3-25-2017

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Recommended Citation

Varnell, P., Malisoff, M., & Zhang, F. (2017). Stability and robustness analysis for human pointing motions with acceleration under feedback delays. *International Journal of Robust and Nonlinear Control, 27* (5), 703-721. https://doi.org/10.1002/rnc.3593

Stability and Robustness Analysis for Human Pointing Motions with Acceleration under Feedback Delays

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April 7, 2016

Abstract

Pointer acceleration is often used in computer mice and other interfaces to increase the range and speed of pointing motions without sacrificing precision during slow movements. However, the effects of pointer acceleration are not yet well understood. We use a system perspective and feedback control to analyze the effects of pointer acceleration. We use a new pointer acceleration model connected in feedback with the vector integration to endpoint (or VITE) model for pointing motions. When there are no feedback delays, we prove global asymptotic stability of the closed loop system for a general class of acceleration profiles. We also prove robustness under delays and perturbations by building Lyapunov-Krasovskii functionals for delay systems, and we find state performance bounds using robust forward invariance with maximal perturbation sets. The results are relevant to designing pointing interfaces, and our simulations illustrate the good performance of our control under realistic operating conditions.

Key Words: Delay, robustness, stability, time-varying.

1 Introduction

It is often desirable in human pointing interfaces (such as mice and joysticks) for users to perform slow, precise movements when making corrections and to perform fast long range movements in a limited workspace with minimal required effort. These goals can be achieved by pointer acceleration, which increases the sensitivity of the pointer as the speed of the pointer increases, leading to a wider range of pointer speeds. Due to its potential benefits,

*Zhang was supported by ONR grants N00014-10-10712 (YIP) and N00014-14-1-0635, and NSF Grants ECCS-0841195, ECCS-0845333 (CAREER), OCE-1032285, IIS-1319874, and CMMI-1436284. Malisoff was supported by NSF-ECCS Grant 1102348.

Some of this work was done in April 2015 during Malisoff's visit to Georgia Tech; he thanks Georgia Tech for the hospitality he enjoyed during the period.

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pointer acceleration has been adopted as the default setting in several operating system mouse interfaces and many other pointing interfaces; see [1–3].

Even though pointer acceleration is used widely, there is not yet a consensus on how much or what type of pointer acceleration should be used for a given task or interface. In practice, real implementations of pointer acceleration are usually designed by experimenting with different acceleration profiles without using specific design principles. There are also some disadvantages to pointer acceleration that are not yet completely understood. For instance, pointer acceleration may make it more difficult for the user to predict the motion of the pointer and to reproduce desired motions; see [3, 4]. This can decrease pointing accuracy and speed and can worsen the user's subjective rating of a device's feel. Our goal is to more rigorously understand the effects of pointer acceleration, so in the future we are able to better design pointing interfaces that keep the positive effects and reduce the negative effects of pointer acceleration.

In this work, we use a systems perspective to approach the problem of analyzing the effects of pointer acceleration. We build a dynamic model of human pointing motion with pointing acceleration, and we analyze its performance and stability properties. Our results provide insights into the impact of pointer acceleration in pointer interfaces, and have the potential to benefit future designs of such interfaces.

Human pointing interfaces have been studied extensively in human-computer interaction, neuroscience, physiology, and psychology; see [18]. Two key results are the Fitts Law, which was was first presented in [11] and explains an invariant in pointing performance for many human pointing interfaces and tasks, and the vector integration to endpoint (or VITE) model, which was first presented in [10] and reproduces the Fitts Law by describing pointing motions. Other papers have analyzed the dynamics of human pointing as described by the VITE model; see [5–9], with some having feedback delays and some not having delays. Our work [19] used dissipativity to design and study pointing systems, and it gave a preliminary analysis of the VITE model in the dissipativity framework using a switching controller. Our conference article [20] presented the new model of pointer acceleration that is used in this paper and analyzed its stability and performance properties when connected in feedback to the VITE model without any feedback delays or perturbations or robust forward invariance.

Our approach is novel because, to the best of our knowledge, we are the first to use a systems perspective to study pointer acceleration. We use a new model that captures most existing implementations of pointer acceleration, and we show that when the VITE dynamics are connected in feedback to the pointer acceleration model, the interconnection is globally asymptotically stable, including in cases with feedback delays. We include a robustness analysis under perturbations, which can arise from discretization errors and inaccuracies in human perception and control. These results significantly extend the results shown in [20], which did not consider feedback delays or perturbations, and our new results address the difficulties arising from infinite dimensional delay systems and nondeterministic perturbations using robust forward invariance methods, which had not previously been used in the pointing literature. We also develop state performance bounds for the closed loop system, as related to the Fitts Law. The strict Lyapunov functions we construct here are useful for proving key properties such as input-to-state stability (or ISS, as defined in [14]) under uncertainties. However, using sublevel sets of ISS Lyapunov functions to find bounds on the allowable perturbations can lead to bounds that are conservative. Therefore, we also use a robust forward invariance approach from [15, 16], to generate state performance bounds that are less conservative.

An advantage of the robust forward invariance approach is that it chooses the state constraints to facilitate finding maximal allowable perturbation sets under which the state constraint sets are strongly forwardly invariant; see our definitions in Section 3. This contrasts with the usual approaches to state constrained problems, where the state space is generally fixed and where the goal was to prevent the state from exiting the given fixed state constraint set; for example, see the interesting robust positive invariance approach in [13], where the goal is to construct sets of initial states for discrete time dynamics such that the given state constraint set is never violated under perturbations, but where no maximal allowable perturbation sets are found under delays. However, the methods from [15, 16] are limited to two- or three-dimensional curve tracking dynamics that arise in marine robotics, so the robust forward invariant sets we provide in this work are completely different from those in [15, 16]. Hence, other advantages of the method of this paper compared with the ones of [15, 16] are that (i) we extend the approaches of [15, 16] to a very different dynamical system and (ii) the new dynamical system we consider here leads to a completely different trapezoidal shaped class of robust forward invariant sets, which could not be handled using the analysis in [16], which was limited to hexagonally shaped robustly forwardly invariant sets for curve tracking dynamics.

2 Pointer Acceleration Background and Dynamics

Considerable research has shown that humans generate similar motions when reaching and pointing with their arms, laser pointers, mouse pointers, and other devices [18]. The VITE model for pointing [10] can be written as

$$\left\{ \begin{array}{rcl} \dot{\nu} & = & \gamma(-\nu + \rho - u) \\ \dot{y} & = & g(t)[\nu]_d^+ \end{array} \right. \tag{1}$$

where the gain $g(t)$ is called the go signal, $\gamma > 0$ is an internal system parameter, ρ is the target position of the pointer, u is the feedback of the perceived position of the pointer, y is the true position of the pointer that is specified by the user, and the state ν is called the difference vector. The operator $[\cdot]_d^+$ is used to switch the pointer motion off when the pointer overshoots its target, and is defined by

$$[\nu]_d^+ = \left\{ \begin{array}{ll} \nu, & \text{if } \langle \nu, d \rangle \geq 0 \\ 0, & \text{otherwise} \end{array} \right. \tag{2}$$

where d is typically defined as the direction from the pointer to the target at the initial time, so $d = \rho(0) - u(0)$. The notation $[\nu]_d^+$ extends the original VITE model to allow arbitrary target locations.

One can use the VITE model to describe motion by a person trying to drive the true position y to the target position ρ for the pointer. If there is an overshoot and y passes over the target pointer position, then the human quickly stops, but does not take corrective action to move back closer to the target. The model parameters are adjusted by the human to fit different types of performance objectives, and these parameters can be viewed as being very slowly time varying. In Section 5, we show how a relationship between γ and the pointer acceleration scaling function can ensure that key state performance bounds are met.

Pointer acceleration aims to make the pointer more sensitive at higher speeds than lower speeds. This enables quick and long pointer motions, as well as slow and precise pointer motions. As we will see below, pointer acceleration can make pointer motion less stable in the presence of feedback delays, in addition to making it more difficult for a user to predict pointer motion. Whether or not pointer acceleration improves a user's subjective feel for the pointing interface appears to be a very personal choice. Although many operating systems have some type of pointer acceleration as their default settings, many modern video games that require exact pointer movement do not use any pointer acceleration.

While there are many different pointer acceleration implementations, most can be described by the following model. We view pointer acceleration as a transformation of the user's pointer position output. If $v \in \mathbb{R}^n$ is the pointer position specified by the user, then the pointer acceleration output w satisfies

$$\dot{w} = G(|\dot{v}|)\dot{v}, \tag{3}$$

where $G : \mathbb{R} \to \mathbb{R}$ is a suitable scaling function and $|\cdot|$ is the usual Euclidean norm; see Section 3 for all of our notation. A few popular choices for the scaling function G are summarized in Table 1 (at the end of this paper), including the Linear and Threshold scaling functions. The form of pointer acceleration (3) is in fact pointer velocity scaling, although we use the term pointer acceleration since this language is commonly used in the literature. We can typically approximate the profiles in such a way that G is positive definite and continuously differentiable, which can help us apply Theorem 3 below. A notable exception to the pointer acceleration model (3) involving actual acceleration is the polynomial scaling function $G_{\text{polynomial}}(|\dot{v}|, |\ddot{v}|) = k_1 + k_2|\dot{v}|^{|\ddot{v}|}$, which adjusts sensitivity using both pointer input acceleration and velocity. While we do not consider scaling functions that are a function of pointer input acceleration, we believe that it may be possible to extend our work to them. In practice, most pointing interfaces do not measure pointer velocity directly. Instead, the pointer velocity is computed via discretization for use in the acceleration system. While this is typically not a major issue, the resulting accelerated pointer output can be very irregular when the pointer moves very quickly, or when the scaling is very large. One typically mitigates this by limiting the maximum pointer speed output, but we do not consider this here; however, see Remark 3 for additive uncertainties on v that can represent discretization errors.

To connect the VITE model (1) with the pointing acceleration model (3), as in Figure 1, we choose the input $u = w$ and the output $v = y$. In later sections, we consider feedback connections that have delays and perturbations, which may be present in real systems due to limitations in the human operator and computer interface. Choosing the overall state as $x = (w, \nu)^\top = (x_1, x_2)^\top$, the closed loop pointing dynamics become

$$\begin{aligned}\dot{x} &= \begin{bmatrix} \dot{w} \\ \dot{\nu} \end{bmatrix} = \begin{bmatrix} G(|\dot{v}|)\dot{v} \\ \gamma(-\nu + \rho - u) \end{bmatrix} = \begin{bmatrix} G(|\dot{y}|)\dot{y} \\ \gamma(-\nu + \rho - w) \end{bmatrix} \\ &= \begin{bmatrix} G(|g(t)[\nu]_d^+|)g(t)[\nu]_d^+ \\ \gamma(-\nu + \rho - w) \end{bmatrix} = \begin{bmatrix} G(|g(t)[x_2]_d^+|)g(t)[x_2]_d^+ \\ \gamma(-x_2 + \rho - x_1) \end{bmatrix}.\end{aligned}$$

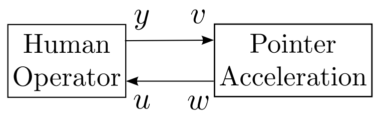


Figure 1: Block diagram of closed loop pointing model.

For simplicity, we take g to be a constant, which is the most commonly considered case in the literature, and only consider one dimensional pointing dynamics, as most pointing motions are largely constrained to the line between the start and goal pointer position, even when there are additional degrees of freedom. We assume without loss of generality that $\rho = 0$ and $x_1(0) = w(0) < 0$, so $d = \rho(0) - u(0) > 0$, which can always be achieved by a coordinate transformation. The closed loop dynamics are then

$$\dot{x} = \begin{bmatrix} \tilde{G}(x_2^+)x_2^+ \\ -\gamma(x_1 + x_2) \end{bmatrix}, \tag{4}$$

where $x = (x_1, x_2)^\top = (w, \nu)^\top$ as defined previously, $\tilde{G}(\cdot) = gG(g\cdot)$, and x_2^+ denotes the positive part, i.e., $p^+ = p$ when $p \geq 0$, and $p = 0$ when $p < 0$. The desired equilibrium set is

$$\mathcal{E} = \{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 = -x_1\}, \tag{5}$$

but see below for more general pointing models with delays, perturbations, and state constraints. To cover these, we need several definitions, which we turn to next.

3 Definitions and Notation

In this section, all dimensions are arbitrary, unless indicated. We use the standard classes of comparison function $\mathcal{K}_\infty$ and $\mathcal{KL}$ [12, Section 4.4]. Let $\mathcal{E} \subseteq \mathbb{R}^n$. A function α defined on some set $\mathcal{O} \subseteq \mathbb{R}^n$ is *positive definite* with respect to $\mathcal{E}$ provided it is zero at all points in $\mathcal{E} \cap \mathcal{O}$ and positive at all points in $\mathcal{O} \setminus \mathcal{E}$. We use C^1 to mean continuously differentiable, and we understand the derivative $f'(0)$ of any function f that is defined on $[0, \infty)$ as the right derivative at 0. By piecewise continuity of a function defined on $[0, \infty)$, we mean that it is continuous except at finitely many points on each bounded interval. Let $\mathcal{U} \subseteq \mathbb{R}^m$ and $\mathcal{M}_\mathcal{U}$ denote the set of all piecewise continuous locally bounded functions $\delta : [0, \infty) \to \mathcal{U}$. For all $\delta \in \mathcal{M}_\mathcal{U}$ and any interval $\mathcal{I} \subseteq [0, \infty)$, let $|\delta|_\mathcal{I}$ be the supremum of the restriction of δ to $\mathcal{I}$. Set $|x|_\mathcal{E} = \inf\{|x - r| : r \in \mathcal{E}\}$ for all $x \in \mathbb{R}^n$.

Given a constant $\tau \geq 0$ (representing an input delay) and a set $\mathcal{S} \subseteq \mathbb{R}^n$, we let $C_{\text{in}}(\mathcal{S})$ denote the set of all continuous functions $x_I : [-\tau, 0] \to \mathcal{S}$, which we write as C_{in} when $\mathcal{S} = \mathbb{R}^n$. We define the operators $x_t(s) = x(t+s)$ for all $s \in [-\tau, 0]$ and $t \geq 0$ and all functions $x : [-\tau, \infty) \to \mathbb{R}^n$. Given a function $\mathcal{F} : C_{\text{in}} \times \mathcal{U} \to \mathbb{R}^n$, we call the system $\dot{x}(t) = \mathcal{F}(x_t, \delta(t))$ *forward complete* provided for each $x_I \in C_{\text{in}}$ and each $\delta \in \mathcal{M}_\mathcal{U}$, the corresponding solution $x(t, x_I, \delta)$ of the system is uniquely defined for all $t \in [0, \infty)$. Given a subset $\mathcal{S} \subseteq \mathbb{R}^n$ and a forward complete system, we say that $\mathcal{S}$ is *robustly forwardly invariant for the system with perturbations valued in* $\mathcal{U}$ provided $x(t, x_I, \delta) \in \mathcal{S}$ for all $t \geq 0$, $x_I \in C_{\text{in}}(\mathcal{S})$, and $\delta \in \mathcal{M}_\mathcal{U}$. We say that $\mathcal{S}$ is *robustly forwardly invariant for the system with the maximal perturbation* *set* $\mathcal{U}$ provided these two conditions hold: (a) $\mathcal{S}$ is robustly forwardly invariant for the system with perturbations valued in $\mathcal{U}$ and (b) for each point $\bar{\delta} \in \mathbb{R}^m \setminus \text{closure}(\mathcal{U})$, there exists an initial function $x_I \in C_{\text{in}}(\mathcal{S})$ such that the trajectory $x(t, x_I, \bar{\delta})$ of the system for the constant perturbation $\delta(t) = \bar{\delta}$ admits a time $\bar{t} > 0$ such that $x(\bar{t}, x_I, \bar{\delta}) \in \mathbb{R}^n \setminus \mathcal{S}$. Maximality of $\mathcal{U}$ therefore roughly means that enlarging $\mathcal{U}$ would allow trajectories to leave $\mathcal{S}$. The special case of robust forward invariance where there are no perturbations δ is the standard strong invariance property.

The system is *input-to-state stable (or ISS)* with respect to $\mathcal{E}$ and $\mathcal{U}$ on $\mathcal{S}$ provided (a) $\mathcal{S}$ contains $\mathcal{E}$ and is robustly forwardly invariant for the system with perturbations valued in $\mathcal{U}$ and (b) there are functions $\beta \in \mathcal{KL}$ and $\alpha \in \mathcal{K}_\infty$ such that for all solutions $x(\cdot, x_I, \delta)$ of the system for all $x_I \in C_{\text{in}}(\mathcal{S})$ and for all $\delta \in \mathcal{M}_\mathcal{U}$, we have $|x(t, x_I, \delta)|_\mathcal{E} \leq \beta(\sup\{|x_I(r)|_\mathcal{E} : -\tau \leq r \leq 0\}, t) + \alpha(|\delta|_{[0,t]})$ for all $t \geq 0$. The special case of ISS where the $\alpha(|\delta|_{[0,t]})$ term is not present in the ISS estimate and $\mathcal{F}$ has no perturbations δ (i.e., $\dot{x}(t) = \mathcal{F}(x_t)$) is *global asymptotic stability (or GAS)* to $\mathcal{E}$ on $\mathcal{S}$.

If $\mathcal{S}$ is robustly forwardly invariant for a system with perturbations valued in $\mathcal{U}$ and $\mathcal{E} \subseteq \mathcal{S}$, then a function $V : C_{\text{in}} \to \mathbb{R}^n$ is an *ISS Lyapunov function* for the system with respect to $\mathcal{E}$ and $\mathcal{U}$ on $\mathcal{S}$ provided (a) $V(x_t)$ is differentiable as a function of t on $[0, \infty)$ for all solutions $x(\cdot)$ of the system with initial functions $x_I \in C_{\text{in}}(\mathcal{S})$ and (b) there are class $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, α_o, and α_1 such that along all trajectories $x(t)$ of the system for all initial functions $x_I \in C_{\text{in}}(\mathcal{S})$ and for all choices of $\delta \in \mathcal{M}_\mathcal{U}$, the following hold for all $t \geq 0$: $\underline{\alpha}(|x(t)|_\mathcal{E}) \leq V(x_t) \leq \overline{\alpha}(\sup\{|x(\ell)|_\mathcal{E} : \ell \in [t - \tau, t]\})$ and $\frac{d}{dt}V(x_t) \leq -\alpha_o(V(x_t)) + \alpha_1(|\delta|_{[0,t]})$. Standard arguments show that the existence of an ISS Lyapunov function with respect to $\mathcal{E}$ and $\mathcal{U}$ on $\mathcal{S}$ implies the ISS property with respect to $\mathcal{E}$ and $\mathcal{U}$ on $\mathcal{S}$ when no delays are present [12], and the same can be shown under time delays, by similar arguments that we omit here.

The special case of an ISS Lyapunov function for systems $\dot{x}(t) = \mathcal{F}(x_t)$ without perturbations is a *strict Lyapunov function* with respect to $\mathcal{E}$ on $\mathcal{S}$. This is a slightly more restrictive definition of strict Lyapunov functions, because α_o is normally allowed to be positive definite with respect to $\{0\}$, without necessarily being of class of $\mathcal{K}_\infty$, but there are techniques for transforming strict Lyapunov functions for cases where α_o is only positive definite into new Lyapunov functions that satisfy our requirements with $\alpha_o \in \mathcal{K}_\infty$; see [14]. The existence of a strict Lyapunov function implies GAS to $\mathcal{E}$ on $\mathcal{S}$; see [12]. Unless indicated, we assume for simplicity in all of what follows that the initial functions $x_I \in C_{\text{in}}(\mathcal{S})$ are constant.

4 Strict Lyapunov Function

To analyze the effects of time delays and perturbations on the closed loop pointing system with acceleration (4), we will examine its ISS properties using a strict Lyapunov function that we construct next. See Proposition 1 for ways to construct the function $\mathcal{H}$ in the following theorem, and see the sections below where we use the ideas from this section to prove results under state constraints and delays to get state performance bounds and robustness results.

Theorem 1. *Let* $\gamma > 0$ *be a constant, and* $\tilde{G} : [0, \infty) \to [0, \infty)$ *be locally Lipschitz and positive valued on* $(0, \infty)$ *and satisfy* $\liminf_{r \to \infty} \tilde{G}(r) > 0$. *Let* $\mathcal{H} : \mathbb{R} \to \mathbb{R}$ *be a* C^1 *function satisfying:*

1. $\mathcal{H}(r) = 0$ *for all* $r \leq 0$, *and* $\mathcal{H}(r) > 0$ *for all* $r > 0$,

2. $\int_0^r \tilde{G}(\ell)\ell \mathrm{d}\ell \geq (2/\gamma)\mathcal{H}^2(r)$ *for all* $r \geq 0$,

3. $r\tilde{G}(r) \geq \mathcal{H}(r)$ *for all* $r \geq 0$, *and*

4. $\mathcal{H}'(0) = 0$, *and* $0 < \mathcal{H}'(r) \leq \frac{\gamma}{4}$ *for all* $r > 0$.

Then the function

$$V_{\text{new}}(x) = \int_0^{x_2} \tilde{G}(\zeta^+)\zeta^+\mathrm{d}\zeta + \frac{\gamma}{2}(x_1+x_2)^2 - \mathcal{H}(x_2)(x_1+x_2) \tag{6}$$

satisfies the estimates

$$\begin{aligned} &\tfrac{1}{10}\int_0^{x_2} \tilde{G}(\zeta^+)\zeta^+\mathrm{d}\zeta + \tfrac{\gamma}{4}(x_1+x_2)^2 \leq V_{\text{new}}(x) \\ &\leq \int_0^{x_2} \tilde{G}(\zeta^+)\zeta^+\mathrm{d}\zeta + \tfrac{\mathcal{H}^2(x_2)}{2} + \tfrac{\gamma+1}{2}(x_1+x_2)^2 \end{aligned} \tag{7}$$

for all $x \in \mathbb{R}^2$ *and*

$$\dot{V}_{\text{new}}(x) \leq -\frac{\gamma^2}{4}(x_1+x_2)^2 - \frac{1}{2}\mathcal{H}(x_2)\tilde{G}(x_2^+)x_2^+ \tag{8}$$

along all trajectories of (4), and so is a strict Lyapunov function for (4) with respect to $\mathcal{E} = \{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 = -x_1\}$ *on* $\mathcal{S} = \mathbb{R}^2$. *Therefore, (4) satisfies GAS with respect to (5) on* $\mathcal{S} = \mathbb{R}^2$.

Proof. We can expand the first term of (6) and apply Condition 2) as

$$\begin{aligned} V_{\text{new}}(x) &= \tfrac{1}{10}\int_0^{x_2} \tilde{G}(\zeta^+)\zeta^+\mathrm{d}\zeta + \tfrac{9}{10}\int_0^{x_2} \tilde{G}(\zeta^+)\zeta^+\mathrm{d}\zeta + \tfrac{\gamma}{2}(x_1+x_2)^2 - \mathcal{H}(x_2)(x_1+x_2) \\ &\geq \tfrac{1}{10}\int_0^{x_2} \tilde{G}(\zeta^+)\zeta^+\mathrm{d}\zeta + \tfrac{9}{5\gamma}\mathcal{H}^2(x_2) + \tfrac{\gamma}{2}(x_1+x_2)^2 - |\mathcal{H}(x_2)(x_1+x_2)| \end{aligned}$$

where we applied Condition 2 with the choice $r = x_2$ and also need the fact that $\mathcal{H} = 0$ on $(-\infty, 0)$ and the lower bound $-\mathcal{H}(x_2)(x_1+x_2) \geq -|\mathcal{H}(x_2)(x_1+x_2)|$. By Young's inequality, we also have $|\mathcal{H}(x_2)(x_1+x_2)| \leq \frac{1}{\gamma}\mathcal{H}^2(x_2) + \frac{\gamma}{4}(x_1+x_2)^2$ for all $x \in \mathbb{R}^2$. Combining the preceding two inequalities gives the lower bound in (7). The upper bound in (7) follows by using the triangle inequality $-\mathcal{H}(x_2)(x_1+x_2) \leq \frac{1}{2}\mathcal{H}^2(x_2) + \frac{1}{2}(x_1+x_2)^2$ to upper bound the term $-\mathcal{H}(x_2)(x_1+x_2)$ in the formula (6).

To prove the decay condition (8), notice that along all trajectories of (4), we can use the triangle inequality to get

$$\begin{aligned} \dot{V}_{\text{new}} &= -\gamma\tilde{G}(x_2^+)x_2^+(x_1+x_2) + \gamma\mathcal{H}'(x_2)(x_1+x_2)^2 \\ &\quad + (\gamma(x_1+x_2) - \mathcal{H}(x_2))\left(\tilde{G}(x_2^+)x_2^+ - \gamma(x_1+x_2)\right) \\ &\leq -\gamma^2(x_1+x_2)^2 - \mathcal{H}(x_2)\tilde{G}(x_2^+)x_2^+ \\ &\quad + \left\{\left(\tfrac{\gamma^2}{2} + \gamma\mathcal{H}'(x_2)\right)(x_1+x_2)^2 + \tfrac{1}{2}\mathcal{H}^2(x_2)\right\}. \end{aligned} \tag{9}$$

Also, by conditions 3)-4), we have

$$\gamma\mathcal{H}'(x_2) \leq \frac{\gamma^2}{4} \text{ and } \frac{1}{2}\mathcal{H}^2(x_2) \leq \frac{1}{2}\mathcal{H}(x_2)x_2^+\tilde{G}(x_2^+) \tag{10}$$

when $x_2 \geq 0$; furthermore, (10) holds when $x_2 < 0$ as well, since in this case $\mathcal{H}(x_2) = \mathcal{H}'(x_2) = 0$ by conditions 1) and 4). Using (10) to upper bound the terms in curly braces in (9) gives (8). Letting $V^l(x)$ and $V^u(x)$ denote the lower and upper bounding functions for $V_{\rm new}(x)$ in (7) respectively, we then choose $\underline{\alpha}(s) = \frac{s}{1+s}\inf\{V^l(x) : |x|_{\mathcal{E}} \geq s\}$, $\overline{\alpha}(s) = s + \sup\{V^u(x) : |x|_{\mathcal{E}} \leq s\}$, and the composition $\alpha_o = \alpha_1 \circ \bar{\alpha}^{-1}$ to satisfy the requirements in our strict Lyapunov function definition with $\tau = 0$, where $\alpha_1(s) = \frac{s}{1+s}\inf\{\gamma^2(x_1+x_2)^2/4 + \mathcal{H}(x_2)\tilde{G}(x_2^+)x_2^+/2 : |x|_{\mathcal{E}} \geq s\}$. □

An important motivation for having an explicit construction for $\mathcal{H}$ from Theorem 1 comes from the possibility of redesigning the output of the pointer acceleration system to make the closed loop system ISS when there is a perturbation. Given $\mathcal{H}$, it is possible to construct a redesigned pointer acceleration output $w^\sharp$ such that when there are perturbations δ in the feedback connection $u = w^\sharp + \delta$, the dynamics

$$\dot{x} = \begin{bmatrix} \tilde{G}(x_2^+)x_2^+ \\ -\gamma(w^\sharp(x) + x_2 + \delta) \end{bmatrix} \tag{11}$$

is ISS with respect to $\mathcal{E}$ and $\mathcal{U} = \mathbb{R}$ on $\mathcal{S} = \mathbb{R}^2$. This is because we can take $w^\sharp(x) = x_1 + \gamma(\partial V_{\rm new}(x)/\partial x_2)$, which can be expressed in terms of $\tilde{G}$ and $\mathcal{H}$ from the strict Lyapunov function $V_{\rm new}$ in (6), and use $\alpha_o \in \mathcal{K}_\infty$ from our proof of Theorem 1 and the triangle inequality to get $-|\gamma(\partial V_{\rm new}(x)/\partial x_2)|^2 - \gamma(\partial V_{\rm new}(x)/\partial x_2)\delta \leq -0.5|\gamma(\partial V_{\rm new}(x)/\partial x_2)|^2 + 0.5|\delta|^2$ and so also $\dot{V}_{\rm new} \leq -\alpha_o(V_{\rm new}(x)) + 0.5|\delta|^2$ along all trajectories of (11), so $V_{\rm new}$ is an ISS Lyapunov function, which implies the ISS property. This motivates finding a formula for $\mathcal{H}$, and under standard conditions on $\tilde{G}$ (which hold for many examples of interest from Section 2), we can readily find a function $\mathcal{H}$ satisfying the requirements from Theorem 1. For instance, we prove the following:

Proposition 1. *Let $\gamma > 0$ be a positive constant, and $\tilde{G} : [0,\infty) \to [0,\infty)$ satisfy the requirements from Theorem 1 and admit a constant $c_a > 0$ such that $\tilde{G}(r) \geq c_a r$ for all $r \geq 0$. Set*

$$\bar{\kappa} = \min\left\{c_a, \sqrt{\tfrac{\gamma}{6}c_a}, \tfrac{\gamma}{8}\right\}. \tag{12}$$

Then for any constant $\kappa_o \in (0, \bar{\kappa}]$, the function

$$\mathcal{H}(\ell) = \kappa_o \tfrac{(\ell^+)^2}{1+(\ell^+)^2} \tag{13}$$

satisfies conditions 1)-4) from Theorem 1, so (6) with the choice (13) is a strict Lyapunov function for (4) with respect to (5) on $\mathcal{S} = \mathbb{R}^2$.

Proof. Condition 2) holds because our lower bound on $\tilde{G}$ gives

$$\int_0^r \tilde{G}(\ell)\ell \mathrm{d}\ell \;\geq\; \int_0^r c_a \ell^2 \mathrm{d}\ell \;=\; \tfrac{c_a}{3}r^3 \;\geq\; \tfrac{c_a}{3}\tfrac{r^4}{(1+r^2)^2} \;\geq\; \tfrac{2}{\gamma}\mathcal{H}^2(r) \tag{14}$$

for all $r \geq 0$, where the second inequality followed by separately considering the cases $r \geq 1$ and $r < 1$, and where the last inequality used the fact that $\kappa_0 \leq \sqrt{\gamma c_a/6}$. To check condition 3), note that

$$\mathcal{H}(r) \;\leq\; c_a \tfrac{r^2}{1+r^2} \;\leq\; c_a r^2 \;\leq\; r\tilde{G}(r) \tag{15}$$

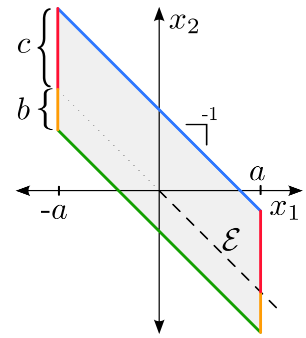


Figure 2: Robustly forwardly invariant set $S_{a,b,c}$.

holds for all $r \geq 0$. Condition 4) holds because $\kappa_0 \leq \gamma/8$, so

$$\mathcal{H}'(r) \;=\; \kappa_o \tfrac{2r}{(1+r^2)^2} \;\leq\; \tfrac{\gamma}{4}\tfrac{r}{(1+r^2)^2} \;\leq\; \tfrac{\gamma}{4}$$

for all $r \geq 0$, which proves the proposition. □

Although (11) is ISS, it is useful to find conditions under which the original closed loop system (4) is ISS with respect to additive uncertainties on the original output $w^\sharp(x) = x_1$ on suitable robustly forwardly invariant sets with maximal perturbation sets. The ISS and invariance properties of the original system characterizes its robustness under different choices of scaling functions and other parameters. We begin this analysis with our next section on robust forward invariance.

5 Robust Forward Invariance

The preceding analysis motivates the problem of finding maximal allowable perturbation sets for robustly forwardly invariant sets, since the perturbation bounds one can obtain from ISS Lyapunov functions may be conservative. The works [15, 16] found robustly forwardly invariant sets with maximal perturbation sets for curve tracking, but they do not apply to other systems. In this section, we first provide an analogous result for the undelayed pointing dynamics

$$\dot{x} = \begin{bmatrix} \tilde{G}(x_2^+)x_2^+ \\ -\gamma(x_1 + x_2 + \delta(t)) \end{bmatrix} \tag{16}$$

with uncertainties δ under suitable conditions on $\tilde{G}$. Later, we use the δ's to represent control uncertainty, or the effects of input delays. For what follows, we set $\tilde{G}^\sharp(r) = r\tilde{G}(r)$, and we use the sets

$$S_{a,b,c} = \{x \in \mathbb{R}^2 : |x_1| \leq a, -b \leq x_1 + x_2 \leq c\} \tag{17}$$

for suitable constants $a > 0$, $b > 0$, and $c > 0$; see Figure 2.

Proposition 2. *Let $\tilde{G} : [0, \infty) \to [0, \infty)$ be locally Lipschitz and positive valued on $(0, \infty)$. Let γ, a, b, and c be any positive constants such that $c > |\tilde{G}^\sharp|_{[0,a+c]}/\gamma$ and $a > \max\{b, c\}$. Then $S_{a,b,c}$ is robustly forwardly invariant for (16) with the maximal perturbation set $\mathcal{D}_{a,b,c} = ((1/\gamma)|\tilde{G}^\sharp|_{[0,a+c]} - c, b)$.*

Proof. To prove robust forward invariance of $S_{a,b,c}$, it suffices to consider continuous δ's, because if $x(t, x_I, \delta)$ was a trajectory for (16) that starts in $S_{a,b,c}$ for a piecewise continuous δ but exits $S_{a,b,c}$, then we could approximate δ by a continuous $\mathcal{D}_{a,b,c}$ valued perturbation δ_c (in the L^1 norm) such that $x(t, x_I, \delta_c)$ also exited $S_{a,b,c}$. Set $c_p = c - |\tilde{G}^\sharp|_{[0,a+c]}/\gamma$, and note that $x_2^+ \in [0, a+c]$ for all $x \in S_{a,b,c}$. If $\delta : [0,\infty) \to \mathcal{D}_{a,b,c}$ is continuous and $t \geq 0$ were such that a corresponding trajectory $x(\cdot)$ of (16) for δ starts in $S_{a,b,c}$ and $x(t)$ is on the top leg of $S_{a,b,c}$, then $x_1(t) + x_2(t) = c$, so

$$\begin{array}{rcl} \dot{x}_1(t) + \dot{x}_2(t) & = & -\gamma\big(c + \delta(t) - \frac{1}{\gamma}\tilde{G}(x_2^+(t))x_2^+(t)\big) \\ & < & -\gamma\big(c - |\tilde{G}^\sharp|_{[0,a+c]}/\gamma\big) + \gamma c_p \ = \ 0. \end{array} \tag{18}$$

This prevents $x(t)$ from exiting $S_{a,b,c}$ through the top leg of $S_{a,b,c}$, except possibly at times t when $x_1(t) = \pm a$, by continuity of $\dot{x}_1(\ell) + \dot{x}_2(\ell)$, which implies that $x_1(\ell) + x_2(\ell)$ is decreasing in an interval of ℓ values of the form $[t, t+\ell_*]$ for some $\ell_* > 0$. (The strictness of the inequality in (18) is needed to ensure that $x_1(\ell) + x_2(\ell)$ is decreasing in such an interval of ℓ values.) Similarly, if $x(t)$ is on the bottom leg of $S_{a,b,c}$, then $x_1(t) + x_2(t) = -b$, so we instead get $\dot{x}_1(t) + \dot{x}_2(t) = \tilde{G}(x_2^+(t))x_2^+(t) - \gamma(-b + \delta(t)) \geq \gamma(b - \delta(t)) > 0$, which prevents $x(t)$ from leaving $S_{a,b,c}$ through the bottom leg of $S_{a,b,c}$, except possibly if $x_1(t) = \pm a$. Since $a > b$, we get $x_2 > 0$ and so also $\dot{x}_1 > 0$ on the left leg of $S_{a,b,c}$. Also, $\dot{x}_1 = 0$ on the right leg of $S_{a,b,c}$, since $a > c$ ensures that $x_2^+ = 0$ on the right leg of $S_{a,b,c}$. (Without the condition $a > c$, we could have $\dot{x}_1 > 0$ on the right leg of $S_{a,b,c}$, which would allow trajectories to exit through the right leg.) This proves the robust forward invariance property.

To prove the maximality of $\mathcal{D}_{a,b,c}$, first note that for each constant $d > b$, the trajectory of (16) for the constant perturbation $\delta(t) = d$ starting at the initial state $(-b, 0)$ on the bottom leg of $S_{a,b,c}$ satisfies $\dot{x}_1(0) + \dot{x}_2(0) = \tilde{G}(x_2^+(0))x_2^+(0) - \gamma(-b + d) < 0$, so the trajectory leaves $S_{a,b,c}$ through $(-b, 0)$. Also, if $d < -c_p$ is any constant, and if we choose $v \in [0, a+c]$ such that $\tilde{G}^\sharp(v) = |\tilde{G}^\sharp|_{[0,a+c]}$, then the trajectory for the constant perturbation $\delta(t) = d$ starting at the point $(c - v, v)$ on the top leg of $S_{a,b,c}$ satisfies

$$\begin{array}{rcl} \dot{x}_1(0) + \dot{x}_2(0) & = & \tilde{G}(v)v - \gamma(c + d) \\ & = & -\gamma(c - |\tilde{G}^\sharp|_{[0,a+c]}/\gamma + d) \ > \ 0, \end{array} \tag{19}$$

so $x(\cdot)$ exits $S_{a,b,c}$. This proves the maximality of $\mathcal{D}_{a,b,c}$. □

Remark 1. *We can replace the bounds $-a \leq x_1 \leq a$ in the definition of $S_{a,b,c}$ by $-a \leq x_1 \leq d$ for any constant $d > c$; the proof of the robust forward invariance with this change is as before. The requirement $c > |\tilde{G}^\sharp|_{[0,a+c]}/\gamma$ from Proposition 2 holds if $\gamma > 0$ is large enough. While robust forward invariance does not imply convergence to $\mathcal{E}$, it can be used with our Lyapunov analysis to prove asymptotic convergence to $\mathcal{E}$ while maintaining suitable state performance bounds; see below. We can choose $(a, b, c) \in (0,\infty)^3$ such that the robustly forwardly invariant set $S_{a,b,c}$ is arbitrary large. Also, even though each set $S_{a,b,c}$ is a proper subset of $\mathbb{R}^2$, we can build a nested sequence $S_1 \subseteq S_2 \subseteq \ldots$ of such sets whose union $\bigcup_i S_i = \mathbb{R}^2$. This allows us to make statements about global system behavior.*

Two pertinent features of Proposition 2 are (a) the compactness of the robustly forwardly invariant sets $S_{a,b,c}$ and (b) the fact that the maximum perturbation sets $\mathcal{D}_{a,b,c}$ are intervals. If

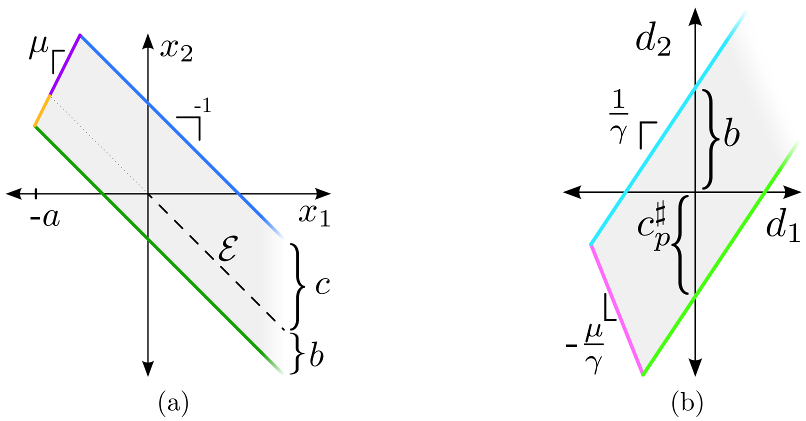


Figure 3: (a) Robustly forwardly invariant set $S^{\mu}_{a,b,c}$ and (b) corresponding perturbation set $\mathcal{D}^{\mu}_{a,b,c}$.

we relax the conditions that the robustly forwardly invariant sets and maximum perturbation sets are bounded, then for each constant $\mu > 1$, we can prove robust forward invariance results for the more general system

$$\dot{x} = \left[\begin{array}{c} \tilde{G}(x_2^+)x_2^+ + \delta_1(t) \\ -\gamma(x_1 + x_2 + \delta_2(t)) \end{array} \right], \tag{20}$$

with perturbations δ_i in both equations, by replacing $S_{a,b,c}$ by

$$S^{\mu}_{a,b,c} = \left\{ \begin{array}{l} x \in \mathbb{R}^2 : -b \leq x_1 + x_2 \leq c, x_1 \geq -a, \\ \text{and } \; x_2 \leq \mu x_1 + (\mu+1)a - b \end{array} \right\} \tag{21}$$

for any constant $a > \max\{b, c\}$; see Figure 3. To see how, let ℓ_μ denote the upper left leg of $S^{\mu}_{a,b,c}$, having the slope μ. Then $\ell_\mu \subseteq (-\infty, 0) \times (0, \infty)$, because $x_1 \leq -a + (b+c)/(1+\mu) < 0$ for all x_1 such that $x \in \ell_\mu$. Let

$$\begin{aligned} \bar{\sigma}_\mu &= \max\left\{ -\tilde{G}^\sharp(x_2) - \tfrac{\gamma}{\mu}(x_1 + x_2) : x \in \ell_\mu \right\}, \\ c_p^\sharp &= c - \tfrac{1}{\gamma} \max\left\{ \tilde{G}^\sharp(s) : s \in \left[0, a + c - \tfrac{c+b}{1+\mu}\right] \right\}, \text{ and} \\ \mathcal{D}^{\mu}_{a,b,c} &= \left\{ d \in \mathbb{R}^2 : \max\left\{ \tfrac{d_1}{\gamma} - c_p^\sharp, \tfrac{\mu}{\gamma}(\bar{\sigma}_\mu - d_1) \right\} < d_2 < \tfrac{d_1}{\gamma} + b \right\}. \end{aligned}$$

We then prove:

Proposition 3. *If $\mu > 1$ is a constant and γ, $\tilde{G}$, a, b, and c satisfy the assumptions from Proposition 2, then $S^{\mu}_{a,b,c}$ is a robustly forwardly invariant set for (20) with the maximum perturbation set $\mathcal{D}^{\mu}_{a,b,c}$.*

Proof. We indicate the changes needed in the proof of Proposition 2. We replace $S_{a,b,c}$, $\mathcal{D}_{a,b,c}$, and c_p by $S^{\mu}_{a,b,c}$, $\mathcal{D}^{\mu}_{a,b,c}$, and $c_p^\sharp$, respectively. For each $d \in \mathcal{D}^{\mu}_{a,b,c}$, we have $-c_p^\sharp < d_2 - d_1/\gamma < b$. Hence, our treatment of the slope -1 legs of $S_{a,b,c}$ in the proof of Proposition 2 (with δ

replaced by $\delta_2-\delta_1/\gamma$), combined with the fact that $x_2 \le a+c-(c+b)/(1+\mu)$ for all $x \in S^{\mu}_{a,b,c}$, imply that no trajectory of (20) starting in $S^{\mu}_{a,b,c}$ for any $\mathcal{D}^{\mu}_{a,b,c}$-valued continuous perturbation can exit through either of the slope -1 legs of $S^{\mu}_{a,b,c}$, except possibly though an endpoint of ℓ_μ. On the other hand, for any trajectory $x(\cdot)$ of (20) for any $\mathcal{D}^{\mu}_{a,b,c}$-valued continuous perturbation $\delta(\cdot)$ and any time $t \ge 0$ such that $x(t) \in \ell_\mu$, our definition of $\mathcal{D}^{\mu}_{a,b,c}$ gives $\dot{x}_2(t)-\mu\dot{x}_1(t) = -\gamma(x_1(t)+x_2(t)+\delta_2(t))-\mu\tilde{G}^\sharp(x_2(t))-\mu\delta_1(t) \le \mu(\bar{\sigma}_\mu-((\gamma/\mu)\delta_2(t)+\delta_1(t))) < 0$. Since $x_2-\mu x_1$ is the x_2 axis intercept of the line through any point x having slope μ, this implies that $x(\cdot)$ cannot exit $S^{\mu}_{a,b,c}$ through ℓ_μ. Finally, the maximality of $\mathcal{D}^{\mu}_{a,b,c}$ follows from the maximality part of the proof of Proposition 2 (with $|\tilde{G}^\sharp|_{[0,a+c]}$ and δ replaced by $|\tilde{G}^\sharp|_{[0,a+c-(c+b)/(1+\mu)]}$ and $\delta_2-\delta_1/\gamma$, respectively), combined with the fact that the maximum in the definition of $\bar{\sigma}_\mu$ occurs at some point $\bar{x} \in \ell_\mu$. Hence, for any $\bar{d}=(\bar{d}_1,\bar{d}_2) \in \mathbb{R}^2$ such that $(\mu/\gamma)(-\bar{d}_1+\bar{\sigma}_\mu) > \bar{d}_2$ and the trajectory $x(\cdot)$ of (20) starting at the maximizing pair $\bar{x}$ for the constant perturbation $\delta(t)=\bar{d}$, we get $\dot{x}_2(0)-\mu\dot{x}_1(0) = \mu(\bar{\sigma}_\mu-((\gamma/\mu)\bar{d}_2+\bar{d}_1)) > 0$, so the trajectory exits $S^{\mu}_{a,b,c}$ through $\bar{x} \in \ell_\mu$. □

Remark 2. *The maximum perturbation sets $\mathcal{D}^{\mu}_{a,b,c}$ are complicated because they are not product sets. However, if the assumptions of Proposition 3 hold, then we can choose μ large enough such that $\bar{\sigma}_\mu < 0$, since $x_2 > 0$ for all $x \in \ell_\mu$. Then for all constants $r_0 \in (0,1)$, the set $\mathcal{D}^{\mu}_{a,b,c}$ contains the open product set neighborhood*

$$\mathcal{D}^{\flat,\mu}_{a,b,c} \;=\; (-(1-r_0)\gamma\mathcal{M}_0,(1-r_0)\gamma\mathcal{M}_0)\times(-r_0\mathcal{M}_0,r_0\mathcal{M}_0) \tag{22}$$

of 0, where $\mathcal{M}_0=\min\{b,c^\sharp_p,-\bar{\sigma}_\mu/\gamma\} > 0$. In fact, for each $d=(d_1,d_2)$ in the set (22), we get $(d_1/\gamma)-c^\sharp_p < (1-r_0)\mathcal{M}_0-c^\sharp_p < d_2+\mathcal{M}_0-c^\sharp_p \le d_2$, $(\mu/\gamma)(\bar{\sigma}_\mu-d_1) < (\mu/\gamma)(\bar{\sigma}_\mu+(1-r_0)\gamma\mathcal{M}_0) < (\mu/\gamma)(\bar{\sigma}_\mu+\gamma\mathcal{M}_0)-r_0\mathcal{M}_0 < d_2$, and $(d_1/\gamma)+b > -(1-r_0)\mathcal{M}_0+b \ge r_0\mathcal{M}_0 > d_2$, which shows that $d \in \mathcal{D}^{\mu}_{a,b,c}$.

Remark 3. *We can use the sets (22) to cover more general pointer acceleration models that could have additive uncertainties on the pointer position measurements, i.e., $v=y+\Delta$, where we assume that Δ is a C^1 perturbation. To see how, note that by a slight variant of the argument that led to the interconnected dynamics (4), we can show that replacing y by $y+\Delta$ produces the new x_1 dynamics $\dot{x}_1 = G(|gx_2^+ +\delta_1|)(gx_2^+ +\delta_1)$, where $\delta_1=\dot{\Delta}$. If $x(\cdot)$ is valued in $S^{\mu}_{a,b,c}$ and the continuous perturbation δ is valued in (22) for some choice of the constant $r_0 \in (0,1)$, and if G is C^1, then we can use the Mean Value Theorem to rewrite our new x_1 subsystem as*

$$\begin{aligned}\dot{x}_1 &= \tilde{G}(x_2^+)x_2^+ + G(gx_2^+)\delta_1 + \left(G(|gx_2^+ +\delta_1|)-G(gx_2^+)\right)(gx_2^+ +\delta_1)\\ &= \tilde{G}(x_2^+)x_2^+ + d_*\delta_1,\end{aligned} \tag{23}$$

where $|d_*|$ is bounded by $\bar{\Delta}=|G'|_{[0,g(a+c)+(1-r_0)\mathcal{M}_0\gamma]}(g(a+c)+(1-r_o)\mathcal{M}_0\gamma)+|G|_{[0,g(a+c)}$, and where we used the fact that $x_2 \le a+c$ if $x \in S^{\mu}_{a,b,c}$. This produces a perturbed system that is covered by Proposition 3 (with δ_1 in the proposition replaced by the scaled perturbation $d_*\delta_1$). Therefore, for each constant $r_0 \in (0,1)$, each set $S^{\mu}_{a,b,c}$ is robustly forwardly invariant for the perturbed dynamics

$$\dot{x} = \begin{bmatrix} G(|gx_2^+ +\delta_1(t)|)(gx_2^+ +\delta_1(t)) \\ -\gamma(x_1+x_2+\delta_2(t)) \end{bmatrix}, \tag{24}$$

when we restrict the perturbations δ to be piecewise continuous locally bounded functions that are valued in

$$\mathcal{D}^{\flat,s,\mu}_{a,b,c} \;=\; \left(-\tfrac{(1-r_0)\gamma\mathcal{M}_0}{\max\{1,\bar{\Delta}\}}, \tfrac{(1-r_0)\gamma\mathcal{M}_0}{\max\{1,\bar{\Delta}\}}\right) \times (-r_0\mathcal{M}_0, r_0\mathcal{M}_0) \tag{25}$$

which is a scaled version of the product set (22) from Remark 2.

6 Robustness to Delays and Perturbations

We have so far shown how the closed loop pointing system is effected by perturbations when there are no delays. We next prove stability properties on robustly forward invariant sets under delays but without perturbations; see Theorem 3 for robustness under both delays and perturbations, under a slightly more restrictive delay bound. In what follows, the delay only occurs in x_1, which corresponds to delays between the output of the pointer acceleration system and the user's perception of the pointer location. See Section 7 for an example showing how the delay bound in the following theorem cannot be removed. Our strategy for handling delay is to add together (i) a Lyapunov type function for the corresponding non-delayed system and (ii) a double integral term whose bounds involve the delay. This so-called Lyapunov-Krasovskii method was developed in [17], and also applied in [16] in the context of curve tracking. However, the preceding references do not apply to our state constrained dynamics, e.g., because as we noted in the introduction, [16] was confined to curve tracking dynamics that are very different from the ones we consider here.

Theorem 2. *(A) For all positive constants b, c, and $a > \max\{b, c\}$, nondecreasing locally Lipschitz functions $\tilde{G} : [0,\infty) \to [0,\infty)$ that are positive on $(0,\infty)$, and constants*

$$\gamma > \tilde{G}(a+c)(a+c)/c \tag{26}$$

and

$$\tau \in \left[0, \tfrac{1}{(a+c)\tilde{G}(a+c)} \min\left\{b, c - \tfrac{1}{\gamma}\tilde{G}(a+c)(a+c)\right\}\right), \tag{27}$$

we have: For all initial functions valued in $S_{a,b,c}$, all solutions of

$$\dot{x}(t) = \begin{bmatrix} \tilde{G}(x_2^+(t))x_2^+(t) \\ -\gamma(x_1(t-\tau) + x_2(t)) \end{bmatrix} \tag{28}$$

asymptotically converge to $\mathcal{E}$. (B) For each bounded nondecreasing locally Lipschitz function $\tilde{G} : [0,\infty) \to [0,\infty)$ that is positive on $(0,\infty)$, each constant $\gamma > 0$, and each constant $\tau \in [0, 1/|\tilde{G}|_{[0,\infty)})$, we have: All solutions of (28) asymptotically converge to $\mathcal{E}$.

Proof. We first prove part (A). We first show that each trajectory $x(\cdot)$ of (28) for any initial value in $S_{a,b,c}$ remains in $S_{a,b,c}$ for all $t \geq 0$. We argue by contradiction. Let $\tau > 0$ be a constant satisfying (27), and pick constants $\tilde{a} > a$, $\tilde{b} > b$, and $\tilde{c} > c$ such that

$$\tilde{a} > \max\{\tilde{b}, \tilde{c}\},\ \gamma > \tfrac{\tilde{G}(\tilde{a}+\tilde{c})(\tilde{a}+\tilde{c})}{\tilde{c}},\ \text{and}\ \tau < \tfrac{\min\{b,c_p\}}{\tilde{G}(\tilde{a}+\tilde{c})(\tilde{a}+\tilde{c})}, \tag{29}$$

where $c_p = c - |\tilde{G}^\sharp|_{[0,a+c]}/\gamma$ as before. We can always find such values $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$ because of the strictness of the inequalities in our assumptions and the continuity of $\tilde{G}$. If $x(\ell)$ did not remain in $S_{a,b,c}$, then set $t_1 = \sup\{s \geq 0 : x(\ell) \in S_{a,b,c} \text{ for all } \ell \in [0,s]\}$, and let $t_2 > t_1$ be such that $x(\ell) \in S_{\tilde{a},\tilde{b},\tilde{c}}$ for all $\ell \in [0, t_2]$. Such a t_2 exists because $S_{a,b,c} \subseteq \text{interior}(S_{\tilde{a},\tilde{b},\tilde{c}})$. Then the restriction of $x(\cdot)$ to $[0, t_2]$ is a solution for the perturbed dynamics (16) starting in $S_{a,b,c}$ for $\delta(\ell) = x_1(\ell - \tau) - x_1(\ell)$, which satisfies

$$\begin{aligned} |\delta(\ell)| &\leq \int_{\ell-\tau}^{\ell} \tilde{G}(x_2^+(r))x_2^+(r)\mathrm{d}r \\ &\leq \tau\tilde{G}(\tilde{a}+\tilde{c})(\tilde{a}+\tilde{c}) \ < \ \min\{b, c_p\}. \end{aligned} \tag{30}$$

Since $\delta(\ell)$ remains in $\mathcal{D}_{a,b,c}$ on $[0, t_2]$, it follows from Proposition 2 that $x(\cdot)$ stays in $S_{a,b,c}$ on $[0, t_2]$, contradicting the definition of t_1.

Since $\tau < 1/\tilde{G}(a + c)$, there is a constant $\varepsilon \in (0, 1)$ such that

$$\tau < \sqrt{\frac{1-\varepsilon}{\tilde{G}(a+c)\left(\frac{\tilde{G}(a+c)}{1-\varepsilon} + \gamma\varepsilon\right)}}, \tag{31}$$

since the right side of (31) converges to $1/\tilde{G}(a + c)$ as $\varepsilon \to 0$ from the right. Fix such a constant $\varepsilon > 0$. We now use the function

$$V_{\mathrm{n}}(x) = \int_0^{x_2} \tilde{G}(\zeta^+)\zeta^+\mathrm{d}\zeta + \frac{\gamma}{2}\{x_1^2 + \varepsilon x_2^2 + 2\varepsilon x_1x_2\}. \tag{32}$$

Note that V_{n} is not a Lyapunov function with respect to $\mathcal{E}$ on $\mathbb{R}^2$, since it is not identically zero on $\mathcal{E}$. For instance, $V_{\mathrm{n}}((1, -1)) = \frac{\gamma}{2}(1 - \varepsilon)$. Nevertheless, we use V_{n} and Barbalat's Lemma to prove our stabilization result under our delay bound (27), as follows.

Along all trajectories of (28) starting in $S_{a,b,c}$, we have

$$\begin{aligned} \tfrac{d}{dt}V_{\mathrm{n}}(x(t)) &= -\gamma\big[\tilde{G}^\sharp(x_2^+(t)) + \tfrac{\gamma}{2}(2\varepsilon x_1(t) + 2\varepsilon x_2(t))\big](x_1(t-\tau) + x_2(t)) \\ &\quad + \tfrac{\gamma}{2}(2x_1(t) + 2\varepsilon x_2(t))\tilde{G}^\sharp(x_2^+(t)) \\ &= -\gamma\big[\tilde{G}^\sharp(x_2^+(t)) + \varepsilon\gamma(x_1(t) + x_2(t))\big]\left(x_1(t) - \int_{t-\tau}^{t}\tilde{G}^\sharp(x_2^+(\ell))\mathrm{d}\ell + x_2(t)\right) \\ &\quad + \gamma(x_1(t) + x_2(t))\tilde{G}^\sharp(x_2^+(t)) + \gamma(\varepsilon - 1)\tilde{G}(x_2^+(t))(x_2^+(t))^2 \\ &= -W(x(t)) + \mathcal{A}(x_t), \end{aligned} \tag{33}$$

where $\tilde{G}^\sharp(r) = \tilde{G}(r)r$ as before, and

$$\begin{aligned} &W(x) = \gamma(1-\varepsilon)\tilde{G}(x_2^+)\left(x_2^+\right)^2 + \gamma^2\varepsilon(x_1 + x_2)^2, \\ &\mathcal{A}(x_t) = \gamma[\tilde{G}(x_2^+(t))x_2^+(t) + \varepsilon\gamma(x_1(t) + x_2(t))]\mathcal{I}_t, \\ &\text{and } \mathcal{I}_t = \int_{t-\tau}^{t}\tilde{G}(x_2^+(\ell))x_2^+(\ell)\mathrm{d}\ell. \end{aligned}$$

Since $x_2 \leq a + c$ for all $x \in S_{a,b,c}$ and $\tilde{G}$ is nondecreasing, we get $\tilde{G}(x_2^+) \leq (\tilde{G}(a+c)\tilde{G}(x_2^+))^{1/2}$ for all $x \in S_{a,b,c}$. Hence, since $x(\ell) \in S_{a,b,c}$ for all $\ell \geq 0$, the Jensen inequality gives

$$\mathcal{I}_t^2 \leq \tau\int_{t-\tau}^{t}\tilde{G}(x_2^+(\ell))(x_2^+(\ell))^2\mathrm{d}\ell\tilde{G}(a+c) \tag{34}$$

so two applications of Young's inequality give

$$\begin{array}{rcl}\mathcal{A}(x_t) & \leq & \frac{1}{2}\gamma(1-\varepsilon)\tilde{G}(x_2^+(t))(x_2^+(t))^2 + \frac{\gamma\tilde{G}(a+c)}{2(1-\varepsilon)}\mathcal{I}_t^2 \\ & & +\frac{1}{2}\gamma^2\varepsilon(x_1(t)+x_2(t))^2 + \frac{\varepsilon\gamma^2}{2}\mathcal{I}_t^2 \\ & \leq & \frac{1}{2}W(x(t)) + \bar{\mathcal{B}}\int_{t-\tau}^t \tilde{G}(x_2^+(\ell))\left(x_2^+(\ell)\right)^2 \mathrm{d}\ell\end{array} \tag{35}$$

where

$$\bar{\mathcal{B}} = \frac{\tau\gamma\tilde{G}(a+c)}{2}\left(\frac{\tilde{G}(a+c)}{1-\varepsilon} + \gamma\varepsilon\right). \tag{36}$$

Using (35) to upper bound $\mathcal{A}(x_t)$ in (33), we get

$$\frac{d}{dt}V_{\mathrm{n}}(x(t)) \quad \leq \quad -\frac{1}{2}W(x(t)) + \bar{\mathcal{B}}\int_{t-\tau}^t \tilde{G}(x_2^+(\ell))\left(x_2^+(\ell)\right)^2 \mathrm{d}\ell. \tag{37}$$

Also, (31) gives $\bar{\mathcal{B}} < \frac{\gamma}{2\tau}(1-\varepsilon)$. Fix any constant $\mathcal{L} > 0$ such that

$$\bar{\mathcal{B}} < \mathcal{L} < \frac{\gamma}{2\tau}(1-\varepsilon). \tag{38}$$

Notice that for all $t \geq 0$, we have

$$\begin{array}{l}\frac{d}{dt}\int_{t-\tau}^t\int_s^t \tilde{G}(x_2^+(\ell))\left(x_2^+(\ell)\right)^2 \mathrm{d}\ell \mathrm{d}s \\ = \tau\tilde{G}(x_2^+(t))(x_2^+(t))^2 - \int_{t-\tau}^t \tilde{G}(x_2^+(\ell))\left(x_2^+(\ell)\right)^2 \mathrm{d}\ell.\end{array}$$

Hence, (37) and our bounds on $\mathcal{L}$ give a constant $c_0 > 0$ such that the time derivative of

$$V^\sharp(x_t) = V_{\mathrm{n}}(x(t)) + \mathcal{L}\int_{t-\tau}^t\int_s^t \tilde{G}(x_2^+(\ell))\left(x_2^+(\ell)\right)^2 \mathrm{d}\ell\mathrm{d}s \tag{39}$$

along all trajectories of (28) starting in $S_{a,b,c}$ satisfies

$$\begin{array}{rcl}\frac{d}{dt}V^\sharp(x_t) & \leq & -\frac{1}{2}W(x(t)) + \mathcal{L}\tau\tilde{G}(x_2^+(t))\left(x_2^+(t)\right)^2 \\ & & +(\bar{\mathcal{B}}-\mathcal{L})\int_{t-\tau}^t \tilde{G}(x_2^+(\ell))\left(x_2^+(\ell)\right)^2 \mathrm{d}\ell \\ & \leq & -c_0 W(x(t)).\end{array} \tag{40}$$

The forward invariance of the compact set $S_{a,b,c}$ ensures boundedness of $x(t)$, and so also absolute continuity of $W(x(t))$. Since W is positive definite with respect to $\mathcal{E}$, and since (40) implies that $\int_0^\infty W(x(\ell))\mathrm{d}\ell < \infty$, Barbalat's Lemma gives $\lim_{t\to\infty} W(x(t)) = 0$, and therefore also convergence of $x(t)$ to $\mathcal{E}$. This proves part (A). To prove part (B), we replace $\tilde{G}(a+c)$ in the preceding argument by $|\tilde{G}|_{[0,\infty)}$, and omit the portion of the argument about bounding $\delta(t)$, since no robust forward invariance is needed when proving a result on $\mathbb{R}^2$ when $\tilde{G}$ is bounded. This proves the theorem. □

Theorem 2 provides a simple bound on the allowable constant delay τ that ensures attractivity properties of the input delayed model (28). It can be applied when one knows a suitable upper bound on τ, even if the exact value of τ is uncertain. However, since Theorem 2 is based on Barbalat's Lemma instead of a strict Lyapunov function, it does not lend itself to proving ISS properties for

$$\dot{x}(t) = \left[\begin{array}{c}\tilde{G}(x_2^+(t))x_2^+(t) \\ -\gamma(x_1(t-\tau)+x_2(t)+\delta(t))\end{array}\right] \tag{41}$$

with uncertainty $\delta(t)$ and constant delay τ. Therefore, we prove the next result on (41), under a slightly more restrictive bound on τ than the one in Theorem 2 (but see Remark 4 for a further generalization with perturbations in both equations). We use a different $\mathcal{H}$ from the one in Proposition 1. Our new choice of $\mathcal{H}$ will not satisfy the requirements 1)-4) from Theorem 1 for all $r \geq 0$, but it will satisfy the requirements 1)-4) for all $r \leq a+c$ and so is a valid choice when we restrict x to any of our robustly forwardly invariant sets $S_{a,b,c}$.

Theorem 3. *Let $\tilde{G} : [0,\infty) \to [0,\infty)$ be C^1, positive definite with respect to $\{0\}$, and nondecreasing. Let $b > 0$, $c > 0$, and $a > \max\{b, c\}$ be constants. Set*

$$\mathcal{H}(r) = r^+ \tilde{G}(r^+), \tag{42}$$

and let the constant $\gamma > 0$ be such that $c > |\tilde{G}^\sharp|_{[0,a+c]}/\gamma$ and

$$\gamma > \sup_{r\in(0,a+c]} \max\left\{4\mathcal{H}'(r), 2\mathcal{H}^2(r)/\int_0^r \tilde{G}(\ell)\ell \mathrm{d}\ell\right\}. \tag{43}$$

Set $c_p = c - \tilde{G}(a+c)(a+c)/\gamma$. Then for all constants τ such that

$$0 \leq \tau < \min\left\{\tfrac{1}{2\gamma}, \tfrac{\min\{b,c_p\}}{(a+c)\tilde{G}(a+c)}\right\} \tag{44}$$

the system (41) is ISS with respect to $\mathcal{E}$ and $\mathcal{U} = (-\bar{\delta}, \bar{\delta})$ on $\mathcal{S} = S_{a,b,c}$, where $\bar{\delta} = \min\{b, c_p\} - \tau\tilde{G}(a+c)(a+c)$.

Proof. Since (44) implies that the bound $\tau < \min\{b, c_p\}/((a+c)\tilde{G}(a+c))$ from Theorem 2 holds, we can use our bound on $|\delta|$ to argue as in the first part of the proof of Theorem 2 (with δ from the earlier proof replaced by the combined disturbance $\delta^\sharp(\ell) = x_1(\ell-\tau) - x_1(\ell) + \delta(\ell)$) to prove that $S_{a,b,c}$ is robustly forwardly invariant for (41) with perturbations valued in $\mathcal{U} = (-\bar{\delta}, \bar{\delta})$, when (44) holds.

In particular, $S_{a,b,c}$ is strongly invariant for (41) when $\tau = 0$ and $\delta = 0$. Therefore, since $x_2^+ \leq a+c$ for all $x \in S_{a,b,c}$, and since (43) implies that (42) satisfies conditions 1)-4) from Theorem 1 when we restrict to values $r \leq a+c$, the decay estimate (8) from Theorem 1 holds along all trajectories of (41) starting in $S_{a,b,c}$ when $\tau = 0$ and $\delta = 0$. Hence, along all trajectories of (41) starting in $S_{a,b,c}$ for all $\tau \geq 0$ satisfying (44) and $\delta = 0$, our choice (42) of $\mathcal{H}$ gives

$$\begin{aligned}\tfrac{d}{dt}V_{\rm new}(x) &\leq -\tfrac{\gamma^2}{4}(x_1+x_2)^2 - \tfrac{1}{2}\mathcal{H}(x_2)\tilde{G}(x_2^+)x_2^+ \\ &\quad + \gamma\left[-\mathcal{H}'(x_2)(x_1+x_2) + \gamma(x_1+x_2)\right]\int_{t-\tau}^t \tilde{G}(x_2^+(\ell))x_2^+(\ell)\mathrm{d}\ell \\ &\leq -\tfrac{\gamma^2}{4}(x_1+x_2)^2 - \tfrac{1}{2}\mathcal{H}(x_2(t))\tilde{G}(x_2^+)x_2^+ \\ &\quad + \left\{\tfrac{5\gamma^2}{4}|x_1+x_2|\int_{t-\tau}^t \tilde{G}(x_2^+(\ell))x_2^+(\ell)\mathrm{d}\ell\right\}\end{aligned} \tag{45}$$

where we used (8), (42), and (43). Using Jensen's inequality, we get

$$\begin{aligned}&\tfrac{5\gamma^2}{4}|x_1(t)+x_2(t)|\int_{t-\tau}^t \tilde{G}(x_2^+(\ell))x_2^+(\ell)\mathrm{d}\ell \\ &\leq \tfrac{1}{5}\gamma^2(x_1(t)+x_2(t))^2 + 2\tau\gamma^2\int_{t-\tau}^t \mathcal{H}^2(x_2^+(\ell))\mathrm{d}\ell,\end{aligned} \tag{46}$$

by our choice (42) of $\mathcal{H}$, since the triangle inequality gives

$$\frac{5}{4}pq \le \frac{5}{4}((1/2)(8/25)p^2 + (1/2)(25/8)q^2)$$

for all $p \ge 0$ and $q \ge 0$.

By our delay bound (44), we can find a constant $J > 0$ such that

$$\tau < \tfrac{\sqrt{1-2J\tau}}{2\gamma}, \tag{47}$$

so $1 - 4\tau^2\gamma^2 - 2J\tau > 0$. Also, our choice (42) of $\mathcal{H}$ implies that $\alpha_0 \in \mathcal{K}_\infty$ from the proof of Theorem 1 satisfies

$$\alpha_0(V_{\text{new}}(x)) \le \tfrac{\gamma^2}{4}(x_1 + x_2)^2 + \tfrac{1}{2}\mathcal{H}^2(x_2) \tag{48}$$

for all $x \in S_{a,b,c}$. Hence, we can use (46) to upper bound the quantity in curly braces in (45) and the relation

$$\tfrac{d}{dt}\int_{t-\tau}^{t}\int_{s}^{t}\mathcal{H}^2(x_2^+(\ell))\mathrm{d}\ell = \tau\mathcal{H}^2(x_2^+(t)) - \int_{t-\tau}^{t}\mathcal{H}^2(x_2^+(\ell))\mathrm{d}\ell \tag{49}$$

to conclude that along all trajectories of (41) for $\delta = 0$ starting in $S_{a,b,c}$ for any constant $\tau > 0$ satisfying (44), the function

$$V_{\text{new}}^\sharp(x_t) = V_{\text{new}}(x(t)) + (2\tau\gamma^2 + J)\int_{t-\tau}^{t}\int_{s}^{t}\mathcal{H}^2(x_2^+(\ell))\mathrm{d}\ell\mathrm{d}s$$

satisfies

$$\begin{aligned}
\tfrac{d}{dt}V_{\text{new}}^\sharp(x_t) &\le -\tfrac{\gamma^2}{20}(x_1(t) + x_2(t))^2 - J\int_{t-\tau}^{t}\mathcal{H}^2(x_2^+(\ell))\mathrm{d}\ell \\
&\quad - \left(\tfrac{1}{2} - 2\tau^2\gamma^2 - J\tau\right)\mathcal{H}^2(x_2^+(t)) \\
&\le -\min\left\{\tfrac{1}{5}, 1 - 4\tau^2\gamma^2 - 2J\tau\right\}\alpha_0(V_{\text{new}}(x(t))) \\
&\quad -\tfrac{J}{\tau}\int_{t-\tau}^{t}\int_{s}^{t}\mathcal{H}^2(x_2^+(\ell))\mathrm{d}\ell\mathrm{d}s \\
&\le -\mathcal{M}_0\left\{\alpha_0(V_{\text{new}}(x(t)))\right. \\
&\quad \left. + (2\tau\gamma^2 + J)\int_{t-\tau}^{t}\int_{s}^{t}\mathcal{H}^2(x_2^+(\ell))\mathrm{d}\ell\mathrm{d}s\right\} \\
&\le -\alpha_0^\flat(V_{\text{new}}^\sharp(x_t)),
\end{aligned}$$

where

$$\mathcal{M}_0 = \min\left\{\tfrac{1}{5}, 1 - 4\tau^2\gamma^2 - 2J\tau, \tfrac{J}{\tau(2\tau\gamma^2 + J)}\right\} \tag{50}$$

and $\alpha_0^\flat(r) = \mathcal{M}_0 \min\left\{\alpha_0\left(\frac{r}{2}\right), \frac{r}{2}\right\}$ is of class $\mathcal{K}_\infty$, and where we used the fact that for the class $\mathcal{K}_\infty$ function $\alpha(\ell) = \min\{\alpha_0(\ell), \ell\}$, we have $\alpha((a+b)/2) \le \alpha(a) + \alpha(b)$ for all $a \ge 0$ and $b \ge 0$. Using the fact that $|\nabla V_{\text{new}}|$ is bounded by some constant $V_{a,b,c}^\star$ on the compact set $S_{a,b,c}$, it follows that along all trajectories of (41) starting in $S_{a,b,c}$, and for all constant delays τ satisfying (44) and all perturbations $\delta \in \mathcal{M}(\mathcal{U})$, we have

$$\tfrac{d}{dt}V_{\text{new}}^\sharp(x_t) \le -\alpha_0^\flat(V_{\text{new}}^\sharp(x_t)) + \gamma V_{a,b,c}^\star|\delta|_{[0,t]}. \tag{51}$$

Therefore, $V_{\text{new}}^\sharp$ is an ISS Lyapunov function for (41) with respect to $\mathcal{E}$ and $\mathcal{U}$ on $\mathcal{S} = S_{a,b,c}$, which implies the ISS property. □

Remark 4. *We can generalize Theorem 3 to provide ISS results for*

$$\dot{x}(t) = \begin{bmatrix} \tilde{G}(x_2^+(t))x_2^+(t) + \delta_1(t) \\ -\gamma(x_1(t-\tau) + x_2(t) + \delta_2(t)) \end{bmatrix} \tag{52}$$

having locally bounded piecewise continuous perturbations δ_i in both equations and constant input delays $\tau \geq 0$, as follows. We assume that $\tilde{G}$, a, b, c, γ and τ satisfy the requirements from Theorem 3, and we let $\mu > 1$ be any positive constant. Fix any constant $\bar{d}_1 > 0$ such that $\tau < \min\{b, c_p\}/(\tilde{G}^\sharp(a+c) + \bar{d}_1)$, which always exists, by (44). Then (52) is ISS with respect to $\mathcal{E}$ and $\mathcal{U}^\sharp$ on $S^\mu_{a,b,c}$, where

$$\mathcal{U}^\sharp = \left\{ \begin{array}{l} d \in \mathbb{R}^2 : \ |d_1| \leq \bar{d}_1, \ \max\left\{\frac{d_1}{\gamma} - c_p^\sharp, \frac{\mu}{\gamma}(\bar{\sigma}_\mu - d_1)\right\} + \\ \tau(\tilde{G}^\sharp(a+c) + \bar{d}_1) < d_2 < \frac{d_1}{\gamma} + b - \tau(\tilde{G}^\sharp(a+c) + \bar{d}_1) \end{array} \right\}.$$

and $S^\mu_{a,b,c}$, $c_p^\sharp$, and $\bar{\sigma}_\mu$ are as defined in Section 5. This follows from Proposition 3, the use of the augmented perturbation $\delta_2^\sharp(\ell) = x_1(\ell-\tau) - x_1(\ell) + \delta_2(\ell)$, and the fact that the gradient $\nabla V_{\text{new}}(x)$ is bounded on the unbounded set $S^\mu_{a,b,c}$, where the terms $\tau(\tilde{G}^\sharp(a+c)+\bar{d}_1)$ were used to bound the terms $x_1(\ell-\tau) - x_1(\ell)$ from $\delta_2^\sharp$. This lets us cover the dynamics (24) with perturbed pointer position measurements. Due to page limitations, we leave the details to the reader.

With this final result, we have shown how the closed loop pointing system is effected by delays and perturbations. These results provide a relationship between the maximum delay, maximal perturbation set, and size of invariant sets, and this relationship depends on the scaling function G and other system parameters. In the future, this relationship can be used to compare the properties of different scaling functions and better design pointing interfaces using acceleration.

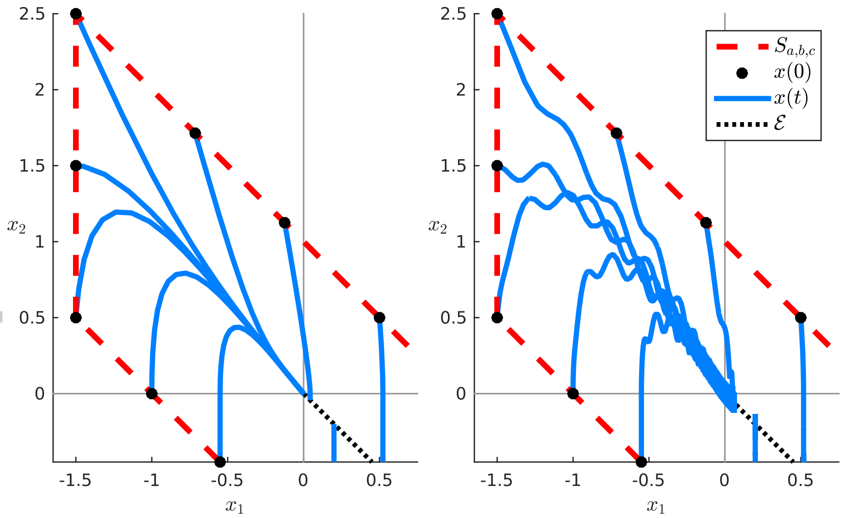
7 Simulations

We use computational simulations to illustrate the results of this paper, including the invariance and stability properties of the closed loop pointing system with perturbations. Figure 4 shows several trajectories of (41) with and without delays and perturbations, with initial conditions selected in $S_{a,b,c}$. For this simulation, we chose the linear scaling function $\tilde{G}(s) = 1 + 0.1s$, $g = 1$, and the parameter values $a = 1.5$, $b = c = 1$, and $\gamma = 7$, which satisfy our requirements

$$\begin{array}{l} 1 = c > \frac{|\tilde{G}^\sharp|_{[0,a+c]}}{\gamma} \approx 0.4 \ \text{ and} \\ 7 = \gamma > \sup\limits_{r\in(0,a+c]} \max\left\{4\mathcal{H}'(r), 2\mathcal{H}^2(r)/\int_0^r \tilde{G}(\ell)\ell \mathrm{d}\ell\right\} = 6 \end{array}$$

from Theorem 3. Using the notation from Theorem 3, we get $c_p = c - \tilde{G}(a+c)(a+c)/\gamma \approx 0.6$, so our delay condition (44) becomes

$$0 \leq \tau < \min\left\{\frac{1}{2\gamma}, \frac{\min\{b, c_p\}}{(a+c)\tilde{G}(a+c)}\right\} \approx 0.1 \tag{53}$$



(a) (b)

Figure 4: Trajectories starting in $S_{a,b,c}$ (a) without delays or perturbations and (b) with significant delays and perturbations.

and the delay bound from Theorem 3 is $\bar{\delta} = \min\{b, c_p\} - \tau\tilde{G}(a + c)(a + c) \approx 0.3$. For our simulation, we chose $\tau = 0.1$ and $\delta(t) = 0.3\sin(10\pi t)$. Without perturbations, the trajectories with initial states in $S_{a,b,c}$ converge asymptotically to the equilibrium set $\mathcal{E}$. When we add perturbations, the trajectories do not always converge to $\mathcal{E}$, but do remain within $S_{a,b,c}$. These figures show that, at least for these choices of parameters, $S_{a,b,c}$ is invariant and $\mathcal{E}$ is attractive.

It is tempting to surmise that our delay conditions such as (27) can be eliminated, so that our robust forwardly invariant conclusions would remain true without the delay bounds. However, we cannot drop our delay conditions, even if there are no perturbations. For instance, if we take $\tilde{G}(s) = 1 + 0.1s$, $g = 1$, and the values $a = 1.5$, and $b = c = 1$ as in our simulation, then the corresponding set $S_{a,b,c}$ has the upper left vertex $(-1.5, 2.5)$. However, the solution of (28) for the initial state $(-1.5, 2.5) \in S_{a,b,c}$ and the delay $\tau = 0.5$ passes through $(0, 1.52) \notin S_{a,b,c}$, so $S_{a,b,c}$ would no longer be forwardly invariant for (28), if we were to allow a larger delay such as $\tau = 0.5$. Hence, our delay bound (27) from Theorem 2 cannot be removed.

8 Conclusions and Future Work

Pointer acceleration has the potential to help improve the performance of computer mice, joysticks, and other important human interfaces. We presented a new model for pointer acceleration and proved key properties for the interconnection of our model with the VITE model of human pointing. This provided ISS estimates under input delays and uncertainty, and robustly forward invariant state performance bounds with maximum perturbation sets. Hence, our work is an interesting analog of our robust forward invariance results from [15,16], which were limited to curve tracking dynamics from marine robotics. Our simulations demonstrated the utility of our theory under realistic operating conditions. In future work, we hope to incorporate input smoothing, pointer prediction, corrective motions, and workspace constraints. More work on the experimental side could also help transition our results into actual pointing devices to improve their effectiveness in human-pointer interfaces.

References

[1] Pointer ballistics for Windows XP. https://web.archive.org/web/20120513022418/http://msdn.microsoft.com/en-us/windows/hardware/gg463319.

[2] Mouse acceleration preference pane for Mac OS X. http://triq.net/articles/mouse-acceleration-preference-pane-mac-os-x.

[3] X.org pointer acceleration development documentation. http://www.x.org/wiki/Development/Documentation/PointerAcceleration/.

[4] How to disable mouse acceleration. https://support.microsoft.com/en-us/kb/149228.

[5] D. Beamish, S. Bhatti, C. Chubbs, I. MacKenzie, J. Wu, and Z. Jing. Estimation of psychomotor delay from the Fitts' law coefficients. *Biological Cybernetics*, 101(4):279-296, 2009.

[6] D. Beamish, S. Bhatti, I. MacKenzie, and J. Wu. Fifty years later: a neurodynamic explanation of Fitts' law. *Journal of the Royal Society Interface*, 3(10):649-654, 2006.

[7] D. Beamish, S. Bhatti, J. Wu, and Z. Jing. Performance limitations from delay in human and mechanical motor control. *Biological Cybernetics*, 99(1):43-61, 2008.

[8] D. Beamish, I. MacKenzie, and J. Wu. Speed-accuracy trade-off in planned arm movements with delayed feedback. *Neural Networks*, 19(5):582-599, 2006.

[9] D. Beamish, C. Peskun, and J. Wu. Critical delay for overshooting in planned arm movements with delayed feedback. *Journal of Mathematical Biology*, 50(1):22-48, 2004.

[10] D. Bullock and S. Grossberg. Neural dynamics of planned arm movements: emergent invariants and speed-accuracy properties during trajectory formation. *Psychological Review*, 95(1):49-90, 1988.

[11] P. Fitts. The information capacity of the human motor system in controlling the amplitude of movement. *Journal of Experimental Psychology*, 47(6):381-391, 1954.

[12] H. Khalil. *Nonlinear Systems, Third Edition*. Upper Saddle River, NJ: Prentice Hall, 2002.

[13] I. Kolmanovsky and E. Gilbert. Theory and computation of disturbance invariant sets for discrete-time linear systems. *Mathematical Problems in Engineering*, 4(4):317-367, 1998.

[14] M. Malisoff and F. Mazenc. *Constructions of Strict Lyapunov Functions*. New York: Springer, 2009.

[15] M. Malisoff, F. Mazenc, and F. Zhang. Stability and robustness analysis for curve tracking control using input-to-state stability. *IEEE Transactions on Automatic Control*, 57(5):1320-1326, 2012.

[16] M. Malisoff and F. Zhang. Robustness of adaptive control under time delays for three-dimensional curve tracking. *SIAM Journal on Control and Optimization*, 53(4):2203-2236, 2015.

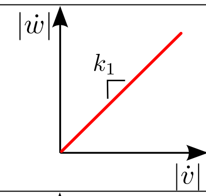
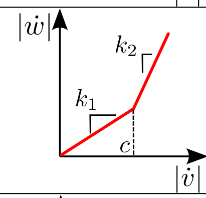
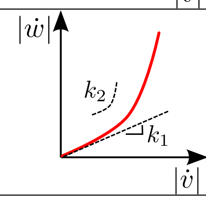
[17] F. Mazenc, M. Malisoff, and Z. Lin. Further results on input-to-state stability for nonlinear systems with delayed feedbacks. *Automatica*, 44(9):2415-2421, 2008.

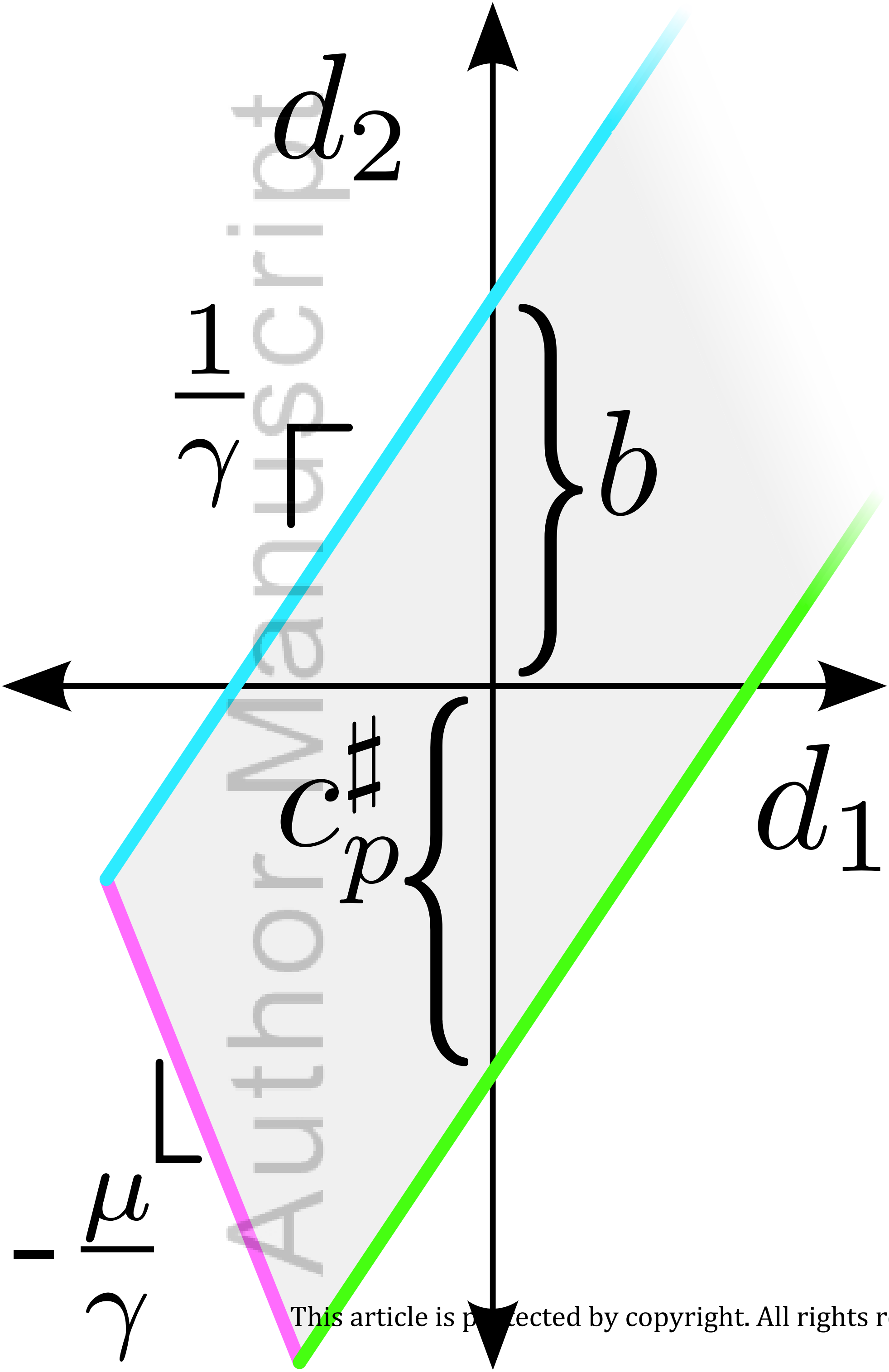
[18] R. Shadmehr and S. Wise. *The Computational Neurobiology of Reaching and Pointing: A Foundation for Motor Learning*. Cambridge, MA: MIT Press, 2005.

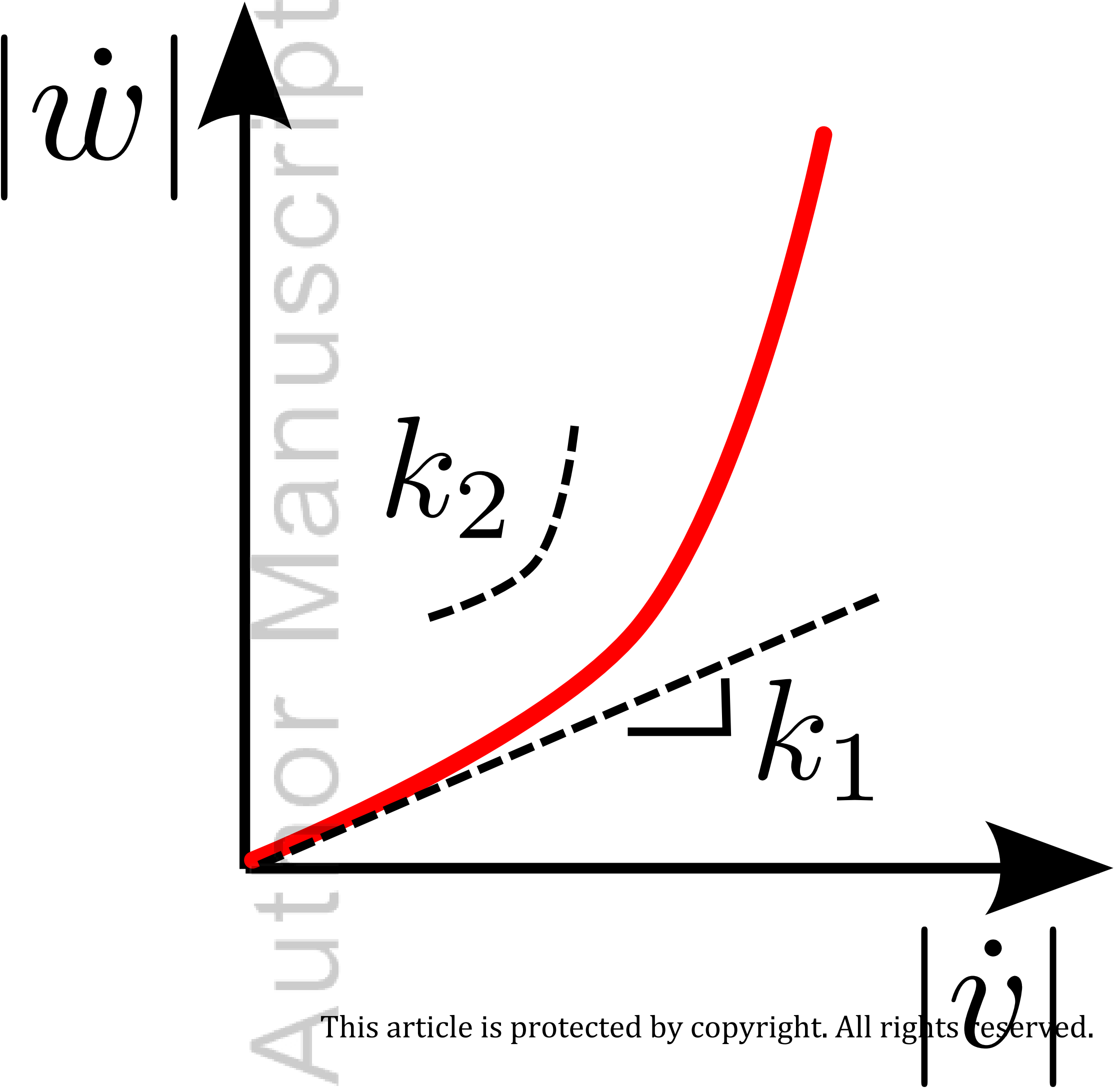
[19] P. Varnell and F. Zhang. Dissipativity-based teleoperation with time-varying communication delays. In *Proceedings of the 4th IFAC Workshop on Distributed Estimation and Control in Networked Systems*, pp. 369-376, 2013.

[20] P. Varnell and F. Zhang. Characteristics of human pointing motions with acceleration. In *Proceedings of the 54th IEEE Conference on Decision and Control*, pp. 5364-5369, 2015.

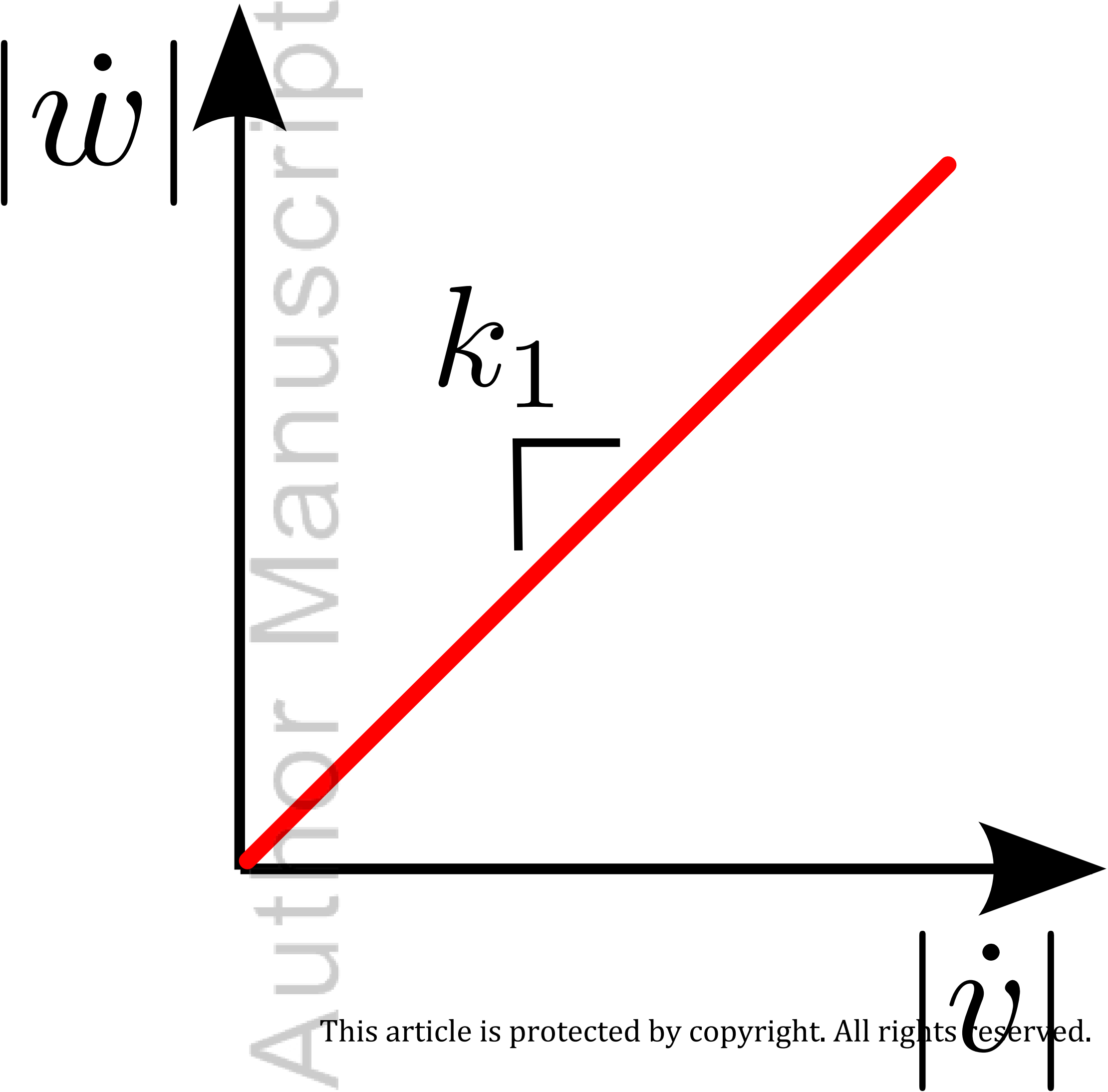
Table 1: Common Pointer Acceleration Profiles

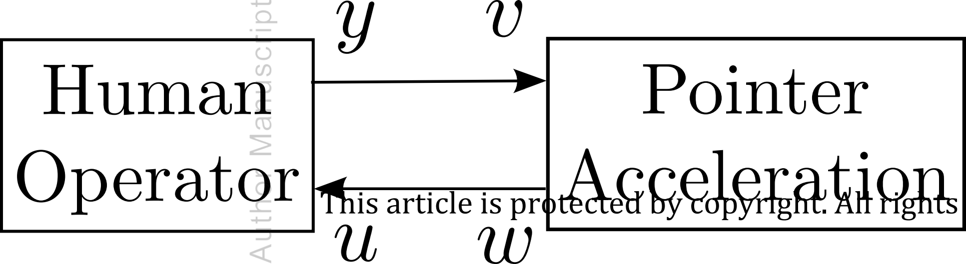
Name	**Scaling Function**	**I/O Velocity plot**
No Accel.	$G(\lvert\dot{v}\rvert) = k_1$	
Threshold	$G(\lvert\dot{v}\rvert) = \begin{cases} k_1, & \text{if } 0 \leq \lvert\dot{v}\rvert < c \\ k_2, & \text{if } \lvert\dot{v}\rvert \geq c \end{cases}$	
Linear	$G(\lvert\dot{v}\rvert) = k_1 + k_2\lvert\dot{v}\rvert$	


d_2
$\frac{1}{\gamma}\Gamma$
b
$c_p^\sharp$
d_1
$-\frac{\mu}{\gamma}L$

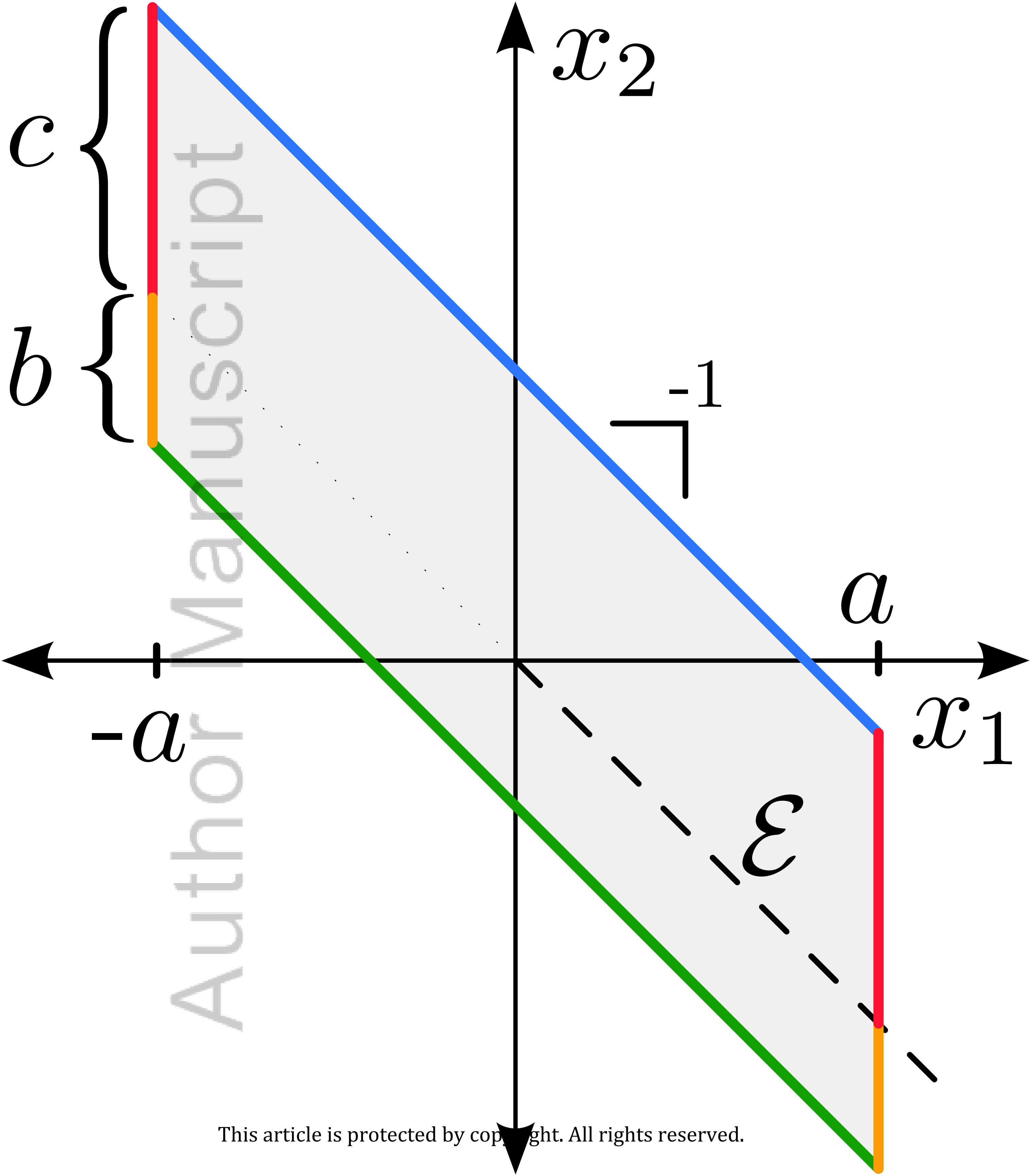


|ẇ|
k2
k1
|v̇|

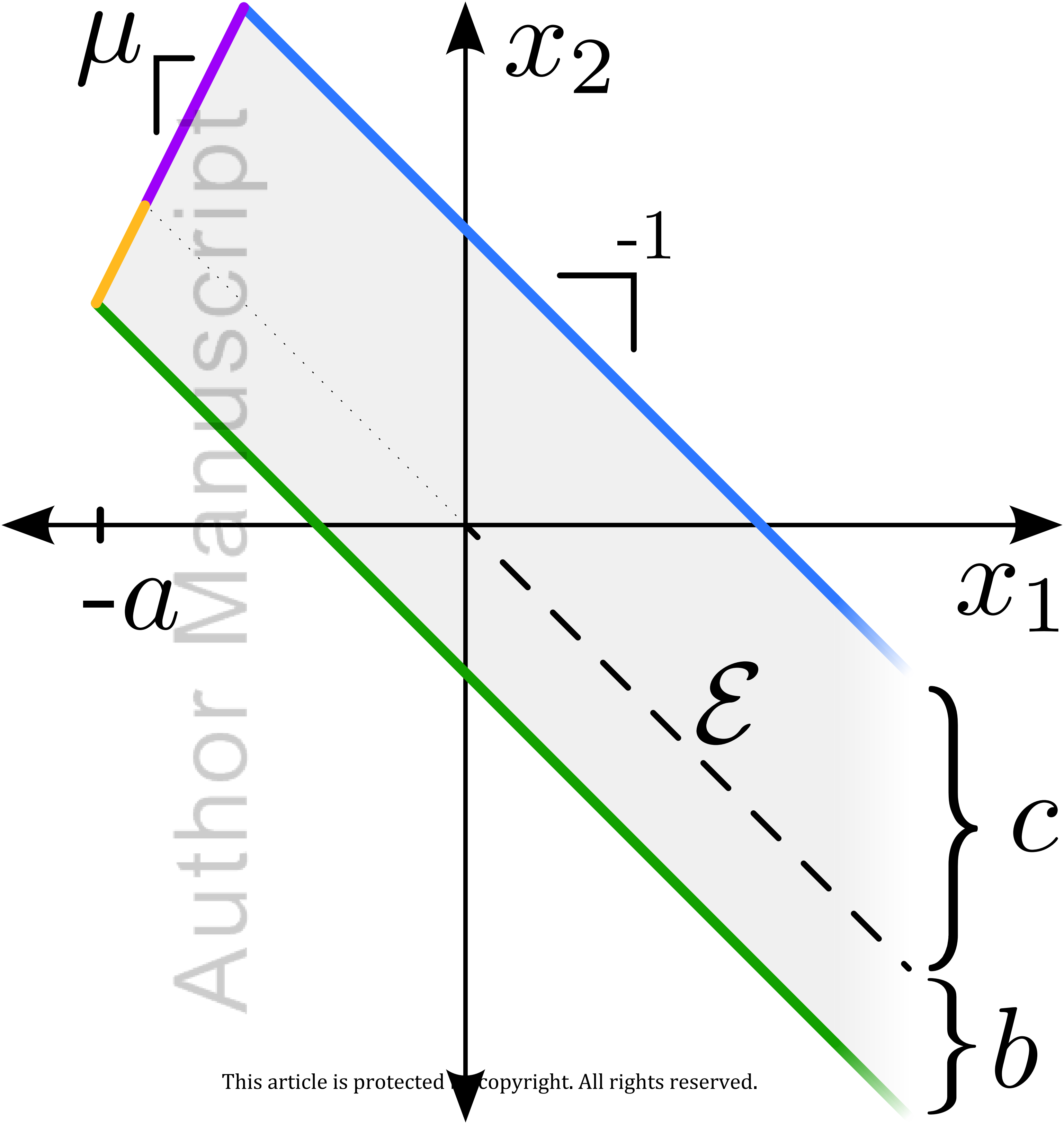

$|\dot{w}|$
k_1
$|\dot{v}|$



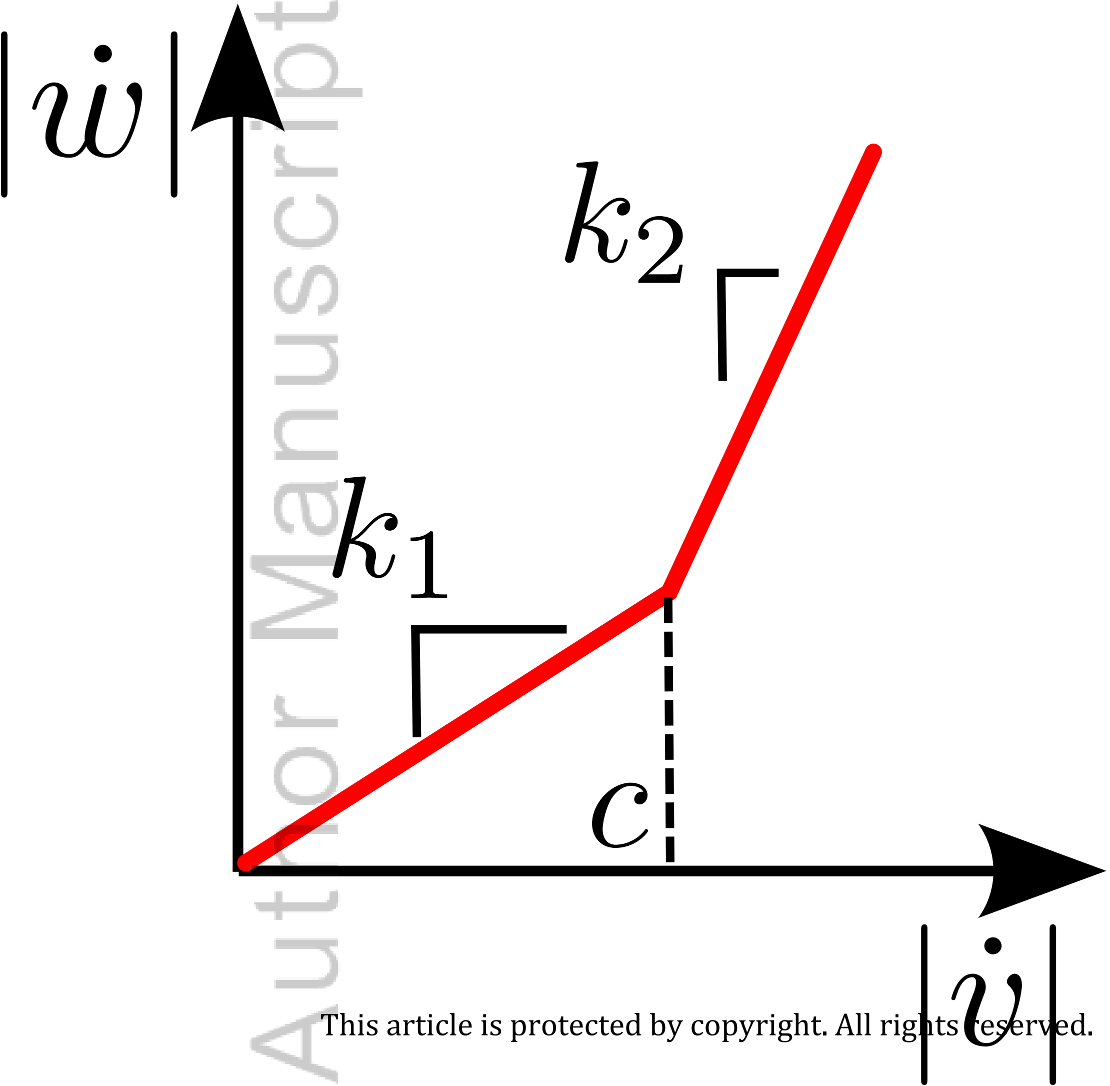
Human Operator
Pointer Acceleration
y
v
u
w



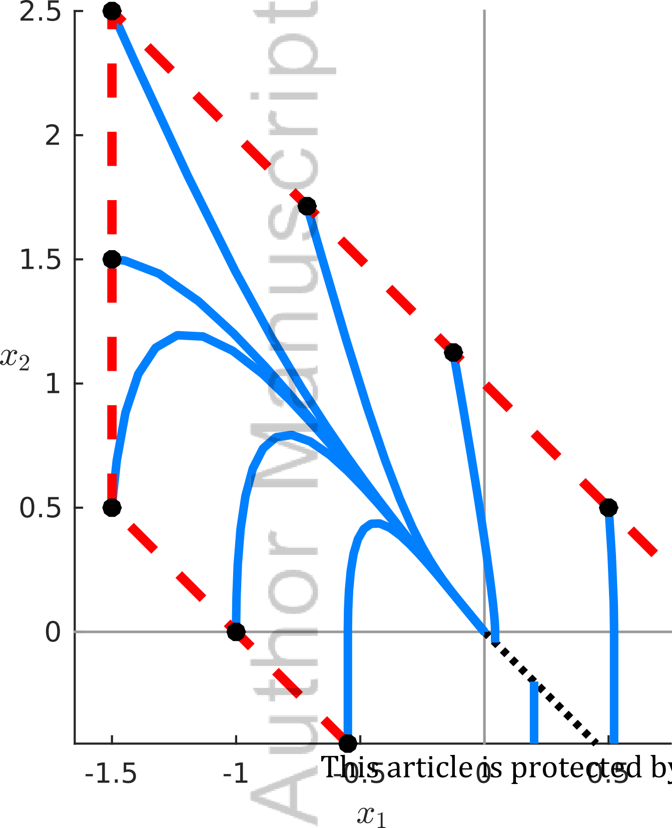
x_2
c
b
-1
a
$-a$
x_1
$\mathcal{E}$



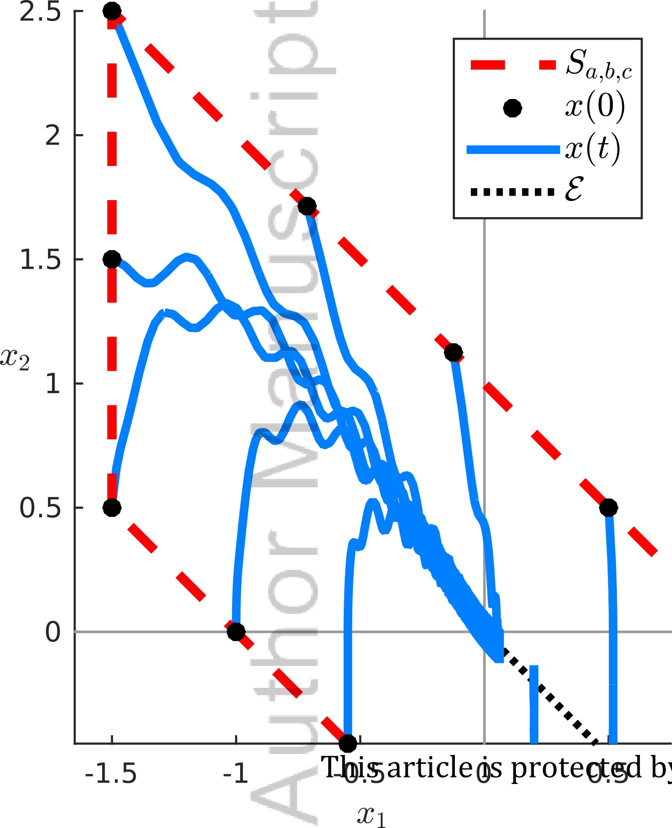
μ
x_2
-1
x_1
-a
$\mathcal{E}$
c
b



$|\dot{w}|$
k_2
k_1
c
$|\dot{v}|$



2.5
2
1.5
1
0.5
0
x_2
-1.5
-1
-0.5
0
0.5
x_1


$S_{a,b,c}$
$x(0)$
$x(t)$
$\mathcal{E}$
x_2
x_1
2.5
2
1.5
1
0.5
0
-1.5
-1
-0.5
0
0.5