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Unavoidable minors in graphs and matroids

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UNAVOIDABLE MINORS IN GRAPHS AND MATROIDS

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Acknowledgments

"The horse is prepared for the day of battle,

but victory is of the LORD." Proverbs 21:31

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Table of Contents

Abstract

It is well known that every sufficiently large connected graph G has either a vertex of high degree or a long path. If we require G to be more highly connected, then we ensure the presence of more highly structured minors. In particular, for all positive integers k, every 2-connected graph G has a series minor isomorphic to a k-edge cycle or $K_{2,k}$. In 1993, Oxley, Oporowski, and Thomas [12] extended this result to 3- and internally 4-connected graphs identifying all unavoidable series minors of these classes. Loosely speaking, a series minor allows for arbitrary edge deletions but only allows edges to be contracted when they meet a degree-2 vertex. Dually, a parallel minor allows for any edge contractions but restricts the deletion of edges to those that lie in 2-edge cycles. This dissertation begins by proving the dual results to those noted above. These identify all unavoidable parallel minors for finite graphs of low connectivity. Following this, corresponding results on unavoidable minors for infinite graphs are proved. The dissertation concludes by finding the unavoidable parallel minors for 3-connected regular matroids, which combines the results for unavoidable series and parallel minors for graphs with Seymour's decomposition theorem for regular matroids.

Chapter 1
Preliminaries

1.1 Definitions

The remainder of this chapter contains a brief outline of some basic graph theory terminology as it will be used in this dissertation. For a more complete introduction to graph theory, see [6].

A *multigraph* G is a pair (V, E), where V is a non-empty set of *vertices* and E is a multiset whose elements are unordered pairs of elements in V. These pairs are called *edges*. We will assume V to be a finite set throughout this chapter. We will remove this restriction in the next chapter to discuss infinite graphs. A *graph*, also called a *simple graph*, is a multigraph in which the edges are distinct pairs of distinct vertices. We define $V(G)$ to be V and $E(G)$ to be E.

Let e be the edge $\{v, w\}$, where v and w are in V. In the literature and this dissertation, the edge $\{v, w\}$ is denoted simply by vw. Then edge e is *between* v and w, and v and w are the *endpoints* of e. Two distinct edges are *adjacent* if they have an endpoint in common, and two distinct vertices are *adjacent* if they are the endpoints of one edge. In the example above, v and w are adjacent to one another. We also say that w is a *neighbor* of v. The *neighborhood* $N_G(v)$ of v in G is the set of neighbors of v. The subscript G may be omitted when it is understood which graph is meant. An edge is *incident* with each of its endpoints, for example, e and v are incident.

The *complement* $\overline{G}$ of a graph G is the graph that has the same vertex set as G such that two vertices are adjacent in $\overline{G}$ exactly when they are non-adjacent in G. The *order* $|G|$ of G is the number of vertices in G. The smallest graph consists of a single vertex with no edges, that is, it has order one.

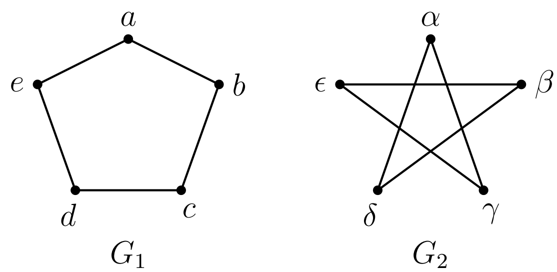

FIGURE 1.1. Two isomorphic graphs.

A multigraph H is a *subgraph* of a multigraph G, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and if each edge of H is an edge of G. If $V(H) = V(G)$, then we say that H *spans* G. Any subgraph H may be obtained from G in the following way. We first *delete* the set of vertices V' in $V(G)$ that are not in $V(H)$. This produces the multigraph $G - V'$, which may also be written as $G\backslash V'$. Each edge incident with a deleted vertex is also deleted, that is, removed from $E(G)$. We then delete the set E' of edges that are not in $E(H)$. We write this as $(G - V') - E'$ or $(G\backslash V')\backslash E'$, and this multigraph is H. For convenience of notation, for a vertex v in $V(G)$, the multigraph $G - \{v\}$ is often written simply as $G - v$. The multigraph G may be obtained from H by *adding* the vertices in G that are not in H and adding the edges in G that are not in H.

A subgraph J of G is an *induced subgraph*, written $J \subseteq_i G$, if J can be obtained from G by deleting some set of vertices of G. The multigraph J is the unique multigraph that G *induces* on the vertex set $V(J)$, and we may write $J = G[V(J)]$ or $J = G[J]$. If H spans G, then $G[H] = G$. Let G be a graph on n vertices with an edge between each pair of distinct vertices in $V(G)$. Then G is a *complete graph*, written K_n. It is worth noting that any induced subgraph of K_n is complete.

Two multigraphs G_1 and G_2 are said to be *isomorphic* to one another if there is a bijection $\phi : V(G_1) \to V(G_2)$ such that vw is an edge in G_1 exactly when $\phi(v)\phi(w)$ is an edge in G_2.

The graphs in Figure 1.1 are clearly isomorphic. To see this formally, consider the bijection $\phi : V(G_1) \to V(G_2)$ that maps $a, b, c, d,$ and e to $\alpha, \gamma, \epsilon, \beta,$ and δ, respectively. Then ϕ preserves adjacency.

A *path* is a graph with vertex set $\{v_0, v_1, \ldots, v_k\}$ and edge set $\{v_0v_1, v_1v_2, \ldots, v_{k-1}v_k\}$. The path is said to have *length* equal to $k+1$ and is denoted by P_{k+1}. The *endpoints* of this path are v_0 and v_k, and the remaining vertices are *interior vertices*. This path is called a v_0-v_k-*path*. A *cycle* is a graph with vertex set $\{v_0, v_1, \ldots, v_k\}$ and edge set $\{v_0v_1, v_1v_2, \ldots, v_{k-1}v_k, v_kv_0\}$. For example, the graphs G_1 and G_2 in Figure 1.1 are *cycles*.

The number of vertices in the neighborhood of a vertex v is equal to the *degree* of v and is written $d(v)$. The maximum degree of the vertices in a graph G is denoted $\Delta(G)$, and the minimum degree is denoted by $\delta(G)$. For the graph G_1 in Figure 1.1, note that the degrees $d(a), d(b), d(c), d(d)$, and $d(e)$ are all equal to 2. In this case, $\Delta(G_1)$ and $\delta(G_1)$ are 2 as well. This graph is 2-regular. In general, a graph is k-*regular* if each of its vertices has degree equal to k. A vertex of a graph of degree one is called a *leaf*. A graph that contains no cycles is called a *forest*, and it is easy to see that every forest that contains at least one edge has at least two leaves.

The graph $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. If each edge in $E(G_2)$ has both endpoints in $V(G_1)$, then we may write $G_1 \cup E(G_2)$ instead of $G_1 \cup G_2$. If a graph has a cycle that spans it, then the cycle is a *Hamilton cycle* and the graph is called *Hamiltonian*. For example, G_1 in Figure 1.1 is a Hamilton graph. Evidently, for a set E' of edges with endpoints in $V(G_1)$, the graph $G_1 \cup E'$ is Hamilton.

1.2 Connectivity

A graph G is *connected* if, for each pair of distinct vertices v and w, there is a v-w-path in G. If a graph is not connected, we say that it is *disconnected*. A graph H is a *component* of G if H is a maximal connected induced subgraph of G. An connected graph containing no cycles is a *tree*. A forest, then, is a disjoint union of trees. Note that the deletion of any edge from a tree results in a disconnected graph. Furthermore, every connected graph has a *spanning tree*, a spanning subgraph that is a tree.

A graph G is k-connected if $|G|$ is at least $k+1$ and, for each set V' of fewer than k vertices of $V(G)$, the graph $G\backslash V'$ is connected. For example, graph G_1 in Figure 1.1 is 2-connected. This definition of connectivity is the definition of what some authors refer to as *vertex connectivity*, which differs from *edge connectivity*. This dissertation will be concerned exclusively with vertex connectivity. Edge connectivity is neither defined nor used. A vertex v in a connected graph G is a *cut vertex* if $G-v$ is disconnected and a vertex set V' is a *cut set* if $G-V'$ is disconnected. A set of vertices V' *separates* vertices v and w in a graph G if v and w are in different components of $G\backslash V'$. Menger's famous theorem [11] from 1927 established another characterization of k-connectivity.

Theorem 1.2.1. *Let G be a graph. For distinct vertices v and w of G, the maximum number of internally disjoint v-w-paths in G is equal to the minimum number of vertices in a vertex cut of G that separates v and w.*

This theorem implies that a graph G is k-connected if and only if, for each pair v and w of distinct vertices, G contains k internally disjoint v-w-paths.

A natural weakening of 4-connectivity is *internal* 4-*connectivity*. A graph G is internally 4-connected if it is 3-connected and, for each set V' of three vertices, either $G-V'$ is connected, or it consists of two components, one of which is a single vertex. A graph that is internally 4-connected is often said to be 4-*connected up to vertices of degree three*. A multigraph is *k-connected* if it contains a k-connected spanning graph.

1.3 Two Important Classes of Graphs

The class of bipartite graphs is a well-known class and has been studied extensively (see, for example, [6]). A graph G is *bipartite* if its vertex set has a partition (A, B) into possibly empty sets such that each edge has one endpoint in A and one endpoint in B; that is, G has no edge having both endpoints in A or both endpoints in B. If the graph that G induces on a vertex set contains no edges, then that set is *stable*. The vertex set of a bipartite graph is the union of two stable sets.

Let G be a bipartite graph with vertex partition (A, B), where $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_l\}$. If each vertex in A is adjacent with every vertex in B, then G is a *complete bipartite graph*, written $K_{k,l}$. Note that any subgraph of G is also bipartite. Thus the class of bipartite graphs is closed under edge and vertex deletion. Since G contains no edge of the form a_ia_j, any path in G must alternate between a-vertices and b-vertices, such as $a_{n_1}b_{n_2}a_{n_3}\ldots b_{n_{m-1}}a_{n_m}$. Clearly any cycle of G also alternates between a-vertices and b-vertices. Thus a bipartite graph contains no odd cycles. This yields an equivalent characterization, that is, a graph is bipartite if and only if it contains no odd cycles.

Another well-known class of graphs is planar graphs. A graph is planar if it can be drawn in the plane so that vertices correspond to points, edges correspond to Jordan curves joining the endpoints, and no edges cross. Such a drawing is a *plane graph*. Note that G_1 and G_2 in Figure 1.1 are planar graphs, but only G_1 is a plane graph. Clearly every subgraph of a plane graph is a plane graph. Thus the class of planar graphs is closed under taking subgraphs.

1.4 Graph Minors

If we *contract* an edge vw in a graph G, then the resulting graph G/vw is obtained from G by adding a vertex vw with neighborhood equal to $N(v) \cup N(w)$, and deleting v and w. This is one of a few definitions of edge contraction. Note that, by this definition, edge contraction sometimes includes identifying two edges, in order to ensure that a graph and not a multigraph is produced. For example, suppose G is a *triangle*, a cycle of length three. Then contracting one edge in G produces a graph with exactly one edge, since the third edge is effectively deleted. The contraction G/Y of a set Y of edges of G is obtained from G by contracting the set Y_F of edges of an edge-maximal spanning forest of Y.

A graph H is a *minor* of G, written $H \preceq G$, if H can be obtained from a subgraph of G by contracting some edges. Equivalently, H is a minor of G if it can be obtained from G by contracting edges in G and taking a subgraph of the result. It is easy to see that the class of planar graphs is closed under taking minors.

A *series edge* is an edge incident with a vertex of degree two. Note that a series edge in G is not necessarily a series edge in a minor of G. If H is obtained from a subgraph of G by contracting, one by one, series edges, then H is called a *topological minor* or *series minor* of G, and we write $H \preceq_t G$. A graph is a *subdivision* of H if it is obtained from H by replacing some edges with paths. We may add vertices to edges in a graph drawing to obtain a drawing of a subdivision of the original graph. Evidently, if $H \preceq_t G$, then G contains a subgraph isomorphic to a subdivision of H. If, H is obtained from G by contracting edges, then H is a *parallel minor* of G, and we write $H \preceq_{\|} G$.

Certain graphs are series minors of a graph exactly when they are minors of that graph, as the following proposition from [6, Proposition 1.7.2(ii)] shows.

Proposition 1.4.1. *If H is a graph in which no vertex has degree greater than three and H is a minor of G, then H is a series minor of G.*

Chapter 2
Excluded Minors and Unavoidable Minors

2.1 Finite Graphs

In this section, we assume all graphs to be finite. Kuratowski [10] proved the following characterization of planar graphs in 1930.

Theorem 2.1.1. *A graph is planar if and only if it has no series minor isomorphic to* K_5 *or* $K_{3,3}$.

In Theorem 2.1.1, K_5 and $K_{3,3}$ are identified as *excluded series minors* of the class of planar graphs. Neither of these graphs is in the class of planar graphs, but every proper series minor of each graph is planar. Furthermore, K_5 and $K_{3,3}$ are the only graphs fitting this description. Kuratowski's Theorem is a well-known result, and much work has been done concerning excluded minors, most notably in the Graph Minors Project.

The Graphs Minors Project of Neil Robertson and Paul Seymour is a set of groundbreaking results published as a series of papers beginning in 1983. This project is widely regarded as the most important work ever done in graph theory. It highlights the deep connections between graph theory and topology. In particular, Robertson and Seymour proved that every class of graphs that is closed under taking minors can be characterized by a finite set of excluded minors, minor-minimal graphs that are not in the class. More generally, the project gave a very general structure theorem for graphs from which much can be derived.

The subject of excluded minors is related to unavoidable minors, which is the subject of this dissertation. The following consequence of Theorem 2.1.1 illustrates this relationship by restating a theorem about excluded minors in terms of unavoidable minors.

Corollary 2.1.2. *Every non-planar graph has a series minor isomorphic to* K_5 *or* $K_{3,3}$.

In Corollary 2.1.2, K_5 and $K_{3,3}$ are identified as *unavoidable series minors* of the set of non-planar graphs.

The work presented in this dissertation is Ramsey-theoretic in nature. Ramsey theory deals with unavoidable structures. The following theorem from 1930 is the simplest of Ramsey's theorems, and will be used in some of the proofs presented in later chapters.

Theorem 2.1.3. *There is an integer-valued function $f_{2.1.3}$ such that, for any positive integer k, every graph of order at least $f_{2.1.3}(k)$ has an induced subgraph isomorphic to K_k or $\overline{K_k}$.*

The following two propositions are simple and well-known results. Their proofs are omitted, but the reader may refer, for example, to [6, Section 9.4]. The first deals with connected graphs.

Proposition 2.1.4. *There is an integer-valued function $f_{2.1.4}$ such that, for positive integers k and l, every connected graph of order at least $f_{2.1.4}(k,l)$ contains $K_{1,k}$, or P_l as a series minor.*

By Proposition 2.1.4, a long path or a vertex of high degree is in every graph that is connected and has sufficient order. A cycle containing exactly k vertices is denoted by C_k. The next proposition deals with 2-connected graphs.

Proposition 2.1.5. *There is an integer-valued function $f_{2.1.5}$ such that, for any positive integer k, every 2-connected graph of order at least $f_{2.1.5}(k)$ contains $K_{2,k}$, or C_k as a series minor.*

Note that the unavoidable minors of connected graphs are all connected and the unavoidable minors of 2-connected graphs are all 2-connected. In 1993, this work was extended by Oporowski, Oxley, and Thomas in [12] to 3-connected graphs and internally 4-connected graphs.

A graph obtained from a cycle C_k by adding a vertex adjacent with all of the vertices in the cycle is called a *wheel* and is denoted W_k. The new edges are called *spokes* and the edges

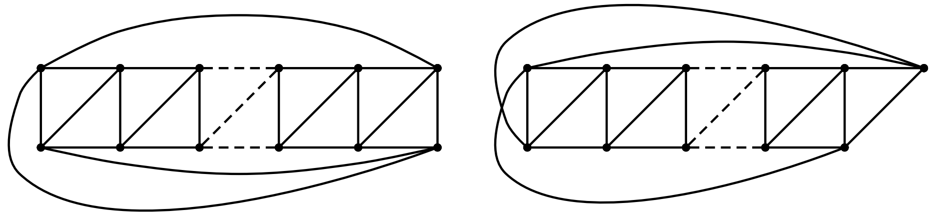

FIGURE 2.1. (a) Quartic planar ladder. (b) Quartic Möbius ladder.

in the cycle are called *rim* edges. The new vertex is called a *hub*. A *double-wheel*, denoted D_k, is obtained from C_k by adding two vertices, called hubs, adjacent with the vertices of the cycle. A *quartic planar ladder* or *zigzag ladder*, denoted Z_k, is obtained from disjoint cycles $v_0v_1 \ldots v_k$ and $w_0w_1 \ldots w_k$ by adding edges v_iw_i and v_iw_{i+1} for $i = 0, 1, \ldots, k$, where addition is modulo $k+1$. If we delete vertex w_k and edge v_0v_k, and add edges v_0w_{k-1} and w_0v_k, then we get a *quartic Möbius ladder* or *Möbius zigzag ladder*, denoted M_k. These two ladders are shown in Figure 2.1.

The two main results of [12] determine the sets of unavoidable topological minors of 3-connected graphs and internally 4-connected graphs. The following two theorems are corollaries of these theorems and will be useful in the work presented in Chapter 3. The theorems determine the sets of unavoidable minors of 3-connected and internally 4-connected graphs.

Theorem 2.1.6. *There is an integer-valued function $f_{2.1.6}$ such that, for any integer k exceeding two, every 3-connected graph of order at least $f_{2.1.6}(k)$ contains a minor isomorphic to W_k, or $K_{3,k}$.*

Theorem 2.1.7. *There is a function $f_{2.1.7}$ such that, for any integers k and l exceeding three, every internally 4-connected graph of order at least $f_{2.1.7}(k, l)$ contains a minor isomorphic to $K_{4,k}$, D_k, M_l, or Z_l.*

Note that the unavoidable minors of 3-connected graphs are 3-connected, and likewise for internally 4-connected graphs. Chapter 3 contains new results that expand on this work.

In particular, the complete lists of unavoidable parallel minors of c-connected graphs are determined for $c = 1, 2, 3$, and 4.

2.2 Infinite Graphs

Let V be an infinite set and let E be a multiset of pairs of elements in V. Then (V, E) is an *infinite multigraph.* If E is a set of distinct pairs of distinct members of V, then (V, E) is an *infinite graph.* We have dealt so far with finite graphs. Most of the operations and terminology defined for finite graphs also apply to infinite graphs. We will use such terms here without redefining them.

This dissertation contains new results that build on two well-known results on unavoidable structure in infinite graphs. The first of these is a result of Ramsey [16] from 1930. If a countably infinite graph has an edge between each pair of vertices, then it is an *infinite complete graph* and is denoted by K_∞. Furthermore, its complement, an infinite stable set, is denoted $\overline{K_\infty}$.

Lemma 2.2.1. *If G is an infinite graph, then G has an induced subgraph isomorphic to K_∞ or $\overline{K_\infty}$.*

A vertex v with an infinite neighborhood is said to have *infinite degree* and is called a *star.* A *ray* is a one-way infinite path such as $v_1v_2\ldots$.

The second well-known result is König's Infinity Lemma.

Lemma 2.2.2 (König's Infinity Lemma). *Every connected infinite graph contains a star vertex or a ray.*

Let c be a natural number. An infinite graph G is *loosely-c*-connected, or ℓ-c-connected, if there is a number d depending on G such that the deletion of fewer than c vertices from G leaves precisely one infinite component and a graph containing at most d vertices. For example, let G_1 be a ray and G_2 be a finite graph such that $V(G_1)$ and $V(G_2)$ are disjoint. Then $G_1 \cup G_2$ is ℓ-1-connected, where d is at most $|G_2|$, but $G_1 \cup G_2$ is not connected. The

ℓ-c-connectivity of an infinite graph discounts the finite flourishes of an infinite graph. Our definition of ℓ-c-connected graphs corresponds to the definition of *essentially c-connected graphs* in [12]. We use our abbreviation since e-c-connectivity could be mistaken for edge connectivity, which will not be discussed in this dissertation, so there will not be any confusion. The two lemmas above provide a basis for determining the sets of unavoidable series minors and parallel minors of infinite ℓ-c-connected graphs in Chapter 4. The set of unavoidable minors of ℓ-c-connected graphs is a corollary of each of these results, and is also a generalization of Oporowski, Oxley, and Thomas given in [12].

Chapter 3
Unavoidable Minors of Finite Graphs

3.1 A Few More Preliminaries

In this chapter, we assume all graphs to be finite. We will find the sets of unavoidable parallel minors of c-connected graphs, for some small values of c. These results are based on joint work with Ding, Oporowski, and Vertigan in [4]. [1]

A minor differs from a parallel minor in that edge deletion and vertex deletion are available operations. If M is a minor of G, then M may be obtained from a parallel minor of G by deleting some edges and vertices. The graph M is therefore contained in a parallel minor N of G that has the same order as M, provided M has exactly one component in each component of G. The unavoidable parallel minors of c-connected graphs should therefore be related to the graphs in the set of unavoidable c-connected minors, except that the parallel minors may have some extra edges.

In this chapter, $\Phi(G, N)$ is the set $\{M \preceq_{\|} G : N \subseteq M \text{ and } |N| = |M|\}$. In order to ensure that $\Phi(G, N)$ is nonempty, N must contain exactly one component in each component of G. Since we will only use this notation in the context of a connected minor of a connected graph, we will not worry about the qualification in the last sentence. Observe that N can be obtained from any member of $\Phi(G, N)$ by deleting edges. Conversely, each member of $\Phi(G, N)$ is the graph N with extra edges.

The sets of unavoidable minors of connected, 2-connected, 3-connected, and internally 4-connected graphs were specified in the preceding chapter. Building on these theorems, the four main results of this chapter give the sets of unavoidable parallel minors of 1-, 2-, 3-, and internally 4-connected graphs. The families of graphs that we introduce in the following

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four theorems are discussed in Section 3.2. The following theorems are the main results of this chapter.

Theorem 3.1.1. *There is an integer-valued function $f_{3.1.1}$ such that, for any positive integer k, every connected graph of order at least $f_{3.1.1}(k)$ contains a parallel minor isomorphic to $K_{1,k}$, C_k, P_k, or K_k.*

For positive integers a and b, the graph $K'_{a,b}$ is obtained from the bipartite graph $K_{a,b}$ by adding a complete graph on the vertices in the class containing a vertices. A *fan*, denoted F_k, is obtained from a path P_k by adding a vertex adjacent with every other vertex.

Theorem 3.1.2. *There is an integer-valued function $f_{3.1.2}$ such that, for any integer k exceeding two, every 2-connected graph of order at least $f_{3.1.2}(k)$ contains a parallel minor isomorphic to $K'_{2,k}$, C_k, F_k, or K_k.*

A *double-fan*, denoted DF_k, is obtained from a fan F_k by adding a new vertex adjacent with every vertex in $V(F_k)$.

Theorem 3.1.3. *There is an integer-valued function $f_{3.1.3}$ such that, for any integer k exceeding three, every 3-connected graph of order at least $f_{3.1.3}(k)$ contains a parallel minor isomorphic to $K'_{3,k}$, W_k, DF_k, or K_k.*

A *double-wheel with axle*, denoted D'_k, is obtained from a double-wheel D_k by adding an edge, called the *axle*, between the hub vertices. A *triple-fan*, denoted TF_k, is obtained from DF_k by adding a new vertex adjacent with every vertex in $V(DF_k)$.

Theorem 3.1.4. *There is an integer-valued function $f_{3.1.4}$ such that, for any integer k exceeding four, every internally 4-connected graph of order at least $f_{3.1.4}(k)$ contains a parallel minor isomorphic to $K'_{4,k}$, D_k, D'_k, TF_k, M_k, Z_k, or K_k.*

Because every parallel minor is also a minor, it is not surprising that the minors listed in Theorem 2.1.6 are closely related to the minors listed in Theorem 3.1.3, and likewise for Theorem 2.1.7 and Theorem 3.1.4.

3.2 Families of 1-, 2-, 3-, and Internally 4-Connected Graphs

This section contains no proofs. Instead, motivation is provided for the specific families of graphs we chose to comprise our sets of unavoidable parallel minors in our variously connected graphs.

We could have included the families of graphs from Theorem 3.1.1 in the list for Theorem 3.1.2, since every 2-connected graph is 1-connected. Observe, however, that each family in the unavoidable set stated in Theorem 3.1.2 is 2-connected. Likewise, Theorem 3.1.3 contains a list of families of 3-connected graphs and Theorem 3.1.4 gives a list of families of internally 4-connected graphs. We will see that each family is necessary among the unavoidable c-connected parallel minors of c-connected graphs.

Consider the 2-connected family $\{F_k\}_{k>2}$ of fans. No large parallel minor of a member of this family is 2-connected, unless it is another member of this same family. This is true of each family of graphs listed in Theorem 3.1.2. Thus no family listed contains another in the list. The same statement can be made with respect to the 1-connected graphs listed in Theorem 3.1.1, the 3-connected graphs listed in Theorem 3.1.3, and the internally 4-connected graphs listed in Theorem 3.1.4.

Evidently, any set of 1-connected graphs that comprise an unavoidable set of parallel minors of large, 1-connected graphs must contain the four families $K_{1,k}$, C_k, P_k, and K_k. Similarly, each family in each of the three other sets is necessary. The rest of this chapter will give a proof that each set stated in Theorem 3.1.1, Theorem 3.1.2, Theorem 3.1.3, and Theorem 3.1.4 is sufficient.

Note that all the unavoidable parallel minors listed in Theorem 3.1.4 are 4-connected. Since a 4-connected graph is internally 4-connected, Theorem 3.1.4 still holds if we replace internal 4-connectivity with 4-connectivity. In other words, the listed graphs are not only unavoidable in large internally 4-connected graphs, they are also unavoidable in large 4-connected graphs.

The unavoidable parallel minors of large, variously connected graphs are significant both because parallel minors are interesting, and because this work complements work done on unavoidable topological minors, the *matroid dual operation* of parallel minor. Matroids will not be addressed again in this dissertation until Chapter 5.

3.3 Some Parallel Minors in Connected Graphs

In this section, we will prove a result for 1-connected graphs, as a step towards proving Theorem 3.1.1 in the next section. Recall that Proposition 2.1.4 gave the set of unavoidable series minors of large, connected graphs. This set consists of a long path and a star vertex. The reader may note that the proposition still holds when the word "minor" is replaced with the word "subgraph." Proposition 2.1.4 will be useful in the proof of the following lemma.

Lemma 3.3.1. *There is an integer-valued function $f_{3.3.1}$ such that, for positive integers k and l, a connected graph G of order at least $f_{3.3.1}(k, l)$ contains a parallel minor isomorphic to $K_{1,k}$, P_k, or K_k; or, G has a 2-connected graph of order at least l as a parallel minor and has no minor isomorphic to $K_{1,r}$, where $r = f_{\ 2.1.3}(k)$.*

Proof. Let k and l be positive integers. We will now select our variables in a particular way to ease the later steps in the proof. Let $f_{2.1.3}$ and $f_{2.1.4}$ be the functions described in Theorem 2.1.3 and Proposition 2.1.4 respectively. Let $r = f_{2.1.3}(k)$, let $q = l(k + 1)$, and let $s = f_{2.1.4}(r, q)$. Set $f_{3.3.1}(k, l) = s$. Let G be a connected graph of order at least s.

By Proposition 2.1.4, we may divide the proof into the following two cases, which are exhaustive.

1. Graph G contains a minor isomorphic to $K_{1,r}$.

2. Graph G contains no minor isomorphic to $K_{1,r}$, and G contains a minor isomorphic to P_q.

If G meets the conditions of case 1, then take $M \preceq G$ such that M is isomorphic to $K_{1,r}$. Fix $H \in \Phi(G, M)$. Take vertex $v \in V(H)$ with degree r. By Theorem 2.1.3, the graph $H - v$ has an induced subgraph isomorphic to K_k or $\overline{K_k}$. If $H - v$ has an induced subgraph isomorphic to K_k, then H has a parallel minor isomorphic to K_k. Assume, therefore, that $H - v$ has an induced subgraph S isomorphic to $\overline{K_k}$. In H, vertex v is adjacent to every other vertex. Contract each edge vu, where $u \notin V(S)$, to obtain a parallel minor isomorphic to $K_{1,k}$, as desired.

If G meets the conditions of case 2, then G has no minor isomorphic to $K_{1,r}$, and we take $M \preceq G$ such that M is isomorphic to P_q. Fix $H \in \Phi(G, M)$. Let V_{cut} be the set of cut vertices of H.

If $|V_{\text{cut}}| \geq k + 1$, then let H' be obtained recursively from H by contracting, one by one, each edge that is incident with a vertex not in V_{cut}. The parallel minor H' is isomorphic to a path of length at least k, hence G has a parallel minor isomorphic to P_k. We are not finished with case 2, since H may have fewer than $k + 1$ cut vertices.

If $|V_{\text{cut}}| < k + 1$, then there is a large piece of H between cut vertices. Let N be a 2-connected subgraph of H of highest order. Subgraph N is an end of H or a piece of H between two vertices of V_{cut}, so there are at most $k + 1$ places in H that N can be. The order $|N|$ is therefore at least $\frac{q}{k+1}$, which is l by definition. Let H' be the parallel minor of G obtained from H by contracting, one by one, each edge not in N. The graph $H' \preceq_{\parallel} G$ is 2-connected and has order at least l. □

3.4 Unavoidable Parallel Minors of 1- and 2-Connected Graphs

We will prove two lemmas before proving the main lemma of this section. For this section, it will be convenient to work with multigraphs. The operations of vertex and edge deletion are identical to those operations in graphs, but edge contraction is different. When contracting edges in a graph, we sometimes identify edges in the result to obtain a graph. If G is a multigraph, we may contract edges in G without identifying all of the edges between a pair of vertices and obtain a multigraph that is a minor of G. To highlight this difference, we will refer to edge contraction in a multigraph as *m-contraction*, which is completed by deleting the edge and identifying its endpoints. For example, m-contracting $k-2$ edges of a k-cycle results in a pair of *parallel edges* and m-contracting $k-1$ edges of a k-cycle results in a single vertex with a *loop*, an edge whose endpoints are equal. To distinguish the operation from contraction within a graph, we denote the m-contraction of edge set X within a multigraph G as $G/^m X$.

For connected multigraphs M and G, let M be a minor of G, where $M = G/^m X \backslash Y$. Take an edge e in M. Two edges are *parallel* if they share the same two endpoints. A *parallel class* is a set of edges in a graph all parallel with a single edge. Let S be the set of edges in the multigraph $M \cup Y$ that are in the parallel class of e.

If M is a Hamiltonian parallel minor of G and C is a Hamilton cycle of M, then the following statements describe an *H-set*. Let P be a path in M along C such that each vertex of P has degree two in M except for one endpoint of P, which may have degree exceeding two. Let e be an edge of P. Let S be a set of edges in G that belong to the same parallel class as edge e in M. The quintuple (M, C, S, P, e) is an *H-set*. Furthermore, we say that the H-set (M', C', S', P', e) is an *H-minor* of the H-set (M, C, S, P, e), which we write as $(M', C', S', P', e) \preceq_H (M, C, S, P, e)$, if the following conditions hold.

1. $E(C') \subseteq E(C)$.

2. The multigraph M' is obtained from M by m-contracting all edges in $E(C - E(C'))$.

Observe that C' is a Hamilton cycle of M', and that the H-minor relation is transitive.

The *weight* of an H-set is the pair $(|S|, |P|)$. The weight $(|S|, |P|)$ is greater than the weight $(|S'|, |P'|)$ if $|S| > |S'|$, or if $|S| = |S'|$ and $|P| > |P'|$.

The following lemma gives the conditions for finding a longer induced path or a larger parallel set in a Hamiltonian graph by using the H-set construction.

Lemma 3.4.1. *For positive integers d and k and a multigraph G, if (M, C, S, P, e) is an H-set where $|M| > dk$ and $\Delta(M) < d$, then $|P| \geq k$, or H-set (M, C, S, P, e) has an H-minor (M', C', S', P', e) of greater weight such that $|M'| > \frac{|M|}{d}$.*

Proof. Let d and k be positive integers. Let (M, C, S, P, e) be an H-set of weight $(|S|, |P|) = (\sigma, \pi)$ such that $|M| = n > dk$, $\Delta(M) < d$, and $\pi < k$. By hypothesis, C is the Hamilton cycle of M, the element e is an edge in P, which in turn is contained in C, and S is a set of σ edges in G that are in a parallel class with e in M. Order the vertices of $C = v_1 v_2 \ldots v_n$ such that the path $P = v_1 v_2 \ldots v_\pi$, where $d_M(v_i) = 2$ for $i = 1, 2, \ldots, (\pi - 1)$. Let $e = v_a v_{a+1}$.

We consider the neighbors of v_π in M. The vertices in $\{v_\pi\} \cup N(v_\pi)$ divide the cycle C into at most d path segments, since v_π has fewer than d neighbors. There must be a path $v_l v_{l+1} \ldots v_{m-1} v_m$ of length at least $\frac{n}{d} > k$ along C, with ends in $\{v_\pi\} \cup N(v_\pi)$ and no other vertices in that set. With the following vertex indices, addition is computed modulo n.

In the case where the long path segment contains $P - v_\pi$, index m is equal to $\pi - 1$, and we do the following operations. Let M' be obtained from M by the m-contraction of the path $v_{\pi+1} v_{\pi+2} \ldots v_{l-1} v_l$ to the vertex v_l; let C' be $C/^m E(v_{\pi+1} v_{\pi+2}$
$\ldots v_l)$; let $S' = S$; and let $P' = v_1 v_2 \ldots v_\pi v_l$. The H-set (M', C', S', P', e) has weight $(\sigma, \pi + 1)$ and $(M', C', S', P', e) \preceq_H (M, C, S, P, e)$, which is what we wanted to show.

We can therefore assume that the long path segment does not meet path P. In this case, take $f \in E(G)$ such that f is represented by the edge $v_\pi v_m$. Let $S' = S \cup \{f\}$. We obtain M' from M by performing the following m-contractions.

1. M-contract the path $v_m v_{(m+1)} \ldots v_{a-1} v_a$ to vertex v_a.

2. M-contract the path $v_{a+1}v_{a+2}\ldots v_\pi$ to v_{a+1}.

3. M-contract the path $v_{\pi+1}v_{\pi+2}\ldots v_l$ to vertex v_l. Note that l is not equal to π, by construction.

Evidently, vertex v_{a+1} has degree two. Let C' be obtained from C by these same contractions, and let $P' = v_a v_{a+1}$. The H-set (M', C', S', P', e) has weight $(\sigma+1, \pi')$ and $(M', C', S', P', e) \preceq_H (M, C, S, P, e)$, which is what we wanted to show. This concludes the proof of Lemma 3.4.1. $\square$

Using this lemma, we will now prove a second lemma.

Lemma 3.4.2. *There is a function $f_{3.4.2}$ such that, for integers k and d exceeding two, any Hamilton cycle of a graph of order at least $f_{3.4.2}(k,d)$ contains edges that may be contracted to obtain either a vertex with d neighbors or a parallel minor isomorphic to C_k.*

Proof. Let k and d be integers greater than two. Let $r_H = d^{(k-1)(d^2-1)+2}$. Set $f_{3.4.2}(k,d) = r_H$. Any Hamiltonian graph with at least r_H vertices has a Hamiltonian minor of order r_H, so it suffices for our lemma to prove that an arbitrary Hamiltonian graph of order r_H will have our desired structure. Let G_H be a Hamiltonian graph of order r_H such that edges of a Hamilton cycle may not be contracted to obtain either a vertex of degree d or a parallel minor isomorphic to C_k.

Let C_H be a Hamilton cycle of G_H. Take vertex v of C_H. Vertex v has degree less than d, so the vertices of $\{v\} \cup N(v)$ divide C_H into at most d path segments. There is some path segment of length at least $\frac{|C_H|}{d} = \frac{d^{(k-1)(d^2-1)+2}}{d} = d^{(k-1)(d^2-1)+1}$. Choose such a path segment, and let C be the cycle obtained from C_H by contracting all edges of C_H that are not in this path segment and that are not incident with v. Let G be the graph obtained from G_H by the same contractions. Observe that C is a Hamilton cycle of G, and $|G| \geq d^{(k-1)(d^2-1)+1}$. Without loss of generality, suppose Hamilton graph G to have order exactly $d^{(k-1)(d^2-1)+1}$. Let $r = d^{(k-1)(d^2-1)+1}$.

Let e and f be the two edges in C incident with v. If $G = C$, then observe that G contains a parallel minor isomorphic to C_k. We assume not. Let $S = \{e\}$ and let P be the path with endpoint v containing the edge e such that each internal vertex of P has degree two and P has an endpoint with degree exceeding two. If $|P| \geq k$, then we may contract edges in $C - E(P) - \{e\}$ to obtain a parallel minor isomorphic to C_k. This is forbidden by our assumptions.

Since G is a graph, it is also a multigraph. We will now treat it like a multigraph and consider the H-set (G, C, S, P, e). If we find an H-set that is an H-minor $(M', C', S', P', e) \preceq_H (G, C, S, P, e)$ such that $|S'| \geq d^2$, then we may contract a path along C in G that contains exactly one end of each edge in S' to obtain a vertex of degree at least d. This is forbidden by our assumptions. Our restrictions also require that P' have fewer than k vertices, for the same reason that path P does.

The H-set (G, C, S, P, e) has weight at least $(1, 1)$, and $|P| < k$. By applying Lemma 3.4.1, we may find an H-set $(M', C', S', P', e) \preceq_H (G, C, S, P, e)$ of greater weight, where $|M'| > \frac{r}{d} = d^{(k-1)(d^2-1)}$. We may do this another $(k-1)(d^2-1)$ times, each time obtaining an H-set with greater weight, each of which is an H-minor of the preceding one. By our assumptions, for each H-set (M'', C'', S'', P'', e) in this sequence, $|P''| < k$. Since this sequence must include at least $(k-1)(d^2-1)+1$ weights greater than $(1, 1)$, none of which may repeat, we may apply the pigeonhole principle to conclude that there must be one H-set $(M''', C''', S''', P''', e)$ among this sequence with weight greater than $(d^2 - 1, k - 1)$, so that $|S'''| > d^2 - 1$.

By transitivity, this H-set is an H-minor of (G, C, S, P, e), as desired. $\square$

To prove the final lemma in this section, we will use Proposition 2.1.5, which gives the set of unavoidable minors in 2-connected graphs. Recall that this set consists of C_k and $K_{2,k}$. This will provide a natural way of dividing into two cases the 2-connected graphs of high order that we will study in this section.

As a next step toward proving our 1- and 2-connected results, Theorems 3.1.1 and 3.1.2, we will now prove a lemma concerning 2-connected graphs.

Lemma 3.4.3. *There is an integer-valued function $f_{3.4.3}$ such that, for integers k and q exceeding two, every 2-connected graph of order at least $f_{3.4.3}(k,q)$ has a parallel minor isomorphic to $K'_{2,k}$, C_k, F_k, K_k, or a 3-connected graph of order at least q.*

Proof. Let k and q be integers exceeding two. Let $f_{3.3.1}$, $f_{2.1.3}$, $f_{3.4.2}$, and $f_{2.1.5}$ be the functions described in Lemma 3.3.1, Theorem 2.1.3, Lemma 3.4.2, and Proposition 2.1.5 respectively. Let $s = f_{3.3.1}(k,q)$, $r = f_{2.1.3}(k+1) + f_{3.4.2}(k,s)$, and $l = f_{2.1.5}(r)$. Set $f_{3.4.3}(k,q) = l$. Let G be a 2-connected graph of order at least l.

Proposition 2.1.5 implies that the following two cases are exhaustive.

1. Graph G has a minor isomorphic to $K_{2,r}$.

2. Graph G has no minor isomorphic to $K_{2,r}$, but G has a minor isomorphic to C_r.

We suppose first that G meets the conditions of case 1. Let M be a minor of G that is isomorphic to $K_{2,r}$. Fix $H \in \Phi(G,M)$. Take v and w in $V(H)$ with degree at least r in M. By Theorem 2.1.3, the graph $H - \{v,w\}$ has an induced subgraph isomorphic to K_{k+1} or $\overline{K_{k+1}}$.

If $H - \{v,w\}$ has an induced subgraph isomorphic to K_{k+1}, then H has a parallel minor isomorphic to K_k. We assume, therefore, that $H - \{v,w\}$ has a stable set X of order $(k+1)$. In H, vertices v and w are adjacent to all other vertices. We contract, one by one, each edge that does not have both ends in $X \cup \{v\}$, to obtain a parallel minor isomorphic to $K_{2,k+1}$ or $K'_{2,k+1}$. We then contract any edge to obtain a parallel minor isomorphic to $K'_{2,k}$. This completes case 1.

We suppose next that G meets the conditions of case 2. Let M be a minor of G that is isomorphic to C_r and fix $H \in \Phi(G,M)$. The graph H is Hamiltonian.

Let C be a Hamilton cycle of H. We may contract edges of C to obtain a parallel minor isomorphic to C_k or a vertex of degree s by Lemma 3.4.2. If the former, then we are done, since C_k is among our list of parallel minors. If the latter, then contract edges of C to find a vertex of degree s. This vertex is contained in a Hamiltonian graph, so we can find a minor N of H isomorphic to F_s. Choose $H' \in \Phi(H, N)$.

Take vertex v of degree s in H'. The graph $H' - v$ is connected, so we may apply Lemma 3.3.1 with the following result. The graph $H' - v$ has a parallel minor isomorphic to $K_{1,k}$, P_k, K_k, or a 2-connected graph of order at least q. Therefore, H' has a parallel minor isomorphic to $K'_{2,k}$, F_k, K_k, or a 3-connected graph of order at least q, respectively. This completes case 2, and the proof of Lemma 3.4.3. $\square$

Using Lemma 3.4.3 with Lemma 3.3.1, we will now prove our first major result of this chapter, Theorem 3.1.1, concerning connected graphs.

Proof of Theorem 3.1.1. Let k be a positive integer. Let $f_{2.1.3}$, $f_{2.1.6}$, $f_{3.4.3}$, and $f_{3.3.1}$ be the functions described in Theorem 2.1.3, Theorem 2.1.6, Lemma 3.4.3, and Lemma 3.3.1 respectively. Let $r = f_{2.1.3}(k)$, $q = f_{2.1.6}(r)$, $l = f_{3.4.3}(2k, q)$, and $s = f_{3.3.1}(k, l)$. Set $f_{3.1.1}(k) = s$. Let G be a connected graph of order at least s.

By Lemma 3.3.1, graph G has a parallel minor isomorphic to $K_{1,k}$, P_k, or K_k; or G has a 2 connected parallel minor of order at least l that has no minor isomorphic to $K_{1,r}$. If G has a parallel minor isomorphic to $K_{1,k}$, P_k, or K_k, then the theorem holds. Suppose that G has a 2-connected parallel minor H of order at least l, and H has no minor isomorphic to $K_{1,r}$.

We apply Lemma 3.4.3 to H to obtain a 3-connected parallel minor of H of order q, or a parallel minor isomorphic to $K'_{2,2k}$, C_{2k}, F_{2k}, or K_{2k}. If $K'_{2,2k}$ is isomorphic to a parallel minor of H, then $K_{1,k}$ is isomorphic to a parallel minor of G. If C_{2k} is isomorphic to a parallel minor of H, then C_k is isomorphic to a parallel minor of G. If F_{2k} is isomorphic to a parallel minor of H, then we contract every other spoke of the fan to obtain a parallel minor of G

isomorphic to $K_{1,k}$. If K_{2k} is isomorphic to a parallel minor of H, then K_k is isomorphic to a parallel minor of G. Therefore, suppose that none of these four parallel minors occur in G.

Let H' be a 3-connected parallel minor of H of order q. By Theorem 2.1.6, the graph H must have a minor isomorphic to W_r or $K_{3,r}$, so H has a minor isomorphic to $K_{1,r}$, which contradicts our assumption. □

We will now prove the 2-connected result.

Proof of Theorem 3.1.2. Let k be an integer exceeding two. Let $f_{2.1.6}$, $f_{3.1.1}$ and $f_{3.4.3}$ be the functions described in Theorem 2.1.6, Theorem 3.1.1, and Lemma 3.4.3, respectively. Let $r = f_{3.1.1}(k+2)$, let $q = f_{2.1.6}(r)$ and let $l = f_{3.4.3}(k,q)$. Set $f_{3.1.2}(k) = l$. Let G be a 2-connected graph of order at least l.

By Lemma 3.4.3, G has a parallel minor isomorphic to $K'_{2,k}$, C_k, F_k, K_k, or a 3-connected graph of order at least q. It remains only to investigate the last possibility. Let G contain a 3-connected graph, G', of order at least q as a parallel minor. Graph G' has a minor isomorphic to W_r or $K_{3,r}$, by Theorem 2.1.6.

Let M be a minor in G' isomorphic to W_r or $K_{3,r}$. Take $H \in \Phi(G,M)$, and take $v \in V(H)$ of degree at least r. The graph H is 3-connected, hence $H - v$ is 2-connected. Since $H - v$ is connected and has order $f_{3.1.1}(k+2)$, the graph $H - v$ has a parallel minor H' isomorphic to $K_{1,k+2}$, C_{k+2}, P_{k+2}, or K_{k+2}, by Theorem 3.1.1. Since v is non-adjacent to at most two other vertices in H', the graph H must have a parallel minor isomorphic to $K'_{2,k}$, F_k, or K_k, as desired. □

3.5 Unavoidable Parallel Minors of 3-Connected Graphs

We will now prove the third result, Theorem 3.1.3, using Theorem 3.1.2. Recall that Theorem 3.1.3 states that, for an appropriate integer k, every 3-connected graph of high enough order contains $K'_{3,k}$, W_k, DF_k, or K_k as a parallel minor.

Proof of Theorem 3.1.3. Let k be an integer exceeding three. Let $f_{3.1.2}$ and $f_{2.1.6}$ be the functions described in Theorem 3.1.2 and Theorem 2.1.6, respectively. Let $r = f_{3.1.2}(k+2)$ and $q = f_{2.1.6}(r)$. Set $f_{3.1.3}(k) = q$. Let G be a 3-connected graph of order at least q. By Theorem 2.1.6, the graph G contains a minor M isomorphic to W_r or $K_{3,r}$. We choose $H \in \Phi(G, M)$.

Take $v \in V(H)$ of highest degree. Graph $H - v$ is 2-connected, and has order at least r, so $H - v$ contains a parallel minor H' isomorphic to $K'_{2,k+2}$, C_{k+2}, F_{k+2}, or K_{k+2}, by Theorem 3.1.2. Evidently, v is adjacent to all but at most two other vertices in H', hence G has a parallel minor isomorphic to $K'_{3,k}$, W_k, DF_k, or K_k, respectively, as desired. $\square$

3.6 Unavoidable Parallel Minors of Internally 4-Connected Graphs

Recall that Theorem 2.1.7 gives the set of unavoidable minors in large, internally 4-connected graphs to be $\{K_{4,k}, D_k, M_l, Z_l\}$, which will provide the basis for this proof. In this section, we will prove the main result of this chapter, Theorem 3.1.4, which states that an internally 4-connected graph of sufficiently high order contains as a parallel minor $K'_{4,k}$, D_k, D'_k, TF_k, M_k, Z_k, or K_k.

Proof of Theorem 3.1.4. Let k be an integer exceeding four. Let $f_{3.1.3}$, $f_{3.4.2}$, and $f_{2.1.7}$ be the functions described in Theorem 3.1.3, Lemma 3.4.2, and Theorem 2.1.7, respectively. Let $q = f_{3.1.3}(k+3)$, $r = f_{3.4.2}(2k+1, 4q)$, and $n = f_{2.1.7}(q, r)$. Set $f_{3.1.4}(k) = n$. Let G be an internally 4-connected graph of order at least n. The graph G has a minor isomorphic to $K_{4,q}$, D_q, M_r, or Z_r, by Theorem 2.1.7.

If G has a minor, M, isomorphic to $K_{4,q}$ or D_q, then choose $H \in \Phi(G, M)$. Take v of highest degree in H. Graph H is 4-connected. Graph $H - v$ is 3-connected, so it has a parallel minor H' isomorphic to $K'_{3,k+3}$, W_{k+3}, DF_{k+3}, or K_{k+3}, by Theorem 3.1.3. Since v is adjacent to all but at most 3 vertices of H', graph H has a parallel minor isomorphic to $K'_{4,k}$, D_k or D'_k, TF_k, or K_k, respectively.

We suppose, then, that G has no minor isomorphic to $K_{4,q}$ or D_q. Then, G has a minor M isomorphic to M_r or Z_r. To deal with these two cases simultaneously, we consider a minor M' of M, which is defined as follows. Let vertices of M be labeled as follows. If $M \cong M_r$, then let it consist of a cycle $u_1u_2 \dots u_{k+1}$, a path $v_1v_2 \dots v_k$, and the edge sets $\{u_1v_1, u_2v_2, \dots, u_kv_k\}$ and $\{u_2v_1, u_3v_2, \dots, u_kv_{k-1}, u_1v_k\}$; and let $M' = M \backslash \{v_1u_2, v_2u_3, \dots, v_{r-1}u_r, v_ru_1\}$. If $M \cong Z_r$, then let it consist of cycles $u_1u_2 \dots u_k$ and $v_1v_2 \dots v_k$ together with edge sets $\{u_1v_1, u_2v_2, \dots, u_kv_k\}$ and $\{u_2v_1, u_3v_2, \dots, u_kv_{k-1}, u_1v_k, u_{k+1}v_k, u_{k+1}v_1\}$; and let $M' = M \backslash \{v_1u_2, v_2u_3, \dots, v_ru_{r+1}, u_{r+1}u_1\} / \{u_ru_{r+1}\}$. Notice that M' is a planar ladder in the first case and a Möbius ladder in the second case. Take $H \in \Phi(G, M')$.

We will work with a collapsed form of H. Let H' be $H/\{v_1u_1, v_2u_2, \dots, v_ru_r\}$ after deleting multiple edges, and let C be the cycle representing the collapsed ladder. We apply Lemma 3.4.2 to conclude that edges of C, a Hamilton cycle, may be contracted to obtain a vertex of degree $4q$ or a parallel minor isomorphic to C_{2k+1}.

Suppose we can obtain a vertex of degree $4q$ from H' by only contracting edges in C. Then we may obtain a graph D from H by the contractions of the corresponding pairs of edges in H. In this case, D contains a vertex x_1 of degree at least $2q$. Observe that D is a parallel minor of H with the same order as the ladder subgraph contained inside it, so it maintains the ladder structure, which we may label the same way we label M', but with x and y vertices instead of u and v vertices, respectively. Let $|V(D)| = 2s$.

The vertex x_1 must be adjacent with at least q vertices in either the x-vertices or the y-vertices of D. If x_1 has q neighbors among the x-vertices, we may contract the path, $y_3y_4 \dots y_{s-2}y_{s-1}$, in D to a vertex y of degree at least q. The vertices y and x_1 are then the two hubs of a minor isomorphic to D_q. If x_1 has more than q neighbors among the y-vertices, we may contract the path $x_2x_3 \dots x_{s-2}x_{s-1}$ in D and then obtain a minor isomorphic to D_q. In both cases we conclude that $D_q \preceq G$, which contradicts our assumptions.

Suppose we cannot obtain a vertex of degree $4q$ from H' by contracting edges in the representative Hamilton cycle. Then, by Lemma 3.4.2, we must be able to find in H' a parallel minor N' isomorphic to C_{2k+1} by contracting edges in the Hamilton cycle. For every edge that we contract in the Hamilton cycle of H' to obtain N', we contract corresponding pair of edges in H to obtain a parallel minor N. Observe that N is simply a Möbius or circular ladder, possibly with extra edges between consecutive rungs. Let us label the vertices of N the same way we label M', but with x and y vertices instead of u and v vertices, respectively. If N is a Möbius ladder, then contracting edges $x_1x_2, x_3x_4, \ldots, x_{2k-1}x_{2k}, x_{2k+1}y_1, y_2y_3, y_4y_5, \ldots, y_{2k}y_{2k+1}$ results in a parallel minor M_k. If N is a circular ladder, then contracting edges $x_1x_2, x_3x_4, \ldots, x_{2k-1}x_{2k}, y_2y_3, y_4y_5, \ldots, y_{2k}y_{2k+1}$, and also $x_{2k+1}x_1$ and $y_{2k+1}y_1$ results in a parallel minor Z_k. This completes the proof. □

3.7 Observations

A graph M is an *induced minor* of G if it can be obtained from a parallel minor of G by deleting vertices. Since a parallel minor is an induced minor, the reader should note that the set of unavoidable parallel minors in a c-connected graph contains the set of unavoidable induced minors. With the exception of C_k in the 1-connected graph case, the families of unavoidable parallel minors and unavoidable c-connected induced minors in c-connected graphs are identical.

The set of unavoidable minors in large 5-connected graphs is currently unknown, so it is likely that the techniques presented here would be largely unhelpful in that case. It remains, however, an open avenue of investigation to identify the sets of topological minors and parallel minors unavoidable in large, 5-connected graphs. It would also be interesting to consider generalizing these results to c-connected graphs for all natural numbers c.

The results in this chapter concern finite graphs, but are related to the results that we will see in the remaining chapters for infinite graphs and for regular matroids.

Chapter 4
Unavoidable Minors in Infinite Graphs

4.1 Infinite Graphs

Recall that an infinite graph is a graph with an infinite vertex set. In the preceding chapter, we saw that the set of unavoidable subgraphs of large connected graphs consists of the long path and the vertex of high degree. The reader should not be surprised then that every infinite connected graph contains a ray or a star vertex as a subgraph. It is easy to see that the only infinite connected minor of a ray is a ray, and likewise for a star. The set of unavoidable minors of 2-connected finite graphs consists of C_k and $K_{2,k}$. To translate this set into the infinite graph context proves a more complex task, since it raises the question of an infinite cycle, which will be addressed later.

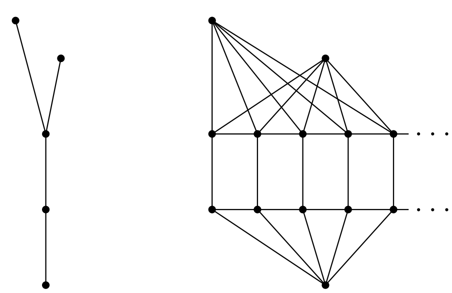
In this chapter, we determine the structure of a set of ℓ-c-connected infinite graphs that are the unavoidable topological minors of ℓ-c-connected infinite graphs. Corresponding results for minors and parallel minors are also obtained. This work is an extension of König's Infinity Lemma. The results in this chapter are based on joint work with Guoli Ding and may be found in [3].

4.2 Finite Characterizations

In the last chapter, some finite graphs were defined. The infinite counterparts for some of those graphs are now presented. An *infinite fan* is the graph of a vertex adjacent to each vertex in a ray. An *infinite ladder* on two rays Y and Z is the graph consisting of the disjoint rays $Y = y_1y_2y_3\ldots$ and $Z = z_1z_2z_3\ldots$, and edges $y_1z_1, y_2z_2, y_3z_3, \ldots$. If the edges $y_2z_1, y_3z_2, \ldots$ are added to this ladder, then the result is an *infinite zigzag ladder* on rays Y and Z. In this chapter, it will be convenient to use *fan* to mean *infinite fan*, *ladder* to mean *infinite ladder*, and *zigzag ladder* to mean *infinite zigzag ladder*. In the zigzag ladder, rays

Y and Z are not symmetric, since Y contains a vertex of degree two and Z does not. We observe, however, that the contraction of the edge y_1y_2 results in a zigzag ladder on rays Z and Y/y_1y_2, where ray Z contains a vertex of degree two and Y/y_1y_2 does not.

The main results of this chapter are aesthetically pleasing, in addition to being useful, since the details of the infinite graphs in each set of unavoidable minors can be completely expressed in a finite tree. We will now define the *expansion* of a finite tree T. If T has one vertex, then its expansion is a ray. If T has two vertices then its expansion is a fan. These are the two special cases of expansion. Recall that a leaf is a vertex with degree one. If T has three or more vertices, then let $t_1, t_2, \ldots, t_m$ be its leaves and $t_{m+1}, t_{m+2}, \ldots, t_n$ be its internal vertices. Then the expansion of T is the graph consisting of vertices $s_1, s_2, \ldots, s_m$ and rays $R_{m+1}, R_{m+2}, \ldots, R_n$, with a ladder on rays R_i and R_j exactly when $t_it_j \in E(T)$, and a fan on vertex s_k and ray R_l exactly when $t_kt_l \in E(T)$. The vertices $s_1, s_2, \ldots, s_m$ are the *stars of the expansion* and $R_{m+1}, R_{m+2}, \ldots, R_n$ are the *rays of the expansion*. When we refer to the rays of the expansion, we mean these particular rays.



(a) (b)

FIGURE 4.1. (a) Tree T. (b) The expansion of T.

An example of expansion is given in Figure 4.1, where tree T in Figure 4.1(a) is expanded in Figure 4.1(b).

The graph $K_{c,\infty}$ is the infinite bipartite graph containing an independent set A with c vertices and an infinite independent set B, such that $A \cup B = V(K_{c,\infty})$ and each vertex in A is adjacent to every vertex in B. Note that $K_{1,\infty}$ is a star. We add an edge between each pair of the c vertices in A to $K_{c,\infty}$ to obtain the graph $K'_{c,\infty}$.

The countable version of part (b) of the following theorem is proved in [12]; part (a) is mentioned without proof.

Theorem 4.2.1. *For each positive integer c, let $\mathcal{M}_c$ be the set of graphs that consists of $K'_{c,\infty}$ and expansions of c-vertex trees. Then the following hold.*

(a) Every graph in $\mathcal{M}_c$ is ℓ-c-connected.

(b) Every ℓ-c-connected graph has a minor that is isomorphic to a graph in $\mathcal{M}_c$.

(c) No graph in $\mathcal{M}_c$ contains another graph in $\mathcal{M}_c$ as a minor.

In the definition of expansion, we could use zigzag ladders instead of ladders. Since zigzag ladders are not symmetric with respect to their two poles, such an expansion would not be unique for a given tree.

The set of unavoidable minors of ℓ-2-connected infinite graphs is $\{F_\infty, K_{2,\infty}\}$, so the fan is the infinite counterpart of the finite cycle. This is appropriate since there is no upperbound on the length of a cycle contained in an infinite fan.

Note that Theorem 4.2.1 completely characterizes all unavoidable (or minimal) minors of ℓ-c-connected graphs, and it generalizes König's Infinity Lemma. In this chapter, we actually prove two stronger results, each of which has Theorem 4.2.1(b) as a corollary.

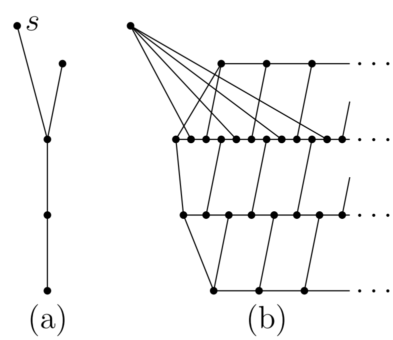


FIGURE 4.2. (a) Tree T. (b) A series expansion of T.

To state the main result we first define a *series expansion* of (T, S), where T is a finite tree, S is a set of leaves of T, and $S \neq V(T)$. Note that S may be empty. A series expansion is basically a subgraph of an expansion of T, except that leaves not in S correspond to rays. The reader may choose to skip the following detailed definition since the idea is clearly illustrated in Figure 4.2.

For the purpose of avoiding notation clutter, we first describe a graph G, from which we will obtain the series expansion of (T, S). Let $V(T) = \{t_1, t_2, \ldots, t_n\}$ with $S = \{t_1, t_2, \ldots, t_m\}$. Let $R_i = r_1^i r_2^i \ldots$ be a ray for $i = m+1, m+2, \ldots, n$. Then G is constructed from vertices $s_1, s_2, ..., s_m$, and disjoint rays R_{m+1}, R_{m+2}, ..., R_n by adding edges $s_i r_i^j, s_i r_{i+n}^j, s_i r_{i+2n}^j, \ldots,$ for each $t_i t_j \in E(T)$ such that $i \leq m < j$, and edges $r_j^i r_i^j, r_{j+n}^i r_{i+n}^j, r_{j+2n}^i r_{i+2n}^j, \ldots,$ for each $t_i t_j \in E(T)$ such that $i, j > m$. Notice that G may have many vertices of degree at most two, all of which are incident only with edges in the rays. The graph obtained from G by contracting, one by one, the edges incident with a vertex of degree at most two is the *cosimplification* of G, which we call a *series expansion* of (T, S). Note that the resulting series expansion depends not only on T and S, but also on how vertices of T are labelled. It is straightforward to verify that all series expansions of the pair (T, S) are *series-equivalent*, meaning that any one contains the other as a series minor. We will refer to vertices in S and $V(T) - S$ as *star vertices* and *ray vertices*, respectively. In the figures, star vertices are labelled with s and ray vertices are unlabelled.

In addition to series expansions of trees, we need to define different versions of $K_{c,\infty}$. A tree is *branching* if it has no vertices of degree two. Let T be a finite branching tree with exactly c leaves, labeled $1, 2, ..., c$, where c is at least three. The *duplication* of T is obtained by taking infinitely many disjoint copies of T and identifying the leaves that have the same label. Note that the duplication of $K_{1,c}$ is exactly $K_{c,\infty}$. For $c = 1, 2$, we consider $K_{1,c}$ a branching tree with c leaves, and its duplication is $K_{c,\infty}$. Each duplication of a branching tree with c leaves is a *version of* $K_{c,\infty}$.

For each positive integer c, let $\mathcal{T}_c$ be the set of graphs that consists of duplications of branching trees with c leaves and series expansions of (T, S) with $|T| = c$. The following is the main result in this chapter, which characterizes a complete set of unavoidable topological minors of ℓ-c-connected graphs.

Theorem 4.2.2. *The following hold for every positive integer c.*

(a) Every graph in $\mathcal{T}_c$ is ℓ-c-connected.

(b) Every ℓ-c-connected graph has a topological minor that is isomorphic to a graph in $\mathcal{T}_c$.

(c) If $M, N \in \mathcal{T}_c$ and $N \preceq_t M$, then M and N are series-equivalent and are isomorphic to the same duplication of $K_{c,\infty}$ or are series expansions of a pair (T, S).

Note that Theorem 4.2.2(c) states that nonequivalent graphs in $\mathcal{T}_c$ are not comparable, which means that, up to equivalence, there is no redundancy in $\mathcal{T}_c$. We could define $\mathcal{T}_c$ by taking one representative from each equivalence class, which would give rise to a formulation similar to Theorem 4.2.1(c). Since no natural representatives are available, we will leave the formulation as it is.

The following figure illustrates all pairs (T, S) for $c \leq 4$. These are finite descriptions of the unavoidable topological minors other than versions of $K_{c,\infty}$.

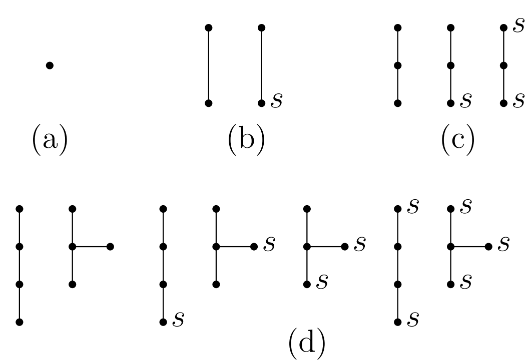


FIGURE 4.3. All possible pairs (T, S) for (a) $c = 1$, (b) $c = 2$, (c) $c = 3$, and (d) $c = 4$.

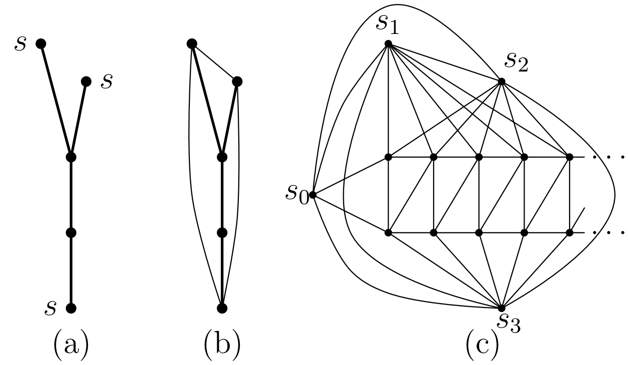


FIGURE 4.4. (a) Tree T with leaves S. (b) Graph $H \supseteq T$. (c) An expansion of (H, S).

The final result is a similar theorem on parallel minors. Since no vertex or edge deletions are allowed, the unavoidable structures are expansions of graphs, instead of trees. A spanning tree T of a finite graph is called *leaf-maximal* if the graph has no spanning tree such that its set of leaves properly contains the set of leaves of T.

We consider pairs (H, S), where H is a connected finite graph and S is a vertex set contained in $V(H)$. Recall that $H[S]$ is the subgraph H induces on S. If H has one or two vertices, we require that $|S| = |H| - 1$, and we define an *expansion* of (H, S) to be a ray or a fan, respectively. If H has three or more vertices, we require that $H - S$ is a tree, $H[S]$ is a clique, and H has a leaf-maximal spanning tree with S as its set of leaves. Let $S = \{t_1, t_2, \ldots, t_m\}$ and $V(H) - S = \{t_{m+1}, t_{m+2}, \ldots, t_n\}$. An *expansion of* (H, S) is a graph consisting of vertices $s_0, s_1, s_2, \ldots, s_m$ and rays $R_{m+1}, R_{m+2}, \ldots, R_n$, with a zigzag ladder on rays R_i and R_j exactly when $t_i t_j \in E(H)$, a fan on vertex s_k and ray R_l exactly when $t_k t_l \in E(H)$, an edge between each pair of vertices in $\{s_0, s_1, ..., s_n\}$, and an edge between s_0 and the first vertex of each ray. Note that there are two ways to put a zigzag ladder onto a pair of rays, therefore there may be several different graphs that are expansions of a pair. For any pair of graphs G and G' in such a set, $G \cong G'/Y$, where Y consists of initial segments of the rays, so we say that the two graphs G and G' are *parallel-equivalent*.

For each positive integer c, let $\mathcal{P}_c$ be the set of graphs that consists of K_∞, $K'_{c,\infty}$, and expansions of (H, S), over all pairs as described in the last paragraph such that $|H| = c$.

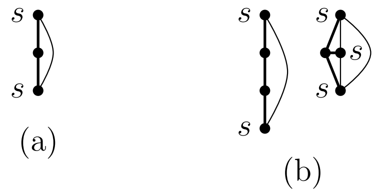


FIGURE 4.5. All possible pairs (H, S) for (a) $c = 3$ and (b) $c = 4$.

The following theorem is the final result of this chapter, a characterization of unavoidable parallel minors of ℓ-c-connected graphs.

Theorem 4.2.3. *The following hold for every positive integer c.*

(a) Every graph in $\mathcal{P}_c$ is ℓ-c-connected.

(b) Every ℓ-c-connected graph has a parallel minor that is isomorphic to a graph in $\mathcal{P}_c$.

(c) If $M, N \in \mathcal{P}_c$ and $N \preceq_{\|} M$, then M and N are parallel-equivalent and are isomorphic to $K'_{c,\infty}$, isomorphic to K_∞, or expansions of a pair (H, S).

It is worth noting that this result also gives a characterization of the set of unavoidable induced minors of ℓ-c-connected graphs: K_∞ and $K'_{c,\infty}$ together with the members of $\mathcal{P}_c - \{K_\infty, K'_{c,\infty}\}$ with s_0 being deleted.

Figure 4.5 contains all possible graphs H for $c = 3$ and $c = 4$. Vertices in S are labelled by s. The darker edges indicate edges in a leaf-maximal spanning tree of H.

The rest of the chapter is organized as follows. Section 4.3 contains proofs of parts (a) and (c) of the main results. In Section 4.4, we will prove a result on augmenting path, which is used in later analysis. In Section 4.5 and Section 4.6 we will prove Theorem 4.2.2(b) and Theorem 4.2.3(b), respectively.

4.3 The Qualification of Unavoidable Sets

We will first prove that all the unavoidable graphs are ℓ-c-connected and then address nonredundancy.

Lemma 4.3.1. *Let T be a tree containing c vertices. Then every series expansion of $(T, \emptyset)$ is ℓ-c-connected.*

Proof. Let T be a tree with c vertices, let G be a series expansion of $(T, \emptyset)$, and let Δ be the maximum degree of the vertices of T. Suppose that G is not ℓ-c-connected. Then, for every integer d, there is a set of fewer than c vertices that divides G into a component and a graph with more than d vertices. Let $d = c(\Delta c)^c$. Take vertex set V' of order at most $c - 1$ that divides G into two graphs H_1 and H_2 both having more than d vertices.

Let R_1, R_2,..., R_c be the rays of the series expansion G. We will see that H_1 meets each of these rays.

An average of $\frac{d}{c}$ vertices of H_1 are in each ray. Therefore at least one ray, say R_1, contains at least $\frac{d}{c} = (\Delta c)^c$ vertices of H_1. Each component of $R_1 \cap H_1$ is adjacent with one or two vertices in $V(R_1) \cap V'$, thus the number of components of $R_1 \cap H_1$ is at most c. Ray R_1 therefore contains a path P_1 with order at least $\frac{(\Delta c)^c}{c} = \Delta^c c^{(c-1)}$. At most Δ rays in G have neighbors in R_1, and each such ray neighboring R_1 contains a path with at least $\frac{\Delta^c c^{(c-1)}}{\Delta} = (\Delta c)^{(c-1)}$ vertices adjacent with P_1. These neighbors are in $V' \cup V(H_1)$, and since $|V'| < c$, there is a path in each ray neighboring R_1 that is in H_1 and has length at least $\frac{(\Delta c)^{(c-1)}}{c} = \Delta^{(c-1)} c^{(c-2)}$.

Ray R_1 contains a path in H_1 with length at least $\Delta^c c^{(c-1)}$. Each ray neighboring R_1 in G contains a path in H_1 with length at least $\Delta^{(c-1)} c^{(c-2)}$. By the same argument, each ray adjacent to a ray neighboring R_1 contains a path in H_1 of length at least $\Delta^{(c-2)} c^{(c-3)}$. Continuing in this fashion, we conclude that a ray in G that is a distance i from R_1 contains a path in H_1 with length at least $\Delta^{(c-i)} c^{(c-1-i)}$. Since G contains c rays, the greatest distance between R_1 and any other ray in G is at most $c - 1$, therefore every ray in G will contain a path in H with length at least Δ. The graph H_1 therefore contains vertices in each of the c rays.

Since $|H_2| \geq d$, we may also conclude that H_2 meets each ray in G. Between a vertex of H_1 and a vertex of H_2 in a ray, there must be a vertex of V', so we conclude that V' meets every ray in G. This contradicts the fact that $|V'| < c$. $\square$

Lemma 4.3.2. *Every graph in $\mathcal{M}_c \cup \mathcal{T}_c \cup \mathcal{P}_c$ is ℓ-c-connected.*

Proof. Clearly K_∞ and every version of $K_{c,\infty}$ is ℓ-c-connected. Furthermore, a ray is ℓ-1-connected and a fan and a ladder are each ℓ-2-connected. Since graphs in $\mathcal{M}_c \cup \mathcal{P}_c$ are obtained from graphs in $\mathcal{T}_c$ by adding edges, it suffices to show that, for $c \geq 3$, every graph in $\mathcal{T}_c$ is ℓ-c-connected. Take a tree T with c vertices. By Lemma 4.3.1, each series expansion of $(T, \emptyset)$ is ℓ-c-connected.

We now assume that each series expansion G of (T, S) is ℓ-c-connected if $|S| = k$, where k is fewer than the number of leaves in T. Take a leaf of T that is a ray vertex and let R be the corresponding ray in G. The vertices $V(R)$ are adjacent with the vertex set of only one other ray of G. We will show that G/R is ℓ-c-connected. Since contracting such a ray does not decrease the connectivity of the graph, we will conclude by induction on k that every member of $\mathcal{T}_c$ is ℓ-c-connected.

We contract R to a vertex r and let $G' = G/R$. We then take $V' \subset V(G')$, a cut set of G' with fewer than c vertices.

If $r \notin V'$, then V' is also a cut set of G. By induction, G is ℓ-c-connected, which implies that $G \backslash V'$ consists of an infinite component X and a graph H with at most d vertices, where d is a number that depends only on G. Ray R is in X, hence, $G' \backslash V'$ consists of the infinite component X/R and graph H. We suppose then that r is in V'. By induction again, $G' - r$ is ℓ-$(c-1)$-connected, so any vertex cut set in $G' - r$ with fewer than $c-1$ vertices separates $G' - r$ into a component and a graph with at most d' vertices for some integer d' depending on $G' - r$.

The graph $G'\backslash V'$ therefore consists of a component and a graph with at most $max\{d, d'\}$ vertices, and we conclude that G' is ℓ-c-connected. $\square$

The following small lemma will be used later in this section.

Lemma 4.3.3. *If P and Q are disjoint rays in graph G joined by an infinite set Π of pairwise disjoint paths, then G contains a subdivision of a ladder with poles contained in $P \cup Q$, with an infinite subset of Π forming the rungs.*

Proof. Let disjoint rays P and Q be disjoint rays $p_1p_2\ldots$ and $q_1q_2\ldots$, respectively, and let them be joined by an infinite set Π of pairwise disjoint paths, $\{P_1, P_2, \ldots\}$, where P_i has ends p_{m_i} and q_{n_i}. The sequence $n_1, n_2, \ldots$ takes infinitely many values, so it contains an infinite subsequence that is strictly increasing. Take such a subsequence, $n_{i_1}, n_{i_2}, \ldots$. The sequence $m_{i_1}, m_{i_2}, \ldots$ takes on infinitely many values, hence it contains a strictly increasing sequence: let S be the set of the indices in this sequence. Let $\Pi' = \{P_i : i \in S\}$. The set $\Pi' \subseteq \Pi$ contains the rungs of a subdivision of a ladder with poles contained in $P \cup Q$. $\square$

The following terminology will be useful in proving Theorem 4.2.2(c), Theorem 4.2.3(c), and Theorem 4.2.1(c). A graph G is *k-disconnected*, for a positive integer k, if there is a set of finite graphs $G_1, G_2, \ldots$ such that G is obtained by identifying V_i, a set of $a_i \leq k$ vertices of G_i, with a_i vertices of G_{i+1} for all positive integers i. Note that, if G is k-disconnected, then it is also k'-disconnected for all $k' > k$. We assume that the edges in $G_i[V_i]$ are identical to the edges in $G_{i+1}[V_i]$. Then G is the *k-path-sum* of $\{G_i\}_{i=1,2,\ldots}$. Since V_i is a cut set for $i = 1, 2, \ldots$, graph G is not ℓ-$(k+1)$-connected.

Note that each minor G' of G is the k-path-sum of some sequence $\{G'_i\}_{i=1,2,\ldots}$ such that G'_i is obtained from G_i by taking a minor of G_i and possibly identifying some of the vertices in the result for $i = 1, 2, \ldots$. The following lemma is the resulting observation.

Lemma 4.3.4. *Every minor of a k-disconnected graph is k-disconnected.*

For any ray R, it is not difficult to see that if R meets some V_i then R meets all V_j with $j > i$. Thus, if a graph is k-disconnected, then it does not have $(k+1)$ pairwise disjoint rays.

Let S be the set of vertices in G that are in infinitely many graphs G_i in the k-path-sum. Let $m = k - |S|$. We will use m, k, and S defined here when stating the remaining lemmas in this section. We make the following observation.

Lemma 4.3.5. *For $S' \subseteq S$, the graph $G \backslash S'$ is $k - |S'|$-disconnected.*

Two rays R and R' are *indistinguishable* if $R \backslash P = R' \backslash P'$ for some finite paths P and P'. Two sets of rays $\{R_1, \ldots, R_m\}$ and $\{R'_1, \ldots, R'_m\}$ are indistinguishable if there is a permutation σ such that R_i is indistinguishable from $R'_{\sigma(i)}$ for all i. The following observation is another consequence of our structure.

Lemma 4.3.6. *Suppose $|V_i| = k$, for all positive integers i, and each graph G_{i+1} contains a unique set of pairwise disjoint paths from the vertices in V_i to the vertices in V_{i+1}. Let R_1, $R_2, \ldots, R_m$ be a set of m pairwise disjoint rays in G. If R'_1, $R'_2, \ldots, R'_m$ are pairwise disjoint rays of M, then $\{R'_1, R'_2, \ldots, R'_m\}$ and $\{R_1, R_2, \ldots, R_m\}$ are indistinguishable.*

We will assume that the assumptions of Lemma 4.3.6 hold for the next three lemmas. We will refer to the assumption that each graph G_i contains a unique set of pairwise disjoint paths from the vertices in V_{i-1} to the vertices in V_i as *uniqueness*.

Let X be a set of edges of G. We now consider the graph $G \backslash X$. Take ray R from a set of m pairwise disjoint rays in G. Let $X' = X \cap E(R)$. Suppose $X' = \{e_1, e_2, \ldots\}$ is infinite. Let G_{i_j} be the graph from which e_j is taken, for $j = 1, 2, \ldots$. It is convenient to assume that $i_1 \leq i_2 \leq \ldots$. By uniqueness, each graph $G_{i_j} - e_j$ contains fewer than m disjoint paths from $V_{i_{j-1}}$ to V_{i_j}. Thus the graph $G_{i_j} - e_j$ contains a cut set with at most $k-1$ vertices. Let V'_2 be the $(k-1)$-vertex cut set in the graph with least index, let V'_3 be the cut set in the graph with next lowest index, and so on. Evidently $G \backslash X'$ may be obtained from some infinite sequence of graphs $G'_1, G'_2, \ldots$ by identifying the vertices V'_j in G'_{j-1} with V'_j in G'_j, for $j = 2, 3, \ldots$. We conclude that $G \backslash X'$ is $(k-1)$-disconnected. By Lemma 4.3.4, $G \backslash X$ is $(k-1)$-disconnected, and we note the following.

Lemma 4.3.7. *The deletion of infinitely many edges from any of the m rays in G results in a $(k-1)$-disconnected graph.*

Take m pairwise disjoint rays in G: R_1, R_2,..., R_m. Let Q be the set of edges in $G[V(R_1) \cup V(R_2) \cup \cdots \cup V(R_m) \cup S]$ that are not in $E(R_1) \cup E(R_2) \cup \cdots \cup E(R_m)$. Take a set Y of edges in G.

Suppose Y contains an infinite set Y' of edges between two rays R_1 and R_2. Since R_1 and R_2 are contained in $G \backslash S$, no vertex is incident with infinitely many edges in Y', hence Y' contains an infinite set of pairwise non-adjacent edges. By Lemma 4.3.3, $(R_1 \cup R_2) \cup Y'$ contains a ladder with rung set Y'' contained in Y'. Let the rungs be $e_1, e_2, \ldots$ in the graphs $G_{i_1}, G_{i_2}, \ldots$, respectively, where $i_1 \leq i_2 \leq \ldots$. By uniqueness, each graph in $G_{i_1}/e_1, G_{i_2}/e_2, \ldots$ contains fewer than m disjoint paths from $V_{i_{j-1}}$ to V_{i_j}. Then each graph G_{i_j}/e_j contains a cut set of G/Y'' with at most $k-1$ vertices, hence G/Y'' is the $(k-1)$-path-sum of a sequence of graphs. Evidently, G/Y'' is $(k-1)$-disconnected, hence, by Lemma 4.3.4, G/Y is $(k-1)$-disconnected.

We suppose then that Y contains an infinite set Y' of edges between a ray R_1 and a vertex in S, say s. Let $e_1, e_2, \ldots$ be the edges of Y' in the graphs $G_{i_1}, G_{i_2}, \ldots$, respectively, where $i_1 \leq i_2 \leq \ldots$. By uniqueness, each graph in $G_{i_1}/e_1, G_{i_2}/e_2, \ldots$ contains fewer than m disjoint paths from $V_{i_{j-1}}$ to V_{i_j}. Then each graph G_{i_j}/e_j contains a cut set of G/Y' with at most $k-1$ vertices, and G/Y' is the $(k-1)$-path-sum of a sequence of graphs. Evidently, G/Y' is not ℓ-k-connected. By Lemma 4.3.4, G/Y is $(k-1)$-disconnected. We make the following observation.

Lemma 4.3.8. *If set $Y \cap Q$ is infinite then G/Y is $(k-1)$-disconnected.*

Let G_Y be the subgraph of G induced by the vertices incident with edges in Y. If G_Y contains a path P between two vertices in S, say s_1 and s_2, then let G/P be obtained from G by contracting P to the vertex s'. Since s_1 and s_2 are incident with infinitely many edges, there is some index z such that $G_z, G_{z+1}, \ldots$ all contain s_1 and s_2. Let G'_z be the k-path-sum of $G_1, G_2, \ldots, G_z$. For integer i at least $z+1$, let G'_i be obtained as follows. If P is in G_i, then let G'_i be obtained from G_i by contracting P to vertex s'. Otherwise, let G'_i be obtained from G_i by identifying s_1 and s_2, and relabeling the vertex s'. Clearly s_1 and s_2 are in each of the sets $V_z, V_{z+1}, \ldots$, hence G/P is the $(k-1)$-path-sum of $G'_z, G'_{z+1}, \ldots$, and G/P is $(k-1)$-disconnected. By Lemma 4.3.4, G/Y is $(k-1)$-disconnected, and we make the following observation.

Lemma 4.3.9. *If any component of G_Y contains two or more vertices of S, then G/Y is $(k-1)$-disconnected.*

The following proof shows nonredundancy among the members of $\mathcal{T}_c$.

Proof of Theorem 4.2.2(c). Take integer c and graphs M and N of $\mathcal{T}_c$ such that $N \preceq_t M$. By Theorem 4.2.2(a), both of these graphs are ℓ-c-connected. Take X and Y in $E(M)$ such that $N = M\backslash X/Y$. Note that each edge in Y is a series element in $M\backslash X$. If M is a version of $K_{c,\infty}$, then it is the duplication of a branching tree T. Then N contains no rays, and is also a version of $K_{c,\infty}$. Since T has no proper topological minor containing c leaves, it is an easy exercise to show that N is also the duplication of T, and the theorem holds.

We assume then that M is the series expansion of (T_M, S_M). If $|T_M| = 1$, then M is a ray. The only ℓ-1-connected minor of M then contains a ray; hence N is a ray and the theorem holds. We assume that the theorem holds if $c < k$ for some integer k at least two. Suppose $c = k$. By construction, M is c-disconnected. Furthermore, M satisfies the conditions of Lemma 4.3.6 and Lemma 4.3.8.

It is useful to note that, since no star vertices are created by deleting edges and contracting series edges, N does not have more star vertices than M. Since c is the sum of the number of stars in N and the number of rays of the expansion of N, the graph $M\backslash X$ has as many pairwise disjoint rays as M does. Let $m = c - |S_M|$. By Lemma 4.3.6, the set of m pairwise disjoint rays $R'_1, R'_2, \ldots, R'_m$ in $M\backslash X$ is indistinguishable from the set of rays $R_1, R_2, \ldots, R_m$ of M. Evidently each of the rays $R''_1, R''_2, \ldots, R''_m$ of N can be obtained by contracting edges in a ray of $M\backslash X$. That is, $R''_i = R'_i/Y_i$, for some edge set Y_i in R'_i, for $i = 1, 2, \ldots, m$. Thus $V(R''_i) \subseteq R'_i$ for each i. If $m = c$, then N is the series expansion of a pair (T_N, S_N) and $S_N = \emptyset$. By Lemma 4.3.8, the set Y contains finitely many edges that are in no ray of M. Evidently, N contains infinitely many edges between rays R'_i and R'_j exactly when M contains infinitely many edges between R_i and R_j. We conclude that $T_N \cong T_M$.

We may assume then that $S_M \neq \emptyset$. Take vertex v in T_M that corresponds to a star vertex s in M. Let w be the vertex in T_M adjacent with v and let R_i be the ray of M corresponding to w. Now M contains a subdivision N' of N. By Lemma 4.3.5, the graph $M - s$ is $(c-1)$-disconnected, and, by Lemma 4.3.4, every minor of a $(c-1)$-disconnected graph is $(c-1)$-disconnected. Therefore N' is not a minor of $M - s$, and we conclude that vertex s is in N'. Now $M - s$ contains $N' - s$. Clearly, the cosimplification of $N' - s$ is a topological minor of $M - s$. It follows that the cosimplification of $N' - s$ is a topological minor of the cosimplification of $M - s$, both of which are members of $\mathcal{T}_{c-1}$. By our induction hypothesis, these two cosimplifications are expansions of the same pair $(T_M - v, S_M - v)$. For edge t_it_j of T_M, let $Q_{t_it_j}$ be the set of edges of M that are between ray R_i or star vertex s_i and ray R_j or star vertex s_j. Now $V(R''_i) \subseteq V(R_i)$ for the ray R''_i of the expansion N, thus N is isomorphic to the graph obtained from M by adding vertex s and a set of edges from Q_{vw} between vertex s and ray R_i, or N is series-equivalent to it. Since adding only a finite set of edges from Q_{vw} results in a $(c-1)$-disconnected graph, N contains an infinite set of edges in Q_{vw}, hence N is the expansion of (T_M, S_M), as desired. $\square$

We will now show nonredundancy among the members of $\mathcal{P}_c$. An *end* of an infinite graph is an equivalence class of rays, where two rays are said to be in the same end of a graph, or equivalent, exactly when they are joined by infinitely many pairwise disjoint paths.

Proof of Theorem 4.2.3(c). Let c be a positive integer. Take M and N in $\mathcal{P}_c$ such that $N \preceq_{\|} M$. Take edge set Y in M such that $N = M/Y$. By Theorem 4.2.3(a), N is ℓ-c-connected. If M is isomorphic to $K'_{c,\infty}$, then N contains no ray, hence $N \cong K'_{c,\infty}$ and the theorem holds.

We assume therefore that M is the expansion of (H_M, S_M). By construction, M is c-disconnected. Furthermore, it satisfies the assumptions of Lemma 4.3.6, Lemma 4.3.8, and Lemma 4.3.9. Take a tree T_M that spans H_M and has the vertices of S_M as leaves. Let $m = |H_M| - |S_M|$ and let $\{R_1, R_2, \ldots, R_m\}$ be the rays of the expansion M.

Suppose Y contains the edge set of a ray R'_i contained in a ray R_i corresponding to vertex t_i in H_M. We first assume that t_i is adjacent to fewer than two vertices in $V(T_M)\backslash S_M$. Since t_i is not a leaf corresponding to a star, it is adjacent to a vertex t_j that corresponds to a star of M. If this star is adjacent with the vertices of a ray R_k of the expansion M other than R_i, then we replace the edge t_it_j in T_M with t_kt_j to obtain a spanning tree of H_M whose leaves properly contain the set S_M, which contradicts the leaf-maximality of T_M. If no vertex other than t_i corresponding to a ray of M is adjacent with t_j, then, to contract R'_i in M, we must delete all but finitely many edges between a star of the expansion M and a ray of the expansion. Clearly this deletion results in a $(c-1)$-disconnected graph, and, by Lemma 4.3.4, N is $(c-1)$-disconnected, a contradiction. Next, we assume that t_i is adjacent with at least two vertices in $V(T_M)\backslash S_M$. We contract R'_i in M to vertex s_{R_i}. Now M/R'_i has $m-1$ rays. The rays are not all in the same end of M/R'_i, however. Take a cut set V' of M/R'_i consisting of the star vertices and a vertex in each of the rays contained in one end. Clearly V' has fewer than c vertices, and each component of $M/R'_i\backslash V'$ is $(c-1)$-disconnected. Since N is

ℓ-c-connected, by Lemma 4.3.4, it is not a minor of any component of $M/R_i'\backslash V'$. It is also easy to see that it is not a minor of M/R_i'. We conclude with the following observation.

4.3.9.1. *For each ray R_i, the set $E(R_i)\backslash Y$ is infinite.*

By Lemma 4.3.8, $Q \cap Y$ is finite, thus $M/(Q \cap Y)$ contains a set of m rays indistinguishable from the rays of M. By 4.3.9.1, Y contains no ray that is contained in a ray of M, hence M/Y contains a set of m rays that are indistinguishable from the rays of M. Take the rays $\{R_1', R_2', \ldots, R_m'\}$ in N and the rays $\{R_1, R_2, \ldots, R_m\}$ of the expansion M such that R_i' is indistinguishable from R_i for each i. Evidently N is not isomorphic to $K_{c,\infty}'$. Furthermore, for each star s_k of M, we take vertex s_k' in N that is s_k or is obtained by contracting the component of G_Y that contains the star s_k. By Lemma 4.3.9, no component of G_Y contains two star vertices, thus exactly $|S_M|$ vertices of N are identified in this way. Since $Q \cap Y$ is finite, R_i' and R_j' have infinitely many edges between them in N exactly when R_i and R_j do in M. Also, R_i' and s_k' have infinitely many edges between them in N exactly when R_i and s_k do in M. By Lemma 4.3.6, the rays of the expansion N are indistinguishable from the set $\{R_1', R_2', \ldots, R_m'\}$. Thus, if R_i' and R_j' have infinitely many edges between them, then N contains a zigzag ladder on R_i' and R_j'. Furthermore, if R_i' and s_k' have infinitely many edges between them, then s_k' is adjacent with all of the vertices of a ray contained in R_i'. We conclude that N must be the expansion of (H_M, S_M), and the theorem holds. $\square$

The remainder of this section contains a proof of the nonredundancy among the members of $\mathcal{M}_c$.

Proof of Theorem 4.2.1(c). Take positive integer c, and take $M, N \in \mathcal{M}_c$ such that $N \preceq M$. Observe that $K_{c,\infty}'$ contains no rays, so if M is isomorphic to $K_{c,\infty}'$, then so is N.

Take M in $\mathcal{M}_c - \{K_{c,\infty}'\}$ and tree T such that M is an expansion of T. Let S be the stars of the expansion M and let $R_1, R_2, \ldots, R_m$ be the rays of the expansion M. By construction of the expansion, we may select G_1, G_2,... such that G is the c-path-sum of this sequence of graphs, each graph in the sequence is a tree, and these graphs are all isomorphic. Observe that each graph G_i contains a unique set of pairwise disjoint paths from the c vertices in V_{i-1} to the c vertices in V_i.

Take N in $\mathcal{M}_c$ that is a minor of M. By Theorem 4.2.1(a), N is ℓ-c-connected. Let $N = M\backslash X/Y$. We apply Lemma 4.3.7 and Lemma 4.3.4 to conclude the following.

4.3.9.2. $X \cap E(R)$ *is finite.*

For edge $e = t_it_j$ of T, let Q_e be the set of edges of M that are between ray R_i and ray R_j or star vertex s_j. Let $X' = Q_e \cap X$ for some edge e in T. Suppose $Q_e\backslash X'$ is finite. Then, for some integer l, each graph in $G_l, G_{l+1}, \ldots$ in the c-path-sum of M contains an edge in X' that is a cut edge in its respective graph. For each integer n at least l, the edge e_n is a cut edge of the tree G_n and V_n has vertices in each component of $G_n - e_n$. If $M\backslash X'$ has one end, then it is clearly $(c-1)$-disconnected and, by Lemma 4.3.4, N is not ℓ-c-conencted, a contradiction. Then $M\backslash X'$ has multiple ends and we take a cut set V' of $M\backslash X'$ consisting of the star vertices and a vertex in each of the rays contained in one end. Clearly V' has fewer than c vertices, and each component of $(M\backslash X')\backslash V'$ is $(c-1)$-disconnected. Since N is ℓ-c-connected, by Lemma 4.3.4, it is not a minor of any component of $(M\backslash X')\backslash V'$. It is easy to see that it is also not a minor of $M\backslash X'$. We conclude with the following observation.

4.3.9.3. *The set* $Q_e\backslash X$ *is infinite for all edges* $e \in E(T)$.

Suppose, for some ray R_i, the set $E(R_i)\backslash Y$ is finite. Let $Y' = E(R_i) \cap Y$ If t_i is adjacent to a leaf t_j of T, then M/Y' requires the deletion of all but a finite set of edges in $Q_{t_it_j}$, contradicting 4.3.9.3. If e is not adjacent to a leaf of T, then M/Y' has multiple ends, each containing at least one ray, and we take a cut set V' of M/Y' consisting of the star vertices and a vertex in each of the rays contained in one end. Clearly V' has fewer than c vertices, and each component of $(M/Y')\backslash V'$ is $(c-1)$-disconnected. Since N is ℓ-c-connected, by

Lemma 4.3.4, it is not a minor of any component of $(M/Y')\backslash V'$. It is easy to see that it is also not a minor of M/Y'. We conclude with the following observation.

4.3.9.4. *For each ray R_i, the set $E(R_i)\backslash Y$ is infinite.*

Evidently, 4.3.9.2 and 4.3.9.4 imply that, for each ray R of the expansion M, there is a ray R' of the expansion N such that a subray of R' consists entirely of edges in R. Then N has m pairwise disjoint rays, hence it is not isomorphic to $K_{c,\infty}$. Also, N has no more than m pairwise disjoint rays, since the M has only m rays. Thus, by Lemma 4.3.6, these rays are indistinguishable from the rays of the expansion N. Take $R'_1, R'_2, \ldots, R'_m$ of the expansion N such that R'_i has its vertices contained entirely in R_i, for $i = 1, 2, \ldots, m$. Furthermore, 4.3.9.4, Lemma 4.3.8, and Lemma 4.3.9 together imply that every component of G_Y is finite, though G_Y may contain infinitely many components, and no two stars of M are in a single component of G_Y. Thus, N has precisely $|S|$ vertices of infinite degree, each obtained by contracting a finite subgraph of M containing a star of M.

If we contract all of the edges in the m pairwise disjoint rays of N then the result is a graph with finitely many vertices. Let Z be its subgraph formed by edges from infinite parallel families. The simplification of Z must be isomorphic to T. For each edge t_it_j in T, by 4.3.9.3, Lemma 4.3.8, and Lemma 4.3.9, t_it_j is an edge in Z. Graph N is therefore not an expansion of any tree other than T. □

4.4 Unavoidable End Behavior in Locally Finite Infinite Graphs

This section contains a result for augmenting paths, which will be essential for finding the unavoidable topological minors in locally finite ℓ-c-connected graphs. We begin with a stronger form of König's Infinity Lemma.

Lemma 4.4.1. *If G is a connected, locally finite infinite graph, then G contains an induced ray.*

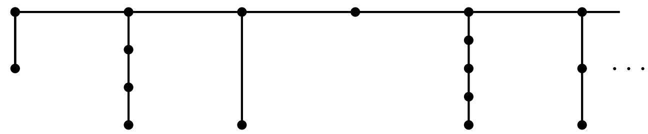


FIGURE 4.6. Example of a comb graph

Proof. Let G be a connected, locally finite infinite graph. Since G is locally finite, by Lemma 2.2.2, G has a ray v_1v_2.... In addition, for each positive integer i, there exists the largest integer $n(i) > i$ such that v_i is adjacent to $v_{n(i)}$. It follows that $v_1v_{n(1)}v_{n(n(1))}$... is an induced ray of G. □

A *comb* is a ray, the *spine* of the comb, combined with an infinite set of pairwise disjoint, finite paths, each containing exactly one vertex in the spine, as shown in Figure 4.6. These finite paths are called *teeth*. Note that a path is a comb, and all its vertices are teeth. The following theorem is proved in [6, Lemma 8.2.2].

Theorem 4.4.2. *If X_1, X_2, ... are pairwises disjoint non-empty sets of vertices in a connected graph G, then G has either a comb containing a tooth that meets X_i for infinitely many of these sets or a subdivided star with leaves in infinitely many of these sets.*

The following theorem is the main result of this section, an essential theorem concerning locally finite ℓ-c-connected infinite graphs.

Theorem 4.4.3. *Suppose G is a locally finite, ℓ-c-connected graph, for some positive integer c. If G contains an end with $c-1$ pairwise disjoint rays, then G contains c pairwise disjoint rays in that end such that infinitely many vertices from each original ray are contained in the set of c rays.*

Proof. Observe that Lemma 2.2.2 implies the result when $c = 1$.

Let c be an integer greater than one. Let G be a locally finite, ℓ-c-connected infinite graph with an end containing $c-1$ pairwise disjoint rays, R_1, R_2,..., R_{c-1}, where $R_i = r_1^i r_2^i \ldots$, for $i = 1, 2, \ldots, c-1$. Take integer d such that any separating set with fewer than c vertices divides G into an infinite component and a graph containing at most d vertices. Let $H = R_1 \cup R_2 \cup \cdots \cup R_{c-1}$. A vertex v *precedes* a vertex w in H if the two vertices are in the same path of H and vertex v has index less than that of w.

We will call each component of $G \backslash V(H)$, together with all edges incident with it in G, a *bridge*. Also, we will call each edge in G that is not in H but has both vertices in H a *bridge*. For a bridge B, we will let the *neighborhood* $N(B)$ be the set of vertices in H incident with B.

Suppose there is a bridge B that contains infinitely many neighbors in H. Then, B has infinitely many neighbors in some ray, say R_1. Let S be the set of vertices in $B \backslash N(B)$ adjacent to vertices in R_1. Since G contains no vertices of infinite degree, $B - N(B)$ is an infinite connected graph and, by Theorem 4.4.2, we obtain a comb, C, with each tooth containing one vertex in S. Let the spine of the comb be $R_c = r_1^c r_2^c \dots$. The teeth of the comb are an infinite set of pairwise disjoint paths between R_1 and R_c, so R_1 and R_c are in the same end of G. Thus, G meets the criteria of the lemma.

Therefore, assume that no bridge has an infinite neighborhood. A vertex pair $\{y, z\}$ *crosses* a vertex pair $\{w, x\}$ if y or z, say y, is in a finite component of $H \backslash \{w, x\}$, and z is in an infinite component, unless y precedes w and x. The vertex set V_1 *crosses* vertex set V_2 if V_1 has a vertex pair that crosses a vertex pair in V_2. A bridge B_1 *crosses* a bridge B_2 if vertex set $N(B_1)$ crosses $N(B_2)$. Observe that bridge B_1 may cross bridge B_2 such that B_2 does not cross B_1. A *directed graph* is a graph in which the edge are ordered pairs, as opposed to unordered pairs, thus each edge has a direction from one endpoint to the other. Let the *crossing graph* of H in a graph G written $\chi_G(H)$ be a directed graph with vertex set equal to the set of bridges and directed edge set $\{(B_k, B_l) : B_l \text{ crosses } B_k\}$.

An infinite *directed path*, or *dipath*, is a path $v_1 v_2 \dots$ such that $(v_i, v_{i+1}) \in E(G)$ for $i = 1, 2, \dots$. We will now see that $\chi_G(H)$ contains an infinite induced dipath.

If S is a set of vertices in H, then let $X(S)$ be the set of vertices of highest index from each of the $c-1$ rays that are in S. The following observation can be easily verified, and the proof is omitted.

4.4.3.1. *If y and z are in an infinite component and a finite component of $H \backslash X(S)$, respectively, then vertex set $\{y, z\}$ crosses S unless z precedes every vertex of $S - \{z\}$.*

We will now prove the following.

4.4.3.2. *There exists a sequence of bridges B_1, B_2,... such that $N(B_i)$ crosses $\{N(B_1) \cup N(B_2) \cup \cdots \cup N(B_{i-1})\}$ for each positive integer i.*

We may assume that r_1^1 is not a cut vertex since, if it is, we may reassign the indices such that r_{d+2}^1 is the first vertex in the ray and path $r_1^1 r_2^1 \ldots r_{d+1}^1$ is in a bridge. The new initial vertex will not be a cut vertex, since it would divide G into a component and a graph with $d+1$ vertices, a contradiction.

If $c = 2$, then take vertex v in R_1 that is in the neighborhood of a bridge of R_1 and precedes every other vertex in R_1 that is in the neighborhood of a bridge. Take vertex w in the neighborhood of a bridge that has v as a neighbor, such that every other vertex in the neighborhood of a bridge with neighbor v precedes w. Let B_1 be the bridge with neighbors v and w.

Since w is not a cut vertex of G, there is some bridge B_2 with neighbors in both components of $R_1 - w$. By our selection of B_1, no neighbor of B_2 precedes v, so B_2 crosses B_1, by 4.4.3.1. Take vertex $z \in N(B_2)$ with highest index in R_1. Since z is not a cut vertex of G, there is a bridge B_3 with neighbors in both components of $R_1 - z$. Observe that the vertices in $N(B_3)$ cross $\{N(B_1) \cup N(B_2)\}$. We may continue in this way to obtain a set of bridges $\{B_1, B_2, \ldots\}$ where each set $N(B_i)$ crosses $\{N(B_1) \cup N(B_2) \cup \cdots \cup N(B_{i-1})\}$. The case $c = 2$ for 4.4.3.2 is complete, so we consider $c > 2$.

Since the rays are in one end of G, if $c > 2$, then there is a bridge, B_1, with neighbors in rays R_1 and R_2. Let $S_1 = X(N(B_1))$. Note that $|S_1| \leq c - 1$. Since S_1 is not a cut set of G, there is a bridge B_2 that has a neighbor in a finite component of $H \backslash S_1$, and a neighbor in an infinite component of $H \backslash S_1$. Observe that B_2 crosses B_1. Let S_2 be the set of vertices in $N(B_1) \cup N(B_2)$ with highest index in each of the $c - 1$ rays of H. There is a bridge B_3 that meets a finite component and an infinite component of $H \backslash S_2$. Bridge B_3 must cross either B_1 or B_2. Let $S_i = X(N(B_1) \cup N(B_2) \cup \cdots \cup N(B_i))$. Choose B_{i+1}, a bridge with neighbors in a finite component and an infinite component of $H \backslash S_i$. This completes the proof of 4.4.3.2. We claim the following.

4.4.3.3. *Bridge B_{i+1} crosses B_1, B_2, $\ldots$, or B_i.*

By the choice of B_i for $c \geq 2$, there are y and z in $N(B_i)$ that belong to an infinite component and a finite component, respectively, of $H \backslash X(N(B_1) \cup N(B_2) \cup \cdots \cup N(B_{i-1}))$. Let j be the smallest index such that z belongs to a finite component of $H \backslash X(N(B_1) \cup N(B_2) \cup \cdots \cup N(B_j))$. Clearly, $j < i$. We claim that $\{y, z\}$ crosses $N(B_j)$. By the minimality of j, vertex z belongs to a finite component of $H \backslash X(N(B_j))$. If my claim is false, then, by 4.4.3.1, z precedes all vertices in $N(B_j) - \{z\}$. Let P be the minimal path in H that contains all of the vertices in $N(B_j)$. By our choice of B_1, we conclude that $j \neq 1$. By induction, B_j crosses some B_k with $k < j$. It follows that some vertex v in $N(B_k)$ belongs to the interior of P, which implies that z precedes v, and thus z belongs to a finite component of $H \backslash X(N(B_k))$, contradicting the minimality of j. This completes the proof of 4.4.3.3.

4.4.3.4. *Each vertex of $\chi_G(H)$ has finitely many outflowing edges.*

Suppose 4.4.3.4 is not true, and vertex $B \in V(\chi_G(H))$ has infinitely many outflowing edges. Then bridge B in G is crossed by infinitely many bridges. These bridges each have an attachment in a finite component of $H \backslash N(B)$, thus a vertex in a finite component of $H \backslash N(B)$ has infinite degree in G. This contradicts our assumption that G is locally finite.

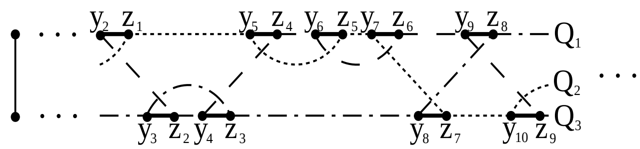


FIGURE 4.7. Continuation of three pairwise disjoint rays in G.

We will now prove the following statement, which states that $\chi_G(H)$ has an infinite dipath.

4.4.3.5. *The sequence $B_1, B_2, \ldots$ contains a subsequence $B_{n_1}, B_{n_2}, \ldots$ such that, for each $i > 1$, the set $N(B_{n_i})$ has two vertices y_i and z_i such that $N(B_{n_{i+1}})$ crosses $\{y_i, z_i\}$, and $\{y_i, z_i\}$ crosses $N(B_{n_{i-1}})$.*

There are outflowing edges from B_1, such as the edge (B_1, B_2). Consider the subgraph χ' of $\chi_G(H)$ that consists of vertices $\{B_i\}$ and, for each $i > 1$, all edges (B_i, B_j) in $E(\chi_G(H)$ such that $j > i$. Note that χ' is a tree with all edges directed away from B_1. By 4.4.3.4, the tree is locally finite. By Lemma 2.2.2, χ' contains the dipath $B_{n_1}, B_{n_2}, \ldots$ we are looking for.

By the choice of the bridges, $(B_{n_{i+1}})$ has a vertex z_{i+1} that belongs to an infinite component of $H \backslash \{X(N(B_{n_1})) \cup \cdots \cup X(N(B_{n_i}))\}$. Clearly, z_{i+1} also belongs to an infinite component of $H \backslash X(N(B_{n_i}))$. Since $B_{n_{i+1}}$ crosses B_{n_i}, there is a vertex y_{i+1} of $N(B_{n_{i+1}})$ that belongs to a finite component of $H \backslash X(N(B_{n_i}))$. Take $z_i \in X(N(B_{n_i}))$ such that y_{i+1} precedes z_i. Since $B_{n_{i+1}}$ has no vertex v that precedes all vertices in $X(N(B_{n_1}) \cup \cdots \cup N(B_{n_{i-2}}))$, vertex z_i must belong to an infinite component of $H \backslash X(N(B_{n_1}) \cup \cdots \cup N(B_{n_{i-1}}))$. Repeating this argument, we can find $y_i \in N(B_{n_i})$ that precedes a vertex $z_{i-1} \in X(N(B_{n_{i-1}}))$. This completes the proof of 4.4.3.5.

Statement 4.4.3.5 implies that we may assume that each B_{n_i} is a path. Since obtaining the paths may require some deletions, we sacrifice our assumption that G is ℓ-c-connected, as we will not need it for the rest of the proof. For the rest of the proof, we assume each bridge to be a path, and relabel the vertices of R to be $P_1 P_2 \ldots$. Let y_j be the neighbor of P_j in a finite component of $H \backslash N(P_{j-1})$, and let z_j be the remaining neighbor of P_j. We will see that this sequence of crossing paths and the rays in H together contain c pairwise disjoint rays. The explanation is quite technical, and the reader may see Figure 4.7 for the general idea when $c = 3$.

Let k be the number of rays in H that are adjacent to vertices in the set of bridges $\{P_1, P_2, \ldots\}$ in G. Without loss of generality, assume these rays to be R_1, R_2, $\ldots$, R_k, and assume that the sequence of bridges P_1, P_2, $\ldots$ meets them in order, that is, if bridge P_i meets ray R_j, then bridges with indices at most i meet rays R_1, R_2,$\ldots$, R_{j-1}. Let ϕ be a function such that $P_{\phi(l)}$ is the bridge with lowest index that has a neighbor in R_l.

We will now see that there are c pairwise disjoint rays, Q_1, Q_2, $\ldots$, and Q_c, and that these rays are in the same end of H. Let q_1^i be the vertex $r^i_{\phi(i)}$ for $i = 1, 2, \ldots, k$, and let Q_i be ray R_i for $i = k+1, k+2, \ldots, c-1$. Let q_1^c be $y_{\phi(k+1)}$. Observe that $z_{\phi(k+1)}$ is in an infinite component of $H \backslash \{r^1_{\phi(1)}, r^2_{\phi(2)}, \ldots, r^k_{\phi(k)}, r_1^{k+1}, r_1^{k+2}, \ldots, r_1^{c-1}\}$. Vertex $y_{\phi(k+2)}$ is in the same ray of H as $y_{\phi(k+1)}$ or $z_{\phi(k+1)}$. If $y_{\phi(k+2)}$ is not in the ray of H with $z_{\phi(k+1)}$, then it is in ray R_k with $y_{\phi(k+1)}$, so $P_{\phi(k+2)}$ crosses a bridge with index lower than that of $P_{\phi(k+1)}$, which contradicts our assumption. Vertex $y_{\phi(k+2)}$ is therefore in the finite component of $H - z_{\phi(k+1)}$, thus $y_{\phi(k+2)}$ precedes $z_{\phi(k+1)}$. Vertex $y_{\phi(k+2)}$ precedes $z_{\phi(k+1)}$, and is proceded by q_1^m for some $i \in \{1, 2, \ldots, k\}$. For the same reason, for integer $i > \phi(k+1)$, vertex y_{i+1} precedes z_i, and y_{i+1} does not precede y_i. Furthermore, y_{i+1} precedes no vertex in $\{z_{i-1}, z_{i-2}, \ldots, z_{\phi(k)}\}$. Let $Q_i = R_i$ for $i = k+1, k+2, \ldots, c-1$. Path Q_i obeys the following rules for $i = 1, 2, \ldots, k, c$. Vertex q_1^i has degree one in Q_i. For any vertex q_m^i, the vertex it immediately precedes is q_{m+1}^i unless $q_m^i = y_j$ for some integer $j > \phi(k)$, in which case the entire path $q_{m+1}^i q_{m+2}^i \ldots q_n^i$ in P_j follows q_m^i, and $q_{n+1}^i = z_j$. Rays $Q_1, Q_2, \ldots, Q_c$ in G are pairwise disjoint and this set of rays contains infinitely many vertices from each ray in $R_1, R_2, \ldots, R_{c-1}$. This completes the inductive argument of this proof.

In summation, of the original rays in H, at least $(c-1) - k$ are contained in H. Ray Q_c includes bridge $P_{\phi(k+1)}$ and vertex $z_{\phi(k+1)}$, which is in a ray R_a of H, but the ray containing first vertex q_1^a includes the bridge that crosses $P_{\phi(k+1)}$, namely $P_{\phi(k+2)}$, and $z_{\phi(k+2)}$ in ray R_b of H. The new ray Q_b that was traveling along R_b includes the bridge $P_{\phi(k+3)}$, so it does not meet Q_a, and so on. This situation may resemble the diagram in Figure 4.7 if $c = 3$, in which one ray is dotted, one dashed, and the third dashed and dotted. $\square$

4.5 Unavoidable Topological Minors of c-connected Infinite Graphs

Let graph G_1 be a subdivision of H_1, a member of $\mathcal{T}_c$, and let graph G_2 be a subdivision of a member of $\mathcal{T}_{c+1}$. Then G_2 is a *direct augmentation* of G_1, written $G_1^{\oplus}$, if G_2 contains a subgraph X of G_1 such that X is isomorphic to a subdivision of H_1.

We will now prove the following theorem, which implies Theorem 4.2.2(b).

Theorem 4.5.1. *For integer c at least two, let G be a ℓ-c-connected infinite graph, and D a subdivision of a graph in $\mathcal{T}_{c-1}$ with the maximal number of star vertices among the subgraphs in one end of G. One of the following occurs:*

(i) D contains a star vertex and G contains a graph $D^{\oplus}$; or

(ii) D is locally finite and G contains a graph Y that is a subdivision of a member of $\mathcal{T}_c$, such that Y contains infinitely many vertices from each ray of D.

Proof. We will prove this theorem by induction on c.

Let $c = 2$, and let G be a ℓ-c-connected infinite graph. Suppose G contains a vertex v adjacent to an infinite set S of vertices. Let D be the star with vertex set $S \cup \{v\}$ and edge set $\{vw\}_{w \in S}$. If $G - v$ contains a subdivision of a star with all of its leaves in S, then observe that G contains a subdivision of $K_{2,\infty}$, which itself contains an infinite subgraph of D and is a direct augmentation $D^{\oplus}$. Suppose not. We apply Theorem 4.4.2 to $N(v)$ in $G - v$ to obtain a comb C with infinitely many teeth that meet S. Observe that $D \cup C$ contains a subdivision of a fan, which is a direct augmentation of D. If G has no vertex of infinite degree, then G is locally finite. By Lemma 2.2.2, we obtain D, a ray. We then apply Theorem 4.4.3 to D in G

to obtain R_1 and R_2, vertex disjoint rays in the same end of G that contain infinitely many vertices in $V(D)$. We then apply Lemma 4.3.3 to R_1 and R_2 and the set of paths between them to obtain a subdivision of a ladder with poles contained in $R_1 \cup R_2$, and (ii) of the theorem holds. Evidently, the theorem holds if $c = 2$.

We now assume the theorem holds if $c < n$ for some integer n at least three. Let $c = n$, and let G be a ℓ-c-connected infinite graph. By the induction hypothesis, we may take D, a subdivision of a member of $\mathcal{T}_{c-1}$ with the maximal number of star vertices such that $D \subseteq G$. As an example, observe that any member of T_c that contains $k < c$ star vertices contains a subdivision of a member of $\mathcal{T}_{c-1}$ with k stars. The following two cases are exhaustive.

1. Graph D contains a vertex of infinite degree; or

2. graph D is locally finite.

We will define some more notation before addressing these cases. For any subdivision of a member of $\mathcal{T}_i$, the *bag graphs* are the components of the graph after the deletion of the star vertices and the edges in each ray. If the member contains a ray, then the bag graphs are ordered by the indices of that ray. If it contains no ray, then the bag graphs are ordered arbitrarily. The *bags* are the vertex sets of the bag graphs.

Suppose case (1) occurs. Graph D contains a star vertex v. We will see that we may augment a subgraph of $D - v$ that will form part of a direct augmentation of D. Let G_v be the subdivided star in G containing v such that each leaf has degree at least three in G and each interior vertex of G_v has degree two in G. Let D' be D after the deletion of the interior vertices of G_v. Graph D' is a subdivision of a member of $\mathcal{T}_{c-2}$. Furthermore, we claim the following.

4.5.1.1. *Graph D' has the maximal number of star vertices of all such subgraphs in the end of $G - v$ that contains D'.*

Suppose not. Then $G - v$ contains a subdivision H of a member of $\mathcal{T}_{c-2}$ with more star vertices than D' in the same end as D'. Then D' has at least one ray. Take star vertex w in H that is not in D'. The vertex w is in the same end as D', so G contains infinitely many pairwise disjoint paths between the neighbors of w and some ray R of D' such that the paths meet no other ray of D'. Evidently, G contains a subdivision of a member of T_{c-1} that does not contain ray R but contains the star vertices in D' and v and w, which contradicts our choice of D. We conclude that 4.5.1.1 holds.

Since graph $G-v$ is ℓ-$(c-1)$-connected, we apply the induction assumption and conclude that $G - v$ contains a graph $D'^{\oplus}$ or $G - v$ contains a subdivision of a member of $\mathcal{T}_{c-1}$ that contains infinitely many vertices from each ray of D'. In either case, $G - v$ contains a graph Y such that Y is a subdivision of a member of $\mathcal{T}_{c-1}$ and Y contains vertices from infinitely many bags of D'. We may delete the edge sets of each bag graph that contains no vertex of Y, so without loss of generality, we assume that each bag meets Y.

We will now see that G contains a graph $Y^{\oplus}$ in $Y \cup G_v$.

We observe that $\{V(G_v) \cap V(D')\}$ is infinite, therefore G_v meets infinitely many bags of D'. Since we may delete some paths in G_v and the edge sets of some bag graphs in D', we assume without loss of generality that each leaf of G_v is contained in exactly one bag of D'. Let G_Y be the extension of the subdivided star G_v through the bag graphs such that $G_Y \cap Y$ is exactly the set of leaves of G_Y. Then G_Y contains infinitely many leaves in a ray R_i of Y, or G_Y contains infinitely many leaves in $Q_{t_it_j}$, the set of paths between star s_i or ray R_i and star s_j or ray R_j. Observe that $G_Y \cup Y$ contains a direct augmentation of Y that is also a direct augmentation of D, as desired.

By the preceding argument, the theorem holds if D contains a vertex of infinite degree. Suppose this is not the case. Then case (2) occurs and D is locally finite.

It follows that G is locally finite, and we apply Lemma 4.4.3 to obtain c rays, R_1, R_2,..., R_c, in G, which contain infinitely many vertices from each ray of D. We conclude this proof with the following lemma.

Lemma 4.5.2. *A series expansion of* $(T, \emptyset)$*, for some* c*-vertex tree* T*, is contained in* G *and has rays contained in* $\{R_1 \cup R_2 \cup \cdots \cup R_c\}$*.*

Proof. Between each pair of rays are infinitely many pairwise disjoint paths, since they are in the same end. Observe that some pair of rays, say R_1 and R_2, is joined by infinitely many pairwise disjoint paths that meet none of the other rays. Let H_1 be the subgraph of G containing R_1, R_2, and an infinite set Π_1 of pairwise disjoint paths that join them but meet none of the other rays. By Lemma 4.3.3, G has a ladder L_1 with poles R_1 and R_2 and rungs in Π_1. There is a ray, say R_3, such that G contains a set Π_2 of infinitely many pairwise disjoint paths between R_3 and L_1 that meet none of the remaining rays. Take a subset Π'_2 of Π_2 such that L_1 contains infinitely many rungs that do not meet members of Π'_2, but each of infinitely many paths in Π'_2 meets a rung of L_1 or meets a pole, say R_1, of L_1. Each such path meeting a rung may be extended into a path that meets R_1. By Lemma 4.3.3, G has a ladder L_2 with poles R_1 and R_3 and rungs in Π'_2.

We continue in this fashion to attach ladder poles and rungs onto the graph, maintaining the pre-existing ladders, until all c rays have been attached. Observe that the resulting graph contains a subdivision of $(T, \emptyset)$ for some c-vertex tree, T. Furthermore, the rays of this graph are contained in $\{R_1 \cup R_2 \cup \cdots \cup R_c\}$, which contain infinitely many vertices from each ray of D, so H contains infinitely many vertices from each ray of D, and the lemma holds. □

□

4.6 Unavoidable Parallel Minors of ℓ-c-connected Infinite Graphs

The following proof of Theorem 4.2.3 will complete this chapter.

Proof of Theorem 4.2.3. Take positive integer c. Let G be a ℓ-c-connected infinite graph that contains no minor isomorphic to K_∞. Graph G contains an infinite component, so we may ignore the finite components of G and assume that G is connected. By Theorem 4.2.1(b), a corollary of Theorem 4.2.2(b), we obtain a minor of G in $\mathcal{M}_c$. Let M be the minor of G in $\mathcal{M}_c$ containing the most star vertices and take edge sets X and Y such that $M = G\backslash X/Y$, where M spans G/Y.

If $M \cong K_{c,\infty}$, then we may add some edges to Y to obtain Y' such that $G\backslash X/Y' = M' \cong K'_{c,\infty}$. Since K_∞ is not a minor of G, we apply Lemma 2.2.1 to obtain an infinite independent set $A \subset V(G/Y')$. Let S be the set of star vertices in M'. Take $s \in S$. We contract the edges in G/Y' between s and each vertex in $V(M')\backslash\{S \cup A\}$ to obtain a parallel minor of G isomorphic to $K'_{c,\infty}$.

Suppose then that M is not isomorphic to $K_{c,\infty}$. Then M is an expansion of some tree T. Let S be the set of leaves of T. We add edges to Y to obtain Y' such that M/Y' is an expansion of (T, S). That is, $G\backslash X/Y'$ is isomorphic to the graph obtained from M by adding a complete graph on the star vertices, a vertex s_0 that is adjacent with each star and the first vertex of each ray, and a zigzag ladder between each pair of ladder poles in M. Now, let $M' = G\backslash X/Y'$. Take H, S, and T such that M' is an expansion of (H, S) and T is a leaf-maximal spanning tree of H with leaf set S. Consider the edges X in G/Y'.

For each vertex pair $\{t_i, t_j\}$ of $V(T)$, let $Q_{t_it_j}$ be the set of edges in G/Y' between R_i or s_i and R_j or s_j. We will say that each edge in $Q_{t_it_j}$ is *between the vertex pair* t_i *and* t_j. Take integer n such that X contains edges between exactly n vertex pairs of $V(H)$. We will prove the theorem by induction on n. If $n = 0$, then $X = \emptyset$ and an expansion of (H, S) is a parallel minor of G and the theorem holds. Suppose the theorem holds for $(n - 1)$.

Suppose that G/Y' contains edges between n vertex pairs of $V(H)$. Take one such vertex pair $\{t_i, t_j\}$.

If $Q_{t_it_j}$ is finite, then take a vertex r_l^k with highest index l for $k \in \{i, j\}$ that is incident with an edge in $Q_{t_it_j}$. Take star vertex s of M'. For each ray R_a, contract the path $sr_1^a r_2^a \dots r_l^a$ to vertex s to eliminate the edges in $Q_{t_it_j}$ and obtain a minor $G\backslash X'/Y''$ of $G\backslash X/Y'$ that contains a copy of M'. Then X' contains edges between fewer than n vertex pairs of $V(H)$. By the inductive hypothesis, the theorem holds.

Suppose then that $Q_{t_it_j}$ is infinite. The following three cases are exhaustive:

1. $t_i = R_i = t_j$;

2. $t_i = R_i$ and $t_j = s_j$; or

3. $t_i = R_i \neq t_j = R_j$.

For the rest of the proof, it will be convenient to let $E(r_l r_{l+1})$ denote the edge set $\{r_l^k r_{l+1}^k : R_k \text{ is a ray of } M'\}$.

Suppose case 1 occurs. Let R' be the graph that $Q_{t_it_j}$ induces on $V(R_i)$. If R' contains a vertex r of infinite degree, then we contract the edge sets $E(r_l r_{l+1})$ if and only if $r_l^i \notin N(r)$, where $N(r)$ is the neighborhood of vertex r. Observe that r is a star of the resulting graph, thus G contains a minor in $\mathcal{M}_c$ with more star vertices than M, a contradiction. We make the following observation, where S is the set of stars of M'.

4.6.0.1. *The graph that edge set $Q_{t_it_j}$ induces in $M'\backslash S$ is locally finite.*

If R' is locally finite, then let $r_1^i = r_{n_1}$. Let r_{n_2} be the vertex with highest index among the neighbors of r_{n_1} in R'. Let r_{n_i} be the vertex with highest index that is a neighbor of a vertex in the path $r_{n_{i-2}} r_{n_{i-2}+1} \dots r_{n_{i-1}}$. Contract the edge set $E(r_l r_{l+1})$ if and only if $l \notin \{n_1, n_2, \dots\}$. Observe that by these contractions in R', we contract each edge of $Q_{t_it_j}$ to a single vertex. In this way, we obtain a parallel minor of G that contains a copy of M' and the remaining edges of X are between at most $(n-1)$ vertex pairs of $V(H)$. By the inductive hypothesis, the theorem holds. We may therefore assume that case 1 does not occur.

Suppose case 2 occurs: $t_i = R_i$ and $t_j = s_j$. Contract the edge set $E(r_l r_{l+1})$ if and only if $l \notin N(s_j)$ to obtain an expansion of $(H \cup t_i t_j, S)$. Tree T is a leaf maximal spanning tree, and we obtain a parallel minor of G that contains a copy of M' and the remaining edges of X are between at most $(n-1)$ vertex pairs of $V(H)$ not in $E(H \cup \{t_i t_j\})$. By the inductive hypothesis, the theorem holds. We also make the following observation.

4.6.0.2. *If a star s is adjacent with infinitely many vertices in a ray R_i in Z, then we may assume s to be adjacent with every vertex in R_i.*

We therefore assume that case 2 does not occur.

Suppose case 3 occurs: $t_i = R_i \neq t_j = R_j$. We apply 4.6.0.1 and conclude that $Q_{t_i t_j}$ contains no infinite set of edges adjacent with a single vertex, thus $Q_{t_i t_j}$ contains an infinite set Π of pairwise non-adjacent edges.

The following argument is technical and amounts to obtaining a zigzag ladder on R_i and R_j. We break up the edge set $E(r_l r_{l+1})$ into two sets. Edge $t_i t_j$ is a cut edge of tree T and divides the graph into a component containing t_i and a component containing t_j. Let $E_i(r_l r_{l+1})$ be the set of edges corresponding to the edges in $E(r_l r_{l+1})$ that are in the rays labelling vertices in the component of $T \backslash t_i t_j$ containing t_i. Let $E_j(r_l r_{l+1})$ be the set of edges $E(r_l r_{l+1}) \backslash E_i(r_l r_{l+1})$. We apply Lemma 4.3.3 to obtain L, a subdivided ladder with poles in R_i and R_j and with rung set ρ in Π. This allows us to assume that, for every integer $k > 0$, we may find a rung in ρ with ends in the infinite components of $R_i - r^i_k$ and $R_j - r^j_k$. Let $i_1 = 1$. Let j_1 be the lowest index such that $r^j_1 r^j_2 \ldots r^j_{j_1}$ has a neighbor in $R_i - r^i_1$ and $j_1 \geq m$ for each vertex r^j_m adjacent with $r^i_{i_1}$. For $n = 2, 3, \ldots$, let i_n be the lowest index such that $i_n > m$ for each vertex r^i_m adjacent with a vertex in $r^j_1 r^j_2 \ldots r^j_{j_{n-1}}$ and $r^i_{i_{n-1}+1} r^i_{i_{n-1}+2} \ldots r^i_{i_n}$ has a neighbor in the infinite component of $R_j - r^j_{n-1}$; and let j_n be the lowest index such that $j_n \geq m$ for each vertex r^j_m adjacent with a vertex in $r^i_1 r^i_2 \ldots r^i_{i_n}$ and $r^j_{j_{n-1}+1} r^j_{j_{n-1}+2} \ldots r^j_{j_n}$ has a neighbor in the infinite component of $R_i - r^i_n$. Contract edge set $E_i(r_l r_{l+1})$ if and only if $l \notin \{i_1, i_2, \ldots\}$ and contract edge set $E_j(r_l r_{l+1})$ if and only if $l \notin \{j_1, j_2, \ldots\}$ to obtain a zigzag ladder on R_i and R_j. Let Z be the resulting graph. Observe that the graph that Z induces on rays R_i and R_j is a zigzag ladder.

If $t_it_j \in E(T)$, then Z is an expansion of (H,S), and the theorem holds.

If $t_it_j \notin E(T)$, then $T \cup R_iR_j$ contains a cycle $C = R_{k_1}R_{k_2}\dots R_{k_l}$ of interior vertices, where $k_1 = i$ and $k_2 = j$. Observe that T is not leaf-maximal in $H \cup t_it_j$. We will see that G contains a member of $\mathcal{M}_c$ with more star vertices than M and obtain a contradiction. We begin by identifying a set of l rays in Z each of which contains infinitely many vertices of each ray in this cycle. Since there are two different ways of expressing a zigzag ladder between two rays, we will have to be careful with this construction. Let $\phi(a)$ be equal to one if $r_1^{k_a}r_2^{k_{a+1}} \in E(Z)$, where we say that $l+1 = 1$, otherwise $\phi(a) = 0$. Let $\Sigma(a) = 1 + \sum_{m=1}^{a}\phi(m)$. Let ray R_1' be $r_1^{k_1}r_{\Sigma(1)}^{k_2}r_{\Sigma(2)}^{k_3}r_{\Sigma(3)}^{k_4}\dots$. For $m = 2, 3, \dots, l$, let

$$R_m' = r_1^{k_m}r_{\Sigma(m)}^{k_{m+1}}r_{\Sigma(m+1)}^{k_{m+2}}r_{\Sigma(m+2)}^{k_{m+3}}\dots$$

Observe that these l rays are pairwise disjoint and each contains infinitely many vertices of each of the l original rays of Z. The graph that Z induces on each pair of rays R_m' and R_{m+1}', where $l+1 = 1$, is a zigzag ladder. We also conclude the following.

4.6.0.3. *Every ray and star labelling a vertex of H with infinitely many neighbors in R_1' contains infinitely many neighbors in R_m' for $m = 2, 3, \dots, l$.*

Finally, we will see that G contains a minor in $\mathcal{M}_c$ with more star vertices than M, a contradiction that will conclude our proof.

Let S_Z be the star set of Z. We will see that R_1' is not a cut set of $Z \backslash S_Z$ and that no star has infinitely many neighbors only in R_1' and conclude that we may contract R_1' without losing ℓ-c-connectivity. Let R be a ray containing infinitely many vertices adjacent with R_1'. Apply 4.6.0.3 and conclude that R has infinitely many neighbors in R_2'. We apply Lemma 4.3.3 and conclude that the graph that Z induces on $R \cup R_1'$ contains a subdivision of a ladder. Let s be a star with infinitely many neighbors in R_1'. We apply 4.6.0.3 and conclude that s is adjacent to an infinite subset of vertices in R_2', and we may apply 4.6.0.2 to this pair and assume that s is adjacent to each vertex in R_1'. We contract ray R_1' in Z to obtain an ℓ-c-connected graph that contains a member of $\mathcal{M}_c$ with more star vertices than M, a contradiction. We may assume, therefore, that case 3 does not occur. $\square$

Chapter 5
Introduction to Matroid Theory

5.1 A Matroid

The reader who is familiar with matroids may turn directly to the next chapter, in which the matroid results are presented. This chapter contains an introduction to some basic matroid theory terminology. For a complete introduction to matroid theory, please refer to [13].

A *matroid* M is an ordered pair $(E, \mathcal{I})$, where $\mathcal{I}$ is a collection of *independent sets* that are subsets of the finite *ground set* E and satisfy the following conditions:

(i) $\emptyset \in \mathcal{I}$.

(ii) If $I \in \mathcal{I}$ and $I' \subseteq I$, then I' is a member of $\mathcal{I}$.

(iii) For I_1 and I_2 in $\mathcal{I}$, if $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup e$ is a member of $\mathcal{I}$.

This defines a matroid by its independent sets. By (iii), every independent set is contained in a maximal independent set. Thus a matroid M may also be defined in terms of its collection $\mathcal{B}(M)$ of maximal independent sets, which are called the *bases* of M. By (iii), the bases of M all have the same cardinality, and this cardinality is equal to the *rank* of M, written $r(M)$.

If M_1 and M_2 are the matroids $(E_1, \mathcal{I}_1)$ and $(E_2, \mathcal{I}_2)$, then M_1 is *isomorphic to* M_2 if there is a bijection $\phi : E_1 \to E_2$ such that a subset X of E_1 is in $\mathcal{I}_1$ if and only if $\phi(X)$ is in $\mathcal{I}_2$.

A matroid M may also be defined in terms of its collection $\mathcal{C}(M)$ of minimal dependent sets, which we call *circuits*. A circuit is not independent, but every proper subset of it is independent. The circuits of a matroid satisfy the following three conditions:

(C1) $\emptyset \notin \mathcal{C}(M)$.

(C2) If C_1 and C_2 are in $\mathcal{C}(M)$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.

(C3) If C_1 and C_2 are distinct members of $\mathcal{C}(M)$ and $e \in C_1 \cap C_2$, then there is a member C_3 of $\mathcal{C}(M)$ such that $C_3 \subseteq (C_1 \cup C_2) - e$.

Let G be a multigraph. A matroid arises from G in the following way. Let $E = E(G)$ and let the circuits be set of edge-sets of the cycles in G. The matroid obtained in this way from G is written $M(G)$ and is called the *cycle matroid of* G. Any matroid that is isomorphic to the cycle matroid of a multigraph is called a *graphic matroid*. Note that the independent sets of $M(G)$ are the edge sets of forests in G.

Let A be a matrix with n labelled columns. Let E be the set of column labels and let $\mathcal{I}$ be the collection of subsets of E that label linearly independent sets of column vectors. It is easy to see that the pair $(E, \mathcal{I})$ satisfies (i), (ii), and (iii). Thus it is a matroid. This matroid is the *vector matroid* of A, written $M[A]$. A matroid is said to be *representable over* $GF(q)$ if it is isomorphic to the vector matroid of a matrix over $GF(q)$.

5.2 Matroid Duals

A plane multigraph G has a *dual multigraph* G^* that is the multigraph whose vertices are the faces of G such that, for each edge e in $E(G)$, there is an edge e' in $E(G^*)$ whose endpoints are the faces that meet e in G.

The *dual* M^* of a matroid M is the matroid with ground set $E(M)$ whose set of bases is $\{E(M) - B : B \in \mathcal{B}(M)\}$. A basis of M^* is a *cobasis* of M, and an independent set in M^* is a *coindependent set* of M. The circuits of M^* are the *cocircuits* of M. A three-element circuit is often called a *triangle*, and a three-element cocircuit is often called a *triad*. No circuit meets a cocircuit in exactly one element, and this property is called *orthogonality*.

Let A be the matrix $[I_r|D]$ and let D^T be the transpose of D. For a vector matroid $M[A]$, the dual M^* is equal to $M\left[I_{|E(M)|-r}|D^T\right]$. It is not difficult to see that, if G is a plane multigraph, then $M^*(G) = M(G^*)$. A matroid is *cographic* if it has a graphic dual.

5.3 Matroid Minors

Let e be an element in the ground set E of a matroid M. Then the *deletion* of e from M, written $M\backslash e$, is the matroid with ground set $E - e$ having $\{C \in \mathcal{C}(M) : e \notin C\}$ as its set of circuits. Note that the deletion of an element e from a vector matroid $M[A]$ gives the vector matroid of $A - e$, the matrix obtained from A by deleting the column labelled by e. Also, for a multigraph G, it is easy to see that $M(G)\backslash e = M(G\backslash e)$ for any edge e in $E(G)$. The *contraction* of e in M, written M/e, results in a matroid with ground set $E - e$ whose circuits are the minimal non-empty members of $\{C - e : C \in \mathcal{C}(M)\}$. When e is an edge of a multigraph G, the contraction $M(G)/e$ equals $M(G/^m e)$, where $G/^m e$ is the m-contraction of e as defined in Section 3.4.

A matroid N is a *minor* of a matroid M if $N = M\backslash X/Y$ for some disjoint subsets X and Y of $E(M)$. A matroid N is a *parallel minor* of a matroid M if N can be obtained from M by a sequence of moves each of which consists of deleting an element that is in a 2-element circuit or contracting an element. A multigraph H is a parallel minor of a multigraph G if H may be obtained from G by a sequence of moves each of which consists of deleting an edge that is parallel with another edge that is present or m-contracting an edge. If N^* is a parallel minor of M^*, we call N a *series minor* of M. A multigraph H is a series minor of a multigraph G if H may be obtained from G by a sequence of moves each of which consists of deleting an edge, deleting a vertex, or m-contracting an edge that is incident with a vertex of degree two. If G and H are multigraphs and H is a parallel minor of G, then $M(H)$ is a parallel minor of $M(G)$. Conversely, when G and H are loopless 3-connected multigraphs, if $M(H)$ is a parallel minor of $M(G)$, then H is a parallel minor of G.

5.4 Matroid Rank, Closure, and Connectivity

A matroid is *connected* if every pair of elements in its ground set is contained in a circuit. If a graphic matroid is connected, note that the multigraph is then 2-connected. The notion of 1-connectivity defined for multigraphs does not correspond to anything meaningful about the matroid.

The rank of a subset X of $E(M)$, written $r(X)$, is the cardinality of a largest independent set of M contained in X. Clearly, $X \in \mathcal{I}$ if and only if $r(X) = |X|$. The rank of X in M^*, written $r^*(X)$, is called the *corank* of X and is equal to the cardinality of a largest coindependent set of M in X. Note that $r^*(E(M)) = |E(M)| - r(E(M))$.

The *closure*, $\mathrm{cl}_M(X)$ or $\mathrm{cl}(X)$, of a subset X of $E(M)$ is the maximal set $X' \subseteq E(M)$ such that $X \subseteq X'$ and $r_M(X') = r_M(X)$. Clearly, the closure of any basis is the entire matroid. A *flat* F is a *closed set*, that is, $\mathrm{cl}_M(F) = F$.

Let M be a matroid with ground set E and rank function r. The *connectivity function* λ_M of M is defined for all subsets X of E by

$$\lambda_M(X) = r(X) + r(E - X) - r(M).$$

Equivalently, $\lambda_M(X) = r(X) + r^*(X) - |X|$. Thus $\lambda_M(X) = \lambda_{M^*}(X)$. A partition (X, Y) of E with $\lambda_M(X) < m$ is an *m-separation* if $\min\{|X|, |Y|\} \geq m$ and is a *vertical m-separation* if $\min\{r(X), r(Y)\} \geq m$. A matroid is *$n$-connected* if, for all $m < n$, it has no m-separations. A 3-connected matroid is *internally 4-connected* if $\min\{|X|, |Y|\} \geq 4$ for each 3-separation (X, Y).

A *loop* in a matroid is an element with rank zero. In a vector matroid, this element corresponds to the zero vector. In a multigraph, this element is a loop edge. Since a loop in M is in no basis of M, it is in every basis of M^*, and it is a *coloop* of M^*.

In this dissertation, a matroid M is *vertically 3-connected* if it is loopless and has no vertical 1-separation and no vertical 2-separation. The reader who is familiar with vertical connectivity may note that this adds the requirement that M be loopless to the usual definition of vertical 3-connectedness.

A pair of elements are *parallel* if they form a circuit. In a vector matroid, these elements correspond to scalar multiples of a single vector. In a multigraph, these elements are in a single parallel class. For an element e in a matroid M, the *parallel class of* e is the set e together with every element parallel with e. The simplification of a matroid M, written $\mathrm{si}(M)$, is obtained from M by deleting all loops and deleting all but one element from each parallel class. Then M is vertically 3-connected if and only if M is loopless and $\mathrm{si}(M)$ is 3-connected.

Theorem 3.1.2, Theorem 3.1.3, and Theorem 3.1.4, which concern unavoidable parallel minors of finite graphs, may be restated in terms of graphic matroids. For example, Theorem 3.1.4 translates as follows.

Theorem 5.4.1. *There is an integer-valued function f such that, for any integer k exceeding four, every internally 4-connected graphic matroid with at least $f(k)$ elements contains a parallel minor isomorphic to the cycle matroid of $K'_{4,k}$, D_k, D'_k, TF_k, M_k, Z_k, or K_k.*

5.5 Regular and Binary Matroids

A *binary matroid* is a matroid that is isomorphic to the vector matroid of a matrix over $GF(2)$. A *regular matroid* is a matroid that is isomorphic to the vector matroid of a totally unimodular matrix over $\mathbb{R}$, that is, to a real matrix all of whose square subdeterminants are in $\{0, 1, -1\}$. It is not difficult to show that every graphic matroid is regular and that every regular matroid is binary.

Binary matroids are a well-studied class of matroids that have many equivalent characterizations. One such characterization from [13, Theorem 9.1.2] is stated as follows.

Theorem 5.5.1. *A matroid M is binary if and only if the symmetric difference of any set of circuits is the disjoint union of circuits.*

In the next chapter, this characterization will be used without explicit reference to this theorem. The *Fano matroid*, F_7, is the vector matroid of the following matrix over $GF(2)$:

$$\left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right)$$

Note that the columns of the matrix consist of all non-zero vectors in the 3-dimensional vector space over $GF(2)$. The following characterization of regular matroids from [13, Theorem 13.1.2] will be used in the next chapter without explicit reference.

Theorem 5.5.2. *A binary matroid is regular if and only if it has no minor isomorphic to F_7 or F_7^*.*

By this theorem, if N is a rank-3 minor of a regular matroid, then $\mathrm{si}(N)$ does not contain seven elements. Note that, for any element e in the Fano matroid, $F_7 \backslash e$ is isomorphic to $M(K_4)$.

The next theorem concerning regular matroids was proved by Seymour [17]. The matroid R_{10} is a special 10-element matroid that will be referenced in the next chapter. It is the vector matroid of the 5x10 matrix over $GF(2)$ whose columns consist of all 5-tuples with exactly three ones. Since every column in this matrix has an odd number of ones, all circuits of R_{10} have even cardinality. In particular, R_{10} contains no triangles. No other information about R_{10} is necessary to understand the work in the next chapter.

Chapter 6
Unavoidable Minors in Matroids

6.1 Preliminaries

The results presented in this chapter are based on joint work [5] with James Oxley.

The collections of unavoidable parallel and unavoidable series minors for 3-connected graphs were determined by Chun, Ding, Oporowski, and Vertigan [4] and Oporowski, Oxley, and Thomas [12]. In this chapter, we combine these results with Seymour's decomposition theorem for regular matroids [17], stated later as Theorem 6.1.6, to determine the unavoidable parallel minors for 3-connected regular matroids. In particular, we prove that the last collection is precisely the union of the 3-connected graphic and 3-connected cographic matroids. The collections of unavoidable minors for binary 3-connected matroids and for all 3-connected matroids were determined in [8, 9]. From the first of these, one can determine the collection of unavoidable minors for regular 3-connected matroids, although this result has been obtained earlier by Ding and Oporowski [7]. The following theorem is the main result of this chapter.

Theorem 6.1.1. *There is a function $f_{6.1.1}$ such that, for each integer k exceeding three, every 3-connected regular matroid with at least $f_{6.1.1}(k)$ elements has a parallel minor isomorphic to $M(K'_{3,k})$, $M^*(K_{3,k})$, $M(\mathcal{W}_k)$, $M(DF_k)$, or $M(K_k)$.*

By using duality, we immediately obtain the set of unavoidable series minors of 3-connected regular matroids. We denote the dual of the double fan DF_k by V_k. It can be obtained from two cycles $v_1v_2v_3\ldots v_k$ and $v_1u_2u_3\ldots u_k$ that share a single vertex by adding the edges $\{v_iu_i : i \in \{2, 3, \ldots, k\}\}$.

Corollary 6.1.2. *There is a function $f_{6.1.2}$ such that, for each integer k exceeding three, every 3-connected regular matroid with at least $f_{6.1.2}(k)$ elements has a series minor isomorphic to $M^*(K'_{3,k})$, $M(K_{3,k})$, $M(\mathcal{W}_k)$, $M(V_k)$, or $M^*(K_k)$.*

From either of the last two results, we can deduce the following result of Ding and Oporowski [7] which shows that the collection of unavoidable minors of 3-connected regular matroids is the union of the collections of unavoidable minors for the classes of 3-connected graphic and 3-connected cographic matroids.

Corollary 6.1.3. *There is a function $f_{6.1.3}$ such that, for each integer k exceeding three, every 3-connected regular matroid with at least $f_{6.1.3}(k)$ elements has a minor isomorphic to $M(K_{3,k})$, $M^*(K_{3,k})$, or $M(\mathcal{W}_k)$.*

By Seymour's decomposition theorem, stated later as Theorem 6.1.6, an internally 4-connected regular matroid with at least eleven elements is graphic or cographic. This means that the sets of unavoidable parallel minors and unavoidable series minors of internally 4-connected regular matroids can be immediately determined by combining Theorem 3.1.4 with the result in [12] that determines the sets of unavoidable series minors of internally 4-connected graphs. The following theorem is this result. The graph $K^-_{4,k}$ is obtained from a pair of graphs both isomorphic to $K_{2,k}$ by adding a set E' of k edges between the classes of k vertices so that no pair of edges in E are adjacent, and ML_k is a *cubic Möbius ladder* obtained from a cycle $v_1v_2 \dots v_{2k}$ by adding the edge set $\{v_1v_{k+1}, v_2v_{k+2}, \dots, v_kv_{2k}\}$.

Theorem 6.1.4. *There is an integer-valued function $f_{6.1.4}$ such that, for any integer k exceeding four, every internally 4-connected regular matroid with at least $f_{6.1.4}(k)$ elements contains a parallel minor isomorphic to $M(K'_{4,k})$, $M(D_k)$, $M(D'_k)$, $M(TF_k)$, $M(M_k)$, $M(Z_k)$, $M(K_k)$, $M^*(ML_k)$, $M^*(K_{4,k})$, or $M^*(K^-_{4,k})$.*

By duality, we immediately obtain the set of unavoidable series minors of internally 4-connected regular matroids.

Theorem 6.1.5. *There is an integer-valued function $f_{6.1.5}$ such that, for any integer k exceeding four, every internally 4-connected regular matroid with at least $f_{6.1.5}(k)$ elements contains a series minor isomorphic to $M^*(K'_{4,k})$, $M^*(D_k)$, $M^*(D'_k)$, $M^*(TF_k)$, $M^*(M_k)$, $M^*(Z_k)$, $M^*(K_k)$, $M(ML_k)$, $M(K_{4,k})$, or $M(K^s_{4,k})$.*

The proof of Theorem 6.1.1 contains numerous technicalities but the basic method is standard. By Seymour's decomposition theorem, a large 3-connected regular matroid can be decomposed in a tree-like fashion into pieces each of which is graphic or cographic. If any of these pieces is large enough, then we can apply the known results on unavoidable parallel minors in 3-connected graphic matroids and in 3-connected cographic matroids. Thus we may assume that all the pieces are small, so the tree is large and therefore contains a long path or a vertex of high degree. In both of these cases, we can find a parallel minor of the desired type.

The remainder of this section is used to introduce some more terminology and prove some lemmas that will be used in the proof of the main theorem, which appears in the next section. Much of what we do here is concerned with finding a tree-like decomposition of a regular matroid. Of particular importance here is the operation of generalized parallel connection of matroids, which was introduced by Brylawski [2]. We will only use one special case of this operation.

For binary matroids M_1 and M_2 with ground sets E_1 and E_2 such that $E_1 \cap E_2 = \Delta$ and $M_1|\Delta$ and $M_2|\Delta$ are triangles, the *generalized parallel connection* of M_1 and M_2 with respect to Δ, written $P_\Delta(M_1, M_2)$, is the matroid with ground set $E_1 \cup E_2$ in which F is a flat if and only if $F \cap E_i$ is a flat of M_i for each i. Then $P_\Delta(M_2, M_1) = P_\Delta(M_1, M_2)$. Moreover, one can show that if cl, cl_1, and cl_2 are the closure operators of $P_\Delta(M_1, M_2)$, M_1, and M_2, then, for every subset X of $E_1 \cup E_2$,

$$\mathrm{cl}(X) = \mathrm{cl}_1([X \cup \mathrm{cl}_2(X \cap E_2)] \cap E_1) \cup \mathrm{cl}_2([X \cup \mathrm{cl}_1(X \cap E_1)] \cap E_2). \qquad (6.1)$$

When M_1 and M_2 both have at least seven elements and Δ does not contain a cocircuit of M_1 or M_2, Seymour [17] defined the 3-*sum*, $M_1 \oplus_\Delta M_2$, of M_1 and M_2 to be the matroid $P_\Delta(M_1, M_2)\backslash\Delta$. In much of what we do, it will be convenient to work with generalized parallel connections rather than 3-sums because of the additional constraints that must be satisfied in order for the latter to be defined. The generalized parallel connection across a triangle of two graphic matroids is easily seen to be graphic. Hence so is their 3-sum. Note, however, that the 3-sum of two cographic matroids need not be cographic. For example, the non-cographic matroid R_{12} can be written as a 3-sum of $M(K_5 - e)$ and $M^*(K_{3,3})$ (see, for example, [13, Exercise 1(ii), p. 440]). When G_1 and G_2 are multigraphs and both have Δ as a vertex bond, $P_\Delta(M^*(G_1), M^*(G_2))$ and $P_\Delta(M^*(G_1), M^*(G_2))\backslash\Delta$ are easily shown to be cographic. Hence so is $M^*(G_1) \oplus_\Delta M^*(G_2)$ when it is defined. The following theorem is Seymour's decomposition theorem.

Theorem 6.1.6. *Let M be a 3-connected regular matroid. Then*

(i) M is graphic;

(ii) M is cographic;

(iii) $M \cong R_{10}$; or

(iv) there are regular matroids M_1 and M_2 such that $E(M_1) \cap E(M_2) = \Delta$, where Δ is a triangle of both M_1 and M_2, and $M = M_1 \oplus_\Delta M_2$; and, for each i in $\{1, 2\}$,

  (a) M_i is 2-connected and, for every 2-separation (X, Y) of it, either X or Y has exactly two elements and meets Δ, so $\mathrm{si}(M_i)$ is 3-connected;

  (b) M_i is isomorphic to a minor of M; and

  (c) $|E(M_i) - \mathrm{cl}_{M_i}(\Delta)| \geq 6$ and $|E(\mathrm{si}(M_i)| \geq 9$.

The proof of our main result will require use to carefully consider both the matroids that are built up by a sequence of generalized parallel connections across disjoint triangles, and the matroids we get by deleting all of these triangles. We now formally describe these constructions. Let M_1 and M_2 be binary matroids with $E(M_1) \cap E(M_2) = \Delta_2$, where Δ_2 is a triangle of both M_1 and M_2. Let $P(M_1, M_2)$ and (M_1, Δ_2, M_2) be $P_{\Delta_2}(M_1, M_2)$ and $P_{\Delta_2}(M_1, M_2)\backslash\Delta_2$, respectively. Now assume, for some $n \geq 3$, that $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{n-1}, M_{n-1})$ and $P(M_1, M_2, \dots, M_{n-1})$ have been defined, that

$$(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{n-1}, M_{n-1}) = P(M_1, M_2, \dots, M_{n-1})\backslash(\Delta_2 \cup \Delta_3 \cup \dots \cup \Delta_{n-1}),$$

and that the flats of $P(M_1, M_2, \dots, M_{n-1})$ are those subsets F of its ground set such that $F \cap E(M_i)$ is a flat of M_i for all i in $\{1, 2, \dots, n-1\}$. Let M_n be a binary matroid whose ground set meets that of $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{n-1}, M_{n-1})$ in a set Δ_n that is a triangle of both $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{n-1}, M_{n-1})$ and M_n. Define

$$(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n) = P_{\Delta_n}((M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{n-1}, M_{n-1}), M_n)\backslash\Delta_n$$

and $P(M_1, M_2, \dots, M_n) = P_{\Delta_n}(P(M_1, M_2, \dots, M_{n-1}), M_n)$. Then one easily checks that $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n) = P(M_1, M_2, \dots, M_n)\backslash(\Delta_2 \cup \Delta_3 \cup \dots \cup \Delta_n)$ and that the flats of $P(M_1, M_2, \dots, M_n)$ are those subsets F of its ground set such that $F \cap E(M_i)$ is a flat of M_i for all i in $\{1, 2, \dots, n\}$. It will be convenient to abbreviate $P(M_1, M_2, \dots, M_n)$ as $M^P_{[n]}$. Observe that the construction guarantees that $\Delta_2, \Delta_3, \dots, \Delta_n$ are disjoint.

Lemma 6.1.7. *If* $\mathrm{si}((M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n))$ *is* 3*-connected, then* $\mathrm{si}(M_i)$ *is* 3*-connected for all* i*.*

Proof. By definition, $\mathrm{si}((M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n))$ is

$$\mathrm{si}(P_{\Delta_n}((M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{n-1}, M_{n-1}), M_n)\backslash\Delta_n).$$

Assume that $\mathrm{si}(P_{\Delta_2}(M_1, M_2)\backslash\Delta_2)$ is 3-connected. If we can show that both $\mathrm{si}(M_1)$ and $\mathrm{si}(M_2)$ are 3-connected, then the result will follow by induction. For some k in $\{1, 2\}$, suppose that (X, Y) is a vertical k-separation of M_1. Without loss of generality, we may assume that

$|X \cap \Delta_2| \geq 2$. Then

$$r(X \cup \Delta_2) + r(Y - \Delta_2) - r(M_1) \leq r(X) + r(Y) - r(M_1) \leq k - 1.$$

Now, by [13, Lemma 8.2.10],

$$\begin{aligned}
&r((X \cup E(M_2) - \Delta_2) + r(Y - \Delta_2) - r(P_{\Delta_2}(M_1, M_2)\backslash\Delta_2) \\
&\leq r(X \cup E(M_2) \cup \Delta_2) + r(Y - \Delta_2) - r(P_{\Delta_2}(M_1, M_2)) \\
&\leq [r(X \cup \Delta_2) + r(M_2) - r(\Delta_2)] + r(Y - \Delta_2) - [r(M_1) + r(M_2) - r(\Delta_2)] \\
&= r(X \cup \Delta_2) + r(Y - \Delta_2) - r(M_1) \leq k - 1.
\end{aligned}$$

Thus $P_{\Delta_2}(M_1, M_2)\backslash\Delta_2$ has a vertical k-separation; a contradiction. Therefore M_1 is vertically 3-connected and, by symmetry, so is M_2. $\square$

The next lemma will be helpful in the proof of Lemma 6.1.9, where we use Seymour's theorem to obtain a sequential decomposition of a regular matroid.

Lemma 6.1.8. *Let M_1 and M_2 be binary matroids whose ground sets meet in a set Δ_2 that is a triangle of both matroids. If Δ_3 is a triangle of $P_{\Delta_2}(M_1, M_2)\backslash\Delta_2$, then, for some $\{i, j\} = \{1, 2\}$, either*

(i) $\Delta_3 \subseteq E(M_i)$; or

(ii) $|\Delta_3 \cap E(M_i)| = 2$ and $|\Delta_3 \cap E(M_j)| = 1$, and the element c of $\Delta_3 \cap E(M_j)$ is parallel to some element g of M_i. Moreover, if M'_j and M'_i are obtained by deleting c from M_j, and adding c in parallel to g in M_i, then $P_{\Delta_2}(M'_1, M'_2) = P_{\Delta_2}(M_1, M_2)$, while $\mathrm{si}(M'_1) = \mathrm{si}(M_1)$ and $\mathrm{si}(M'_2) = \mathrm{si}(M_2)$.

Proof. Let $E_1 = E(M_1)$ and $E_2 = E(M_2)$. We may assume that $|\Delta_3 \cap E_1| = 2$ and $|\Delta_3 \cap E_2| = 1$. Then, in $P_{\Delta_2}(M_1, M_2)$, the intersection of $\mathrm{cl}(E_1)$ and $\mathrm{cl}(E_2)$ is $\mathrm{cl}(\Delta_2)$. Thus the element c of $\Delta_3 \cap E(M_2)$ is parallel to some element of $\mathrm{cl}(\Delta_2)$, and the lemma follows. $\square$

Lemma 6.1.9. *Let M be a vertically 3-connected regular matroid such that* $\mathrm{si}(M)$ *has at least six elements and is not isomorphic to R_{10}. Then either M is graphic or cographic, or, for some $n \geq 2$, there is a sequence $M_1, M_2, \ldots, M_n$ of graphic and cographic matroids such that $M = (M_1, \Delta_2, M_2, \Delta_3, \ldots, \Delta_n, M_n)$ where, for all i with $2 \leq i \leq n$, the triangle $\Delta_i \subseteq E(M_j)$ for some $j < i$, and all of* $\mathrm{si}(M_1), \mathrm{si}(M_2), \ldots, \mathrm{si}(M_n)$ *are 3-connected having at least nine elements.*

Proof. We will assume that M is simple since it suffices to prove the lemma in that case. We proceed by induction on $|E(M)|$. Since M is regular, if $|E(M)| \leq 9$, then either M is graphic, or M is isomorphic to $M^*(K_{3,3})$ and so is cographic. In both cases, the lemma holds. Now suppose that the lemma holds for matroids with fewer than k elements and let $|E(M)| = k \geq 10$.

Assume that M is neither graphic nor cographic. Then, by Theorem 6.1.6, M is the 3-sum of some matroids N_1 and N_2, where both $\mathrm{si}(N_1)$ and $\mathrm{si}(N_2)$ are 3-connected having at least nine elements. Choose such a 3-sum decomposition in which $|E(N_2)|$ is minimized. Let Δ be the common triangle of N_1 and N_2. We may assume that $\Delta \subseteq E(\mathrm{si}(N_i))$ for each i.

Since N_2 has a triangle, it is not isomorphic to R_{10}. Suppose $\mathrm{si}(N_2)$ is not graphic or cographic. Then, by Theorem 6.1.6, N_2 is the 3-sum of matroids N_2' and N_2'' across a common triangle Δ' where each of $\mathrm{si}(N_2')$ and $\mathrm{si}(N_2'')$ is 3-connected and contains at least nine elements. As Δ is a triangle of $P_{\Delta'}(N_2', N_2'')\backslash\Delta'$, Lemma 6.1.8 implies that, without altering $\mathrm{si}(N_2')$ or $\mathrm{si}(N_2'')$, we can assume that $\Delta \subseteq E(N_2')$. By comparing flats, we can show that $P_\Delta(N_1, P_{\Delta'}(N_2', N_2'')) = P_{\Delta'}(P_\Delta(N_1, N_2'), N_2'')$, so $M = (N_1 \oplus_\Delta N_2') \oplus_{\Delta'} N_2''$. By Lemma 6.1.7, $\mathrm{si}(N_1 \oplus_\Delta N_2')$ is 3-connected; a contradiction, since $|E(N_2)|$ was chosen to be minimal.

We may now assume that $\mathrm{si}(N_2)$ is graphic or cographic. Hence so is N_2. By the inductive hypothesis, $N_1 = (M_1, \Delta_2, M_2, \Delta_3, \ldots, \Delta_n, M_n)$ and the desired conditions hold. Now Δ is a triangle of N_1. Pick the smallest integer k such that $\Delta \subseteq E((M_1, \Delta_2, M_2, \Delta_3, \ldots, \Delta_k, M_k))$. Then Δ meets $E(M_k)$.

Suppose that $|\Delta \cap E(M_k)| \geq 2$. Then, by moving at most one element of Δ from being parallel to an element of Δ_k in $(M_1, \Delta_2, M_2, \Delta_3, \ldots, \Delta_{k-1}, M_{k-1})$ to being parallel to that element of Δ_k in M_k, we ensure that $\Delta \subseteq E(M_k)$, as desired.

It remains to consider when $\Delta \cap E(M_k)$ contains a single element, say c. Then, by Lemma 6.1.8 again, we move c from being parallel to an element of Δ_k in M_k to being parallel with that element in $(M_1, \Delta_2, M_2, \Delta_3, \ldots, \Delta_{k-1}, M_{k-1})$. We now have $\Delta \subseteq E((M_1, \Delta_2, M_2, \Delta_3, \ldots, \Delta_{k-1}, M_{k-1}))$ and we can repeat the above process until we eventually obtain $\Delta \subseteq E(M_i)$ for some i. Thus the lemma holds. $\square$

Let M be a vertically 3-connected regular matroid having at least six elements. If $M = (M_1, \Delta_2, M_2, \Delta_3, \ldots, \Delta_n, M_n)$ for some $n \geq 2$, we call $(M_1, \Delta_2, M_2, \Delta_3, \ldots, \Delta_n, M_n)$ a *good decomposition* of M if, for all i with $2 \leq i \leq n$, the triangle $\Delta_i \subseteq E(M_j)$ for some $j < i$. Also, we view (M) as a good decomposition of M.

Two disjoint triangles X_1 and X_2 in a binary matroid are *parallel* if $r(X_1 \cup X_2) = 2$. Recall that a regular matroid M is vertically 3-connected if $\mathrm{si}(M)$ is 3-connected and M is loopless. For a good decomposition $(M_1, \Delta_2, M_2, \Delta_3, \ldots, \Delta_n, M_n)$ of a vertically 3-connected regular matroid, define the *associated tree* T to have vertex set $\{M_1, M_2, \ldots, M_n\}$ and edge set $\{\Delta_2, \Delta_3, \ldots, \Delta_n\}$ where Δ_i joins M_i to the vertex M_j with $j < i$ such that $\Delta_i \subseteq E(M_j)$. We will sometimes write M_T for M. Note that this labelling means that, for every path $M_{i_1} M_{i_2} \ldots M_{i_k}$ in T, there is a j in $\{1, 2, \ldots, k\}$ such that $i_1 > i_2 > \cdots > i_j$ and $i_j < i_{j+1} < \cdots < i_k$. The reader may find some features of the tree disconcerting. For example, the matroids labelling two non-adjacent vertices may contain triangles that are parallel in $M^P_{[n]}$. In spite of this apparent shortcoming, this tree will be adequate for our needs.

Lemma 6.1.10. *Let M be a vertically 3-connected regular matroid for which $|E(\mathrm{si}(M))| \geq 9$ and $\mathrm{si}(M) \not\cong R_{10}$. Let $(M_1, \Delta_2, M_2, \Delta_3, \ldots, \Delta_n, M_n)$ be a good decomposition of M and $M_i M_j$*

be an edge of the associated tree with $j < i$. Then

$$(M_1, \Delta_2, M_2, \Delta_3, \ldots, \Delta_j, (M_j, \Delta_i, M_i), \Delta_{j+1}, \ldots, M_{i-1}, \Delta_{i+1}, M_{i+1}, \ldots, \Delta_n, M_n)$$

is a good decomposition of M. Moreover, $\mathrm{si}((M_j, \Delta_i, M_i))$ *is* 3*-connected.*

Proof. We will show first that

$$\begin{aligned}(M_1, \Delta_2, M_2, \Delta_3, \ldots, \Delta_j, (M_j, \Delta_i, M_i), \Delta_{j+1}, \ldots, \Delta_{i-1}, M_{i-1})& \\ = (M_1, \Delta_2, M_2, \Delta_3, \ldots, \Delta_i, M_i).& \quad (6.2)\end{aligned}$$

Now $(M_1, \Delta_2, M_2, \Delta_3, \ldots, \Delta_i, M_i)$ is obtained from $P(M_1, M_2, \ldots, M_i)$ by deleting $\Delta_2 \cup \Delta_3 \cup \cdots \cup \Delta_i$. Moreover, $P(M_1, M_2, \ldots, M_i)$ has, as its flats, those sets F such that $F \cap E(M_s)$ is a flat of M_s for all s with $1 \leq s \leq i$. The matroid on the left-hand side of (6.2) is obtained from $P(M_1, M_2, \ldots, M_{j-1}, P_{\Delta_i}(M_j, M_i)\backslash\Delta_i, M_{j+1}, \ldots, M_{i-1})$ by deleting $\Delta_2 \cup \Delta_3 \cup \cdots \cup \Delta_{i-1}$. Thus it is obtained from $P(M_1, M_2, \ldots, M_{j-1}, P_{\Delta_i}(M_j, M_i), M_{j+1}, \ldots, M_{i-1})$ by deleting $\Delta_2 \cup \Delta_3 \cup \cdots \cup \Delta_i$. The flats of the last parallel connection coincide with the flats of $P(M_1, M_2, \ldots, M_i)$. Hence (6.2) holds. It follows that M has the decomposition specified in the lemma, and one easily checks that this decomposition is good. Finally, $\mathrm{si}((M_j, \Delta_i, M_i))$ is 3-connected by Lemma 6.1.7. □

We will repeatedly use the following routine consequence of the last lemma.

Corollary 6.1.11. *Let T be a tree associated with a vertically* 3*-connected matroid M. Delete an edge M_aM_b of T and let T_a be the component of the resulting forest that contains M_a. A new tree associated with M can be obtained from T by contracting the edges of T_a, one by one, each time labelling the composite vertex that results from contracting the edge Δ joining M_i and M_j by (M_j, Δ, M_i).*

When we have a good decomposition of a regular matroid M, the next two lemmas will be useful in obtaining good decompositions of certain minors of M.

Lemma 6.1.12. *Let $(M_1, \Delta_2, M_2, \Delta_3, \ldots, \Delta_n, M_n)$ be a good decomposition of a regular matroid M. For e in $E(M_i) - (\Delta_2 \cup \Delta_3 \cup \cdots \cup \Delta_n)$, if $e \in \mathrm{cl}_{M^P_{[n]}}(\Delta_j)$ for some j, then $e \in \mathrm{cl}_{M_i}(\Delta_k)$ for some k in $\{2, 3, \ldots, n\}$ where $\Delta_k \subseteq E(M_i)$.*

Proof. Choose j to be the smallest integer t for which $e \in \mathrm{cl}_{M^P_{[n]}}(\Delta_t)$. If $\Delta_j \subseteq E(M_i)$, then the result holds with $j = k$. Thus we may assume that $\Delta_j \not\subseteq E(M_i)$ so $\Delta_j \cap E(M_i) = \emptyset$ and $j \neq i$. Now e is parallel in $M^P_{[n]}$ to some element of Δ_j.

Assume $j < i$. Then $e \in \mathrm{cl}_{M^P_{[i]}}(\Delta_j)$ so, in $M^P_{[i]}$, the element e is in the intersection of $\mathrm{cl}(E(M_i))$ and $\mathrm{cl}(E(P(M_1, M_2, \ldots, M_{i-1}))$. Hence $e \in \mathrm{cl}_{M^P_{[i]}}(\Delta_i)$. Thus $e \in \mathrm{cl}_{M_i}(\Delta_i)$ and the result holds with $k = i$.

We may now assume that $j > i$ so $j \geq 2$. We know that $\Delta_j \subseteq E(M_j)$ and $\Delta_j \subseteq E(M_s)$ for some $s < j$. If $s < i$, then, it follows, as above, that $e \in \mathrm{cl}_{M_i}(\Delta_i)$. Hence we may assume that $s > i$. Then $e \in \mathrm{cl}_{M^P_{[s]}}(\Delta_j)$ so $e \in \mathrm{cl}_{M^P_{[s]}}(\Delta_s)$ and hence $e \in \mathrm{cl}_{M^P_{[n]}}(\Delta_s)$. But $s < j$; a contradiction. □

Lemma 6.1.13. *Let $(M_1, \Delta_2, M_2, \Delta_3, \ldots, \Delta_n, M_n)$ be a good decomposition of a regular matroid M. For e in $E(M_i) - (\Delta_2 \cup \Delta_3 \cup \cdots \cup \Delta_n)$, if $e \in \mathrm{cl}_{M^P_{[n]}}(E(M_j))$ for some $j \neq i$, then $e \in \mathrm{cl}_{M_i}(\Delta_k)$ for some k in $\{2, 3, \ldots, n\}$ where $\Delta_k \subseteq E(M_i)$.*

Proof. First we show the following.

6.1.13.1. *The lemma holds if $e \in \mathrm{cl}_{M^P_{[q+1]}}(E(M_j)) - \mathrm{cl}_{M^P_{[q]}}(E(M_j))$ for some q with $j \leq q < n$.*

By definition, $M^P_{[q+1]} = P_{\Delta_{q+1}}(M^P_{[q]}, M_{q+1})$. Suppose $E(M_j) \cap E(M_{q+1}) \neq \emptyset$. Then the construction of M means that $E(M_j) \cap E(M_{q+1}) = \Delta_{q+1}$. Thus, by (6.1), $\mathrm{cl}_{M^P_{[q+1]}}(E(M_j)) = \mathrm{cl}_{M^P_{[q]}}(E(M_j)) \cup \mathrm{cl}_{M_{q+1}}(\Delta_{q+1})$, so $e \in \mathrm{cl}_{M_{q+1}}(\Delta_{q+1})$. Hence $e \in \mathrm{cl}_{M^P_{[q+1]}}(\Delta_{q+1})$, so $e \in \mathrm{cl}_{M^P_{[n]}}(\Delta_{q+1})$ and the lemma follows by Lemma 6.1.12. Hence 6.1.13.1 holds.

Now assume that $j > i$. If $e \notin \mathrm{cl}_{M^P_{[j]}}(E(M_j))$, then, since $e \in \mathrm{cl}_{M^P_{[n]}}(E(M_j))$, the lemma follows by 6.1.13.1. Hence we may assume that $e \in \mathrm{cl}_{M^P_{[j]}}(E(M_j))$. Then $e \in E(M_i) \cap$

$\mathrm{cl}_{M^P_{[j]}}(E(M_j))$. Hence $e \in \mathrm{cl}_{M^P_{[j]}}(\Delta_j)$, so $e \in \mathrm{cl}_{M^P_{[n]}}(\Delta_j)$ and again the lemma follows by Lemma 6.1.12.

Finally, assume that $j < i$. By 6.1.13.1, we may assume that $e \in \mathrm{cl}_{M^P_{[i]}}(E(M_j))$. But $e \in E(M_i)$, so $e \in \mathrm{cl}_{M^P_{[i]}}(E(M_j)) \cap \mathrm{cl}_{M^P_{[i]}}(E(M_i)) \subseteq \mathrm{cl}_{M^P_{[i]}}(\Delta_i)$. Thus $e \in \mathrm{cl}_{M^P_{[n]}}(\Delta_i)$ and the lemma follows by Lemma 6.1.12. □

Corollary 6.1.14. *Let $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_n, M_n)$ be a good decomposition of a regular matroid M. For some i in $\{1, 2, \dots, n\}$, let N_i be a minor of M_i such that if $\Delta_j \subseteq E(M_i)$ for some j in $\{2, 3, \dots, n\}$, then Δ_j is a triangle of N_i. Then*

$$(M_1, \Delta_2, M_2, \Delta_3, \dots, M_{i-1}, \Delta_i, N_i, \Delta_{i+1}, M_{i+1}, \dots, \Delta_n, M_n)$$

is a good decomposition of a minor of M.

Proof. It suffices to prove this when N_i is $M_i \backslash e$ or M_i / e for some element e. In this case, the result follows without difficulty using the last lemma and properties of the generalized parallel connection [2] summarized in [13, Proposition 12.4.16]. □

Let A and B be parallel triangles in a loopless binary matroid N. Then $N|(A \cup B)$ is a *double triangle*. We call N a *multi-K_4 with respect to A and B* if $\mathrm{si}(N) = M(K_4)$; and we call N a *multi-triangle with respect to A and B* if $r(N) = 2$ and N contains at least one element not in $A \cup B$.

The following lemma result is an immediate consequence of the Scum Theorem. For the statement of the Scum Theorem, a well-known result, see [13, Section 3.3].

Lemma 6.1.15. *If a binary matroid M has as a minor a multi-triangle or a multi-K_4 with respect to two parallel triangles A and B, then $E(M)$ has a subset Y such that M/Y is, respectively, a multi-triangle or a multi-K_4 with respect to A and B.*

The next lemma [14] was proved by Jim Geelen and is useful for finding a double triangle as a parallel minor of a 3-connected graphic or cographic matroid. If X and Y are disjoint subsets of the ground set of a matroid M, we define $\kappa_M(X, Y)$ to be $\min\{\lambda_M(Z) : X \subseteq Z \subseteq E(M) - Y\}$.

Lemma 6.1.16. *Let C and X be disjoint sets in a matroid M such that C is a circuit and $\kappa_M(C, X) = 2$. Then there are elements $a, b,$ and c of C and a minor N of M that has $\{a, b, c\}$ as a circuit and $X \cup \{a, b, c\}$ as its ground set such that $\kappa_N(\{a, b, c\}, X) = 2$.*

A connected subgraph of a tree associated with the decomposition of a matroid has some useful properties, as elucidated in the following lemma.

Lemma 6.1.17. *Let M be a vertically 3-connected regular matroid for which $|E(\mathrm{si}(M))| \geq 9$ and $\mathrm{si}(M) \not\cong R_{10}$. Let $(M_1, \Delta_2, M_2, \Delta_3, \ldots, \Delta_n, M_n)$ be a good decomposition of M such that each $\mathrm{si}(M_i)$ has at least nine elements. Let T be the tree associated with this decomposition. Let T' be a connected subgraph of T. Then T' is a tree associated with the matroid M' that labels the one vertex that results after all the edges of T' are contracted. Moreover, $\mathrm{si}(M')$ is a 3-connected matroid that is isomorphic to a parallel minor of M.*

Proof. It suffices to prove the lemma in the case that $T' = T - M_i$ for some vertex M_i of degree one. Let M_j be the neighbor of M_i in T and let Δ_k be the triangle common to M_i and M_j. By Corollary 6.1.11, $M = P_{\Delta_k}(M_i, M'_j)\backslash\Delta_k$ where M'_j labels the vertex other than M_i in the graph that is obtained by contracting every edge of T other than M_iM_j. By Lemma 6.1.7, $\mathrm{si}(M'_j)$ is 3-connected. We may assume that the only 2-circuits of M_i meet $\mathrm{cl}_{M_i}(\Delta_k)$.

Because the vertex M_i has degree one in T, in $M^P_{[n]}$, the intersection of the closures of $E(M_i)$ and $E(M_1) \cup \cdots \cup E(M_{i-1}) \cup E(M_{i+1}) \cup \cdots \cup E(M_n)$ is the closure of Δ_k. Let $Y_i = E(M_i) - \mathrm{cl}_{M_i}(\Delta_k)$. Then $|Y_i| \geq 6$ so, as M_i is regular and cosimple, $r^*(Y_i) \geq 3$. Now $2 = \lambda_{M_i}(\Delta_k) = \lambda_{M_i}(Y_i) = r(Y_i) + r^*(Y_i) - |Y_i|$. Thus $r(Y_i) < |Y_i|$ so Y_i contains a circuit C. By Lemma 6.1.16, there are elements $a, b,$ and c of C and a minor N_i of M_i that has $\{a, b, c\}$ as a circuit and $\Delta_k \cup \{a, b, c\}$ as its ground set such that $\kappa_{N_i}(\{a, b, c\}, \Delta_k) = 2$. Thus

$2 = \lambda_{N_i}(\{a,b,c\}) = r(\{a,b,c\}) + r(\Delta_k) - r(N_i) \leq r(\Delta_k) \leq 2$, so equality holds throughout and $r(N_i) = 2$. Therefore N_i is a double triangle that is a minor of M_i. Hence, by the Scum Theorem, since M_i is binary, N_i is a parallel minor of M_i. Then (N_i, Δ_k, M'_j) is isomorphic to M'_j and the lemma follows without difficulty using Corollary 6.1.14. $\square$

The next lemma is from an unpublished paper [7] of Ding and Oporowski. The proof is given here for completeness.

Lemma 6.1.18. *Let G be a 3-connected simple multigraph containing distinct 3-element bonds S_1 and S_2. Then one of the following occurs.*

(i) S_1 and S_2 are both vertex bonds.

(ii) G has a subgraph H that is a subdivision of K_4 such that H has a degree-three vertex v so that $S_1 \cup S_2$ is contained in the union of the minimal paths in H from v to the other degree-three vertices of H.

Proof. Let $S_1 = \{e_1, f_1, g_1\}$ and $S_2 = \{e_2, f_2, g_2\}$. Either $S_1 \cap S_2 = \emptyset$ or $|S_1 \cap S_2| = 1$. In each case, since G is 3-connected, $S_2 - S_1$ is a bond of $G \backslash S_1$, and $S_1 - S_2$ is a bond of $G \backslash S_2$. Let A be the component of $G \backslash S_1$ that does not contain $S_2 - S_1$, and let C be the component of $G \backslash S_2$ that does not contain $S_1 - S_2$. Then A and C are vertex disjoint.

Suppose A contains no cycles. Then A is a tree and, since G is 3-connected, all the leaves of A must meet edges of S_1. Assume that A contains an edge. Then A has at least two vertices of degree one, so G has a vertex of degree at most two; a contradiction. Hence A contains no edges, and S_1 is a vertex bond. Likewise, if C contains no cycles, then S_2 is a vertex bond.

We may now assume that A or C, say A, contains a cycle D, otherwise (i) holds. Take a vertex v in $V(C)$. By Theorem 1.2.1, G contains three paths from v to $V(D)$ that have no internal vertices in $V(D)$ and that are disjoint except that all contain v. Each such path contains exactly one edge of S_1 and exactly one edge of S_2. The union of these three paths with D is a subdivision of K_4 satisfying (ii). $\square$

6.2 The Proof of the Main Theorem

The following theorem is the main result of [12].

Theorem 6.2.1. *There is an integer-valued function $f_{6.2.1}$ such that, for each integer k exceeding two, every 3-connected graph with at least $f_{6.2.1}(k)$ vertices has a subgraph that is isomorphic to a subdivision of V_k, $\mathcal{W}_k$, or $K_{3,k}$.*

We will also use the following result of Oxley [15].

Lemma 6.2.2. *Let N be a 3-connected binary matroid having rank and corank at least three and suppose $\{x, y, z\} \subseteq E(N)$. Then N has a minor isomorphic to $M(K_4)$ whose ground set contains $\{x, y, z\}$.*

The proof of our main result will occupy the rest of this chapter.

Proof of Theorem 6.1.1. Let k be an integer exceeding three. Let $f_{3.1.3}$ and $f_{6.2.1}$ be the functions described in Theorems 3.1.3 and 6.2.1, respectively. Let $s = f_{3.1.3}(k) + f_{6.2.1}(k) + 11$. Let $m = \lceil (k+2)\frac{1}{3} f_{6.2.1}(k) \rceil$ and let $l = \max\{\binom{s}{3}(k+3), 2(2m+1)\}$. Let $t = (s-1) f_{2.1.4}(l)$. Set $f_{6.1.1}(k) = t$. Let M be a 3-connected regular matroid with at least t elements. Then $t \geq 11$.

By Lemma 6.1.9, M has a good decomposition into matroids each of which is graphic or cographic and has a 3-connected simplification with at least nine elements. By Lemma 6.1.10, we retain a good decomposition satisfying these additional conditions if we contract, one by one, the edges between vertices labelling graphic matroids. Let the resulting good decomposition be $(M_1, \Delta_2, M_2, \Delta_3, \ldots, \Delta_n, M_n)$, and let T be the tree associated with this decomposition.

By Lemma 6.1.7, for each i, the matroid $\mathrm{si}(M_i)$ is 3-connected. Suppose that some such $\mathrm{si}(M_i)$ has at least s elements. By Lemma 6.1.17, $\mathrm{si}(M_i)$ is isomorphic to a parallel minor

N of M. If N is graphic, then, by Theorem 3.1.3, M has a parallel minor isomorphic to $M(K'_{3,k})$, $M(\mathcal{W}_k)$, $M(DF_k)$, or $M(K_k)$, and the theorem holds. If, instead, N is cographic, then, by Theorem 6.2.1, N^* has a series minor isomorphic to $M(K_{3,k})$, $M(V_k)$, or $M(\mathcal{W}_k)$. Thus N, and hence M, has a parallel minor isomorphic to $M^*(K_{3,k})$, $M(DF_k)$, or $M(\mathcal{W}_k)$, and again the theorem holds.

We may now assume that no vertex of T labels a matroid whose simplification has at least s elements. As $|E(M)| \leq \sum_{i=1}^{n} |E(\mathrm{si}(M_i))|$, we have $n > \frac{t}{s-1} = f_{2.1.4}(l)$.

Suppose next that T contains a vertex M_i of degree at least l. We will show that M has a parallel minor isomorphic to $M(K'_{3,k})$. Since $\mathrm{si}(M_i)$ has fewer than s elements, $\mathrm{si}(M_i)$ has fewer than $\binom{s}{3}$ triangles. As M_i has degree at least l, for some triangle S in $\mathrm{si}(M_i)$, at least $l/\binom{s}{3}$ of the matroids labelling vertices adjacent with the vertex M_i have a triangle that is parallel to S in $M^P_{[n]}$. Clearly $j > i$ for all but at most one neighbor M_j of M_i in T. By definition, $l/\binom{s}{3} \geq k+3$. Take a subgraph T' of T induced by M_i and $k+2$ of its higher-indexed neighbors, $M_{i_1}, M_{i_2}, \ldots, M_{i_{k+2}}$, that contain triangles parallel to S. By Lemma 6.1.17, the simplification of the matroid M' associated with T' is isomorphic to a parallel minor Q of M. For convenience, we relabel M_i, M_{i_j}, and Δ_{i_j} as M_0, M_j, and Δ_j. Then $V(T') = \{M_0, M_1, \ldots, M_{k+2}\}$.

By Lemma 6.2.2, for all j in $\{1, 2, \ldots, k+2\}$, the matroid M_j has an $M(K_4)$-minor M'_j having Δ_j as a triangle. Because M_j has no Fano-minor, by the Scum Theorem, M'_j is a parallel minor of M_j. Take two distinct elements s_1 and s_2 in S and extend $\{s_1, s_2\}$ to a basis B of M_0. Let $M'_0 = M_0/(B - \{s_1, s_2\})$. Now, by Corollary 6.1.14, Q has $(M'_0, \Delta_1, M'_1, \Delta_2, \ldots, \Delta_{k+2}, M'_{k+2})$ as a parallel minor N. Moreover, $\mathrm{si}(N)$ can be obtained by identifying a triangle in each of $k+2$ matroids isomorphic to $M(K_4)$ and then possibly deleting some of the elements of the identified triangle. When all three elements of this triangle are deleted, we get $M(K_{3,k+2})$. Thus N, and hence M, has a parallel minor isomorphic to $M(K'_{3,k})$.

We may now suppose that every vertex of T has degree at most $l-1$. By Theorem 2.1.4, T contains a path $M_{i_1}M_{i_2}\dots M_{i_l}$ with l vertices. By construction, there is some index j such that $i_1 > i_2 > \cdots > i_j$ and $i_j < i_{j+1} < \cdots < i_l$. By definition, $\frac{l}{2} \geq 2m+1$, so T contains a path T' with at least $2m+1$ vertices such that the indices on the vertices are increasing. Since no two adjacent vertices of this path label graphic matroids, by removing vertices from the ends of the path, we can get a path T' with $2m$ vertices such that the first vertex of T' labels a non-graphic matroid. For convenience, we relabel the vertices of T' so that $T' = M_1M_2\dots M_{2m}$ and we relabel each edge M_iM_{i+1} as Δ_{i+1}. Let $M' = (M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{2m}, M_{2m})$ and $\bar{M} = \mathrm{si}(M')$. By Lemma 6.1.17, $\bar{M}$ is 3-connected and is isomorphic to a parallel minor of M. We can modify the decomposition we have for M' to obtain a good decomposition for $\bar{M}$ by *deleting superfluous parallel elements*. Specifically, we replace each M_i by its restriction to the set $(E(\bar{M} \cap E(M_i)) \cup (\Delta_i \cup \Delta_{i+1}))$. Note that Δ_1 and Δ_{2m+1} do not exist so we take these sets to be empty. This process gives us a good decomposition of $\bar{M}$ for which we will retain the labelling $(M_1, \Delta_2, M_2, \Delta_3, \dots, \Delta_{2m}, M_{2m})$.

Next we prove two lemmas to deal with this kind of situation. Let N be a 3-connected regular matroid having $(N_1, \Delta_2, N_2, \Delta_3, \dots, \Delta_d, N_d)$ as a good decomposition such that the associated tree is a path $N_1N_2\dots, N_d$; each $\mathrm{si}(N_i)$ has at least nine elements and is graphic or cographic, with no two consecutive matroids being graphic; and N_1 is not graphic. We call such a good decomposition a *fine decomposition* of N. Note that, in a fine decomposition, every non-trivial parallel class of each N_i meets Δ_i or Δ_{i+1}. When $(N_1, \Delta_2, N_2, \Delta_3, \dots, \Delta_d, N_d)$ is a fine decomposition of N, if $1 < i < d$, we denote $(N_1, \Delta_2, N_2, \dots, \Delta_{i-i}, N_{i-1})$ and $(N_{i+1}, \Delta_{i+2}, N_{i+2}, \dots, \Delta_d, N_d)$ by $\hat{N}_{i-1}$ and $\check{N}_{i+1}$. As a graph, the *triangular prism* consists of the vertices and edges of the triangular prism polyhedron. This graph is the planar dual of the graph $K_5\backslash e$.

Lemma 6.2.3. *Let $(N_1, \Delta_2, N_2, \Delta_3, \dots, \Delta_d, N_d)$ be a fine decomposition of a 3-connected regular matroid. For all i with $1 < i < d$, one of the following occurs:*

(i) N_i *is graphic and* $E(N_i)$ *has a subset* Y_i *such that* N_i/Y_i *is a multi-triangle with respect to* Δ_i *and* Δ_{i+1}*;*

(ii) N_i *is the cycle matroid of a triangular prism, and* N_{i-1} *and* $\hat{N}_{i-1}$ *have no triads meeting* Δ_i*, while* N_{i+1} *and* $\check{N}_{i+1}$ *have no triads meeting* Δ_{i+1}*;*

(iii) N_i *is not graphic and* $N_i = M^*(G_i)$ *for some multigraph* G_i *where* Δ_i *and* Δ_{i+1} *are vertex bonds of* G_i*; or*

(iv) N_i *is cographic but not graphic and* $E(N_i)$ *has a subset* Y_i *such that* N_i/Y_i *is a multi-*K_4 *with respect to* Δ_i *and* Δ_{i+1}*.*

Proof. If Δ_i and Δ_{i+1} are parallel in N_i, then Lemma 6.2.2 implies that $E(N_i)$ has a subset Y_i such that N_i/Y_i is a multi-K_4 with respect to Δ_i and Δ_{i+1}. Thus (i) or (iv) holds depending on whether or not N_i is graphic. We may now assume that Δ_i and Δ_{i+1} are not parallel in N_i.

Suppose that N_i is graphic and let G_i be the 3-connected multigraph such that $M(G_i) = N_i$. By Theorem 1.2.1, G_i has three vertex-disjoint paths, P_1, P_2, and P_3, from $V(\Delta_i)$ to $V(\Delta_{i+1})$.

We assume first that $G_i\backslash(E(\Delta_i) \cup E(\Delta_{i+1}))$ has a component C that contains at least two of the chosen paths. Then $G_i\backslash(E(\Delta_i) \cup E(\Delta_{i+1}))$ contains a path R with ends in two different chosen paths and no other vertices in any chosen path. Evidently, G_i has a multi-triangle as a minor whose restriction to each of $E(\Delta_i)$ and $E(\Delta_{i+1})$ is a triangle. By Lemma 6.1.15, $E(N_i)$ contains a set Y_i such that N_i/Y_i is a multi-triangle with respect to Δ_i and Δ_{i+1}, and (i) holds.

We may now assume that $G_i\backslash(E(\Delta_i) \cup E(\Delta_{i+1}))$ has three disjoint components each containing one chosen path. Since G_i is 3-connected, no P_i has an internal vertex since its ends do not form a vertex cut. Thus $V(G_i) = V(P_1) \cup V(P_2) \cup V(P_3)$. If G_i has a non-trivial parallel class, then this class meets Δ_i or Δ_{i+1}, and (i) holds with $Y_i = P_1 \cup P_2 \cup P_3$. Thus we may assume that G_i is simple. Then $|E(G_i)| = |E(\mathrm{si}(N_i))| \geq 9$, and it follows that G_i is a triangular prism.

Let $\{x_1, x_2, x_3\} = E(N_i) - (\Delta_i \cup \Delta_{i+1})$. By Lemma 6.1.10, $N_{i-1} \oplus_{\Delta_i} N_i$ and $\hat{N}_{i-1} \oplus_{\Delta_i} N_i$ have no series pairs. Thus N_{i-1} and $\hat{N}_{i-1}$ have no triads meeting Δ_i. Similarly, N_{i+1} and $\check{N}_{i+1}$ have no triads meeting Δ_{i+1}, and (ii) holds.

We may now assume that N_i is not graphic. Then N_i is cographic and so too is $\mathrm{si}(N_i)$. Hence $\mathrm{si}(N_i) = M^*(H_i)$ for some 3-connected simple multigraph H_i. Now Δ_i and Δ_{i+1} are not parallel in N_i. Thus $r(\Delta_i \cup \Delta_{i+1})$ is 3 or 4. Hence we can choose H_i so that either both Δ_i and Δ_{i+1} label bonds of it, or so that Δ_i and $(\Delta_{i+1} - e_{i+1}) \cup e_i$ label bonds of it where $\{e_i, e_{i+1}\}$ is a circuit of N_i with each e_j in Δ_j. Consider the bonds Δ_i and Δ'_{i+1} of H_i where Δ'_{i+1} is Δ_{i+1} or $(\Delta_{i+1} - e_{i+1}) \cup e_i$. Suppose first that both Δ_i and Δ'_{i+1} are vertex bonds. Then, by replacing edges of H_i by paths if necessary, we can get a multigraph G_i such that $N_i = M^*(G_i)$ and Δ_i and Δ_{i+1} are both vertex bonds of G_i. Thus (iii) holds.

It remains to consider when Δ_i or Δ'_{i+1} is not a vertex bond of H_i. By Lemma 6.1.18, H_i has a subgraph J that is a subdivision of K_4 such that J has a degree-three vertex v so that $\Delta_i \cup \Delta'_{i+1}$ is contained in the union of the minimal paths in J from v to the other degree-three vertices of J. If $\Delta'_{i+1} \neq \Delta_{i+1}$, then form J' from J by replacing e_i by a 2-edge path $\{e_i, e_{i+1}\}$; otherwise let J' be J. Then $M^*(J')$ is a minor of N_i. By Lemma 6.1.15, $E(N_i)$ has a subset Y_i such that N_i/Y_i is a multi-K_4 with respect to Δ_i and Δ_{i+1}, and (iv) holds. $\square$

We will say that N_i is *type (i)* if it meets the conditions of (i) in the preceding lemma. Likewise, we will say that N_i is *type (ii)*, *type (iii)*, or *type (iv)* if it meets the conditions of (ii), (iii), or (iv), respectively. The goal of the next lemma is to eliminate the graphic matroids in a fine decomposition. The strategy of the proof is as follows. Suppose that $(N_1, \Delta_2, N_2, \Delta_3, \ldots, \Delta_d, N_d)$ is a fine decomposition of a 3-connected regular matroid and that N_i is graphic for some i other than 1 or d. By the preceding lemma, N_i is type (i) or type (ii). In the latter case, it is straightforward to eliminate N_i by replacing it by a double-triangle. But if N_i is type (i), then replacing it with the multi-triangle N_i/Y_i may create a series pair in the underlying matroid. In particular, this will occur if every pair of elements in Δ_i is in a triad in both N_{i-1} and N_{i+1} and N_i/Y_i contains exactly seven elements. When such a pair arises, we will need to contract an element, say a, from this pair to preserve the vertical 3-connectivity of the matroid we are working with.

Lemma 6.2.4. *Let $(N_1, \Delta_2, N_2, \Delta_3, \ldots, \Delta_d, N_d)$ be a fine decomposition of a 3-connected regular matroid N. For some i with $2 \leq i \leq d-1$, suppose $N_1, N_2, \ldots, N_{i-1}$ are not graphic. When N_i is type (i), let N'_i be a contraction of N_i that is a multi-triangle with respect to Δ_i and Δ_{i+1}. When N_i is type (ii), let N'_i be the double triangle obtained by contracting each element not in a triangle of N_i. Then either $(N_1, \Delta_2, N_2, \Delta_3, \ldots, \Delta_i, N'_i, \Delta_{i+1}, \ldots, \Delta_d, N_d)$ is vertically 3-connected, or there is an element a of $E(N_j) - (\mathrm{cl}_{N_j}(\Delta_j) \cup \mathrm{cl}_{N_j}(\Delta_{j+1}))$ for some $j \leq i-1$ such that*

$$(N_1, \Delta_2, N_2, \ldots, \Delta_j, N_j/a, \Delta_{j+1}, \ldots, N_{i-1}, \Delta_i, N'_i, \Delta_{i+1}, \ldots, \Delta_d, N_d)$$

is vertically 3-connected, and N_j/a is not graphic.

Proof. By Lemma 6.1.17, both $\hat{N}_{i-1}$ and $\check{N}_{i+1}$ are vertically 3-connected. We show first that:

6.2.4.1. *Either $(\hat{N}_{i-1}, \Delta_i, N'_i, \Delta_{i+1}, \check{N}_{i+1})$ is vertically 3-connected, or that there is an element a of $E(\hat{N}_{i-1}) - \Delta_i$ such that $(\hat{N}_{i-1}/a, \Delta_i, N'_i, \Delta_{i+1}, \check{N}_{i+1})$ is vertically 3-connected.*

Now N'_i is either a double triangle with ground set $\Delta_i \cup \Delta_{i+1}$, or it is obtained from this matroid by adding some elements in parallel with elements of Δ_{i+1}. In both cases, we let $\hat{N}'_{i-1} = \hat{N}_{i-1} \oplus_{\Delta_i} N'_i$. Then $\hat{N}'_{i-1}$ may be obtained from $\hat{N}_{i-1}$ by relabelling the elements of Δ_i by the appropriate elements in Δ_{i+1} and, when N'_i is type (i), adding some non-empty set of elements in parallel with those of Δ_{i+1}. Let $\bar{N}$ be the matroid $P_{\Delta_{i+1}}(\hat{N}'_{i-1}, \check{N}_{i+1})$. Then every non-trivial parallel class of $\bar{N}$ meets Δ_{i+1}. Let $\Delta_{i+1} = \{x, y, z\}$. We will distinguish the following two cases:

(a) no element of Δ_{i+1} is in a non-trivial parallel class of $\bar{N}$; and

(b) some element, say z, of Δ_{i+1} is in a non-trivial parallel class of $\bar{N}$.

Observe that if N_i is type (i), then (b) holds.

Assume first that (a) holds. Then N_i is type (ii), so $\hat{N}'_{i-1}$ has no triad meeting Δ_{i+1} because $\hat{N}_{i-1}$ has no triad meeting Δ_i. Moreover, $\bar{N}$ is simple and, since it is the generalized parallel connection across a triangle of two 3-connected matroids, it too is 3-connected. Let C^* be a cocircuit of $\bar{N}$ meeting Δ_{i+1}. Then $|C^* \cap \Delta_{i+1}| = 2$. Furthermore, as $C^* \cap E(\hat{N}'_{i-1})$ and $C^* \cap E(\check{N}_{i+1})$ contain cocircuits of $\hat{N}'_{i-1}$ and $\check{N}_{i+1}$, it follows that both $|C^* \cap E(\hat{N}'_{i-1})|$ and $|C^* \cap E(\check{N}_{i+1})|$ exceed 3, so $|C^*| \geq 6$. Thus, if $Z \subseteq \Delta_{i+1}$, then $\bar{N}\backslash Z$ has no 2-cocircuits. Since $\bar{N}/x$ has a non-minimal 2-separation, it follows, by a well-known result of Bixby [1] (see also [13, Proposition 8.4.6]), that $\bar{N}\backslash x$ is 3-connected. Similarly, $\bar{N}\backslash x/y$ and $\bar{N}\backslash x, y/z$ have non-minimal 2-separations, so $\bar{N}\backslash x, y$ is 3-connected and then so is $\bar{N}\backslash x, y, z$. Hence, in case (a), $(\hat{N}_{i-1}, \Delta_i, N'_i, \Delta_{i+1}, \check{N}_{i+1})$ is vertically 3-connected.

Now assume that (b) holds. Then $\bar{N}$ has $\{e, z\}$ as a 2-circuit for some element e, so $\mathrm{si}(\bar{N}\backslash z)$ is 3-connected. We will show next that $\mathrm{si}(\bar{N}\backslash z, y)$ is 3-connected. Suppose not. Then y is not in a 2-circuit of $\bar{N}$. Clearly $\mathrm{si}(\bar{N}\backslash z)/y$ has a non-minimal 2-separation. Thus, by Bixby's Lemma, $\mathrm{co}(\mathrm{si}(\bar{N}\backslash z)\backslash y)$ is 3-connected, that is, $\mathrm{co}(\mathrm{si}(\bar{N}\backslash z, y))$ is 3-connected. As $\mathrm{si}(\bar{N}\backslash z, y)$ is not 3-connected, $\mathrm{si}(\bar{N}\backslash z)\backslash y$ has a 2-cocircuit. Thus $\mathrm{si}(\bar{N}\backslash z)$ has a triad C^* containing y. As each of $\mathrm{si}(\hat{N}'_{i-1})$ and $\mathrm{si}(\check{N}_{i+1})$ is a restriction of $\mathrm{si}(\bar{N}\backslash z)$, and either $C^* \cap E(\mathrm{si}(\hat{N}'_{i-1}))$ or $C^* \cap E(\mathrm{si}(\check{N}_{i+1}))$ has exactly two elements, we deduce that $\mathrm{si}(\hat{N}'_{i-1})$ or $\mathrm{si}(\check{N}_{i+1})$ has a cocircuit with at most two elements; a contradiction. Thus $\mathrm{si}(\bar{N}\backslash z, y)$ is indeed 3-connected.

Now $\mathrm{si}(\bar{N}\backslash z, y)/x$ has a non-minimal 2-separation. Thus, by Bixby's Lemma again, $\mathrm{co}(\mathrm{si}(\bar{N}\backslash z, y)\backslash x)$ is 3-connected. As $\mathrm{si}(\bar{N}\backslash z, y, x) \cong \mathrm{si}(P(\hat{N}'_{i-1}, \check{N}_{i+1})\backslash \Delta_{i+1})$, we assume that $\mathrm{si}(\bar{N}\backslash z, y, x)$ is not 3-connected, otherwise the lemma holds. Then

6.2.4.2. *$\bar{N}$ has no 2-circuit containing x or y.*

As $\operatorname{si}(\bar{N}\backslash z, y)$ is 3-connected, $\bar{N}$ has no 2-circuit containing x. By symmetry, $\bar{N}$ has no 2-circuit containing y.

Now $\operatorname{si}(\bar{N}\backslash z, y)$ must have a triad containing x. Assume that $\{a, b, x\}$ and $\{c, d, x\}$ are such triads. Then their symmetric difference is a disjoint union of cocircuits of $\operatorname{si}(\bar{N}\backslash z, y)$. Thus $\{a, b\} \cap \{c, d\} = \emptyset$. Now $\operatorname{si}(\bar{N}\backslash z)\backslash y$ is 3-connected. Therefore $\{a, b, x, y\}$ and $\{c, d, x, y\}$ contain cocircuits of $\operatorname{si}(\bar{N}\backslash z)$ containing $\{a, b, x\}$ and $\{c, d, x\}$. By considering the intersections of these cocircuits with $E(\operatorname{si}(\hat{N}'_{i-1}))$ and $E(\operatorname{si}(\check{N}_{i+1}))$, we see that each such cocircuit has four elements. Moreover, we may assume that the first contains $\{a, c\}$ and the second contains $\{b, d\}$. Thus $\{a, x, y\}$ and $\{c, x, y\}$ are cocircuits of $\operatorname{si}(\hat{N}'_{i-1})$. Hence $\operatorname{si}(\hat{N}'_{i-1})$ has a cocircuit contained in $\{a, c\}$; a contradiction. We deduce that $\operatorname{si}(\bar{N}\backslash z, y)$ has exactly one triad, say $\{a, b, x\}$, containing x. Moreover, we may assume that $\{a, x, y\}$ and $\{b, x, y\}$ are triads of $\operatorname{si}(\hat{N}'_{i-1})$ and $\operatorname{si}(\check{N}_{i+1})$, respectively.

6.2.4.3. *$\hat{N}'_{i-1}$ has no 2-circuit containing a.*

If a is in a 2-circuit of $\hat{N}'_{i-1}$, then, by 6.2.4.2, a is parallel to z. Thus $\{a, x, y\}$ is both a triangle and a triad of $\operatorname{si}(\hat{N}'_{i-1})$; a contradiction.

By 6.2.4.2 and 6.2.4.3, $\{a, x, y\}$ is a triad of $\hat{N}'_{i-1}$. Since $\{a, b\}$ is the only 2-cocircuit of $\operatorname{si}(\hat{N}'_{i-1} \oplus_{\Delta_{i+1}} \check{N}_{i+1})$, the matroid $\operatorname{si}(\hat{N}'_{i-1} \oplus_{\Delta_{i+1}} \check{N}_{i+1})/a$ is 3-connected, so $\operatorname{si}((\hat{N}'_{i-1}/a) \oplus_{\Delta_{i+1}} \check{N}_{i+1})$ is 3-connected. This completes the proof of 6.2.4.1.

Observe that the construction of $\hat{N}'_{i-1}$ means that we can label the triangle Δ_i of N_{i-1} by $\{x_i, y_i, z_i\}$ where $\{x, x_i\}$, $\{y, y_i\}$, and $\{z, z_i\}$ are circuits of N'_i. Clearly $\hat{N}_{i-1}$ can be obtained from $\hat{N}'_{i-1}$ by first relabelling the elements x, y, and z of the latter as x_i, y_i, and z_i and then deleting some elements that are parallel to x_i, y_i, or z_i. By 6.2.4.2 and 6.2.4.3, none of a, x, or y is in a 2-circuit of $\hat{N}'_{i-1}$. Hence none of a, x_i, or y_i is in a 2-circuit of $\hat{N}_{i-1}$. Moreover, as $\{a, x, y\}$ is a triad of $\hat{N}'_{i-1}$, and $\operatorname{si}(\hat{N}_{i-1})$ is 3-connected, $\{a, x_i, y_i\}$ is a triad of $\hat{N}_{i-1}$.

For all p with $2 \leq p \leq i-1$, let $\Delta_p = \{x_p, y_p, z_p\}$. Now $\hat{N}_{i-1} = P_{\Delta_{i-1}}(\hat{N}_{i-2}, N_{i-1})\backslash\Delta_{i-1}$. Since $\{a, x_i, y_i\}$ is a triad of $\hat{N}_{i-1}$, either $\{a, x_i, y_i\}$ is a triad of N_{i-1}; or $\{a, x_i, y_i\}$

$\cup Z$ is a cocircuit of $P_{\Delta_{i-1}}(\hat{N}_{i-2}, N_{i-1})$ for some 2-element subset Z of Δ_{i-1}. In the latter case, we may assume that $Z = \{x_{i-1}, y_{i-1}\}$. Then $\{a, x_{i-1}, y_{i-1}\}$ contains, and so is, a cocircuit of $\hat{N}_{i-2}$. By repeating this argument, we deduce that, for some j with $1 \leq j \leq i-1$, after possibly relabelling the elements of Δ_{j+1}, we have $\{a, x_{j+1}, y_{j+1}\}$ as a triad of N_j.

Next we will show that a is not in the closure of Δ_j or Δ_{j+1} in N_j. Note that, when $j = 1$, the set Δ_j is empty. We have $\{a, x_{j+1}, y_{j+1}\}$ as a triad of N_j. If N_j has a circuit containing a and contained in $a \cup \Delta_j$, then we contradict orthogonality. If N_j has a circuit containing a and contained in $a \cup \Delta_{j+1}$, then a is parallel to some element of Δ_{j+1}. Thus $\mathrm{si}(N_j)$ has a 2-cocircuit, a contradiction since $\mathrm{si}(N_j)$ is 3-connected having at least nine elements.

We now show that N_j/a is not graphic. Assume it is and let G be a multigraph such that $M(G) = N_j^*$. Since $\{a, x_{j+1}, y_{j+1}\}$ is a triad of N_j, it is a triangle of G. As $\{x_{j+1}, y_{j+1}, z_{j+1}\}$ is a triad of $M(G)$, the vertex v common to x_{j+1} and y_{j+1} has degree 3. Since N_j is not graphic, G has a minor isomorphic to K_5 or $K_{3,3}$. Assume first that G has a $K_{3,3}$-minor. Since $K_{3,3}$ is cubic, G contains a subgraph H that is a subdivision of $K_{3,3}$. As $M^*(G\backslash a)$ is graphic, $G\backslash a$ has no $K_{3,3}$-minor. Thus a is in H. Since H has no triangles, at most one of x_{j+1} and y_{j+1} is in H. Either v has degree two in H, or v is not in $V(H)$. In each case, by interchanging x_{j+1} and y_{j+1} if necessary, we get that G/x_{j+1} has a $K_{3,3}$-minor. But $\{a, y_{j+1}\}$ is a cycle of G/x_{j+1}, so $G/x_{j+1}\backslash a$ has a $K_{3,3}$-minor. Hence so does $G\backslash a$; a contradiction.

We may now assume that G has a K_5-minor. Then G has five disjoint connected subgraphs G_1, G_2, G_3, G_4, and G_5 that together contain all of the vertices in G and such that G has at least one edge between every pair of these subgraphs. Suppose first that a is in G_1. Then two of the three neighbors of v are in G_1, and we may assume that v is in G_1. Hence x_{j+1} and y_{j+1} are in G_1. Then $G_1\backslash a$ is connected, since $\{a, x_{j+1}, y_{j+1}\}$ is a triangle, and $G\backslash a$ contains a minor isomorphic to K_5; a contradiction. Finally, assume that a is a G_1-G_2-edge. If x_{j+1} or y_{j+1} is a G_1-G_2-edge, then $G\backslash a$ has a minor isomorphic to K_5. In the exceptional case, without loss of generality, we may assume that x_{j+1} is a G_2-G_3-edge and y_{j+1} is a G_3-G_1-

edge. Then v is in G_3. Since v has degree three in G, it has degree one in the multigraph G_3. Hence $G_3 - v$ is a connected multigraph and, for each i in $\{4, 5\}$, there is an edge of G with one end in $G_3 - v$ and the other in G_i. We contract the subgraphs G_1, G_2, $G_3 - v$, G_4, and G_5 to vertices v_1, v_2, v_3, v_4, and v_5, respectively, and delete the edge a. The resulting 6-vertex multigraph has $K_{3,3}$ as a subgraph, where the vertex classes are $\{v_1, v_2, v_3\}$ and $\{v, v_4, v_5\}$. Thus $G\backslash a$ has a $K_{3,3}$-minor; a contradiction. We conclude that N_j/a is not graphic and the lemma is proved. $\square$

Now returning to the proof of the main theorem, recall that, immediately before Lemma 6.2.3, we showed that we could obtain a fine decomposition $(M_1, \Delta_2, M_2, \Delta_3, \ldots, \Delta_{2m}, M_{2m})$ of a 3-connected matroid $\bar{M}$ that is isomorphic to a parallel minor of M. Each M_i with $1 < i < 2m$ satisfies one of (i)-(iv) of Lemma 6.2.3.

Suppose that some matroid in the path $M_1 M_2 \ldots M_{2m}$ is graphic. In that case, let M_i be the lowest-indexed graphic matroid. Then $i > 1$, so M_i labels a type (i) or type (ii) matroid. By Lemma 6.2.4, we may contract elements from M_i to obtain a matroid M_i' that is a double triangle or a multi-triangle containing Δ_i and Δ_{i+1}, and we may contract at most one element of some M_j with $j \leq i - 1$ to obtain a non-graphic matroid M_j'' such that

$$(M_1, \Delta_2, M_2, \ldots, \Delta_j, M_j'', \Delta_{j+1}, \ldots, M_{i-1}, \Delta_i, M_i', \Delta_{i+1}, \ldots, \Delta_{2m}, M_{2m}) \tag{6.3}$$

is vertically 3-connected. Now let M_{i-1}'' be M_j'' when $j = i - 1$ and let $M_{i-1}'' = M_{i-1}$ when $j < i - 1$. Then M_{i-1}'' is cographic but not graphic. Hence $(M_{i-1}'', \Delta_i, M_i')$ is also cographic but not graphic. Thus, in (6.3), we replace $M_{i-1}'', \Delta_i, M_i'$ by $(M_{i-1}'', \Delta_i, M_i')$. This gives a good decomposition of a vertically 3-connected matroid whose simplification is a parallel minor $\bar{M}'$ of M. We can convert this good decomposition into a fine decomposition for $\bar{M}'$ by deleting superfluous parallel elements. This means that we can repeat the above process. Thus, from our original fine decomposition, we eliminate graphic matroids one by one, beginning with the lowest-indexed such matroid. After each such move, we recover a fine decomposition of a 3-connected parallel minor of M. Since no two consecutive matroids in $M_1, M_2, \ldots, M_{2m}$ are graphic and M_1 is non-graphic, we eventually obtain a fine decomposition for which the corresponding path has at least $m+1$ vertices, where each vertex except possibly the last labels a cographic matroid that is not graphic. If this path ends in a graphic matroid, that matroid has been unaltered in the above process and so its simplification has at least nine elements. Hence we can apply Lemma 6.1.17 and remove at least one vertex from the end of this path to obtain a path Q with m vertices each of which is labelled by a cographic matroid that is not graphic. Again by deleting superfluous parallel elements, we may assume that M_Q, which is a parallel minor of M, is simple. Relabel Q as $N_1N_2\ldots N_m$. By Lemma 6.2.3, each N_i is type (iii) or type (iv).

Recall that $m = \lceil (k+2)\frac{1}{3}f_{6.2.1}(k)\rceil$. Suppose that Q contains a subpath Q' of at least $\lfloor \frac{1}{3}f_{6.2.1}(k)\rfloor$ vertices each of which is labelled by a matroid that is type (iii). Then it is not difficult to check that the associated matroid $M_{Q'}$ is cographic. Because each $\mathrm{si}(N_i)$ has at least nine elements, $\mathrm{si}(M_{Q'})$ has at least $f_{6.2.1}(k)$ elements and, by Lemma 6.1.17, $M_{Q'}$ is vertically 3-connected. Recalling that DF_k is the dual of V_k, we deduce by Theorem 6.2.1, that M has a parallel minor isomorphic to $M(DF_k)$, $M(\mathcal{W}_k)$, or $M^*(K_{3,k})$. Hence, in this case, Theorem 6.1.1 holds.

We may now assume that every subpath of Q with at least $\frac{1}{3}f_{6.2.1}(k)$ vertices contains a vertex labelled by a type (iv) matroid. Thus Q has at least $\lfloor m/(\frac{1}{3}f_{6.2.1}(k))\rfloor$ vertices that are labelled by type (iv) matroids, so Q has at least $k+2$ such vertices.

We now modify each N_i in Q to produce N_i' as follows. If N_i is type (iv), then it contains a set Y_i such that N_i/Y_i is a multi-K_4 with respect to Δ_i and Δ_{i+1}. In this case, we let $N_i' = N_i/Y_i$. Now suppose N_i is type (iii). Then $N_i = M^*(G_i)$ for some multigraph G_i that has Δ_i and Δ_{i+1} as vertex bonds. By Theorem 1.2.1, G_i has a subgraph H_i that is a subdivision of $K_{2,3}$ where Δ_i and Δ_{i+1} are vertex bonds of H_i. Thus N_i has, as a minor, a double triangle with ground set $\Delta_i \cup \Delta_{i+1}$. Hence, by the Scum Theorem, for some subset

Y_i of $E(N_i)$, the matroid N_i/Y_i is either this double triangle or a multi-triangle with respect to Δ_i and Δ_{i+1}. In this case, we take N_i' to be N_i/Y_i.

Let $R = N_2'N_3' \ldots N_m'$. Using Corollary 6.1.14 and Lemma 6.1.17, we can show that $\mathrm{si}(M_R)$ is a parallel minor of $\mathrm{si}(M_Q)$. Furthermore, M_R may be obtained by identifying at least $k+2$ copies of $M(K_4)$ across a triangle and either deleting elements from the common triangle or adding elements parallel with the elements in the common triangle. Evidently M_R, and hence M, has a parallel minor isomorphic to $M(K_{3,k}')$, and this completes the proof of the theorem. $\square$

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Vita

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