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The Structure of Convolution Measure Algebras.

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THE STRUCTURE OF CONVOLUTION MEASURE ALGEBRAS

A Dissertation

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ABSTRACT

The object of this dissertation is to study the structure of special types of Banach algebras which we have labeled convolution measure algebras. There are several important examples of such algebras. $L_1(G)$, where G is a locally compact abelian group, is a convolution measure algebra which has been extensively studied. $M(G)$, the algebra of all regular Borel measures on G , is a far more complex convolution measure algebra and little is known about its structure. $M(G)$ may be considered the motivating example for this work.

The basic result of this work is a representation of the maximal ideal space of a convolution measure algebra. It is well known that the maximal ideal space of $L_1(G)$ is $\hat{G}$, the group of characters on G . We show that in general the maximal ideal space of a convolution measure algebra may be represented as the semigroup $\hat{S}$ of all semicharacters on some compact topological semigroup S . When applied to $M(G)$ this representation makes certain pathological properties of $M(G)$ appear more natural and leads to a partial result on the Shilov boundary question for $M(G)$.

Chapter I is purely measure theoretic in nature and may be considered background material for the later chapters. We consider a particular type of Banach space $\mathcal{M}$ of measures, and prove that $\mathcal{M}^*$ can be represented as $C(X)$, the space of all continuous functions on a compact, extremely disconnected Hausdorff space X . Most of the

results of this chapter are contained in Cunningham's work on abstract L-spaces (Cf. [1]). However, we include proofs of these results for completeness.

In Chapter II we define convolution measure algebra and show that the maximal ideal space of such an algebra $\mathcal{M}$, can be represented as the semigroup $\widehat{S}$ of all semicharacters on some compact topological semigroup S .

In Chapter III we investigate the relationships between the structures of $\mathcal{M}$, S , and $\widehat{S}$. We identify a subset H of $\widehat{S}$ on which the Gelfand transform of every element of $\mathcal{M}$ attains its maximum modulus.

In Chapter IV we consider the case where $\mathcal{M}$ is $M(G)$ for some locally compact group G . We show that the kernel of S is the Bohr compactification of G and identify the natural imbedding of $\widehat{G}$ into $\widehat{S}$. We also show that, in this case, H (vide supra) is a proper subset of $\widehat{S}$ but is not closed in the Gelfand topology. It is not known whether the closure of H is proper in $\widehat{S}$. The solution of the Shilov boundary problem may depend on the answer to this question.

In Chapter V we list several unsolved problems which arise in connection with this work.

INTRODUCTION

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compact, extremely disconnected Hausdorff space X . Most of the results of this chapter are contained in Cunningham's work on abstract L -spaces (cf. [1]). However, we include proofs of these results for completeness.

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CHAPTER I

In this chapter we define a quite general type of Banach space of measures and prove that the adjoint of such a space may be represented as the algebra of all continuous functions on a compact Hausdorff space. In our use of Banach algebra terminology we follow [8].

Throughout this chapter $\mathcal{B}$ will denote a boolean algebra where $\wedge$ and $\vee$ denote the lattice operations in $\mathcal{B}$, $\sim$ denotes complementation, $\emptyset$ is the zero element of $\mathcal{B}$, and E is the identity element (cf. [7], pp. 81-82).

Definition 1.1. (a) A measure μ on $\mathcal{B}$ is a function from $\mathcal{B}$ to the complex numbers, such that $\mu(A \vee B) = \mu(A) + \mu(B)$ for every pair A, B of elements of $\mathcal{B}$ for which $A \wedge B = \emptyset$.

(b) If μ is a measure on $\mathcal{B}$ we set

$$|\mu|(A) = \sup \sum_1 |\mu(A_i)|$$

where the supremum is taken over all finite subsets $\{A_i\}_{i=1}^n$ of $\mathcal{B}$ for which $A_i \wedge A_j = \emptyset$ if $i \neq j$ and $A_1 \vee A_2 \vee \dots \vee A_n = A$. $|\mu|$ is called the variation function of μ and is itself a measure on $\mathcal{B}$ provided it is finite for each A in $\mathcal{B}$.

(c) $M(\mathcal{B})$ will denote the set of all measures μ on $\mathcal{B}$ for which $|\mu|(E) < \infty$. For μ in $M(\mathcal{B})$ we set $\|\mu\| = |\mu|(E)$. $\|\mu\|$ is called the total variation of μ .

$M(\mathcal{B})$ is a Banach space under the usual addition and scalar multiplication of functions and the total variation norm. $M(\mathcal{B})$ is a partially ordered Banach space if we introduce the order relation:

$\mu \leq \nu$ if $(\nu - \mu)(A) \geq 0$ for each A in $\mathcal{B}$.

Definition 1.2. (a) A simple step function on $\mathcal{B}$ is a pair $(\{A_i\}_{i=1}^n, \{a_i\}_{i=1}^n)$ where each A_i is in $\mathcal{B}$, $A_i \cap A_j = \emptyset$ if $i \neq j$, and each a_i is a complex number.

(b) If $f = (\{A_i\}_{i=1}^n, \{a_i\}_{i=1}^n)$ is a simple step function on $\mathcal{B}$, μ is in $M(\mathcal{B})$, and B is in $\mathcal{B}$, then we define

$$T_f \mu(B) = \int_B f d\mu = \sum_i a_i \mu(B \cap A_i).$$

It is easy to verify that $T_f \mu$ is again a measure in $M(\mathcal{B})$ and in fact, T_f is a bounded linear operator on $M(\mathcal{B})$ for each simple step function f .

Definition 1.3. (a) If μ and ν are elements of $M(\mathcal{B})$ and for each $\epsilon > 0$ there exists a $\delta > 0$, such that $|\mu|(A) < \epsilon$ whenever A is in $\mathcal{B}$ and $|\nu|(A) < \delta$, then μ is said to be ν -continuous.

(b) If μ and ν are elements of $M(\mathcal{B})$ and for each $\epsilon > 0$ there exists an element A of $\mathcal{B}$, such that $|\mu|(A) < \epsilon$ and $|\nu|(\tilde{A}) < \epsilon$, then μ is said to be ν -singular. We will occasionally denote this by $\mu \perp \nu$.

(c) A closed linear subspace $\mathcal{M}$ of $M(\mathcal{B})$ will be called an L -space over $\mathcal{B}$ if whenever μ is in $\mathcal{M}$ and ν is μ -continuous,

ν is also in $\mathfrak{M}$.

Throughout the remainder of this chapter $\mathfrak{M}$ will denote a fixed L-space over $\mathcal{B}$. If $\mathfrak{N}$ is an L-space over $\mathcal{B}$ which is contained in $\mathfrak{M}$ then $\mathfrak{N}$ will be called an L-subspace of $\mathfrak{M}$; $\mathfrak{N}^\perp$ will denote the elements of $\mathfrak{M}$ which are μ -singular for each μ in $\mathfrak{N}$. It follows immediately from the definitions that $\mathfrak{N}^\perp$ is also an L-subspace of $\mathfrak{M}$.

The primary object of this chapter is to characterize the adjoint space of $\mathfrak{M}$. We shall require three basic tools of measure theory which are summarized in the following theorem.

Theorem 1.1. (a) If μ is a real valued measure in $M(\mathcal{B})$, then there exist unique non-negative measures μ^+ and μ^- in $M(\mathcal{B})$, such that $\mu = \mu^+ - \mu^-$ and $\mu^+ \perp \mu^-$. It follows that

$$|\mu| = \mu^+ + \mu^-.$$

(b) If μ and ν are measures in $M(\mathcal{B})$, then there exist unique measures ν_1 and ν_2 in $M(\mathcal{B})$, such that $\nu = \nu_1 + \nu_2$, ν_1 is μ -continuous, and ν_2 is μ -singular.

(c) If μ and ν are measures in $M(\mathcal{B})$, then ν is μ -continuous if and only if for each $\epsilon > 0$, there exists a simple step function f on $\mathcal{B}$, such that $\|\nu - T_f \mu\| < \epsilon$.

Part (a) of the above theorem is the well known Jordan decomposition theorem (cf. [5], pp. 122-123). Part (b) is the Lebesgue decomposition theorem. In this generality, it was proved by

Darst (cf. [2]). Part (c) is the Radon-Nikodym theorem in a more general setting. It was also proved by Darst (cf. [2]).

The first step in our characterization of the adjoint space $\mathcal{M}^*$ of $\mathcal{M}$ is to represent $\mathcal{M}^*$ as an algebra of operators on $\mathcal{M}$. To this end we denote by R_0 the algebra of all operators T_f where f ranges over the simple step functions on $\mathcal{B}$. R_0 is the linear span of the projection operators P_A where A is an element of $\mathcal{B}$ and P_A is defined by the equation $P_A \mu(B) = \mu(A \cap B)$. It is immediate from the definitions, that if T is in R_0 and μ is in $\mathcal{M}$, then $T\mu$ is μ -continuous. Thus $T\mu$ is again in $\mathcal{M}$ and in fact, every L -subspace of $\mathcal{M}$ is an invariant subspace for the operators in R_0 .

We denote by R the Banach algebra of all bounded linear operators on $\mathcal{M}$ which commute with all the operators in R_0 . If F is an element of the adjoint space $\mathcal{M}^*$ of $\mathcal{M}$ we define an operator T on $\mathcal{M}$ in the following manner: $T\mu(A) = F(P_A \mu)$ for each μ in $\mathcal{M}$ and each A in $\mathcal{B}$.

Theorem 1.2. The correspondence $F \mapsto T$, defined above, is an isometry from $\mathcal{M}^*$ onto R .

Proof. $T\mu$ is again a measure in $M(\mathcal{B})$, for if $A \cap B = \emptyset$ then $P_{A \cup B} = P_A + P_B - P_{A \cap B} = P_A + P_B$, so that $T\mu(A \cup B) = F(P_A \mu) + F(P_B \mu) = T\mu(A) + T\mu(B)$. T is clearly a linear operator from $\mathcal{M}$ to $M(\mathcal{B})$. T is bounded; in fact if $\{A_i\}_{i=1}^n \subset \mathcal{B}$ and $A_i \cap A_j = \emptyset$ for $i \neq j$, then

$$\sum_1 |T\mu(A_1)| = \sum_1 |F(P_{A_1}\mu)| \leq \|F\| \sum_1 \|P_{A_1}\mu\| \leq \|F\| \|\mu\|.$$

Hence $\|T\| \leq \|F\|$ and T is a bounded operator of norm at most $\|F\|$.

Note that if A and B are in $\mathcal{B}$, then

$$P_A T\mu(B) = T\mu(A \wedge B) = F(P_{A \wedge B}\mu) = F(P_B P_A \mu) = TP_A \mu(B); \text{ hence}$$

T commutes with all the operators P_A . It follows that T commutes with all operators in R_0 . To show that T is in R , it remains to show that $T\mu$ is in $\mathcal{M}$ for each μ in $\mathcal{M}$. In fact $T\mu$ is μ -continuous, since $|T\mu|(A) = \|P_A T\mu\| = \|TP_A \mu\| \leq \|T\| \|P_A \mu\| = \|T\| |\mu|(A)$ for each A in $\mathcal{B}$.

We have shown that the correspondence $F \rightarrow T$ carries $\mathcal{M}^*$ into R and $\|T\| \leq \|F\|$. It remains to show that the map is linear, has an inverse, and $\|F\| \leq \|T\|$. The map is clearly linear, since $(T_1 + T_2)\mu(A) = T_1\mu(A) + T_2\mu(A) = F_1(P_A \mu) + F_2(P_A \mu) = (F_1 + F_2)(P_A \mu)$. If T is in R , then $F(\mu) = T\mu(E)$ defines an element F of $\mathcal{M}^*$, such that $\|F\| \leq \|T\|$ and $F(P_A \mu) = TP_A \mu(E) = P_A T\mu(E) = T\mu(A)$. The Map $T \rightarrow F$ is the required inverse and the proof is complete.

Theorem 1.3. If $\mathcal{N}$ is a closed subspace of $\mathcal{M}$, then the following statements are equivalent:

- (a) $\mathcal{N}$ is an L -subspace;
- (b) $\mathcal{N}$ is an invariant subspace for R ;

(c) $\mathcal{N}$ is an invariant subspace for R_0 .

Proof. If $\mathcal{N}$ is an L-subspace of $\mathcal{M}$, μ is in $\mathcal{N}$, and T is in R , then $T\mu$ is μ -continuous; hence $T\mu$ is in $\mathcal{N}$. Thus (a) implies (b).

Since R_0 is contained in R , (b) clearly implies (c).

If $\mathcal{N}$ is invariant under R_0 , μ is in $\mathcal{N}$, and ν is μ -continuous, then, by Theorem 1.1 (c), ν is the norm limit of measures of the form $T_f \mu$ where T_f is in R_0 . Each $T_f \mu$ is in $\mathcal{N}$; hence ν is in $\mathcal{N}$. Thus (c) implies (a).

Theorem 1.4. If $\mathcal{N}$ is an L-subspace of $\mathcal{M}$, then there exists a unique idempotent operator P in R , such that $P\mathcal{M} = \mathcal{N}$. Furthermore, if P is any idempotent operator in R , then $P\mathcal{M}$ is an L-subspace, $(P\mathcal{M})^\perp = (I-P)\mathcal{M}$, and P commutes with each operator in R .

Proof. If μ and ν are in $\mathcal{M}$, then, by Theorem 1.1 (b), there exist unique measures ν_1 and ν_2 , such that $\nu = \nu_1 + \nu_2$, ν_1 is μ -continuous, and ν_2 is μ -singular. Then clearly $\nu_1 \perp \nu_2$. It follows from the definitions that if ρ and ω are bounded measures and $\rho \perp \omega$, then $\|\rho + \omega\| = \|\rho\| + \|\omega\|$. Hence $\|\nu\| = \|\nu_1\| + \|\nu_2\|$. We set $P_\mu \nu = \nu_1$; that is, $\nu = P_\mu \nu + (\nu - P_\mu \nu)$ is the unique decomposition of a measure ν into the sum of a μ -continuous part $P_\mu \nu$ and a μ -singular part $\nu - P_\mu \nu$.

If ρ is another measure in $\mathfrak{M}$, then $\nu + \rho = (P_\mu \nu + P_\mu \rho) + ((\nu - P_\mu \nu) + (\rho - P_\mu \rho))$ and $P_\mu \nu + P_\mu \rho$ is μ -continuous whereas $((\nu - P_\mu \nu) + (\rho - P_\mu \rho))$ is μ -singular. Hence $P_\mu(\nu + \rho) = P_\mu \nu + P_\mu \rho$. Also, if c is a scalar, then $P_\mu c\nu = cP_\mu \nu$. Hence P_μ is a linear operator on $\mathfrak{M}$. P_μ is bounded of norm one, since $\|P_\mu \nu\| = \|\nu\| - \|\nu - P_\mu \nu\| \leq \|\nu\|$. Note that $P_\mu \mu = \mu$; in fact, if ν is μ -continuous, then $P_\mu \nu = \nu$. It follows that $P_\mu(P_\mu \nu) = P_\mu \nu$, so that P_μ is an idempotent operator. Also, if ν is μ -continuous, and ρ is in $\mathfrak{M}$, then $P_\nu \rho$ is μ -continuous; hence $P_\mu P_\nu \rho = P_\nu \rho$.

Now if T is in R and μ and ν are in $\mathfrak{M}$, then $T\nu = TP_\mu \nu + T(\nu - P_\mu \nu)$; $TP_\mu \nu$ is $P_\mu \nu$ -continuous and hence μ -continuous; $T(\nu - P_\mu \nu)$ is $(\nu - P_\mu \nu)$ -continuous and hence μ -singular. Thus $P_\mu T\nu = TP_\mu \nu$. That is, P_μ commutes with every operator in R . Thus P_μ itself must be in R .

The range of P_μ is clearly the L -space of all measures in $\mathfrak{M}$ which are μ -continuous. We denote this space by $\mathcal{L}(\mu)$. The preceding discussion proves the first statement of Theorem 1.4 in the case where $\mathfrak{N}$ is $\mathcal{L}(\mu)$ for some fixed μ in $\mathfrak{M}$. We now establish the general case.

If $\mathfrak{N}$ is any L -subspace of $\mathfrak{M}$, let $\mathfrak{P}$ denote the collection of all operators P_μ for μ in $\mathfrak{N}$. We set $P_\nu \leq P_\mu$ if $P_\mu P_\nu = P_\nu$; this happens if and only if ν is μ -continuous. Under this partial ordering, $\mathfrak{P}$ is a directed set; for if μ and ν are in $\mathfrak{N}$, then $\rho = |\mu| + |\nu|$ is in $\mathfrak{N}$, $P_\mu \leq P_\rho$, and $P_\nu \leq P_\rho$.

If $P_\nu \leq P_\mu$ then $P_\mu = P_\nu + (P_\mu - P_\nu) = P_\nu + P_\mu(I - P_\nu)$ and since $(I - P_\nu)\rho$ is ν -singular for each ρ in $\mathfrak{M}$, so is

$P_\mu (I - P_\nu) f$. Thus $P_\nu f \perp (P_\mu - P_\nu) f$ for each f in $\mathcal{M}$; hence $\|P_\mu f\| = \|P_\nu f\| + \|(P_\mu - P_\nu) f\|$. A simple induction shows that if $P_{\mu_1} \leq P_{\mu_2} \leq \dots \leq P_{\mu_n}$ are in $\mathcal{B}$ and f is in $\mathcal{M}$, then

$$\|P_{\mu_n} f\| = \|P_{\mu_1} f\| + \sum_{i=1}^{n-1} \|(P_{\mu_{i+1}} - P_{\mu_i}) f\| \leq \|f\|.$$

It follows that for any tower of elements P_μ of $\mathcal{B}$, the corresponding tower of elements $P_\mu f$ has at most countably many distinct members and satisfies the Cauchy condition for each f in $\mathcal{M}$. It follows that for each f in $\mathcal{M}$, the directed set $\{P_\mu f : P_\mu \text{ is in } \mathcal{B}\}$ converges in norm to an element Pf of $\mathcal{N}$. Thus the directed set $\mathcal{B}$ converges in the strong operator topology to an operator P on $\mathcal{M}$.

Since $P_\nu \leq P_\mu$ implies $P_\mu P_\nu = P_\nu$, we have $P P_\nu = P_\nu$ for every ν in $\mathcal{N}$. Hence $P P = P$. The range of P is clearly $\mathcal{N}$, since μ in $\mathcal{N}$ implies $P\mu = P P_\mu \mu = P_\mu \mu = \mu$. Since P is the limit in the strong operator topology of operators that commute with all operators in R , P has the same property.

If P_1 is any idempotent in R and $P_1 \mathcal{M} = \mathcal{N}$, then $P P_1 = P_1$; but P commutes with P_1 ; hence $P P_1 = P_1 P = P$. Thus $P_1 = P$ and P is the unique idempotent in R which projects on $\mathcal{N}$.

Since $P_\mu \nu = 0$ for all μ in $\mathcal{N}$ if and only if ν is in $\mathcal{N}^\perp$, the operator $(I - P)$ is an idempotent in R with range $\mathcal{N}^\perp$.

Now if P is any idempotent operator in R , μ is in $P\mathcal{M}$, and T is in R_0 , then $P T \mu = T P \mu = T \mu$. Hence $T \mu$ is in $P\mathcal{M}$. Thus $P\mathcal{M}$ is a closed invariant subspace for R_0 . By Theorem 1.3,

$P\mathcal{M}$ is an L -subspace. This completes the proof.

Corollary. If P_1 and P_2 are idempotents in R , then $P_1P_2 = P_2P_1$ is an idempotent and $P_1P_2\mathcal{M} = P_1\mathcal{M} \cap P_2\mathcal{M}$.

The two previous theorems characterize the idempotents in R . To complete characterization of R , we require a method of determining the elements of R in terms of the idempotents of R . To this end, we develop an analogue of the spectral decomposition theory for rings of operators in a Hilbert space.

We will say that an operator T in R is real if $T\mu$ is a real valued measure for every real valued measure μ in $\mathcal{M}$. Each operator T in R can be written as $T_1 + i T_2$ for some pair of real operators T_1 and T_2 in R ; in fact, it suffices to set $T_1\mu = \operatorname{Re}(T \operatorname{Re}\mu) + i \operatorname{Re}(T \operatorname{Im}\mu)$ and $T_2\mu = \operatorname{Im}(T \operatorname{Re}\mu) + i \operatorname{Im}(T \operatorname{Im}\mu)$.

Theorem 1.5. If T is a real operator in R , then for each real number t there exists an idempotent operator P_t in R , such that the following conditions hold:

$$(1) \quad t_1 \leq t_2 \text{ implies } P_{t_1} P_{t_2} = P_{t_1};$$

$$(2) \quad P_t = 0 \text{ for } t < -\|T\| \text{ and}$$

$$P_t = I \text{ for } t \geq \|T\|;$$

(3) the function $t \rightarrow P_t$ is right continuous in the strong operator topology; and

(4) $T = \int_{-\|T\|}^{\|T\|} t \, dP_t$ where the integral converges in the operator norm.

Proof. If t is any real number, we define $\mathcal{N}_t$ to be the subset of $\mathcal{M}$ consisting of all measures μ in $\mathcal{M}$ for which $T|\mu| \leq t|\mu|$. We shall show that $\mathcal{N}_t$ is an L-subspace of $\mathcal{M}$.

$\mathcal{N}_t$ is clearly a closed subset of $\mathcal{M}$. Also, if μ is in

$\mathcal{N}_t$ and $T_0 = \sum_{i=1}^n a_i P_{A_i}$ is in R_0 , with $A_i \wedge A_j = \emptyset$ for $i \neq j$, then

$$|T_0 \mu| = \sum_{i=1}^n |a_i| |P_{A_i} \mu| = \sum_{i=1}^n |a_i| P_{A_i} |\mu|.$$

Hence,

$$T|T_0 \mu| = \sum_{i=1}^n |a_i| P_{A_i} T|\mu| \leq \sum_{i=1}^n |a_i| P_{A_i} t|\mu| = t|T_0 \mu|,$$

and $T_0 \mu$ is in $\mathcal{N}_t$. It follows from Theorem 1.1 (c) that if μ is in $\mathcal{N}_t$ and ν is μ -continuous, then ν is in $\mathcal{N}_t$. Also, if μ_1 and μ_2 are in $\mathcal{N}_t$ and a and b are scalars, then $a\mu_1 + b\mu_2$ is $|\mu_1| + |\mu_2|$ -continuous, and $|\mu_1| + |\mu_2|$ is clearly in $\mathcal{N}_t$; hence $a\mu_1 + b\mu_2$ is in $\mathcal{N}_t$. Thus $\mathcal{N}_t$ is an L-subspace of $\mathcal{M}$.

By Theorem 1.4 there exists a unique idempotent P_t in R , such that $P_t \mathcal{M} = \mathcal{N}_t$ and $(I - P_t) \mathcal{M} = \mathcal{N}_t^\perp$. Note that if μ is a non-negative measure in $\mathcal{M}$, then $P_t \mu = \mu$ if and only if $T\mu \leq t\mu$.

Suppose μ is a non-negative measure in $\mathcal{N}_t^\perp$; then $T\mu - t\mu$ is a real valued measure in $\mathcal{N}_t^\perp$. If $T\mu - t\mu$ is not non-negative, then by Theorem 1.1 (a), $T\mu - t\mu = \nu_1 - \nu_2$ where ν_1 and ν_2 are

non-negative measures in $\mathcal{N}_t^\perp$, $\nu_2 \neq 0$, and $\nu_1 \perp \nu_2$. By Theorem 1.1 (b), $\mu = \mu_1 + \mu_2$ where μ_1 is ν_1 -continuous and μ_2 is ν_1 -singular. Then $\nu_1 - \nu_2 = T\mu - t\mu = (T\mu_1 - t\mu_1) + (T\mu_2 - t\mu_2)$, where $T\mu_1 - t\mu_1$ is ν_1 -continuous and $T\mu_2 - t\mu_2$ is ν_1 -singular. It follows that $\nu_1 = T\mu_1 - t\mu_1$ and $-\nu_2 = T\mu_2 - t\mu_2$. Thus $T\mu_2 \leq t\mu_2$ and, hence, μ_2 is in $\mathcal{N}_t$. This is a contradiction, since μ_2 is in $\mathcal{N}_t^\perp$ and $\mu_2 \neq 0$. Thus $T\mu - t\mu$ must be non-negative. We have proved that $T\mu \geq t\mu$ for each non-negative measure μ in $\mathcal{N}_t^\perp$.

We now verify that the operators P_t , for t real, satisfy the four conditions of the theorem. Clearly if $t_1 \leq t_2$, then $\mathcal{N}_{t_1} \subset \mathcal{N}_{t_2}$, so that $P_{t_1} P_{t_2} = P_{t_1}$. If $t < -\|T\|$ and μ is a non-negative measure in $\mathcal{M}$, then $t\mu(E) = t\|\mu\| < -\|T\|\|\mu\| \leq -\|T\mu\| \leq T\mu(E)$, so that μ is not in $\mathcal{N}_t$. Thus $\mathcal{N}_t$ is empty and $P_t = 0$. If $t \geq \|T\|$ and μ is any non-negative measure in $\mathcal{M}$, then $T\mu(A) \leq \|P_A T\mu\| \leq \|T\| \|P_A \mu\| \leq \|T\| \mu(A) \leq t\mu(A)$, so that μ is in $\mathcal{N}_t$. Thus $\mathcal{N}_t = \mathcal{M}$ and $P_t = I$. Note that $\bigcap_{t_0 < t} \mathcal{N}_t = \mathcal{N}_{t_0}$; hence for each μ in $\mathcal{M}$, $\lim_{t \rightarrow t_0^+} P_t \mu = P_{t_0} \mu$. Thus $t \rightarrow P_t$ is right continuous in the strong operator topology. We have established (1), (2), and (3) of the theorem.

To prove (4), note that if $t_1 \leq t_2$ and μ is a non-negative measure in $\mathcal{M}$, then $t_1(P_{t_2} - P_{t_1})\mu \leq T\mu \leq t_2(P_{t_2} - P_{t_1})\mu$. This is because $(P_{t_2} - P_{t_1})\mu$ is in $\mathcal{N}_{t_2} \cap \mathcal{N}_{t_1}^\perp$. Thus if $-\|T\| = t_0 \leq t_1 \leq \dots \leq t_n = \|T\|$ is a partition of the interval $[-\|T\|, \|T\|]$ of norm less than ε , and μ is a non-negative measure in $\mathcal{M}$, then

$$\sum_{i=1}^n t_{i-1} (P_{t_i} - P_{t_{i-1}}) \mu \leq T \mu \leq \sum_{i=1}^n t_i (P_{t_i} - P_{t_{i-1}}) \mu, \text{ and}$$

$$\left\| \sum_{i=1}^n t_{i-1} (P_{t_i} - P_{t_{i-1}}) \mu - \sum_{i=1}^n t_i (P_{t_i} - P_{t_{i-1}}) \mu \right\| \leq$$

$$\sum_{i=1}^n |t_{i-1} - t_i| \| (P_{t_i} - P_{t_{i-1}}) \mu \| < \delta \|\mu\|.$$

It follows that $\left\| \sum_{i=1}^n t_{i-1} (P_{t_i} - P_{t_{i-1}}) - T \right\| < \delta.$

Thus $T = \int_{-\|T\|}^{\|T\|} t \, dP_t$ where the integral converges in norm. This completes the proof.

Corollary. The closed linear span of the idempotents in R is R . Furthermore, R is a commutative algebra.

Proof. Every operator in R can be written as a linear combination of two real operators and by Theorem 1.5, every real operator is the norm limit of linear combinations of idempotent operators in R . It follows that R is commutative, since by Theorem 1.4 the idempotents in R commute.

The fact that R is a commutative Banach algebra allows us to define an involution in R and then apply the Gelfand representation theory.

If μ is in $\mathfrak{M}$ let $\bar{\mu}$ denote the complex conjugate of μ . For μ in $\mathfrak{M}$ and T in R , we set $T^*\mu = \overline{T\bar{\mu}}$. This equation clearly defines a new operator T^* in R . T is a real operator if and only if $T^* = T$.

Theorem 1.6. The map $T \rightarrow T^*$ on R satisfies the following properties:

- (1) $T^{**} = T$,
- (2) $(T_1 + T_2)^* = T_1^* + T_2^*$,
- (3) $(aT)^* = \bar{a} T^*$ if a is a scalar,
- (4) $(T_1 T_2)^* = T_2^* T_1^*$, and
- (5) $\|TT^*\| = \|T\|^2$.

Proof. Properties (1) - (3) follow immediately from the definition. To verify (4) note that $(T_1 T_2)^*\mu = \overline{T_1 T_2 \bar{\mu}} = \overline{T_1 (T_2^* \bar{\mu})} = T_1^* T_2^* \mu$; then $(T_1 T_2)^*\mu = T_2^* T_1^* \mu$, since R is commutative.

We shall show that (5) holds whenever T is an operator of the form $T = \sum_{i=1}^n a_i P_i$, where each P_i is a non-zero idempotent in R and $P_i P_j = 0$ for $i \neq j$; then by the corollary to Theorem 1.5, it will follow that (5) holds for all T in R . For an operator T of the above form and a measure μ in $\mathfrak{M}$,

$$\|T\mu\| \leq \sum_{i=1}^n |a_i| \|P_i \mu\| \leq \sup_i |a_i| \|\mu\|.$$

Hence $\|T\| \leq \sup_i |a_i|$. On the other hand, if $|a_j| = \sup_i |a_i|$ and μ is a measure in $\mathcal{M}$ such that $P_j \mu \neq 0$, then $\|T P_j \mu\| = |a_j| \|P_j \mu\| = \sup_i |a_i| \|P_j \mu\|$. Hence $\|T\| = \sup_i |a_i|$. Now if P is an idempotent in R , then P^* is an idempotent in R having the same range as P . Hence $P^* = P$ by the uniqueness of P (cf. Theorem 1.4). Thus,

$$T^* = \left(\sum_{i=1}^n a_i P_i \right)^* = \sum_{i=1}^n \overline{a_i} P_i.$$

Then since $P_i P_j = 0$ for $i \neq j$, we have

$$TT^* = \sum_{i=1}^n |a_i|^2 P_i \quad \text{and} \quad \|TT^*\| = \sup_i |a_i|^2 = \|T\|^2.$$

This completes the proof.

Theorem 1.7. There is a compact Hausdorff space X and an isomorphism-isometry $T \longrightarrow f$ of R onto $C(X)$, such that $T^* \longrightarrow \overline{f}$.

Proof. This is a direct consequence of Theorem 1.6 and the well known Gelfand representation theory (cf. [8], pp. 87-88).

We are finally in a position to prove the main theorem of this chapter. Let $M(X)$ denote the Banach space of all regular Borel measures on X (cf. [5], Chapter X).

Theorem 1.8. There is an isometry $F \longrightarrow f$ of $\mathcal{M}^*$ onto

$C(X)$ and an isometry $\mu \rightarrow \mu_X$ of $\mathfrak{M}$ into $M(X)$, such that:

$$(1) \quad F(\mu) = \int f \, d\mu_X, \quad \text{and}$$

(2) if T is the operator in R corresponding to F , as in Theorem 1.2, then

$$(T\mu)_X(U) = \int_U f \, d\mu_X$$

for every Borel set U of X .

Proof. By Theorem 1.2 there is an isometry $F \rightarrow T$ of $\mathfrak{M}^*$ onto R and, by Theorem 1.7, an isometry $T \rightarrow f$ of R onto $C(X)$; the composition $F \rightarrow f$ is the required isometry of $\mathfrak{M}^*$ onto $C(X)$. The adjoint of this map is an isometry of $M(X)$ onto $\mathfrak{M}^{**}$; its inverse, when restricted to $\mathfrak{M}$, is the map $\mu \rightarrow \mu_X$. By the definition of adjoint map we have

$$F(\mu) = \int f \, d\mu_X$$

for each μ in $\mathfrak{M}$.

To prove (2), note that the map $F \rightarrow T$ satisfies

$$T\mu(E) = F(\mu) = \int f \, d\mu_X.$$

If F_1 is another element of $\mathfrak{M}^*$, corresponding to f_1 in $C(X)$ and T_1 in R , then

$$\int f_1 \, d(T\mu)_X = F_1(T\mu) = T_1 T\mu(E) = \int f_1 f \, d\mu_X,$$

since the map $T \rightarrow f$ is an isomorphism. This equation holds for any f_1 in $C(X)$; hence

$$(T\mu)_X(U) = \int_U f \, d\mu_X$$

for each Borel set U of X . This completes the proof.

X will be called the standard domain of $\mathfrak{M}$. We shall show that the map $\mu \rightarrow \mu_X$ preserves all the measure theoretic properties that we have discussed in this chapter. The only significant difference between $\mathfrak{M}$ and its image $\mathfrak{M}_X$ is in the boolean algebras on which they are defined. We first need the following theorem.

Theorem 1.9. If h is a bounded Borel function on X , then there exists an operator T in R , such that

$$(T\mu)_X(U) = \int_U h \, d\mu_X$$

for each μ in $\mathfrak{M}$ and each Borel set U of X .

Proof. The equation $F(\mu) = \int h \, d\mu_X$ defines an element F of $\mathfrak{M}^*$. Let T be the operator in R corresponding to F . If T_1 is any operator in R and f_1 is the continuous function on X

$$\text{corre: } \int f_1 h \, d\mu_X = F(T_1\mu) = TT_1\mu(E) = T_1T\mu(E) = \int f_1 d(T\mu)_X.$$

$$\int f_1 h \, d\mu_X = F(T_1\mu) = TT_1\mu(E) = T_1T\mu(E) = \int f_1 d(T\mu)_X.$$

It follows that

$$(T\mu)_X(U) = \int_U h \, d\mu_X$$

for each Borel set U of X and each μ in $\mathfrak{M}$.

Corollary. If h is a bounded Borel function on X , then there exists a continuous function f on X , such that $f(x) = h(x)$ except on a set N which is of measure zero for each μ_X in $\mathfrak{M}_X$.

Proof. Let T be the operator corresponding to h , as in Theorem 1.9, and let f be the continuous function on X corresponding to T . Then

$$\int_U f d\mu_X = (T\mu)_X(U) = \int_U h d\mu_X$$

for each μ in $\mathfrak{M}$ and each Borel set U of X . It follows that $N = \{x \text{ in } X : f(x) \neq h(x)\}$ has μ_X -measure zero for each μ in $\mathfrak{M}$.

Theorem 1.10. (a) $\mathfrak{M}_X$ is an L -space of measures on the Borel algebra of X ,

(b) if μ is in $\mathfrak{M}$ then $|\mu_X| = |\mu|_X$,

(c) if μ and ν are in $\mathfrak{M}$ then ν is μ -continuous if and only if ν_X is μ_X -continuous, and

(d) ν is μ -singular if and only if ν_X is μ_X -singular.

Proof. By Theorem 1.9 $\mathfrak{M}_X$ is invariant under all the operators T_X of the form $T_X \lambda(U) = \int_U h d\lambda$ for h a Borel function on X . $\mathfrak{M}_X$ is closed, since $\mu \rightarrow \mu_X$ is an isometry. This implies that $\mathfrak{M}_X$ is an L -space (cf. Theorem 1.3).

To prove (b), let U be a Borel set of X and let π_U be the characteristic function of U . Then there exists an operator P (which must be idempotent) in R , such that

$$(P\mu)_X(V) = \int_V \pi_U d\mu_X$$

for every Borel set V of X . Then $|\mu_X|(U) = \|(P\mu)_X\| = \|P\mu\| =$

$$|P\mu|(E) = P|\mu|(E) = (P|\mu|)_X(X) = \int \pi_U d|\mu|_X = |\mu|_X(U).$$

Hence $|\mu_X| = |\mu|_X$.

By Theorem 1.3 ν is μ -continuous if and only if there exists a sequence $\{T_i\}_{i=1}^{\infty}$ of operators in R , such that $\lim_i T_i \mu = \nu$ in norm. This is equivalent, by Theorem 1.8 and the Radon-Nikodym theorem, to ν_X being μ_X -continuous.

By Theorem 1.4 ν is μ -singular if and only if there exists an idempotent P in R , such that $P\mu = \mu$ and $P\nu = 0$. By Theorems 1.8 and 1.9 this is equivalent to the existence of a Borel set U of X , such that $|\mu_X|(X \setminus U) = 0$ and $|\nu_X|(U) = 0$; this is equivalent to ν_X being μ_X -singular.

In the next three theorems we investigate the topology of X and its close relationship to the structure of $\mathfrak{M}$.

If $\mathfrak{N}$ is an L -subspace of $\mathfrak{M}$ and $\mathfrak{N}_X$ is its image in $M(X)$, then $K(\mathfrak{N}_X)$ will denote the smallest closed subset K of X for which $|\mu_X|(X \setminus K) = 0$ for every μ in $\mathfrak{N}$.

Theorem 1.11. (a) $K(\mathfrak{N}_X)$ is an open-compact set for each L -subspace $\mathfrak{N}$ of $\mathfrak{M}$.

(b) If K is any open-compact subset of X and

$$\mathfrak{N} = \{\mu \text{ in } \mathfrak{M} : |\mu_X|(X \setminus K) = 0\}, \text{ then } K = K(\mathfrak{N}_X).$$

$$(c) \quad K(\mathfrak{N}_X^\perp) = X \setminus K(\mathfrak{N}_X)$$

Proof. Let P be the idempotent in R whose range is $\mathfrak{N}$. The function in $C(X)$ corresponding to P is the characteristic function of an open-compact set K , where $(P\mu)_X(V) = \mu_X(V \cap K)$ for each Borel set V of X . Thus $|\mu_X|(X \setminus K) = 0$ if and only if $(P\mu)_X = \mu_X$, that is if and only if μ is in $\mathfrak{N}$. Hence $K(\mathfrak{N}_X) \subseteq K$. $K \setminus K(\mathfrak{N}_X)$ is an open subset of X ; so if $K \setminus K(\mathfrak{N}_X)$ is non-empty, Urysohn's lemma yields a continuous function f which is zero off $K \setminus K(\mathfrak{N}_X)$ but not identically zero. f corresponds to a non-zero operator T in R ,

such that $|T\mu_X|(X \setminus K) = \int_{X \setminus K} |f| d|\mu_X| = 0$ for each

μ in $\mathfrak{M}$. Hence $T\mu$ is in $\mathfrak{N}$ for each μ in $\mathfrak{M}$, but then

$$|T T^* \mu| = \int |f|^2 d|\mu_X| = \int_{X \setminus K(\mathfrak{N}_X)} |f| d|T\mu_X| + \int_{K(\mathfrak{N}_X)} |f| d|T\mu_X| = 0,$$

since $|f| = 0$ on $K(\mathfrak{N}_X)$ and $|(T\mu)_X|(X \setminus K(\mathfrak{N}_X)) = 0$. Then $T T^* = 0$, which contradicts the assumption that f is not zero. Hence

$K \setminus K(\mathfrak{N}_X)$ is empty and $K = K(\mathfrak{N}_X)$. This establishes part (a).

Part (c) follows immediately, since the idempotent which projects on $\mathfrak{N}^\perp$ is $I - P$, which corresponds to the characteristic function of the complement of $K(\mathfrak{N}_X)$.

If K is any open-compact subset of X , then the characteristic function of K is a continuous function which corresponds to an idempotent P in R . If $\mathfrak{N} = P\mathfrak{M}$ then μ is in $\mathfrak{N}$ if and only if $P\mu = \mu$, that is, if and only if $|\mu_X|(X \setminus K) = 0$. Part (b) follows.

The carrier of a regular Borel measure λ is the smallest closed set K for which $|\lambda|(X \setminus K) = 0$. If μ is in $\mathfrak{M}$ then clearly $\text{carrier}(\mu_X) = K(\mathcal{L}(\mu)_X)$. Hence the following corollary is an immediate consequence of Theorem 1.11.

Corollary. If μ and ν are in $\mathfrak{M}$ the carriers of μ_X and ν_X are open-compact sets, ν is μ -continuous if and only if $\text{carrier}(\nu_X) \subset \text{carrier}(\mu_X)$, and ν is μ -singular if and only if $\text{carrier}(\nu_X)$ and $\text{carrier}(\mu_X)$ are disjoint.

Theorem 1.12. (a) A Borel subset K of X is a set of measure zero for each μ_X in $\mathfrak{M}_X$ if and only if K has no interior.

(b) If λ is in $M(X)$ then λ is in $\mathfrak{M}_X$ if and only if $\lambda(K) = 0$ for each Borel subset K of X with no interior.

Proof. It follows directly from Urysohn's lemma and Theorem 1.8 that if K is a Borel set with interior, then $\mu_X(K) \neq 0$ for some μ in $\mathfrak{M}$.

Conversely, if K is a Borel set with no interior, then no compact subset of K has interior. Let K_1 be a compact subset of K . If μ_X is in $\mathfrak{M}_X$ and $\mu_X(K_1) \neq 0$, then the measure λ , defined by $\lambda(V) = \mu_X(V \cap K_1)$, is a non-zero measure in $\mathfrak{M}_X$. By the corollary to Theorem 1.11, $\text{carrier}(\lambda)$ is an open-compact set. However, $\text{carrier}(\lambda) \subset K_1$ and hence $\text{carrier}(\lambda)$ is empty. Thus $\mu_X(K_1) = 0$ for every μ in $\mathfrak{M}$. Then since each μ_X is a regular Borel measure, $\mu_X(K) = 0$ for every μ in $\mathfrak{M}$. This establishes part (a).

To prove part (b), we apply Theorem 1.4 to the L-space $M(X)$.

Since $\mathfrak{M}_X$ is an L-subspace of $M(X)$, there is an idempotent operator Q on $M(X)$ which leaves all L-subspaces of $M(X)$ invariant, and such that $Q M(X) = \mathfrak{M}_X$ and $(I-Q) M(X) = \mathfrak{M}_X^\perp$. The set of all measures in $M(X)$ that are zero on Borel sets with no interior is an L-subspace of $M(X)$; hence it is invariant under Q and $I-Q$. Thus if λ is a measure in $M(X)$ which is zero on all Borel sets without interior, then $(I-Q)\lambda$ has the same property and is μ_X -singular for each μ in $\mathfrak{M}$. Let K be the carrier of $(I-Q)\lambda$. If K has non-empty interior K° , then there is a non-zero measure μ in $\mathfrak{M}$, such that $\text{carrier}(\mu_X) \subset K^\circ$. Since $(I-Q)\lambda$ is μ_X -singular, there exist Borel sets U and V , such that $U \cup V = X$, $|(I-Q)\lambda|(U) = 0$, and $|\mu_X|(V) = 0$. Then $U \cap K$ has no interior, and $U \cap \text{carrier}(\mu_X) \subset U \cap K$. Hence $|\mu_X|(U) = 0$ by part (a). Then $\mu_X = 0$, since $U \cup V = X$. The resulting contradiction shows that K has no interior and $|(I-Q)\lambda|(K) = 0$. Thus $(I-Q)\lambda = 0$ and λ is in $\mathfrak{M}_X$. This establishes part (b).

Theorem 1.13. X is extremely disconnected; that is, the closure of every open set is open.

Proof. If U is an open subset of X , then by the corollary to Theorem 1.11 there exists a continuous function f on X , such that $f = \pi_U$ except on a set N of μ_X -measure zero for every μ in $\mathfrak{M}$. Let $K = \{x \text{ in } X : f(x) = 1\}$, and $V = \{x \text{ in } X : |f(x)| > 0\}$. $U \setminus K$ and $V \setminus \bar{U}$ are both open sets contained in N , but N has no interior, so $U \setminus K$ and $V \setminus \bar{U}$ are empty. Thus $U \subset K \subset V \subset \bar{U}$; but since K is a closed set, $K = V = \bar{U}$. Hence $\bar{U}$ is open.

The preceding results were developed independently as tools for the study of convolution measure algebras. However, similar results have been published by several authors in connection with Kakutani's abstract L-spaces (cf. [6]). Perhaps the most complete results of this nature are due to Cunningham (cf. [1]). Also see [3] in this connection.

To complete Chapter I, we prove some homomorphism theorems for L-spaces which are essential in obtaining the results in Chapter II.

Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be L-spaces and X_1 and X_2 be the standard domains of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ respectively. The boolean algebras on which $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are defined are not important for our purposes; by Theorem 1.10 we may as well assume that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are L-spaces over the Borel algebras of X_1 and X_2 respectively. That is, we may identify $\mathfrak{M}_1$ with $(\mathfrak{M}_1)_{X_1}$ and $\mathfrak{M}_2$ with $(\mathfrak{M}_2)_{X_2}$.

Definition 1.4. If θ is a bounded linear map from $\mathfrak{M}_1$ to $\mathfrak{M}_2$, then θ will be called an L-homomorphism if the following three conditions are satisfied:

- (1) If μ is in $\mathfrak{M}_1$, then $\theta\mu(X_2) = \mu(X_1)$;
- (2) if μ is a non-negative measure in $\mathfrak{M}_1$, then $\theta\mu$ is non-negative; and
- (3) if μ is a non-negative measure in $\mathfrak{M}_1$ and ν is a measure in $\mathfrak{M}_2$, such that $0 \leq \nu \leq \theta\mu$, then there exists a non-negative measure $\omega \leq \mu$, in $\mathfrak{M}_1$, such that $\theta\omega = \nu$.

We are particularly interested in the adjoint map θ^* of a bounded linear map θ . In this case, θ^* is a bounded linear map from $C(X_2)$ to $C(X_1)$ and is defined by the equation

$$\int \theta^* f \, d\mu = \int f \, d\theta\mu .$$

Theorem 1.14. If θ is a bounded linear map from $\mathcal{M}_1$ to $\mathcal{M}_2$, then the following statements are equivalent:

- (a) θ is an L-homomorphism;
- (b) there exists a continuous map α from X_1 to X_2 , such that $\theta\mu(V) = \mu(\alpha^{-1}(V))$ for each Borel set V of X_2 and each μ in $\mathcal{M}_1$.

(c) θ^* is a homomorphism of the algebra $C(X_2)$ into the algebra $C(X_1)$, such that $\theta^* \overline{f} = \overline{\theta^* f}$ for each f in $C(X_2)$ and $\theta^* 1 = 1$.

Proof. To prove that (a) implies (b), we first show that if θ is an L-homomorphism, μ is a non-negative measure in $\mathcal{M}_1$, and ν is μ -continuous, then $\theta\nu$ is $\theta\mu$ -continuous. For such a μ and ν and each $\epsilon > 0$, there exists a simple step function

$$f = \sum_{i=1}^n a_i \chi_{V_i}, \text{ such that if } \nu'_\epsilon(V) = \int_V f \, d\mu, \text{ then } \|\nu'_\epsilon - \nu\| < \epsilon.$$

For each i let $\mu_i(V) = \mu(V \cap V_i)$; then $\mu - \mu_i \geq 0$ for each i ,

and $\nu'_\epsilon = \sum_{i=1}^n a_i \mu_i$. By condition (2) for an L-homomorphism,

$\theta\mu_i \leq \theta\mu$ for each i . Hence $\theta\mu_i$ is $\theta\mu$ -continuous for each i .

It follows that $\theta\nu'_\epsilon$ is $\theta\mu$ -continuous. Since this holds for each $\epsilon > 0$, we conclude that $\theta\nu$ is $\theta\mu$ -continuous.

If $\mathcal{N}$ is any L-subspace of $\mathcal{M}_2$, we define $\mathcal{N}' = \{\mu \text{ in } \mathcal{M}_1 : \theta\mu \text{ is in } \mathcal{N}\}$. If μ is in $\mathcal{N}'$ and ν is μ -continuous, then $|\nu|$ is

$|\mu|$ -continuous and, by the above paragraph, $\theta|\nu|$ is $\theta|\mu|$ -continuous. Hence $\theta|\nu|$ is in $\mathcal{N}$ and ν is in $\mathcal{N}'$. If μ_1 and μ_2 are in $\mathcal{N}'$, then $|\mu_1| + |\mu_2|$ is in $\mathcal{N}'$ and $\mu_1 + \mu_2$ is $|\mu_1| + |\mu_2|$ -continuous; hence $\mu_1 + \mu_2$ is in $\mathcal{N}'$. $\mathcal{N}'$ is clearly a closed subset of $\mathcal{M}_1$. It follows that $\mathcal{N}'$ is an L-subspace of $\mathcal{M}_1$.

If $\mu \perp \mathcal{N}'$ then $|\mu| \perp \mathcal{N}'$ and, by Theorem 1.4, there exist non-negative measures ν_1 and ν_2 , such that $\nu_1 \leq \theta|\mu|$, $\nu_2 \leq \theta|\mu|$, $\theta|\mu| = \nu_1 + \nu_2$, ν_1 is in $\mathcal{N}$, and ν_2 is in $\mathcal{N}^\perp$. By condition (3) for an L-homomorphism, there exists a non-negative measure $\omega \leq |\mu|$, such that $\theta\omega = \nu_1$. But $\omega \perp \mathcal{N}'$ since $\mu \perp \mathcal{N}'$, and ω is in $\mathcal{N}'$ since $\theta|\omega| = \theta\omega$ is in $\mathcal{N}$. Hence $\omega = 0$ and $\nu_1 = 0$. Thus if $\mu \perp \mathcal{N}'$ then $\theta\mu \perp \mathcal{N}$. It follows that $(\mathcal{N}^\perp)' = (\mathcal{N}')^\perp$.

If μ is in $\mathcal{M}_1$, $\mathcal{N}$ is an L-subspace of $\mathcal{M}_2$, and P and P' are the idempotents of Theorem 1.4 which project on $\mathcal{N}$ and $\mathcal{N}'$ respectively, then $P'\mu(X_1) = \theta P'\mu(X_2)$, by condition (1) for an L-homomorphism. However, $\theta P'\mu$ is in $\mathcal{N}$ and $\theta(I-P')\mu$ is in $\mathcal{N}^\perp$ by the above paragraph. Then $P\theta\mu = \theta P'\mu$ since $\theta\mu = \theta P'\mu + \theta(I-P')\mu$. Hence $P\theta\mu(X_2) = P'\mu(X_1)$. Now P corresponds to an open-compact subset V of X_2 and P' to an open-compact subset V' of X_1 , such that $P'\mu(X_1) = \mu(V')$ and $P\theta\mu(X_2) = \theta\mu(V)$ for each μ in $\mathcal{M}_1$. The correspondence $V \rightarrow V'$ is a map from all open-compact subsets of X_2 to open-compact subsets of X_1 , which satisfies $X_1 \setminus V' = (X_2 \setminus V)'$, $(V_1 \cap V_2)' = V_1' \cap V_2'$, and $\theta\mu(V) = \mu(V')$ for each μ in $\mathcal{M}_1$. X_2 is extremely disconnected by Theorem 1.13; hence the

topology in X_2 has a basis of open-compact sets. If $\{V_\rho\}_{\rho \in D}$ is a basis of open-compact sets at the point x in X_2 , let

$\alpha^{-1}(x) = \bigcap_{\rho \in D} V_\rho'$. It is easily seen from the properties of the map $V \rightarrow V'$, that each y in X_1 is in $\alpha^{-1}(x)$ for exactly one x in X_2 ; we denote this unique x by $\alpha(y)$. Clearly

$\alpha^{-1}(x) = \{y \text{ in } X_1 : \alpha(y) = x\}$ for each point x in X_2 . α is a map from X_1 to X_2 , such that $\alpha^{-1}(V) = V'$ for each open-compact subset V of X_2 . It follows that α is continuous and $\theta\mu(V) = \mu(\alpha^{-1}(V))$ for each μ in $\mathcal{M}_1$ and each open-compact subset V of X_2 . Then the same equality holds for each Borel set of X_2 . Thus (a) implies (b).

To prove that (b) implies (c) note that if α satisfies statement (b) of the theorem, then

$$\int_{\mathcal{M}_1} f(\alpha(y)) d\mu(y) = \int_{\mathcal{M}_2} f(x) d\theta\mu(x)$$

for each μ in $\mathcal{M}_1$ and each f in $C(X_2)$. Hence $\theta^*f(y) = f(\alpha(y))$ for each y in X_1 . It follows that θ^* is an algebraic homomorphism, $\theta^*\bar{f} = \overline{\theta^*f}$, and $\theta^*1 = 1$.

To complete the argument, we show that (c) implies (a). If $\theta^*1=1$ then clearly $\theta\mu(X_2) = \mu(X_1)$ for each μ in $\mathcal{M}_1$. If θ^* is a homomorphism and $\theta^*\bar{f} = \overline{\theta^*f}$ for each f in $C(X_2)$, then for each non-negative μ in $\mathcal{M}_1$ and each f in $C(X_2)$

$$\int |f|^2 d\theta\mu = \int |\theta^*f|^2 d\mu \geq 0.$$

Hence $\theta\mu$ is non-negative whenever μ is non-negative. If μ is a non-negative measure in $\mathcal{M}_1$ and $0 \leq \nu \leq \theta\mu$, then there is a Borel

function h on X_2 , such that $0 \leq h \leq 1$ and

$$\int g \, d\nu = \int g h \, d\theta\mu$$

for each g in $C(X_2)$. By the corollary to Theorem 1.9 there is a continuous function f on X_2 , such that $f = h$ except on a set of λ -measure zero for each λ in $\mathfrak{M}_2$. Then $0 \leq f \leq 1$, and

$$\int g \, d\nu = \int g f \, d\theta\mu$$

for each g in $C(X_2)$. Let

$$\omega(V) = \int_V \theta^* f \, d\mu.$$

Then $0 \leq \omega \leq \mu$, since $0 \leq \theta^* f \leq 1$. Also

$$\int g \, d\theta\omega = \int (\theta^* g)(\theta^* f) \, d\mu = \int \theta^*(gf) \, d\mu = \int g f \, d\theta\mu = \int g \, d\nu$$

for each g in $C(X_2)$; hence $\theta\omega = \nu$. Thus conditions (1), (2), and (3) of Definition 1.4 are met and θ is an l -homomorphism. This completes the proof.

Remark. If X and Y are compact Hausdorff spaces and α is a continuous function from X into Y , then the equation $\theta\mu(V) = \mu(\alpha^{-1}(V))$, for μ in $M(X)$ and V a Borel set of Y , defines a homomorphism θ of $M(X)$ into $M(Y)$. The adjoint map θ^* , restricted to the bounded Borel functions on Y , has the form $\theta^*f(x) = f(\alpha(x))$ and is clearly a homomorphism which preserves conjugation and takes 1 onto 1 . A glance at the last part of the proof of Theorem 1.14 shows that this is enough to ensure that θ is an

L-homomorphism.

Condition (3) in the definition of an L-homomorphism may be replaced by a formally weaker condition, namely:

(3') There exists a set $\mathcal{F}$ of non-negative measures in $\mathcal{M}_1$, such that every non-negative measure in $\mathcal{M}_1$ is the norm limit of linear combinations, with positive coefficients, of elements of $\mathcal{F}$; and if μ is any element of $\mathcal{F}$ and $0 \leq \nu \leq \theta\mu$, then there exists a sequence $\{\mu_i\}_{i=1}^{\infty}$ of measures in $\mathcal{M}$, such that $0 \leq \mu_i \leq \mu$ for each i and $\lim_i \theta\mu_i = \nu$.

Theorem 1.15. If θ is a bounded linear map from $\mathcal{M}_1$ to $\mathcal{M}_2$ which satisfies (1) and (2) of Definition 1.4 and (3') as stated above, then θ is an L-homomorphism.

Proof. We must show that if θ satisfies (1), (2), and (3') then θ satisfies (3). Let $\mathcal{F}$ be the subset of $\mathcal{M}_1$ given by condition (3'). If μ is any non-negative measure in $\mathcal{M}_1$, then by condition (3'), there exists a sequence $\{\mu_i\}_{i=1}^{\infty}$ where

$$\mu_i = \sum_{j=1}^{n_i} a_{ij} \mu_{ij}, \quad a_{ij} \geq 0 \quad \text{and} \quad \mu_{ij} \text{ is in } \mathcal{F}$$

for each i and j , and $\lim_i \mu_i = \mu$.

If $0 \leq \nu \leq \theta\mu$, then since $\lim_i \theta\mu_i = \theta\mu$ there exists a sequence $\{\nu_i\}_{i=1}^{\infty}$, such that $0 \leq \nu_i \leq \theta\mu_i$ for each i and $\lim_i \nu_i = \nu$. Furthermore, each ν_i may be written in the form

$$\nu_i = \sum_{j=1}^{n_i} a_{ij} \nu_{ij} \quad ,$$

where $0 \leq \nu_{ij} \leq \theta \mu_{ij}$ for each i and j . These statements follow from the fact that the non-negative measures in $M(X_2)$ form a lattice under the order $\leq$.

By condition (3'), there exists for each i and j a sequence $\{\omega_{ijk}\}_{k=1}^{\infty}$ of measures in $\mathfrak{M}_1$, such that $0 \leq \omega_{ijk} \leq \mu_{ij}$ and $\lim_i \theta \omega_{ijk} = \nu_{ij}$.

Let $\omega_{ik} = \sum_{j=1}^{n_i} a_{ij} \omega_{ijk}$. Then

$0 \leq \omega_{ik} \leq \mu_i$ and $\lim_k \theta \omega_{ik} = \nu_i$. Since $\lim_i \nu_i = \nu$, there exists a subsequence $\{\omega_{ik_i}\}_{i=1}^{\infty}$, such that $\lim_i \theta \omega_{ik_i} = \nu$.

We now apply some results of Porcelli concerning weak convergence in spaces of measures. Since the sequence $\{\mu_i\}_{i=1}^{\infty}$ converges in norm, it also converges weakly. By Theorem 4.3 of [10], this implies that the sequence $\{\mu_i\}_{i=1}^{\infty}$ is uniformly ρ -continuous, where

$$\rho = \sum_{i=1}^{\infty} (2^i (1 + \|\mu_i\|))^{-1} \mu_i .$$

That is, for each $\epsilon > 0$ there exists a $\delta > 0$, such that if V is a Borel set of X_1 and $\rho(V) < \delta$, then $\mu_i(V) < \epsilon$ for each i . Since $0 \leq \omega_{ik_i} \leq \mu_i$ for each i , the sequence $\{\omega_{ik_i}\}_{i=1}^n$ is also uniformly ρ -continuous. Then since ρ is a bounded,

countably additive measure,

$$\lim_{i,n} \sum_{j \geq n} \omega_{ik_i}(V_j) = 0$$

for each sequence $\{V_j\}_{j=1}^{\infty}$ of pairwise disjoint Borel sets of X_1 . By Theorem 3.2 of [10], this implies that the sequence $\{\omega_{ik_i}\}_{i=1}^{\infty}$ has a weakly convergent subsequence $\{\omega'_p\}_{p=1}^{\infty}$. Let ω be the weak limit of this subsequence. Then $0 \leq \omega \leq \mu$ and $\theta \omega = \nu$. Thus θ satisfies condition (3). This completes the proof.

Theorem 1.16. If θ is an L-homomorphism from $\mathfrak{M}_1$ to $\mathfrak{M}_2$ and θ is one to one, then θ is an isometry.

Proof. We apply Theorem 1.14. If θ is one to one, then θ^* is onto which implies that α is one to one. Hence α is a homeomorphism. Since θ is defined by the equation $\theta \mu(V) = \mu(\alpha^{-1}(V))$, θ is clearly an isometry.

CHAPTER II

In this chapter we define the concept of convolution measure algebra and develop a representation of the maximal ideal ideal space of such an algebra.

Throughout the chapter $\mathfrak{M}$ will denote an L-space and X will denote the standard domain of $\mathfrak{M}$. Unless otherwise specified, we shall identify $\mathfrak{M}$ with $\mathfrak{M}_X$ and drop the subscript.

Definition 2.1. A convolution measure algebra is an L-space $\mathfrak{M}$ together with a multiplication $\cdot$ on $\mathfrak{M}$, such that $(\mathfrak{M}, \cdot)$ forms a commutative Banach algebra and the following conditions are satisfied:

- (1) $\mu \cdot \nu(X) = \mu(X)\nu(X)$ for each μ and ν in $\mathfrak{M}$;
- (2) if μ and ν are non-negative measures in $\mathfrak{M}$, then $\mu \cdot \nu$ is non-negative; and
- (3) if μ , ν , and ω are non-negative measures in $\mathfrak{M}$ and $\omega \leq \mu \cdot \nu$, then for each $\epsilon > 0$ there exists sets $\{\mu_i\}_{i=1}^n$ and $\{\nu_j\}_{j=1}^m$ of non-negative measures in $\mathfrak{M}$, and a set $\{a_{ij}\}_{i,j=1}^{n,m}$ of numbers in the interval $[0,1]$, such that

$$\sum_{i=1}^n \mu_i \leq \mu, \quad \sum_{j=1}^m \nu_j \leq \nu, \quad \text{and} \quad \left\| \sum_{i,j=1}^{n,m} a_{ij} \mu_i \cdot \nu_j - \omega \right\| < \epsilon.$$

Let S be any locally compact abelian topological semigroup

(in particular S may be a group). $M(S)$ denotes the L -space of all bounded regular Borel measures on S . Convolution in S is a multiplication defined by the equation

$$\int f(s) d(\mu \cdot \nu)(s) = \iint f(st) d\mu(s) d\nu(t)$$

for each bounded Borel function f on S and each μ and ν in $M(S)$. It is well known that $M(S)$ is a commutative Banach algebra under this multiplication (Cf. [14]).

Theorem 2.1. If S is a locally compact abelian topological semigroup, then $M(S)$ is a convolution measure algebra.

Proof. Conditions (1) and (2) in the definition of convolution measure algebra obviously hold for convolution in $M(S)$.

To show that (3) holds, let μ , ν , and ω be non-negative measures in $M(S)$, such that $\omega \leq \mu \cdot \nu$. By the Radon-Nikodym Theorem, there exists a Borel function f on S , such that $0 \leq f \leq 1$ and $\omega(V) = \int_V f d\mu \cdot \nu$ for each Borel set V of X . Let h be the function on $S \times S$ defined by $h(s, t) = f(st)$. Since multiplication in S is continuous, h is a Borel function. Clearly $0 \leq h \leq 1$. It follows that for each $\epsilon > 0$ there exists k on $S \times S$ of the form

$$k = \sum_{i,j=1}^{n,m} a_{ij} \pi_{U_i \times V_j},$$

where $0 \leq a_{ij} \leq 1$ for each i and j , $\{U_i\}_{i=1}^n$ and $\{V_j\}_{j=1}^m$ are

pairwise disjoint collections of Borel sets of S , $\pi_{U_i \times V_j}$ is the characteristic function of the set $U_i \times V_j$, and

$$\int |h-k| d\mu \times \nu < \epsilon .$$

Let $\mu_i(V) = \mu(V \cap U_i)$ and $\nu_j(V) = \nu(V \cap V_j)$

for each Borel set V of S and each i and j . Then $\{\mu_i\}_{i=1}^n$ and $\{\nu_j\}_{j=1}^m$ are collections of non-negative measures in $M(S)$, such that

$$\sum_{i=1}^n \mu_i \leq \mu \quad \text{and} \quad \sum_{j=1}^m \nu_j \leq \nu .$$

If g is any continuous function on S and

$$\rho = \sum_{i,j=1}^{n,m} a_{ij} \mu_i \cdot \nu_j ,$$

then by Fubini's Theorem and the definition of convolution,

$$|\int g d(\rho - \omega)| = \left| \sum_{i,j=1}^{n,m} \int \int a_{ij} g(st) d\mu_i(s) d\nu_j(t) - \int \int g(st) h(st) d\mu(s) d\nu(t) \right|$$

$$= \left| \int g(k-h) d\mu \times \nu \right| \leq \int |g| |k-h| d\mu \times \nu < \|g\| \epsilon . \quad \text{It follows that}$$

$$\|\rho - \omega\| < \epsilon . \quad \text{This completes the proof.}$$

Definition 2.2. If $\mathcal{M}$ is an L -space, let $\mathcal{M} \otimes \mathcal{M}$ denote the set of all regular Borel measures ρ on $X \times X$, such that ρ is $\mu \times \nu$ -continuous for some μ and ν in $\mathcal{M}$.

Theorem 2.2. $\mathcal{M} \otimes \mathcal{M}$ is an L-space and every non-negative measure in $\mathcal{M} \otimes \mathcal{M}$ is the norm limit of linear combinations, with non-negative coefficients, of measures of the form $\mu \times \nu$ where μ and ν are non-negative measures in $\mathcal{M}$.

Proof. Obviously if ρ is in $\mathcal{M} \otimes \mathcal{M}$ and ω is ρ -continuous then ω is in $\mathcal{M} \otimes \mathcal{M}$. If ρ_1 and ρ_2 are in $\mathcal{M} \otimes \mathcal{M}$ then there exist measures μ_1, μ_2, ν_1 , and ν_2 in $\mathcal{M}$, such that ρ_1 is $\mu_1 \times \nu_1$ -continuous and ρ_2 is $\mu_2 \times \nu_2$ -continuous; hence $\rho_1 + \rho_2$ is $(|\mu_1| + |\mu_2|) \times (|\nu_1| + |\nu_2|)$ -continuous and $\rho_1 + \rho_2$ is in $\mathcal{M} \otimes \mathcal{M}$. If $\{\rho_i\}_{i=1}^{\infty}$ is a sequence in $\mathcal{M} \otimes \mathcal{M}$ which converges to a measure ρ in norm, then ρ_i is $\mu_i \times \nu_i$ -continuous for some μ_i and ν_i in $\mathcal{M}$, for each i . Let

$$\mu = \sum_{i=1}^{\infty} 2^{-i} |\mu_i| \quad \text{and} \quad \nu = \sum_{i=1}^{\infty} 2^{-i} |\nu_i| ;$$

then each ρ_i is $\mu \times \nu$ -continuous; and hence ρ is $\mu \times \nu$ -continuous. Thus ρ is in $\mathcal{M} \otimes \mathcal{M}$. This completes the proof that $\mathcal{M} \otimes \mathcal{M}$ is an L-space.

If ρ is a non-negative measure in $\mathcal{M} \otimes \mathcal{M}$ and ρ is $\mu \times \nu$ -continuous, where μ and ν are in $\mathcal{M}$, then ρ is $|\mu| \times |\nu|$ -continuous. Hence there exist for each $\epsilon > 0$ Borel sets $\{U_i\}_{i=1}^n$ and $\{V_i\}_{i=1}^n$ and non-negative numbers $\{a_i\}_{i=1}^n$, such that if $\mu_i(V) = |\mu|(V \cap U_i)$ and $\nu_i(V) = |\nu|(V \cap V_i)$ for each Borel set V of X , then

$$\left\| \sum_{i=1}^n a_i \mu_i \times \nu_i - \rho \right\| < \epsilon .$$

This proves the second part of the theorem.

Theorem 2.3. If $(\mathfrak{M}, \cdot)$ is a convolution measure algebra, then there exists an L-homomorphism θ from $\mathfrak{M} \otimes \mathfrak{M}$ to $\mathfrak{M}$, such that $\theta(\mu \times \nu) = \mu \cdot \nu$ for each μ and ν in $\mathfrak{M}$.

Proof. Fix μ and ν in $\mathfrak{M}$; we define θ first on $\mathcal{L}(\mu \times \nu)$. Let $\{U_i \times V_i\}_{i=1}^n$ be a pairwise disjoint collection of Borel rectangles in $X \times X$. Set $\mu_i(V) = \mu(V \cap U_i)$ and $\nu_i(V) = \nu(V \cap V_i)$ for each Borel set V of X ; then $\mu_i \times \nu_i \perp \mu_j \times \nu_j$ if $i \neq j$.

If ρ is a measure in $\mathcal{L}(\mu \times \nu)$ of the form $\sum_{i=1}^n a_i \mu_i \times \nu_i$, we define

$$\theta \rho = \sum_{i=1}^n a_i \mu_i \cdot \nu_i. \text{ Then } \|\theta \rho\| \leq \sum_{i=1}^n |a_i| \|\mu_i \cdot \nu_i\| \leq \sum_{i=1}^n |a_i| \|\mu_i\| \|\nu_i\| =$$

$$\sum_{i=1}^n |a_i| \|\mu_i \times \nu_i\| = \|\rho\|.$$

If $\{U_{ij} \times V_{ik}\}_{i,j,k=1}^{n, m_i, p_i}$ is a refinement of the subdivision $\{U_i \times V_i\}_{i=1}^n$,

$\bigcup_{j=1}^{m_i} U_{ij} = U_i$ and $\bigcup_{k=1}^{p_i} V_{ik} = V_i$ for each i , and $U_{ij} \cap U_{ij'} = \emptyset$ and

$V_{ik} \cap V_{ik'} = \emptyset$ for $j \neq j'$ and $k \neq k'$, then ρ is also represented

as

$$\sum_{ijk} a_i \mu_{ij} \times \nu_{ik}$$

where $\mu_{ij}(V) = \mu(V \cap U_{ij})$ and $\nu_{ik}(V) = \nu(V \cap V_{ik})$.

However,

$$\mu_i = \sum_{j=1}^{m_i} \mu_{ij} \quad \text{and} \quad \nu_i = \sum_{k=1}^{p_i} \nu_{ik}, \quad \text{so that}$$

$$\mu_i \cdot \nu_i = \sum_{j,k} \mu_{ij} \cdot \nu_{ik} \quad \text{and} \quad \theta \rho = \sum_{ijk} a_i \mu_{ij} \cdot \nu_{ik}.$$

Since any two subdivisions of $X \times X$ by Borel rectangles have a common refinement of the above form, the definition of $\theta \rho$ is independent of the subdivision chosen. We shall call a measure ρ of the form described above a simple $\mu \times \nu$ -continuous measure. We have defined θ on the set of all simple $\mu \times \nu$ -continuous measures and shown that $\|\theta \rho\| \leq \|\rho\|$ for each such measure ρ .

If ρ and ω are simple $\mu \times \nu$ -continuous measures and a and b are scalars, we may assume (by taking refinements) that

$$\rho = \sum_{i=1}^n a_i \mu_i \times \nu_i \quad \text{and} \quad \omega = \sum_{i=1}^n b_i \mu_i \times \nu_i. \quad \text{Then}$$

$$a\rho + b\omega = \sum_{i=1}^n (aa_i + bb_i) \mu_i \times \nu_i. \quad \text{Hence}$$

$$\theta(a\rho + b\omega) = \sum_{i=1}^n (aa_i + bb_i) \mu_i \cdot \nu_i = a \theta \rho + b \theta \omega.$$

Hence θ is a bounded linear map defined on the simple $\mu \times \nu$ -continuous measures.

The simple $\mu \times \nu$ -continuous measures form a dense subspace of $\mathcal{L}(\mu \times \nu)$; hence θ may be extended to all of $\mathcal{L}(\mu \times \nu)$. Clearly $\theta(\mu_1 \times \nu_1) = \mu_1 \cdot \nu_1$ for each μ -continuous measure μ_1 and ν -continuous measure ν_1 in $\mathcal{M}$. In fact θ is uniquely determined by this condition.

If μ_1, μ_2, ν_1 , and ν_2 are measures in $\mathcal{M}$ and θ_1 and θ_2 are defined on $\mathcal{L}(\mu_1 \times \nu_1)$ and $\mathcal{L}(\mu_2 \times \nu_2)$ respectively, by the above construction, then the uniqueness of the definition implies that θ_1 and θ_2 agree on $\mathcal{L}(\mu_1 \times \nu_1) \cap \mathcal{L}(\mu_2 \times \nu_2)$. It follows that θ may be defined on all of $\mathcal{M} \otimes \mathcal{M}$, such that $\theta(\mu \times \nu) = \mu \cdot \nu$ for each μ and ν in $\mathcal{M}$.

To prove that θ is an L-homomorphism note that (1) in Definition 2.1 implies that $\theta(\mu \times \nu)(X) = \mu \cdot \nu(X) = \mu(X)\nu(X) = (\mu \times \nu)(X \times X)$ for each μ and ν in $\mathcal{M}$, and hence $\theta f(X) = f(X \times X)$ for every f in $\mathcal{M} \otimes \mathcal{M}$. Thus θ satisfies (1) of Definition 1.4. Similarly, (2) of Definition 2.1 and Theorem 2.2 imply that θ satisfies (2) of Definition 1.4. Also, if we set $\mathcal{F} = \{\mu \times \nu : \mu \text{ and } \nu \text{ are non-negative measures in } \mathcal{M}\}$, then (3) of Definition 2.1 and Theorem 2.2 imply that $\mathcal{F}$ and θ satisfy condition (3') as used in Theorem 1.15. Hence, by that theorem, θ is an L-homomorphism.

Theorem 2.4. If Y is the standard domain of $\mathcal{M} \otimes \mathcal{M}$, then there is an isomorphism-isometry $\mathcal{Q}$ of $C(X \times X)$ into $C(Y)$, such that $\mathcal{Q}\bar{f} = \overline{\mathcal{Q}f}$ for each f in $C(X \times X)$, $\mathcal{Q}1 = 1$, and

$$\int f \, d\rho = \int \mathcal{Q}f \, d\rho_Y \text{ for each } f \text{ in } C(X \times X) \text{ and } \rho \text{ in } \mathcal{M} \otimes \mathcal{M}.$$

Proof. Each f in $C(X \times X)$ determines an operator T on

$\mathcal{M} \otimes \mathcal{M}$ by the equation $T\rho(V) = \int_V f d\rho$. T clearly commutes

with all the operators on $\mathcal{M} \otimes \mathcal{M}$ determined by simple step functions on the Borel algebra of $X \times X$. Each set of the form $U \times V$ in $X \times X$, where U and V are open-compact sets, carries a measure of the form

$\mu \times \nu$ in $\mathcal{M} \otimes \mathcal{M}$. The sets of this form comprise a neighborhood

basis in $X \times X$. It follows that $\mathcal{M} \otimes \mathcal{M}$ is a weak- $*$ dense subspace of $M(X \times X)$. Hence $\|T\| = \sup\{\|T\rho\| : \rho \text{ is in } \mathcal{M} \otimes \mathcal{M} \text{ and } \|\rho\| = 1\}$

$$= \sup\left\{\int |f| d|\rho| : \rho \text{ is in } \mathcal{M} \otimes \mathcal{M} \text{ and } \|\rho\| = 1\right\}$$

$$= \sup\left\{\int |f| d|\rho| : \rho \text{ is in } M(X \times X) \text{ and } \|\rho\| = 1\right\} = \|f\|.$$

The theorem now follows immediately from Theorems 1.7 and 1.8.

Definition 2.3. If $(\mathcal{M}, \cdot)$ is a convolution measure algebra, we denote by Δ the collection of all functions f in $C(X)$, such that

$$\int f d\mu \cdot \nu = \int f d\mu \int f d\nu \text{ for each } \mu \text{ and } \nu \text{ in } \mathcal{M}.$$

Since $\mathcal{M}^* = C(X)$, the maximal ideal space of $(\mathcal{M}, \cdot)$ is by definition Δ with the weak- $*$ topology of $\mathcal{M}^*$ (cf. [8], P. 69).

Theorem 2.5. If f is in $C(X)$, then f is in Δ if and only if $\theta^* f$ is in $\mathcal{C}(X \times X)$ and $\mathcal{Q}^{-1} \theta^* f(x, y) = f(x)f(y)$ for each x and y in X , where θ and $\mathcal{Q}$ are as defined in Theorems 2.3 and 2.4.

Proof. Let h denote the continuous function on $X \times X$ defined by $h(x,y) = f(x)f(y)$ for each x and y in X . By Theorem 2.4 $\varphi h = \theta^* f$ if and only if

$$\int h \, d\rho = \int \varphi h \, d\rho_Y = \int f \, d\theta\rho$$

for each ρ in $\mathcal{M} \otimes \mathcal{M}$. This equation holds for each ρ in $\mathcal{M} \otimes \mathcal{M}$ if and only if it holds for each $\mu \times \nu$ with μ and ν in $\mathcal{M}$. Then $\varphi h = \theta^* f$ if and only if

$$\int f \, d\mu \cdot \nu = \int f \, d\theta(\mu \times \nu) = \int h \, d\mu \times \nu = \int f \, d\mu \int f \, d\nu$$

for each μ and ν in $\mathcal{M}$. Hence $\varphi h = \theta^* f$ if and only if f is in Δ .

Corollary. (a) 1 is in Δ ,

(b) if f is in Δ then $\bar{f}$ is in Δ , and

(c) if f and h are in Δ then fh is in Δ .

Proof. By Theorem 2.3, θ is an L-homomorphism and hence, by Theorem 1.14, θ^* is a homomorphism, $\theta^* \bar{f} = \overline{\theta^* f}$, and $\theta^* 1 = 1$. By Theorem 2.4, φ is an isometry having these same properties. The corollary now follows immediately from the characterization of Δ in Theorem 2.5.

Definition 2.4. If S is a topological semigroup, a semicharacter on S is a bounded continuous function f on S , such that $f(st) = f(s)f(t)$ for each s and t in S . The collection of all semicharacters on S is denoted by $\widehat{S}$.

Theorem 2.6. If $(\mathcal{M}, \cdot)$ is a convolution measure algebra, then there exists a compact, abelian topological semigroup S and a continuous map σ from X onto S , such that

- (a) $\widehat{S}$ separates points in S and
- (b) the correspondence $h \mapsto h\sigma$ is a one to one map of $\widehat{S}$ onto Δ .

Proof. Let $\bigwedge$ be the closed linear span of Δ in $C(X)$. By the corollary to Theorem 2.5, $\bigwedge$ is a subalgebra of $C(X)$ which is closed under conjugation and contains the constant functions. For each x in X we set $\sigma(x) = \{y \text{ in } X : f(x) = f(y) \text{ for each } f \text{ in } \bigwedge\}$, S is the collection of all such equivalence classes $\sigma(x)$. A subset U of S is said to be open if $\sigma^{-1}(U)$ is open in X . It is well known that with this topology on S the map σ is continuous, S is a compact Hausdorff space, and the map $h \mapsto h\sigma$ is an isomorphism-isometry of $C(S)$ onto $\bigwedge$ (Cf. [7], p. 69, and [16]).

By Theorem 2.5 $\theta^* \Delta \subset \mathcal{U}C(X \times X)$, and since $\bigwedge$ is the closed linear hull of Δ , we conclude that $\theta^* \bigwedge \subset \mathcal{U}C(X \times X)$. Also, if f is in Δ , then by Theorem 2.5, $\mathcal{U}^{-1} \theta^* f(x, y) = f(x)f(y)$, so that $\mathcal{U}^{-1} \theta^* f$ is constant on sets of the form $\sigma(x) \times \sigma(y)$ in $X \times X$; hence this last statement also holds for each f in $\bigwedge$. Thus each f in $\bigwedge$ defines a function $(\mathcal{U}^{-1} \theta^* f)(\sigma \times \sigma)^{-1}$ in $C(S \times S)$ through the equation $(\mathcal{U}^{-1} \theta^* f)(\sigma \times \sigma)^{-1}(\sigma(x), \sigma(y)) = \mathcal{U}^{-1} \theta^* f(x, y)$ for each x and y in X . If h is in $C(S)$ and $f = h\sigma$, we set $\gamma h = (\mathcal{U}^{-1} \theta^* f)(\sigma \times \sigma)^{-1}$. This defines a map γ from $C(S)$ to $C(S \times S)$. On the basis of Theorems 2.4 and 2.3 we conclude that γ is

a homomorphism, $\gamma \bar{h} = \overline{\gamma h}$, and $\gamma 1 = 1$. Note that, by Theorem 2.5, if h is in $C(S)$ then $h\sigma$ is in Δ if and only if $\gamma h(s,t) = h(s)h(t)$. Since the closed linear span of Δ is $\bigwedge$, this condition completely determines γ .

Each point (s,t) of $S \times S$ determines a non-zero homomorphism $\delta_{(s,t)}$ of $C(S \times S)$ into the complex numbers by the equation $\delta_{(s,t)}(f) = f(s,t)$. Since γ is a homomorphism and $\gamma 1 = 1$, the equation $\delta(h) = \delta_{(s,t)}(\gamma h)$ determines a non-zero homomorphism δ of $C(S)$ into the complex numbers. It is well known that each such homomorphism arises from a point st of S through the formula $\delta(h) = h(st)$ (Cf. [8], p. 54). That is, there is a map $(s,t) \longrightarrow st$ of $S \times S$ into S , such that $\gamma h(s,t) = h(st)$ for each h in $C(S)$ and each s and t in S . The sets of the form $U(h,\epsilon) = \{s \text{ in } S : |h(s)| < \epsilon\}$ for h in $C(S)$ form a sub-basis for the topology of S . The inverse image of such a set $U(h,\epsilon)$ under the map $(s,t) \longrightarrow st$ is precisely $\{(s,t) \text{ in } S \times S : |\gamma h(s,t)| < \epsilon\}$, which is open in $S \times S$; hence $(s,t) \longrightarrow st$ is continuous. By the previous paragraph, if h is in $C(S)$ then $h\sigma$ is in Δ if and only if $\gamma h(s,t) = h(s)h(t)$, that is, if and only if $h(st) = h(s)h(t)$. This condition uniquely determines the map $(s,t) \longrightarrow st$. Thus if $h\sigma$ is in Δ , s,t , and u are in S , then $h((st)u) = h(st)h(u) = h(s)h(t)h(u) = h(s(tu))$; hence $(st)u = s(tu)$. By a similar argument, $st = ts$ for each s and t in S . Thus S forms a compact, abelian topological semigroup under the multiplication $(s,t) \longrightarrow st$.

The above paragraph also shows that h is in $\widehat{S}$ if and only if $h\sigma$ is in Δ . The closed linear space of Δ is $\bigwedge$. Hence the

closed linear span of $\widehat{S}$ is $C(S)$. Thus $\widehat{S}$ separates points in S . this completes the proof.

We shall call S the structure semigroup of $(\mathfrak{M}, \cdot)$. According to the preceding theorem, we may consider $\widehat{S}$ to be the maximal ideal space of $(\mathfrak{M}, \cdot)$. The Gelfand transform $\hat{\mu}$ of an element μ of $\mathfrak{M}$ is a function on $\widehat{S}$ defined by the equation

$$\hat{\mu}(h) = \int h \sigma d\mu$$

for each h in $\widehat{S}$. The Gelfand topology on $\widehat{S}$ is the weak topology generated by the functions $\hat{\mu}$ for μ in $\mathfrak{M}$. $\widehat{S}$ is always locally compact in this topology and is compact if and only if $\mathfrak{M}$ has an identity. The map $\mu \rightarrow \hat{\mu}$ is a norm decreasing homomorphism of $(\mathfrak{M}, \cdot)$ into $C_0(\widehat{S})$, where $C_0(\widehat{S})$ denotes the continuous functions on $\widehat{S}$ which vanish at infinity. If $\mu \rightarrow \hat{\mu}$ is one to one then $(\mathfrak{M}, \cdot)$ is said to be semisimple. For a discussion of the Gelfand theory see Chapter IV of [8].

Definition 2.5. A C -homomorphism of a convolution measure algebra $(\mathfrak{M}, \cdot)$ is a bounded homomorphism of $(\mathfrak{M}, \cdot)$ which is also an L -homomorphism of $\mathfrak{M}$.

Theorem 2.7. If $(\mathfrak{M}, \cdot)$ is a convolution measure algebra, S is the structure semigroup of $(\mathfrak{M}, \cdot)$, X is the standard domain of $\mathfrak{M}$, and σ is the natural map from X to S , then there is a C -homomorphism $\mu \rightarrow \mu_S$ of $(\mathfrak{M}, \cdot)$ into $M(S)$, such that

$$(a) \quad \int h d\mu_S = \int h \sigma d\mu \quad \text{for each } h \text{ in } C(S) \text{ and}$$

each μ in $\mathfrak{M}$,

(b) the image $\mathfrak{M}_S$ of $\mathfrak{M}$ under the map $\mu \rightarrow \mu_S$ is weak-* dense in $M(S)$, and

(c) $\mu \rightarrow \mu_S$ is an isometry if and only if $\mathfrak{M}$ is semisimple.

Proof. If μ is in $\mathfrak{M}$, let $\mu_S(V) = \mu(\sigma^{-1}(V))$ for each Borel set V of S . The remark following Theorem 1.14 shows that the map $\mu \rightarrow \mu_S$ is an L-homomorphism of $\mathfrak{M}$ into $M(S)$. Clearly

$$\int_S h(s) d\mu_S(s) = \int_X h(\sigma(x)) d\mu(x) .$$

To show that $\mu \rightarrow \mu_S$ is an algebraic homomorphism, note that if h is in $\hat{S}$ and μ and ν are in $\mathfrak{M}$ then

$$\begin{aligned} \int h d(\mu \cdot \nu)_S &= \int h \sigma d\mu \cdot \nu = \int h \sigma d\mu \int h \sigma d\nu = \\ \int h d\mu_S \int h d\nu_S &= \iint h(s)h(t) d\mu_S(s) d\nu_S(t) = \\ \iint h(st) d\mu_S(s) d\nu_S(t) &= \int h d\mu_S \cdot \nu_S . \end{aligned}$$

It follows that for each h in $C(S)$

$$\int h d(\mu \cdot \nu)_S = \int h d\mu_S \cdot \nu_S$$

since the linear span of $\hat{S}$ is dense in $C(S)$. Hence

$$(\mu \cdot \nu)_S = \mu_S \cdot \nu_S \text{ for each } \mu \text{ and } \nu \text{ in } \mathfrak{M} .$$

Since $\mathfrak{M}$ separates points in $C(X)$, $\mathfrak{M}_S$ separates points in $C(S)$; hence $\mathfrak{M}_S$ is weak-* dense in $M(S)$.

The map $\mu \rightarrow \mu_S$ is one to one if and only if the image $\bigwedge$ of

$C(S)$ under the map $h \mapsto h\sigma$ separates points in $\mathfrak{M}$. This happens if and only if the image Δ of $\hat{S}$ separates points in $\mathfrak{M}$, that is, if and only if the Gelfand transform $\mu \mapsto \hat{\mu}$ is one to one, i.e., if and only if $\mathfrak{M}$ is semisimple. By Theorem 1.16, $\mu \mapsto \mu_S$ is one to one if and only if it is an isometry. This completes the proof.

To complete this chapter, we prove a theorem which characterizes C -homomorphisms. $(\mathfrak{M}_1, \cdot)$ and $(\mathfrak{M}_2, \cdot)$ will denote semisimple convolution measure algebras with structure semigroups S_1 and S_2 respectively.

Theorem 2.8. If θ is a bounded linear map from $\mathfrak{M}_1$ to $\mathfrak{M}_2$, then the following statements are equivalent:

- (a) θ is a C -homomorphism.
- (b) θ^* restricted to $\hat{S}_2$ is a homomorphism of $\hat{S}_2$ into $\hat{S}_1$, such that $\theta^* \bar{h} = \overline{\theta^* h}$ and $\theta^* 1 = 1$.
- (c) There is a continuous homomorphism α from S_1 into S_2 , such that $(\theta\mu)_{S_2}(V) = \mu_{S_1}(\alpha^{-1}(V))$ for each μ in $\mathfrak{M}_1$ and each Borel set V of S_2 .

Proof. That (a) implies (b) follows from Theorem 1.14 and the fact that if θ is a homomorphism, then θ^* carries complex homomorphisms of $(\mathfrak{M}_2, \cdot)$ into complex homomorphisms of $(\mathfrak{M}_1, \cdot)$.

Let X_1 and X_2 be the standard domains of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ respectively, and σ_1 and σ_2 the natural maps of X_1 onto S_1 and X_2 onto S_2 . Since $(\mathfrak{M}_2, \cdot)$ is semisimple, the linear span

of $\widehat{S}_2$ is weak-* dense in $\mathcal{M}_2^*$. It follows that if θ^* satisfies (b), then θ^* is a homomorphism of $C(X_2)$ into $C(X_1)$, $\theta^* \overline{f} = \overline{\theta^* f}$, and $\theta^* 1 = 1$. Hence by Theorem 1.14, there exists a continuous function α' from X_1 into X_2 , such that $\theta \mu(V) = \mu(\alpha'^{-1}(V))$ for each μ in $\mathcal{M}_1$ and each Borel set V of X_2 . Since θ^* carries $\widehat{S}_2$ into $\widehat{S}_1$, it follows that α' carries equivalence classes on which σ_1 is constant into equivalence classes on which σ_2 is constant. That is, there exists a continuous map α from S_1 to S_2 , such that $\alpha'(\sigma_1^{-1}(s)) \subset \sigma_2^{-1}(\alpha(s))$ for each s in S_1 . α satisfies $\theta^* h(s) = h(\alpha(s))$ for each h in $\widehat{S}_2$ and s in S_1 . Hence $h(\alpha(st)) = \theta^* h(st) = \theta^* h(s) \theta^* h(t) = h(\alpha(s)) h(\alpha(t))$ for each h in $\widehat{S}_2$ and s and t in S_1 . Thus $\alpha(st) = \alpha(s)\alpha(t)$ for each s and t in S_1 . Clearly $(\theta \mu)_{S_2}(V) = \mu_{S_1}(\alpha^{-1}(V))$ for each μ in $\mathcal{M}_1$ and each Borel set V of S_2 . Hence (b) implies (c).

That (c) implies (a) follows immediately from the remark following Theorem 1.14.

CHAPTER III

The results of Chapter II open up the possibility of using semigroup theory to investigate the structure of convolution measure algebras. In this chapter we study some of the structural relationships between convolution measure algebras and their structure semigroups. However, this is only a beginning; a great deal more information is needed on the subject.

$(\mathfrak{M}, \cdot)$ will denote a fixed semisimple convolution measure algebra. Theorem 2.7 allows us to identify $\mathfrak{M}$ with $\mathfrak{M}_S$ and drop the use of the subscript. Thus in this chapter $\mathfrak{M}$ will be considered a subalgebra of $M(S)$. R will denote the algebra of bounded linear operators on $\mathfrak{M}$ which commute with the operators determined by simple step functions, as in Theorem 1.2.

Theorem 3.1. An operator T in R is an algebraic homomorphism of $\mathfrak{M}$ if and only if T corresponds to a function f in $\hat{S}$, such that

$$T\mu(V) = \int_V f d\mu$$

for each μ in $\mathfrak{M}$ and each Borel set V of S .

Proof. Let f be an element of $\hat{S}$, μ and ν elements of $\mathfrak{M}$, and T the operator in R defined by $T\mu(V) = \int_V f d\mu$. Then

$$\begin{aligned}
T(\mu \cdot \nu)(V) &= \int_V f d\mu \cdot \nu = \iint \pi_V(st) f(st) d\mu(s) d\nu(t) \\
&= \iint \pi_V(st) f(s) f(t) d\mu(s) d\nu(t) = \iint \pi_V(st) dT\mu(s) dT\nu(t) \\
&= (T\mu) \cdot (T\nu)(V). \text{ Hence } T \text{ is an algebraic homomorphism.}
\end{aligned}$$

Conversely, if T is an algebraic homomorphism in R , then T corresponds to an element F of $\mathcal{M}^*$, by Theorem 1.2, such that $F(\mu \cdot \nu) = T(\mu \cdot \nu)(S) = (T\mu \cdot T\nu)(S) = T\mu(S)T\nu(S) = F(\mu)F(\nu)$ for each μ and ν in $\mathcal{M}$. Hence F is a complex homomorphism of $\mathcal{M}$ and by Theorems 2.6 and 2.7, F corresponds to an element f of $\widehat{S}$, such that

$$T\mu(V) = \int_V f d\mu.$$

This completes the proof.

Theorem 3.2. If h is a bounded Borel function on S and $h(st) = h(s)h(t)$ for each s and t in S , then there exists an element f of $\widehat{S}$, such that $f = h$ except on a set N which has μ -measure zero for each μ in $\mathcal{M}$.

Proof. We define $T\mu(V) = \int_V h d\mu$ for each μ in $\mathcal{M}$. Then T is an operator in R and the argument used in the preceding theorem shows that T is a homomorphism. Hence there exists f in $\widehat{S}$, such that

$$T\mu(V) = \int_V f d\mu = \int_V h d\mu$$

for each μ in $\mathcal{M}$ and each Borel set V of S . It follows that

$N = \{s \text{ in } S : f(s) \neq h(s)\}$ is a set of μ -measure zero for each μ in $\mathcal{M}$.

Definition 3.0: (a) An ideal of $\mathcal{M}$ which is also an L-subspace of $\mathcal{M}$ will be called an L-ideal. A subalgebra of $\mathcal{M}$ which is also an L-subspace of $\mathcal{M}$ will be called an L-subalgebra.

(b) An L-ideal $\mathcal{I}$ of $\mathcal{M}$ will be called prime if $\mathcal{I}^\perp$ is a subalgebra. An L-subalgebra $\mathcal{N}$ of $\mathcal{M}$ will be called prime if $\mathcal{N}^\perp$ is an ideal.

(c) An ideal J of S will be called prime if $S \setminus J$ is a sub-semigroup of S . A sub-semigroup S' of S will be called prime if $S \setminus S'$ is an ideal of S .

The following theorem characterizes the prime L-subalgebras of $\mathcal{M}$ in terms of the open-compact prime sub-semigroups of S and in terms of the idempotents in $\widehat{S}$.

Theorem 3.3. If $\mathcal{N}$ is an L-subspace of $\mathcal{M}$, then the following statements are equivalent:

(a) $\mathcal{N}$ is a prime L-subalgebra of $\mathcal{M}$.

(b) The idempotent P in R , which projects on $\mathcal{N}$, is a homomorphism.

(c) There is an idempotent π in $\widehat{S}$, such that

$$\mathcal{N} = \{\mu \text{ in } \mathcal{M} : \int \pi d|\mu| = |\mu|(S)\}.$$

(d) There is an open-compact prime sub-semigroup S' of S , such that $\mathcal{N} = \{\mu \text{ in } \mathcal{M} : |\mu|(S \setminus S') = 0\}.$

Proof. Let $\mathcal{N}$ be a prime L-subalgebra of $\mathcal{M}$ and P the idempotent in R which projects on $\mathcal{N}$. Then $\mu \cdot \nu = (P\mu) \cdot (P\nu) + (P\mu) \cdot ((I-P)\nu) + (P\nu) \cdot ((I-P)\mu) + ((I-P)\mu) \cdot ((I-P)\nu)$ for each μ and ν in $\mathcal{M}$. Since $\mathcal{N}$ is a subalgebra and $\mathcal{N}^\perp$ is an ideal, $(P\mu) \cdot (P\nu)$ is in $\mathcal{N}$ while $(P\mu) \cdot ((I-P)\nu) + (P\nu) \cdot ((I-P)\mu) + ((I-P)\mu) \cdot ((I-P)\nu)$ is in $\mathcal{N}^\perp$. Hence $P(\mu \cdot \nu) = (P\mu) \cdot (P\nu)$ and P is a homomorphism. Thus (a) implies (b).

If P is a homomorphism, then by Theorem 3.1, P corresponds to a function π in $\widehat{S}$, such that $P\mu(V) = \int_V \pi d\mu$ for each μ in $\mathcal{M}$. Since P is an idempotent of R , π is an idempotent of $\widehat{S}$. If μ is in $\mathcal{M}$, then μ is in $\mathcal{N}$ if and only if $P\mu = \mu$, that is if and only if μ is carried on the set where π is one. Hence μ is in $\mathcal{N}$ if and only if $\int \pi d|\mu| = |\mu|(S)$. Thus (b) implies (c).

If π is an idempotent in $\widehat{S}$, then π is the characteristic function of an open-compact set S' . If s and t are in S , then st is in S' if and only if $\pi(st) = \pi(s)\pi(t) = 1$, that is if and only if s and t are both in S' . It follows that S' is a sub-semigroup and $S \setminus S'$ is an ideal. Clearly

$$\{\mu \text{ in } \mathcal{M} : \int \pi d|\mu| = |\mu|(S)\} = \{\mu \text{ in } \mathcal{M} : |\mu|(S \setminus S') = 0\}.$$

Hence (c) implies (d).

To prove that (d) implies (a) we use the fact that if μ and ν are in $M(S)$ then $\text{carrier}(\mu \cdot \nu) \subset \text{carrier}(\mu) \cdot \text{carrier}(\nu)$. This follows readily from the definition of convolution multiplication. Let

S' be an open-compact prime sub-semigroup of S and let $\mathcal{M} = \{\mu \text{ in } \mathcal{M} : |\mu|(S \setminus S') = 0\}$. Clearly μ is in $\mathcal{M}$ if and only if $\text{carrier}(\mu) \subset S'$ and μ is in $\mathcal{M}^\perp$ if and only if $\text{carrier}(\mu) \subset S \setminus S'$. Then since S' is a sub-semigroup and $S \setminus S'$ is an ideal, it follows that $\mathcal{M}$ is a subalgebra and $\mathcal{M}^\perp$ is an ideal. Thus (d) implies (a) and the proof is complete.

Theorem 3.4. (a) If S' is an open prime sub-semigroup of S and $\overline{S'}$ is the closure of S' , then $\overline{S'}$ is an open-compact prime sub-semigroup and $\overline{S'} \setminus S'$ is a set of μ -measure zero for each μ in $\mathcal{M}$.

(b) If S' is a closed prime sub-semigroup of S and $(S')^\circ$ is the interior of S' , then $(S')^\circ$ is an open-compact prime sub-semigroup of S and $S \setminus (S')^\circ$ is a set of μ -measure zero for each μ in $\mathcal{M}$.

Proof. If S' is an open prime sub-semigroup of S and h is the characteristic function of S' , then h is a bounded Borel function and $h(st) = h(s)h(t)$ for each s and t in S . Hence by Theorem 3.2, there is a function π in $\hat{\mathcal{S}}$, such that $\pi = h$ except on a set N of μ -measure zero for each μ in $\mathcal{M}$. It follows by an argument similar to the one used in Theorem 1.13, that π is the characteristic function of $\overline{S'}$ and $N = \overline{S'} \setminus S'$. By Theorem 3.3, $\overline{S'}$ is an open-compact prime sub-semigroup. This completes the proof of Part (a).

Part (b) follows by an analogous argument.

Further information concerning the open-compact prime sub-semigroups of S can be obtained in terms of certain idempotents in S .

If p is an idempotent in S , let $S(p)$ denote the collection of all elements s of S for which there exists a t in S , such that $st = p$. $S(p)$ is easily seen to be a closed prime sub-semigroup of S . $pS = \{ps : s \text{ is in } S\}$ is a closed ideal of S . If $G(p) = pS \cap S(p)$ then $G(p)$ is a compact group containing p as its identity. $G(p)$ is the maximal group in S containing p .

Every compact semigroup contains a minimal ideal called its kernel. The kernel of a compact abelian semigroup is a compact group. If p is an idempotent in S then $G(p)$ is the kernel of $S(p)$.

The above statements are well known aspects of topological semigroup theory. For a general discussion of this theory see [15].

$\widehat{S}$ is also a semigroup, and we may define $\widehat{S}(\pi)$ and $\widehat{G}(\pi)$ for any idempotent π in $\widehat{S}$, as above. $\widehat{S}(\pi)$ is a prime sub-semigroup of $\widehat{S}$, $\widehat{G}(\pi)$ is the maximal group in $\widehat{S}$ containing π , and $\widehat{G}(\pi)$ is the kernel of $\widehat{S}(\pi)$. However, multiplication in $\widehat{S}$, as a function from $\widehat{S} \times \widehat{S}$ to $\widehat{S}$, may not be continuous in the Gelfand topology. Hence $\widehat{S}(\pi)$ and $\widehat{G}(\pi)$ may not be closed.

Definition 3.1. An idempotent p in S is said to be essential if p is an interior point of $S(p)$.

Theorem 3.5. If p is an idempotent of S , then the following statements are equivalent:

- (a) p is an essential idempotent.

(b) $S(p)$ is open.

(c) There exists a net $\{\mu_\alpha\}_{\alpha \text{ in } D}$ of non-negative measures of norm one in $\mathcal{M}$, such that $|\mu_\alpha|(S \setminus S(p)) = 0$ for each α and $\{\mu_\alpha\}_{\alpha \text{ in } D}$ converges in the weak-* topology of $M(S)$ to the point measure δ_p at p .

Proof. If p is an essential idempotent and s is in $S(p)$, then there is an element t of S , such that $st = p$. Since $S \setminus S(p)$ is an ideal, t is in $S(p)$. p is an interior point of $S(p)$, so there exists an open set U containing p and contained in $S(p)$. Since multiplication is continuous in S , there exists an open set V containing s , such that $Vt \subset U$. Then V is contained in $S(p)$, since $S \setminus S(p)$ is an ideal. Hence $S(p)$ is open. Thus (a) implies (b).

Since $\mathcal{M}$ is weak-* dense in $M(S)$, no open subset of S can be a set of μ -measure zero for each μ in $\mathcal{M}$. It then follows from the fact that $\mathcal{M}$ is an L-space, that for every open set U of S there is a non-negative measure μ of norm one in $\mathcal{M}$, such that $|\mu|(S \setminus U) = 0$. If $S(p)$ is an open set then there is a neighborhood basis $\{U_\alpha\}_{\alpha \text{ in } D}$ at p , such that U_α is contained in $S(p)$ for each α . Choose a non-negative measure μ_α of norm one in $\mathcal{M}$ for each α , such that $\mu_\alpha(S \setminus U_\alpha) = 0$. Then clearly

$$\lim_{\alpha} \int f d\mu_\alpha = f(p) = \int f d\delta_p$$

for each continuous function f in $C(S)$. That is $\{\mu_\alpha\}_{\alpha \text{ in } D}$ converges to δ_p in the weak-* topology of $M(S)$. Hence (b) implies (c).

To show that (c) implies (a), let $\{\mu_\alpha\}_{\alpha \text{ in } D}$ be a net of non-negative measures of norm one in $\mathcal{M}$, such that $\mu_\alpha(S \setminus S(p)) = 0$ for each α and $\lim_\alpha \mu_\alpha = \delta_p$ in the weak-* topology of $M(S)$. By Theorem 3.4, $S(p) \setminus S(p)^\circ$ is a set of μ -measure zero for each μ in $\mathcal{M}$ and $S(p)^\circ$ is open and compact. Hence $\mu_\alpha(S \setminus S(p)^\circ) = 0$ for each α and the characteristic function of $S(p)^\circ$ is continuous. Hence

$$\pi(p) = \lim_\alpha \int \pi d\mu_\alpha = 1$$

and p is in $S(p)^\circ$. Thus p is an essential idempotent. This completes the proof.

Corollary. If p and q are essential idempotents of S then pq is an essential idempotent of S .

Proof. We use statement (c) of Theorem 3.5. Let $\{\mu_\alpha\}_{\alpha \text{ in } C}$ and $\{\nu_\beta\}_{\beta \text{ in } D}$ be nets of non-negative measures of norm one in $\mathcal{M}$, such that $\mu_\alpha(S \setminus S(p)) = 0$ for each α , $\nu_\beta(S \setminus S(q)) = 0$ for each β , and $\{\mu_\alpha\}_{\alpha \text{ in } C}$ and $\{\nu_\beta\}_{\beta \text{ in } D}$ converge to δ_p and δ_q respectively in the weak-* topology of $M(S)$. It is well known that convolution multiplication is weak-* continuous on the unit ball of $M(S)$ (Cf. [4], p. 53). Hence the net $\{\mu_\alpha \cdot \nu_\beta\}_{(\alpha, \beta) \text{ in } C \times D}$ converges to $\delta_p \cdot \delta_q = \delta_{pq}$ in the weak-* topology of $M(S)$. Clearly $S(p) \cdot S(q)$ is contained in $S(pq)$ and $\mu_\alpha \cdot \nu_\beta(S \setminus (S(p) \cdot S(q))) = 0$ for each α and β . Hence $\mu_\alpha \cdot \nu_\beta(S \setminus S(pq)) = 0$ for each α and β . Thus pq satisfies (c) of Theorem 3.5 and pq is an essential idempotent. This completes the proof.

Theorem 3.6. If S' is an open-compact prime sub-semigroup of S , π is the characteristic function of S' , and p is the idempotent in the kernel of S' , then

- (a) $S' = S(p)$,
- (b) $\widehat{S}(\pi) = \{f \in \widehat{S} : |f| = 1 \text{ on } S(p)\} = \{f \in \widehat{S} : f(p) \neq 0\}$,
- (c) $\widehat{G}(\pi) = \{f \in \widehat{S}(\pi) : f = 0 \text{ on } S \setminus S(p)\} = \{f \in \widehat{S} : |f| = \pi\}$, and
- (d) the restriction map $f \rightarrow f|_{G(p)}$ is an isomorphism of $\widehat{G}(\pi)$ onto $\widehat{G(p)}$, the character group of $G(p)$.

Proof. If s is in S' then ps is in the kernel of S' . The kernel of S' is a group, so ps has an inverse t relative to p . Then $t(ps) = (tp)s = p$ and by definition, s is in $S(p)$. Hence $S' \subset S(p)$. If s is in $S(p)$ then there exists t in S , such that $st = p$. Hence s is in S' since $S \setminus S'$ is an ideal. Thus $S' = S(p)$.

If f is in $\widehat{S}$ then f is in $\widehat{S}(\pi)$ if and only if there exists h in $\widehat{S}$, such that $fh = \pi$. Since $|f| \leq 1$ for any semicharacter f , $fh = \pi$ for some semicharacter h in $\widehat{S}$ implies that $|f| = 1$ on $S(p)$. Conversely, if $|f| = 1$ on $S(p)$ then $f(\bar{f}\pi) = \pi$. Hence f is in $\widehat{S}(\pi)$ if and only if $|f| = 1$ on $S(p)$. By definition of $S(p)$, $|f| = 1$ on $S(p)$ if and only if $|f(p)| = 1$. However, since p is an idempotent, $f(p) = 0$ or 1 . Hence $|f| = 1$ on $S(p)$ if and only if $f(p) \neq 0$.

$\widehat{G}(\pi) = \pi\widehat{S} \cap \widehat{S}(\pi) = \{f \text{ in } \widehat{S}(\pi) : \pi f = f\} = \{f \text{ in } \widehat{S}(\pi) : f = 0 \text{ on } S \setminus S(p)\}$ since π is the characteristic function of $S(p)$. By the previous paragraph, if f is in $\widehat{S}$ then

f is in $\widehat{S}(\pi)$ if and only if $|f| = 1$ on $S(p)$. Hence f is in $\widehat{G}(\pi)$ if and only if $|f| = 1$ on $S(p)$ and 0 on $S \setminus S(p)$, that is, if and only if $|f| = \pi$.

Clearly the restriction $f|_{G(p)}$ of each element f of $\widehat{G}(\pi)$ is a character on $G(p)$ and the map $f \rightarrow f|_{G(p)}$ is a homomorphism. If χ is any character on $G(p)$, $f(s) = \chi(ps)$ if s is in $S(p)$, and $f(s) = 0$ if s is in $S \setminus S(p)$, then f is an element of $\widehat{G}(\pi)$ and $f|_{G(p)} = \chi$. Thus $f \rightarrow f|_{G(p)}$ maps $\widehat{G}(\pi)$ onto the character group of $G(p)$. If $h|_{G(p)} = 1$ for some h in $\widehat{G}(\pi)$, then $h(s) = h(p)h(s) = h(ps) = 1$ if s is in $S(p)$ and $h(s) = 0$ if s is in $S \setminus S(p)$. Hence $h = \pi$ and the map $f \rightarrow f|_{G(p)}$ is an isomorphism. This completes the proof.

We now show how each of the maximal groups $\widehat{G}(\pi)$ may be characterized in terms of a homomorphism of $\mathfrak{M}$ into the measure algebra on a group.

Definition 3.2. A homomorphism θ of $\mathfrak{M}$ will be called a partial C-homomorphism if there is a prime L-subalgebra $\mathfrak{N}$ of $\mathfrak{M}$, such that θ restricted to $\mathfrak{N}$ is a C-homomorphism and θ restricted to $\mathfrak{N}^\perp$ is zero. In this case $\mathfrak{N}$ will be called the L-kernel of θ .

Definition 3.3. If p is an essential idempotent of S , $\mathfrak{N}_p$ is the prime L-subalgebra corresponding to $S(p)$ by Theorem 3.3, and P is the idempotent in R which projects on $\mathfrak{N}_p$, then we define $\theta_p \mu = (P\mu) \cdot \delta_p$ for each μ in $\mathfrak{M}$.

Theorem 3.7. (a) θ_p is a partial C-homomorphism of $\mathcal{M}$ into $M(G(p))$, with L-kernel $\mathcal{N}_p$.

(b) If π is the characteristic function of $S(p)$, then the elements of $\widehat{G}(\pi)$, are precisely those functions h which arise from characters χ on $G(p)$ through the formula

$$\int h \, d\mu = \int \chi \, d\theta_p \mu.$$

Here χ is the restriction of h to $G(p)$.

(c) If θ is any partial C-homomorphism of $\mathcal{M}$ into $M(G)$ for some compact abelian group G , and if $\mathcal{N}_p$ is the L-kernel of θ , then there is a C-homomorphism φ of $M(G(p))$ into $M(G)$, such that $\theta = \varphi \theta_p$.

Proof. θ_p is clearly a bounded linear map of $\mathcal{M}$ into $M(S)$. If μ is in $\mathcal{M}$ then $P\mu$ is in $\mathcal{N}_p$ and hence $\text{carrier}(P\mu) \subset S(p)$ by Theorem 3.1. Then $\text{carrier}(P\mu) \cdot \delta_p \subset (\text{carrier}(P\mu)) \cdot p \subset G(p)$. Hence $(P\mu) \cdot \delta_p$ may be considered a measure in $M(G(p))$. If μ and ν are in $\mathcal{M}$, then $P(\mu \cdot \nu) = (P\mu) \cdot (P\nu)$ by Theorem 3.3 and $P(\mu \cdot \nu) \cdot \delta_p = (P\mu) \cdot \delta_p \cdot (P\nu) \cdot \delta_p$. Hence θ_p is a homomorphism of $\mathcal{M}$ into $M(S)$. If μ is in $\mathcal{N}_p^\perp$ then $P\mu = 0$ and hence $\theta_p \mu = 0$. To show that θ_p restricted to $\mathcal{N}_p$ is a C-homomorphism, we note that if V is a Borel set of $G(p)$ and μ is in $\mathcal{N}_p$, then $\theta_p \mu(V) = \mu \cdot \delta_p(V) = \mu(W)$ where $W = \{s \text{ in } S : ps \text{ is in } V\}$. Thus θ_p restricted to $\mathcal{N}_p$ arises from a continuous map $s \rightarrow sp$ of $S(p)$ into $G(p)$. By the remark following Theorem 1.14, θ_p restricted to $\mathcal{N}_p$ is an L-homomorphism and hence a C-homomorphism. This completes

the proof of part (a).

If h is in $\widehat{G}(\pi)$ then by Theorem 3.6 h is zero on $S \setminus S(p)$ and h restricted to $G(p)$ is a character χ on $G(p)$. Hence if μ is in $\mathcal{M}_p^\perp$, then

$$\int h \, d\mu = 0 = \int \chi \, d\theta_p \mu,$$

and if μ is in $\mathcal{M}_p$, then

$$\int h(s) \, d\mu(s) = \int h(ps) \, d\mu(s) = \int \chi(ps) \, d\mu(s) = \int \chi(s) \, d\theta_p \mu(s).$$

Hence $\int h \, d\mu = \int \chi \, d\theta_p \mu$ for each μ in $\mathcal{M}$.

Conversely, if h is in $C(S)$ and χ is a character on $G(p)$, such that

$$\int h \, d\mu = \int \chi \, d\theta_p \mu$$

for each μ in $\mathcal{M}$, then $\int h \, d\mu = 0$ for each μ in $\mathcal{M}_p^\perp$ and

$$\int h(s) \, d\mu(s) = \int \chi(ps) \, d\mu(s)$$

for each μ in $\mathcal{M}_p$. Hence h is zero on $S \setminus S(p)$ and

$h(s) = \chi(ps)$ for each s in $S(p)$. It follows from Theorem 3.6 that h is in $\widehat{G}(\pi)$ and χ is the restriction of h to $G(p)$. This completes the proof of part (b).

If G is a compact topological group and θ is a partial C -homomorphism of $\mathcal{M}$ into $M(G)$, with L -kernel $\mathcal{M}_p$, then the adjoint map θ^* of θ , when restricted to the character group $\widehat{G}$ of G , is a homomorphism of $\widehat{G}$ into $\widehat{S}$. Clearly $\theta^*1 = \pi$ and hence $(\theta^*\chi_1)(\overline{\theta^*\chi_1}) = \theta^*\chi_1\overline{\chi_1} = \theta^*\chi_1\chi_1^{-1} = \theta^*1 = \pi$ for each χ_1 in $\widehat{G}$. By

Theorem 3.6, $\theta^* \chi_1$ is in $\widehat{G}(\pi)$ for each χ_1 in $\widehat{G}$. The restriction of $\theta^* \chi_1$ to $G(p)$ is a character χ on $G(p)$ and, by Theorem 3.6, the map $\theta^* \chi_1 \rightarrow \chi$ is an isomorphism. Hence the map $\chi_1 \rightarrow \chi$ is a homomorphism of $\widehat{G}$ into $\widehat{G(p)}$. We note that

$$\int \chi_1 d \theta \mu = \int \theta^* \chi_1 d \mu = \int \chi d \theta_p \mu$$

for each χ_1 in $\widehat{G}$ and each μ in $\mathcal{M}$. The map $\chi_1 \rightarrow \chi$ induces a continuous homomorphism α of $G(p)$ into G , such that

$\chi_1(\alpha(s)) = \chi(s)$ for each s in $G(p)$. This follows from the fact that $G(p)$ is the character group of $\widehat{G(p)}$ and G is the character group of $\widehat{G}$ (Cf. [12], pp. 27-28). It follows that

$$\int f(\alpha(s)) d \theta_p \mu(s) = \int f(g) d \theta \mu(g)$$

for every μ in $\mathcal{M}$ and f in $C(G)$. If we set $\varphi \lambda(V) = \lambda(\alpha^{-1}(V))$ for each λ in $M(G(p))$ and each Borel set V of G , then on the basis of the remark following Theorem 1.14, we conclude that φ is a C -homomorphism of $M(G(p))$ into $M(G)$. If μ is in $\mathcal{M}$ and f is in $C(G)$, then

$$\int f(g) d \theta \mu(g) = \int f(\alpha(s)) d \theta_p \mu(s) = \int f(g) d \varphi \theta_p \mu(g).$$

Hence $\theta = \varphi \theta_p$. This completes the proof.

Part (c) of Theorem 3.7 shows that each of the maps θ_p is a maximal partial C -homomorphism having a fixed L -kernel. Each of the maximal groups $\widehat{G}(\pi)$ in $\widehat{S}$ is characterized by such a map.

Definition 3.4. H will denote the collection of all semicharacters h in $\widehat{S}$, such that $|h(s)| = 0$ or 1 for each s in S .

By Theorem 3.6 H is precisely the union of the maximal groups $\widehat{G}(\pi)$ in $\widehat{S}$. Much of the pathology in convolution measure algebras arises when H is a proper subset of $\widehat{S}$. The remaining theorems of the chapter are devoted to an investigation of this situation.

Theorem 3.8. If f is in $\widehat{S}$ then there is a function h in H and a non-negative function r in $\widehat{S}$, such that $f = rh$.

Proof. We set $r = |f|$. Then r is clearly a non-negative semicharacter. We then define $h_1(s) = |f(s)|^{-1} f(s)$ if $f(s) \neq 0$, and $h_1(s) = 0$ if $f(s) = 0$. Note that h_1 is a bounded Borel function on S , and if s and t are in S , then $h_1(st) = |f(s)f(t)|^{-1} f(s)f(t) = h_1(s)h_1(t)$ if $f(s) \neq 0 \neq f(t)$ and $h_1(st) = 0 = h_1(s)h_1(t)$ if $f(s) = 0$ or $f(t) = 0$. Hence by Theorem 3.2, there is a semicharacter h in $\widehat{S}$, such that $h = h_1$ except on a set N of μ -measure zero for each μ in $\mathcal{M}$. Since $\int h_1(s) d\mu(s) = 1$ for each μ in $\mathcal{M}$ and h is continuous, it follows that $|h(s)| = 0$ or 1 for each s in S . Hence h is in H . Clearly $f = rh$ except possibly on N , but since f and rh are continuous functions, $f = rh$ everywhere on S .

Definition 3.5. If r is a non-negative semicharacter in $\widehat{S}$ and z is a complex number with $\operatorname{Re} z > 0$, we define r^z to be the function in $C(S)$, such that $r^z(s) = (r(s))^z$ for each s in S .

Theorem 3.9. If r is a non-negative semicharacter in $\widehat{S}$, then r^z is in $\widehat{S}$ for each complex number z with positive real part. The map $z \rightarrow r^z$ is an analytic vector valued function from $\{z : \operatorname{Re} z > 0\}$ into $C(S)$.

Proof. If s and t are in S then $r^z(st) = (r(st))^z = (r(s))^z(r(t))^z = r^z(s)V^z(t)$. Hence r^z is in $\widehat{S}$.

If $\operatorname{Re} z_0 > 0$ then $(z_0 - z)^{-1}(x^{z_0} - x^z)$ converges uniformly for x in $[0,1]$ to $x^{z_0} \ln x$ as z converges to z_0 . Hence $(z_0 - z)^{-1}(r^{z_0} - r^z)$ converges uniformly on S to $r^{z_0} \ln r$ as z converges to z_0 . Thus r^z has a derivative $r^{z_0} \ln r$ at each point z_0 in $\{z : \operatorname{Re} z > 0\}$. By definition, $z \rightarrow r^z$ is an analytic vector valued function (Cf. [9], pp. 65-66).

Theorem 3.10. If μ is in $\mathfrak{M}$ then the Gelfand transform $\hat{\mu}$ of μ attains its maximum modulus on H .

Proof. Since $\hat{\mu}$ is in $C_0(\widehat{S})$ it attains its maximum modulus somewhere on $\widehat{S}$. Suppose $\hat{\mu}$ attains its maximum modulus at f in $\widehat{S}$. By Theorem 3.8, we may write f as rh where h is in H and r is a non-negative semicharacter. We define

$$\xi(z) = \int r^z h \, d\mu = \hat{\mu}(r^z h)$$

for $\operatorname{Re} z > 0$. Since $z \rightarrow r^z$ is an analytic vector valued function on $\{z : \operatorname{Re} z > 0\}$, ξ is an analytic function on $\{z : \operatorname{Re} z > 0\}$ (Cf. [9], p. 66). However, ξ attains its maximum modulus at $z = 1$ since $\xi(1) = \hat{\mu}(f)$. Hence, by the maximum modulus principle, ξ is

a constant. Thus $\hat{\mu}(r^z h) = \hat{\mu}(f)$ for each z in $\{z : \operatorname{Re} z > 0\}$.

Let $k(s) = \lim_n r^n(s)$ for each s in S . Note that $k(s) = 0$ if $r(s) < 1$ and $k(s) = 1$ if $r(s) = 1$. If $S_1 = \{s \text{ in } S : r(s) = 1\}$ then clearly S_1 is a closed prime sub-semigroup of S . By Theorem 3.4, S_1^0 is an open-compact prime sub-semigroup of S and $S_1 \setminus S_1^0$ is a set of $|\mu|$ -measure zero. If π is the characteristic function of S_1^0 , then π is in H and

$$\hat{\mu}(\pi h) = \int \pi h \, d\mu = \int k h \, d\mu = \lim_n \int r^n h \, d\mu = \hat{\mu}(f).$$

Hence $\hat{\mu}$ attains its maximum modulus at πh , which is in H . This completes the proof.

The Shilov boundary of $\mathcal{M}$ is the smallest subset of $\hat{S}$ which is closed in the Gelfand topology and on which every $\hat{\mu}$ for μ in $\mathcal{M}$ attains its maximum modulus. A point f of $\hat{S}$ is called a strong boundary point if for every open subset U of $\hat{S}$ which contains f , there is an element μ of $\mathcal{M}$, such that $\hat{\mu}$ attains its maximum modulus at f but not at any point outside U . The collection of strong boundary points is a dense subset of the Shilov boundary provided $\mathcal{M}$ has an identity (Cf. [11], p. 138).

Theorem 3.10 shows that the Shilov boundary of $\mathcal{M}$ is contained in the closure of H . Furthermore, we have the following corollary:

Corollary. H contains the set of strong boundary points in $\hat{S}$.

Proof. If f is a semicharacter in $\hat{S}$, let πh be constructed for f as in the proof of Theorem 3.10. If $f \neq \pi h$ then there is an

open subset U of $\widehat{S}$ containing f and not containing πh . If $\hat{\mu}$ is any element of $\widehat{M}$ which attains its maximum modulus at f , then as in Theorem 3.10, $\hat{\mu}$ attains its maximum modulus at πh , which is not in U . Hence f is not a strong boundary point. If $f = \pi h$ then f is in H . Hence every strong boundary point is in H .

Definition 3.6. We denote by K the union of the maximal groups $G(p)$ for p an idempotent of S .

Theorem 3.11. If s is in S then s is in K if and only if $|f(s)| = 0$ or 1 for each f in $\widehat{S}$.

Proof. If s is in K then s is in some maximal group $G(p)$ in S . Hence if f is in $\widehat{S}$, f restricted to $G(p)$ is either zero or a character on $G(p)$. Hence $|f(s)| = 0$ or 1 .

To prove the converse, suppose s is in S and $|f(s)| = 0$ or 1 for each f in $\widehat{S}$. For each f in $\widehat{S}$ we let U_f be the closed unit disc and C_f the closed unit circle in the complex plane. Each U_f is a compact topological semigroup under ordinary multiplication of complex numbers. The direct product

$$\Gamma = \prod_{f \text{ in } \widehat{S}} U_f$$

is a compact topological semigroup under coordinate-wise multiplication and the product topology. For each t in S we define $\alpha(t)$ to be the element of Γ whose f -coordinate is $f(t)$ for each f in $\widehat{S}$. Each f in $\widehat{S}$ is a continuous homomorphism of S into U_f and $\widehat{S}$ separates points in S ; hence α is a continuous isomorphism

of S into Γ . Since $|f(s)| = 0$ or 1 for each f in $\widehat{S}$, $\alpha(s)$ is contained in

$$\Gamma_s = \prod_{|f(s)|=1} C_f \times \prod_{|f(s)|=0} \{0\}.$$

Γ_s is a compact group in Γ . Since α is a continuous homomorphism, $\Gamma_s \cap \alpha(S)$ is a compact sub-semigroup of Γ_s . It is well known that a compact sub-semigroup of a compact group is a group (Cf. [15], p. 99). Hence $\Gamma_s \cap \alpha(S)$ is a group. It follows that $\alpha^{-1}(\Gamma_s \cap \alpha(S))$ is a group containing s . Hence s is in some maximal group $G(p)$. This completes the proof.

Corollary. $K = S$ if and only if $H = \widehat{S}$.

Proof. This is a direct consequence of Theorem 3.11 and the definitions of K and H .

It is well known that the union of all maximal groups in a compact topological semigroup is a compact sub-semigroup. Thus K is a compact sub-semigroup of S and $M(K)$ is an L -subalgebra of $M(S)$.

Definition 3.7. If s is in K then s^{-1} will denote the inverse of s in the maximal group containing s . If μ is in $M(K)$ and V is a Borel set of K , then we define $\tilde{\mu}(V) = \bar{\mu}(V^{-1})$.

It is known that $s \rightarrow s^{-1}$ is a continuous isomorphism of K onto K and hence the map $\mu \rightarrow \tilde{\mu}$ defines an involution on $M(K)$ (Cf. [15], p. 98).

$\mathfrak{M}$ is said to be symmetric if $\hat{\mathfrak{M}}$, the algebra of Gelfand transforms of elements of $\mathfrak{M}$, is closed under conjugation. The next theorem shows how the symmetry of $\mathfrak{M}$ is related to the sub-semigroup K and the involution $\sim$.

Theorem 3.12. If μ is in $\mathfrak{M}$ then $\hat{\mu}$ is in $\hat{\mathfrak{M}}$ if and only if μ is in $M(K)$ and $\tilde{\mu}$ is in $\mathfrak{M}$. In this case $\hat{\tilde{\mu}} = \bar{\hat{\mu}}$. $\mathfrak{M}$ is symmetric if and only if $K = S$ and $\mathfrak{M}$ is closed under the involution $\sim$.

Proof. Suppose μ is in $\mathfrak{M}$ and there is a measure ν in $\mathfrak{M}$, such that $\hat{\mu} = \hat{\nu}$. We shall show first that $\text{carrier}(\mu) \subset K$. If not, then there is a point s of $\text{carrier}(\mu)$ which is not in K and by Theorem 3.11, there is an f in $\hat{S}$, such that $0 < |f(s)| < 1$. Let $U = \{t \text{ in } S : 0 < |f(t)| < 1\}$ and let ρ be the measure in $\mathfrak{M}$ defined by $\rho(V) = \mu(U \cap V)$ for each Borel set V of S . ρ is not the zero measure since U intersects $\text{carrier}(\mu)$. Hence there is some h in $\hat{S}$, such that $\int |f|h d\rho \neq 0$.

We define

$$\xi(z) = \int |f|^z h d\mu + \int |f|^z h d\rho + C \quad \text{where} \quad C = \lim_n \int |f|^n h d\mu$$

for $\text{Re } z > 0$. By Theorem 3.9, ξ is an analytic function. ξ is not constant since $\xi(1) \neq C$ and $\lim \xi(n) = C$. Hence $\bar{\xi}$ is not an analytic function. However, $\bar{\xi}(z) = \hat{\tilde{\mu}}(|f|^z h) = \hat{\nu}(|f|^z h)$ and $\bar{\xi}$ must be analytic by Theorem 3.9. The resulting contradiction shows that $\text{carrier}(\mu) \subset K$ and μ may be considered a measure in $M(K)$.

Also

$$\int f \, d\nu = \int_K \overline{f(s)} \, d\overline{\mu(s)} = \int_K f(s^{-1}) \, d\overline{\mu(s)} = \int f \, d\tilde{\mu}$$

for each f in $\widehat{S}$. Hence $\nu = \tilde{\mu}$, $\tilde{\mu}$ is in $\mathfrak{M}$ and $\hat{\tilde{\mu}} = \bar{\mu}$.

Conversely, if μ is in $M(K) \cap \mathfrak{M}$ and $\tilde{\mu}$ is in $\mathfrak{M}$, then

$$\bar{\tilde{\mu}}(f) = \int_K \overline{f(s)} \, d\overline{\mu(s)} = \int_K f(s^{-1}) \, d\overline{\mu(s)} = \int_K f \, d\tilde{\mu} = \hat{\tilde{\mu}}(f)$$

for each f in $\widehat{S}$. Hence $\bar{\tilde{\mu}}$ is in $\hat{\mathfrak{M}}$ and $\bar{\tilde{\mu}} = \hat{\tilde{\mu}}$.

$\mathfrak{M}$ is symmetric if and only if $\bar{\tilde{\mu}}$ is in $\hat{\mathfrak{M}}$ for each μ in $\mathfrak{M}$. Hence $\mathfrak{M}$ is symmetric if and only if $\mathfrak{M}$ is contained in $M(K)$ and $\mathfrak{M}$ is closed under the involution $\sim$. $\mathfrak{M}$ is weak-* dense in $M(S)$ and $M(K)$ is a weak-* closed subspace of $M(S)$. Hence if $\mathfrak{M} \subset M(K)$ then $M(K) = M(S)$ and $K = S$. This completes the proof.

CHAPTER IV

In this chapter G will denote a locally compact abelian topological group. S will denote the structure semigroup of $M(G)$ and $\widehat{G}$ will denote the character group of G .

Each χ in $\widehat{G}$ determines a homomorphism F of $M(G)$ into the complex numbers through the formula

$$F(\mu) = \int \chi d\mu .$$

Hence for each χ in $\widehat{G}$ there is an element h_χ of $\widehat{S}$, such that

$$\int \chi d\mu = \int h_\chi d\mu_S$$

for each μ in $M(G)$ (Cf. Theorems 2.6 and 2.7). It is known that $\widehat{G}$ separates points in $M(G)$ (Cf. [12], p. 17). Hence $M(G)$ is semisimple and $\mu \rightarrow \mu_S$ is an isometry.

Theorem 4.1. The map $\chi \rightarrow h_\chi$ is an isomorphism of $\widehat{G}$ onto $\widehat{G}(1)$ (Cf. p. 52).

Proof. The fact that $\chi \rightarrow h_\chi$ is a homomorphism of $\widehat{G}$ into $\widehat{S}$ follows directly from Theorems 1.7 and 3.1. The map is clearly one to one, since $M(G)$ separates points in $\widehat{G}$. Also, $h_\chi \overline{h_\chi} = h_{\chi\bar{\chi}} = h_1 = 1$ and by Theorem 3.6, h_χ is in $\widehat{G}(1)$ for each χ in $\widehat{G}$. We shall show that each h in $\widehat{G}(1)$ is h_χ for some χ in $\widehat{G}$.

If λ is Haar measure on G , then $\mathfrak{L}(\lambda)$ is an ideal of $M(G)$ which is isomorphic-isometric to $L_1(G)$ (Cf. [12], p. 16). Each homomorphism F of $\mathfrak{L}(\lambda)$ onto the complex numbers is of the form

$$F(\mu) = \int \chi d\mu$$

for some character χ in $\widehat{G}$. We let h be an element of $\widehat{G}(1)$ and define

$$F(\mu) = \int h d\mu_S$$

for each μ in $M(G)$. Since $h\bar{h} = 1$ there is an element ν of $\mathfrak{L}(\lambda)$, such that $F(\nu) \neq 0$, and we may assume that $F(\nu) = 1$. There is a χ in $\widehat{G}$, such that

$$\int h d\mu_S = F(\mu) = \int \chi d\mu$$

for each μ in $\mathfrak{L}(\lambda)$. If μ is any element of $M(G)$, then $\mu \cdot \nu$ is in $\mathfrak{L}(\lambda)$ and

$$\begin{aligned} \int h d\mu_S &= F(\mu) = F(\mu)F(\nu) = F(\mu \cdot \nu) = \int \chi d\mu \cdot \nu = \int h \chi d(\mu \cdot \nu)_S = \\ &= \int h \chi d\mu_S \int h \chi d\nu_S = \int h \chi d\mu_S. \end{aligned}$$

Hence $h = h\chi$. This completes the proof.

The Bohr compactification $\bar{G}$ of G is defined to be the character group of $(\widehat{G}, d)$, where $(\widehat{G}, d)$ is $\widehat{G}$ with the discrete topology. $\bar{G}$ is characterized by the fact that it is a compact group whose character group is isomorphic to $\widehat{G}$ (Cf. [12], pp. 30-32).

We denote the idempotent in the kernel of S by k . Then

$G(k)$ is the kernel of S . We then have the following corollary to Theorem 4.1:

Corollary. $G(k)$ is the Bohr compactification of G .

Proof. Clearly $S(k) = S$. Hence 1 is the idempotent in $\widehat{S}$ corresponding to the essential idempotent k . By Theorem 3.6 the restriction map $f \rightarrow f|_{G(k)}$ is an isomorphism of $\widehat{G}(1)$ onto $\widehat{G(k)}$. Hence by Theorem 4.1, $\widehat{G(k)}$ is isomorphic to $\widehat{G}$. Since $G(k)$ is a compact group, it follows that $G(k)$ is the Bohr compactification of G .

Definition 4.1. $M_c(G)$ denotes the collection of measures μ in $M(G)$, such that $\mu(\{g\}) = 0$ for each point g of G . $M_d(G)$ denotes the closed linear span of the point measures δ_g for g in G .

It is well known that $M_c(G)$ is an L -ideal of $M(G)$ and $M_d(G)$ is an L -subalgebra of $M(G)$. Also $M_c(G)^\perp = M_d(G)$ (Cf. [12], pp. 16-17). $M_d(G)$ may be identified with $M((G,d))$ where (G,d) is G with the discrete topology.

Theorem 4.2. S has an identity e and if e' is the identity of G , then $(\delta_{e'})_S = \delta_e$. $G(e)$ is the Bohr compactification of (G,d) .

Proof. A regular Borel measure is concentrated at a single point if and only if the L -space it generates is one dimensional. An

L-homomorphism clearly preserves this property. Hence $(\delta_e)_S$ is a point measure in $M(S)$. Since δ_e is the identity of $M(G)$, its Gelfand transform is the identically one function on $\hat{S}$. The only point measure in $M(S)$ having this property is δ_e . Hence $(\delta_e)_S = \delta_e$. It follows that if μ is any measure in $M_d(G)$ then $\text{carrier}(\mu_S) \subset G(e)$. By Theorem 3.5 e is an essential idempotent. Hence $G(e) = S(e)$ is the open-compact prime sub-semigroup of S corresponding to $M_d(G)$. Since $M_d(G) = M((G,d))$, an argument similar to the one used in Theorem 4.1 shows that the character group of (G,d) is isomorphic to the character group of $G(e)$. Hence $G(e)$ is the Bohr compactification of (G,d) .

In the previous two theorems we have characterized the kernel of S and the maximal group at the identity of S in terms of the group G . There is as yet no similar characterization of all maximal groups at essential idempotents of S . This problem is related to the problem of determining all prime L-subalgebras of $M(G)$ in terms of the structure of G . Some work has been done on this question (Cf. [13], pp. 14-19).

The action of the maps θ_p of Definition 3.3. is one thing which distinguishes $M(G)$ from some other convolution measure algebras.

Theorem 4.3. If p is an essential idempotent of S , then θ_p restricted to $\mathcal{N}_p$ is an isometry.

Proof. Since θ_p restricted to $\mathcal{N}_p$ is a $\mathbb{C}$ -homomorphism, it suffices to show that θ_p is one to one on $\mathcal{N}_p$. Note that if μ is in $\mathcal{N}_p$, then $\theta_p \mu = \mu \cdot \delta_p$ and $\theta_k \mu = \mu \cdot \delta_k = \mu \cdot \delta_{kp} = \mu \cdot \delta_k \cdot \delta_p = \theta_k \theta_p \mu$. Hence it suffices to show that θ_k is one to one on $M(G)_S$. If μ is in $M(G)$ and $\theta_k \mu_S = 0$, then

$$\int \chi d\mu = \int \chi d\mu_S = \int \chi d\delta_k \cdot \mu_S = \int \chi d\theta_k \mu_S = 0$$

for each χ in $\widehat{G}$. Hence $\mu = 0$ since $\widehat{G}$ separates points in $M(G)$. This completes the proof.

It is well known that if G is non-discrete, then $M(G)$ is not symmetric (Cf. [12], p. 107). The next theorem, in conjunction with Theorem 3.12, shows how the asymmetry of $M(G)$ is related to the structure of S .

Theorem 4.4. If G is non-discrete then there is a compact subset V of S which carries non-zero continuous measures in $(M(G))_S$, such that every continuous function of norm less than or equal to one on V is the restriction to V of a semicharacter in $\widehat{S}$. It follows that H is a proper subset of $\widehat{S}$ and K is a proper subset of S .

Proof. Rudin has shown that every non-discrete locally compact abelian group G contains a Cantor set Q , such that if $M_c(Q)$ denotes $M_c(G) \cap M(Q)$, then every linear functional F on $M_c(Q)$, such that $\|F\| \leq 1$, is the restriction to $M_c(Q)$ of a complex homomorphism of

$M(G)$ (Cf. [12], pp. 108-112). We denote by V the smallest closed subset of S containing $\text{carrier}(\mu_S)$ for each μ in $M_c(Q)$. If f_1 is a continuous function on V and $\|f_1\| \leq 1$, then the linear functional F_1 , defined by

$$F_1(\mu) = \int f_1 d\mu_S$$

for μ in $M_c(Q)$, is the restriction to $M_c(Q)$ of a complex homomorphism F of $M(G)$. Hence there is an f in $\widehat{S}$, such that

$$\int f d\mu_S = F(\mu) = \int f_1 d\mu_S$$

for each μ in $M_c(Q)$. Since f and f_1 are continuous functions, it follows that $f = f_1$ on V .

A continuous function on V which takes on values strictly between zero and one must be the restriction to V of a semicharacter not in H . Hence H is a proper subset of $\widehat{S}$ and K is a proper subset of S , by the corollary to Theorem 3.11.

Theorem 4.5. If G is a non-discrete group, then H is not a closed subset of $\widehat{S}$. There are points of the Shilov boundary of $M(G)$ which are not in H .

Proof. Let Q , $M_c(Q)$, and V be chosen as in Theorem 4.4, and let f_1 be a continuous function of norm less than or equal to one on V , such that $0 < |f_1(s)| < 1$ for some point s in V . Theorem 5.4.1 of [12] shows that f_1 is the restriction to V of a semicharacter f which may be chosen as the limit in the Gelfand

topology of elements of the Shilov boundary. Hence f itself is in the Shilov boundary. However, f is not in H since $0 < |f(s)| < 1$. Since H is dense in the Shilov boundary by Theorem 3.10, H is not closed. This completes the proof.

In a compact topological group the union of all maximal groups is closed. Hence the above theorem shows that, in this case, $\widehat{S}$ is not a topological group in the Gelfand topology. For each f_1 in $\widehat{S}$ the map $f \rightarrow f_1 f$ is continuous on $\widehat{S}$ with the Gelfand topology. However, multiplication as a function from $\widehat{S} \times \widehat{S}$ to $\widehat{S}$ is not continuous.

Whether or not the closure of H is a proper subset of $\widehat{S}$ is still an open question in the case of $M(G)$ where G is non-discrete.

Example. If μ is any element of $M(S)$, then μ defines a function $\hat{\mu}$ on $\widehat{S}$ through the formula

$$\hat{\mu}(f) = \int f d\mu$$

for f in $\widehat{S}$. If μ is in $M(G)_S$ then $\hat{\mu}$ is the Gelfand transform of μ and is a continuous function on $\widehat{S}$. The question arises as to whether or not this characterizes those elements of $M(S)$ which are in $M(G)_S$. That is, if μ is in $M(S)$ and $\hat{\mu}$ is continuous, then must μ be in $M(G)_S$. The following example shows that the answer to this question is no.

Let G be the unit circle and λ be Haar measure on G . Srieder has shown the existence of a prime L -subalgebra $\mathcal{N}$ which does not contain $\mathcal{L}(\lambda)$, and a measure μ in $\mathcal{N}$, such that the Fourier-Stieltjes transform of μ vanishes at infinity (Cf. [13], pp. 25-26).

That is, $\hat{\mu}$ vanishes on the boundary of $\widehat{G}(1)$ in $\widehat{S}$, by Theorem 4.1. If k is the idempotent in the kernel of S and $\nu = \mu_S \cdot \delta_k$, then

$$\hat{\nu}(f) = \int f(s) f(k) d\mu_S = \int f(s) d\mu_S = \hat{\mu}(f)$$

for f in $\widehat{G}(1)$, and

$$\hat{\nu}(f) = \int f(s) f(k) d\mu_S = 0$$

if f is in $\widehat{S} \setminus \widehat{G}(1)$. This is because $f(k) = 1$ if f is in $\widehat{G}(1)$ and $f(k) = 0$ if f is in $\widehat{S} \setminus \widehat{G}(1)$. Since $\hat{\mu}$ is zero on the boundary of $\widehat{G}(1)$, it follows that $\hat{\nu}$ is continuous. Since μ is in a proper prime L-subalgebra $\mathfrak{N}$ of $M(G)$, $\text{carrier}(\mu_S)$ is contained in a proper open-compact prime sub-semigroup of S . But $\text{carrier}(\nu)$ is contained in $G(k)$, the kernel of S . Hence $\mu_S \neq \nu$. $\hat{\mu}$ and $\hat{\nu}$ agree on $\widehat{G}(1)$ and $\widehat{G}(1)$ separates points in $M(G)_S$. Hence ν is not in $M(G)_S$.

CHAPTER V

There are several interesting unsolved problems which arise in connection with convolution measure algebras. In this chapter we list and discuss some of these problems.

(1) Is there an intrinsic characterization of those elements of $M(S)$ which are in $\mathcal{M}_S$? Theorem 1.12 gives a topological characterization of the elements of $M(X)$ which are in $\mathcal{M}_X$. The lack of similar results for $\mathcal{M}_S$ has made it difficult to relate the structure of $\mathcal{M}$ to the structure of S .

(2) Is $M(K) \cap \mathcal{M}_S$ always closed under the involution $\sim$ (Cf. Definition 3.7)? If the answer to this question were yes, then Theorem 3.12 could be strengthened to read: $\mathcal{M}$ is symmetric if and only if $K = S$.

(3) What may be said about $\mathcal{M}$ in the case where S is a group? If S is a group then $\mathcal{M}$ may be imbedded in $M(G)$, where G is the character group of $\widehat{S}$ with the Gelfand topology. $\mathcal{M}$ then becomes a subalgebra of $M(G)$ which has the same maximal ideal space as $L_1(G)$. One might ask how close $\mathcal{M}$ comes to being $L_1(G)$.

(4) Is there a characterization of the Gelfand topology on $\widehat{S}$

in terms of the algebraic and topological structure on S ?

(5) In the case where $\mathcal{M} = M(G)$, for some locally compact abelian group G , is H dense in $\widehat{S}$? By Theorem 3.10, if H is not dense in $\widehat{S}$, then the Shilov boundary of $M(G)$ is a proper subset of $\widehat{S}$.

(6) Is there a characterization of all prime L -subalgebras of $M(G)$ in terms of the structure of G ?

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