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Introduction of Advanced Math Concepts through a Geometry Project

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by

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Geometric Basis for Advanced Math

The nation is taking drastic steps in furthering the education of students of all ages. With new policies and standards, officials are attempting to create an equitable environment and make it possible for all students to attain a proper education. Education researchers are constantly creating new teaching methods that address all types of learners. The latest methods go along with the new Common Core standards in attempt to make the curriculum relatable and meaningful for students. Teachers are no longer expected to teach content directly, but instead they are linking new material to what students already know.

This project takes these ideas even further. In this project, advanced math concepts will be taught through a geometry project that students already understand. Teaching students through this project will not just allow them to make connections between advanced math concepts and geometry, but it will show students that these concepts are actually inseparable.

Linking new concepts to students' prior knowledge or to something relatable has become a focus in the classroom, and has proven to be extremely effective. However, after high school, advanced concepts often seem to be disjoint from those learned in high school. They may have been taught how certain concepts are related but they lack the skill to make the connections themselves. This is extremely apparent in the STEM fields. In fact, 38% of students entering college as a STEM major do not end up graduating as one. This is disappointing because in these fields, instructors have an opportunity not only to build on what students already know, but to show them just how connected everything is.

In this project, a problem solvable by geometry is introduced. This problem can easily be solved using trigonometry, complex variables, and calculus, but instead students will solve it geometrically and it will serve as a platform for these advanced concepts. There are eight lessons, three of which have a corresponding script, that guide students through this problem and only require students to have a background in geometry. During these lessons, the instructor is setting the framework for the introduction of advanced math topics while providing students with a model that they can always go back to when they use these concepts in

upper level math classes. The sample scripts provide examples of probing questions that the instructor may ask throughout the lesson to help students arrive at conclusions, instead of simply telling them the answers. These lessons are meant to be inquiry-based, and students should be constantly encouraged to make their own observations and reflect on what they have learned.

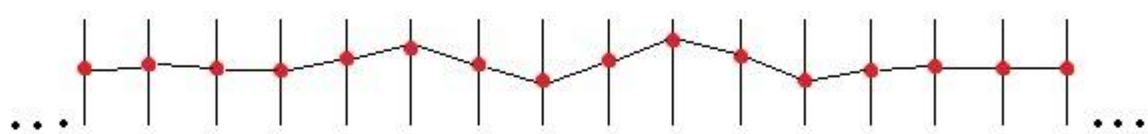
This project includes lessons that outline the introduction of trigonometry and complex variables. Students only need a background in geometry to solve this problem, so most likely students will not have experience with these advanced concepts. The goal of this project is to show students that these advanced topics are actually inseparable from the geometry concepts that they already know. The advanced math lessons introduce new concepts simply as new notation for what they students already did while solving the problem. This way, the topics are introduced in a context that students already understand and can easily be expanded on.

A Problem Solvable by Geometry

This project explores the idea of using concrete geometrical problems to teach higher math. A project on the motions of a string of connected beads is solved first by geometry alone. Then, higher math topics will flow naturally from the geometric solution.

The problem is quickly solvable using advanced math concepts such as derivatives, complex variables, and trigonometry. However, without a deep understanding of these concepts, arriving at the solution only requires symbolic manipulation. In reality, the symbols are describing geometric phenomena in a sophisticated way.

The students will envision a line of beads connected by strings, where each bead is on a rod. They will be asked to describe the forces that affect a single bead, which will result in them finding a formula that describes propagating waves in the system.



Once they understand the problem geometrically, the new concepts will be introduced simply as symbolic representations of the geometry. This way students don't see these advanced math concepts as scary new material, they just see it as notation that describes succinctly what they already understand. This model provides students with a relatable model that they can easily envision, and they can always go back to affirm their understanding of the geometry behind these advanced concepts.

Lessons on Solving the Problem by Geometry

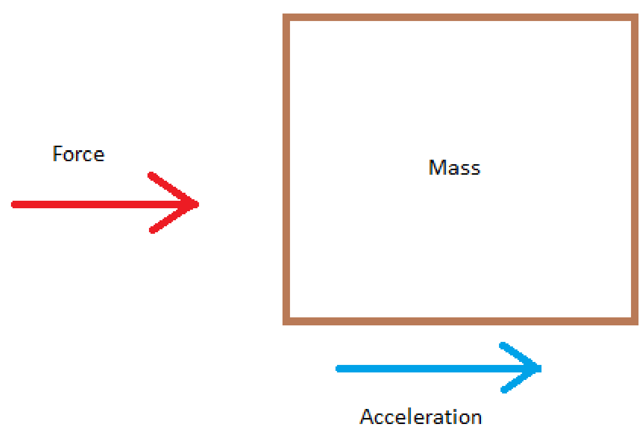
Lesson 1: Newton's 2nd Law

Topic: Newton's 2nd Law Grade Level: High School Created by: Emelie Mativi

Concepts: Students will apply their knowledge on force, velocity, and acceleration to understand Newton's 2nd Law.
Objectives: • Discuss everyday situations in which force, velocity, and acceleration are involved. • Understand the relationship between force, mass, and acceleration through experimentation.

Introduction

- Brief class discussion on meaning of **force, mass,** and **acceleration**
- Thought Experiment with a Box
    - Have students imagine a situation where they are pushing a box on a frictionless surface. Facilitate class discussion on what would happen if the force or the mass were changed.
    - Draw the situation on the board



Group Work

- Students will work in groups to finish the activity (attached) and discuss the relationships between force, mass, and acceleration
- Depending on classes' prior knowledge
    - Give class a few different possible relationships between force, mass and acceleration and ask them to choose OR
    - Have groups attempt to write a relationship themselves

Class Discussion

- Compile classes' conclusions from the group activity
- Discuss the relationship F=ma
    - Discuss why other relations do not work
- Ask class for real world situations that involve Newton's second law

Group Activity

Considering the box situation that was just discussed, answer the following questions

1. Suppose another box with the same mass is stacked on top of the original box, and the person pushing does not change the force at wish they are pushing. What happens to the acceleration?

2. Suppose two people begin pushing the box, each one pushing with the same force as before, and the mass stays constant. What happens to the acceleration?

3. Suppose the person pushing begins pushing a new box with the same force, but the acceleration is double that from the original situation. What can we conclude about the mass of the box?

After considered these situations, what is the correct relationship?

a. $Force = Mass \div Acceleration$
b. $Force = Mass + Acceleration$
c. $Force = Mass \times Acceleration$
d. $Force = Acceleration \div Mass$

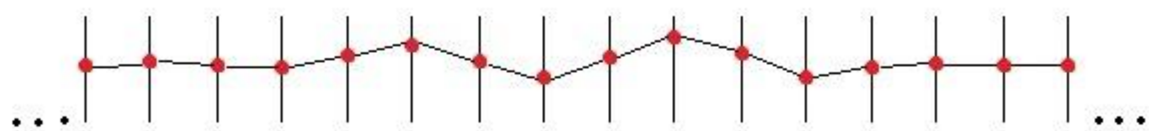
Lesson 2: Introduction to Lattices

Topic: Lattice System Grade Level: High School Created by: Emelie Mativi

Concepts:
Students will apply their knowledge on force, velocity, and acceleration to envision and try to describe motions that occur in a lattice system, specifically an infinite chain of beads connected by taut strings.
Objectives: • Discuss everyday situations in which force, velocity, and acceleration are involved. • Envision motions, simple and complex, would occur in a lattice model.

Introduction

- Review Newton's 2nd law
- Introduce the lattice system
    - Describe system in detail
        - Infinite lattice
        - Beads on rods- only one degree of freedom
        - Strings between each bead
    - Draw on board



Discussion

- Goal: to intuitively describe the factors that affect the motion of each bead.
    - We are going to explore a system where beads and strings are interacting and pulling on one another
    - We want to analyze what kind of movements we can set up in an infinite system
- Explain that we are looking at the factors that affect the motion of a single bead
- Have class make a list of these factors
    - Neighboring beads
        - How far apart they are
    - Tension in a string

    - Mass of the bead
- Tell class to begin thinking of how we would represent this in an equation

End of Class

- Begin labeling the lattice system if there is time left

Note that this lesson needs to be thorough, because this is the foundation of the unit. More time can be spent on this lesson if needed. See sample script

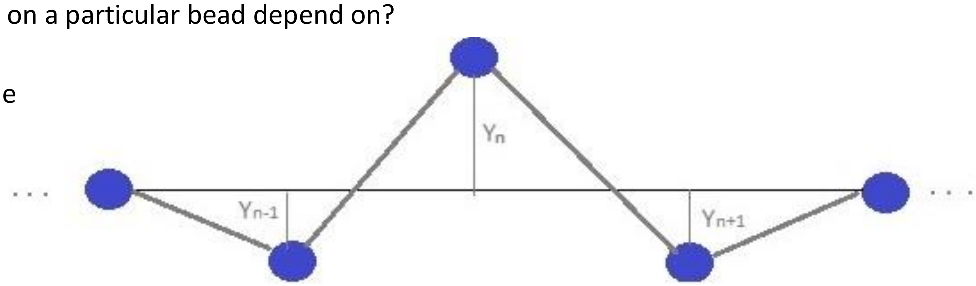
Lesson 3: Force Equations in a Lattice

Topic: Equations of Motion in Lattices   Grade Level: High School   Created by: Emelie Mativi

Concepts: Students will apply their knowledge from previous lessons to understand the relationship between position, velocity, acceleration and forces in a lattice system
Objectives: • Envision motions, simple and complex, in a lattice • Propose law to describe these motions mathematically

Introduction

- Draw and discuss the lattice system
  - What does the force on a particular bead depend on?
- Label lattice system
  - Explain every variable



Group Discussion

- Write down the formula for the force on a particular bead that comes from the relative displacement of neighboring beads

$$F_n = \tau(Y_{n+1} - Y_n) - \tau(Y_n - Y_{n-1})$$

- Don't discuss the formula, instead split class up into groups and have them discuss why this formula makes sense
  - Ask groups to come up with a statement to explain why the formula makes sense (they will be explaining this statement to the class).
  - Also ask them to discuss any questions they have about the lattice system. If they can't come up with an answer as a group then they may also ask this question to the whole class.

Class discussion

- Each group will read their explanation to the class
  - Class will discuss their explanations to decide which groups explained it well
- Each group asks any questions they have
  - Class will discuss the answers to the questions before the instructor answers
  - If class can't come to an answer the instructor will help restate the question more clearly
- Ask class what other equation we discussed that described force

    - Have students write a relationship between position and acceleration using the nearest neighbor relationship and Newton's law- this is the **master equation**
        - $mA_n = \tau(Y_{n+1} + Y_{n-1} - 2Y)$

End of Class (if time permits)

- Students should go home and think about how complicated motions could get in this system

Lesson 4: Using the Turntable to Describe Oscillatory Motion

Topic: Circular Motion   Grade Level: High School   Created by: Emelie Mativi

Concepts: Students will use turntable motion to simplify discussion of oscillatory, or constant frequency, motion.
Objectives: • Discuss turntable motion and envision its projection onto a wall • Explain why converting the motions of the beads in the lattice to circular motion simplifies the problem

Introduction

- When students walk in
  - Students must write the master equation and write a short explanation of the formula
- Discuss how the motions in the lattice system could get very difficult
- In order to simplify the motions in the lattice we are going to set some restrictions
  - We want all beads to oscillate at the same frequency
  - The beads can have different amplitude and phase shifts
  - We can use a concept called turntable motion to make oscillatory motion simple and geometric
- Introduce turntable motion
  - Ask class what they think of when they hear the word turntable
  - Have a brief class discussion on turntable motion and come up with a class definition

Teacher Demonstration/Explanation

- Explain why we are using turntable motion
  - We want to look at simple motions in the system
    - Don't explain exactly how this is going to simplify the system. Leave that for class discussion at the end
- If a turntable is available do an actual demonstration for the class. If not use an animation.
  - Tell class to imagine a bead glued to a turntable
  - Ask them to predict what the projection of the bead on the wall will look like
  - Demonstrate that the projection shows the bead moving up and down in a straight line

- Explain that in a lattice we have these harmonic (up and down) motions that look like our projections. In order to be able to geometrically describe the motions we want to think of each bead as its own turntable
- Discuss why this simplifies the system

Lesson 5: Position, Velocity, and Acceleration

Topic: Turntable Motion Grade Level: High School Created by: Emelie Mativi

Concepts: Students will apply their knowledge of position, velocity, and acceleration to discover a relationship between position and acceleration.
Objectives: • Discuss how turntable motion applies to position, velocity, and acceleration • Express the relationship between position and acceleration • Use the relationship to rewrite the master equation

Introduction

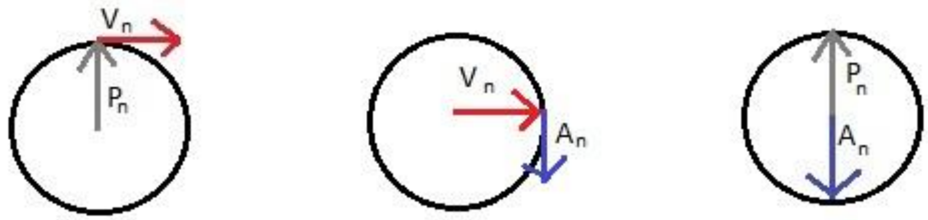
- When students walk in
  - Students must write the equation for the force on a particular bead and explain it
  - Students must also write a brief explanation of how turntable motion simplifies the system
- Briefly review the projection of a bead on a turntable discussed last class
- Review velocity and acceleration

Group Discussion

- Split students into groups and have them discuss the relationship between position and velocity
- Example activity
  - Give each group a transparency with a turntable on it
  - Draw a bead on four different places on each groups turntable
  - Instruct each group to draw an arrow where they think the bead would go if it suddenly became unglued in each of the spots marked on the turntable
  - When each group is finished and has done it correctly, stack the papers on top of each other, with the tail of velocity vectors at the same point, and show class
  - This will help them see that the velocity can be drawn on its own turntable
- Tell groups that acceleration can also be drawn on its own turntable
  - Have them predict what it will look like in relation to position
  - To help have them imagine they are on a carousal
    - Ask them to tell what way they feel like the carousal is pushing them
    - Ask them which way they pull to stay on

Class Discussion

- Draw the three turntable on the board
- Discuss why acceleration is the opposite of position
    - Relate this to the carousel discussion
- Show animation of turntables
- End class by asking class to think about how they would write a relationship between acceleration and position



Lesson 6: Position, Velocity, and Acceleration 2

Topic: Turntable Motion   Grade Level: High School   Created by: Emelie Mativi

Concepts: Students will apply what they learned about turntable motion during the previous lesson discover a relationship between position and acceleration.
Objectives: • Discuss how turntable motion can be used to describe position, velocity, and acceleration • Express the relationship between position and acceleration • Use the relationship to rewrite the master equation

Introduction

- When students walk in
  - Students must write the equation for the force on a particular bead and explain it
  - Students must state Newton's law
- Discuss the three different turntables drawn yesterday
  - Have 3 students come up and draw each turntable
    - Position with velocity tangent to circle
    - Velocity with acceleration tangent to circle
    - Acceleration with position as well
- Discuss the relationship between position and acceleration that students should have explored in the previous lesson

Lecture/Class Discussion

- Discussion on vectors and their projections

  ***Depth of this discussion depends on previous knowledge of the class

- Discuss how we can write the projection of velocity in terms of position
  - Call the magnitude of the position Y_n
  - The magnitude of the velocity vector is some frequency, ω, times Y_n
  - Relate back to the Merry-go-Round
- Discuss how the projection of acceleration can be written in terms of position
  - The projection of acceleration is ω times the projection of the velocity vector
  - So the projection of the acceleration vector is $-\omega^2 Y_n$
- Discuss how acceleration relates to position of a particular bead
  - Since acceleration is the opposite of position
    - $A_n = -\omega^2 Y_n$

End of Class

- Ask students to write a new master equation using the relationship between acceleration and position
- Tell students to bring their answers next class

Lesson 7: Propagating Motion

Topic: Turntable Motion Grade Level: High School Created by: Emelie Mativi

Concepts: Students will use circular motion to discover the frequencies at which propagation occurs in the lattice system
Objectives: • Envision the projection of the beads when each bead is displaced a certain angle from the one before it • Discuss propagation and discover the frequencies at which it occurs

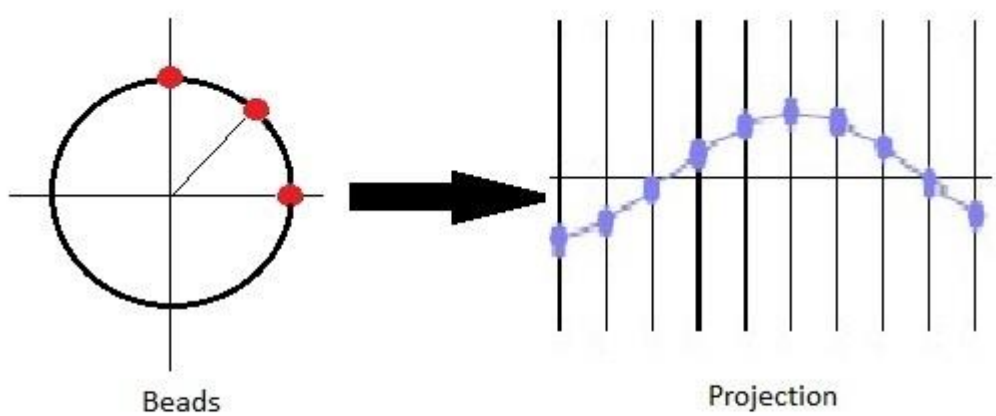
Introduction

- Discuss the master equation (This discussion needs to be thorough)
  - We had two equations that describe the force on a particular bead
    - $F_n = mA_n$
    - $F_n = \tau(Y_{n+1} + Y_{n-1} - 2Y_n)$
    - We can equate this two equations
  - We also had a relationship between acceleration and position
    - $A_n = -\omega^2 Y_n$
  - So we can write a sub-master equation that describes oscillatory motion at a constant frequency
    - $\omega^2 Y_n = -\frac{\tau}{m}(Y_{n+1} + Y_{n-1} - 2Y_n)$

Group/Class Discussion

- We thought of Y_n as a position vector with attributes amplitude and phase
- What will happen if we change the phase?
- Split class into groups
  - Imagine we have a lot of turntables with a bead on them. Each one is spinning at the same speed, but each bead is displaced by a certain angle θ from the previous bead
  - Predict what the turntables and their projection will look like if all these turntables are stacked on top of each other.
  - Now apply it to the lattice
    - What will it look like if we take a snapshot of the lattice?
- Give each group a whiteboard and marker and have them draw their predictions
- Have each group show their prediction and facilitate a class discussion on the correct answer
- Discuss propagation in the system

- There are certain frequencies at which waves are allowed to move through the lattice
    - Think of a jump rope with many knots in it all a certain distance apart (similar to our bead and string system)
    - If you are holding one end of the rope and the other end is tied to a pole, there are some frequencies that you can move the rope at so that the wave will go all the way through the rope. If you shake the rope too hard or too soft, then the wave will stop before it reaches the end of the rope.
    - The same thing happens in our lattice. We want to find those frequencies at which waves make it all the way through.



End of Class

- Tell class to think about how we could prove that solutions like this will work

Lesson 8: Solving the Master Equation Geometrically

Topic: Turntable Motion Grade Level: High School Created by: Emelie Mativi

Concepts:
Students will use basic vector addition to solve for a frequency band in which propagation is allowed

Objectives:

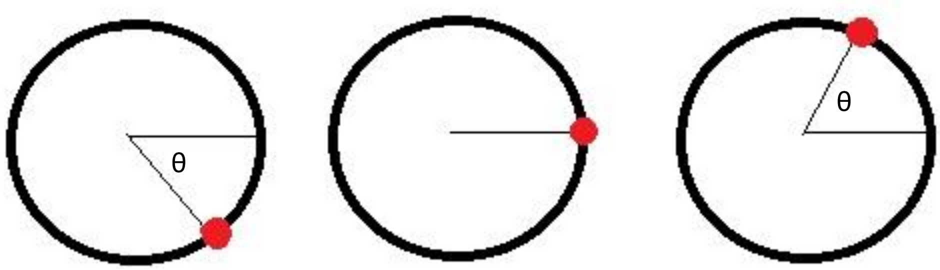
- Envision the projection of the beads when each bead is a particular angle θ away from the previous bead
- Discuss propagation and discover the frequencies at which it occurs

Introduction

- Recall: Last class we discussed the possibility of having multiple turntables, all displaced by a certain angle
    - What did the projection look like?
- We want to discover where propagation will occur.
    - Will it occur at really high frequencies? Really low frequencies? Is there an interval of frequencies in which propagation will occur?

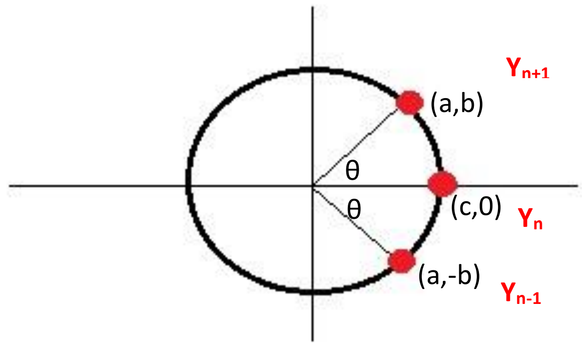
Discussion

- Draw a picture of three different turntables, each off by a phase of θ. We can label them Y_n, Y_{n-1}, Y_{n+1}.

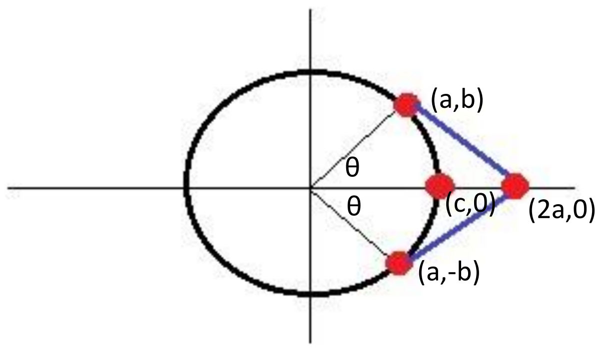


Y_{n-1} Y_n Y_{n+1}

- We want to use our master equation to find the frequencies that satisfy this kind of motion. Recall the master equation:
    - $\omega^2 Y_n = -\frac{\tau}{m}(Y_{n+1} + Y_{n-1} - 2Y_n)$
    - If we rearrange this it becomes easier to work with
        - $Y_{n+1} + Y_{n-1} + (\frac{\omega^2}{\alpha} - 2)Y_n = 0 \quad (\alpha = \frac{\tau}{m})$
    - Remember that the Y_n's are vectors, they had a horizontal and a vertical component. We can add these vectors to figure out the first part of this equation, $Y_{n+1} + Y_{n-1}$
    - First we will put all of these on one graph, and then give coordinate names to each of the points



- What would the coordinates of $Y_{n+1} + Y_{n-1}$ be?
  - (a+a, b-b) = (2a,0)
- What conclusions can be made?
  - Adding $Y_{n+1} + Y_{n-1}$ results in a point on the same line as Y_n
  - Since a<c, then 2a<2c, so $Y_{n+1}+Y_{n-1}$ will be never be greater than $2Y_n$



- If we use the same reasoning, we can also find what Y_n must always be greater than. What would this be?
  - $Y_{n+1} + Y_{n-1}$ will always be greater than $-2Y_n$
- Now we can use these inequalities along with the sub-master equation to get an interval of frequencies

Conclusion

- $Y_{n+1} + Y_{n-1} + (\frac{\omega^2}{\alpha} - 2)Y_n = 0$
  - $Y_{n+1} + Y_{n-1} = (2 - \frac{\omega^2}{\alpha})Y_n$
  - $-2Y_n < Y_{n+1} + Y_{n-1} < 2Y_n$
  - $-2Y_n < (2 - \frac{\omega^2}{\alpha})Y_n < 2Y_n$
- Propagation is allowed in the frequency interval $-2\sqrt{\alpha} < \omega < 2\sqrt{\alpha}$

Sample Scripts

Sample Script for Newton's Law Lesson

Introduction

- **I want you to imagine a situation in which you are pushing a box on a frictionless surface- let's say ice.**
- **What is supplying the force in this situation?** The person pushing it
- **What happens when we push the box?** The box changes speed. ***Teacher: The rate at which the velocity changes over time is called the acceleration. Prompt class so that they know that we are dealing with the velocity of the box.
- **What would happen if the box became heavier but you were still pushing with the same force?** The acceleration would decrease.
- **What if you want the box to have the same acceleration as it had before the box was made heavier?** You have to push harder.
- **What if the box became really light and you pushed it really hard?** The acceleration would get really big

Discussion

- **We can see that in this situation, force, mass, and acceleration are the main factors. We want to find out how they are related.**
- **What factors affect the acceleration of the box?** The mass of the box and the force exerted on it
- **What would need to happen if we wanted to keep the box moving with a constant acceleration, but we doubled the mass? What kind of relationship does this establish between force and mass?** We would need to double the force exerted. The force and mass are directly proportional.
- **What if the force stayed constant, but we doubled the mass? What kind of relationship does this establish between mass and acceleration?** The acceleration would be cut in half. The mass and acceleration are inversely proportional.
- **What about the relationship between acceleration and force?** They are directly proportional.
- **We want to write a relationship between force, mass, and acceleration that is consistent with this data. What would the relationship be?** F=ma
- **Why when we push a car with a constant force does it not accelerate?**

Sample Script for Introduction to Lattices Lesson

Introduction

- Discuss Newton's law and ask class to give a few examples
  What does Newton's law state? Force = Mass times Acceleration
  What situation did we use last class to discuss this law? We discussed pushing a box on a frictionless surface. We changed the mass of the box and the force at which we were pushing to find a relationship between force, mass, and acceleration.

- Introduce the lattice system
  We are going to be exploring a system where beads connected by strings are interacting and pulling on one another. Imagine a string of beads that goes on and on forever. Each bead is on a rod so it can only move up and down. (Draw the lattice on the board while describing it).
  - Describe system in great detail
    - Infinite lattice
      What does it mean for the lattice to be infinite? It goes on forever in both directions
    - Beads on rods
      If each bead is on a rod, what direction can the beads move in? The beads can only move up and down
    - Strings between each bead
      If I pull two neighboring beads in opposite vertical directions, what would happen to the string? The string will be very tight. The string will prevent you from stretching the beads too far.
  - Draw on board

Discussion

- Goal: to explore the motions in this system
  - We want to analyze what kind of motions can occur in an infinite system
  - We are going to explore a system where beads and strings are interacting and pulling on one another
- Explain that we are looking at the forces on a single bead
  To better understand the motions in the lattice we are going to focus on one bead. What could be affecting the movement of a single bead?
- Have class make a list of what affects the force on a single bead
  - Neighboring beads

        - How far apart they are
    - Tension in a string
    - Mass of the bead

End of Class

- Begin labeling the lattice system if there is time left
  In order to work with this system, we are going to have to introduce some notation. I could label each bead with numbers.
  (Begin labeling with number 1. Start this is the middle of the lattice)
  What should I name the bead on the left of number 1? And on the left of that one? 0, then -1, then keep going into the negatives
  Does each bead have to be at the same height? No
  Since each bead can have a different height, we will just name the height Y and associate it with each bead. So the height of the first bead is Y_1 and the height of the second bead Y_2 etc.

Note that this lesson needs to be thorough, because this is the foundation of the unit. More time can be spent on this lesson if needed. See sample script

Sample Script for Turntable Motion Lesson

Introduction

- Entrance slip
    - Students must write the formula for the force on a particular bead and write a short explanation of the formula
- Introduce turntable motion
    - **Ask class what they think of when they hear the word turntable** Records, DJ, a circle spinning
    - Have a brief class discussion on turntable motion and come up with a class definition

      How could we define turntable motion? The movement of an object along a circular path

Teacher Demonstration/Explanation

- Explain why we are using turntable motion

  We are restricting the beads to a specific type of motion. We want them to be oscillating at a fixed frequency.

  How do the beads in the lattice system move? Up and down

  Today we are going to see how we can relate that up and down motion to a circular, or turntable, motion.
    - We want to look at simple motions in the system

      Turntable motion is very simple because it rotates around at a fixed frequency. We are hoping that there is a relation between circular motion and the up and down motion of our beads.
- If a turntable is available do an actual demonstration for the class. If not use an animation.
    - **Tell class to imagine a bead glued to a turntable**
    - **Ask them to predict what the projection of the bead on the wall will look like** The projection looks like the bead is moving up and down. It looks like it is changing speed.
    - Demonstrate that the projection shows the bead moving up and down in a straight line
- **Explain that we want to think of each bead as its own turntable**

  How are the movements in the lattice related to the turntable motion? The projection of the turntable motion is the up and down motion that we see in the string of beads.

  So now, if we think of each bead as being on its own turntable, we know that the projection will look like all the beads moving up and down.

Class Discussion

- Discuss why this would simplify the system
  Why is turntable motion easier to work with than the up and down motion? Using turntable motion allows us to relate a motion with a constant speed to the motion of our beads which are not oscillating at a constant frequency.
- Tell class to imagine what the projection of multiple turntables moving at the same frequency would look like if they were all started at a different time. Have them draw their predictions on a piece of paper.

Advanced Math Lessons

Name of Lesson: Introduction of Trigonometry

Topic: Turntable Motion   Grade Level: High School   Created by: Emelie Mativi

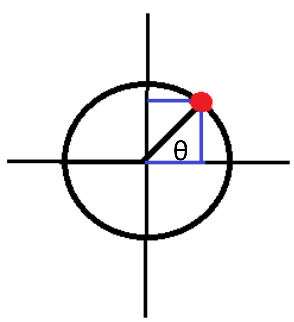
With circular motion, angles arise naturally as the distance θ the bead traveled around the unit circle. Students already understand vertical and horizontal projections of the bead, so rename the vertical projection Sine of θ and the horizontal projection Cosine of θ. So the position of each bead can be described by its coordinates, called sine and cosine.
Objectives: • Introduce Sine and Cosine as names of concepts students have already discovered

Introduction

- Look back at when we found the propagation band. We described the position of each bead with a vector.
- What were the two components of each vector?
    - The vertical and horizontal projections
- We are going to use this information to introduce the foundation of trigonometry.

Discussion

- Draw a picture of a bead on a turntable and label the vertical and horizontal projections as a class
- There is an angle formed when the bead moves around the turntable. We will just call this angle θ.



- Now we will introduce new names for these projections
    - Call the horizontal projection Cosine of θ
    - Call the vertical projection Sine of θ
- Now each position on the turntable can be described as its coordinates, sine and cosine.

Group Work

- Explore a few interesting angles
    - Let's start at (1,0)
        - What angle is formed at this point?
            - 0°

        - So we could write this coordinate as (Cos(0), Sin(0))
            - So we see that Cos(0) is equal to 1 and Sin(0) is equal to 0
    - Instruct students to the find the values of Sine and Cosine at (0,1), (-1,0), and (0,-1)
    - Use right triangles to explore other interesting angles (45°,30°,60°)

Name of Lesson: Introduction of Complex Variables

Topic: Turntable Motion   Grade Level: High School   Created by: Emelie Mativi

Introduce complex exponents to show students that a phase shift is just multiplication of complex variables.
Objectives: • Continue exploring trigonometry to discover foundational concepts of concept variables

Introduction

- Introduce complex numbers
    - Can anyone tell me the definition of i?
    - $i = \sqrt{-1}$
        - Continue to work with i to find i^2, i^3, i^4
    - A complex number is represented in the form $a + ib$
        - We call a the real part and b the imaginary part
        - Do some problems that involve addition and multiplication of complex variables
- Introduce the complex plane and show students how to plot points on it
    - $(a, b) = a + ib$

  *******Note: If students have never seen complex numbers, go more in depth with this introduction********

Discussion

- Introduce Euler's formula
    - What would happen if we were to raise the number e to a complex number?
    - Euler's formula states $e^{i\theta} = cos\theta + isin\theta$
- Examples
    - What is $e^{i\frac{\pi}{2}}$, $e^{i\pi}$?
- Consider a turntable with a bead on it. The bead travels around the circle a distance of $\frac{\pi}{2}$ and then a distance of $\frac{\pi}{4}$.
    - What far has the bead traveled now?
        - $\frac{3\pi}{4}$
    - How could we represent the position of this bead?
        - $\cos\left(\frac{3\pi}{4}\right) + isin\left(\frac{3\pi}{4}\right)$ or $e^{i\frac{3\pi}{4}}$

- What if, instead, we multiplied the values of the position of the bead at $\frac{\pi}{2}$ and $\frac{\pi}{4}$?
  - $\cos\left(\frac{\pi}{2}\right) + isin\left(\frac{\pi}{2}\right)$ and $\cos\left(\frac{\pi}{4}\right) + isin\left(\frac{\pi}{4}\right)$
- We get the same answer! So shifting the bead by $\frac{\pi}{4}$ was the same thing as multiplying the original position by $e^{i\frac{\pi}{4}}$

* Let's see if we can prove this for all angles, θ.
  - Show: $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$
* This is the trigonometric identity for the sum of angles. This tells us that when have a phase shift it is just multiplication by a complex exponent.