(e_1 + e_2) = ||e_0||^2(e_1e_1 + e_2e_2)$. Since $e_1e_1 \in I_1$ and $e_2e_2 \in I_2$, we have that $e_1 = ||e_0||e_1e_1$ and $e_2 = ||e_0||e_2e_2$. It follows that $e_1$ and $e_2$ are self-adjoint and thus $||e_0||e_1$ and $||e_0||e_2$ are self-adjoint idempotents and $(||e_0||e_1)(||e_0||e_2) = 0$. However, $f = ||e_0||e_1 + ||e_0||e_2$ and since $f$ is irreducible, $||e_0||e_1 = 0$ or $||e_0||e_2 = 0$ and therefore $e_1 = 0$ or $e_2 = 0$. It was assumed that $P_k e \neq 0$ and $P_k e = e_2$, so that $e_1 = 0$. Therefore $e = e_2 \in I_2$ and \( \{e \in E : P_k e \neq 0\} \subset P_k(H) \) which is finite dimensional.

Therefore $P_k e$ is finitely non-zero and hence $\varnothing(P_k)(e) = (P_k e,e)$ is finitely non-zero. This gives us that $\varnothing(P_k) \in C_0(E)$ or that $P_k \in \mathcal{L}_0(H)$ for each $k$. Hence $B = \sum_{k=1}^{\infty} a_k P_k$ is an element of the operator closure of $\mathcal{L}_0(H)$ which is $\mathcal{L}_\infty(H)$. A similar argument shows that $C$ is also a member of $\mathcal{L}_\infty(H)$. Since $C_\infty(E)$ is a subalgebra of $C(E)$ and $\varnothing$ is an isomorphism, we have that $A = B + iC \in \mathcal{L}_\infty(H)$.

Therefore $I_0(H) \cap \mathcal{L}(H) \subset \mathcal{L}_\infty(H)$ and $\mathcal{L}_\infty(H) = I_0(H) \cap \mathcal{L}(H)$. This concludes the theorem.

It was noted in the proof of Corollary 2.4.1 that $\mathcal{L}(H)$ is both weak and strong operator dense in $R(H)$. In the case that $H$ is commutative, it is easy to see that the operator norm closure of $\mathcal{L}(H)$ is properly contained in $R(H) = \mathcal{L}(H)$ if $H$ is not finite dimensional. It is
only necessary to notice that the mapping, $\mathcal{L}(H)$, of Theorem 2.8 restricted to $\mathcal{L}(H)$ yields the Gelfand transform.

Also, for $x \in H$, the Gelfand transform, $\hat{x}$, is a member of $C_\infty(E)$. Since $C_\infty(E)$ is supremum norm closed in $C(E)$, we have that the supremum norm closure of $\mathcal{L}(H)$ is a subset of $C_\infty(E)$ which is properly contained in $C(E)$ unless $E$ is finite. By tracing back to $\mathcal{L}(H)$ an element of $C(E)$ which is not in $C_\infty(E)$, we can obtain a centralizer which is not in the operator norm closure of $\mathcal{L}(H)$ and so conclude that the operator norm closure of $\mathcal{L}(H)$ is properly contained in $\mathcal{L}(H)$ whenever $E$ is not finite (or equivalently, whenever $H$ is not finite dimensional).

We will now direct our attention to a particular commutative $H^*_*$-algebra, namely $L^2(G)$, the space of square-integrable functions over a compact abelian topological group $G$. If we give $L^2(G)$ the norm, $||f||_2 = (\int |f(x)|^2 dx)^{1/2}$, where $dx$ is Haar measure on $G$, normalized in such a way that the measure of $G$ is unity, then $L^2(G)$ is a Banach space. Define convolution between two functions by $(f \ast g)(x) = \int f(x - y)g(y)dy$ and involution by $f^*(x) = f(-x)$. Under these definitions, $L^2(G)$ is a commutative $H^*_*$-algebra with regular maximal ideal space $\hat{G}$, the group of continuous characters on $G$ (cf. [10;156]).

Denote by $M(G)$, the space of all complex-valued regular Borel measures on $G$. For $\mu, \lambda \in M(G)$ and $B$ a Borel set in $G$, define $(\mu \ast \lambda)(B) = (\mu \times \lambda)(B(2))$ where $B(2) = \{(x,y) \in G \times G : x + y \in B\}$ (see Rudin [14;13]). With total variation for norm and convolution as defined above,
$M(G)$ is a commutative Banach algebra with unit, $[14;14]$. It is possible to imbed $L^2(G)$ in $M(G)$ by corresponding $f$ to $\mu_f$ where $\mu_f(B) = \int_B f(x) \, dx$. Under this imbedding $L^2(G)$ becomes an ideal of $M(G)$ and multiplication is $(\mu \cdot f)(x) = \int f(x-y) \mu(y) \, dy$ for $\mu \in M(G)$ and $f \in L^2(G)$. Hence we can regard the measure $\mu$ in $M(G)$ as an operator on $L^2(G)$ by $L_\mu(f) = \mu \cdot f$. It is clear that if $\mu \in M(G)$, then $L_\mu \in \mathcal{L}(L^2(G))$. Notice that the mapping, $\emptyset$, of Theorem 2.8 restricted to $L_\mu$ is simply the Fourier-Stieltjes transform, $\hat{\mu}$, defined by $\hat{\mu}(\gamma) = \int \overline{\gamma(x)} \mu(x) \, dx$ for each $\gamma \in \hat{G}$, the group of characters of $G$: $\emptyset(L_\mu)(\gamma) = (L_\mu(\gamma), \gamma) = (\mu \cdot \gamma, \gamma) = \int (\mu \cdot \gamma)(x) \overline{\gamma(x)} \, dx = \int \int \gamma(x-y) \overline{\gamma(x)} \mu(y) \, dx = \int \gamma(-y) \mu(y) \, dy = \hat{\mu}(\gamma)$.

In our previous discussion of $H^*$-algebras, we saw that the operator norm closure of $\mathcal{L}(H)$ is properly contained in $\mathcal{L}(H)$. It seems appropriate to ask the same question concerning the left multiplication operators defined by members of $M(G)$. The question is answered by the following theorem.

**Theorem 2.10.** If $G$ is a compact abelian topological group such that $\hat{G}$ has an element of infinite order, then the operator norm closure of the set of operators defined by left multiplication by members of $M(G)$ is properly contained in $\mathcal{L}(L^2(G))$.

**Proof.** By our previous discussion, $\emptyset(L_\mu) = \hat{\mu}$ and hence we may focus our attention on $\hat{M(G)} = \{ \hat{\mu} : \mu \in M(G) \}$. Let $\gamma_0$ denote an element of $\hat{G}$ of infinite order. Define
the function \( h \) from \( \hat{G} \) into the complex numbers by

\[
h(\gamma) = \begin{cases} 
1, & \text{if } \gamma = \gamma_0^n \text{ for some } n \geq 0 \\
0, & \text{otherwise} 
\end{cases}
\]

Denote by \( W(\hat{G}) \) the space of weakly almost periodic functions on \( \hat{G} \); i.e., \( W(\hat{G}) = \{ f \in C(\hat{G}) : \text{the weak closure of } O(f) \text{ is weakly compact} \} \), where \( O(f) = \{ f\gamma : \gamma \in \hat{G} \} \) and \( f\gamma'(\gamma') = f(\gamma'\gamma) \). It is known that \( W(\hat{G}) \) is a supremum norm closed subspace of \( C(\hat{G})(\text{cf.}[2]) \) and it has been shown in [3] that \( M(\hat{G}) \subset W(\hat{G}) \). Since the function \( h \), defined above, is clearly an element of \( M(L^2(G)) = C(\hat{G}) \), it will suffice to show that it is not in \( W(\hat{G}) \). In order to do this, we recall that a necessary and sufficient condition that a function \( f \) be in \( W(\hat{G}) \) is that if \( (\sigma_i)_{i=1}^{\infty} \) and \( (\tau_j)_{j=1}^{\infty} \) are sequences in \( \hat{G} \) such that

\[
\lim_{i \to \infty} \lim_{j \to \infty} f(\sigma_i \tau_j) \text{ and } \lim_{j \to \infty} \lim_{i \to \infty} f(\sigma_i \tau_j)
\]
both exist, then the two iterated limits must be the same ([5;183]). Now, let \( \sigma_i = \gamma_0^i \) for \( i = 1, 2, 3, \ldots \), and let \( \tau_j = \gamma_0^{-j} \) for \( j = 1, 2, 3, \ldots \). Then

\[
\lim_{i \to \infty} \lim_{j \to \infty} h(\sigma_i \tau_j) = 0 \text{ and } \lim_{j \to \infty} \lim_{i \to \infty} h(\sigma_i \tau_j) = 1
\]

and thus \( h \not\in W(\hat{G}) \). Since \( W(\hat{G}) \) is supremum norm closed in \( C(\hat{G}) \), we have that \( h \) is not in the supremum norm closure of \( M(\hat{G}) \). Hence \( \varphi^{-1}h \) is not an element of the operator norm closure of the left multiplication operators defined by \( M(G) \). This completes the proof of the theorem.

It seems natural to ask in what sense does \( L^2(G) \) determine the group \( G \). For example, it is known, [14;92],
that if there is an isomorphism from $L^1(G)$ onto $L^1(H)$ with norm less than or equal to one, then $G$ and $H$ are isomorphic. The space $L^2(G)$ is not as closely related to the group structure of $G$, as indicated by the following example.

**Example 2.1.** Let $G$ be the Klein 4-group and $H$ be the cyclic group on four elements, each endowed with the discrete topology. Then, $G$ and $H$ are not isomorphic but $L^2(G)$ is isometric $^*$-algebra isomorphic to $L^2(H)$.

**Proof.** We first give the multiplication tables for $G$ and $H.$

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A measure $\mu \in M(G)$ or $M(H)$ can be regarded as $\mu = (a,b,c,d)$, where $a$, $b$, $c$ and $d$ are complex numbers. That is, the measure $\mu$ is determined by attaching weights $a$, $b$, $c$ and $d$ to the points 0, 1, 2, and 3 respectively.

Convolution in $M(G)$ and $M(H)$, respectively, is easily seen to be $(a,b,c,d)(x,y,z,w) = (ax + by + cz + dw$, $ay + bx + cw + dz, az + bw + cx + dy, aw + bz + cy + dx)$ and $(a,b,c,d)(x,y,z,w) = (ax + bw + cz + dy$, $ay + bx + cw + dz, az + by + cx + dw, aw + bz + cy + dx)$. Involution in $L^2(G)$ is defined by $(a,b,c,d)^* = (\bar{a},\bar{b},\bar{c},\bar{d})$ and in $L^2(H)$ by $(a,b,c,d)^* = (\bar{a},\bar{d},\bar{c},\bar{b})$. The $L^2$-norm in each case has the form $\|(a,b,c,d)\|_2 =$.
\[ 2\sqrt{a^2 + b^2 + c^2 + d^2}. \]

Notice that \( L^2(G) = M(G) \) and \( L^2(H) = M(H) \) and the total variation norm in each of \( M(G) \) and \( M(H) \) is \( \| (a,b,c,d) \| = |a| + |b| + |c| + |d| \).

Consider the following mapping, \( \theta \), from \( L^2(G) \) into \( L^2(H) \) : for \( (a,b,c,d) \in L^2(G) \), let \( \theta(a,b,c,d) = (a, \frac{(c + d) + (c - d)i}{2}, b, \frac{(c + d) + (d - c)i}{2}) \). Then \( \theta \) is clearly linear and preserves the involution. A straightforward verification, which is quite lengthy (and thus will be omitted), will show that \( \theta \) is a homomorphism. The proof that \( \theta \) is an isometry depends only on the fact that
\[
|x - y|^2 + |x + y|^2 = 2(|x|^2 + |y|^2)
\]
for \( x \) and \( y \) complex numbers. Hence, it follows that \( L^2(G) \) and \( L^2(H) \) are isometric \( * \)-algebra isomorphic while \( G \) and \( H \) are not isomorphic.

This example and the proof of Theorem 2.8 provide the motivation and method of proof for the following theorem.

**Theorem 2.11.** Let \( H_1 \) and \( H_2 \) be commutative \( H^* \)-algebras. Then there is a mapping, \( \theta \), from \( H_1 \) onto \( H_2 \) which is a \( * \)-algebra isomorphism and a topological mapping if and only if there is a one-to-one correspondence between \( E_1 \) and \( E_2 \) (i.e. if and only if \( H_1 \) and \( H_2 \) have the same dimension, as Hilbert spaces).

**Proof.** We will take \( E_i, i = 1,2 \), as the set of irreducible (non-normalized) self-adjoint idempotents, in contrast to the normalization employed in Theorem 2.8. Denote the elements of \( E_1 \) by \( e_a \) and the elements of \( E_2 \) by \( f_b \). Let \( E_1 \) and \( E_2 \) be in one-to-one correspondence.
and denote the element of $E_2$ corresponding to $e_a \in E_1$ by $f_a$. We can assume that they both have the same set of indices. Now, for $x \in H_1$, we have that $x = \sum_a(x,e_a)e_a/\|e_0\|^2$, where $\|e_0\|$ is the constant norm of each $e_a \in E_1$. Define $\theta$ from $H_1$ into $H_2$ by $\theta(x) = \sum_a(x,e_a)f_a/\|e_0\|^2$. Since $\sum_a |(x,e_a)e_d/\|e_0\|^2|^2 < \infty$, we have that $\theta(x) \in H_2$. Also, $\theta(cx + dy) = \sum_a (cx + dy,e_a)f_a/\|e_0\|^2 = c\theta(x) + d\theta(y)$ for $c$ and $d$ complex numbers and $x, y \in H_1$. Notice that $(xy,e_a) = (x[\sum_b (y,e_b)e_b/\|e_0\|^2],e_a) = (\sum_b (y,e_b)x_e_b/\|e_0\|^2,e_a) = \sum_b (y,e_b)\|e_0\|^2(x_e_b,e_a) = (y,e_a)(x,e_a)/\|e_0\|^2$ for $x, y \in H_1$ and $e_a \in E_1$. Therefore $\theta(xy) = \sum_a (xy,e_a)f_a/\|e_0\|^2 = \sum_a (x,e_a)/\|e_0\|^2 (y,e_a)/\|e_0\|^2 f_a$ and $\theta(x)\theta(y) = (\sum_a (x,e_a)f_a/\|e_0\|^2)(\sum_b (y,e_b)f_b/\|e_0\|^2) = \sum_a \sum_b (x,e_a)/\|e_0\|^2(y,e_b)/\|e_0\|^2 f_a f_b = \sum_a (x,e_a)/\|e_0\|^2 (y,e_a)/\|e_0\|^2 f_a$ and hence $\theta(xy) = \theta(x)\theta(y)$.

If $z \in H_2$, then $z = \sum_a (z,f_a)f_a/\|f_0\|^2$ and if we let $z' = \sum_a (z,f_a)e_a/\|f_0\|^2$, then $z' \in H_1$, $\theta(z') = \sum_a (z,f_a)f_a/\|f_0\|^2 = z$, and $\theta$ is onto. For $x \in H_1$, $x^* = \sum_a (x^*,e_a)e_a/\|e_0\|^2 = \sum_a (e_a,x)e_a/\|e_0\|^2$ and hence $\theta(x^*) = \sum_a (e_a,x)f_a/\|e_0\|^2 = \sum_a [(x,e_a)f_a]^*/\|e_0\|^2 = [\sum_a (x,e_a)f_a/\|e_0\|^2]^* = \theta(x)^*$. Also, $\|x\|^2 = \sum_a $
\[ \sum_a |(x, e_a)/\|e_0\||^2 \quad \text{and} \quad \|\theta(x)\|^2 = \|\sum_a (x, e_a)f_a/\|e_0\|^2\|^2 = \sum_a \|f_a\|(x, e_a)/\|e_0\|^2 \|^2 = \|f_o\|^2/\|e_0\|^2 \left[ \sum_a |(x, e_a)/\|e_0\|^2 \right] = \|x\|^2/\|f_o\|^2/\|e_0\|^2. \] Hence \[ \|\theta(x)\| = \|x\|\|f_o\|/\|e_0\| \] and the mapping is topological. It then follows that \( \theta \) is a topological \( * \)-algebra isomorphism from \( H_1 \) onto \( H_2 \).

Conversely suppose that we have a mapping, \( \theta \), from \( H_1 \) onto \( H_2 \) with the above properties. We will now show that the restriction of \( \theta \) is a one-to-one correspondence between \( E_1 \) and \( E_2 \). We will first show that \( \theta(e_a) \in E_2 \) for \( e_a \in E_1 \). Since \( e_a \) is a self-adjoint idempotent, it is clear that \( \theta(e_a) \) is a self-adjoint idempotent.

Suppose now that \( \theta(e_a) = x + y \), where \( xx = x \), \( yy = y \) and \( xy = 0 \). Since \( \theta \) is an isomorphism, \( \theta^{-1} \) is also an isomorphism and thus \( e_a = \theta^{-1}(x) + \theta^{-1}(y) \) where

\[ \theta^{-1}(x)\theta^{-1}(x) = \theta^{-1}(x), \quad \theta^{-1}(y)\theta^{-1}(y) = \theta^{-1}(y) \text{ and} \quad \theta^{-1}(x)\theta^{-1}(y) = 0. \]

Therefore, since \( e_a \) is irreducible, we have that \( \theta^{-1}(x) = 0 \) or \( \theta^{-1}(y) = 0 \) and thus \( x = 0 \) or \( y = 0 \). Hence \( \theta(e_a) \) is an irreducible self-adjoint idempotent. Since \( \theta(e_a) \) is irreducible and \( E_2 \) is a basis for \( H_2 \), it follows that \( \theta(e_a) \) is some member of \( E_2 \), call it \( f_a \). Hence the restriction of \( \theta \) is a one-to-one mapping of \( E_1 \) into \( E_2 \). Applying a dual argument to the isomorphism \( \theta^{-1} \), we can conclude that its restriction is a one-to-one mapping of \( E_2 \) into \( E_1 \) and combining these two facts, we get that the restriction of \( \theta \) is a one-to-one correspondence between \( E_1 \) and \( E_2 \).
Corollary 2.11.1. Let $H_1$ and $H_2$ be commutative $H^*$-algebras. Then $H_1$ is isometric $*$-algebra isomorphic to $H_2$ if and only if the cardinality of $E_1$ is the same as the cardinality of $E_2$ and the irreducible self-adjoint idempotents of $H_1$ and $H_2$ have the same norm.

Proof. Just as in Theorem 2.11, define $\theta$ by $\theta(x) = \sum_a (x, e_a)f_a/\|e_0\|^2$, for $x \in H_1$. Then $\theta$ has all of the properties of the mapping in Theorem 2.11. The additional assumption on the norms of the irreducible self-adjoint idempotents, namely $\|e_0\| = \|f_0\|$, gives us that $\|\theta(x)\| = \|x\|\|f_0\|/\|e_0\| = \|x\|$ and thus $\theta$ is an isometry. The converse is the same as Theorem 2.11 and the isometry of the mapping yields that the irreducible self-adjoint idempotents of $H_1$ and $H_2$ have the same norm.

Theorem 2.12. Let $H$ be a commutative $H^*$-algebra. Then there exists a compact abelian topological group $G$ and a mapping $\theta$ from $H$ onto $L^2(G)$ which is a topological $*$-algebra isomorphism. Further, for the same group $G$, there is a mapping $\theta'$ from $H$ onto $L^2(G)$ which is a linear isomorphism, an isometry, and preserves involution.

Proof. Let $E_d$ denote $E$ with the discrete topology and any abelian group structure. Note that it is always possible to introduce on $E$ an operation which makes $E$ into an abelian group by imbedding $E$ in the direct sum (weak direct product) of the integers modulo two, where the index set ranges over $E$. Let $G = \hat{E_d}$, the group of continuous characters on $E_d$. Then $G$ is a compact abelian
topological group with $\hat{G} = E^a$, [12;134]. Furthermore, $L^2(G)$ is a commutative $H^*-\text{algebra}$ with regular maximal ideal space $\hat{G}$. There is a one-to-one correspondence between $\hat{G}$ and $E$ which are the respective sets of irreducible self-adjoint idempotents for $L^2(G)$ and $H$. Therefore, by Theorem 2.11, we have a mapping $\theta$ from $H$ onto $L^2(G)$ with the desired properties. If we define $\theta'$ from $H$ into $L^2(G)$ by $\theta'(x) = \sum_a (x, e_a) f_a / ||e_0||$, where $f_a$ denotes the character of $G$ corresponding to $e_a$, then $\theta'$ will have the linear and involution properties of $\theta$.

Also, since $||f_a||_2 = 1$, we have $||\theta'(x)||_2 = ||e_0|| \theta(x) = \|e_0\| ||\theta'(x)||_2 = \|e_0\| \theta(x) = (||e_0|| ||f_o||_2 / ||e_0||) ||x|| = ||x||$, and hence $\theta'$ is an isometry. Since $\theta'$ is trivially onto, we have the linear isomorphism which is an isometry and preserves involution from $H$ onto $L^2(G)$.

**Theorem 2.13.** Let $G$ and $H$ be compact abelian topological groups. Then $L^2(G)$ is isometric and $*-$algebra isomorphic to $L^2(H)$ if and only if there is a one-to-one correspondence between $\hat{G}$ and $\hat{H}$, the respective dual groups of $G$ and $H$.

**Proof.** Since $\hat{G}$ and $\hat{H}$ are the regular maximal ideal spaces of $L^2(G)$ and $L^2(H)$, respectively, a one-to-one correspondence between them yields, via Theorem 2.11, a mapping, $\theta$, which is a $*-$algebra isomorphism onto. Also, since the irreducible self-adjoint idempotents in each of $L^2(G)$ and $L^2(H)$ have norm one, we see, by Corollary 2.11.1, that $\theta$ is actually an isometry. The
converse follows directly from Theorem 2.11, which concludes the proof of the theorem.

We include one more application of Theorem 2.8 in order to obtain a characterization of the bounded complex-valued functions on a set.

**Theorem 2.14.** Let $A$ be a commutative $B^*$-algebra with unit. Then a necessary and sufficient condition that there exist a set $X$ such that $A$ "is" the set of bounded complex-valued functions on $X$ is that there exist a compact abelian topological group $G$ such that $A$ "is" the algebra of centralizers on $L^2(G)$, where "is" means isometric $^*$-algebra isomorphic.

**Proof.** First, assume that there is a set $X$ such $A$ is the set of all bounded complex-valued functions on $X$. Then, denote by $X_d$, the set $X$ with the discrete topology and any abelian group structure. If we let $G = \hat{X}_d$, then $G$ is a compact abelian topological group with dual group $X_d$. By Theorem 2.8, the algebra of centralizers of $L^2(G)$ is precisely the set of all bounded continuous complex-valued functions on $\hat{G} = X_d$ which is the same as the set of all bounded complex-valued functions on $X$. Conversely, suppose that there exists a compact abelian topological group $G$ such that $A$ is the algebra of centralizers on $L^2(G)$. By Theorem 2.8, we have that $\mathcal{Z}(L^2(G)) = C(\hat{G})$ and hence $A$ is the set of all bounded complex-valued functions on the set $\hat{G}$. 
CHAPTER III

This chapter will be devoted to a study of the centralizers of the Banach algebras $L^p(G)$, for $1 \leq p < \infty$, where $G$ is a compact abelian topological group. For $p$ distinct from two, the algebras of multipliers (see Definition 1.5) are somewhat easier to work with than the corresponding algebras of centralizers. The alternate proof of Theorem 2.8 shows that $M(L^2(G)) = C(\hat{G})$ and it is shown in [7] that $M(L^1(G)) = \hat{M}(G) = \{ \hat{\mu} : \mu \in M(G) \}$, where $\hat{\mu}$ is the Fourier-Stieltjes transform of the complex-valued, regular Borel measure $\mu$.

We will use, without proof, the fact that the regular maximal ideal space of $L^p(G)$, for $1 \leq p < \infty$, is (homeomorphic to) $\hat{G}$, the group of continuous characters on $G$ whenever $G$ is a compact abelian topological group. The reason we make the restriction $p < \infty$ is that $L^\infty(G)$ does not have this property.

It should be noted that the Wang mapping from $\mathcal{C}(X)$ onto $M(X)$ has the form $\theta(A)(\gamma) = \hat{A}\gamma(\gamma)$, for $\gamma \in \hat{G}$, when applied to $X = L^p(G)$. Since the Wang mapping will, in general, depend on $p$, we will denote it here by $\theta_p$. Since $L^p(G)$ is a convolution ideal in $M(G)$, each $L\mu$, for $\mu \in M(G)$, is an element of $\mathcal{C}(L^p(G))$. Also

$\theta_p(L\mu) \in M(L^p(G))$ and $\theta_p(L\mu)(\gamma) = \hat{L\mu}\delta(\gamma) = \hat{\mu} \cdot \delta(\gamma) = \hat{\mu}(\gamma)$. Hence, we have that $\hat{\mu} \in M(L^p(G))$ for

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\[ 1 \leq p < \infty \] and \( \mu \in \mathbb{M}(G) \). We might also note that \( \theta_p \) is, in a very real sense, an extension of the Fourier–Stieltjes transform, since \( \theta_p \) applied to \( L\mu \) is \( \hat{\mu} \). Since \( \mathbb{M}(L^p(G)) \) is a subset of the bounded, continuous, complex-valued functions on the regular maximal ideal space of \( L^p(G) \), we now have the following chain of containments:

\[
\widehat{\mathbb{M}(G)} = \mathbb{M}(L^1(G)) \subset \mathbb{M}(L^p(G)) \subset \mathbb{M}(L^2(G)) = C(\widehat{G}) \text{ for every } p \text{ such that } 1 \leq p < \infty.
\]

The main purpose of this section will be to show that \( \mathbb{M}(L^p(G)) \) is properly between these extremities for a large class of compact abelian topological groups \( G \). Our result will require some rather deep theorems from summability theory (e.g., see [17]) and the general theory of Fourier series on groups.

The following theorem collects together the general operator theoretic properties of \( \mathcal{L}(L^p(G)) \).

**Theorem 3.1.** Let \( G \) be a compact abelian topological group. Then, for \( 1 \leq p < \infty \), \( \mathcal{L}(L^p(G)) \) is a commutative Banach \(*\)-algebra with unit. \( \mathcal{L}(L^p(G)) \) is also a reduced algebra in the sense that if \( A \in \mathcal{L}(L^p(G)) \) and if \( F(A^*A) = 0 \) for every positive functional \( F \), then \( A = 0 \).

**Proof.** By Theorem 1.1, \( \mathcal{L}(L^p(G)) \) is a commutative Banach subalgebra of \( B(L^p(G)) \) and contains the identity operator as its unit. For \( f \in L^p(G) \), we define involution by \( f^*(x) = \overline{f(-x)} \). Then \( ** \) is a mapping from \( L^p(G) \) into \( L^p(G) \) with the following properties:

a). \( (f \cdot g)^* = f^* \cdot g^* = g^* \cdot f^* \)

b). \( f^{**} = f, (af + bg)^* = \overline{af} + \overline{bg}^* \)
c). \( \hat{f}^* = \overline{f} \)

for all \( f, g \in L^p(G) \) and \( a \) and \( b \) complex numbers. Since

\[
(f \cdot g)^*(x) = \overline{(f \cdot g)(-x)} = \int f(-x-y)g(y)dy = \int f(-x+y)\overline{g(-y)}dy = \int f^*(x-y)\overline{g^*(y)}dy = (f^* \cdot g^*)(x),
\]

property a) is satisfied. Property b) is clearly satisfied and \( \hat{f}^*(\zeta) = \int f^*(x)\overline{\zeta(x)}dx = \int f(-x)\overline{\zeta(x)}dx = \int f(x)\overline{\zeta(-x)}dx = \int f(x)\overline{\zeta(x)}dx = \hat{f}(\zeta) \) shows that property c) is satisfied. Now define \( '\cdot'^* \) on \( \mathcal{S}(L^p(G)) \) by \( A \cdot f = (A^*f)^* \). We will now show that \( '\cdot'^* \) is an involution for \( \mathcal{S}(L^p(G)) \). The mapping \( A^* \) clearly maps \( L^p(G) \) into \( L^p(G) \) and, for \( f, g \in L^p(G) \), \( f \cdot (A^*g) = f^*(A^*g)^* = [f^*(A^*)^* = [(A^*f)^*]^* = (A^{**}f)^* = (A^{**}f)^* = (A^*)^*g = (A^*f)g \) implies that \( A^* \in \mathcal{S}(L^p(G)) \). Also, for \( A, B \in \mathcal{S}(L^p(G)) \), \( a \) and \( b \) complex numbers and \( f \in L^p(G) \), we have \( (aA + bB)^*(f) = [(aA + bB)(f^*)]^* = [aA^* + bB^*]^* = \overline{a(A^*f)^*} + B(B^*f)^* = (\overline{a} A^* + B B^*)(f) \), \( A^{**}f = (A^*)^* \) and \( (AB)^*f = [(AB)^*(f^*)] = [A(B^*f)^* = A^*(B^*f)] = (A^*B^*)f \). Hence \( '\cdot'^* \) is an involution for \( \mathcal{S}(L^p(G)) \) which also satisfies \( \|A\|_0 = \|A^*\|_0 \) since \( \|A^*\|_0 = \sup\{|A^*f|^p : \|f\|^p \leq 1\} = \sup\{|(A^*)^*|^p : \|f^*\|^p \leq 1\} = \sup\{|A^*f|^p : \|f^*\|^p \leq 1\} = \|A\|_0 \). Now let \( A \in \mathcal{S}(L^p(G)) \) be such that \( A \neq 0 \). Since the linear span of \( \hat{G} \) is dense in \( L^p(G) \), [14;24], and \( A \) is a bounded operator, we have that there is \( \eta_0 \in \hat{G} \) such that \( A \eta_0 \neq 0 \). Define \( F \) on \( \mathcal{S}(L^p(G)) \) into the complex numbers by \( F(B) = \overline{B^*}(\eta_0) \). The correspondence \( F \) is obviously a bounded linear functional and \( F(B^*B) = B^*B(\eta_0 \eta_0) = B^*B(\overline{\eta_0 \cdot \eta_0})(\eta_0) = \overline{B^*B}(\eta_0 \eta_0) \).
positive functional. Furthermore, since \( A \gamma_y = A(\gamma_y \cdot \gamma_y) \cdot \gamma_y = A \gamma_y (\gamma_y) \gamma_y \) and \( A \gamma_y \neq 0 \), we have that \( |A \gamma_y (\gamma_y)|^2 > 0 \) and thus \( F(A \star A) \neq 0 \). Hence \( \mathcal{L}(L^p(G)) \) is a reduced algebra, concluding the theorem.

We will need the following lemma as a computational device in some of the theorems which follow.

**Lemma 3.2.1.** Suppose that \( G \) is a compact abelian topological group and \( 1 < p < \infty \). If \( A \in \mathcal{L}(L^p(G)) \), then \( A(f \gamma) = (Af) \gamma \) for each \( f \in L^p(G) \) and \( \gamma \in G \), where by \( g_\gamma \) we mean the function \( g_\gamma(x) = g(x - \gamma) \).

**Proof.** Since \( L \mu \in \mathcal{L}(L^p(G)) \) for each \( \mu \in M(G) \) and \( \mathcal{L}(L^p(G)) \) is a commutative algebra, we have that \( L \mu A = AL \mu \). In particular, we have \( L \gamma A = AL \gamma \), where \( \gamma \) is the point measure concentrated at \( \gamma \in G \). Also, if \( g \in L^p(G) \), then \( (L_\gamma g)(x) = (\gamma \cdot g)(x) = \int g(x - t)d\gamma(t) = g(x - \gamma) = g_\gamma(x) \). Combining these results, we see that \( A(f \gamma) = (AL_\gamma f) = (L_\gamma A)f = (Af) \gamma \) and the lemma is proved.

The following theorem (besides being of independent interest) is helpful in dealing with the multiplier algebras of \( L^p(G) \), for \( 1 < p < \infty \).

**Theorem 3.2.** Let \( G \) be a compact abelian topological group, with \( p \) and \( q \) two numbers satisfying \( 1 < p < \infty \), \( 1 < q < \infty \), and \( 1/p + 1/q = 1 \). Then \( M(L^p(G)) = M(L^q(G)) \).

**Proof.** For \( f \) a fixed element of \( M(L^p(G)) \), by Theorem 1.2, there is an operator \( A \) in \( \mathcal{L}(L^p(G)) \) such that \( \hat{A}h = f \hat{h} \) for all \( h \in L^p(G) \). Since \( A \) is a bounded
A linear operator on $L^p(G)$, it defines an adjoint operator $A'$ on $L^q(G)$, the adjoint space of $L^p(G)$, by the formula 

$$(A'g)(k) = \int Ak(x)\overline{g(x)}dx,$$

for each $g \in L^q(G)$ and $k \in L^p(G)$. We will now show that $A' \in \mathcal{L}(L^q(G))$. If $g, h \in L^q(G)$ and $k \in L^p(G)$, then 

$$[A'(g*h)](k) = \int Ak(x)\overline{g(x)h(x)}dx = \int \int Ak(x)\overline{g(x-y)h(y)}dydx. $$

On the other hand, 

$$[(A'g)*h](k) = \int k(x)[(A'g)*h](x)dx = \int \int h(y)[\int k(x)A'g(x-y)dx]dy = \int \int h(y)[\int (Ak)(x)g(x-y)dx]dy = \int \int h(y)\overline{g(x-y)}dx]dy,$$

Hence the action of $(A'g)*h$ and $A'(g*h)$ is the same on $L^p(G)$ and thus they must be the same as elements of $L^q(G)$. Therefore $A' \in \mathcal{L}(L^q(G))$ and, by applying Theorem 1.2 again, we get a function $f' \in M(L^q(G))$ such that $\widehat{A^q} = f'\widehat{g}$ for each $g \in L^q(G)$. Also, $f' \in M(L^q(G))$ implies that $f' \in M(L^q(G))$ and we will show that $f$ and $f'$ are the same function. For $\gamma \in \hat{G}$, we have that $\gamma \in L^p(G) \cap L^q(G)$ and since $\hat{A}h = fh$ for each $h \in L^p(G)$, we have $\hat{A}\gamma = f\overline{\gamma}$ and $\hat{A}\gamma (\gamma) = f(\gamma)$. Similarly, $\hat{A'}\gamma (\gamma) = \overline{f'(\gamma)}$ for each $\gamma \in \hat{G}$. Furthermore, 

$$\widehat{A'}\gamma (\gamma) = \int A' \gamma (x)\overline{\gamma(x)}dx = \int \gamma(x)A'\overline{\gamma(x)}dx = (A'\gamma)(\gamma) = \int A\gamma (x)\overline{\gamma(x)}dx = \overline{A\gamma (\gamma)}.$$ 

Therefore $f(\gamma) = \widehat{A}\gamma (\gamma) = \overline{f'(\gamma)}$, $f = \overline{f'}$, $f \in M(L^q(G))$, and $M(L^p(G)) \subset M(L^q(G))$. A dual argument gives the other containment and hence $M(L^p(G)) = M(L^q(G))$. 

Corollary 3.2.1. Let $G$ be a compact abelian topological group, with $p$ and $q$ two numbers satisfying $1 < p < \infty$, $1 < q < \infty$, and $1/p + 1/q = 1$. Then $\mathcal{L}(L^p(G))$ is isometric $*$-algebra isomorphic to $\mathcal{L}(L^q(G))$.

Proof. In the proof of Theorem 3.2, we had that if $A \in \mathcal{L}(L^p(G))$ and if $A'$ is the adjoint operator defined by $A$, then $A^\vee (\gamma) = A'^\vee (\gamma)$ and hence $A^\vee (\gamma) = A'^\vee (\gamma)$. If we consider the Wang mappings, $\theta_p$ and $\theta_q$, we see that $\theta_p(A)(\gamma) = A^\vee (\gamma) = A'^\vee (\gamma) = \theta_q(A'^\vee)(\gamma)$. Therefore the mapping which sends $A$ into $\theta_p^{-1}(A) = A'^\vee$ is an algebra isomorphism from $\mathcal{L}(L^p(G))$ onto $\mathcal{L}(L^q(G))$. Also, $\theta_p(A^*) (\gamma) = A'^\vee (\gamma) = A'^\vee (\gamma) = \theta_q(A'^\vee)(\gamma)$, and hence $\theta_p^{-1}(A^*) = A' = [\theta_p^{-1}(A)]^*$. Furthermore, $\|A'^\vee\|_0 = \|A'\|_0 = \|A\|_q$ which gives us the isometry. Hence $\theta_p^{-1}$ is an isometry $*$-algebra isomorphism from $\mathcal{L}(L^p(G))$ onto $\mathcal{L}(L^q(G))$ and the proof is complete.

The next theorem will essentially follow Section 2.7.3 of [14,54]. We are able to obtain the analogous results for $L^p(G)$ relatively easy in view of the compactness of $G$. This theorem will enable us to extend some results for the circle group to a considerably larger class of compact abelian topological groups. We will use the notation $H^\perp$ to denote the annihilator of the closed subgroup $H$ of $G$; i.e., $H^\perp = \{ \gamma \in \hat{G} : \gamma(x) = 1 \text{ for all } x \in H\}$.

Theorem 3.3. Let $G$ be a compact abelian topological group and $H$ a closed subgroup of $G$ and $1 \leq p < \infty$. Then the functions belonging to $\overline{L^p(G/H)}$ are precisely the restrictions to $H^\perp$ of the functions belonging to $\overline{L^p(G)}$. 
Proof. Denote by $m_G$, $m_H$, and $m_{G/H}$ the Haar measures of $G$, $H$, and $G/H$ respectively, all normalized to have total variation one. Let $N$ denote the natural map from $G$ onto $G/H$ where $N(x) = x + H$. For $f \in L^p(G)$, the integral $\int_H f(x+y)dm_H(y)$ is not changed if we replace $x$ by $x + h$, where $h \in H$. Hence we can regard the correspondence $x + H \mapsto \int_H f(x+y)dm_H(y)$ as a function on $G/H$. It follows from the Fubini Theorem, [6;148], that this function is an element of $L^p(G/H)$ whose norm is less than or equal $\|f\|_p$. Further, if $f \in C(G)$, then it is in $C(G/H)$ and if we let $L(f) = \int_{G/H} \left[ \int_H f(x+y)dm_H(y) \right] dm_{G/H}(\xi)$, where $\xi$ is the coset of $H$ containing $x$, then $L$ is a positive translation invariant linear functional on $C(G)$ in the sense that $L(f_x) = L(f)$ for each $x \in G$. The uniqueness of Haar measure then gives us that $\int_G f(x)dm_G(x) = \int_G \left[ \int_H f(x+y)dm_H(y) \right] dm_G(x)$ for all $f \in C(G)$.

Let $F(\xi) = \int_H f(x+y)dm_H(y)$ for each $f \in L^p(G)$ and define $T$ on $L^p(G)$ by $Tf = F$. From our remarks above, $T$ is a bounded linear transformation from $L^p(G)$ into $L^p(G/H)$. Let $F \in C(G/H)$ and define $f$ on $G$ by $f(x) = F(N(x))$. Since $F$ and $N$ are continuous, $f$ is a continuous function on $G$. Furthermore, $\|f\|_p^p = \int_G |f(x)|^p dm_G(x) = \int_G |F(N(x))|^p dm_G(x) = \int_{G/H} \left[ \int_H |F(N(x+y))|^p dm_H(y) \right] dm_{G/H}(\xi) =$
\[
\int_{G/H} \left[ \int_{H} |F(N(x))| P_{d\mu_H(y)} \right] d\mu_{G/H}(\xi) = \int_{G/H} |F(N(x))| P_{d\mu_{G/H}(\xi)}
\]

= \|F\|_P, and Tf = F. So, for F \in L^P(G/H), there is a sequence \( \{F_n\} \) contained in \( C(G/H) \) such that

\[ \|F_n - F\|_P \to 0. \]

Further, \( F_n \in C(G/H) \) implies that there exists \( f_n \in C(G) \) such that \( Tf_n = F_n \) and \( \|f_n\|_P = \|F_n\|_P \).

Since \( \|F_n - F\|_P \to 0 \), \( \{F_n\} \) is a p-Cauchy sequence and hence \( \{f_n\} \) is also a p-Cauchy sequence. Therefore, there exists \( f \in L^P(G) \) such that \( \|f_n - f\|_P \to 0. \) Since \( T \) is bounded, \( \|Tf_n - Tf\|_P \to 0 \) and \( Tf_n = F_n \) implies that \( Tf = F. \)

Hence the mapping \( T \) is onto \( L^P(G/H) \). Since \( T \) is the restriction of the mapping \( \pi \) of Section 2.7.2 of \([14;53]\) to \( L^P(G) \) imbedded in \( M(G) \), we see that \( \hat{F}(\gamma) = \hat{f}(\gamma) \), for \( \gamma \in \mathbb{H}^1 \). The theorem now follows.

**Theorem 3.4.** If \( G \) is the circle group, then the union of all the sets \( M(L^p(G)) \), for \( p \neq 2, 1 \leq p < \infty \), is properly contained in \( M(L^2(G)) \).

**Proof.** In view of Theorem 3.2, it is sufficient to consider values of \( p \) satisfying \( 1 \leq p < \infty \). Furthermore, since \( M(L^1(G)) \subset M(L^p(G)) \), for \( 1 \leq p \leq \infty \), it is sufficient to consider \( 1 \leq p < 2 \). Define \( f \) on \( G \) by \( f(x) = \sum_{n=1}^{\infty} \left( \frac{2}{\sqrt{n+1}} \right) \cos nx \), for \( 0 \leq x \leq 2\pi \), wherever this series converges. Since \( \frac{2}{\sqrt{n+1}} \to 0 \), we have, by (2.6) of \([17;4]\), that \( \sum_{n=1}^{\infty} \left( \frac{2}{\sqrt{n+1}} \right) \cos nx \) converges uniformly on any interval \( 0 < \varepsilon < x < 2\pi - \varepsilon \), for each \( \varepsilon > 0 \). If \( 1 < p < 2 \), then a direct computation yields
that \(2 - p/2 > 1\) \(and\) \(S_p = \sum_{n=1}^{\infty} (1/\sqrt{n})^p n^{p-2} = \sum_{n=1}^{\infty} (1/n^p/2)(1/n^{2-p}) = \sum_{n=1}^{\infty} 1/n^{2-p}/2\). Since \(2 - p/2 > 1\), \(S_p\) \(is\) a convergent series. Therefore, by Lemma 6.6 \(\[18;129\], \\sum_{n=1}^{\infty} (1/\sqrt{n})\cos nx\) \(is\) an \(L^P(G)\) function \(for\) \(1 < p < 2\). \(In\) a similar manner, we get that \(\sum_{n=1}^{\infty} (1/\sqrt{n})\sin nx\) \(is\) an \(L^P(G)\) function \(for\) \(1 < p < 2\), \(and\) \(we\) can write \(f(x) = \sum_{n=1}^{\infty} (2/\sqrt{n})\cos nx = \sum_{n=1}^{\infty} (1/\sqrt{n})\cos nx + i \sum_{n=1}^{\infty} (1/\sqrt{n})\sin nx + \sum_{n=1}^{\infty} (1/\sqrt{n})\cos nx - i \sum_{n=1}^{\infty} (1/\sqrt{n})\sin nx = \sum_{n=1}^{\infty} (1/\sqrt{n})e^{inx} + \sum_{n=1}^{\infty} (1/\sqrt{n})e^{-inx} = \sum_{n=1}^{\infty} (1/\sqrt{n})e^{inx}\) \(where\) the term \(for\) \(n = 0\) \(is\) omitted. \(Now,\) let \(a_0 = b_n = 0\) \(for all\) \(n > 0\) \(and let\) \(a_n = 1/\sqrt{n}\) \(for\) \(n > 0\). \(Then,\) \(for\) \(the\) sequences \(\{a_n\}_{n=0}^{\infty}\) \(and\) \(\{b_n\}_{n=0}^{\infty}\) \(define\) above, \(the\) series \(a_0^2/4 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = \sum_{n=1}^{\infty} 1/n\) \(is\) divergent. Therefore, by a result of Zygmund \([17;215]\), almost all of the series \(\pm a_0/2 + \sum_{n=1}^{\infty} \pm (a_n \cos nx + b_n \sin nx) = \sum_{n=1}^{\infty} \pm (1/\sqrt{n})\cos nx\) \(fail\) \(to\) be Fourier series. \(To\) be precise, \(if\) we let \(\phi_n(t) = \text{sign} [\sin(2n+1\pi t)], for 0 \leq t \leq 1, t \neq p/2^q, then there is a set \(E\) contained \(in\) \([0,1]\) \(with\) Lebesgue measure \(one\) \(such\) \(that\) if \(t \in E, then\)
\[ \sum_{n=1}^{\infty} (\phi_n(t)/\sqrt{n}) \cos nx \] is not a Fourier series. A series
\[ \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx, \]
is said to be a Fourier series if there is an integrable function with Fourier coefficients \( \{a_n\}_{n=1}^{\infty} \) and \( \{b_n\}_{n=1}^{\infty} \). We might note that the functions \( \phi_n \) are the well-known Rademacher functions, [17; 6]. Choose any such \( t \in \mathbb{E} \), and define \( g \) from \( \hat{\mathbb{G}} \) (in this case, the integers) into the complex numbers by \( g(n) = \begin{cases} \phi_n(t) & \text{for } n > 0 \\ 0 & \text{for } n \leq 0 \end{cases} \). We will show \( g \) is not an element of \( M(L^p(\mathbb{G})) \) for any \( p \) such that \( 1 < p < 2 \). Notice that \( g \in M(L^2(\mathbb{G})) \) since it has values only 1 and \(-1\) and hence is a bounded (continuous) function on \( \hat{\mathbb{G}} \) which implies that \( g \in M(L^2(\mathbb{G})) \) (see Theorem 2.8). Now suppose there is a \( p \) such that \( 1 < p < 2 \) and \( g \in M(L^p(\mathbb{G})) \). Then, by definition, \( gL^p(\mathbb{G}) \subseteq \hat{L}^p(\mathbb{G}) \) and since \( f(x) = \sum_{n=1}^{\infty} (2/\sqrt{n}) \cos nx \) is a member of \( L^p(\mathbb{G}) \), we must have that \( g f \in \hat{L}^p(\mathbb{G}) \). That is, there exists \( h \in L^p(\mathbb{G}) \) with the property that \( g(n)\hat{f}(n) = \hat{h}(n) \) for all \( n \in \hat{\mathbb{G}} \). Hence \( \hat{h}(n) = \begin{cases} \phi_n(t)/\sqrt{n} & \text{for } n > 0 \\ 0 & \text{for } n \leq 0 \end{cases} \). Since \( h \in L^p(\mathbb{G}) \), it has a Fourier series given by \( \sum_{n=-\infty}^{\infty} \hat{h}(n)e^{inx} \) or
\[ \sum_{n=1}^{\infty} (\phi_n(t)/\sqrt{n})e^{inx}. \] Also, the real part of \( h \) is in \( L^p(\mathbb{G}) \) and has a Fourier series given by
\[ \sum_{n=1}^{\infty} (\phi_n(t)/\sqrt{n})\cos nx. \] However, we have noted earlier
that \( \sum_{n=1}^{\infty} (\phi_n(t)/\sqrt{n}) \cos nx \) is not a Fourier series.

Thus, we have arrived at a contradiction and we can conclude that \( g \notin M(L^p(G)) \), for \( 1 < p < 2 \). In view of our remarks at the first of the proof, we now have that \( g \) is not an element of \( M(L^p(G)) \) for all \( p \) distinct from two. Therefore, we have that the union of all the sets \( M(L^p(G)) \), for \( 1 \leq p < \infty, p \neq 2 \), is properly contained in \( M(L^2(G)) \), concluding the theorem.

Theorem 3.5 will combine the results of Theorem 3.3 and Theorem 3.4 in order to obtain the same conclusion as Theorem 3.4 for any compact abelian topological group which has an element of infinite order in its dual group.

**Theorem 3.5.** Let \( G \) be a compact abelian topological group such that \( \hat{G} \) has an element of infinite order. Then the union of all the sets \( M(L^p(G)) \), for \( p \neq 2, 1 \leq p < \infty \), is properly contained in \( M(L^2(G)) \).

**Proof.** Suppose that \( \gamma_0 \) is an element of \( \hat{G} \) with infinite order. Let \( \Gamma \) be the subgroup of \( G \) generated by \( \gamma_0 \). Note that \( \Gamma \) is topologically isomorphic to the integers and hence there exists \( H \), a closed subgroup of \( G \), such that \( G/H \) is topologically isomorphic to \( T \), the circle group, and \( H^\perp = \Gamma \), [14;35]. By Theorem 3.4, there is \( g' \in C(H^\perp) \) such that \( g' \) is not an element of \( M(L^p(G/H)) \) for any \( p \) satisfying \( 1 < p < 2 \). Define \( g \) on \( \hat{G} \) into the complex numbers by \( g(\gamma) = \begin{cases} g'(\gamma) & \text{for } \gamma \in H^\perp \\ 0 & \text{for } \gamma \notin H^\perp \end{cases} \).

We will show that \( g \) fails to be in \( M(L^p(G)) \), for \( 1 < p < 2 \).
In order to do this, suppose there is a \( p \) for which 

\[ 1 < p < 2 \]

and \( g \in \mathcal{M}(L^p(G)) \); thus \( gL^p(G) \subset L^p(G) \). For this \( p \), \( g' \) is not in \( \mathcal{M}(L^p(G/H)) \) which implies that there is a function \( f' \in L^p(G/H) \) such that \( g'f' \notin L^p(G/H) \).

Since \( f' \in L^p(G/H) \), \( f' \in L^p(G/H) \) and Theorem 3.3 gives us a function \( f \in L^p(G) \) such that \( \hat{f} \) restricted to \( H^\perp \) agrees with \( \hat{f}' \). Since \( gL^p(G) \subset L^p(G) \) and \( f \in L^p(G) \), we have that \( g\hat{f} \in L^p(G) \). By again applying Theorem 3.3, we conclude that the restriction of \( g\hat{f} \) to \( H^\perp \) is a member of \( L^p(G/H) \). However, for \( \gamma \in H^\perp \), \( g(\gamma)\hat{f}(\gamma) = g'(\gamma)\hat{f}'(\gamma) \) and thus the restriction of \( g\hat{f} \) to \( H^\perp \) is \( g'\hat{f}' \) which is not in \( L^p(G/H) \). Therefore, we have a contradiction and can conclude that \( g \notin \mathcal{M}(L^p(G)) \) for any \( p \) satisfying \( 1 < p < 2 \).

Again, by applying Theorem 3.2 and noticing that \( \mathcal{M}(L^1(G)) \) is a subset of \( \mathcal{M}(L^p(G)) \), we have that \( g \notin \mathcal{M}(L^p(G)) \) for any \( p \) distinct from two. Thus the union of all the sets \( \mathcal{M}(L^p(G)) \), for \( p \neq 2 \), \( 1 \leq p < \infty \), is properly contained in \( \mathcal{M}(L^2(G)) \). This completes the proof of the theorem.

The following is a discussion of the material found in Section 8.7.1 of [14;216]. Suppose that \( u \) is a trigonometric polynomial on a compact connected abelian topological group \( G \). Then \( u(x) = \sum_{\gamma \in \hat{G}} C_\gamma \gamma(x) \), where \( C_\gamma \) is finitely non-zero. The analytic contraction of \( u \) is, by definition, the function \( F(x) = \sum_{\gamma \in E} C_\gamma \gamma(x) \) where \( E = \{ \gamma \in \hat{G} : \gamma \geq 0 \} \). Note that since \( G \) is connected, it is possible to order \( \hat{G} \), and "\( \gamma \)" refers to any such fixed order on \( \hat{G} \). For \( u \) a trigonometric polynomial,
consider the mapping, denoted by $\emptyset$, which maps $u$ into its analytic contraction $F$. We will need the following theorem concerning the mapping $\emptyset$.

**Theorem 3.6.** [14;217] Let $G$ be a compact connected abelian topological group and $p$ a number such that $1 < p < \infty$. Then there exists a constant $A_p$ such that $\left\|\emptyset u\right\|_p \leq A_p \left\|u\right\|_p$ holds for every trigonometric polynomial $u$ on $G$, hence $\emptyset$ can be extended to a bounded linear operator on $L^p(G)$, which we shall denote by $\emptyset_p$.

We are now prepared to investigate the properness condition for $M(L^p(G))$ at the other extremity. That is, we will show that $M(L^1(G))$ is properly contained in $M(L^p(G))$ for $1 < p < \infty$ and a large class of groups $G$.

**Theorem 3.7.** Let $G$ be a compact abelian topological group such that $\hat{G}$ has an element of infinite order. Then $M(L^1(G))$ is properly contained in the intersection of all the sets $M(L^p(G))$, for $1 < p < \infty$.

**Proof.** Suppose that $\gamma_o$ is an element of $\hat{G}$ which has infinite order. Just as in Theorem 3.5, we will let $\Gamma$ denote the subgroup of $\hat{G}$ generated by $\gamma_o$ and note that $\Gamma$ is topologically isomorphic to the integers. As in the proof of Theorem 3.5, there exists $H$, a closed subgroup of $G$, such that $G/H$ is topologically isomorphic to $T$, the circle group, and $H^\perp = \Gamma$. Since $G/H$ is topologically isomorphic to $T$, $G/H$ is a compact connected abelian topological group which satisfies the hypothesis of Theorem 3.6. Hence, $\emptyset$ is a mapping from the trigonometric polynomials of $G/H$ into the trigonometric
polynomials of $G/H$ and its extension, $\varnothing_p$, is a bounded linear operator from $L^p(G/H)$ into $L^p(G/H)$. We will first show that $\varnothing$ has the centralizer property on the trigonometric polynomials. If $f$ and $g$ are trigonometric polynomials, then $f(x) = \sum_{\gamma \in B} c_\gamma y(x)$ and $g(x) = \sum_{\gamma \in D} d_\gamma y(x)$, where $B$ and $D$ are finite subsets of $G/H$. Further, if $E = \{ \gamma \in G/H : \gamma \geq 0 \}$, then $f \cdot g = \sum_{\gamma \in B \cap D} c_\gamma d_\gamma y$ and $\varnothing(f \cdot g) = \sum_{\gamma \in B \cap D \cap E} c_\gamma d_\gamma y$.

Also, $\varnothing f = \sum_{\gamma \in B \cap E} c_\gamma y$ and thus $(\varnothing f) \cdot g = \sum_{\gamma \in B \cap E \cap D} c_\gamma d_\gamma y$. Therefore $\varnothing(f \cdot g) = (\varnothing f) \cdot g$ for $f$ and $g$ trigonometric polynomials. We will now show that $\varnothing_p \in \mathcal{L}(L^p(G/H))$ for $1 < p < \infty$. If $f, g \in L^p(G/H)$, then there exist sequences $(f_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$ of trigonometric polynomials such that $\|f_n - f\|_p \to 0$ and $\|g_n - g\|_p \to 0$.

Thus, $\|\varnothing_p(f \cdot g) - (\varnothing_p f) \cdot g\|_p \leq \|\varnothing_p(f \cdot g) - \varnothing_p(f_n \cdot g_n)\|_p + \|(\varnothing_p f_n) \cdot g - (\varnothing_p f) \cdot g\|_p + \|(\varnothing_p f_n) \cdot g_n - (\varnothing_p f_n) \cdot g\|_p + \|(\varnothing_p f_n) \cdot g_n - g\|_p + Ap\|f \cdot g - f_n \cdot g_n\|_p + \|\varnothing_p f_n\|_p \|g_n - g\|_p + Ap\|g\|_p \|f_n - f\|_p$ which converges to zero with $n$. Hence $\varnothing_p(f \cdot g) = (\varnothing_p f) \cdot g$ and $\varnothing_p \in \mathcal{L}(L^p(G/H))$ for $1 < p < \infty$. If we let $g_p$ denote the multiplier corresponding to $\varnothing_p$, then we have that $g_p(\gamma) = \varnothing_p(\delta(\gamma)) = \varnothing(\delta(\gamma))$ which is independent of $p$.

Hence $g(\gamma) = \varnothing(\delta(\gamma))$ is a function which is in $M(L^p(G/H))$ for every $p$ satisfying $1 < p < \infty$. Note also, $g(\gamma) = \begin{cases} 1 & \text{if } \gamma = \gamma_n^0 \text{ for some } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$ and thus is the
function $h$ constructed in Theorem 2.10. In the proof of
Theorem 2.10, we showed that $g \not\in \overline{M(G/H)}$. We thus have
that $g$ is an element of all the sets $M(L^p(G/H))$ for
$1 < p < \infty$ and that $g \not\in \overline{M(L^1(G/H))}$. We will now use
Theorem 3.3 to obtain the same result for the group $G$.
Define $\overline{g}$ from $\hat{G}$ into the complex numbers by $\overline{g}(\gamma) =
\begin{cases}
g(\gamma) & \text{for } \gamma \in \overline{H^1} \\
0 & \text{for } \gamma \not\in \overline{H^1}
\end{cases}
$. We will first show that
$\overline{g} \not\in M(L^1(G))$ : if $\overline{g} \in M(L^1(G)) = \overline{M(G)}$, then, by 2.7.2
of [14, 53], $\overline{g}$ restricted to $H^\perp$ is a function in $\overline{M(G/H)}$.
However, $\overline{g}$ restricted to $H^\perp$ is precisely $g$ which we
known is not in $\overline{M(G/H)}$. Hence $\overline{g} \not\in M(L^1(G))$. We will
now show that $\overline{g} \in M(L^p(G))$ for $1 < p < \infty$. If $f \in L^p(G)$,
then $(\overline{g^*})(\gamma) =
\begin{cases}
g(\gamma)\hat{f}(\gamma) & \text{for } \gamma \in \overline{H^1} \\
0 & \text{for } \gamma \not\in \overline{H^1}
\end{cases}
$. Since
$f \in L^p(G)$, we have, by Theorem 3.3, that $\hat{f}$ restricted
to $H^\perp$ is an element of $\overline{L^p(G/H)}$. Thus, there exists
$h \in L^p(G/H)$ such that, for $\gamma \in \overline{H^1}$, $\hat{h}(\gamma) = \hat{f}(\gamma)$. Since $\hat{h} \in \overline{L^p(G/H)}$ and $g \in M(L^p(G/H))$, $g\hat{h} \in \overline{L^p(G/H)}$ and hence
there exists $k \in L^p(G/H)$ such that $g\hat{h} = \hat{k}$. Now, for
$x \in G$, define $\overline{K}(x) = k(N(x))$, where $N$ is the natural map
from $G$ onto $G/H$. Since $k \in L^p(G/H)$ implies that
$\overline{K} \in L^p(G)$, $\overline{K}$ is the pre-image of $k$ constructed in Theorem
3.3. Thus $T\overline{K} = k$ and we have that $\overline{K}$ restricted to $H^\perp$ is
$\hat{k}$. Hence, for $\gamma \in H^\perp$, $\hat{\overline{K}}(\gamma) = \overline{K}(\gamma) = g(\gamma)\hat{h}(\gamma) =
g(\gamma)\hat{f}(\gamma)$ and thus $\overline{K}$ agrees with $\overline{g^*}$ on $H^\perp$. Now, for
$\gamma \not\in H^\perp$, $\hat{\overline{K}}(\gamma) = \int \overline{K}(x)\overline{\gamma}(x)dx = \int \overline{K}(x+h)\overline{\gamma}(x)\overline{\gamma}(h)dx$
for all $h \in H$. From the definition of $\overline{K}$ we have that
\[ E(x + h) = k(N(x+h)) = k(N(x)) = E(x). \] Thus \( \hat{E}(\gamma) = \frac{\mathcal{F}(h)}{\mathcal{F}(h)} \int E(x) \mathcal{F}(x) \, dx = \frac{\mathcal{F}(h)}{\mathcal{F}(h)} \hat{E}(\gamma) \) for each \( h \in H \). Since \( \gamma \not\in H^\perp \), there is at least one \( h \in H \) for which \( \mathcal{F}(h) \neq 1 \) and thus the only way that the equality \( \hat{E}(\gamma) = \frac{\mathcal{F}(h)}{\mathcal{F}(h)} \hat{E}(\gamma) \) can hold is that \( \hat{E}(\gamma) = 0 \). Hence \( \hat{E} \) agrees with \( \hat{g} \hat{f} \) outside \( H^\perp \) and we now have \( \hat{E} = \hat{g} \hat{f} \) on \( \mathcal{G} \). Since \( \hat{E} \in L^p(\mathcal{G}) \), we conclude that \( \hat{g} \hat{f} \in L^p(\mathcal{G}) \) and hence \( \bar{g} \in M(L^p(\mathcal{G})) \) for \( 1 < p < \infty \). Thus \( \bar{g} \) is in the intersection of all the sets \( M(L^p(\mathcal{G})) \) for \( 1 < p < \infty \) and \( \bar{g} \not\in M(L^1(\mathcal{G})) \), which concludes the theorem.
CHAPTER IV

In this chapter, we list some unsolved problems and conjectures which have arisen in connection with this work.

(1) Is there a characterization of \( \mathcal{L}(H) \) if \( H \) is a non-commutative \( H^* \)-algebra? It appears possible to represent the operators in \( \mathcal{L}(H) \) as infinite matrices which are composed of finite square matrices along the diagonal and zeroes in the other entries.

(2) It is conjectured that the algebra of centralizers of an \( H^* \)-algebra is \( (* \)-algebra isomorphic isometric with) the algebra of centralizers of the center of \( H \).

(3) Although the methods of proof used in the latter part of Chapter III do not seem to lend themselves to groups where each character has finite order, it appears likely that these results will go through in an arbitrary infinite compact abelian topological group.

(4) It is conjectured that for \( 1 \leq p \leq s \leq 2 \), the multiplier algebra of \( L^p(G) \) (\( G \) a compact abelian topological group) is contained in the multiplier algebra of \( L^s(G) \), with proper containment if and only if \( p < s \).

Remark. It has quite recently been brought to our attention that Figà-Talamanca in [4] has answered the first part of (4) in the affirmative.
(5) In Chapter III, we made the restriction \( p < \infty \) in discussing algebras of multipliers. This fact raises a very interesting problem, namely that of characterizing the multipliers of \( L^\infty(G) \).

(6) In Chapter II, we showed that the supremum norm closure of \( M(L^1(G)) = \widehat{M(G)} \) was properly contained in \( C(\hat{G}) = M(L^2(G)) \) for a large class of groups. Does \( M(L^p(G)) \) have the same property for \( p > 1 \)? It seems likely.

(7) In the same line of thought, is the supremum norm closure of \( M(L^p(G)) \) properly contained in \( M(L^s(G)) \) whenever \( 1 \leq p < s \leq 2 \)?

(8) When is \( M(L^p(G)) \) supremum norm closed in \( C(\hat{G}) \)? It may well be that this happens only when \( p = 2 \).

(9) When is \( \mathcal{L}(L^p(G)) \) a \( B^* \)-algebra? It appears that this is true only when \( p = 2 \).

(10) Is there a corresponding theory for \( \mathcal{L}(I_{L^p}(G)) \) or \( M(L^p(G)) \) for \( G \) a compact topological group (non-commutative)?

(11) If \( G \) is a locally compact abelian topological group, then \( \mathcal{L}^p(G) = L^1(G) \cap L^p(G) \) with \[ \| \cdot \| = \| \cdot \|_1 + \| \cdot \|_p \] is a commutative Banach algebra. Is there a natural generalization of the centralizer theory to this class of Banach algebras?

(12) If \( G \) is a compact connected abelian topological group, is there a corresponding theory for the centralizers of \( H^p(G) \) where \( H^p(G) = \)
\[ \{ f \in L^p(G) : \hat{f}(\gamma) = 0 \text{ for } \gamma \in \mathbb{G} \} \text{? For example, we can show that if } 1 < p < \infty, 1 < q < \infty \text{ and } \frac{1}{p} + \frac{1}{q} = 1, \text{ then } M(H^p(G)) = M(H^q(G)). \]

(13) Is there a characterization of the maximal ideal space of \( M(L^p(G)) \)? In the case \( p = 2 \), it is easily seen to be the Stone-Čech compactification of \( \mathbb{G} \). In the case \( p = 1 \), it is a very difficult problem as \( M(L^p(G)) = \widehat{M(G)} \).
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BIOGRAPHY

Charles N. Kellogg was born on June 29, 1938 in Albuquerque, New Mexico. He attended public schools in Estancia, New Mexico, graduating from high school there in May, 1956. He did his undergraduate work in the field of mathematics at New Mexico Institute of Mining and Technology and received his Bachelor of Science degree in June, 1960.

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Candidate: Charles N. Kellogg

Major Field: Mathematics - Analysis

Title of Thesis: On Certain Algebras of Centralizers

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Date of Examination:

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