11-1-2014

Local stabilization of nonlinear systems through the reduction model approach

Frederic Mazenc
CNRS Centre National de la Recherche Scientifique

Michael Malisoff
Louisiana State University

Follow this and additional works at: https://digitalcommons.lsu.edu/mathematics_pubs

Recommended Citation

Mazenc, F., & Malisoff, M. (2014). Local stabilization of nonlinear systems through the reduction model approach. *IEEE Transactions on Automatic Control*, 59 (11), 3033-3039. <https://doi.org/10.1109/TAC.2014.2317292>

This Article is brought to you for free and open access by the Department of Mathematics at LSU Digital Commons. It has been accepted for inclusion in Faculty Publications by an authorized administrator of LSU Digital Commons. For more information, please contact ir@lsu.edu.

Local Stabilization of Nonlinear Systems through the Reduction Model Approach

Frédéric Mazenc, Michael Malisoff

► To cite this version:

Frédéric Mazenc, Michael Malisoff. Local Stabilization of Nonlinear Systems through the Reduction Model Approach. IEEE Transactions on Automatic Control, Institute of Electrical and Electronics Engineers, 2014, 59 (11), pp.3033-3039. 10.1109/tac.2014.2317292 . hal-01098433

HAL Id: hal-01098433

<https://hal.inria.fr/hal-01098433>

Submitted on 14 Jan 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Local Stabilization of Nonlinear Systems Through the Reduction Model Approach

Frederic Mazenc and Michael Malisoff

Abstract—We study a general class of nonlinear systems with input delays of arbitrary size. We adapt the reduction model approach to prove local asymptotic stability of the closed loop input delayed systems, using feedbacks that may be nonlinear. Our Lyapunov-Krasovskii functionals make it possible to determine estimates of the basins of attraction for the closed loop systems.

Index Terms—Delay, nonlinear, reduction model, stabilization.

I. INTRODUCTION

The reduction model approach is a well-known stabilization technique for systems with input delays. It originated in [1] and has been studied in many works, e.g., [2]–[6]. It is effective for stabilizing linear time-invariant systems with arbitrarily long pointwise or distributed input delays. However, the approach does not directly apply to nonlinear systems; it is extended by introducing an extra dynamic (which gives the ‘state predictor’) whose initial condition is given by an implicit equation (as is done in [7]–[9], and [6, Chapt. 6, p. 128]), and only a few recent works adapt it to time varying systems [10]. This is a limitation, because many systems are nonlinear and lead to the stabilization of time varying nonlinear systems when a trajectory has to be tracked. Moreover, the work [11] is limited to globally Lipschitz nonlinear systems, and it has a restriction on the size of the delays. See also [12] and [13] for stabilization of nonlinear systems with arbitrarily long input delays when the systems have special structures, and [14] for compensation of arbitrarily long input delays under input sampling based on prediction.

These remarks motivate our work. We show that the reduction model approach can be used to locally asymptotically stabilize a large family of nonlinear time varying systems of the form $\dot{x}(t) = A(t)x(t) + B(t)u(t - \tau) + F(t, x(t))$, with arbitrarily long constant known input delays τ , where F is of order 2 in x at the origin. Our key assumption is the stabilizability of a linear approximation of the closed loop system at 0. Under this assumption, the result seems intuitively obvious. However, to the best of the authors’ knowledge, it has never been rigorously established. In particular, the stability of the closed loop system we obtain cannot be proven by applying the Hartman-Grobman theorem, which only applies to ordinary differential equations; see [15, Chapt. 1]. One of the crucial benefits offered by our result is that it yields asymptotically stable closed loop systems for which one can determine a suitable subset of the basin of attraction of the closed loop systems. This information is valuable, because it gives a guarantee that some solutions converge to the origin. We estimate the basin of attraction by building a Lyapunov-Krasovskii functional. It is different from the one in [16], but can be combined with it to establish ISS results. See also [17] for estimates

of the basins of attraction for time invariant nonlinear systems with 48 predictor feedbacks, under an ISS assumption on the closed loop 49 systems with undelayed controllers. The predictor feedbacks in [17] 50 can be implemented using numerical methods but are totally different 51 from ours, so our work can be viewed as complementary to [17]. 52 Our work is mainly a methodological development, rather than a 53 specific real-world application or experiments. However, input delays 54 naturally arise from measurement and transport phenomena, and our 55 assumptions are very general, so we anticipate that our work can 56 have considerable benefits when applied to mechanical systems where 57 latencies commonly occur. 58

The rest of this note is organized as follows. We give our definitions 59 in Section II. In Section III, we show how the class of systems we 60 study naturally arises in tracking problems. We state our main result in 61 Section IV, and we prove it in Section V. In Section VI, we discuss a 62 large class of examples where the estimates of the basins of attraction 63 become arbitrarily large when the input delays converge to zero. In 64 Section VII, we illustrate our result in a worked out example. We 65 conclude in Section VIII with a summary of our findings. 66

II. DEFINITIONS AND NOTATION

We let $n \in \mathbb{N}$ be arbitrary and I_n denote the identity matrix in $\mathbb{R}^{n \times n}$, and $\|\cdot\|$ be the usual Euclidean norm of matrices and vectors. For square matrices M_1 and M_2 of the same size, we write $M_1 \geq M_2$ to mean that $M_1 - M_2$ is nonnegative definite. For each integer $r \geq 1$, let C^r denote the set of all functions whose partial derivatives up to order r exist and are continuous, and C^0 denotes the set of all continuous functions, when the domains and ranges are clear from the context. When we want to emphasize the domains and ranges, we use $C^r(\mathcal{U}, \mathcal{V})$ to denote the set of all C^r functions having domain $\mathcal{U}$ and range $\mathcal{V}$, where $\mathcal{U}$ and $\mathcal{V}$ are suitable subsets of Euclidean spaces. For any constant $\tau \geq 0$ and any continuous function $\varphi : [-\tau, \infty) \rightarrow \mathbb{R}^n$ and all $t \geq 0$, we define the function φ_t by $\varphi_t(\theta) = \varphi(t + \theta)$ for all $\theta \in [-\tau, 0]$, i.e., the translation operator. Let $\mathcal{K}_\infty$ be the set of all C^0 functions $\gamma : [0, \infty) \rightarrow [0, \infty)$ such that $\gamma(0) = 0$ and γ is strictly increasing and unbounded. Given subsets S_1 and S_2 of Euclidean spaces, we say that a function $J : S_1 \times S_2 \rightarrow \mathbb{R}^p$ is locally Lipschitz with respect to its second argument provided for each compact set $E \subseteq S_2$, there is a constant L_E such that $\|J(p, x) - J(p, y)\| \leq L_E \|x - y\|$ for all $p \in S_1$ and all $x \in E$ and $y \in E$. We say that J is strictly increasing in its second argument provided the function $Y(x) = J(p, x)$ is strictly increasing for each $p \in S_1$; we define strictly increasing and nondecreasing in either argument in a similar way. We say that J has order 2 in y at the origin provided there is a continuous function α such that $\|J(p, y)\| \leq |y|^2 \alpha(|y|)$ for all $(p, y) \in S_1 \times S_2$. We sometimes omit arguments of functions when the arguments are clear from the context.

III. MOTIVATION: TRACKING PROBLEM

In this section, we explain how the problem of tracking a trajectory may lead to the problem we solve in the next section. Consider a time varying nonlinear system

$$\dot{\xi}(t) = g(t, \xi(t)) + B(t)\mu(t - \tau) \quad (1)$$

where the state ξ is valued in $\mathbb{R}^n$, the control μ is valued in $\mathbb{R}^p$, $\tau \geq 0$ is a known constant delay, $g = (g_1, g_2, \dots, g_n)^\top$ is a nonlinear function of class C^2 , and B is a continuous function. The dimensions n and p are arbitrary. We assume that (1) is forward complete for all measurable locally essentially bounded choices for μ , so $\xi(t)$ is

Manuscript received July 4, 2013; revised December 1, 2013; accepted March 31, 2014. This technical note appeared in part in the American Control Conference, 2014. This work was supported by NSF-ECCS Grant 1102348. Recommended by Associate Editor P. Pepe.

F. Mazenc is with EPI DISCO INRIA-Saclay, L2S, CNRS-Supelec, Gif-sur-Yvette 91192, France (e-mail: frederic.mazenc@lss.supelec.fr).

M. Malisoff is with the Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803-4918 USA (e-mail: malisoff@lsu.edu).

Digital Object Identifier 10.1109/TAC.2014.2317292

defined for all nonnegative times for all such μ 's. We also assume that there is a nondecreasing function γ such that

$$\max \left\{ \left| \frac{\partial^2}{\partial \xi^2} g_i(t, \xi) \right| : |\xi| \leq q, t \geq 0, i \in \{1, 2, \dots, n\} \right\} \leq \gamma(q) \quad (2)$$

for all $q \geq 0$, which exists when g is C^2 and periodic in t .

The objective is to follow an admissible trajectory ξ_r of class C^1 , meaning the dynamics for $x = \xi - \xi_r$ should be asymptotically stable. By admissible, we mean that there is a known continuous function $\mu_r(t)$ such that $\dot{\xi}_r(t) = g(t, \xi_r(t)) + B(t)\mu_r(t)$ for all $t \geq 0$. In particular, this means that $\xi_r(t)$ is defined for all $t \geq 0$. We assume that ξ_r is a known bounded function.

Let $x(t) = \xi(t) - \xi_r(t)$ and $\mu(t - \tau) = u(t - \tau) + \mu_r(t)$. Then the error equation is

$$\dot{x}(t) = G(t, x(t)) + B(t)u(t - \tau) \quad (3)$$

where $G(t, x) = g(t, x + \xi_r(t)) - g(t, \xi_r(t))$. Notice that $G(t, x) = \int_0^1 (\partial g / \partial x)(t, \ell x + \xi_r(t)) x d\ell$, so $G(t, x) = (\partial g / \partial x)(t, \xi_r(t))x + F(t, x)$, where

$$F(t, x) = \int_0^1 \left(\frac{\partial g}{\partial x}(t, \ell x + \xi_r(t)) - \frac{\partial g}{\partial x}(t, \xi_r(t)) \right) x d\ell \quad (4)$$

holds for all t and x .

Applying the Mean Value Theorem and using (2) and the boundedness of ξ_r , we can find a function $\alpha \in C^0$ such that $|F(t, x)| \leq |x|^2 \alpha(|x|)$. Since ξ_r can depend on t , the system (3) is time varying, even if g is time-invariant and B is constant. This motivates the study of systems of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t - \tau) + F(t, x(t)) \quad (5)$$

where F is of order 2 in x at the origin, which will be our focus for the rest of this note.

IV. STATEMENT OF MAIN RESULT

We state our main result for (5), where x is valued in $\mathbb{R}^n$, the control u is valued in $\mathbb{R}^p$ and is to be specified, $\tau \geq 0$ is a given constant delay, and F is a nonlinear function. The dimensions n and p are arbitrary. The functions A , B and F are continuous, and F is locally Lipschitz with respect to x . The set of all initial conditions we consider is $E_0 = \{(\phi_x, \phi_u) \in C^0([-\tau, 0], \mathbb{R}^n \times \mathbb{R}^p)\}$, so the initial times for our trajectories are always 0. Let $\lambda : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ be the fundamental solution associated with A . Then $\lambda(t_0, t_0) = I_n$ and $(\partial \lambda / \partial t)(t, t_0) = A(t)\lambda(t, t_0)$ hold for all real numbers t and t_0 . We introduce the following assumptions:

Assumption 1:

(i) There is a continuous, positive valued, nondecreasing function h such that

$$|\lambda(t, l)B(l)| \leq h(\tau) \text{ for all } t \in \mathbb{R} \text{ and } l \in [t, t + \tau]. \quad (6)$$

(ii) There is a constant $a^+ \geq 0$ such that $\sup_{t \in \mathbb{R}} |A(t)| \leq a^+$. $\square$

Assumption 1 always holds when B is bounded and A is constant, so for instance, it holds for the one-dimensional system

$$\dot{x}(t) = x(t) + u(t - \tau) + lx^2(t) \sin(x(t)) \quad (7)$$

where $u \in \mathbb{R}$ is the input, τ is a positive constant delay, and l is a positive constant. In the case of (7), we can take $A = 1$, $B = 1$,

$\lambda(t, t_0) = e^{t-t_0}$, and $F(t, x) = lx^2 \sin(x)$, so Assumption 1 holds with $h(\tau) = 1$. To ease the readability of our technical assumptions, we will explain how the example (7) satisfies our assumptions, after we introduce each of our three assumptions. Our next assumption is:

Assumption 2: There are a continuous function $K : [0, \infty)^2 \rightarrow \mathbb{R}^{p \times n}$, a nondecreasing continuous function $k : [0, \infty) \rightarrow (0, \infty)$, an everywhere positive definite and symmetric function $Q : [0, \infty)^2 \rightarrow \mathbb{R}^{n \times n}$ of class C^1 with respect to its first argument, and continuous functions $q_i : [0, \infty) \rightarrow (0, \infty)$ for $i = 1, 2, 3$ such that $|K(t, \tau)| \leq k(\tau)$ for all $(t, \tau) \in [0, \infty)^2$, and such that with the choices $H(t, \tau) = A(t) + \lambda(t, t + \tau)B(t + \tau)K(t, \tau)$ and $R(t, \tau, s) = s^\top Q(t, \tau)s$, the following two conditions are satisfied for all $\tau \geq 0$: (i) Along all trajectories of $\dot{s}(t) = H(t, \tau)s(t)$, we have $\dot{R}(t, \tau, s(t)) \leq -q_1(\tau)R(t, \tau, s(t))$ and (ii) the bounds

$$q_2(\tau)I_n \leq Q(t, \tau) \text{ and } |Q(t, \tau)| \leq q_3(\tau) \quad (8)$$

are satisfied for all $t \geq 0$. $\square$

Assumption 2 holds for (7) as well. In fact, by choosing $K(t, \tau) = -2e^\tau$, we obtain $H(t, \tau) = 1 - e^{-\tau}2e^\tau = -1$, so Assumption 2 is satisfied with $Q(t, \tau) = 1/2$, $q_1(\tau) = 2$, $q_2(\tau) = q_3(\tau) = 1/2$, and $k(\tau) = 2e^\tau$. Finally, we assume:

Assumption 3: There are two continuous functions f_1 and f_2 that are locally Lipschitz with respect to their last argument, and continuous functions α_1 and α_2 , such that

$$F(t, x) = \lambda(t, t + \tau)B(t + \tau)f_1(t, \tau, x) + f_2(t, x) \text{ and} \quad (9)$$

$$|f_1(t, \tau, x)| \leq |x|^2 \alpha_1(\tau, |x|^2) \text{ and}$$

$$|f_2(t, x)| \leq |x|^2 \alpha_2(|x|^2) \quad (10)$$

for all $t \in \mathbb{R}$, $\tau \geq 0$, and $x \in \mathbb{R}^n$. Also, $\beta_3(\tau, m) = m\alpha_1(\tau, m^2)$ is strictly increasing and unbounded in m , and $\beta_4(m) = m\alpha_2(m^2)$ is nondecreasing in m . Finally, there are continuous functions $\theta_1 : [0, \infty)^2 \rightarrow (0, \infty)$ and $\theta_2 : [0, \infty) \rightarrow (0, \infty)$ such that

$$|\alpha_1(\tau, b + c) - \alpha_1(\tau, c)| \leq b\theta_1(\tau, b + c) \quad (11)$$

$$|\alpha_2(b + c) - \alpha_2(c)| \leq b\theta_2(b + c) \quad (12)$$

are satisfied for all $b \geq 0$ and $c \geq 0$. $\square$

To see why (7) satisfies Assumption 3, note that for (7), the fact that $\lambda(t, t + \tau)B$ is invertible implies that one can choose $f_2 = 0$ and $f_1(t, \tau, x) = le^\tau x^2 \sin(x)$. Then we can satisfy Assumption 3 for (7) by taking $\alpha_1(\tau, m) = le^\tau$ and $\alpha_2(m) = 0$ for all m and τ .

Returning to the general system (5), it follows from Assumptions 2–3 that for any constant $\tau > 0$ and:

$$\alpha_3(\tau, m) = \frac{q_3(\tau)}{\sqrt{q_2(\tau)}} \alpha_2(m) + 2a\alpha_1(\tau, m), \quad (13)$$

where a is any constant such that

$$0 < a \leq \frac{q_1(\tau)\sqrt{q_2(\tau)}}{8k(\tau)} \quad (14)$$

there are unique positive values $v_1(\tau)$ and $v_2(\tau)$ (which also depend on a) such that

$$v_1(\tau)\alpha_3\left(\tau, \frac{4}{q_2(\tau)}v_1^2(\tau)\right) = \frac{q_1(\tau)q_2(\tau)}{16} \text{ and} \quad (15)$$

$$v_2(\tau)\alpha_3\left(\tau, \frac{4h^2(\tau)}{a^2}v_2^2(\tau)\right) = \frac{a^2}{4\tau h^2(\tau)}. \quad (16)$$

180 The existence of unique values $v_1(\tau)$ and $v_2(\tau)$ follows because
 181 $\beta_3(\tau, m)$ is strictly increasing and unbounded in m and $\beta_4(m)$ is
 182 nondecreasing, so $m\alpha_3(\tau, m^2)$ is strictly increasing and unbounded
 183 in m . The choice of α_3 in (13) will become clear when we prove:

184 *Theorem 1:* Let $\tau > 0$ be any constant and Assumptions 1–3
 185 hold. Let a be any constant satisfying (14), and set $v(\tau) =$
 186 $\min\{v_1(\tau), v_2(\tau)\}$ where v_1 and v_2 are as above. Then, for each
 187 initial function $(\phi_x, \phi_u) \in C^0([-\tau, 0], \mathbb{R}^n \times \mathbb{R}^p)$ satisfying

$$\sqrt{q_3(\tau)} \left| \phi_x(0) + \int_{-\tau}^0 \lambda(0, m + \tau) B(m + \tau) \phi_u(m) dm \right| + \frac{a}{\tau} \int_{-\tau}^0 (m + 2\tau) |\phi_u(m)| dm < v(\tau) \quad (17)$$

188 the unique solution of (5), in closed loop with

$$u(t) = -f_1(t, \tau, x(t)) + K(t, \tau) \left[x(t) + \int_{t-\tau}^t \lambda(t, m + \tau) B(m + \tau) u(m) dm \right] \quad (18)$$

189 converges to 0 as $t \rightarrow \infty$. Moreover, (18) locally asymptotically
 190 stabilizes (5) to 0. $\square$

191 *Remark 1:* We comment that our control (18) agrees with the
 192 standard predictor controller in the linear time invariant case where
 193 $f_1 = f_2 = 0$ and A and B are constant. The extra term $-f_1(t, \tau, x(t))$
 194 is used to compensate part of the nonlinearity of the system (5).
 195 Assumption 2 is a generalization of the standard assumption that
 196 (A, B) is a stabilizable pair, which is the special case of Assumption
 197 2 where $\tau = 0$, A and B are constant, and where K and Q can also be
 198 taken to be constant. However, we allow the delay $\tau > 0$ to be as large
 199 as we want. On the other hand, since the q_i 's are continuous positive
 200 valued functions of the delay, they have positive upper and lower
 201 bounds over all $\tau \in [0, \tau_M]$ for any constant τ_M . Also, the function
 202 k from Assumption 2 is nondecreasing in τ . Hence, if we are only
 203 concerned with a bounded set $[0, \tau_M]$ of possible values for τ , then we
 204 can assume in Assumption 2 that the q_i 's and k are all positive con-
 205 stants, by replacing them by the constants $\min\{q_1(\tau) : 0 \leq \tau \leq \tau_M\}$,
 206 $\min\{q_2(\tau) : 0 \leq \tau \leq \tau_M\}$, $\max\{q_3(\tau) : 0 \leq \tau \leq \tau_M\}$, and $k(\tau_M)$
 207 without relabeling. These observations will be key to our proof in
 208 Section VI that for important special cases, our estimate of the domain
 209 of attraction becomes arbitrarily large when $\tau \rightarrow 0^+$. $\square$

210 *Remark 2:* Assumptions 1–2 always hold when A and B are con-
 211 stant provided (A, B) is stabilizable. Indeed, in that case $\lambda(t, t_0) =$
 212 $e^{(t-t_0)A}$, so the stabilizability of (A, B) is equivalent to the stabiliz-
 213 ability of $(A, \lambda(t, t + \tau)B)$. Also, when the α_i 's are C^1 , the existence
 214 of functions θ_i satisfying the requirements from Assumption 3 follows
 215 from the Mean Value Theorem, since Assumption 2 only requires
 216 (11), (12) for nonnegative b 's and c 's. Since F is of order 2 in x
 217 at 0, we can always satisfy Assumption 3 with $f_1 = 0$ and $f_2 = F$.
 218 However, these choices may lead to a conservative estimate of the size
 219 of the basin of attraction; see the example in Section VII. Our use
 220 of a feedback control with distributed terms is motivated by the facts
 221 that τ is arbitrary and $\xi(t) = A(t)\xi(t)$ may be exponentially unstable.
 222 In general, the explicit expression for λ is unknown, but it can be
 223 computed in many important cases. This is the case in particular if
 224 A is constant or $n = 1$. We illustrate Theorem 1 in Section VII. $\square$

225 *Remark 3:* In conjunction with our local asymptotic stability result,
 226 we have boundedness of the control from Theorem 1, along all of the
 227 closed loop trajectories. $\square$

V. PROOF OF THEOREM 1

228

Throughout the proof, we consider any solution of (5) in closed loop 229
 with (18) for any initial condition satisfying the requirements (17) of 230
 Theorem 1, and any constant delay $\tau \geq 0$. 231

First Part: New Representation of the System: Let t_e be any positive 232
 real number or ∞ such that the solution is defined over $[-\tau, t_e]$. Such 233
 a $t_e > 0$ exists, because the dynamics (5) grows linearly in x in any 234
 bounded open neighborhood of $x(0)$. Later we show that t_e can always 235
 be taken to be ∞ for all of the trajectories we are considering. We 236
 introduce the operators 237

$$z(t) = x(t) + \Gamma(t, u_t), \text{ where}$$

$$\Gamma(t, u_t) = \int_{t-\tau}^t \lambda(t, m + \tau) B(m + \tau) u(m) dm. \quad (19)$$

In all of what follows, we assume that $t \in [0, t_e]$ is arbitrary, unless 238
 otherwise noted, and we omit some of the arguments of the time 239
 derivatives when they are clear, so $\dot{\Gamma}(t)$ means $(d/dt)\Gamma(t, u_t)$. Then 240
 the properties of the fundamental matrix give $\dot{\Gamma}(t) = A(t)\Gamma(t, u_t) +$ 241
 $\lambda(t, t + \tau)B(t + \tau)u(t) - B(t)u(t - \tau)$. Using the formula (5) and 242
 our decomposition (9) for $F(t, x)$, we obtain 243

$$\dot{z}(t) = A(t)z(t) + \lambda(t, t + \tau)B(t + \tau)[u(t) + f_1(t, \tau, x(t))] + f_2(t, x(t)). \quad (20)$$

Also, our feedback (18) satisfies $u(t) = -f_1(t, \tau, x(t)) +$ 244
 $K(t, \tau)z(t)$. Consequently, in terms of our function H from 245
 Assumption 2, (20) becomes 246

$$\dot{z}(t) = H(t, \tau)z(t) + f_2(t, x(t)). \quad (21)$$

Assumption 2 ensures global asymptotic stability of the linearizations 247
 $\dot{z}(t) = H(t, \tau)z(t)$ of (21) at 0. Moreover, the equality 248

$$x(t) = z(t) - \int_{t-\tau}^t \lambda(t, m + \tau) B(m + \tau) u(m) dm \quad (22)$$

is satisfied. 249

Second Part: Decay Conditions: We study the stability of the closed 250
 loop system using its representation as (21) coupled with (22). We 251
 introduce the operator 252

$$\Xi(u_t) = \frac{1}{\tau} \int_{t-\tau}^t (m - t + 2\tau) |u(m)| dm. \quad (23)$$

Observe for later use that 253

$$\int_{t-\tau}^t |u(m)| dm \leq \Xi(u_t) \leq 2 \int_{t-\tau}^t |u(m)| dm. \quad (24)$$

Then, for all $t \geq 0$, we have 254

$$\dot{\Xi}(t) \leq 2 |u(t)| - \frac{1}{\tau} \int_{t-\tau}^t |u(m)| dm. \quad (25)$$

Also, we can use the upper bound on f_1 from (10), the bound for 255
 K given in Assumption 2 and the formula $u(t) = -f_1(t, \tau, x(t)) +$ 256
 $K(t, \tau)z(t)$ to get $|u(t)| \leq k(\tau)|z(t)| + |x(t)|^2 \alpha_1(\tau, |x(t)|^2)$. More- 257
 over, (8) implies that for all $t \geq 0$ and all $z \in \mathbb{R}^n$, we have $q_2(\tau)|z|^2 \leq$ 258

259 $z^\top Q(t, \tau)z$. Taking square roots of both sides of the preceding in-
 260 equality and the dividing by $\sqrt{q_2(\tau)} > 0$ gives

$$|z| \leq \frac{1}{\sqrt{q_2(\tau)}} \sqrt{R(t, \tau, z)}. \quad (26)$$

261 Combining the last two estimates with (25) gives

$$\begin{aligned} \dot{\Xi}(t) \leq & -\frac{1}{\tau} \int_{t-\tau}^t |u(m)| dm + \frac{2k(\tau)}{\sqrt{q_2(\tau)}} \sqrt{R(t, \tau, z(t))} \\ & + 2|x(t)|^2 \alpha_1(\tau, |x(t)|^2). \end{aligned} \quad (27)$$

262 We deduce from Assumptions 2–3 that the time derivative of R
 263 along all trajectories of (21) satisfies

$$\begin{aligned} \dot{R}(t) \leq & -q_1(\tau)R(t, \tau, z(t)) + 2z(t)^\top Q(t, \tau)f_2(t, x(t)) \\ \leq & -q_1(\tau)R(t, \tau, z(t)) + 2|z(t)|q_3(\tau)|f_2(t, x(t))|. \end{aligned} \quad (28)$$

264 From (26), we deduce that

$$\begin{aligned} \dot{R}(t) \leq & -q_1(\tau)R(t, \tau, z(t)) \\ & + 2q_3(\tau) \frac{\sqrt{R(t, \tau, z(t))}}{\sqrt{q_2(\tau)}} |x(t)|^2 \alpha_2(|x(t)|^2). \end{aligned} \quad (29)$$

265 Consider the family of functions $S_\varepsilon(t, \tau, z) = \sqrt{R(t, \tau, z)} + \varepsilon -$
 266 $\sqrt{\varepsilon}$ parameterized by the constant $\varepsilon \in [0, 1]$ and let $S = S_0$. Since R
 267 is of class C^1 with respect to t and z , it follows that for all $\varepsilon \in (0, 1)$,
 268 the function S_ε is of class C^1 with respect to t and z , while S is only
 269 continuous. Also, (29) and Lemma 1 in the Appendix (applied with the
 270 choice $r = R(t, \tau, z)$) give

$$\begin{aligned} \dot{S}_\varepsilon(t) \leq & -q_1(\tau) \frac{R(t, \tau, z(t))}{2\sqrt{R(t, \tau, z(t))} + \varepsilon} \\ & + q_3(\tau) \frac{\sqrt{R(t, \tau, z(t))} |x(t)|^2 \alpha_2(|x(t)|^2)}{\sqrt{R(t, \tau, z(t))} + \varepsilon\sqrt{q_2(\tau)}} \\ \leq & -\frac{q_1(\tau)}{2} S(t, \tau, z(t)) \\ & + \frac{q_3(\tau)}{\sqrt{q_2(\tau)}} |x(t)|^2 \alpha_2(|x(t)|^2) \\ & + \frac{q_1(\tau)}{2} \varepsilon^{\frac{1}{4}} [1 + S(t, \tau, z(t))] \end{aligned} \quad (30)$$

271 along all trajectories of (21).

272 *Third Part: Lyapunov-Krasovskii Functionals:* Let us consider the
 273 family of functions

$$V_\varepsilon(t, z, u_t) = a\Xi(u_t) + S_\varepsilon(t, \tau, z) \quad (31)$$

274 where the constant a satisfies (14) and we omit the argument τ in V_ε
 275 to simplify the notation. Then, (27) and (30) give

$$\begin{aligned} \dot{V}_\varepsilon(t) \leq & \left(\frac{2ak(\tau)}{\sqrt{q_2(\tau)}} - \frac{q_1(\tau)}{2} \right) S(t, \tau, z(t)) \\ & + \frac{q_3(\tau)}{\sqrt{q_2(\tau)}} |x(t)|^2 \alpha_2(|x(t)|^2) \end{aligned}$$

$$\begin{aligned} & -\frac{a}{\tau} \int_{t-\tau}^t |u(m)| dm + 2a|x(t)|^2 \alpha_1(\tau, |x(t)|^2) \\ & + \frac{q_1(\tau)}{2} \varepsilon^{\frac{1}{4}} [1 + S(t, \tau, z(t))]. \end{aligned} \quad (32)$$

Since a satisfies (14), we get

$$\begin{aligned} \dot{V}_\varepsilon(t) \leq & -\frac{q_1(\tau)}{4} S(t, \tau, z(t)) + |x(t)|^2 \alpha_3(\tau, |x(t)|^2) \\ & -\frac{a}{\tau} \int_{t-\tau}^t |u(m)| dm + \frac{q_1(\tau)}{2} \varepsilon^{\frac{1}{4}} [1 + S(t, \tau, z(t))] \end{aligned} \quad (33)$$

where α_3 was defined in (13).

Next, we find a suitable upper bound on the term
 $|x(t)|^2 \alpha_3(\tau, |x(t)|^2)$ from (33). Our formula (22) for $x(t)$,
 Assumption 1, and our bound (26) on $|z|$ give

$$\begin{aligned} |x(t)| \leq & |z(t)| + h(\tau) \int_{t-\tau}^t |u(m)| dm \\ \leq & \frac{1}{\sqrt{q_2(\tau)}} S(t, \tau, z(t)) + h(\tau) \int_{t-\tau}^t |u(m)| dm. \end{aligned} \quad (34)$$

Recall that our monotonicity properties of β_3 and β_4 from Assumption
 3 imply that $m\alpha_3(\tau, m^2)$ is strictly increasing as a function of m
 for each τ . Therefore, by separately considering the cases where
 $S(t, \tau, z(t))/\sqrt{q_2(\tau)} \leq h(\tau) \int_{t-\tau}^t |u(m)| dm$ and where the reverse
 inequality holds, we get

$$\begin{aligned} |x(t)|^2 \alpha_3(\tau, |x(t)|^2) & \leq \frac{4}{q_2(\tau)} S^2(t, \tau, z(t)) \alpha_3\left(\tau, \frac{4}{q_2(\tau)} S^2(t, \tau, z(t))\right) \\ & + 4h^2(\tau) \left[\int_{t-\tau}^t |u(m)| dm \right]^2 \alpha_3\left(\tau, 4h^2(\tau) \left[\int_{t-\tau}^t |u(m)| dm \right]^2\right). \end{aligned} \quad (35)$$

We can combine this inequality with (33) to get

$$\begin{aligned} \dot{V}_\varepsilon(t) \leq & \left[-\frac{q_1(\tau)}{4} + \frac{4}{q_2(\tau)} S(t, \tau, z(t)) \alpha_3 \right. \\ & \times \left(\tau, \frac{4}{q_2(\tau)} S^2(t, \tau, z(t)) \right) \left. \right] S(t, \tau, z(t)) \\ & + \left[-\frac{a}{\tau} + 4h^2(\tau) \int_{t-\tau}^t |u(m)| dm \alpha_3 \right. \\ & \times \left(\tau, 4h^2(\tau) \left[\int_{t-\tau}^t |u(m)| dm \right]^2 \right) \left. \right] \\ & \times \int_{t-\tau}^t |u(m)| dm + \frac{q_1(\tau)}{2} \varepsilon^{\frac{1}{4}} [1 + S(t, \tau, z(t))]. \end{aligned}$$

287 Also, $V_\varepsilon(t, z(t), u_t) \geq \sqrt{R(t, \tau, z(t)) + \varepsilon} - \sqrt{\varepsilon} \geq S(t, \tau, z(t)) -$
 288 $\sqrt{\varepsilon}$ and $V_\varepsilon(t, z(t), u_t) \geq a \int_{t-\tau}^t |u(m)| dm$ hold for all $\varepsilon \in [0, 1]$, by
 289 (24). Since $m\alpha_3(\tau, m^2)$ is increasing in m for each τ , it follows that:

$$\begin{aligned} S(t, \tau, z(t)) \alpha_3 \left(\tau, \frac{4}{q_2(\tau)} S^2(t, \tau, z(t)) \right) \\ \leq [V_\varepsilon(t, z(t), u_t) + \sqrt{\varepsilon}] \alpha_3 \\ \times \left(\tau, \frac{4}{q_2(\tau)} [2V_\varepsilon(t, z(t), u_t) \sqrt{\varepsilon} + \varepsilon] + \frac{4}{q_2(\tau)} V_\varepsilon^2(t, z(t), u_t) \right). \end{aligned}$$

290 We now apply (11), (12), with $b = (4/q_2(\tau))(2V_\varepsilon(t, z(t), u_t)\sqrt{\varepsilon} +$
 291 $\varepsilon)$ and $c = 4V_\varepsilon^2(t, z(t), u_t)/q_2(\tau)$, and use the fact that $\varepsilon \leq \sqrt{\varepsilon} \leq$
 292 $\varepsilon^{1/4}$ for all $\varepsilon \in [0, 1]$, to find a continuous positive valued and non-
 293 decreasing function φ_c (also depending on τ , but independent of ε)
 294 such that

$$\begin{aligned} \dot{V}_\varepsilon(t) \leq & \left[-\frac{q_1(\tau)}{4} + \frac{4}{q_2(\tau)} V_\varepsilon(t, z(t), u_t) \right. \\ & \times \alpha_3 \left(\tau, \frac{4}{q_2(\tau)} V_\varepsilon^2(t, z(t), u_t) \right) \left. \right] S(t, \tau, z(t)) \\ & + \frac{1}{a} \left[-\frac{a^2}{\tau} + 4h^2(\tau) V_\varepsilon(t, z(t), u_t) \right. \\ & \times \alpha_3 \left(\tau, 4h^2(\tau) \frac{V_\varepsilon^2(t, z(t), u_t)}{a^2} \right) \left. \right] \int_{t-\tau}^t |u(m)| dm \\ & + \varepsilon^{\frac{1}{4}} \varphi_c(V_\varepsilon(t, z(t), u_t)). \end{aligned} \quad (36)$$

295 Next, recall that our assumption (17) implies that $\sqrt{q_3(\tau)}|z(0)| +$
 296 $(a/\tau) \int_{-\tau}^0 (m+2\tau)|u(m)| dm < v(\tau)$, where $v(\tau) = \min\{v_1(\tau),$
 297 $v_2(\tau)\}$ as before. Then (8) from Assumption 2 gives $V_0(0, z(0), u_0) <$
 298 $v(\tau)$. Since $\sqrt{c_1} + \sqrt{c_2} \leq \sqrt{c_1} + \sqrt{c_2}$ holds for all nonnegative con-
 299 stants c_1 and c_2 , we know that $V_\varepsilon \leq V_0$ holds pointwise for all
 300 $\varepsilon \in (0, 1]$. It follows that $V_\varepsilon(0, z(0), u_0) \leq V_0(0, z(0), u_0) < \bar{v}$ hold
 301 for all $\varepsilon \in (0, 1]$, where $\bar{v} = [V_0(0, z(0), u_0) + v(\tau)]/2 > 0$. Then
 302 $\bar{v} < v(\tau)$.

303 Set $\bar{v}_a = (v(\tau) + \bar{v})/2$. Since $m\alpha_3(\tau, m^2)$ is strictly increasing in
 304 m , and since $\bar{v}_a < v(\tau) = \min\{v_1(\tau), v_2(\tau)\}$, it follows from our
 305 conditions (15), (16) on $v_1(\tau)$ and $v_2(\tau)$ that the constants:

$$\begin{aligned} \bar{p}_1 &= \frac{q_1(\tau)}{4} - \frac{4}{q_2(\tau)} \bar{v}_a \alpha_3 \left(\tau, \frac{4}{q_2(\tau)} \bar{v}_a^2 \right) \text{ and} \\ \bar{p}_2 &= \frac{a^2}{\tau} - 4h^2(\tau) \bar{v}_a \alpha_3 \left(\tau, 4h^2(\tau) \frac{\bar{v}_a^2}{a^2} \right) \end{aligned} \quad (37)$$

306 are positive for all $\tau > 0$. Fix any value of $\varepsilon \in (0, 1]$ satisfying

$$\varepsilon \in \left(0, \left(\frac{\min\{\bar{p}_1, \bar{p}_2\} \bar{v}}{4\varphi_c(\bar{v}_a) \max\{a^2, 1\}} \right)^4 \right) \quad (38)$$

307 where the left endpoint is omitted because we need $\varepsilon > 0$.
 308 Next, we prove by contradiction that $V_\varepsilon(t, z(t), u_t) \leq \bar{v}$ for all
 309 $t \geq 0$. Assume that this property does not hold. Then, since
 310 $\bar{v}_a > \bar{v}$ and $V_\varepsilon(0, z(0), u_0) < \bar{v}$, we can find a $t_2 > 0$ such that
 311 $V_\varepsilon(t, z(t), u_t) \leq \bar{v}_a$ for all $t \in [0, t_2]$ and $V_\varepsilon(t_2, z(t_2), u_{t_2}) > \bar{v}$. Set
 312 $t_1 = \inf\{t \leq t_2 : V_\varepsilon(p, z(p), u_p) \geq \bar{v} \text{ for all } p \in [t, t_2]\}$. Then, since
 313 $t \mapsto V_\varepsilon(t, z(t), u_t)$ is continuous, we get $V_\varepsilon(t, z(t), u_t) \in [\bar{v}, \bar{v}_a]$ for
 314 all $t \in [t_1, t_2]$, $V_\varepsilon(t_1, z(t_1), u_{t_1}) = \bar{v}$, and $\dot{V}_\varepsilon(t_1) \geq 0$.

By (36) and the fact that $l\alpha_3(\tau, l^2)$ is strictly increasing in l

315

$$\dot{V}_\varepsilon(t) \leq -\bar{p}_1 S(t, \tau, z(t)) - \frac{1}{a} \bar{p}_2 \int_{t-\tau}^t |u(m)| dm + \varepsilon^{\frac{1}{4}} \varphi_c(\bar{v}_a) \quad (39)$$

for all $t \in [t_1, t_2]$. It follows from our lower bound on Ξ from (24) 316
 that: 317

$$\begin{aligned} \dot{V}_\varepsilon(t) &\leq -\frac{1}{2 \max\{a^2, 1\}} \min\{\bar{p}_1, \bar{p}_2\} V_0(t, z(t), u_t) + \varepsilon^{\frac{1}{4}} \varphi_c(\bar{v}_a) \\ &\leq -\frac{1}{2 \max\{a^2, 1\}} \min\{\bar{p}_1, \bar{p}_2\} V_\varepsilon(t, z(t), u_t) + \varepsilon^{\frac{1}{4}} \varphi_c(\bar{v}_a) \end{aligned} \quad (40)$$

for all $t \in [t_1, t_2]$. Since $V_\varepsilon(t, z(t), u_t) \in [\bar{v}, \bar{v}_a]$ for all $t \in [t_1, t_2]$, we 318
 deduce that 319

$$\begin{aligned} \dot{V}_\varepsilon(t) &\leq -\frac{1}{2 \max\{a^2, 1\}} \min\{\bar{p}_1, \bar{p}_2\} \bar{v} + \varepsilon^{\frac{1}{4}} \varphi_c(\bar{v}_a) \\ &\leq -\frac{\min\{\bar{p}_1, \bar{p}_2\}}{4 \max\{a^2, 1\}} \bar{v} < 0 \end{aligned} \quad (41)$$

for all $t \in [t_1, t_2]$ when ε satisfies (38). It follows that $\dot{V}_\varepsilon(t_1) < 0$. 320
 This yields a contradiction with the choice of t_1 . Hence, when (38) 321
 holds, we get $V_\varepsilon(t, z(t), u_t) \leq \bar{v}$ for all $t \geq 0$, which implies that we 322
 can choose $t_e = \infty$. Also, arguing as we did before, we get 323

$$\dot{V}_\varepsilon(t) \leq -\frac{\min\{\bar{p}_1, \bar{p}_2\}}{2 \max\{a^2, 1\}} V_\varepsilon(t, z(t), u_t) + \varepsilon^{\frac{1}{4}} \varphi_c(\bar{v}). \quad (42)$$

for all $t \geq 0$. This gives a value $t_c > 0$ such that for all $t \geq t_c$, we have 324

$$V_\varepsilon(t, z(t), u_t) \leq \frac{4\varphi_c(\bar{v})\varepsilon^{\frac{1}{4}}}{\min\{\bar{p}_1, \bar{p}_2\}} \max\{a^2, 1\} \quad (43)$$

(since V_ε is nonnegative valued), and therefore also 325

$$\begin{aligned} \Xi(u_t) &\leq \frac{4\varphi_c(\bar{v})\varepsilon^{\frac{1}{4}}}{a \min\{\bar{p}_1, \bar{p}_2\}} \max\{a^2, 1\} \text{ and} \\ S_\varepsilon(t, \tau, z) &\leq \frac{4\varphi_c(\bar{v})\varepsilon^{\frac{1}{4}}}{\min\{\bar{p}_1, \bar{p}_2\}} \max\{a^2, 1\} \end{aligned} \quad (44)$$

Since $S_\varepsilon(t, \tau, z) = \sqrt{R(t, \tau, z) + \varepsilon} - \sqrt{\varepsilon} \geq \sqrt{q_2(\tau)}|z| - \sqrt{\varepsilon}$ holds 326
 for all t, τ , and z , (24) gives 327

$$\begin{aligned} \max \left\{ \int_{t-\tau}^t |u(m)| dm, \sqrt{q_2(\tau)}|z| \right\} \\ \leq \sqrt{\varepsilon} + \frac{4\varphi_c(\bar{v})\varepsilon^{\frac{1}{4}}}{\min\{a, 1\} \min\{\bar{p}_1, \bar{p}_2\}} \max\{a^2, 1\} \end{aligned} \quad (45)$$

for all $t \geq t_c$. Set 328

$$\Delta = \max \left\{ \frac{1}{\sqrt{q_2(\tau)}}, 1 \right\} \left(1 + \frac{4\varphi_c(\bar{v})}{\min\{a, 1\} \min\{\bar{p}_1, \bar{p}_2\}} \max\{a^2, 1\} \right). \quad (46)$$

Then, since $\varepsilon \in [0, 1]$, it follows that for all $t \geq t_c$, the inequalities 329

$$|z(t)| \leq \Delta \varepsilon^{\frac{1}{4}} \text{ and } \int_{t-\tau}^t |u(m)| dm \leq \Delta \varepsilon^{\frac{1}{4}} \quad (47)$$

are satisfied. Since ε is arbitrarily small, we deduce that $|z(t)|$ and 330
 $\int_{t-\tau}^t |u(m)| dm$ converge to zero when $t \rightarrow \infty$. This and the first 331
 inequality in (34) imply that $x(t) \rightarrow 0$ as $t \rightarrow \infty$. Also, by letting 332
 ε depend on the maximum of V_0 on a suitable neighborhood of the 333
 origin, we can prove the local stability part. This proves the theorem. 334

VI. ARBITRARILY LARGE DOMAINS OF ATTRACTION

Theorem 1 applies for all $\tau > 0$. On the other hand, consider the special case where $f_2 = 0$ in the decomposition (9) of F . Then, setting $\tau = 0$ in (9) and in our control (18) produces the uniformly globally asymptotically stable closed loop system $\dot{x}(t) = [A(t) + B(t)K(t, 0)]x(t)$ from Assumption 2. This suggests that the domain of attraction should become arbitrarily large as $\tau \rightarrow 0^+$ when $f_2 = 0$. Our next theorem implies that this is indeed the case. We will assume that the functions q_i and k from Assumption 2 are constant, so we omit their arguments τ . This is not restrictive, since now we only need to consider τ 's on a bounded interval; see Remark 1.

Corollary 1: Let Assumptions 1–3 hold with $f_2 = 0$ and the q_i 's and k all constant. Then for each constant $v_* > 0$, we can find values $a \in (0, q_1\sqrt{q_2}/(8k))$ and $\tau_M > 0$ (both depending on v_*) such that: For each initial condition $(\phi_x, \phi_u) \in C^0([-\tau, 0], \mathbb{R}^n \times \mathbb{R}^p)$ satisfying

$$\left| \sqrt{q_3(\tau)} \left| \phi_x(0) + \int_{-\tau}^0 \lambda(0, m + \tau) B(m + \tau) \phi_u(m) dm \right| + \frac{a}{\tau} \int_{-\tau}^0 (m + 2\tau) |\phi_u(m)| dm \right| < v_* \quad (48)$$

and each constant delay $\tau \in (0, \tau_M)$, the trajectory of (5) in closed loop with (18) converges to 0 as $t \rightarrow \infty$. $\square$

Proof: We set $\alpha_2 = 0$, so we have $\alpha_3 = 2a\alpha_1$. Then (15), (16) become

$$\begin{aligned} v_1(\tau)\alpha_1 \left(\tau, \frac{4}{q_2} v_1^2(\tau) \right) &= \frac{q_1 q_2}{32a} \text{ and} \\ v_2(\tau)\alpha_1 \left(\tau, \frac{4h^2(\tau)}{a^2} v_2^2(\tau) \right) &= \frac{a}{8\tau h^2(\tau)}. \end{aligned} \quad (49)$$

For each constant $\tau_M > 0$, Assumption 3 provides a function $\bar{\gamma}$ of class $\mathcal{K}_\infty$ such that $m\alpha_1(\tau, m^2) \leq \bar{\gamma}(m)$ for all $\tau \in [0, \tau_M]$ and $m \geq 0$. Then, replacing $\alpha_1(\tau, m^2)$ in (49) by $\bar{\gamma}(m)/m$ gives

$$\begin{aligned} \bar{\gamma} \left(\sqrt{\frac{4}{q_2}} v_1(\tau) \right) &= \frac{q_1 \sqrt{q_2}}{16a} \text{ and} \\ \bar{\gamma} \left(\frac{2h(\tau)}{a} v_2(\tau) \right) &= \frac{1}{4\tau h(\tau)} \end{aligned} \quad (50)$$

for all $\tau \in (0, \tau_M)$. Our proof of Theorem 1 shows that the conclusions of that theorem remain true when $v_1(\tau)$ and $v_2(\tau)$ are defined to be the solutions of (50). Therefore

$$\begin{aligned} v_1(\tau) &= \frac{\sqrt{q_2}}{2} \bar{\gamma}^{-1} \left(\frac{q_1 \sqrt{q_2}}{16a} \right) \text{ and} \\ v_2(\tau) &= \frac{a}{2h(\tau)} \bar{\gamma}^{-1} \left(\frac{1}{4\tau h(\tau)} \right). \end{aligned} \quad (51)$$

Also, when τ is sufficiently small, the choice

$$a = \frac{1}{\sqrt{\bar{\gamma}^{-1} \left(\frac{1}{4\tau h(\tau)} \right)}} \quad (52)$$

will satisfy our requirements (14) on a , because (52) converges to 0 as $\tau \rightarrow 0^+$ and because we are now assuming that the q_i 's and k are positive constants. Then (51) become

$$\begin{aligned} v_1(\tau) &= \frac{\sqrt{q_2}}{2} \bar{\gamma}^{-1} \left(\frac{q_1 \sqrt{q_2}}{16} \sqrt{\bar{\gamma}^{-1} \left(\frac{1}{4\tau h(\tau)} \right)} \right) \text{ and} \\ v_2(\tau) &= \frac{1}{2h(\tau)} \sqrt{\bar{\gamma}^{-1} \left(\frac{1}{4\tau h(\tau)} \right)}. \end{aligned} \quad (53)$$

Therefore, both $v_1(\tau)$ and $v_2(\tau)$ converge to ∞ when $\tau \rightarrow 0^+$. It follows that $v(\tau) \rightarrow \infty$ as $\tau \rightarrow 0^+$, so we can satisfy (48) for small enough $\tau > 0$ by choosing τ such that $v(\tau) > v_*$. The corollary now follows from Theorem 1. $\blacksquare$

VII. ILLUSTRATIVE EXAMPLE

We illustrate Theorem 1 using the 1 dimensional system from (7), which is

$$\dot{x}(t) = x(t) + u(t - \tau) + lx^2(t) \sin(x(t)) \quad (54)$$

where $u \in \mathbb{R}$ is the input, τ is a positive constant delay, and l is a positive constant. This system is not globally Lipschitz in the state x . With the notation of the previous sections, we have $A = 1$, $B = 1$, $\lambda(t, t_0) = e^{t-t_0}$, and $F(t, x) = lx^2 \sin(x)$. As we saw in Section IV, (54) satisfies our assumptions with $h(\tau) = 1$, $K(t, \tau) = -2e^\tau$, $Q(t, \tau) = 1/2$, $q_1(\tau) = 2$, $q_2(\tau) = q_3(\tau) = 1/2$, $k(\tau) = 2e^\tau$, $f_2 = 0$, $f_1(t, \tau, x) = le^\tau x^2 \sin(x)$, $\alpha_1(\tau, m) = le^\tau$ and $\alpha_2(m) = 0$. According to (14), the inequalities $0 < a \leq 1/(8\sqrt{2}e^\tau)$ have to be satisfied and, by the expression of α_3 in (13), $\alpha_3(\tau, m) = 2ale^\tau$. Choosing

$$a = \frac{1}{8\sqrt{2}e^\tau} \quad (55)$$

we can straightforwardly derive an estimate of the basin of attraction from Theorem 1 by using $v = \min\{v_1, v_2\}$, where

$$v_1(\tau) = \frac{1}{2\sqrt{2}l} \quad (56)$$

and

$$v_2(\tau) = \frac{1}{64\sqrt{2}\tau e^{2\tau} l} \quad (57)$$

which converge to ∞ as $l \rightarrow 0$ for each $\tau > 0$. On the other hand, when $\tau \in (0, 1]$, we can take

$$a = \frac{\sqrt{\tau}}{8\sqrt{2}e^\tau} \quad (58)$$

to obtain the values

$$v_1(\tau) = \frac{1}{2l\sqrt{2}\tau} \quad (59)$$

and

$$v_2(\tau) = \frac{1}{64le^{2\tau}\sqrt{2}\tau} \quad (60)$$

so $v(\tau) = \min\{v_1(\tau), v_2(\tau)\}$ converges to ∞ as l converges to zero for fixed $\tau > 0$, or as τ converges to zero for fixed l , so the basin of attraction becomes arbitrarily large. This gives convergence of the closed loop solution to 0.

If, on the other hand, we had chosen, $f_1 = 0$ and $f_2(t, x) = l x^2 \sin(x)$, then one could choose $\alpha_1 = c_0$ for any constant $c_0 > 0$ and $\alpha_2(m) = l$. This gives $\alpha_3(\tau, m) = 2ac_0 + (1/\sqrt{2})l$. Then the corresponding solutions of (15), (16) with the choice

$$a = \frac{1}{8\sqrt{2}e^\tau} \quad (61)$$

satisfy

$$v_1(\tau) \leq \frac{\sqrt{2}}{16l} \quad (62)$$

and

$$v_2(\tau) \leq \frac{1}{256\sqrt{2}e^{2\tau}\tau l} \quad (63)$$

which would mean that $v(\tau) = \min\{v_1(\tau), v_2(\tau)\}$ does not converge to ∞ as τ goes to zero. Thus, the choice $f_1 = 0$ and $f_2(t, x) = l x^2 \sin(x)$ is conservative.

VIII. CONCLUSION

Stabilization of nonlinear systems with input delays is a central problem that has been studied by many authors using model reduction, prediction, and other methods. Here we adapted the reduction model approach to the problem of locally asymptotically stabilizing the origin of time varying nonlinear systems with pointwise input delays. Our method of proof makes it possible to determine an estimate of the basin of attraction. The result can be adapted to the case where the delay in the input is distributed. Our results can be combined with those of [5] and [10].

APPENDIX TECHNICAL LEMMA

We used the following to get (30) in the second part of the proof of Theorem 1:

Lemma 1: Let $\varepsilon \in (0, 1]$ be a positive real number. Then

$$-\frac{r}{\sqrt{r+\varepsilon}} \leq -\sqrt{r} + \varepsilon^{\frac{1}{4}}[1 + \sqrt{r}] \quad (64)$$

holds for all $r \geq 0$.

Proof: Let $r \geq 0$ be given. We first prove that

$$\frac{r}{\sqrt{r+\varepsilon}} \geq \frac{1}{\sqrt{1+\sqrt{\varepsilon}}} \sqrt{r} - \varepsilon^{\frac{1}{4}}. \quad (65)$$

If $\sqrt{r}/(\sqrt{1+\sqrt{\varepsilon}}) - \varepsilon^{1/4} \leq 0$, then (65) is satisfied. On the other hand, if $\sqrt{r}/(\sqrt{1+\sqrt{\varepsilon}}) - \varepsilon^{1/4} \geq 0$, then $r \geq (1+\sqrt{\varepsilon})\sqrt{\varepsilon}$. It follows that $(\sqrt{\varepsilon}+1)r \geq (1+\sqrt{\varepsilon})\varepsilon + r \geq \varepsilon + r$. Consequently, $r/(r+\varepsilon) \geq 1/(\sqrt{\varepsilon}+1)$. Taking the square root, and then multiplying through by $\sqrt{r}$, we obtain

$$r\sqrt{\frac{1}{r+\varepsilon}} \geq \frac{\sqrt{r}}{\sqrt{\sqrt{\varepsilon}+1}}. \quad (66)$$

Therefore, (65) holds in both cases. Next, observe that (65) implies that

$$\begin{aligned} -\frac{r}{\sqrt{r+\varepsilon}} &\leq -\sqrt{r} + \left[1 - \frac{1}{\sqrt{1+\sqrt{\varepsilon}}}\right] \sqrt{r} + \varepsilon^{\frac{1}{4}} \\ &\leq -\sqrt{r} + \left[\sqrt{1+\sqrt{\varepsilon}} - 1\right] \sqrt{r} + \varepsilon^{\frac{1}{4}}. \end{aligned} \quad (67)$$

Hence, the relation $\sqrt{b+c} \leq \sqrt{b} + \sqrt{c}$ for nonnegative values b and c gives $-r/\sqrt{r+\varepsilon} \leq -\sqrt{r} + \varepsilon^{1/4}\sqrt{r} + \varepsilon^{1/4}$. This gives the conclusion. $\square$

REFERENCES

- [1] D. Mayne, "Control of linear systems with time delay," *Electron. Lett.*, vol. 4, no. 20, pp. 439–440, 1968.
- [2] Z. Artstein, "Linear systems with delayed controls: A reduction," *IEEE Trans. Autom. Control*, vol. 27, no. 4, pp. 869–879, 1982.
- [3] W. Kwon and A. Pearson, "Feedback stabilization of linear systems with delayed control," *IEEE Trans. Autom. Control*, vol. 25, no. 2, pp. 266–269, 1980.
- [4] A. Manitius and A. Olbrot, "Finite spectrum assignment problem for systems with delays," *IEEE Trans. Autom. Control*, vol. 24, no. 4, pp. 541–553, 1979.
- [5] F. Mazenc and D. Normand-Cyrot, "Reduction model approach for linear systems with sampled delayed inputs," *IEEE Trans. Autom. Control*, vol. 58, no. 5, pp. 1263–1268, 2013.
- [6] E. Witrant, "Stabilisation des systèmes commandés par réseaux," Ph.D. dissertation, INPG, Laboratoire d'Automatique de Grenoble, Grenoble, France, 2005.
- [7] D. Georges, G. Besançon, Z. Benayache, and E. Witrant, "A nonlinear state feedback design for nonlinear systems with input delay," in *Proc. Eur. Control Conf.*, Kos, Greece, 2007, pp. 348–353.
- [8] M. Krstic, *Delay Compensation for Nonlinear, Adaptive, PDE Systems*. Boston, MA: Birkhauser, 2009.
- [9] M. Krstic and N. Bekiaris-Liberis, "Compensation of infinite-dimensional input dynamics," *IFAC Annu. Rev.*, vol. 34, no. 2, pp. 233–244, 2010.
- [10] F. Mazenc, M. Malisoff, and S.-I. Niculescu, "Reduction model approach for linear time-varying systems with delays," *IEEE Trans. Autom. Control*, to be published.
- [11] F. Mazenc and S.-I. Niculescu, "Generating positive and stable solutions through delayed state feedback," *Automatica*, vol. 47, no. 3, pp. 525–533, 2011.
- [12] M. Krstic, "Input delay compensation for forward complete and feedforward nonlinear systems," *IEEE Trans. Autom. Control*, vol. 55, no. 2, pp. 287–303, 2010.
- [13] F. Mazenc, S. Mondié, and R. Francisco, "Global asymptotic stabilization of feedforward systems with delay in the input," *IEEE Trans. Autom. Control*, vol. 49, no. 5, pp. 844–850, 2004.
- [14] I. Karafyllis and M. Krstic, "Nonlinear stabilization under sampled and delayed measurements, with inputs subject to delay and zero-order hold," *IEEE Trans. Autom. Control*, vol. 57, no. 5, pp. 1141–1154, 2012.
- [15] M. Malisoff and F. Mazenc, *Constructions of Strict Lyapunov Functions, ser. Communications and Control Engineering Series*. New York: Springer, 2009.
- [16] F. Mazenc, S.-I. Niculescu, and M. Krstic, "Lyapunov-Krasovskii functionals and application to input delay compensation for linear time-invariant systems," *Automatica*, vol. 48, no. 7, pp. 1317–1323, 2012.
- [17] N. Bekiaris-Liberis and M. Krstic, "Compensation of state-dependent input delay for nonlinear systems," *IEEE Trans. Autom. Control*, vol. 58, no. 2, pp. 275–289, 2013.

AUTHOR QUERY

NO QUERY.

IEEE
Proof