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Stock price modeling and insider trading theory

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STOCK PRICE MODELING AND INSIDER TRADING THEORY

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Table of Contents

Abstract

The mathematical study of stock price modeling using Brownian motion and stochastic calculus is a relatively new field. The randomness of financial markets, geometric brownian motions, martingale theory, Ito's lemma, enlarged filtrations, and Girsanov's theorem provided the motivation for a simple characterization of the concepts of stock price modeling.

This work presents the theory of stochastic calculus and its use in the financial market. The problems on which we focus are the models of an investor's portfolio of stocks with and without the possibility of insider trading, opportunities for fair pricing of an option, enlarged filtrations, consumptions, and admissibility.

This survey has two parts. The first part explores the theoretical aspects of stochastic calculus, and the second part shows its application in predicting stock prices and the wealth of an investor's portfolio.

Chapter 1. Introduction

The subject of money is fascinating to many people. Whether one has a little or a lot, a good portion of one's time is spent trying to figure out how to make more, spend less, or stretch it as far as possible.

The world we live in has never been very predictable. Randomness has always been a part of daily activity. Time, change, and uncertainty encompass some of the most important business, economical, and financial decisions. Stochastic models become the tools required to deal with such problems. Stochastic calculus grew out of the need to assign meaning from ordinary differential equations to continuous-time stochastic processes.

Discussions of stochastic models in the financial market presented in this paper focus on several topics:

- Motivation of stochastic calculus and its use in stock price modeling.
- Portfolio analysis of an investor.
- Information of an insider and its effects on the market.
- The fair price of an option.

Chapter 2. Stochastic Calculus

A range of problems came under the field of the theory of functions of a real variable when Newton and Leibniz invented calculus. The main focus of this invention was the use of differentiation to describe rates of change and limits in approximating sums. Similar concepts are needed when dealing with stochastic environments.

In financial markets, pricing assets deal with stochastic variables. Therefore, the notion of risk becomes an important factor. Can the same differential rules apply when dealing with stochastic variables?

Consider a one-dimensional continuous stochastic process $S(t)$ where, $t \in [0, T]$. We define the stochastic differential equation

$$dS(t) = a(S(t), t)dt + b(S(t), t)dB(t)$$

where $B(t)$ represents unpredictable events occurring in the interval dt. Since $dS(t)$, and $dB(t)$ are random increments, they have to be justified by means other than deterministic calculus.

2.1 Geometric Brownian Motion

In 1828, botanist Robert Brown observed the irregular movements of pollen suspended in water, a phenomenon which is now known as Brownian movements. The range of applications of Brownian motion goes further than the study of particles suspended in water. The use of Brownian motion is the basis for modeling certain aspects of the financial market [14].

Definition 2.1.1. A *Brownian Motion*, $B(t,\omega)$, is a process defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ satisfying the following properties:

1. $P(\omega; B(0,\omega) = 0) = 1$.

2. For any $0 \leq s < t$, the random variable $B(t) - B(s)$ has distribution $N(0, t-s)$.

3. For any $0 \leq t_1 < t_2 < \cdots < t_n$, the random variables $B(t_1), B(t_2) - B(t_1), \cdots, B(t_n) - B(t_{n-1})$ are independent.

4. $P(\omega; B(\cdot\,,\omega)$is a continuous function$) = 1$.

The first quantitative work on Brownian motion was introduced in 1900 by the French mathematician Bachelier who used it in his dissertation to model the price movements of stocks and commodities [20]. However, Brownian motion has two major flaws that does not allow this model to function properly in an financial market setting.

1. Stock prices are always positive, and since the price of a stock is a normal random variable, it can theoretically become negative.

2. Fluctuations in the price are proportional to the price of the stock.

Instead we introduce a nonnegative functional of Brownian motion called *Geometric Brownian motion* which is defined as

$$S(t) = S_0 \exp X(t),$$

where $X(t) = \sigma B_t + (\mu - \frac{1}{2}\sigma)t$ is a Brownian motion.

2.2 Martingales

One of the most fundamental concepts of the theory of finance are martingales. Consider a real-valued process $S = S_t : 0 \leq t \leq \infty$ on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

Definition 2.2.1. A *filtration*, $\mathcal{F}$, is an increasing family of σ-fields, i.e:

$$\mathcal{F} = (\mathcal{F}_t : t \in [0,T]), \quad \forall \quad s \leq t, \quad \mathcal{F}_s \subseteq \mathcal{F}_t.$$

For $0 \leq t_1 < t_2 < \cdots < t_n$ this family of information sets will satisfy

$$\mathcal{F}_{t0} \subseteq \mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2} \subseteq \cdots \subseteq \mathcal{F}_{t_n} \subseteq \cdots$$

In the financial perspective, one can think of $\mathcal{F}$ as containing the information available to the investor at time t. The investor keeps all past and present information about the market and we assume there is no foreknowledge of events.

Definition 2.2.2. A process, $S(t)$, is a *martingale* with respect to the filtration $\mathcal{F}$ if for any $t > 0$, $S(t)$ is $\mathcal{F}_t$-measurable and satisfies:

1. $E|S(t)| < \infty$;
2. $E(S(u)|\mathcal{F}_t) = S(t) \quad$ whenever $\quad t \leq u$.

Based on the definition of a *martingale*, the expected value of a stock price in the future given the information known at the present time t, is equal to the present stock price. That is, stock prices that are martingales are risk neutral.

Definition 2.2.3. A process, $S(t)$, is a *submartingale*(respectively, a *supermartingale*) with respect to the filtration $\mathcal{F}$ if for any $t > 0$, $S(t)$ is $\mathcal{F}_t$-measurable and satisfies:

1. $E|S(t)| < \infty$;
2. $E[S(u)|\mathcal{F}_t] \geq S(t)$ (respectively, $E[S(u)|\mathcal{F}_t] \leq S(t)$) $\quad$ whenever $\quad t \leq u$.

However, stock prices are not completely unpredictable. In general, the price of a stock is expected to increase over time, therefore stock prices are expected to follow *submartingale* properties.

2.3 Motivation of the Stochastic Derivative

From standard calculus we know the derivative of a function f can be defined as

$$lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f_x. \tag{2.3.1}$$

Suppose f is a function of a stochastic process x. A Taylor series expansion of $f(x)$ around x_0 will give

$$f(x) = f(x_0) + f_x(x_0)[x-x_0] + \frac{1}{2}f_{xx}(x_0)[x-x_0]^2 + \frac{1}{3!}f_{xxx}(x_0)[x-x_0]^3 + \cdots + R. \tag{2.3.2}$$

Rewrite $f(x)$ as $f(x_0+\Delta x)$, where $\Delta x = x - x_0$. Hence, the Taylor series can be written as

$$f(x_0+\Delta x) - f(x_0) = f_x(x_0)[\Delta x] + \frac{1}{2}f_{xx}(x_0)[\Delta x]^2 + \frac{1}{3!}f_{xxx}(x_0)[\Delta x]^3 + \cdots + R. \tag{2.3.3}$$

If the variable x were deterministic, one would say that the term $(\Delta x)^2$ was small, therefore negligible. However, we assumed x was a random variable, therefore, changes in x are random.

What happens to $(\Delta x)^3$,$(\Delta x)^4$,...? We will show that the quadratic variations of the random process x converge to a meaningful random variable, while the higher ordered variations approximate to zero.

Definition 2.3.1. A *continuous semimartingale* $X = \{X_t, \mathcal{F}_t; 0 \le t \le \infty\}$ is an adapted stochastic process which has the decomposition

$$X_t = M_t + A_t, \qquad 0 \le t < \infty,$$

where M_t is a continuous local martingale and A_t is a continuous stochastic process with paths of finite variation a.s.

Lemma 2.3.2. For a continuous semimartingale, $X = \{X_t, \mathcal{F}_t; 0 \le t \le \infty\}$ has quadratic variation that converges to a random variable, while higher-order variations of X converges to 0.

Proof. Let $[A]_t = \lim_{n\to\infty} \sum_{i=1}^{n} |A_{ti+1} - A_{ti}|^2$, where limit is in the sense of probability.

$\lim_{n\to\infty} \sum_{i=1}^{n} |A_{ti+1} - A_{ti}|^2 \le \lim_{n\to\infty} \sum_{i=1}^{n} |A_{ti+1} - A_{ti}| \cdot \max_i |A_{ti+1} - A_{ti}|$.

Since A_t has finite variations, there exist $K > 0$ such that $\lim_{n\to\infty} \sum_{i=1}^{n} |A_{ti+1} - A_{ti}| < K$

By the continuity of A_t,

$$\max_i |A_{ti+1} - A_{ti}| < \epsilon, \quad \text{when} \quad |t_{i+1} - t_i| < \delta, \quad \text{for some} \quad \delta > 0. \qquad (2.3.4)$$

Hence, $\lim_{n\to\infty} \sum_{i=1}^{n} |A_{ti+1} - A_{ti}|^2 < K \cdot \epsilon$.

Allowing $\epsilon \to 0$, we have $[A]_t = \lim_{n\to\infty} \sum_{i=1}^{n} |A_{ti+1} - A_{ti}|^2 \to 0$.

$[X]_t = [M]_t + 2[M, A]_t + [A]_t$.

By *Cauchy-Schwartz inequality*:

$$[M, A]_t \le [M]_t^{\frac{1}{2}} \cdot [A]_t^{\frac{1}{2}}.$$

Hence, $[M, A]_t = 0$. Therefore, $[X]_t = [M]_t$.

Consider,

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n}|X_{ti+1}-X_{ti}|^3 &\leq [X]_t\cdot\max_i|X_{ti+1}-X_{ti}| \\
&= [M]_t\cdot\max_i|X_{ti+1}-X_{ti}| \\
&= [M]_t\cdot\max_i|M_{ti+1}-M_{ti}+A_{ti+1}-A_{ti}| \\
&\leq [M]_t\cdot\left(\max_i|M_{ti+1}-M_{ti}|+\max_i|A_{ti+1}-A_{ti}|\right).
\end{aligned}$$

Continuity of M_t implies that for any $\omega\in\Omega$, there exist an $\epsilon>0$ such that

$$|M_u(\omega)-M_v(\omega)|<\epsilon \quad \text{when} \quad |u-v|<\delta \quad \text{for} \quad \delta>0$$

From 2.3.4 we know $\max_i|A_{ti+1}-A_{ti}|\to 0$, hence

$$[M]_t\cdot\left(\max_i|M_{ti+1}-M_{ti}|+\max_i|A_{ti+1}-A_{ti}|\right)\to 0 \qquad \text{as} \quad \epsilon\to 0.$$

Therefore,

$$\lim_{n\to\infty}\sum_{i=1}^{n}|X_{ti+1}-X_{ti}|^3\to 0.$$

□

Notation 2.3.3. Similar arguments hold for higher-ordered variations.

As a result of 2.3.2, the higher-ordered terms of δx can be approximated by zero. The Taylor approximation can be written as

$$f(x_0+\Delta x)-f(x_0)\cong f_x(x_0)[\Delta x]+\frac{1}{2}f_{xx}(x_0)[\Delta x]^2. \tag{2.3.5}$$

Therefore, the total change in $f(x_0)$ is given by (2.3.5).

Theorem 2.3.4 (The Ito Formula). *Let $F(S(t),t)$ be a twice differentiable function of t and of the random process $S(t)$ where*

$$dS(t)=a(t)dt+\sigma(t)dB(t), \quad t\geq 0,$$

with drift and diffusion parameters, $a(t),\sigma(t)$. Then we have

$$dF(S(t),t)=\frac{\partial F}{\partial S(t)}dS(t)+\frac{\partial F}{\partial t}dt+\frac{1}{2}\frac{\partial^2 F}{\partial S(t)^2}\sigma^2(t)dt,$$

2.4 Market Efficiency

A fundamental assumption about the market known as the *Efficient Market hypothesis*, claims that asset prices quickly reflect all available information. This hypothesis states that a market digests new information in such an efficient way that all the current information about the market development, including past prices, is at all times contained in the present price. The basic idea behind any test for market efficiency is that if investors can use the information to earn abnormal profits, the market is not efficient.

Definition 2.4.1. The *weak-form* of the efficient market hypothesis states that all information contained in past prices are reflected in current stock prices.

If the weak-form of market efficiency were not true, investors would make above-average returns by interpreting the past history of stock prices. Unfortunately, the competition in the marketplace ensures that the weak-form of the efficient market hypothesis holds by responding to the increasing demand and price.

Definition 2.4.2. Let d be a positive integer and $\mathbf{P}$ be a probability measure on $(\Re^d, \mathcal{B}(\Re^d))$. A d-dimensional process $S = (S_t, \mathcal{F}_t; t \geq 0)$ on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is said to be a *Markov process* if:

- The stochastic process S_n is adapted to the filtration $\mathcal{F}_n$ and
- $\mathbf{P}(S_{k+1} \in A|\mathcal{F}_k) = \mathbf{P}(S_{k+1} \in A|S_k)$ for all $A \in \mathcal{B}(\Re^d)$, $\mathbf{P}$- a.s.

According to this definition, future movements in S_k, given what we observe until time k, are likely to be the same as starting the process at time k. That is, no advantage is gained by taking into account any of the previous price evolutions. Therefore, the Markov property of stock prices is consistent with the weak-form of market efficiency.

Chapter 3. Portfolio of an Investor

3.1 Stock Price Process

Consider a non-dividend-paying risky asset, S_t, with the following parameters:

- $\mu \in \Re$, the expected rate of return.
- $\sigma > 0$, the volatility.
- $S_0 > 0$, the initial stock price.

The stock price follows a continuous-time model:

$$S_t = S_0 e^{\sigma B_t + (\mu - \frac{1}{2}\sigma^2)t} \qquad t \in [0, \infty) \tag{3.1.1}$$

Example 3.1.1. *Deriving the Stock Price model:*

The stock price model is given by the linear stochastic differential equation:

$$dS(t) = \mu S dt + \sigma S dB(t).$$

Integrating both sides we get,

$$\int_0^t \frac{dS(u)}{S(u)} = \mu t + \sigma B(t). \tag{3.1.2}$$

Since $S(t)$ is a random variable, Ito's formula is used. Let $F(S) = ln(S)$, by Ito's formula (2.3.4):

$$\begin{aligned} dF(S) &= \frac{\partial F}{\partial S} dS + \frac{1}{2}\frac{\partial^2 F}{\partial S^2} dS^2 \\ &= \frac{1}{S} dS + \frac{1}{2}(\frac{-1}{S^2})(S^2\sigma^2 dt) \\ &= \frac{dS}{S} - \frac{1}{2}\sigma^2 dt \end{aligned}$$

Therefore,

$$\int_0^t dF(S) = \int_0^t \frac{dS(u)}{S(u)} - \int_0^t \frac{1}{2}\sigma^2 du.$$

We can rewrite this as

$$\begin{aligned}
\int_0^t \frac{dS(u)}{S(u)} &= \int_0^t dF(S) + \int_0^t \frac{1}{2}\sigma^2 du \\
&= ln(S(t) - ln(S(0)) + \frac{1}{2}\sigma^2 t.
\end{aligned}$$

Combining 3.1.2 and 3.1.3:

$$\begin{aligned}
\mu t + \sigma B(t) &= ln(\frac{S(t)}{S(0)}) + \frac{1}{2}\sigma^2 t. \\
ln(\frac{S(t)}{S(0)}) &= \mu t + \sigma B(t) - \frac{1}{2}\sigma^2 t.
\end{aligned}$$

Hence,

$$S(t) = S(0)e^{\sigma B(t) + (\mu - \frac{1}{2}\sigma_2)t}.$$

3.1.1 Multivariate Case

In general, most investors will have more than one risky asset. Consider a set of n-risky assets. We can use the linear stochastic differential equation (3.1.1), where

$$S(t) = \begin{bmatrix} S_1(t) \\ S_2(t) \\ \vdots \\ S_n(t) \end{bmatrix}, \mu(t) = \begin{bmatrix} \mu_1(t) \\ \mu_2(t) \\ \vdots \\ \mu_n(t) \end{bmatrix}, \sigma = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ & \ddots & \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{bmatrix}.$$

Hence the price $S_i(t)$ for a share of the ith stock at time t is modeled by the linear stochastic differential equation:

$$dS_i(t) = \mu_i(t)S_i(t)dt + S_i(t)\sum_{n=1}^{j} \sigma_{ij} dB_i(t).$$

3.2 Portfolio Analysis of a Small Investor

The structure of a portfolio of an investor relies on the theory of stochastic calculus and differential equations to model the risky assets in continuous time.

Definition 3.2.1. A *market* is d+1-valued stochastic process $S(t) = S_0(t), \cdots, S_d(t)$.

Definition 3.2.2. Let $\theta_i(t)$ denote the number of shares of asset i owned by the investor at time t. The *value*, (*wealth*), $\pi(t)$ of a portfolio $\pi(t)$ for the market $S(t)$ is given by

$$X(t) = \sum_{i=0}^{d} \theta_i(t) S_i(t) \tag{3.2.1}$$

Definition 3.2.3. A *portfolio process* $\pi = \pi(t) = (\pi_1(t), \cdots, \pi_d(t)), \mathcal{F}_t$ is a stochastic process where $\pi_j(t)$'s are non anticipating and

$$\sum_{i=1}^{d} \int_T^0 \pi_i^2(t) dt < \infty, a.s. \tag{3.2.2}$$

We denote $\pi_i(t) = \theta_i(t) S_i(t)$ as the amount invested in the i-th stock, $1 \leq i \leq d$, at time t.

Definition 3.2.4. A portfolio π is said to be *self-financing* if

$$dX(t) = d\pi = \sum_{i=0}^{d} \theta_i(t) dS_i(t)$$

Definition 3.2.5. A *consumption process* $C = C_t, \mathcal{F}_t$ is a measurable , adapted process such that, $C \in [0, \infty)$ and

$$\int_T^0 C_t < \infty, a.s. \tag{3.2.3}$$

Example 3.2.6 (Wealth of a small investor). Consider an investor who has $d+1$-asset from a given market where these assets are traded continuously. One of these assets is riskless, and has a price $S_0(t)$ which evolves according to the differential equation

$$dS_0(t) = r(t) S_0(t) dt, \qquad 0 \leq t \leq T. \tag{3.2.4}$$

The remaining risky assets have a price modeled by the linear stochastic differential equation

$$dS_i(t) = \mu_i(t) S_i(t) dt + S_i(t) \sum_{j=1}^{n} \sigma_{ij} dB_i(t) \qquad i = 1 \cdots d. \tag{3.2.5}$$

If the investor choses at time $t+\epsilon$ to consume an amount $C_{t+\epsilon}$, then the rate of change of the reduced wealth would be

$$dX(t) = \sum_{i=0}^{d} \theta_i(t) dS_i(t) - C dt \tag{3.2.6}$$

The wealth of the portfolio can be written as

$$dX(t) = [r(t)X(t) - C_t]dt + \sum_{i=1}^{d} [\mu_i(t) - r(t)]\pi_i(t)d(t) + \sum_{j=1}^{d}\sum_{i=1}^{d} \pi_i(t)\sigma_{ij} dB_j(t).$$

Which has the solution:

$$X(t) = e^{\int_0^t r(s)ds} \left[X_0 + \int_0^t e^{-\int_0^s r(u)du} \left[\pi(s)^T(\mu(s) - r(s)\mathbf{1}) - C_s \right] ds + \int_0^t e^{-\int_0^s r(u)du} \pi(s)^T \sigma(s) dB(s) \right] \tag{3.2.7}$$

Definition 3.2.7. A pair (π, C) is said to be *admissible* for the initial endowment $X(0)$ if the wealth process 3.2.2 satisfies

$$X(t) \geq 0; \qquad 0 \leq t \leq T, \quad \text{a.s.}$$

Definition 3.2.8. An *arbitrage* is a portfolio that starts with $X(0) = 0$ and ends with

$$\mathbf{P}[X(t) \geq 0] = 1, \qquad \mathbf{P}[X(t) > 0] > 0 \text{ a.s.}$$

Chapter 4. Insider Trading

Suppose we have a natural filtration, $\mathcal{F}_t$ on a trading interval $[0,1]$. We let the financial market model consist of a mean rate of return process μ and a volatility process σ which determines the stock price process given by

$$dS(t) = \mu(t)dt + \sigma dB(t).$$

There are two type of traders that can act on the market:

- The regular trader whose information levels correspond to the natural information flow of the market $\mathcal{F}_t$

- The inside trader, who has access to information beforehand.

This knowledge consists of a random variable G that is $\mathcal{F}_1$-measurable. Hence the insider's filtration is given by $(\mathcal{G}_t), t \in [0,1]$ where $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(G)$.

4.1 Enlargement of a Filtration: An Example of $G = B(1)$

Let $G = B(1)$. The process $\{B(t)\}$ is an $\mathcal{F}_t$-adapted Brownian motion and does not remain as a Brownian motion under the enlarged filtration $\mathcal{G}_t$. In order to find the semimartingale decomposition of $\{B(t)\}$ under the new filtration, we need the following lemmas:

Lemma 4.1.1. For any $T > t$, the conditional distribution of $B(t)$ given $B(T)$ is the $N(\frac{B(T)t}{T}; \frac{t(T-t)}{T})$ distribution.

Proof.

$$\begin{aligned}
f_{B(t)|B(T)}(x,y) &= \frac{f(x,y)}{\int_{-\infty}^{\infty} f(x,y)dx} \\
&= \frac{f_{B(t)|B(T)-B(t)}(x,y-x)}{f_{B(T)}(y)}. \\
\text{By iid of increments:} &= \frac{f_{B(t)}(x)\cdot f_{B(T)-B(t)}(y-x)}{f_{B(T)}(y)} \\
&= \frac{\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}\frac{1}{\sqrt{2\pi(T-t)}}e^{-\frac{(y-x)^2}{2(T-t)}}}{\frac{1}{\sqrt{2\pi T}}e^{-\frac{y^2}{2T}}} \\
&= \sqrt{\frac{T}{2\pi t(T-t)}}e^{-\frac{(x-y)^2 T}{2t(T-t)}}
\end{aligned}$$

Hence, $B(t)|B(T) \sim N(\frac{B(T)t}{T}; \frac{t(T-t)}{T})$. □

Lemma 4.1.2. For any $0 < s < r < t < 1$,

$$E\left[B(t) - B(s)|\mathcal{G}_s\right] = \frac{t-s}{1-s}(B(1) - B(s).$$

Proof. Let $s < t$ be fixed in the interval $[0,1]$.

$$\begin{aligned}
E\left[B(t) - B(s)|\mathcal{G}_s\right] &= E\left[B(t) - B(s)|B(u) : u \leq s, B(1)\right] \\
&= E\left[B(t) - B(s)|B(u) : 0 \leq u \leq s, B(1) - B(s)\right] \\
&= E\left[B(t) - B(s)|B(1) - B(s)\right] \text{ by independence of increments.}
\end{aligned}$$

Define a processes $\hat{B}$ as follows: $\hat{B}(r) = B(s+r) - B(s)$ for all $r \geq 0$, and a fixed s. $\{\hat{B}\}_{r\geq 0}$ is a standard Brownian motion. By lemma 4.1.1, the conditional distribution of $\hat{B}(t-s)$ given $\hat{B}(1-s)$ is the normal distribution

$$N\left(\frac{\hat{B}(1-s)(t-s)}{1-s}; \frac{(t-s)(1-t)}{(1-s)}\right).$$

Thus,

$$E\left[\hat{B}(t-s)|\hat{B}(1-s)\right] = \frac{\hat{B}(1-s)(t-s)}{(1-s)}.$$

That is,

$$E\left[B(t) - B(s)|B(1) - B(s)\right] = \frac{B(1) - B(s)(t-s)}{(1-s)}.$$

□

Hence, the expected value of $B(t)$ under the enlarged filtration is

$$E(B(t)|\mathcal{G}_s) = B(s) + \frac{t-s}{1-s}(B(1) - B(s)). \tag{4.1.1}$$

Theorem 4.1.3. *The process* $\{B(t) - \int_0^t \frac{1}{1-u}(B(1) - B(u))du\}_{0\le t\le 1}$ *is a* $\{\mathcal{G}\}_{0\le t\le 1}$*-adapted Brownian motion.*

Proof. Suppose $\mathcal{G}_s = F_s \vee \sigma(B(1))$. Let $s < r < t < 1$, then $\mathcal{G}_s \subset \mathcal{G}_r$.

$$\begin{aligned} E\left(B(t)|\mathcal{G}_s\right) &= E\left[E(B(t)|\mathcal{G}_r)|\mathcal{G}_s\right] \\ \text{By (4.1.1), } E\left(B(t)|\mathcal{G}_s\right) &= E[B(r) + \frac{t-r}{1-r}(B(1) - B(r))|\mathcal{G}_s] \\ &= E(B(r)|\mathcal{G}_s) + \frac{t-r}{1-r}E(B(1) - B(r)|\mathcal{G}_s). \end{aligned}$$

Suppose we partition the interval $s < t < 1$ such that

$$s = r_{n+1} < r_n < r_{n-1} \cdots < r_1 < r_0 = t < 1.$$

Using equation (4.1.2) with $r = r_1, r_2, ...$

$$\begin{aligned}
E\left(B(t)|\mathcal{G}_s\right) &= E(B(r_1)|\mathcal{G}_s) + \frac{t-r_1}{1-r_1}E(B(1)-B(r_1)|\mathcal{G}_s) \\
&= E(B(r_2)|\mathcal{G}_s) + \frac{r_1-r_2}{1-r_2}E(B(1)-B(r_2)|\mathcal{G}_s) + \frac{t-r_1}{1-r_1}E(B(1)-B(r_1)|\mathcal{G}_s) \\
&\vdots \\
&= E(B(r_n)|\mathcal{G}_s) + \sum_{j=1}^{n}\frac{r_{j-1}-r_j}{1-r_j}E(B(1)-B(r_j)|\mathcal{G}_s) \\
&= B(s) + \sum_{j=1}^{n+1}\frac{r_{j-1}-r_j}{1-r_j}E(B(1)-B(r_j)|\mathcal{G}_s) \\
&= B(s) + E\left(\sum_{j=1}^{n+1}\frac{r_{j-1}-r_j}{1-r_j}(B(1)-B(r_j)|\mathcal{G}_s\right) \\
&= B(s) + E\left(\int_s^t \frac{1}{1-u}(B(1)-B(u))du|\mathcal{G}_s\right), \quad \text{as } n\to\infty \text{ with } \lim_{j\to\infty}|r_{j-1}-r_j| \\
&= B(s) + E\left(\int_0^t \frac{1}{1-u}(B(1)-B(u))du|\mathcal{G}_s\right) - \int_0^s \frac{1}{1-u}(B(1)-B(u))du,
\end{aligned}$$

since $B(1)-B(u)$ is $\mathcal{G}$-measurable if $u \leq s$.

$$E(B(t)|\mathcal{G}_s = B(s)+E\left(\int_0^t \frac{1}{1-u}(B(1)-B(u))du|\mathcal{G}_s\right)-\int_0^s \frac{1}{1-u}(B(1)-B(u))du.$$

Therefore,

$$E(B(t) - \int_0^t \frac{1}{1-u}(B(1)-B(u))du|\mathcal{G}_s) = B(s) + \int_0^s \frac{1}{1-u}(B(1)-B(u))du.$$

Hence $B(t) - \int_0^t \frac{1}{1-u}(B(1)-B(u))du$ is a $\mathcal{G}$-martingale. Since the quadratic variation of $\{B(t) - \int_0^t \frac{1}{1-u}(B(1)-B(u))du\}_{t\geq 0}$ is t, we get that $\{B(t) - \int_0^t \frac{1}{1-u}(B(1)-B(u))du\}$ is a $\mathcal{G}$-Brownian motion. □

We define the new stock price process in terms of the enlarged filtration as

$$dS(t) = \mu S(t)dt + \sigma S(t)dM(t) + \sigma S(t)\frac{B(1)-B(t)}{1-t}dt. \tag{4.1.2}$$

where $M(t) = B(t) - \int_0^t \frac{1}{1-u}(B(1)-B(u))du$ is a $\mathcal{G}$-martingale.

Remark 4.1.4. As an example the filtration $\mathcal{G}$ was an enlargement of $\mathcal{F}$ by $B(1)$. In general, an enlarged filtration has the form

$$\int_0^1 f(u)dB(u), \qquad \text{where} f \in L^2[0,1].$$

Let $\int_0^1 f^2(u)du > 0$, for all $t < 1$, then under enlargement of filtration,

$$B(t) - \int_0^t \frac{\int_u^1 f(v)dB(v)}{\int_u^1 f^2(v)dv} f(u)du$$

is a $\mathcal{G}$ -Brownian Motion [15].

4.2 Change of Measure

Theorem 4.2.1 (Girsanov Theorem). *Let $\{B(t)\}_{0\le t\le T}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$. Let $\{\mathcal{F}_t\}_{0\le t\le T}$ be the accompanying filtration, and let $\{X(t)\}$ be an adapted process such that $E\left[e^{\frac{1}{2}\int_0^T |X(u)|^2 du}\right] < \infty$.*

For $0 \le t \le T$ let $M(t)$ be the martingale defined by

$$M(t) = e^{-\int_0^t X(u)dB(u) - \frac{1}{2}\int_0^t X(u)^2 du}.$$

Let $\mathbf{Q_t}$ be a new probability measure on $(\Omega, \mathcal{F}_t)$ given by

$$\mathbf{Q}_t(A) = \int_A M(t) d\boldsymbol{P}, \qquad A \in \mathcal{F}_t.$$

If $\mathbf{Q}_T$ is denoted by $\mathbf{Q}$, then $\mathbf{Q}|_{\mathcal{F}_t} = \mathbf{Q}_t$, and the stochastic process $\hat{B}(t)$ defined by $\hat{B}(t) = B(t) + \int_0^t X(u)du, t \in [0,T]$ is a Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{Q})$.

In general, bonds and stocks exhibit submartingale (2.2.3) characteristics. To find a fair market value of an asset, $\frac{S(t)}{S_0(t)}$ should be converted to a martingale. Using the Girsanov Theorem, one could easily change the measure $\mathbf{P}$ so that the drift of $\frac{S(t)}{S_0(t)}$ is zero.

Recall the following equation for the wealth of an investor:

$$dX(t) = [r(t)X(t) - C_t]dt + \sum_{i=1}^{d}[\mu_i(t) - r(t)]\pi_i(t)dt + \sum_{j=1}^{d}\sum_{i=1}^{d}\pi_i(t)\sigma_{ij}dB_j(t) \quad (4.2.1)$$

Girsanov theorem allows us to remove the drift $\sum_{i=1}^{d}[\mu_i(t) - r(t)]$ by considering a new Brownian motion, $\hat{B}$, under a different probability measure $\mathbf{Q}$. Let us introduce the process

$$\eta(t) = \sigma(t)^{-1}\left[\mu(t) - r(t)\mathbf{1}\right].$$

Let

$$Z(t) = e^{-\sum_{j=1}^{d}\int_0^t \eta(u)dB(u) - \frac{1}{2}\int_0^t \|\eta(u)\|^2 du}.$$

We define the probability measure

$$\mathbf{Q}_T(A) = E^{\mathbf{P}}\left[Z(t)1_A\right]; \qquad A \in \mathcal{F}_t$$

and the process

$$\hat{B}(t) = B(t) + \int_0^t \eta(u)du$$

is a Brownian motion under $\mathbf{Q_T}$.

Hence we can write the wealth process as

$$X(t)e^{-\int_0^t r(s)ds} + \int_0^t e^{-\int_0^s r(u)du}C_s ds = X_0 + \int_0^t e^{-\int_0^s r(u)du}\pi(s)^T\sigma(s)d\hat{B}(s).$$

Definition 4.2.2. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}_{0 \le t \le \infty}$. A random time T is a *stopping time* of the filtration if the event $\{\omega, T(\omega) \le t\}$ belongs to the filtration $\{\mathcal{F}_t\}$, for every $t \ge 0$.

Definition 4.2.3. Let $X = \{X_t : \mathcal{F}_t, 0 \le t \le \infty\}$ be a stochastic process. If there exist a nondecreasing sequence $\{T_n\}_{n=1}^{\infty}$ of stopping times of $\{\mathcal{F}_t\}$ such that $X_{t \wedge T_n}$ is a martingale and $\mathbf{P}[lim_{n\to\infty}T_n = \infty] = 1$, then X is a *local martingale*.

Lemma 4.2.4. If for an admissible pair (π, C), and $X_0+\int_0^t e^{-\int_0^s r(u)du}\pi(s)^T\sigma(s)d\hat{B}(s)$ a $\mathbf{Q}$-local martingale then $X(t)e^{-\int_0^t r(s)ds}+\int_0^t e^{-\int_0^s r(u)du}C_s ds$ is a nonnegative super-martingale and $E_{\mathbf{Q}}\left[X(t)e^{-\int_0^t r(s)ds} + \int_0^t e^{-\int_0^s r(u)du}C_s ds\right] \le X_0$

Proof. Let $Y(t) = X(t)e^{-\int_0^t r(s)ds} + \int_0^t e^{-\int_0^s r(u)du}C_s ds$ be nonnegative. Since $Y(t)$ is a local martingale, it follows that $Y(t \wedge T_n)_{t\ge0}$ is a martingale. For $s \le t$, $E_{\mathbf{Q}}\left[Y(t \wedge T_n)|\mathcal{F}_s\right] = Y(s \wedge T_n)$. Allowing $n \to \infty$, by Fatou's Lemma

$$\begin{aligned} E_{\mathbf{Q}}\left[\liminf_{n\to\infty} Y(t\wedge T_n)|\mathcal{F}_s\right] &\le \lim_{n\to\infty} E_{\mathbf{Q}}\left[Y(t\wedge T_n)|\mathcal{F}_s\right] \\ &= \lim_{n\to\infty} Y(s\wedge T_n) \\ &= Y(s) \end{aligned}$$

Therefore $E[Y(t)|\mathcal{F}_s] \le Y(s)$. Hence $Y(t)$ is a nonnegative super-martingale. Since $E[Y(t)|\mathcal{F}_s] \le Y(s)$, it follows that

$$\begin{aligned} E\left[E(Y(t)|\mathcal{F}_s\right] &\le E[Y(s)] \\ E[Y(t)] &\le E[Y(s)] \end{aligned}$$

Let $s = 0$, then

$$\begin{aligned} E[Y(t)] &\le E[Y(0)] \\ E[Y(t)] &\le X_0 \end{aligned}$$

$\square$

4.3 Detecting Insider Trading

Definition 4.3.1. Recall from 4.1.3 we can define a martingale in terms of an enlarged filtration. Using Girsanov Theorem we defined a new probability measure. Is it possible to have a risk-neutral processes with insider information?

Let $\zeta(s) = \frac{B(1)-B(s)}{1-s}$ and $\eta(t) = \sigma(t)^{-1}\left[\mu(t) - r(t)\mathbf{1}\right]$.

By Girsanov Theorem, 4.2.1, we define a new probability measure. Let $M(t)$ be a $(\mathcal{F}, \mathbf{P})$-martingale defined as:

$$e^{-\int_0^t [\zeta(s)+\eta(s)]dB(u) - \frac{1}{2}\int_0^t [\zeta(s)+\eta(s)]^2 du}.$$

Let $\mathbf{Q}$ be a probability measure on $(\Omega, \mathcal{F})$ given by

$$\mathbf{Q}_t(A) = E^{\mathbf{P}}\left[M(t)1_A\right], A \in \mathcal{F}_t.$$

The stochastic process $\breve{B}(t)$ defined by

$$\breve{B}(t) = B(t) + \int_0^t [\zeta(u) + \eta(u)]du, \quad t \in [0, T]$$

is a Brownian motion on $(\mathcal{G}, \mathbf{Q})$.

Proposition 4.3.2. Let (π, C) be an admissible pair and the final wealth associated with the pair is $X^{\pi,C}$. Then there exist a positive function h for which

$$E_{\mathbf{Q}}\left[X_T^{\pi,C} e^{-\int_0^T r(s)ds} + \int_0^T C(t)e^{-\int_0^t r(s)ds}dt|\mathcal{G}_0\right] \leq h(B(1)),$$

such that $X_0^{\pi,C} = h(B(1))$.

Theorem 4.3.3 (Martingale Representation Theorem). *Suppose Z is a $\mathbf{Q}$-martingale adapted to $\mathcal{G}_t$ then there exist an adapted process $\lambda(t)$ such that*

$$Z(t) = Z(0) + \int_0^t \lambda(t)d\breve{B}(t).$$

Proposition 4.3.4. Given an initial wealth $h(B(1))$, a consumption C, and a random variable $Z(t)$ on the probability space $(\Omega, \mathcal{G}, \mathbf{Q})$ such that

$$E_{\mathbf{Q}}\left[Z_T e^{-\int_0^T r(s)ds} + \int_0^T C(t)e^{-\int_0^t r(s)ds}dt|\mathcal{G}_0\right] = h(B(1)),$$

there exist a portfolio π_Z that is $\mathcal{G}$-adapted such (π_Z, C) is admissible and

$$X_T^{\pi_Z,C} = Z(T).$$

4.4 Fair Pricing of an Option with Additional Information

Suppose at time $t = 0$ we sign an option which gives us the right to buy, at the maturity time T, a share of the stock $S_k(t)$ at a strike price q. At maturity, if the price $P_k(T)$ of the share is below the strike price q, the option is not exercised; on the other hand, if $P_k(t) > q$, we can exercise the option at time T to buy the stock at price q, and then sell the share immediately in the market for $P_k(t)$.

This option is known as a *European call option* which has a payment of $(P_k(T) - q)^+$ at T.

Let $X(0) \geq 0$, and (π, C) be a portfolio/consumption process which is admissible, 3.2.7, for the initial endowment $X(0)$. The pair (π, C) is called a *hedging strategy against the contingent claim* provided that

- C_t, the payoff rate, and
- $X(T)$, the terminal payoff at maturity

holds a.s., where X is the wealth process (3.2.2) associated with the pair (π, C).

Suppose an investor knows privileged information about a stock that he wanted to sell. To reduce the risk of loosing money on this stock transaction he needs to find an appropriate hedging strategy such that it is risk neutral.

Consider a contingent claim consisting of a payment Z at time T. By Proposition 4.3.4 there exist a hedging strategy against the contingent claim, (π_Z, C), whose corresponding wealth process satisfies

$$X_T^{\pi_Z, C} = Z(T).$$

Therefore

$$X_0^{\pi_Z, C} = E_{\mathbf{Q}} \left[Z_T e^{-\int_0^T r(s)ds} + \int_0^T C(t) e^{-\int_0^t r(s)ds} dt | \mathcal{G}_0 \right].$$

If Z is a contingent claim with $\{C_t\}$ as the accompanying consumption process, and if $h(B(1))$ is not given a priori, then

$$E_{\mathbf{Q}}\left[Z_T e^{-\int_0^T r(s)ds}|\mathcal{G}_0\right] \tag{4.4.1}$$

gives the price of this contingent claim at time 0. Hence we can conclude there is a fair price for the contingent claim to be paid given insider information.

For instance, the seller of a European call option with strike price q and maturity date T will sell this claim at time $t = 0$ for the price give by (4.4.1) if C_s is the dividend paid at time s and $Z = \max[S_k(T) - q, 0]$ is the final pay-off. The effect of additional information known in advance is reflected in the definition of the measure $\mathbf{Q}$.

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Appendix: Black-Scholes

In the early 1970's a major breakthrough in the pricing of stock options was developed known as the Black-Scholes model. This model has had such an influence on the way traders price and hedge options that Scholes later won the Nobel price for economics [13]. The Black-Scholes formula gives the price of a call option, $F(t, S_1(t))$, when the following conditions apply:

- The riskless interest rate, r, is constant over the life of the option.
- The call options is European. Therefore it cannot be exercised before the expiration date T.
- There are no transaction costs, and the security pays no dividend before the option matures.
- The price $S_1(t)$ follows the stock price process of a geometric Brownian motion.

Example .0.1 (Black-Scholes option pricing). Let the price of a bond obey the process

$$dS_0(t) = rS_0(t)dt,$$

and the price of a stock obeys

$$dS_1(t) = \mu S_1 dt + \sigma S_1 dB(t) = rS_1 dt + \sigma S_1 d\hat{B}(t).$$

The option to buy one share of the stock at time T at the price q follows the wealth processes corresponding to the fair price of the contingent claim:

$$V_t = E_{\mathbf{Q}}\left[e^{-r(T-t)}(S_1(t) - q)^+ | \mathcal{F}_t\right]; \qquad 0 \leq t \leq T.$$

Theorem .0.2 (Black-Scholes formula). *The price of a European call option with exercise price q and time of maturity T is given by :*

$$F(S_1(t), t) = S_1(t)N(d_1) - qe^{-r(T-t)}N(d_2), \qquad F(S_1(T), T) = \max[S_1(T) - q, 0]$$

where

$$d_1 = \frac{ln\left(\frac{S_1(t)}{q}\right) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t},$$

and

$$N(d_i) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_i} e^{-\frac{1}{2}x^2} dx \qquad i = 1, 2.$$

The Black-Scholes formula explicitly interprets the valuation process V_t into $F(S_1(t), t)$, making all security buying and selling bets fair under the previous conditions.

Vita

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