On the Dimensions of Certain Spaces of Homeomorphisms.

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ABSTRACT

In this paper we investigate the dimensions of the groups of all homeomorphisms of certain metric continua, and also certain subgroups of these. The problem is divided into the following three (overlapping) parts:

(1) continua with non-zero dimensional groups of homeomorphisms
(2) one-dimensional continua
and (3) continua with finite, positive dimensional groups of homeomorphisms.

Our main result for part (1) asserts that the group of homeomorphisms of any locally setwise homogeneous continuum (in a technical sense) is non-zero dimensional. In particular, the universal plane curve and universal curve are such continua. We note that the groups of homeomorphisms of these two continua are also totally disconnected, and thus the non-zero dimensionality of these groups is especially interesting. We also indicate how a proof, similar to the given proof of the above theorem, will show that the group of those homeomorphisms of an n-manifold, $\mathbb{M}^n$, $n > 1$, which carry a fixed, countable, dense subset of $\mathbb{M}^n$ onto itself, is also non-zero dimensional. (This, too, is totally disconnected.)

For part (2) we obtain the following results about the groups of all homeomorphisms of 1-dimensional continua:
(a) The group of all homeomorphisms, \( G \), of a regular curve, \( X \), (a continuum in which every point has arbitrarily small neighborhoods with finite boundaries) is either zero dimensional or infinite dimensional. \( G \) is infinite dimensional if \( X \) contains a free arc; it is zero dimensional otherwise.

(b) There exist rational curves (continua in which every point has arbitrarily small neighborhoods with countable boundaries) whose groups of homeomorphisms are of finite positive dimension. (The proof of this is outlined after some of the results for part (3) are obtained.)

We note that, in the definitions, the difference between regular curves and rational curves seems minor; the rational curves are just slight generalizations of the regular curves. However, they are vastly different in structure — one indication of this being the difference in the possible dimensions of their groups of homeomorphisms, as shown by the above results.

Finally, for part (3) we obtain (or indicate how to obtain) the following results:

(a) There exist continua whose full groups of homeomorphisms are totally disconnected, abelian, and exactly one-dimensional.

(b) For each positive integer \( n \), there exists a continuum, \( M_n \), whose full group of homeomor-
phisms, \( G(M_n) \), is homeomorphic to the product of \( n \) 1-dimensional groups, and thus
\[ 1 \leq \dim G(M_n) \leq n. \]

(c) The continua of (a) and (b) may be taken to be rational curves.

The proof, in each of these cases, is by construction. To get (b), we construct continua such that the groups of all homeomorphisms of these continua are products of \( n \) 1-dimensional groups obtained in (a). But the continua are like unions of continua constructed for (a), rather than products of these.

These results suggest many other questions which are raised, but unanswered, in this paper.
CHAPTER I
PRELIMINARIES

INTRODUCTION:
The purpose of this paper is to study the dimensions of the groups of homeomorphisms of metric continua and also the dimensions of certain of their subgroups. It is well-known, and easy to prove, that the group of all homeomorphisms of an arc is infinite dimensional; in fact, the group of all homeomorphisms of any manifold is infinite dimensional. (We prove this in Theorem 1.1.) The component of the identity in this case is large. However, it is not hard to show that the group of all homeomorphisms of the universal plane curve is totally disconnected. It therefore seems likely that it is zero dimensional. In fact, such a result, for the group of homeomorphisms of the universal curve, was announced (erroneously) by R. D. Anderson [1], who called it to my attention and suggested the general problem of this paper to me. His argument was for total disconnectivity instead of zero dimensionality. However, corollaries of our main theorem of Chapter II show that each of these groups is at least 1-dimensional. By methods similar to those used in Theorem 1.1, we see that the group of homeomorphisms of the universal plane curve contains an

1 See Chapter III for definition.
2 Ibid.
infinite product of 1-dimensional sets, and therefore is quite possibly infinite dimensional. However, at present, we are unable to determine its dimension; that is, we know it is at least 1-dimensional, but we cannot say any more.

In view of the fact that the group of homeomorphisms of an object as simple as an arc is infinite dimensional, and the fact that the group of homeomorphisms of the universal plane curve, while totally disconnected, is at least 1-dimensional, we might ask whether or not there exist continua whose groups of homeomorphisms are finite dimensional. It is known that there are dendrons which admit only the identity homeomorphism, and these are easy to construct. Using this result, it is easy to construct other continua whose groups of homeomorphisms are larger, but still zero dimensional. So we ask: Do there exist continua whose groups of homeomorphisms are of finite, positive dimension? In Chapter V, we answer this question in the affirmative, by constructing such continua. Our main result is the following: For each positive integer \( n \), there exists a continuum, \( M_n \), such that

1. \( G(M_n)^3 \) is the product of \( n \) 1-dimensional groups
2. \( 1 \leq \dim G(M_n) \leq n \)
3. \( G(M_n) \) is totally disconnected and abelian.

We first construct continua whose full groups of homeo-
morphisms are exactly 1-dimensional. We then modify these continua by taking, in some cases unions, in other cases sets which are like unions, of these continua, and thus obtain continua whose full groups of homeomorphisms are products of 1-dimensional groups. This gives us the above stated result. We also construct rational curves with these properties.

In Chapter IV, we study the dimensions of the groups of homeomorphisms of 1-dimensional continua. Our main result asserts that regular curves\(^4\) may have zero or infinite dimensional groups of homeomorphisms, and only these, while rational curves\(^5\) may have finite, positive dimensional groups.

Finally, in each chapter, we raise some questions which the results of that chapter suggest.

**Definitions and Conventions:**

All spaces are separable metric.

If X is a compact, metric space, G(X) (or G when no confusion arises) will denote the group of all homeomorphisms of X. H or H(X) will denote a subgroup of G.

The metric on X will be denoted by d, and d(x,y) denotes the distance between x and y.

\(^4\)See Chapter IV for definition.

\(^5\)Ibid.
The metric in $G(X)$ will also be denoted by $d$.

We define the metric in $G(X)$ as follows:

$$d(g,h) = \text{lub} \{d(g(x), h(x)) : x \in X\}.$$  

If $\mathcal{U}$ is a collection of sets, "$\bigcup \mathcal{U}$" denotes the union of the elements of $\mathcal{U}$. If $\mathcal{V}$ is also a collection of sets, $\mathcal{V}$ is said to refine $\mathcal{U}$ if each element of $\mathcal{V}$ is a subset of some element of $\mathcal{U}$. $\mathcal{V}$ is called a closed refinement of $\mathcal{U}$ if the closure of each element of $\mathcal{V}$ is a subset of some element of $\mathcal{U}$. Mesh of $\mathcal{U}$ means the least upper bound of the diameters of the elements of $\mathcal{U}$.

$C(U)$ denotes the complement of the set $U$.

"Bd $U$" denotes the boundary of $U$.

If $g$ is a homeomorphism, "$g$ is supported on $U$" means "$g$ is the identity on $C(U)$".

In "$f : X \rightarrow X$" the double headed arrow means $f$ is a function from $X$ onto $X$.

If $f : X \rightarrow Y$ is a function, and $A \subseteq X$, then "$f \upharpoonright A$" means "$f$ restricted to $A$".

"map" means "continuous function".

$X^\circ$ denotes the interior of $X$.

"iff" means "if and only if".

---

6 The following is also a metric for $G(X)$:

$$\rho(g,h) = d(g,h) + d(g^{-1}, h^{-1})$$

where $d'$ denotes the metric defined above. Then $\rho$ is a complete metric for $G(X)$ and is equivalent to $d$. However, we prefer to use the metric $d$, since it is intuitively easier to understand.

It is well-known that $G(X)$ is separable.
The empty set is denoted by \( \emptyset \).

The dimension of a set \( X \) is denoted by "dim \( X \)". We use the inductive definition of dimension (for separable metric spaces) defined as follows:

The empty set has dimension -1.

A non-empty set \( X \) is **0-dimensional at a point** \( x \), if there exist arbitrarily small neighborhoods of \( x \) with empty boundaries.

A non-empty set \( X \) is **0-dimensional**, that is, \( \dim X = 0 \), if \( X \) is 0-dimensional at each of its points.

A space \( X \) has dimension \( \leq n \) at a point \( x \) if \( x \) has arbitrarily small neighborhoods whose boundaries have dimension \( \leq n - 1 \).

\( X \) has dimension \( \leq n \), that is, \( \dim X \leq n \), if \( X \) has dimension \( \leq n \) at each of its points.

\( X \) has dimension \( n \) at a point \( x \) if it is true that \( X \) has dimension \( \leq n \) at \( x \), and it is false that \( X \) has dimension \( \leq n - 1 \) at \( x \).

\( X \) has dimension \( n \), that is, \( \dim X = n \), if it is true that \( \dim X \leq n \), but it is false that \( \dim X \leq n - 1 \).

\( X \) has dimension \( \infty \) if \( \dim X \leq n \) is false for each \( n \).

See [6] for a more thorough discussion of inductive dimension.
Preliminary Theorems:

**THEOREM 1.1 (Well-known):** The group of all homeomorphisms of a compact n-manifold, M, is infinite dimensional.

**PROOF:** Let $\{C_i\}_{i=1}^{\infty}$ be a sequence of disjoint n-cells of M, converging to a point of M. For each $i$, $1 \leq i < \infty$, let $\sigma_i$ be an arc of homeomorphisms of M onto itself, supported on $C_i$, and denoted by $\{h_i^\alpha : 0 \leq \alpha \leq \frac{1}{i}\}$. Let $H = \{h \in G(M) : h \text{ is supported on } \bigcup_{i=1}^{\infty} C_i\}$, and, for each $i$, there is an $\alpha_i$ such that $h|_{C_i} = h_i^{\alpha_i}|_{C_i}$. Then H is homeomorphic to a Hilbert cube, since each element $h \in H$ can be identified with an infinite sequence $(h_1^{\alpha_1}, h_2^{\alpha_2}, \ldots, h_n^{\alpha_n}, \ldots)$ where $\alpha_n \in [0, \frac{1}{n}]$. Therefore H, and thus $G(M)$, is infinite dimensional.

**THEOREM 1.2 (R. D. Anderson):** The group of homeomorphisms, G, of the universal plane curve, M, is totally disconnected.

**PROOF:** We will show that any pair of elements of G can be separated; that is, given $g, h \in G$, there exist two

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6This proof was outlined to me by R. D. Anderson.
mutually separate sets $A$ and $B$ such that $g \in A$, $h \in B$, and $A \cup B = G$. We first note that $M$ can be thought of as a circle $C^0$ plus its interior in $E^2$, with the interiors of a countable dense set of pairwise disjoint circles, $\{C_i\}_{i=1}^{\infty}$, removed, where each $C_i$ is in the interior of $C^0$, $\text{diam } C_i \to 0$, and no $C_i$ is a subset of any $C_j^0$. This follows from the main result of [3].

Let $g_o, h_o \in G, g_o \neq h_o$. Since $\bigcup_{i=1}^{\infty} C_i$ is dense in $M$, and $\text{diam } C_i \to 0$, we shall show that there exists an $n_o$ such that $h_o(C^o_{n_o}) \cap g_o(C^o_{n_o}) = \emptyset$. There exists a point $x$ such that $h_o(x) \neq g_o(x)$. Let $V, W$ be 2 disjoint neighborhoods of $h_o(x)$ and $g_o(x)$, respectively. Then $U = h_o^{-1}(V) \cap g_o^{-1}(W)$ is a neighborhood of $x$ whose images under $h_o$ and $g_o$, respectively, are disjoint. Since $\bigcup_{i=1}^{\infty} C_i$ is dense, there is an integer $n_o$ such that $C^o_{n_o} \subset U$, and we have $h_o(C^o_{n_o}) \cap g_o(C^o_{n_o}) = \emptyset$.

Let $A = \{h \in G : h(C^o_{n_o}) = h_o(C^o_{n_o})\}$ and let $B = G - A$. Then $h_o \in A$, $g_o \in B$, and $A \cup B = G$.

We will show that $A$ and $B$ are mutually separate, by showing that there is a $\delta > 0$ such that $d(A, B) \geq \delta$. Let $\delta$ be the diameter of $h_o(C^o_{n_o})$, and let $h_1$ be any element of $A$, and let $g_1$ be any element of $B$. Since $h_1(C^o_{n_o})$ and $g_1(C^o_{n_o})$ are 2 disjoint circles in $E^2$, neither containing the other, there exists a straight line $L$ in $E^2$ separating them. (We
can obtain $L$ by first constructing the line $L'$ joining the centers $a$ and $b$ of $h_1(C_{n_0})$ and $g_1(C_{n_0})$, respectively, and then constructing the line $L$ perpendicular to $L'$, with $L$ going through the midpoint of the smallest segment of $[a,b]$ which joins $h_1(C_{n_0})$ and $g_1(C_{n_0})$.

We note that $h_1(C_{n_0}) = h_0(C_{n_0})$.

Now let $y$ be the point of $L' \cap h_0(C_{n_0})$ which is farther from $L$. Then $y = h_1(x)$ for some $x$ in $C_{n_0}$. Since $g_1(x) \notin g_1(C_{n_0})$, and therefore on the other side of $L$, $d(h_1(x), g_1(x)) > \delta$. Therefore $d(h_1, g_1) > \delta$.

We have thus shown that the distance between any element of $A$ and any element of $B$ is greater than $\delta$. It follows that $d(A, B) \geq \delta$. 

CHAPTER II

CONTINUA WITH NON-ZERO DIMENSIONAL GROUPS

OF HOMEOMORPHISMS

In this chapter we establish sufficient conditions for continua to have non-zero dimensional groups of homeomorphisms. Our main result of this chapter is the following: Let $X$ be a continuum which is locally setwise homogeneous\(^1\) under the closed subgroup $H$ of $G(X)$. Then $H$ is at least one dimensional.

Among the continua which are locally setwise homogeneous\(^2\) are the universal plane curve,\(^3\) universal curve,\(^4\) and manifolds of all positive dimensions. Thus the groups of all homeomorphisms of these continua are at least one dimensional. However, it is easy to show (see Chapter I) that the groups of all homeomorphisms of the manifolds are infinite dimensional.

As a corollary of the proof of the above theorem, it also follows that if $H$ is the group of those homeomorphisms of an $n$-manifold, $n > 1$, which carry a countable dense subset of the manifold onto itself, then $H$ is at

\(^1\)See below for definition.

\(^2\)Ibid.

\(^3\)See Chapter III for definition.

\(^4\)Ibid.
least one dimensional.

We note that, in Chapter V, we will use a very much simplified version of the proof of Theorem 2.1 to show non-zero dimensionality for the groups of homeomorphisms of the examples of that chapter, specifically in Theorem 5.1.

**DEFINITION 2.1:** A chain of open sets \( \mathcal{U} : U_1, \ldots, U_n \), is a finite, ordered set of open sets such that (1) \( U_1 \cap U_j \neq \emptyset \) iff \( |i - j| \leq 1 \), (2) \( \overline{U}_1 \cap \overline{U}_j \neq \emptyset \) iff \( U_1 \cap U_j \neq \emptyset \), and (3) \( U_i \neq U_j \) for any pair \( i, j \).

The elements of the chain are called its links.

A subset of \( \mathcal{U} \) will be called a subchain if its elements form a chain with the order inherited from \( \mathcal{U} \).

A chain whose links are of diameter \( < \epsilon \) is called an \( \epsilon \)-chain.

**DEFINITION 2.2:** Let \( \mathcal{U} : U_1, \ldots, U_n \) be a chain of open sets. Then a set \( A = \{ a_i \}_{i=0}^n \) is called a chain set for \( \mathcal{U} \) if (1) \( a_0 \in U_1 - U_2 \), (2) \( a_i \in U_i \cap U_{i+1} \), for \( 1 \leq i \leq n - 1 \), and (3) \( a_n \in U_n - U_{n-1} \).

**DEFINITION 2.3:** Let \( X \) be a continuum, and let \( \mathcal{U} : U_1, \ldots, U_n \) be a chain of connected open sets of \( X \). Then a chain \( \mathcal{V} : V_1, \ldots, V_m \) of connected open sets of \( X \) is called a straight refinement of \( \mathcal{U} \) if (1) there exists a chain set \( A = \{ a_i \}_{i=0}^n \) for \( \mathcal{U} \) such that
(a) \( V \) is a minimal chain from \( a_0 \) to \( a_n \) with \( a_0 \in V_1 \) and \( a_n \in V_m \).

(b) \( V^* \) contains \( A \).

(c) \( V^*_i \subseteq U_i \) where \( V_i \) is the shortest subchain of \( V \) from \( a_{i-1} \) to \( a_i \).

(d) each \( a_i \) is an element of the intersection of 2 links of \( V \)

and (2) exactly one link of \( V_i \) meets the boundary of \( U_{i+1} \).

A is called an associated chain set for \( \{U, V\} \).

NOTE 1: (c) of (1) implies that \( V \) is a closed refinement of \( U \).

NOTE 2: The chain \( V \) in reverse order need not be a straight refinement of the chain \( U \) in reverse order.

DEFINITION 2.4: The chain \( V \) is called a uniform refinement of the chain \( U \) if

(1) \( V \) is a straight refinement of \( U \)

and (2) if \( k_i \) denotes the number of links in \( V_i \) (of definition 3.1.c), then \( k_i = k_j \) for 

\[ 1 \leq i, j \leq n. \]

The chain \( V_i \) is called the \( i^{th} \) uniform subchain of \( V \).

We note that \( V_i \) depends on the associated chain set for \( \{U, V\} \) chosen, only if \( V \) is not a uniform refinement.

DEFINITION 2.5: Let \( X \) be a continuum, \( U:U_1, \ldots, U_n \) be a chain of connected open sets of \( X \). A homeomorphism
h of X onto itself, h supported on $U^*$, is called a pseudo translation of k units on $U$ if there exists a chain set $A = \{a_i\}_{i=0}^{n}$ for $U$ such that

1. $h(a_i) = a_{i+k}$, for $0 \leq i \leq n - k$
   $h(a_i) \subseteq U_{n}$, for $i > n - k$

2. $h(\overline{U}_i) \subseteq \left(\bigcup_{j=i-1}^{i+k} \overline{U}_j\right) - \overline{U}_{i-2}$, for $1 \leq i \leq n - k$
   $h(\overline{U}_i) \subseteq \left(\bigcup_{j \geq i-1} \overline{U}_j\right) - \overline{U}_{i-2}$, for $i > n - k$.

$A$ is called a chain set for $U$ associated with $h$.

For any $\varepsilon > 0$, we call $h$ an $\varepsilon$-pseudo translation if (2) can be strengthened to:

(2*) $h(\overline{U}_i) \subseteq \left(\bigcup_{j=i-1}^{i+k} \overline{U}_j\right) \cup \text{N}_\varepsilon(U_i)$, for $1 \leq i \leq n - k$

and $h(\overline{U}_i) \subseteq \left(\bigcup_{j \geq i-1} \overline{U}_j\right) \cup \text{N}_\varepsilon(U_i)$, for $i > n - k$,

where $\text{N}_\varepsilon(U_i)$ denotes the $\varepsilon$-neighborhood of $U_i$.

**DEFINITION 2.6:** Let $X$ be a continuum, and let $U : U_1, \ldots, U_n$ be a chain of connected open sets of $X$.

Then a homeomorphism $h$ of $X$ onto itself, $h$ supported on $U^*$, is called a sliding homeomorphism of k units on $U$ if there exists a chain $D : D_1, \ldots, D_{n-k}$ of connected open sets such that

1. $D_i \subseteq U_{i+k}$
2. $h^{-1}(D_i) \subseteq U_i$
and (3) \( h(\overline{U}_i) \subseteq \left[ \left( \bigcup_{j=i-1}^{i+k} \overline{U}_j \right) \setminus \overline{U}_{i-2} \right] \), for \( 1 \leq i \leq n-k \)

\[ h(\overline{U}_i) \subseteq \left[ \left( \bigcup_{j=i-1}^{i+k} \overline{U}_j \right) \setminus \overline{U}_{i-2} \right] \), for \( i > n-k \).

\( \mathcal{D} \) is called an associated chain for \( h \).

For any \( \varepsilon > 0 \), we call \( h \) an \( \varepsilon \)-sliding homeomorphism of \( k \) units on \( \mathcal{U} \) if (3) can be strengthened to:

\[ (3^\ast) \quad h(\overline{U}_i) \subseteq \left[ \left( \bigcup_{j=i}^{i+k} \overline{U}_j \right) \cup N_\varepsilon(U_i) \right] \), for \( 1 \leq i \leq n-k \)

and \( h(\overline{U}_i) \subseteq \left[ \left( \bigcup_{j=i}^{i+k} \overline{U}_j \right) \cup N_\varepsilon(U_i) \right] \), for \( i > n-k \),

where \( N_\varepsilon(U_i) \) denotes the \( \varepsilon \)-neighborhood of \( U_i \).

We call \( h \) a weak sliding homeomorphism if it satisfies condition (3) above, but not necessarily (1) and (2).

**DEFINITION 2.7:** Let \( X \) be a continuum, and let \( H \) be a subgroup of \( G(X) \). Then \( X \) is called locally setwise homogeneous under \( H \) if there exist both a basis, \( \mathcal{E} \), of connected open subsets of \( X \), and also a dense subset, \( B \), of \( X \) such that for any element \( E \) of \( \mathcal{E} \), and any 2 points \( a \) and \( b \) of \( B \cap E \), there exists a homeomorphism \( h \in H \), \( h \) supported on \( E \), such that \( h(a) = b \).

If \( X \) is locally setwise homogeneous under \( G(X) \), then \( X \) is called locally setwise homogeneous.

We will call \( \{ X, \mathcal{E}, B, H \} \) a locally setwise homogeneous structure.
REMARK: A locally setwise homogeneous continuum must be locally connected.

NOTATION: If $C_1, \ldots, C_n$ is a chain of subsets of $X$, then $h(C)$ will represent the chain of subsets of $X$ whose links are the sets $h(C_i)$, $1 \leq i \leq n$.

**Lemma 2.1:** Let $X$ be a locally connected continuum, and let $U:U_1, \ldots, U_n$ be a chain of connected open sets of $X$. Then for each $\varepsilon > 0$, there exists an $\varepsilon$-chain $\mathcal{V}$ which is a straight refinement of $U$. The elements of $\mathcal{V}$ can be chosen from any given basis for $X$.

**Proof:** Let $\mathcal{E}$ be a basis of connected open sets of $X$. Let $a_0 \in U_1 - U_2$, and let $a_1'' \in U_1 \cap U_2$. Let $\mathcal{W}_1$ be a chain of elements of $\mathcal{E}$ of mesh $< \varepsilon$ from $a_0$ to $a_1''$, whose closures are subsets of $U_1$. Let $\mathcal{V}_1$ be the subchain of $\mathcal{W}_1$ from the first link of $\mathcal{W}_1$ to the first link of $\mathcal{W}_1$ which meets $U_2$. Let $a'_1$ be any point of the intersection of $U_2$ with the last link of $\mathcal{V}_1$, and let $a_2''$ be any point of $U_2 \cap U_3$. Let $\mathcal{W}_2$ be a chain of elements of $\mathcal{E}$ of mesh $< \varepsilon$ from $a'_1$ to $a_2''$, whose closures are subsets of $U_2$. Let $\mathcal{V}_2$ be the subchain of $\mathcal{W}_2$ from the last link of $\mathcal{W}_2$ which meets the last link of $\mathcal{V}_1$ to the first link of $\mathcal{W}_2$ which meets $U_3$. Let $a'_1$ be any point of the intersection of the last link of $\mathcal{V}_1$ and the first link of $\mathcal{V}_2$. Let $a'_2$ be any point of the...
intersection of the last link of $\mathcal{V}_2$ with $U_3$.

We continue the above process inductively until we obtain $\mathcal{V}_{n-1}$. Then let $a_{n-1}^i$ be any point of the intersection of the last link of $\mathcal{V}_{n-1}$ with $U_n$, and let $a_n^i$ be any point of $U_n - U_{n-1}$. Let $\mathcal{W}_n$ be any chain of elements of $\mathcal{E}$ of mesh $< \varepsilon$ from $a_{n-1}^i$ to $a_n^i$ whose closures are in $U_n$. Let $\mathcal{V}_n$ be the subchain of $\mathcal{W}_n$ from the last link of $\mathcal{W}_n$ which meets the last link of $\mathcal{V}_{n-1}$ to the last link of $\mathcal{W}_n$, and let $a_{n-1}^i$ be any point of the intersection of the last link of $\mathcal{V}_{n-1}$ with the first link of $\mathcal{V}_n$.

Let $\mathcal{V}$ be the chain whose links are the links of the $\mathcal{V}_i$'s. Then $\mathcal{V}$ is a straight refinement of $\mathcal{U}$, and $A = \{a_i^j\}_{j=0}^n$ is a chain set for $\{\mathcal{U}, \mathcal{V}\}$.

**Lemma 2.2:** Let $X$ be a locally connected continuum, and let $\mathcal{U}: U_1, \ldots, U_n$ be a chain of connected open sets of $X$. Then for each $\varepsilon > 0$, there exists a uniform $\varepsilon$-refinement $\mathcal{W}$ of $\mathcal{U}$.

**Proof:** Let $\delta = \min \left\{ d(U_i, U_{i+2}) \right\}$ and let $\gamma < \min \left\{ \frac{\delta}{3}, \varepsilon \right\}$. By Lemma 2.1, there exists a $\gamma$-chain $\mathcal{V}$ which is a straight refinement of $\mathcal{U}$. Let $A = \{a_i^j\}_{j=0}^n$ be an associated chain set for $\{\mathcal{U}, \mathcal{V}\}$.

Let $k_i$ be the number of links in $\mathcal{V}_i$, the $i^{th}$ uniform subchain of $\mathcal{V}$, and let $k = \max \{k_i\}$. We will modify $\mathcal{V}$ to a chain $\mathcal{W}$ which has the property that, for all $i$, the number of links in $\mathcal{W}_i$ is $k$. 
For each $i$ for which $\mathcal{F}_i$ contains less than $k$ links, we choose a link $V_{t_i}$ such that $V_{t_i} \subset U_i$, but $V_{t_i} \cap U_j = \emptyset$ for $j \neq i$. Let $\alpha = \min \{d(V_{i-1}, V_{i+1})\}$. Let $k' = \max \{k - k_i\}$, and let $b_i \in V_{t_i-1} \cap V_{t_i} \quad b_i' \in V_{t_i} \cap V_{t_i+1}$. Let $\mathcal{C}_i$ be a chain of connected open sets from $b_i$ to $b_i'$ of mesh $< \frac{\alpha}{k'}$, whose closures are in $V_{t_i}$. Let $\mathcal{C}_i'$ be the subchain of $\mathcal{C}_i$ from the last link of $\mathcal{C}_i$ which meets $V_{t_i-1}$ to the first link of $\mathcal{C}_i$ which meets $V_{t_i+1}$. Then $\mathcal{C}_i'$ contains at least $k' + 1$ elements. Next, we modify $\mathcal{C}_i'$ to a chain $\mathcal{C}_i''$ whose links are obtained from $\mathcal{C}_i'$ by taking, as one link, the union of all but the first $k - k_i$ links, and by taking the first $k - k_i$ links of $\mathcal{C}_i'$ as links of $\mathcal{C}_i''$ also. Then let $\mathcal{W}_i$ be the chain consisting of all links of $\mathcal{V}_i$ except $V_{t_i}$ which is to be replaced by $\mathcal{C}_i''$. Then $\mathcal{W}_i$ has exactly $k$ links. Let $\mathcal{W}$ be the chain consisting of the collection of links of the $\mathcal{W}_i$'s. Then $\mathcal{W}$ is the desired chain.

**Lemma 2.3:** Let $\{X, E, B, H\}$ be a locally setwise homogeneous structure, let $\mathcal{U}: U_1, \ldots, U_n$ be a chain of connected open sets of $X$, and let $\varepsilon > 0$. Then for each integer $k$, $1 \leq k \leq n$, there exists an $\varepsilon$-pseudo translation in $H$ of $k$ units on $\mathcal{U}$. 
PROOF: Let \( \mathcal{S} = \min \left\{ d(U_i, U_{i+2}), \in \mathcal{J} \right\} \). Let \( \mathcal{V} \) be a straight refinement of \( \mathcal{U} \) of mesh \( \frac{\mathcal{S}}{n} \) and whose links are elements of \( \mathcal{E} \). By Lemma 2.1, \( \mathcal{V} \) exists. Let \( A = \{a_i\}_{i=0}^n \) be a chain set for \( \{\mathcal{U}, \mathcal{V}\} \), with \( A \subset B \). (Since \( B \) is dense and the \( a_i \)'s are chosen in open sets, we can get \( A \subset B \).)

Let \( \mathcal{V}_1:V_{i1}, \ldots, V_{ik_i} \) be the shortest subchain of \( \mathcal{V} \) from \( a_{i-1} \) to \( a_i \) in \( U_i \). For each \( i, 1 \leq i \leq n \), let \( \phi_{i_1} \) be a homeomorphism supported on \( V_{i1}, \phi_{i_1} \in \mathcal{H} \), and carrying \( a_{i-1} \) to some point of \( V_{i2} \). We define \( \phi_{i_j} \) for \( 1 < j < k_i \) inductively by: \( \phi_{i_j} \) is a homeomorphism \( \in \mathcal{H} \) supported on \( V_{ij} \) and carrying \( \phi_{i_{j-1}} \ldots \phi_{i_2}(a_{i-1}) \) to some point of \( V_{ij+1} \). For \( j = k_i, \phi_{i_j} \in \mathcal{H} \) is supported on \( V_{ij} \) and \( \phi_{i_j}(\phi_{i_{j-1}} \ldots \phi_{i_2}(a_{i-1})) = a_i \).

Now let \( h_i = \phi_{i_{s_i-1}} \ldots \phi_{i_k} \), where \( V_{is_i} \) is the first link of \( \mathcal{V}_i \) meeting \( U_{i+1} \), for \( 1 \leq i < n \), and let \( h_n = \phi_{n_k} \ldots \phi_{n_2} \phi_{n_1} \). Let \( g_i = \phi_{i_{k_i}} \ldots \phi_{i_{s_i}} \), for \( 1 \leq i < n \). Let \( h = \prod_{i=1}^{n-1} g_i \prod_{i=1}^{n} h_i \). Then \( h \) is a \( \frac{\mathcal{S}}{n} \)-pseudo translation of 1 unit in \( \mathcal{H} \). Further, for this \( h \), \( h^k \) is a \( \left[ \frac{k \mathcal{S}}{n} (\epsilon) \right] \) pseudo translation of \( k \) units on \( \mathcal{U} \), since by this choice of \( h_i \), no point can be moved back more than 1 link of \( \mathcal{V} \).
**Lemma 2.4:** Let \( X \) be a locally setwise homogeneous continuum, \( \mathcal{U} : U_1, \ldots, U_n \) be a chain of connected open sets of \( X \), and \( h \) be a pseudo translation of \( k \) units on \( \mathcal{U} \). Then there exists a chain \( \mathcal{D} : D_1, \ldots, D_{n-k} \), of connected open sets of \( X \) such that (1) \( D_1 = U_1 \) and (2) \( h(D_1) \subseteq U_{i+k} \). Thus it follows that the chain \( h(\mathcal{D}) \) satisfies the following conditions: (1) \( h(D_1) \subseteq U_{i+k} \) and (2) \( h^{-1}(h(D_1)) \subseteq U_i \); that is, \( h \) is a sliding homeomorphism of \( k \) units on \( \mathcal{U} \).

**Proof:** Let \( A = \{ a_i \}_{i=0}^n \) be a chain set for \( \mathcal{U} \) associated with \( h \). For \( i \) such that \( 1 \leq i \leq n-k \), let 
\[
C_i = \overline{U_i} \cap h^{-1} \left( \bigcup_{j=-1}^{k-1} U_{i+j} \right) - U_{i+k}.
\]
Then \( C_i \) is a closed subset of \( X \) and \( C_i = C_i \cap U_i \) is a closed subset of \( U_i \). Let \( D_i' = U_i - C_i' \). Then \( D_i' \) is an open subset of \( U_i \), and therefore of \( X \), and \( h(D_i') \subseteq U_{i+k} \). We note that \( D_i' \) contains both \( a_{i-1} \) and \( a_i \). Further, \( C_i' \) cannot separate \( a_{i-1} \) from \( a_i \), for if it did, then \( h(C_i') \) would separate \( h(a_{i-1}) \) from \( h(a_i) \); that is, \( a_{i+k-1} \) from \( a_{i+k} \). But \( h(C_i') \cap U_{i+k} = \emptyset \). This is a contradiction. Let \( D_i \) be the component of \( D_i' \) which contains both \( a_{i-1} \) and \( a_i \). Then \( D_i \subseteq U_i \) and \( h(D_i) \subseteq U_{i+k} \).

**Lemma 2.5:** Let \( X \) be a continuum, and let \( \mathcal{U} : U_1, \ldots, U_n \) be a chain of connected open sets of \( X \) of mesh \(< \varepsilon \). Let \( h \) be a sliding homeomorphism of \( k \)
units on $\mathcal{U}$, and let $\mathcal{D}: D_1, \ldots, D_{n-k}$ be an associated chain for $h$; that is, (1) $D_i \subset U_{i+k}$ and 
(2) $h^{-1}(D_i) \subset U_i$. Let $\varphi$ be a $\mathcal{S}$-sliding homeomorphism of 1 unit on $\mathcal{D}$, where $2\mathcal{S}$ is less than 
$\min \left\{ d(U_{i-2}, h(U_i) \cup U_1) \right\}$, and $\varphi$ is constructed as in Lemma 2.3.

Then (1) $\varphi h$ is a sliding homeomorphism of $k + 1$ units 
on $\mathcal{U}$,
and (2) $d(h, \varphi h) < 3 \varepsilon$.

**Proof:** (1) Let $\mathcal{E}: E_1, \ldots, E_{n-k-1}$ be an associated chain for $\varphi$. Then $\mathcal{E}$ is also an associated chain for $\varphi h$.

(2) Clear.

**Lemma 2.6:** Let $X$ be a continuum, and let $h$ be a sliding homeomorphism of $k$ units on a chain $\mathcal{U}: U_1, \ldots, U_n$ of connected open sets of $X$. Let $\mathcal{D}: D_1, \ldots, D_{n-k}$ be an associated chain for $h$. Let $2\varepsilon = \min \left\{ d(U_{i-2}, h(U_i) \cup U_1) \right\}$.

Let $\mathcal{V}: V_1, \ldots, V_{s \cdot (n-k)}$ be a uniform refinement of $\mathcal{D}$ of mesh $< \varepsilon$, and let $\varphi$ be a sliding homeomorphism of $s$ units on $\mathcal{V}$. Then $\varphi h$ is a sliding homeomorphism of $k + 1$ units on $\mathcal{U}$.

**Proof:** Let $\mathcal{E}: E_1, \ldots, E_{s \cdot (n-k-1)}$ be an associated chain for $\varphi$. We form a new chain $\mathcal{F}: F_1, \ldots, F_{n-k-1}$ by an amalgamation process from $\mathcal{E}$, by letting 

$$F_i = \bigcup_{j=[s \cdot (i-1)]+1}^{s \cdot i} E_j.$$ 

Then $F_i \subset U_{i+k+1}$ and $\mathcal{F}$ is an
associated chain for $\varphi h$. This is easy to see since

$$\varphi^{-1}(F_i) \subset D_i \subset U_{i+k}$$

and therefore $h^{-1}(\varphi^{-1}(F_i)) \subset U_i$; that is, $(\varphi h)^{-1}(F_i) \subset U_i$. It is also clear that

$$\varphi h(U_i) \subset \left( \bigcup_{j=i-1}^{i+k+1} U_j \right) - U_{i-2}$$

Thus $\varphi h$ is a sliding homeomorphism of $k + 1$ units on $U$.

**Lemma 2.7:** Let $X$ be a continuum, and let $U: U_1, \ldots, U_n$ be a chain of connected open subsets of $X$ of mesh $< \varepsilon$. Then if $h$ is any homeomorphism of $X$ onto itself, and $\varphi$ is a homeomorphism supported on $\mathcal{U}^*$ such that

$$\varphi(U_i) \subset (U_{i-1} \cup U_i \cup U_{i+1}),$$

then $d(h, \varphi h) < 3 \varepsilon$.

**Proof:** Clear.

**Lemma 2.8:** Let $X$ be a continuum, let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a decreasing sequence of positive numbers such that $\sum \varepsilon_n$ exists, and let $\{s_n\}_{n=1}^{\infty}$ and $\{k_n\}_{n=1}^{\infty}$ be sequences of integers. Also let $\{C_n\}_{n=1}^{\infty}$, $\{\varphi_n\}_{n=1}^{\infty}$, and $\{h_n\}_{n=1}^{\infty}$ be sequences such that

1. $C_n; C_{n+1}, \ldots, C_n, s_n$ is an $\varepsilon_n$-chain of $s_n$
2. $C_{n+1}$ is a uniform refinement of a subchain of $C_n$
3. for $n \geq 2$, $\varphi_{n-1}$ is a sliding homeomorphism
of $k_n$ units on $C_n$, and is a weak sliding homeomorphism of 1 unit on $C_{n-1}$

(4) $h_1$ is a sliding homeomorphism of $k_1$ units on $C_1$, and for $n \geq 2$, $h_n = \phi_{n-1} h_{n-1}$

(5) $h_{n-1}^{-1}(C_n)$ has mesh $< \varepsilon_n$, for $n \geq 2$

and (6) $\phi_{n-1}^{-1}(C_{n+1,i+k_n}) \subseteq C_n,i$, for $1 \leq i \leq s_n - k_n$,

where $C_{n+1,i+k_n}$ is the $(i + k_n)^{th}$ uniform subchain of $C_{n+1}$.

Then (1) $\{h_n\}$ is Cauchy

and (2) the limit map $h$ is a homeomorphism of $X$ onto itself.

PROOF: To show that $\{h_n\}$ is Cauchy, we note that

$h_n = \phi_{n-1} h_{n-1}$ where $\phi_{n-1}$ is supported on $C_n$ and is a weak sliding homeomorphism of 1 unit on $C_{n-1}$. By Lemma 2.7, $d(h_{n-1}, \phi_{n-1} h_{n-1}) < 3 \varepsilon_{n-1}$, and as $h_n = \phi_{n-1} h_{n-1}$, we have $d(h_{n-1}, h_n) < 3 \varepsilon_{n-1}$. But $\leq \varepsilon_n < \infty$; therefore $\{h_n\}$ is Cauchy. Since this is a Cauchy sequence of homeomorphisms of a compact space, $\{h_n\}$ has a limit map $h$ [6, pg 56]. We note that $h$ is the infinite product $\ldots \phi_3 \phi_2 \phi_1 h_1$.

We show that $h$ is onto. Let $x_0 \in X$, and let $a_n = h_n^{-1}(x_0)$. We assume that $\{a_n\}$ converges to some point $a$ of $X$; (if not, we choose a convergent subsequence). Then $h_n(a_n) = x_0$. We show that $h(a) = x_0$ also. Let $\gamma_i \rightarrow 0$. Since $h_n \rightarrow h$, for each $\gamma_i$
there exists an integer $n_1$ such that $d(h_{n_1}, h) < \gamma_1$. Therefore $d(h_{n_1}(a_{n_1}), h(a_{n_1})) < \gamma_1$; that is,

$$d(x_0, h(a_{n_1})) < \gamma_1.$$ But this means that $h(a_{n_1}) \to x_0$.

Now $\{a_{n_1}\}$ is a subsequence of $\{a_n\}$. Therefore $a_{n_1} \to a$. By the continuity of $h$, $h(a_{n_1}) \to h(a)$.

It follows that $x_0 = h(a)$. Therefore $h$ is onto.

To show that $h$ is a homeomorphism, it is sufficient, since $X$ is compact, to show that $h$ is 1-1.

We first observe that $\bigcap_n C_n^*$ is an arc [7, pg 84]. We then note that $h$ carries $\bigcap_n h^{-1}_n(C_n^*) = \bigcap_n h^{-1}_n(C_n^*)$ onto $\bigcap_n C_n^*$. This is true since for each $n$ and each $m > n$, $h_m$ as well as $h_n$, carries $h^{-1}_n(C_n^*)$ onto $C_n^*$. Thus $h$ also carries $h^{-1}_n(C_n^*)$ onto $C_n^*$. It follows that $h$ carries $\bigcap_n h^{-1}_n(C_n^*)$ onto $\bigcap_n C_n^*$, and as $\bigcap_n h^{-1}_n(C_n^*) = \bigcap_n h^{-1}_n(C_n^*)$, $h[\bigcap_n h^{-1}_n(C_n^*)] = \bigcap_n C_n^*$.

We next observe that if $x_0 \notin \bigcap_n h^{-1}_n(C_n^*)$ then $h(x_0)$ is determined at a finite stage. For there exists an integer $N$ such that $x_0 \notin h_{N-1}^{-1}(C_N^*)$. Therefore $h_{N-1}(x_0) \in C_N^*$. But $h = \varphi h_{N-1}$ where $\varphi$ is supported on $C_N^*$. Thus $h(x_0) = h_{N-1}(x_0) = h_n(x_0)$ for all $n > N - 1$, and $h(x_0)$ is determined at a finite stage.

Now, to show that $h$ is 1-1, we consider 3 cases. If $x \neq y$, $x, y \notin \bigcap_n h^{-1}_n(C_n^*)$, then by the above comments,
$h(x)$ and $h(y)$ are determined at finite stages; that is, there exist integers $M, N$ such that $h(x) = h_M(x) = h_m(x)$ for $m > M$, and $h(y) = h_N(y) = h_n(y)$ for $n > N$. Let $R = \max\{M, N\}$. Then $h_R(x) = h(x)$ and $h_R(y) = h(y)$. Since $h_R$ is a homeomorphism, $h_R(x) \neq h_R(y)$; that is, $h(x) \neq h(y)$.

If $x \neq y$, $x \not\in h_{n-1}^{-1}(C_n^*)$ and $y \not\in h_{n-1}^{-1}(C_n^*)$, then by the above comments, $h(x) \not\in C_n^*$, but $h(y) \in C_n^*$. So again, $h(x) \neq h(y)$.

Finally, if $x \neq y$, $x, y \not\in h_{n-1}^{-1}(C_n^*)$, let $\delta = d(x, y)$. Since for each $n$, mesh $h_{n-1}^{-1}(C_n^*) < \epsilon_n$, and $\epsilon_n \to 0$, there is an $N$ such that $x$ and $y$ are separated by at least 7 elements of $h_{N-1}^{-1}(C_N^*)$. Therefore $h_{N-1}(x)$ and $h_{N-1}(y)$ are separated by at least 7 elements of $C_N$. Now, $h_N = \varphi_{N-1} h_{N-1}$ where $\varphi_{N-1}$ is a sliding homeomorphism of $k_N$ units on $C_N$ and $\varphi_{N-1}(C_{N+1,i}) = C_{N,i+k_N}$. But this implies that $h_N(x)$ and $h_N(y)$ are separated by at least 4 elements of $C_N$, since $h_{N-1}(x)$ moves at most $k_N+1$ units to the right, and $h_{N-1}(y)$ moves at least $k_N-2$ units to the right. (For $k_N=1$, this reduces to a movement, at worst, of 2 units to the right for $h_{N-1}(x)$, and 1 unit to the left for $h_{N-1}(y)$.) Now $h = \varphi h_N$ where $\varphi$ is supported on $C_{N+1}$ and is a weak sliding homeomorphism of at most 1 unit on $C_N$. (This is true since $\varphi = \ldots \varphi_{N+2} \varphi_{N+1} \varphi_N$ where $\varphi_i$ is a weak sliding
homeomorphism of 1 unit on $C_i$, and any finite product $\varphi_{N+k} \cdot \ldots \cdot \varphi_{N+1} \varphi_N$ is a weak sliding homeomorphism of at most 1 unit on $C_N$.) Therefore $\varphi$ moves no point, $z$, outside the closure of the union of the link of $C_N$ that contains $z$, its predecessor, and its successor. It follows that $h(x)$ and $h(y)$ are separated by at least one element of $C_N$, and therefore $h(x) \neq h(y)$.

Therefore $h$ is 1-1, and thus a homeomorphism of $X$ onto itself.

**Theorem 2.1:** Let $X$ be a continuum which is locally set-wise homogeneous under the closed subgroup $H$ of $G$. Then $H$ is at least 1-dimensional.

**Proof:** We will show that for sufficiently small $\varepsilon > 0$, every neighborhood $U$ of the identity has a point $h \in H$ on its boundary. We do this by obtaining a sequence of pairs of homeomorphisms of $H$, $\{h_n, g_n\}$, such that

(1) $h_n \in U$, $g_n \in C(U)$

(2) $d(h_n, g_n) < 3 \varepsilon_n$, $\varepsilon_n \to 0$

and (3) $\{h_n\}$ satisfies the conditions of Lemma 2.7.

Then (1) $\{h_n\}$ is Cauchy

(2) the limit map $h$ is a homeomorphism

and (3) $h \in Bd U$.

Since $H$ is closed, $h \in H$, and therefore $H$ is at least 1-dimensional.

Let $g$ be any homeomorphism of $X$ onto itself,
$g \neq e$. Let $x_0$ be a point of $X$ such that $g(x_0) \neq x_0$, and let $\varepsilon = d(x_0, g(x_0))$. Let $U$ be a neighborhood of the identity such that $\text{diam } U < \frac{\varepsilon}{3}$. We will show that $H \cap \text{Bd } U \neq \emptyset$.

Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a decreasing sequence of positive numbers such that (1) $\varepsilon_1 < \frac{\varepsilon}{3}$ and (2) $\sum \varepsilon_n < \infty$.

Let $C_1: C_1, 1, C_1, 2, \ldots , C_1, s_1$ be an $\varepsilon_1$-chain of connected open sets of $X$ from $x_0$ to $g(x_0)$, and let $s_1$ be the number of links of $C_1$. Let $h_1$ be a sliding homeomorphism in $H$ of the largest possible number of units ($k_1$) on $C_1$ such that $h_1 \in U$. (We observe that $k_1 < s_1 - 2$, for any sliding homeomorphism of $s_1 - 2$ units moves some point of $C_1, 1$ to some point of $C_1, s_1$, and therefore is of distance more than $\frac{\varepsilon}{3}$ from the identity. Thus such a homeomorphism is in $C(U)$.)

Since $h$ is a sliding homeomorphism of $k_1$ units on $C_1$, by Lemma 2.4, there exists a chain $D_1: D_1, 1, \ldots , D_1, s_1 - k_1$, whose links are connected open sets such that

(1) $D_1, i \subseteq C_1, i + k_1$

and (2) $h_1^{-1}(D_1, i) \subseteq C_1, i$.

Let $\psi_1$ be a $\gamma_1$-sliding homeomorphism in $H$ of 1 unit on $D_1$, where $2\gamma_1 < \min_i \{d(\overline{C}_1, i - 2, h(\overline{C}_1, i)) \cup \overline{C}_1\}$, as in Lemma 2.5. Then $\psi_1 h_1$ is a sliding homeomorphism of $k_1 + 1$ units on $C_1$, and thus is in $C(U)$. Let $g_1 = \psi_1 h_1$. By Lemma 2.7, $d(h_1, g_1) < 3 \in_1$. Then $h_1 \in U, g_1 \in C(U)$,
and \(d(h_1, g_1) < 3 \epsilon_1\).

Now \(\epsilon_2 > 0\). By the uniform continuity of \(h_1^{-1}\), there exists a \(\delta_1 > 0\) such that \(h_1^{-1}\) takes any set of diameter less than \(\delta_1\) to a set of diameter less than \(\epsilon_2\). Let \(C_2 : C_{2,1}, \ldots, C_{2,s_2}\) (where \(s_2 = t_2 \cdot (s_1 - k_1)\)) be a chain of connected open sets of mesh less than \(\min \{\gamma_1, \delta_1, \epsilon_2\}\), which is a uniform refinement of \(D_1\).

Let \(\phi_1\) be a sliding homeomorphism in \(H\) of the largest possible number of units \(k_2 < t_2\) on \(C_2\) such that \(\phi_1 h_1\) is an element of \(U\). We note that, by Lemma 2.6, if \(\phi\) is a sliding homeomorphism of \(t_2\) units on \(C_2\), then \(\phi h_1\) is a sliding homeomorphism of \(k_1 + 1\) units on \(C_1\), and therefore must be in \(C(U)\).

Let \(h_2 = \phi_1 h_1\).

Since \(\phi_1\) is a sliding homeomorphism of \(k_2\) units on \(C_2\), there exists a chain \(D_2 : D_{2,1}, \ldots, D_{2,s_2-k_2}\), whose links are connected open sets, such that

1. \(D_{2,i} \subseteq C_{2,i+k_2}\)

and (2) \(\phi_1^{-1}(D_{2,i}) \subseteq C_{2,i}\).

Let \(\psi_2\) be a \(\gamma_2\)-sliding homeomorphism in \(H\) of 1 unit on \(D_2\), where \(2 \gamma_2 < \min \{d(C_{2,i-2}, h_2(C_{2,i}), \epsilon_2), \epsilon_2 \}\) as in Lemma 2.5. Then \(\psi_2 \phi_1\) is a sliding homeomorphism of \(k_2 + 1\) units on \(C_2\), and thus \(\psi_2 \phi_1 h_1 = \psi_2 h_2\) is in \(C(U)\).

Let \(g_2 = \psi_2 h_2\). By Lemma 2.5, \(d(h_2, g_2) < 3 \epsilon_2\). Then \(h_2 \in U, g_2 \in C(U)\), and \(d(h_2, g_2) < 3 \epsilon_2\).
We continue the above process inductively, thus obtaining sequences of integers, \( \{k_n\} \) and \( \{s_n\} \), and sequences \( \{c_n\} \), \( \{\varphi_n\} \), and \( \{h_n\} \) which satisfy the hypotheses of Lemma 2.8. Then \( \{h_n\} \) is Cauchy, and converges to a limit homeomorphism \( h \). Since \( d(h_n, g_n) \to 0 \), \( g_n \to h \) also. Therefore \( h \in \text{Bd} \ U \). Since \( H \) is closed, and each \( h_n \in H \), \( h \in H \). It follows that \( H \) is at least 1-dimensional.

**COROLLARY 2.1.1**: If \( X \) is a locally setwise homogeneous continuum, then \( G(X) \) is at least 1-dimensional.

**PROOF**: Clear.

**COROLLARY 2.1.2**: Let \( X \) be a continuum which contains an open set \( U \) whose closure \( \overline{U} \) is locally setwise homogeneous. Then \( G(X) \) is at least 1-dimensional.

**PROOF**: Let \( G' \) be the group of all homeomorphisms of \( \overline{U} \) onto itself, and let \( H' \) be the subgroup of \( G' \) which consists of those homeomorphisms of \( \overline{U} \) onto itself which are supported on \( U \) — that is, which are the identity on \( \overline{U} - U \). Then it follows easily that \( H' \) is a closed subgroup of \( G' \), and \( \overline{U} \) is locally setwise homogeneous under \( H' \). Thus, by Theorem 2.1, \( H' \) is at least 1-dimensional.

Let \( H(X) \) be the subgroup of \( G(X) \) which consists of those homeomorphisms of \( X \) onto itself which are supported on \( U \). Then \( H(X) \) can be identified with \( H' \),
and it follows that $H(X)$, and therefore $G(X)$, is at least 1-dimensional.

**QUESTION:** Let $X$ be a locally setwise homogeneous continuum. What is the dimension of $G(X)$? Must $G(X)$ be infinite dimensional? Can it be of finite positive dimension?
CHAPTER III

APPLICATIONS OF THEOREM 2.1

In this chapter we first show that the universal plane curve\(^1\) and the universal curve\(^2\) are each locally setwise homogeneous. Thus it follows, as a corollary of Theorem 2.1, that the groups of all homeomorphisms of these continua are at least 1-dimensional.

We next assert that the group, \(H\), of those homeomorphisms of a compact \(n\)-manifold \(M^n\), \(n > 1\), which carry a countable dense subset of \(M^n\) onto itself is at least 1-dimensional. We note in Corollary 2 of Lemma 3.3 that \(M^n\) is locally setwise homogeneous under \(H\). However, by Corollary 1 of Lemma 3.3, \(H\) is dense in \(G(M^n)\). Thus Theorem 2.1 does not apply directly, since the limit homeomorphism \(h\), of the sequence of homeomorphisms \(\{h_i\}\) of \(H\), may not be in \(H\). However, in the latter half of this chapter, we will indicate how the proof of Theorem 2.1 can be modified so that \(h\) must be in \(H\).

DEFINITION 3.1: The standard construction for the universal plane curve is the following: Let \(S\) be a square plus its interior in the plane. Divide \(S\) into 9 equal

\(^1\)See below for definition.
\(^2\)Ibid.
squares, and remove the interior of the middle ninth. Break each of the remaining 8 squares into 9 equal squares, and remove the interiors of their middle ninths. Continue the process inductively. The set which remains is a continuum called the universal plane curve.

This continuum has been characterized by G. T. Whyburn [3] as the only locally connected, 1-dimensional, plane continuum with no local cut points.

If $M$ is any such continuum imbedded in the plane, then the simple closed curve which bounds the unbounded complementary domain of $M$ is called the outer boundary of $M$ and is denoted by $C_0$. The collection of simple closed curves, different from $C_0$, and which bound complementary domains of $M$, will be denoted by $\{C_i\}_{i=1}^{\infty}$. The points of $M$ not on any complementary bounding simple closed curve are called its interior points.

**DEFINITION 3.2:** The standard construction for the universal curve is the following: Let $F_1$, $F_2$, and $F_3$ be 3 faces of the solid cube $C$ in $E^3$ such that no 2 of these faces are opposite each other. We remove portions of $C$ by punching out, to the opposite side, the interiors of the middle ninths of $F_1$, $F_2$, and $F_3$. We next punch out, to the opposite side, the interiors of the middle ninths of each of the remaining 8 squares in $F_1$, $F_2$, and $F_3$, and we continue the process inductively. The remaining set is a continuum called the universal curve.
This continuum has been characterized by R. D. Anderson, Theorem 12 of [2], as the only locally connected, 1-dimensional continuum with no local cut points, such that no open subset is imbeddable in the plane.

**DEFINITION 3.3:** Let \( M \) be a separable metric space such that, for some \( n \), each point of \( M \) has a neighborhood whose closure is homeomorphic to an \( n \)-cell. Then \( M \) is called a manifold.

**LEMMA 3.1:** Let \( M \) be the universal plane curve. Then there exists a basis, \( \mathcal{E} \), of open sets of \( M \) such that the closure of each element of \( \mathcal{E} \) is homeomorphic to the universal plane curve.

**PROOF:** We think of \( M \) as having its standard construction in the plane. First we will decompose \( M \) into 4 "equal" subcontinua, which intersect, pairwise, in Cantor sets or in the empty set. We accomplish this by chopping up \( M \) by means of a vertical and a horizontal line in \( \mathbb{E}^2 \) through the midpoints of 2 adjacent sides. Each of the 4 subcontinua is the closure of an open set and is also a universal plane curve, whose outer boundary meets both the interior of \( M \) and also points of some of the complementary bounding simple closed curves of \( M \).

We next break up each of the 4 universal plane curves in a similar manner, thus obtaining 16 "equal" universal plane curves, each of which is the closure of an open set,
and such that the intersection of any 2 is empty or is a Cantor set. Continue the process inductively. We observe that the diameters of these sets have limit 0.

We now enlarge each of these sets slightly, accomplishing this again by means of vertical and horizontal lines near the lines determining these sets, and intersecting M in Cantor sets. Then the collection of interiors of each of these new sets forms the desired basis.

**Lemma 3.2:** Let M be a universal plane curve, and let a and b be interior points of M. Then there exists an arc N in M, such that N contains a and b, and N lies in the interior of M, except for its endpoints which are in C0.

**Proof:** By a theorem of R. L. Moore [7, pg 363] there exists a monotone map ϕ of M onto a disk D, (which we think of as the unit disk in E2), such that the non-degenerate inverses of M are the elements of ∪{Ci}i=1∞. Let C = ∪ Ci, and let A = ϕ(C). Then A is a countable dense subset of the interior of D, and ϕ(a) and ϕ(b) are 2 points of D° - A.

Let L1 and L2 be 2 parallel lines in E2 containing ϕ(a) and ϕ(b) respectively, and let L1' = L1 ∩ D, and let L2' = L2 ∩ D. Let D1 be the portion of D bounded by Bd D and L1', let D2 be the portion of D bounded by Bd D, L1', and L2', and let
be the portion of \( D \) bounded by \( \text{Bd } D \) and \( L_2' \).

Now there are uncountably many polygonal arcs joining \( \varphi(a) \) and a point of \( \text{Bd } D \), and which lie in \( D_1^0 \) except for their endpoints. Let \( P_1 \) be such a polygonal arc which misses \( A \). For similar reasons, there is a polygonal arc, \( P_2 \), which joins \( \varphi(a) \) and \( \varphi(b) \), and lies, except for its endpoints, in \( D_2^0 - A \).

Also, there is a polygonal arc, \( P_3 \), joining \( \varphi(b) \) and some point of \( \text{Bd } D \), and which lies, except for its endpoints, in \( D_3^0 - A \). Thus \( P = P_1 \cup P_2 \cup P_3 \) is a (polygonal) arc which lies, except for its endpoints, in \( D^0 - A \).

Let \( N = \varphi^{-1}(P) \). Then \( N \subseteq M \), and is the desired arc.

**THEOREM 3.1**: The universal plane curve, \( M \), is locally setwise homogeneous.

**PROOF**: Let \( B \) be the set of interior points of \( M \). Then \( B \) is a dense subset of \( M \). Let \( \mathcal{E} \) be the basis of open sets constructed in Lemma 3.1. Let \( E \in \mathcal{E} \), and let \( a, b \in B \cap E \). We will show that there exists an \( h \in G(M) \) such that \( h \) is supported on \( E \) and \( h(a) = b \).

By Lemma 3.2, there exists an arc \( N_o \) in \( \overline{E} \) such that \( N_o \) contains \( a \) and \( b \), and \( N_o \) lies in the interior of \( \overline{E} \), except for its endpoints, which are in the outer boundary of \( \overline{E} \).

The outer boundary of \( \overline{E} \), together with \( N_o \), form
a $\varphi$-curve, $N$. Let $N = N_1 \cup N_2$ where $N_1$ and $N_2$ are simple closed curves such that $N_1 \cap N_2 = N_0$, and $N_1$ is the outer boundary of the universal plane curve $Q_1$, and $N_2$ is the outer boundary of the universal plane curve $Q_2$.

Let $\varphi$ be any homeomorphism of $N$ onto itself such that $\varphi$ carries $a$ to $b$ and is the identity on $N - N_0$. We note that $\varphi$ is the identity on the outer boundary of $\overline{E}$. Let $\varphi_1 = \varphi|_{N_1}$, and let $\varphi_2 = \varphi|_{N_2}$. By Theorem 1 of [3], there exist extensions $\varphi_1^*$ of $\varphi_1$ to $Q_1$ and $\varphi_2^*$ of $\varphi_2$ to $Q_2$. Let $h$ be the homeomorphism of $M$ onto itself which is the identity outside $\overline{E}$, $\varphi_1^*$ on $Q_1$, and $\varphi_2^*$ on $Q_2$. Then $h$ is a homeomorphism of $M$ onto itself, $h$ is supported on $E$, and $h(a) = b$. Thus, the universal plane curve is locally setwise homogeneous.

**COROLLARY 3.1.1:** The group of all homeomorphisms of the universal plane curve is at least 1-dimensional.

**PROOF:** This follows immediately from Theorem 2.1 and Theorem 3.1 above.

**THEOREM 3.2:** The universal curve is locally setwise homogeneous.

**PROOF:** This is just a rewording of a statement given in the proof of Theorem 16 of [2].
**COROLLARY 3.2.1:** The group of all homeomorphisms of the universal curve is at least 1-dimensional.

**PROOF:** This follows immediately from Theorem 2.1 and Theorem 3.2 above.

We next consider the group, $H$, of those homeomorphisms of a compact $n$-manifold, $M^n$, $n > 1$, which carry a countable dense subset, $A = \{a_i\}_{i=1}^{\infty}$, of $M^n$ onto itself. We indicate how the proof of Theorem 2.1 can be modified to show that $H$ is at least 1-dimensional.

**Modification of Proof of Theorem 2.1**

We will establish below, in Lemmas 3.3, 3.4, and 3.5, the existence in $H$ of sliding homeomorphisms of $k$ units, $1 \leq k \leq m$, on a chain $C: C_1, \ldots, C_m$, in $M^n$. Thus we may choose the sequences of sliding homeomorphisms, $\{h_i\}$ and $\{g_i\}$ of Theorem 2.1, to be elements of $H$.

Since the manifold is of dimension greater than 1, we may further require that $C_i^\ast$ miss $a_i$. Thus $\bigcap_i C_i^\ast$ misses $A$, and the limit homeomorphism, $h$, carries $A$ onto itself. It follows that $h \in H$, and thus $H$ is non-zero dimensional.

We now prove the necessary lemmas.

**Lemma 3.3:** Let $U$ be a connected open subset of a compact
manifold, \( M \), and let \( A = \{a_i\}_{i=1}^{\infty} \) be a countable dense subset of \( U \). Let \( h \) be a homeomorphism supported on \( U \). Then for each \( \varepsilon > 0 \), there exists a homeomorphism \( g \) supported on \( U \) such that \( d(h, g) < \varepsilon \) and \( g(A) = A \).

**Proof:** Let \( \{\varepsilon_i\} \) be a decreasing sequence of positive numbers such that \( \sum \varepsilon_i < \varepsilon \). Let \( b_1 \) be the first point in the sequence \( \{a_i\} \) such that at least one of \( h^{-1}(b_1) \) and \( h(b_1) \) is not in \( A \). Let \( U_1 \) and \( V_1 \) be disjoint \( \varepsilon_1 \)-neighborhoods of \( b_1 \) and \( h(b_1) \), respectively, whose boundaries miss \( A \cup h(A) \), and each of which misses \( a_i \) and \( h(a_i) \) for each \( a_i \) which precedes \( b_1 \) in the ordering of \( A \). Let \( \varphi_1 \) be a homeomorphism supported on \( U_1 \) and carrying some point of \( h(A) \) to \( b_1 \), if \( h^{-1}(b_1) \) is not in \( A \). Let \( \psi_1 \) be a homeomorphism supported on \( V_1 \) and carrying \( h(b_1) \) to some point of \( A \), if \( h(b_1) \) is not in \( A \). Let \( g_1 = \psi_1 \varphi_1 h \). Then \( d(h, g_1) < \varepsilon_1 \), and \( g_1^{-1}(b_1) \) and \( g_1(b_1) \) are in \( A \) as well as \( g_1^{-1}(a_i) \) and \( g_1(a_i) \) for each \( a_i \) preceding \( b_1 \). The homeomorphism \( g_1 \) is a first approximation to \( g \).

We again note that \( g_1^{-1}(a_i) \) and \( g_1(a_i) \) are elements of \( A \) for all \( a_i \) which precede \( b_1 \), as well as for \( b_1 \). Let \( b_2 \) be the first point in the ordering of \( A \) such that at least one of \( g_1^{-1}(b_2) \) and \( g_1(b_2) \) is

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3 This proof was outlined to me by R. D. Anderson.
not in A. Since $g_1^{-1}$ is uniformly continuous, there is a $\delta_1 > 0$ such that $g_1^{-1}$ takes any $\delta_1$-set to a set of diameter $< \varepsilon_2$. Let $U_2$ and $V_2$ be 2 disjoint, connected, open sets of diameter less than $\min \{ \delta_1, \varepsilon_2 \}$ and containing $b_2$ and $g_1(b_2)$ respectively, with the following restrictions:

1. if $b_2 \in U_1$ (or $V_1$) then $\overline{U}_2 \subseteq U_1$ (or $V_1$)
2. if $g_1(b_2) \in U_1$ (or $V_1$) then $\overline{V}_2 \subseteq U_1$ (or $V_1$)
3. if $b_2 \in C(U_1 \cup V_1)$ then $\overline{U}_2 \subseteq C(U_1 \cup V_1)$
4. if $g_1(b_2) \in C(U_1 \cup V_1)$ then $\overline{V}_2 \subseteq C(U_1 \cup V_1)$

We also require that $\text{Bd } U_2$ and $\text{Bd } V_2$ miss $A \cup g_1(A)$ (a countable set) and that $U_2$ and $V_2$ each miss all the (finitely many) points $a_i$ and $h(a_i)$ for $a_i$ preceding $b_2$. Let $\varphi_2$ be a homeomorphism supported on $U_2$ and carrying some point of $g_1(A)$ to $b_2$, if $g_1^{-1}(b_2)$ is not in $A$; let $\psi_2$ be a homeomorphism supported on $V_2$ and carrying $g_1(b_2)$ to some point of $A$, if $g_1(b_2)$ is not in $A$. Let $g_2 = \psi_1 \varphi_1 g_1$. Then $d(g_1, g_2) < \varepsilon_2$, and $g_2^{-1}(b_2)$ and $g_2(b_2)$ are in $A$, as well as $g_2^{-1}(a_i)$ and $g_2(a_i)$ for all $a_i$ preceding $b_2$.

We continue the above process inductively, with the following additional restriction: we require that $\overline{U}_n$ and $\overline{V}_n$ be entirely contained in any $U_i$ or $V_i$, $i < n$, which it meets. Thus we obtain a sequence of homeomorphisms, $\{g_i\}$, such that $d(g_{i-1}, g_i) < \varepsilon_i$. Since $\sum \varepsilon_i < \infty$, $\{g_i\}$ is Cauchy, and thus converges to a map $g$. We will show that $g$ is 1-1, and therefore a
homeomorphism.

Let \( x \neq y \), and let \( \delta = d(x, y) \). There exists an integer \( N \) such that for \( n > N \), the diameter of \( g_{n-1}^{-1}(U_n) < \delta \) and the diameter of \( g_{n-1}^{-1}(V_n) < \delta \). Then not both \( g_{n-1}(x) \) and \( g_{n-1}(y) \) can be in the same one of these sets \( U_n \) or \( V_n \). Thus if \( g_{n-1}(x) \in U_n \) (or \( V_n \)) and \( g_{n-1}(y) \notin U_n \) (or \( V_n \)), then because of the (partial) towering of the \( U_i \)'s (or \( V_i \)'s), \( g(x) \in U_n \) (or \( V_n \)) and \( g(y) \notin U_n \) (or \( V_n \)); that is, \( g(x) \neq g(y) \). If neither \( g_{n-1}(x) \) nor \( g_{n-1}(y) \) is in any \( U_n \) or \( V_n \) for \( n > N \), then each is determined at a finite stage; that is, \( g(x) = g_N(x) \) and \( g(y) = g_N(y) \) for some \( N \), and since \( g_N \) is a homeomorphism, \( g_N(x) \neq g_N(y) \). So again we have \( g(x) \neq g(y) \). Thus \( g \) is 1-1 and therefore a homeomorphism.

Since \( d(h, g_1) < \varepsilon_1 \), and \( d(g_{i-1}, g_i) < \varepsilon_i \), it follows that \( d(h, g) < \Sigma \varepsilon_i < \varepsilon \). Thus \( g \) is the desired homeomorphism.

**REMARK:** In the above proof, the sequence \( \{\varepsilon_i\} \) could have been chosen to be a decreasing sequence of numbers with \( \varepsilon_1 < \varepsilon \), and so that \( \varepsilon_i \to 0 \). The limit map \( g \) would still exist and be the desired homeomorphism.

**COROLLARY 3.3.1:** The group of those homeomorphisms of a compact manifold, \( M \), taking a countable dense subset of \( M \) onto itself, is dense in the group of all homeomorphisms of the manifold.
PROOF: Clear.

**COROLLARY 3.3.2:** Let $M$ be a compact manifold, $A$ be a countable dense subset of $M$, and $H$ be the group of those homeomorphisms of $M$ onto itself, which carry $A$ onto itself. Then $M$ is locally setwise homogeneous under $H$.

**PROOF:** Let $U$ be a connected open set in $M$, let $a, b \in U \cap A$. It is easy to show that there is a homeomorphism $h \in G(M)$ such that $h(a) = b$, and $h$ is supported on $U$. By Lemma 3.3, $h$ can be modified to a homeomorphism $g$ such that $g$ is supported on $U$ and $g(A) = A$. We can insure that $g(a) = b$, simply by choosing each of the $U_i$'s and $V_i$'s of Lemma 3.3 so that they miss the point $b$.

Thus $g$ is a homeomorphism supported on $U$ such that $g(a) = b$, $g(A) = A$, and $g \in H$. It follows that $M$ is locally setwise homogeneous under $H$.

**REMARK:** We observe that if $M$ is a 1-manifold, then $H$ is zero dimensional. (This is easy to prove.) Thus local setwise homogeneity of a space under a group, $H$, is not sufficient to guarantee that $H$ is at least 1-dimensional. Therefore the hypothesis of Theorem 2.1, that $H$ be closed, cannot be omitted without substituting some other condition to insure that $H$ is at least 1-dimensional.
LEMMA 3.4: Let $C : C_1, ..., C_n$ be a chain of connected open spheres of the compact manifold, $M$, whose boundaries miss a countable dense set $A$. Let $h$ be a sliding homeomorphism of $k$ units on $C$, let $\delta = \min \{d(C_{i-2}, h(C_i)) \}$, and let $\varepsilon < \delta$. Then there exists a homeomorphism $g$ such that $d(g, h) < \varepsilon$, $g(A) = A$, and $g$ is also a sliding homeomorphism of $k$ units on $C$.

PROOF: Let $D : D_1, ..., D_{n-k}$ be a chain so that
(1) $D_i \subset C_{i+k}$ and (2) $h^{-1}(D_i) \subset C_i$. We may further require that $\bigcup_i \text{Bd } D_i$ miss $A \cup h(A)$, for, if not, we can always choose a chain which is a straight refinement of $D$ and such that the boundary of each of its links misses $A \cup h(A)$.

We construct $g$ as a limit of a sequence $\{g_i\}$, as in Lemma 3.3, with the following additional restriction: each $\overline{U_i}$ and $\overline{V_i}$ is a subset of either $D^*$ or $C(D^*)$, and lies entirely in any $C_j$ which it meets. Then $d(g, h) < \varepsilon$, $g(A) = A$, and $g$ is also a sliding homeomorphism of $k$ units on $C$.

LEMMA 3.5: Let $M$ be a manifold, let $C : C_1, ..., C_n$ be a chain of connected open sets of $M$, and let $A$ be a countable dense subset of $M$. Let $H$ be the group of those homeomorphisms of $M$ which carry $A$ onto itself. Then for each integer $k$, $1 \leq k \leq n$, there exists a
sliding homeomorphism, \( h \), of \( k \) units on \( C \), and such that \( h \in H \).

**Proof:** This follows from Lemma 3.4 above, and Lemma 2.4 of Chapter 2, which asserts the existence of sliding homeomorphisms.

Thus, with the proofs of these lemmas, we have completed (the outline of) the proof of the following:

**Theorem 3.3:** The group of those homeomorphisms of a compact \( n \)-manifold, \( M^n \), \( n > 1 \), which carry a countable dense subset of \( M \) onto itself, is at least \( 1 \)-dimensional.

**Question 1:** What is the dimension of the group of homeomorphisms of the universal plane curve? universal curve?

**Question 2:** What is the dimension of the group of those homeomorphisms of a compact \( n \)-manifold, \( n > 1 \), which carry a countable dense subset of the \( n \)-manifold onto itself?
CHAPTER IV

ONE-DIMENSIONAL CONTINUA

In this section we investigate the dimensions of the groups of homeomorphisms of 1-dimensional continua. It is known and easy to show (see Theorem 1.1) that the group of homeomorphisms of an arc is infinite dimensional. On the other hand, there are dendrons and other 1-dimensional continua [8] which admit no homeomorphisms onto themselves, other than the identity.

Our main results are the following:

(1) The group of homeomorphisms, $G$, of a regular curve, $^1 M$, is either 0-dimensional or infinite dimensional. $G$ is infinite dimensional if $M$ contains a free arc, $^2 0$-dimensional otherwise.

(2) For each positive integer $n$, there exists a rational curve, $^3 M_n$, such that $1 \leq \dim G(M_n) \leq n$, and $G(M_n)$ is homeomorphic to the product of $n$ 1-dimensional groups.

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$^1$A regular curve is a continuum in which every point has arbitrarily small neighborhoods with finite boundaries [5, pg 96]. A regular curve is locally connected [4, pg 99].

$^2$A free arc is an arc whose interior is an open subset of the space.

$^3$A rational curve is a continuum in which every point has arbitrarily small neighborhoods with countable boundaries [5, pg 96]. A rational curve need not be locally connected.
DEFINITION 4.1: Let $M$ be a regular curve, $B$ an order basis for $M$. Let $W : W_1, \ldots, W_n$ be a cover of $M$ of mesh $< \epsilon$ with the following properties for each $i$:

1. $W_i$ is a continuum which is the closure of an open set,
2. $Bd W_i$ is a finite subset of $B$, and
3. $W_i \cap W_j = Bd W_i \cap Bd W_j$ for each $j \neq i$. Then $W$ is called a regular $\epsilon$-cover with respect to $B$.

LEMMA 4.1: Let $M$ be a regular curve and let $B$ be an order basis for $M$ which contains all the local cut points of order $\geq 3$. Then for each $\epsilon > 0$, there exists a regular $\epsilon$-cover with respect to $B$.

PROOF: Let $U$ be a finite $\frac{\epsilon}{3}$-cover of $M$ whose elements have finite boundaries. We will modify the cover $U$ to obtain a cover $U'$, each of whose elements has a finite boundary in $B$. Let $B'$ be the set of boundary points of elements of $U$. This is a finite set, and therefore there exists a $\delta > 0$ such that $d(b_i, b_j) > \delta$ for each pair $b_i, b_j \in B'$. Let $b \in B'$ on the boundary of some

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4 $B$ is an order basis for a regular curve $M$ if every point $x \in M$ has arbitrarily small neighborhoods $U_x$ with boundaries in $B$ such that the number of points on $Bd U_x$ is less than or equal to the order of $x$. [5, pg 277]. (See footnote 5 for definition of "order").

5 The order of a point $x$ of $M$ is the smallest number $n$ for which there exist arbitrarily small neighborhoods with boundaries consisting of less than or equal to $n$ points, if such exists. If not, we say the point is of infinite order [5, pg 97-99].
U \in \mathcal{U}. \text{ If } b \text{ is also in } B, \text{ we leave it alone. If not, there are two possibilities: (1) } b \text{ is not a local cut point, or (2) } b \text{ is a local cut point of order 2. In the first case } U \cup \{b\} \text{ is also open, and we add } b \text{ to } U. \text{ In the second case, there exists a neighborhood } N_b \text{ of } b \text{ of diameter } < \min \left\{ \delta, \frac{\varepsilon}{3} \right\} \text{ with } Bd N_b \text{ containing at most 2 points in } B, \text{ since } B \text{ is an order basis for } M. \text{ We add such a neighborhood to } U. \text{ Thus we have modified each element of } \mathcal{U} \text{ a finite number of times. We now have a new cover } \mathcal{U}' \text{ of open sets of diameter } < \varepsilon, \text{ each of which has a finite boundary which is a subset of } B. \text{ We choose a minimal subcover } \mathcal{U}' \text{ of } \mathcal{U}'.

Let \mathcal{U}' \text{ have the elements } U'_1, U'_2, \ldots, U'_m. \text{ Then we obtain a new cover } \mathcal{V} \text{ in the following way: let } V_1 = U'_1, V_k = U'_k - \bigcup_{i=1}^{k-1} U'_i \text{ for } k > 1. \text{ Then } \mathcal{V} \text{ has at most } m \text{ elements. } \mathcal{V} \text{ is a finite cover of } M \text{ of mesh } < \varepsilon \text{ whose elements are the closures of open sets, and whose boundaries are finite subsets of } B. \text{ Let } \mathcal{W} \text{ be the collection of components of the elements of } \mathcal{V}. \text{ Since each element of } \mathcal{V} \text{ has only a finite number of boundary points, and each of these can belong to at most 1 component of that element, } \mathcal{W} \text{ is finite. Then } \mathcal{W} \text{ is the desired cover.
LEMMA 4.2: Let $M$ be a regular curve, $b$ a local cut point of $M$, and $N$ a connected neighborhood of $b$ such that $N - b = A_1 \cup A_2$ where $A_1$ and $A_2$ are mutually separate sets. Let $S$ be a sphere about $b$ (the set of all points a fixed distance from $b$) such that $S \subset N$ and $S \cap A_1 \neq \emptyset$ and $S \cap A_2 \neq \emptyset$. Then there exists an arc $C$ from some point of $S \cap A_1$ to some point of $S \cap A_2$ such that $C^0$ is a subset of the "interior" of $S$.

PROOF: Let $x$ be any point of $S \cap A_1$ and let $y$ be any point of $S \cap A_2$. Since $N$ is locally connected and connected, there exists an arc $C'$ in $N$ from $x$ to $y$. Since $b$ separates $A_1$ from $A_2$, $C'$ must contain $b$. Further, the subarc of $C'$ from $x$ to $b$ contains no point of $A_2$, and the subarc of $C'$ from $b$ to $y$ contains no point of $A_1$.

Let $x_1$ be the first point of $S \cap A_1$ on the arc from $b$ to $x$, and let $y_1$ be the first point of $S \cap A_2$ on the arc from $b$ to $y$. The subarc $C$ of $C'$ from $x_1$ to $y_1$ is an arc from some point of $S \cap A_1$ to some point of $S \cap A_2$ and we show that $C^0$ lies in the interior of $S$. Since $S$ separates its interior from its exterior, and $C^0$ contains no point of $S$, $C^0$ is in the interior of $S$ or the exterior of $S$. But $C^0$ contains $b$, and therefore $C^0$ is in the interior of $S$. 


LEMMA 4.3: Let $M$ be a regular curve, $U$ be a connected open subset of $M$ such that $\overline{U}$ contains no local cut points of $M$ of order $> 2$ and such that $\text{Bd} \ U$ is a finite set of local cut points of $M$. Then $\overline{U}$ is a regular curve containing no local cut points of order $> 2$ (in $\overline{U}$).

PROOF: A subcontinuum of a regular curve is a regular curve [4, pg 99]. If a point of $U$ is a local cut point of order $> 2$ in $\overline{U}$, then it is also a local cut point of order $> 2$ in $M$. Let $b \in \overline{U} - U$. Then $b$ is a local cut point of $M$. If it is also a local cut point of $\overline{U}$, then it is not of order $> 2$ in $\overline{U}$. For if it were, it would be a local cut point of $M$ of order $> 2$, but this is a contradiction.

LEMMA 4.4: Let $M$ be a regular curve containing no local cut points of order $> 2$. Then $M$ is an arc or a simple closed curve.

PROOF: Let $B$ be a countable dense set of local cut points of order $2$ of $M$. Then $B$ is an order basis for $M$ [5, pg 278]. Therefore, for each $\varepsilon > 0$, there exists a regular $\varepsilon$-cover of $M$ with respect to $B$. It follows that every point of $M$ is a point of order at most $2$. The points of $B$ are of order $2$. Therefore $M$ is a
curve of the second order,\(^6\) and it follows that \(M\) is an arc or a simple closed curve [5, pg 267].

**THEOREM 4.1:** Let \(M\) be a regular curve with a dense set of local cut points of order \(> 2\). Then \(G(M)\) is 0-dimensional.

**PROOF:** It is sufficient to show that for each \(\varepsilon > 0\), there is an open and closed set \(H\) of diameter \(< 4\varepsilon\) which contains the identity.

Let \(\varepsilon > 0\), and let \(B\) be the set of local cut points of order \(> 2\) of \(M\). We know that \(B\) is countable and dense and therefore an order basis for \(M\) [5, pg 164, 278]. By Lemma 4.1, there exists a cover \(W\) of \(M\) which is a regular \(\varepsilon\)-cover with respect to \(B\).

Let \(B' = \{b \in B : b\) is a point of \(W_i \cap W_j, W_i, W_j \in W,\) for some \(i, j; i \neq j\}\). Since \(M\) is locally connected, and \(B'\) is a finite set, we can find about each \(b \in B'\) a locally connected neighborhood \(N_b\) such that the set \(\{N_b : b \in B'\}\) satisfies the following property: \(\overline{N}_{b_i} \cap \overline{N}_{b_j} = \emptyset\), if \(b_i \neq b_j\). Let \(S_b\) be a sphere about \(b\) (the set of all points a fixed distance from \(b\)) which is contained in \(N_b\) and which contains no point of \(B\). We further require that \(S_b\) be small enough

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\(^6\)A curve of the \(n\)th order is a continuum which contains no points of order \(> n\), but at least 1 point of order \(n\) [5, pg 98].
so that it contains points of each $W_i$ which contains $b$. Such a sphere exists, since $B$ is countable. Let $S = \bigcup_{b \in B} S_b$, and let $A_i = S \cap W_i$. We note that $A_i$ is a closed subset of $W_i^0$.

Let $H = \{ h \in G(M) : h(A_i) \subseteq W_i \}$. It is clear that $e \in H$, since $e(A_i) = A_i \subseteq W_i$. We show that $H$ is open, closed, and of diameter $< 4\varepsilon$.

We first show that $H$ is open. Let $h \in H$. Then $h(A_i)$ is a closed subset of $W_i^0$. Let $\varepsilon = \min \{ d(h(A_i)) \}$, complement of $W_i^0$. Since these are disjoint closed sets, $\varepsilon$ is positive. Let $K$ be a $\varepsilon$-sphere of homeomorphisms about $h$. We show that $K \subseteq H$. Let $g \in K$. Then $d(h(x), g(x)) < \varepsilon$ for all $x$, in particular for $x \in A_j$. Therefore $g(A_j) \subseteq W_j$ and it follows that $g \in H$.

We next show that $H$ is closed. Let $\{ h_n \}$ be a sequence of homeomorphisms of $H$ such that $\{ h_n \}$ converges to some homeomorphism $h$. We show that $h \in H$. Let $x \in A_k$. Then $h_n(x) \in W_k$ for all $n$. Since $W_k$ is closed, $h(x) \in W_k$ also. Therefore $h(A_k) \subseteq W_k$; that is, $h \in H$.

Finally we show that the diameter of $H$ is less than $4\varepsilon$. Clearly $e \in H$. It is sufficient to show that if $h \in H$, $d(h, e) < 2\varepsilon$. It then follows by the triangle inequality that diameter $H < 4\varepsilon$. To this end, let $h \in H$ and suppose that $x \in W_i$ and $h(x) \in W_j$ where $W_i \cap W_j = \emptyset$. We will show that this is
impossible; that is, $$d(x, h(x)) < 2 \varepsilon$$.

Let $$B'' = \{b_1, \ldots, b_r\}$$ be the points of $$\partial \omega_{w_2}$$; that is, the points of $$B' \cap \omega_{w_2}$$. Let $$C_j$$ be an arc from some point of $$A_i$$ to some point of $$A_i$$ for $$i \neq i_2$$, such that $$C_j \subseteq (S_{b_j}^\circ$$ plus its interior). By Lemma 4.2, such $$C_j$$'s exist. We note that $$C_j$$ contains $$b_j$$. We therefore have a finite collection of pairwise disjoint arcs $$C_1, \ldots, C_r$$ whose union contains $$B''$$. Since each of these arcs $$C_i$$ has an endpoint whose image is in $$\omega_{w_2}^\circ$$ and an endpoint whose image is not in $$\omega_{w_2}^\circ$$, and since $$B''$$ separates $$\omega_{w_2}^\circ$$ from $$C(\omega_{w_2})$$, $$h(C_i^\circ)$$ must contain at least 1 point of $$B''$$. But we have a total of $$r$$ disjoint arcs (and therefore their images are disjoint) and these must contain at least $$r$$ points of $$B''$$. Therefore $$h(\bigcup_{i=1}^{r} C_i^\circ)$$ contains $$B''$$.

We will construct another arc $$C'$$ which misses $$\bigcup_{i=1}^{r} C_i^\circ$$, but such that $$h(C')$$ must also contain a point of $$B''$$. Let $$y \in \omega_{w_2}$$. Since $$M$$ is locally connected, there exists an arc $$C$$ from $$x$$ to $$y$$. Let

$$S' = \left[\left(\bigcup_{i=1}^{r} S_{b_i}\right) \cap C(\omega_{w_2})\right]$$. Since $$B''$$ separates $$\omega_{w_2}^\circ$$ from $$C(\omega_{w_2})$$, $$S'$$ separates $$\omega_{w_2}^\circ$$ from all elements of $$\omega$$ not meeting $$\omega_{w_2}$$. Therefore $$C$$ must contain a point of $$S'$$. Let $$z$$ be the first point of $$S'$$ on $$C$$, and let $$C'$$ be the subarc of $$C$$ from $$x$$ to $$z$$. Then
\( C' \cap \left( \bigcup_{i=1}^{r} C_i^0 \right) = \emptyset, \) \( h(z) \in W_j \) for some \( j \neq i_2, \) and \( h(x) \in W_{i_2}. \) But the last 2 conditions imply that \( h(C') \) must contain some point of \( B'' \). However, as noted above, \( B'' \subset h\left( \bigcup_{i=1}^{r} C_i^0 \right) \) and \( C' \cap \bigcup_{i=1}^{r} C_i^0 = \emptyset, \) and therefore \( h(C') \cap h\left( \bigcup_{i=1}^{r} C_i^0 \right) = \emptyset. \) This is a contradiction.

**THEOREM 4.2:** The group of homeomorphisms of a regular curve \( M \) is either 0- or \( \infty \)-dimensional. It is \( \infty \)-dimensional if \( M \) contains a free arc. It is 0-dimensional otherwise.

**PROOF:** If \( M \) contains a free arc \( A, \) let \( H \) be the set of homeomorphisms of \( M \) onto itself, supported on \( A. \) Then \( H \) can be identified with the group of homeomorphisms of an arc keeping the endpoints fixed. But this is \( \infty \)-dimensional. (See Theorem 1.1.)

If \( M \) does not contain a free arc, we show that \( M \) must contain a dense set of local cut points of order \( > 2, \) and therefore by Theorem 4.1, \( G(M) \) is 0-dimensional. We do this by proving the contrapositive.

Suppose \( M \) does not contain a dense set of local cut points of order \( > 2. \) Then there exists a connected, open set \( U \subset M \) such that \( \overline{U} \) contains no local cut point of \( M \) of order \( > 2, \) and such that \( \text{Bd} \ U \) is a finite set of local cut points of \( M. \) (We choose \( U \) so that \( \text{Bd} \ U \) is contained in some order basis.) By Lemma 4.3, \( \overline{U} \) is a
regular curve containing no local cut points of order \(> 2\).

By Lemma 4.4, \(\overline{U}\) is an arc or a simple closed curve. It follows that \(M\) contains a free arc.

Thus we have shown that if \(M\) does not contain a free arc, it contains a dense set of local cut points of order \(>2\), and thus \(G(M)\) is 0-dimensional.

**THEOREM 4.3:** There exist rational curves whose full groups of homeomorphisms are exactly one-dimensional.

**DISCUSSION OF PROOF:** We will construct such continua in more detail in Chapter V. The properties of their groups of homeomorphisms will be proved directly, or will be similar to, or follow from, other theorems in that section. Briefly, the continua are constructed by use of a sequence of dendrons \(\{D_i\}\) converging to a limit arc \(D\), in such a way that exactly 1 endpoint of each \(D_i\) meets the left-hand endpoint of \(D\), and exactly 1 other endpoint of each \(D_i\) meets the right-hand endpoint of \(D\). This is constructed carefully so that each \(D_i\) admits only countably many homeomorphisms onto itself, and any homeomorphism of such a continuum onto itself, when restricted to the limit arc \(D\), is just a "translation" of \(D\) onto itself.

**THEOREM 4.4:** For each positive integer \(n\), there exist rational curves, \(R_n\), whose full groups of homeomorphisms, \(G_n\), are at least one and at most \(n\) dimensional. In fact,
$G_n$ is homeomorphic to the product of $n$ 1-dimensional sets.

**DISCUSSION OF PROOF:** We construct these continua in detail in Chapter V also. They are obtained by taking the union, $M$, of $n$ different continua, $M^1_i$, constructed for Theorem 4.3, in such a way that the $i^{\text{th}}$ and $(i + 1)^{\text{th}}$ meet only at an endpoint of their limit arcs. Then the group of homeomorphisms of $M$ is just the product of the groups of homeomorphisms of the $M^1_i$'s.

**REMARK:** If a 1-dimensional continuum is not a rational curve, then its group of homeomorphisms may be 0-dimensional, for example a dendron with a dense set of branch points of different orders, or it may be at least 1-dimensional, as in the case of the universal plane curve and the universal curve. Certainly it may be infinite dimensional; simply attach a free arc to a point of the outer boundary of a universal plane curve.

**QUESTION 1:** Do there exist 1-dimensional non-rational curves whose groups of homeomorphisms are of finite positive dimension?

**QUESTION 2:** Do there exist one-dimensional, locally connected continua whose groups of homeomorphisms are of finite positive dimension?
In this chapter we will construct continua such that the groups of all homeomorphisms of these continua are exactly 1-dimensional. We then generalize these to continua, $M_n$, such that $1 \leq \dim G(M_n) \leq n$, and $G(M_n)$ is the product of $n$ 1-dimensional sets. We also show that these continua may be rational curves.

We make the constructions in such a way that we can use a vastly simplified version of the proof that we already have for non-zero dimensionality (the proof of Theorem 2.1). We also insure that the continuum contains an arc or simple closed curve, $T$, with the property that every homeomorphism, $h$, of the continuum onto itself, must carry $T$ onto itself, and also must move "almost everything". This is designed to produce the result that the group of homeomorphisms of the continuum does not contain products of 1-dimensional sets of homeomorphisms, and thus suggests that its group of homeomorphisms may be at most 1-dimensional. This, in fact, is the case for our examples, and will be proved in Theorem 5.2.

Our method of procedure is the following. We will first construct an auxiliary space — a compact set, $K$, such that $G(K)$ is exactly 1-dimensional. We then
"extend" $K$ to a continuum, $M$, such that $G(M)$ is both topologically and algebraically the same as $G(K)$. Thus $M$ is a continuum such that $G(M)$ is exactly 1-dimensional.

**Construction of Certain Dendrons:**

It is well-known that there exist dendrons which admit no homeomorphisms onto themselves other than the identity. Any dendron which is constructed so that it contains a dense set of branch points, no 2 of which have the same order, is such an example.

We show that there are uncountably many such dendrons. Let $f$ be a function from the set of natural numbers greater than or equal to 3 to the set $\{0, 1\}$. Let $D_f$ be a dendron which includes exactly 1 cut point of order $n$ iff $f(n) = 1$. Since there are uncountably many such functions, there are uncountably many such dendrons.

**Construction of the Continua $K_n$:**

It is easy to show, using the dendrons of the previous section, that for each positive integer $n$, there is a continuum, (in fact, a regular curve$^1$), $K_n$, such that $G(K_n)$ is cyclic of order $n$.$^2$ We construct these

$^1$See Chapter IV for definition.

$^2$J. de Groot in [8] and [9] has constructed continua with these properties, but from a somewhat different point of view.
continua for use in the next construction.

Let $C$ be a circle of radius $r$ about the origin in $E^2$, and consider the set of points of $C$ with polar coordinates $(r, \frac{2\pi k}{n})$, for each integer $k$, $0 \leq k < n$. These will be called the "vertices" of $K_n$. Let $D$ be a dendron admitting exactly one homeomorphism such that

1. the branch points of $D$ form a dense set of points, no two of which have the same order
2. $D$ lies in a neighborhood of the arc $A$ of $C$ from $(r, 0)$ to $(r, \frac{2\pi}{n})$ and contains this arc, $A$
3. the endpoints of $A$ are endpoints of $D$
4. if $P$ is a point of $D$, other than the endpoints of $A$, and the polar coordinates of $P$ are $(r, \varphi)$, then $0 < \varphi < \frac{2\pi}{n}$.

Let $r_n$ be the counterclockwise, period $n$ rotation, about the origin, of $E^2$ onto itself. We note that $[r_n(D)] \cap D = (r, \frac{2\pi}{n})$. We define $K_n$ as follows:

Let $K_n = \bigcup_{i=0}^{n-1} r_n^i(D)$.

Clearly, $G(K_n)$ contains a cyclic group of order $n$. We wish to show that these are the only homeomorphisms of $K_n$ onto itself. We first note that the set of vertices of $K_n$ must go onto itself, since any point, other than a vertex, has a neighborhood which contains no branch point whose order is that of any branch point near the vertex.

Further, we see that the set of vertices must go onto itself in an order-preserving, orientation-preserving
manner. For the arc joining 2 adjacent vertices, and which contains no other vertex, must go onto an arc joining 2 vertices and which also contains no other vertex, since the image must contain branch points of the same order as the original set. Also, this arc must go onto its image arc with the same orientation, because the branch points to the left of a vertex are of orders different from the orders of the branch points to the right of a vertex.

It follows that the only homeomorphisms of $K_n$ onto itself are the $n$ "rotations".

**THEOREM (See Theorems 2.3 and 2.6):** There exists a compact set $K$ such that $G(K)$ is exactly 1-dimensional.

**Construction of $K$:**

Let $C_n$ be a circle of radius $1 - \frac{1}{n}$ about the origin in $E^2$. Then $C_n \rightarrow T$, where $T$ is the unit circle about the origin. We construct, in a small neighborhood of each $C_n$, a regular curve, $K_n$, as constructed above, so that $K_n \cap K_m = \emptyset$, for $n \neq m$.

Since, for our construction, $K_m \supset C_m$, for all $m$, in particular for $m > n$, $K_n \cap T = \emptyset$. Let $K = \bigcup K_n \cup T$.

Then $K$ is the desired set.

To show this, we first prove some lemmas.
DEFINITION 5.1: The set \( V_n = \left\{ \left( 1 - \frac{1}{n}, \frac{2\pi k}{n} \right) : 0 \leq k < n \right\} \) will be called the set of vertices of \( K_n \).

Let \( V = \bigcup_n V_n \). Then \( V \) is called the set of vertices of \( K \).

LEMMA 5.1: Let \( h \) be a homeomorphism of \( K \) onto itself such that, for some point \( x_0 \) in \( T \), \( h(x_0) = x_0 \). Then \( h | T \) is the identity.

PROOF: Let \( \{ v_i \} \), \( v_i \in V_i \), be a sequence of vertices of \( K \) converging to \( x_0 \). Then \( h(v_i) \to x_0 \) also. Since \( h | K_i \) is a rotation, it follows that \( \{ h | K_i \} \) converges to a rotation of \( T \), which must be the 0-rotation (since \( h(x_0) = x_0 \)); that is, \( h | T \) must be the identity.

DEFINITION 5.2: Let \( r \) be a rotation of \( T \) onto itself. An extension \( h \) of \( r \), \( h : K \to K \), is called a \( \ast \)-extension if \( d(h, e) = d(r, e | T) \).

LEMMA 5.2: Let \( r \) be a rotation of \( T \) onto itself with \( r < \pi \). Then there exists a \( \ast \)-extension \( h \) of \( r \).

PROOF: Let \( r_n \) be the counterclockwise period \( n \) rotation about the origin in \( E^2 \). For each positive integer \( n \), let \( k_n \) be the largest non-negative integer less than \( n \), for which \( r_n^{k_n} \) is a rotation which is less than or equal to \( r \). Let \( h : K \to K \) be defined by \( h | K_n = r_n^{k_n} | K_n \); \( h | T = r \). Then \( h \) is a \( \ast \)-extension of \( r \).
**LEMMA 5.3:** Let \( h \) be a homeomorphism of \( K \) onto itself. Then \( h \upharpoonright T \) is a rotation.

**PROOF:** Suppose \( h \upharpoonright T \) is not a rotation. Let \( x_0 \in T \) such that \( h(x_0) \neq x_0 \), and let \( r \) be a rotation of \( T \) onto itself such that \( r(x_0) = h^{-1}(x_0) \). By Lemma 5.1, there exists an extension \( r' \), of \( r \), to \( K \). Let \( g = hr' \). Then \( g(x_0) = x_0 \), for \( g(x_0) = hr'_0(x_0) = h(r'(x_0)) = h(h^{-1}(x_0)) = x_0 \). However, by Lemma 5.1, \( g \upharpoonright T \) must be the identity; that is, \( h \upharpoonright T = r^{-1} \), and therefore \( h \upharpoonright T \) is a rotation. This is a contradiction.

**LEMMA 5.4:** \( G(K) \) is complete.

**PROOF:** Let \( \{h_i\} \) be a Cauchy sequence of homeomorphisms on \( K \). Since \( K \) is compact, \( \{h_i\} \) converges to a limit map, \( h \). Since \( K_n \) admits only finitely many homeomorphisms, there exists an integer, \( I_n \), such that for \( i, j > I_n \), \( h_i \upharpoonright K_n = h_j \upharpoonright K_n \). Therefore, for each \( n \), \( h \upharpoonright K_n \) is a homeomorphism, and thus \( h \) is 1-1 off \( T \). Now, by Lemma 5.3, \( h_n \upharpoonright T \) is a rotation for each \( n \).

Since \( \{h_n\} \) is Cauchy, we have a Cauchy sequence of rotations on \( T \), and this must converge to a rotation. Therefore \( h \) is 1-1 on \( T \). Since \( h \) is 1-1, and \( K \) is compact, it follows that \( h \) is a homeomorphism.

**LEMMA 5.5:** Let \( W = \{h \in G(K) : h \upharpoonright T \text{ is the identity}\} \).

Let \( h_0 \) be any homeomorphism in \( G(K) \) such that
h_o \mid T \neq e \mid T, \text{ and let } W' = \{ h \in G(K) : h \mid T = h_o \mid T \}.

Then W and W' are homeomorphic.

PROOF: Define a function, \( \varphi \), on W, by \( \varphi(h) = h h_o \).

Clearly \( \varphi \) is 1-1. We show that \( \varphi(W) = W' \). Since \( h \mid T \) is the identity, \( h h_o \mid T = h_o \mid T \), and therefore
\( \varphi(W) \subseteq W' \). Now let \( g \in W' \). Then \( g \mid T = h_o \mid T \).

Therefore \( g h_o^{-1} \mid T \) is the identity. Then \( g h_o^{-1} \in W \), and \( \varphi(g h_o^{-1}) = (g h_o^{-1}) h_o = g \). Therefore \( \varphi(W) \) contains \( W' \). It follows that \( \varphi(W) = W' \).

We next show that \( \varphi \) is continuous. Let \( h_i \rightarrow h \), \( h_i, h \in W \). Then \( \varphi(h_i) = h_i h_o \) and \( \varphi(h) = h h_o \).

Now \( d(h_i h_o, h h_o) = d(h_i, h) \rightarrow 0 \). Therefore \( h_i h_o \rightarrow h h_o \); that is, \( \varphi(h_i) \rightarrow \varphi(h) \), so that \( \varphi \) is continuous.

We show that \( \varphi^{-1} \) is also continuous. If \( g_i \in W' \) and \( g_i \rightarrow g \in W' \), let \( g_i = h_i h_o \), and let \( g = h h_o \). We wish to show that \( \varphi^{-1}(g_i) \rightarrow \varphi^{-1}(g) \);
that is, \( h_i \rightarrow h \). Since \( d(g_i, g) = d(h_i h_o, h h_o) = d(h_i, h) \), it is clear that \( h_i \rightarrow h \); that is,
\( \varphi^{-1}(g_i) \rightarrow \varphi^{-1}(g) \).

It follows that \( \varphi \) is a homeomorphism.

LEMMA 5.6: For each \( \varepsilon > 0 \), there exists a \( \delta \), \( 0 < \delta < \varepsilon \), such that if \( h \in G(K) \) for which \( d(h, e) = \delta \), then the only points of \( K \) which move a distance \( \delta \) under \( h \) are the points of \( T \).
**PROOF:** We first observe that since there are only a finite number of homeomorphisms on each $K_n$, $\left\{ h \mid K_n : h \in G(K) \right\}$ is finite; therefore there are only a finite number of distances between $h \mid K_n$, $e \mid K_n$. It follows that $D = \{d : d = d(h \mid K_n, e \mid K_n) \text{ for some } n\}$ is countable. Let $\delta < \varepsilon$, $\delta \notin D$, and let $h \in G(K)$ for which $d(h, e) = \delta$. By Lemma 5.2, we know such an $h$ exists.

We show that the only points of $K$ which move a distance $\delta$ under $h$ are the points of $T$. Since $d(h, e) = \delta$, no point moves more than a distance $\delta$ away from itself. Further, no point of any $K_n$ can move a distance $\delta$ away from itself, for then $d(h \mid K_n, e \mid K_n) > \varepsilon$. But $d(h, e) = \delta$ and $\delta \notin D$.

It follows that the only points which move a distance $\delta$ under $h$ are the points of $T$.

**THEOREM 5.1:** $G(K)$ is at least 1-dimensional.

**PROOF:** Let $U$ be a neighborhood of the identity of diameter $< \frac{3}{2}$. Let $\{\varepsilon_j\}$ be a decreasing sequence of numbers such that $\varepsilon_j < \frac{\pi}{2^{j-1}}$. We will obtain a sequence of pairs of homeomorphisms $\{h_j, g_j\}$ such that $h_j \in U$, $g_j \in C(U)$, $d(h_j, g_j) < \varepsilon_j$, and $h_j \to h$, a homeomorphism in $G(K)$. Then $h \in Bd U$.

In the following, $r_j$ denotes a rotation of $\frac{\pi}{2^j}$ radians on $T$. Let $k_1$ be the smallest non-negative integer such that $r_{-k_1}$ has a $*$-extension $h_1 \in U$, but
for which any *-extension of $r_1^{k_1+1} \in C(U)$. Let $\psi_1$ be a *-extension of $r_1$, and let $g_1 = \psi_1 h_1$. We note that $g_1$ is a *-extension of $r_1^{k_1+1}$. Then $h_1 \in U$, $g_1 \in C(U)$, and $d(h_1, g_1) < \epsilon_1$.

We proceed by induction. Assume that we have $h_{j-1}, g_{j-1}$ such that $h_{j-1} \in U$, $g_{j-1} \in C(U)$, $g_{j-1} = \psi_{j-1} h_{j-1}$ where $\psi_{j-1}$ is a *-extension of $r_{j-1}$, and $\psi h_{j-1} \in C(U)$ for any *-extension $\psi$ of $r_{j-1}$.

Let $k_j$ be the smallest non-negative integer for which there exists a *-extension $\varphi_{j-1}$ of $r_j^{k_j}$ such that

$$\varphi_{j-1} h_{j-1} \in U,$$

but any *-extension $\psi$ of $r_j^{k_j+1} \in C(U)$.

Let $h_j = \varphi_{j-1} h_{j-1}$, and let $g_j = \psi_j h_j$ where $\psi_j$ is a *-extension of $r_j$.

We now have a sequence of pairs of homeomorphisms $\{h_j, g_j\}$ such that $d(h_{j-1}, h_j) < \epsilon_{j-1}$. Therefore $\{h_j\}$ is Cauchy, and by Lemma 5.4, $h_j \rightarrow h$, where $h$ is a homeomorphism of $K$ onto itself. Since $d(h_j, g_j) < \epsilon_j$, and $\epsilon_j \rightarrow 0$, $g_j \rightarrow h$ also. Therefore $h \in \text{Bd} U$.

It follows that $G(K)$ is at least 1-dimensional.

**Lemma 5.7:** Let $\epsilon$ be a number such that $0 < \epsilon < 1$ and $d(h \vert K_n, e \vert K_n) \neq \epsilon$, for all $h, n$. Let $G_\epsilon = \{h \in G(K) : d(h, e) = \epsilon\}$. Then $G_\epsilon$ is 0-dimensional.

**Proof:** By Lemma 5.6, the hypothesis can be satisfied; it also follows that $G_\epsilon$ is a subset of the set of
homeomorphisms, $G_\varepsilon'$, which are extensions of a rotation $T$ which moves points of $T$ a distance $\varepsilon$. By Lemma 5.5, $G_\varepsilon'$ is homeomorphic to $W = \{ h \in G(K) : h|_T$ is the identity $\}$.

We show that $W$ is 0-dimensional. It is sufficient to show that $W$ is 0-dimensional at the identity, $e$.

To this end, let $\{\varepsilon_i\}$ be a sequence of positive numbers, $\varepsilon_i \to 0$, such that if $h \in G(K)$, then $d(h|_{K_n}, e|_{K_n}) \neq \varepsilon_i$, for all $i, n$. Let $S_{\varepsilon_i} = \{ h \in W : d(h, e) \leq \varepsilon_i \}$. $\text{Bd } S_{\varepsilon_i}$ is a subset of $H = \{ h \in W : d(h, e) = \varepsilon_i \}$. But $H = \emptyset$, since the only homeomorphisms which are a distance $\varepsilon_i$ from $e$, must move points on $T$, but $H$ is a set of extensions of the identity on $T$. Therefore $\text{Bd } S_{\varepsilon_i} = \emptyset$. Thus we have a sequence of spheres about $e$, converging to $e$, and whose boundaries are empty. It follows that $W$, and therefore $G_\varepsilon'$, is 0-dimensional.

**Theorem 5.2:** $G(K)$ is at most 1-dimensional.

**Proof:** Let $\{\varepsilon_i\}$ be a sequence of positive numbers such that $\varepsilon_i \to 0$ and $d(h|_{K_n}, e|_{K_n}) \neq \varepsilon_i$, for all $n, i$. Let $S_{\varepsilon_i} = \{ h \in G(K) : d(h, e) \leq \varepsilon_i \}$. Then $\text{Bd } S_{\varepsilon_i} \subseteq G_{\varepsilon_i} = \{ h \in G(K) : d(h, e) = \varepsilon_i \}$. By Lemma 5.7, $G_{\varepsilon_i}$ is 0-dimensional. Therefore $\text{Bd } S_{\varepsilon_i}$ is 0-dimensional. It follows that $G(K)$ is at most 1-dimensional at the
identity. Therefore $G(K)$ is at most 1-dimensional.

**THEOREM 5.3:** $G(K)$ is exactly 1-dimensional.

**PROOF:** By Theorem 5.1, $\dim G(K) \geq 1$, and by Theorem 5.2, $\dim G(K) \leq 1$. Therefore $\dim G(K) = 1$.

**REMARK 1:** Since there are uncountably many different dendrons which could have been used in the construction, there are uncountably many such compact spaces.

**REMARK 2:** By using continua $K_{p^n}$, we obtain a set $K$ for which rotations of period $p^n$ on $T$ can be extended to rotations of period $p^n$ on $K$. However, if we use continua $K_{\alpha_n}$, where the sequence $\{\alpha_n\}$ includes an infinite sequence of relatively prime integers, then no rotation of $T$ of any finite positive period can be extended to a homeomorphism of $K$ of that period.

**THEOREM 5.4:** $G(K)$ is totally disconnected.

**PROOF:** We show that any pair of homeomorphisms of $G(K)$ can be separated. Let $g_1, g_2 \in G(K), g_1 \neq g_2$. Then there exists an integer $n > 0$, such that $g_1|_{K_n} \neq g_2|_{K_n}$. Let $A = \{h \in G(K) : h|_{K_n} = g_1|_{K_n}\}$, and let $B = G(K) - A$. Then $g_1 \notin A, g_2 \notin B$, and

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3 This proof is similar to the proof of Theorem 1.2, suggested by R. D. Anderson to show that the group of homeomorphisms of the universal plane curve is totally disconnected.
A ∪ B = G(K). Let \( h_0 \) be any element of \( A \), and let \( g_0 \) be any element of \( B \). Then \( h_0 \) and \( g_0 \) differ on \( K_n \) and therefore must be a positive distance apart — at least \( d(e \mid K_n, r_n \mid K_n) \) where \( r_n \) is the rotation of \( \frac{2\pi}{n} \) radians on \( K_n \). Therefore \( A \) and \( B \) are mutually separate, and \( G(K) \) is totally disconnected.

**Theorem 5.5:** \( G(K) \) is abelian.

**Proof:** Let \( h, g \in G(K) \). We show that \( hg = gh \). Since \( K_n \) goes onto itself under any homeomorphism in \( G(K) \), and \( h \mid K_n \) and \( g \mid K_n \) are elements of a cyclic group, \( hg \mid K_n = gh \mid K_n \). Since \( T \) goes onto itself under all elements of \( G(K) \), and \( h \mid T \) and \( g \mid T \) are rotations, \( hg \mid T = gh \mid T \). Therefore, for any \( x \in K \), \( hg(x) = gh(x) \); that is \( hg = gh \).

Thus \( G(K) \) is abelian.

Thus we have completed construction of the auxiliary space \( K \). Our next objective is to construct a continuum \( M \) such that \( G(M) \) is topologically and algebraically the same as \( G(K) \).

**Construction of \( M \):**

We will "extend" the set \( K \). We first modify the set \( K \) in the following way: Let \( C = T \times [0, 1] \) in \( E^3 \). Now consider the vertical unit interval in \( E^3 \) above the point in \( E^2 \), the \( z = 0 \) plane, with polar coordinates
Put a dendron $D_n$, with a dense set of branch points of different orders greater than 3, in a small neighborhood of this interval in such a way that

1. $D_n$ contains this interval
2. $D_n \cap E^2$ is an endpoint of $D_n$
3. $D_n$ intersects the plane $z = 1$ in an endpoint of $D_n$
4. $D_n \cap r_n^i(D_n) = \emptyset$, where $r_n$ is the period $n$ positive rotation about the $z$-axis in $E^3$, and $1 \leq i < n$
5. $r_n^i(D_n) \cap r_m^j(D_m) = \emptyset$, for $n \neq m$, and all $i, j$ such that $0 \leq i < n$ and $0 \leq j < m$

and (6) $D_n \cap C = \emptyset$, for all $n$.

Let $K_n' = K_n \cup \bigcup_{i=0}^{n-1} r_n^i(D_n)$. Let $M' = (\bigcup K_n') \cup C$. Let $h$ be a level preserving map of $E^3$ onto itself, which is

1. the identity on $E^2$
2. a homeomorphism on each level other than $z = 1$

and (3) carries the unit disk of the plane $z = 1$ to the point $(0, 0, 1)$.

Let $M = h(M')$. Then $M$ is a continuum with the desired properties.

**Theorem 5.6**: There exists a continuum $M$ such that $G(M)$ is totally disconnected, abelian, and exactly 1-dimensional.
PROOF: Clearly the continuum $M$ constructed above is a continuum in which every homeomorphism $g$ of $K$ onto itself can be extended to a homeomorphism $h$ of $M$ onto itself. Further, this extension is unique, and if $g_i \rightarrow g$, then the extensions $h_i$ of $g_i$ converge to the extension $h$ of $g$. Therefore $G(K)$ and $G(M)$ are topologically the same.

It is also clear from the construction that they are algebraically the same.

Thus it follows from Theorems 5.3, 5.4, and 5.5 that $G(M)$ is totally disconnected, abelian, and exactly 1-dimensional.

The next portion of this chapter is devoted to generalizing the results of Theorem 5.6 above. We wish to construct continua $M_n$ such that $G(M_n)$ is homeomorphic to a product of $n$ 1-dimensional groups. Again we begin by making some auxiliary constructions.

Construction of the Continua $P_{n_1, p_1, \ldots, p_n}$.

For each positive integer $n$, and each $n$ positive relatively prime, integers $p_1, \ldots, p_n$, we construct a continuum $P_{n_1, p_1, \ldots, p_n}$ such that $G(P_{n_1, p_1, \ldots, p_n})$ is the
product of $n$ cyclic groups of orders $p_1, \ldots, p_n$ respectively.

Let $T_n$ be the $n$-dimensional torus, thought of as the product of $n$ circles, $C_1, \ldots, C_n$; that is, $T_n = \prod_{i=1}^{n} C_i$. Let $r_i$ be a rotation of order $p_i$ on $C_i$, and let $h(r_1^i, r_2^i, \ldots, r_n^i)$ be a homeomorphism of $T_n$ onto itself which is the product of the rotations $r_j^i$ on $C_j$. Then $H_n = \{ h(r_1^i, r_2^i, \ldots, r_n^i) : 0 \leq i_j < p_j \}$ is a set of homeomorphisms of $T_n$ onto itself which is the product of $n$ cyclic groups, $G_i$, of order $p_i$. We assume that we have a "rectangular" grid of $T_n$ in such a way that if $R$ is any fixed $n$-rectangle, then $\cup \{ h(R) : h \in H_n \}$ is $T_n$.

Now let $D$ be a dendron with a dense set of branch points of different orders constructed in a neighborhood of a $1$-simplex of the $1$-dimensional skeleton of the rectangular grid, in such a way that $h_1(D) \cap h_2(D)$ is either empty or a single point, if $h_1, h_2 \in H_n$, $h_1 \neq h_2$.

Let $P_n, p_1, \ldots, p_n = \cup \{ h(D) : h \in H_n \}$. Then $P_n, p_1, \ldots, p_n$ satisfies the conditions of the first paragraph.

THEOREM 5.7: For each positive integer $n$, there exists a compact set $Q_n$ such that $G(Q_n)$ is homeomorphic to

\[ \text{See footnote 2.} \]
the product of \( n \) 1-dimensional groups, and thus
\[ 1 \leq \dim G(Q_n) \leq n. \]

Outline of the Construction of \( Q_n \):

Let \( \{p_1, i\}, \{p_2, i\}, \ldots, \{p_n, i\} \) be \( n \) increasing sequences of positive integers such that
\[ \lim_{i \to \infty} p_k, i = \infty, \]
and such that, for each \( i \), each pair \( p_k, i \) and \( p_l, i \) are relatively prime. Let \( \{p_j, p_1, \ldots, p_j\} \) be a continuum as constructed above. Let \( \{p_j, p_1, \ldots, p_j\} \) be a towered sequence of these sets converging (from the interior) to a limit \( n \)-torus, \( T_n \). Then
\[ Q_n = (\bigcup_{j} \bigcup_{p_j, p_1, \ldots, p_j} \bigcup T_n) \]
is the continuum of the theorem, since \( G(Q_n) \) will just be the product of \( n \) 1-dimensional groups, each of which is obtained by considering the homeomorphisms which move points in "one direction" only.

**THEOREM 5.8:** For each positive integer \( n \), there exists a continuum \( M_n \) such that \( G(M_n) \) is homeomorphic to the product of \( n \) 1-dimensional groups, and thus
\[ 1 \leq \dim G(M_n) \leq n. \]

Outline of Construction of \( M_n \):

The compact set \( Q_n \) is a subset of \( E^{n+1} \). We think of the join of the limit torus \( T_n \) and the set of vertices \( V \) of \( Q_n \) with a point \( q \) in
\((E^{n+2} - \text{the } E^{n+1} \text{ hyperplane containing } Q_n)\), and construct, in a neighborhood of each arc from \(q\) to a vertex of \(Q_n\), a dendron (containing this arc) with a dense set of branch points of different orders, much like we did for the continuum \(M\) of Theorem 5.6, which had a 1-dimensional group. Then \(G(M_n)\) is the same as \(G(Q_n)\).

Our final objective of this chapter is to show, as promised in Chapter IV, that there are rational curves with 1-dimensional groups of homeomorphisms. Once again, we begin with some auxiliary constructions.

Outline of the Construction of the Continuum \(Q\):

We construct a continuum, \(Q\), such that the group of all homeomorphisms of \(Q\) is infinite cyclic.\(^5\) Consider the interval \(I = [-1, 1]\) on the x-axis in \(E^2\) and the partition, \(A\), of \(I\) into an infinite number of subintervals by means of the points \(1 - \frac{1}{n}\) and \(-1 + \frac{1}{n}\), \(n > 1\), on \(I\). Let \(D\) be a dendron with a dense set of branch points of different orders, with \(D\) constructed in a neighborhood of \([-\frac{1}{2}, \frac{1}{2}]\), containing this interval, and not meeting \([-1, 1]\) elsewhere. We also require that \(D\) meet each of the lines \(x = -1\) and \(x = 1\), in exactly one point, an end point of \(D\), and that the points of \(D\)

\(^5\)See footnote 2.
lie in the set of points of $E^2$ such that $-\frac{1}{2} \leq x \leq \frac{1}{2}$.

Let $h$ be any homeomorphism of $I \times [-\frac{1}{2}, \frac{1}{2}]$ onto itself carrying each of the subintervals of $I$, determined by $A$, onto the next subinterval, and carrying vertical lines onto vertical lines.

Let $Q = \bigcup_{i=-\infty}^{\infty} h^i(D)$. Clearly, $G(Q)$ is infinite cyclic.

**Theorem 5.9:** There exists a rational curve $R$ such that $G(R)$ is exactly 1-dimensional.

**Outline of Construction of $R$ (and proof):**

Let $\{d_{1,i}\}, \{d_{2,i}\}, \ldots, \{d_{n,i}\}, \ldots$ be an infinite sequence of increasing sequences of integers $> 2$, such that $d_{i,j} \neq d_{k,l}$ whenever $i \neq k$.

Let $I_j$ be the interval from $(-1, \frac{1}{j})$ to $(1, \frac{1}{j})$ in $E^2$. Then $I_j \rightarrow I = [-1, 1]$ on the x-axis.

Let $A_{j,1}$ be the infinite partition of $I_j$ determined by the vertical lines $x = 1 + \frac{1}{n}$ and $x = 1 - \frac{1}{n}$, for all $n > 1$, as in the construction of $Q$ above. We refine $A_{j,1}$ to obtain a partition, $A_{j,2}$, by subdividing into halves each interval of $I_j$ determined by $A_{j,1}$. We continue inductively, until we obtain the partition $A_{j,j}$.

Construct a continuum $Q_j$, on $I_j$, as in the construction of $Q$ above, by using a dendron $D_j$ whose
branch points are of orders $d_{j,i}$ of the sequence $\{d_{j,i}\}$ above, and by using the partition $A_{j,j}$ for $I_j$. We further require that $Q_i \cap Q_j = \emptyset$ for $i \neq j$.

Let $R' = \bigcup Q_j \cup I$.

Let $h$ be a map of $E^2$ onto itself carrying the vertical lines $x = \alpha$, for each real number $\alpha$, onto themselves, so that $h$ is a homeomorphism everywhere, except that $h([a, b]) = (0, 0)$ and $h([c, d]) = (0, 1)$, where $[a, b]$ is the vertical interval from $(0, 0)$ to $(0, 1)$, and $[c, d]$ is the vertical interval from $(1, 0)$ to $(1, 1)$.

Let $R = h(R')$. Then clearly $R$ is a rational curve. The proof that $G(R)$ is 1-dimensional is similar to the proof that $G(M)$ (of Theorem 5.6) is 1-dimensional.

REMARK: We make the observation that if the partitions of the $I_n$'s are chosen arbitrarily with only the provision that the mesh of the infinite partitions has limit 0, it is possible to obtain a rational curve such that no homeomorphism of the limit arc onto itself, except the identity, can be extended, and therefore $G(R)$ would be 0-dimensional.

**COROLLARY 5.9.1:** For each positive integer $n$, there exists a rational curve, $R_n$, such that $G(R_n)$ is homeomorphic to the product of $n$ 1-dimensional groups, and thus
$1 \leq \dim G(R_n) \leq n.$

Outline of Construction and Proof:

Let $R_n$ be the union of $n$ different (that is, non-homeomorphic) continua, $R'_i$, constructed like the continuum $R$ above, $R'_i$ meeting $R'_{i+1}$ at the right- and left-hand endpoints of their limit arcs respectively, $1 \leq i < n$. Then any homeomorphism of $R_n$ onto itself carries each $R'_i$ onto itself, and the homeomorphisms on the different $R'_i$'s are independent of each other. Therefore $G(R_n)$ is the same as $\prod_{i=1}^{n} G(R'_i)$. Since each $G(R'_i)$ is exactly 1-dimensional, and since the product of $n$ 1-dimensional sets is at most 1-dimensional, $1 \leq \dim G(R_n) \leq n.$

REMARK: We make the following observations (without proof) about the continua $M_n$ and their groups of homeomorphisms:

1. There are uncountably many such continua.
2. Every homeomorphism, $r$, which is a product of rotations on the limit torus $T_n$, can be extended to a homeomorphism, $h$, of $M_n$ onto itself, such that $d(r, e|T_n) = d(h, e)$. 
3. If $h \in G(M_n)$, then $h|T_n$ is a product of rotations.
4. Let $r$ be a product of rotations on $T_n$, let $A = \{ h \in G(M_n) : h|T_n = r \}$, and let $B = \{ h \in G(M_n) : h|T_n$ is the identity\}. 

Then $A$ and $B$ are homeomorphic.

(5) Let $x_0 \in T_n$, and let $\varphi : G(M_n) \rightarrow T_n$ be defined by $\varphi(g) = g(x_0)$. Then $\varphi$ is a continuous, open, homomorphism of $G(M_n)$ onto the $n$-dimensional torus.

**QUESTIONS:** In view of the theorems of this chapter, and the statements above, we raise the following questions:

1. Is $\dim G(M_n) = n$?
2. Is $\dim G(R_n) = n$? ($R_n$ is the curve of Corollary 5.9.1.)

**REMARK:** We note that if the answer to Question (1) is "no", then we have examples of continuous, open, dimension-raising homomorphisms from the groups of homeomorphisms of continua onto compact groups.
BIBLIOGRAPHY


VITA

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