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Backstepping Design for Output Feedback Stabilization for a Class of Uncertain Systems using Dynamic Extension ⋆

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Abstract: We revisit the backstepping approach. We show how bounded globally asymptotically stabilizing output feedbacks can be constructed for a family of nonlinear systems. The approach relies on the introduction of a dynamic extension and a converging-input-converging-state assumption. The technique presents several advantages. It provides control laws whose expressions are simple. It makes it possible to stabilize systems in the presence of certain types of uncertain terms which prevent the use of the classical backstepping technique. It applies in notable cases where only part of the state variable is measured.

Keywords: backstepping, nonlinear, output feedback, stabilization

1. INTRODUCTION

The backstepping approach is useful for constructing globally asymptotically stabilizing control laws for nonlinear systems in feedback form, i.e., having a lower triangular structure; see Dixon et al. (2000). Since the pioneering contributions in Coron and Praly (1991) and Tsinias (1997), it has been developed in many contributions and is successfully used in many applications, as illustrated for instance by Jiang and Nijmeijer (1999), Pettersen and Nijmeijer (2002), Smaoui et al. (2006), de Queiroz and Dawson (1996) and Lee et al. (2011). Presentations of the backstepping technique can be found in Khalil (2002), Mazenc and Bowong (2004), and many other research monographs and papers.

One of the drawbacks of this technique is the complexity of the formulas it sometimes provides, notably when it is applied repeatedly and when size constraints on the control laws have to be respected. This is a limitation of the applicability of classical backstepping and of the bounded backstepping results of Mazenc and Iggidr (2004) and Mazenc and Bowong (2004). Another limitation of the approach is due to the fact that in general, it does not apply when only a part of the state is measured. In addition, it relies on the existence of a fictitious feedback which typically has to be of class C^k when the backstepping is applied k times. The presence of uncertainties in the dynamics and in the output may also be an obstacle.

To overcome these drawbacks, a new technique has been proposed recently, based on control laws in which delays are artificial, meaning there are delays in the state values that are used in the controls even when there are no input delays in the original systems. This technique has been initiated in Mazenc and Malisoff (2016) and developed and applied in Mazenc et al. (2016) and Mazenc et al. (2017b), where the case of systems with delays in the input is also considered and where the initial fictitious control law is not required to be of class C^1. A first adaptation of this technique to uncertain outputs has also been studied in Mazenc et al. (2017a), where, due to the use of visual information, only imprecise measurement of the first backstepping variable was available.

In the present paper, we propose a new backstepping design for globally uniformly asymptotically stabilizing control laws for partially linearizable systems; see Isidori (1995) for an introduction to partially linearizable systems. Its fundamental new aspect with respect to the contributions that use artificial delays is that, instead of introducing delays, a finite dimensional dynamic extension is designed, making it possible to obtain feedbacks without delays, which offer the following advantages: (i) they are bounded in the cases where bounded feedbacks can be expected (which contrasts with Mazenc et al. (2016) where the controls obtained for the original systems are not bounded controls), (ii) they are given by simple formulas, (iii) they apply in cases where the subsystem from which the backstepping begins contains unknown vector fields, and (iv) they apply in cases where only a part of the state variable is measured and where some parts of the output are not known with certainty. The family of partially linearizable systems we consider encompasses many systems that are relevant from an applied point of view, as shown for instance by Spong (1994). Also, with respect to the backstepping with artificial delay methods from the works

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Mazenc and Malisoff (2016), Mazenc et al. (2016), and Mazenc et al. (2018, to appear), the main drawback of the main result of the present paper is that it relies on a fictitious feedback of class C^1 (although these prior works did not cover systems with outputs or with uncertain vector fields). Comparisons between the performances of the control laws of Mazenc and Malisoff (2016) and Mazenc et al. (2016) and those of the present paper still have to be made and will be the subject of future studies. Our converging-input-converging-state condition provides a novel and promising approach and was not present in the well known works such as Liu et al. (1996) and Sussmann et al. (1994) for linear systems.

This paper is organized as follows. The main result is shown in Section 2. Section 3 provides a technical lemma that can facilitate checking our assumptions from Section 2. An example based on a direct drive manipulator is given in Section 4. Concluding remarks are drawn in Section 5. Our notation will be simplified whenever no confusion can arise from the context. The dimensions of our Euclidean spaces are arbitrary unless other indicated. The Euclidean norm in $\mathbb{R}^a$, and the induced norm of matrices, are denoted by $|\cdot|$, and $|\cdot|_\infty$ is the usual sup norm. Given a constant $J > 0$, sat_J denotes the symmetric saturation function defined by $\mathrm{sat}_J(x) = \max\{-J, \min\{J, x\}\}$ for all $x \in \mathbb{R}$, where J is called the saturation level. We use the usual class of comparison functions $\mathcal{K}_\infty$ from Khalil (2002).

2. MAIN STABILIZATION RESULT

2.1 The studied system

We consider the nonlinear time-varying system

$$\left\{\begin{array}{rl} \dot{x} &= f(t,x,g_1) \\ \dot{g}_1 &= a_{1,1}g_1 + a_{1,2}g_2 \\ \dot{g}_2 &= a_{2,1}g_1 + a_{2,2}g_2 + a_{2,3}g_3 \\ \vdots & \\ \dot{g}_n &= a_{n,1}g_1 + a_{n,2}g_2 + \cdots + a_{n,n}g_n \\ & \quad + u + \Omega(\mathcal{Y}(t,x), G) \end{array}\right. \tag{1}$$

where x is valued in $\mathbb{R}^p$, $G = (g_1, ..., g_n)$ is valued in $\mathbb{R}^n$, f is a nonlinear function that is locally Lipschitz with respect to (x, g_1) and piecewise continuous with respect to t, u is the scalar-valued input, $\mathcal{Y} : \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}^l$ is a continuous function, each $a_{i,j} \in \mathbb{R}$ is a constant, $a_{i,i+1} \neq 0$ holds for $i = 1, 2...n-1$ (with no sign constraint on the a_{ij}'s), and the function Ω is locally Lipschitz. This family of systems has been studied in many papers because systems of this type may result from partial linearization and are frequently encountered in practice.

In this section, we assume that the $\mathbb{R}^{l+n}$ valued output is

$$Y(t) = (\mathcal{Y}(t, x(t)), g_1(t), \ldots, g_n(t))^\top, \tag{2}$$

which is realistic in practice. Our first assumption is:

Assumption 1. There exist a locally Lipschitz scalar valued function ψ that is bounded by a known constant $\overline{\psi} \geq 0$, and a constant $k > 0$, such that for any continuous function $d : [0, \infty) \to \mathbb{R}$ that converges exponentially to the origin, all solutions $(\xi, \lambda_1, ..., \lambda_n) : [0, \infty) \to \mathbb{R}^{p+n}$ of

$$\left\{\begin{array}{rl} \dot{\xi} &= f(t, \xi, \lambda_1 + d(t)) \\ \dot{\lambda}_1 &= k[-\lambda_1 + \lambda_2] \\ \vdots & \\ \dot{\lambda}_n &= k[-\lambda_n + \psi(t, \mathcal{Y}(t, \xi))] \end{array}\right. \tag{3}$$

converge to the origin as $t \to \infty$. Moreover the function Ω in (1) is bounded by a known constant $\overline{\Omega} \geq 0$.

In terms of the matrices

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & 0 & \ldots & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & \ldots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \ldots & \ldots & a_{n,n} \end{bmatrix} \in \mathbb{R}^{n\times n} \tag{4}$$

for $n > 1$ and $A = a_{1,1}$ for $n = 1$ and $B = (0, 0, \ldots, 1)^\top \in \mathbb{R}^n$, our second assumption is:

Assumption 2. There is a locally Lipschitz function $\varpi : \mathbb{R}^n \to \mathbb{R}$ that is bounded by a known constant $\overline{\varpi} \geq 0$ such that the origin of the system

$$\dot{\Gamma}(t) = A\Gamma(t) + B\varpi(\Gamma(t)) \tag{5}$$

is a globally asymptotically and locally exponentially stable equilibrium.

2.2 Statement of and discussion on theorem

We are ready to state and prove the following result, where sat is as defined in Section 1:

Theorem 1. Let the system (1) satisfy Assumptions 1-2 and let $\epsilon > 0$ be any constant. Then there exist a constant matrix L and constants c_j for $j = 1$ to $n + 1$ such that all solutions $(x, G) : [0, \infty) \to \mathbb{R}^{p+n}$ of (1), in closed loop with the dynamic output feedback

$$\begin{array}{rl} u(G,Z,x) &= -\mathrm{sat}_{\overline{Z}}\left(\displaystyle\sum_{j=1}^{n} c_j z_j + c_{n+1}\psi(t, \mathcal{Y}(t,x))\right) \\ & \quad +\varpi\left(G + LZ\right) - \Omega\left(\mathcal{Y}(t,x), G\right) \\ \dot{z}_1 &= k[-z_1 + z_2] \\ \vdots & \\ \dot{z}_{n-1} &= k[-z_{n-1} + z_n] \\ \dot{z}_n &= k[-z_n + \psi(t, \mathcal{Y}(t,x))] \end{array} \tag{6}$$

with $Z = (z_1, ..., z_n)$ and the saturation level

$$\overline{Z} = \overline{\psi}(1+\epsilon)\sum_{j=1}^{n+1}|c_j|,$$

asymptotically converge to 0 as $t \to \infty$. □

Before turning to the proof of Theorem 1, we make several remarks on its motivation and value.

1) The feedback (6) is bounded by the constant $\overline{Z}+\overline{\varpi}+\overline{\Omega}$.

2) The formula (6) for the control law does not incorporate the first derivative of $\psi(t, \mathcal{Y}(t, x(t)))$, which plays the role of the fictitious control of the classical backstepping approach. Hence, it applies even when $\psi(t, \mathcal{Y}(t, x))$ is not of class C^1. However, in practice Theorem 1 can only be applied when ψ is of class C^1 because checking that Assumption 1 is satisfied frequently necessitates that ψ be of class C^1, as we show in the next section. By contrast, in Mazenc et al. (2016) and Mazenc et al. (2018, to appear), fictitious feedbacks of class C^0 can be used.

3) One can use changes of variables and an appropriate choice of feedback to transform the system (1) into a system of the form (3) with $d = 0$ whose exponential stability property is ensured by Assumption 1. However, in a sense, this result is not satisfactory because the feedback obtained that way may possess inappropriate properties. For instance, in general they would be unbounded. This motivates our alternative approach, based on dynamic extensions.

5) None of the assumptions of this paper imply that f has to be known with accuracy. Also, we only require measurements of $\psi(t, \mathcal{Y}(t,x))$ and $\Omega(\mathcal{Y}(t,x), G)$, instead of $\mathcal{Y}$ itself.

2.3 Proof of Theorem 1

A key aspect of the proof consists of making $g_1(t) - z_1(t)$ converge to zero instead of making $g_1(t) - \psi(t, \mathcal{Y}(t, x(t)))$ converge to zero, as traditionally done in classical backstepping. To achieve our goal, several changes of coordinates are needed, as follows. We first assume that $n > 1$, and we explain how these changes of variables can be constructed by induction. Set $l_{1,1} = -1$ and $r_1 = g_1 - z_1$.

Induction assumption: For all $j \in \{1, ..., i\}$ and $1 < i < n$, there are constants $l_{j,m}$ for $m = 1$ to j such that the variables

$$r_j = g_j + \sum_{m=1}^{j} l_{j,m} z_m \tag{7}$$

for $j = 1$ to i satisfy

$$\begin{cases} \dot{r}_1 = a_{1,1} r_1 + a_{1,2} r_2 \\ \dot{r}_2 = a_{2,1} r_1 + a_{2,2} r_2 + a_{2,3} r_3 \\ \quad \vdots \\ \dot{r}_{i-1} = a_{i-1,1} r_1 + a_{i-1,2} r_2 + \cdots + a_{i-1,i} r_i. \end{cases} \tag{8}$$

First step: $i = 2$. Since $r_1 = g_1 - z_1$, we have

$$\begin{aligned} \dot{r}_1 &= a_{1,1} g_1 + a_{1,2} g_2 - k[-z_1 + z_2] \\ &= a_{1,1} r_1 + a_{1,2} \left[g_2 + \frac{a_{1,1} + k}{a_{1,2}} z_1 - \frac{k}{a_{1,2}} z_2 \right]. \end{aligned} \tag{9}$$

Then the variable

$$r_2 = g_2 + \frac{a_{1,1} + k}{a_{1,2}} z_1 - \frac{k}{a_{1,2}} z_2 \tag{10}$$

satisfies

$$\dot{r}_1 = a_{1,1} r_1 + a_{1,2} r_2 \tag{11}$$

Thus the induction assumption is satisfied at the first step.

Step i: Assume that the induction assumption is satisfied at a step i with $1 < i < n$. Then since (7) holds for $j = 1, 2, \ldots, i$ we have

$$\begin{aligned} \dot{r}_i &= a_{i,1} g_1 + a_{i,2} g_2 + ... + a_{i,i+1} g_{i+1} \\ &\quad + \sum_{m=1}^{i} l_{i,m} k[-z_m + z_{m+1}] \\ &= a_{i,1} r_1 + a_{i,2} r_2 + ... + a_{i,i} r_i + a_{i,i+1} g_{i+1} \\ &\quad + a_{i,1} z_1 - a_{i,2}(l_{2,1} z_1 + l_{2,2} z_2) - ... \\ &\quad - a_{i,i} \sum_{m=1}^{i} l_{i,m} z_m + \sum_{m=1}^{i} l_{i,m} k[-z_m + z_{m+1}]. \end{aligned} \tag{12}$$

Thus, taking

$$\begin{aligned} r_{i+1} &= g_{i+1} + \tfrac{a_{i,1}}{a_{i,i+1}} z_1 - \tfrac{a_{i,2}}{a_{i,i+1}} (l_{2,1} z_1 + l_{2,2} z_2) - ... \\ &\quad - \tfrac{a_{i,i}}{a_{i,i+1}} \sum_{m=1}^{i} l_{i,m} z_m + \tfrac{1}{a_{i,i+1}} \sum_{m=1}^{i} l_{i,m} k[-z_m + z_{m+1}] \end{aligned} \tag{13}$$

we obtain

$$\dot{r}_i = a_{i,1} r_1 + a_{i,2} r_2 + ... + a_{i,i+1} r_{i+1}, \tag{14}$$

so the induction assumption is satisfied at the step $i + 1$.

Taking the time derivative of (13) with the choice $i = n-1$, we can then find a linear change of coordinates $R = G+LZ$ with $R = (r_1, ..., r_n)^\top$ that transforms the system (1) into

$$\begin{cases} \dot{x} = f(t, x, z_1 + r_1) \\ \dot{r}_1 = a_{1,1} r_1 + a_{1,2} r_2 \\ \dot{r}_2 = a_{2,1} r_1 + a_{2,2} r_2 + a_{2,3} r_3 \\ \quad \vdots \\ \dot{r}_n = a_{n,1} r_1 + a_{n,2} r_2 + \cdots + a_{n,n} r_n + u \\ \qquad + \sum_{j=1}^{n} l_{n,j} z_j + l_{n,n+1} \psi(t, \mathcal{Y}(t,x)) \\ \qquad + \Omega(\mathcal{Y}(t,x), G). \end{cases} \tag{15}$$

If instead $n = 1$, then we again obtain (15) with only the dynamics for x and r_n present. Let $c_j = l_{n,j}$ for $j = 1$ to $n + 1$. Then the closed-loop system is

$$\begin{cases} \dot{x} = f(t, x, z_1 + r_1) \\ \dot{R} = AR + B \Bigg[\varpi(R) \\ \qquad - \mathrm{sat}_{\overline{Z}} \left(\sum_{j=1}^{n} c_j z_j + c_{n+1} \psi(t, \mathcal{Y}(t,x)) \right) \\ \qquad + \sum_{j=1}^{n} c_j z_j + c_{n+1} \psi(t, \mathcal{Y}(t,x)) \Bigg] \\ \dot{z}_1 = k[-z_1 + z_2] \\ \quad \vdots \\ \dot{z}_n = k[-z_n + \psi(t, \mathcal{Y}(t,x))]. \end{cases} \tag{16}$$

Consider any solution $(x, R, z) : [0, \infty) \to \mathbb{R}^{p+2n}$ of (16).

Since Assumption 1 ensures that ψ is bounded by $\overline{\psi}$, it follows from (16) that there is a finite value $t_a \geq 0$ such that for all $t \geq t_a$, the inequality

$$\left| \sum_{j=1}^{n} c_j z_j(t) + c_{n+1} \psi(t, \mathcal{Y}(t, x(t))) \right| \leq \overline{Z} \tag{17}$$

is satisfied. Hence, when $t \geq t_a$, the closed-loop system is

$$\begin{cases} \dot{x} = f(t, x, z_1 + r_1) \\ \dot{R} = AR + B\varpi(R) \\ \dot{z}_1 = k[-z_1 + z_2] \\ \quad \vdots \\ \dot{z}_n = k[-z_n + \psi(t, \mathcal{Y}(t,x))]. \end{cases} \tag{18}$$

Assumption 2 ensures that the R-subsystem of the system (18) is globally asymptotically and locally exponentially stable. Then Assumption 1 allows us to conclude.

3. TECHNICAL LEMMA

Assumption 2 is a simple classical assumption that can often be checked easily; see our illustration below. However, checking Assumption 1 can be nontrivial. In this section, we give conditions ensuring that Assumption 1 holds.

3.1 Assumptions and lemma

Consider the system

$$\left\{\begin{array}{rl}\dot{\xi} &= f(t,\xi,\lambda_1+d(t))\\ \dot{\lambda}_i &= k[-\lambda_i+\lambda_{i+1}],\ 1\le i\le n-1\\ \dot{\lambda}_n &= k[-\lambda_n+\psi(t,y_\xi)]\end{array}\right. \tag{19}$$

where $y_\xi = C\xi$ and $C \in \mathbb{R}^{q\times p}$ is a constant matrix, the constant $k>0$ will be further restricted below, and $d:[0,\infty)\to\mathbb{R}$ exponentially converges to 0. Then (19) is a special case of (3). We add these assumptions on (19):

Assumption 3. There is a function $\kappa_1 \in \mathcal{K}_\infty$ such that

$$|\psi(t,y_\xi)| + |f(t,\xi,w)| \le \kappa_1(|(\xi,w)|) \tag{20}$$

for all $t\ge 0$, $\xi\in\mathbb{R}^p$ and $w\in\mathbb{R}$. Also, ψ is of class C^1 and bounded by a known constant $\overline{\psi}$. Finally, there are functions κ_2 and κ_3 of class $\mathcal{K}_\infty$, a function V of class C^1, and a uniformly continuous positive definite W such that

$$\kappa_2(|\xi|) \le V(t,\xi) \le \kappa_3(|\xi|) \text{ for all } (t,\xi)\in[0,\infty)\times\mathbb{R}^p \tag{21}$$

and such that for any choice of the piecewise continuous function δ, the time derivative of V along all solutions of

$$\dot{\xi}(t) = f(t,\xi(t),\psi(t,C\xi(t))+\delta(t)) \tag{22}$$

satisfies the inequality $\dot{V}(t) \le -W(\xi(t)) + |\delta(t)|^2$ for all $t\ge 0$, and f is locally Lipschitz.

Assumption 4. There are positive constants b_f and b_s such that for the function W from Assumption 3, the inequality

$$\begin{array}{l}\left|\frac{\partial\psi}{\partial t}(t,C\xi) + \frac{\partial\psi}{\partial y_\xi}(t,C\xi)Cf(t,\xi,\psi(t,C\xi)+\gamma)\right|^2\\ \le b_f W(\xi) + b_s|\gamma|^2\end{array} \tag{23}$$

holds for all $t\ge 0$, $\xi\in\mathbb{R}^p$ and $\gamma\in\mathbb{R}$.

The inclusion of the γ term on the right side of (23) is justified because γ represents uncertainty and overshoot terms involving uncertainties commonly arise, e.g., from input-to-state stability estimates. We prove:

Lemma 1. Let the system (19) satisfy Assumptions 3 and 4. Then for any $k\ge \underline{k}$ with

$$\underline{k} = 3n\left(n + \tfrac{n+1}{2}\left(b_f + \tfrac{b_s}{2}\right)\right) \tag{24}$$

all solutions of (19) converge to 0 as $t\to\infty$.

3.2 Proof of Lemma 1

Set $s_i = \lambda_i - \psi(t,y_\xi)$ for $i=1,2,\dots,n$ and $S=(s_1,...,s_n)$. Then (19) can be transformed into

$$\left\{\begin{array}{rl}\dot{\xi} &= f(t,\xi,\psi(t,y_\xi)+s_1+d(t))\\ \dot{s}_1 &= k[-s_1+s_2] - \dfrac{\partial\psi}{\partial t}(t,y_\xi)\\ &\quad -\dfrac{\partial\psi}{\partial y_\xi}(t,y_\xi)Cf(t,\xi,\psi(t,y_\xi)+s_1+d(t))\\ \vdots & \\ \dot{s}_n &= -ks_n - \dfrac{\partial\psi}{\partial t}(t,y_\xi)\\ &\quad -\dfrac{\partial\psi}{\partial y_\xi}(t,y_\xi)Cf(t,\xi,\psi(t,y_\xi)+s_1+d(t))\ .\end{array}\right. \tag{25}$$

From Assumption 3, it follows that $\dot{V}(t) \le -W(\xi(t)) + |s_1(t)+d(t)|^2$ for all $t\ge 0$. Assumption 4 gives

$$\begin{array}{l}\left|\dfrac{\partial\psi}{\partial t}(t,y_\xi) + \dfrac{\partial\psi}{\partial y_\xi}(t,y_\xi)Cf(t,\xi,\psi(t,y_\xi)+s_1+d(t))\right|^2\\ \le b_f W(\xi) + b_s|s_1+d(t)|^2\end{array} \tag{26}$$

for all $t\ge 0$. Consider the positive definite function

$$Q(S) = \tfrac{1}{2}\sum_{i=1}^{n} i s_i^2\ . \tag{27}$$

Then we can use the triangle inequality to obtain

$$\begin{array}{l}\displaystyle\sum_{i=1}^{n-1} i s_i s_{i+1} \le \frac{1}{2}\sum_{i=1}^{n-1} i(s_i^2 + s_{i+1}^2)\\ \displaystyle\quad = \frac{1}{2}\left(\sum_{i=1}^{n-1} i s_i^2 + \sum_{i=2}^{n}(i-1)s_i^2\right)\end{array} \tag{28}$$

and then the subadditivity of the square root to conclude that along all solutions of (25) for all $t\ge 0$, we have

$$\begin{array}{rl}\dot{Q}(t) &\displaystyle\le k\sum_{i=1}^{n-1} i s_i(-s_i+s_{i+1}) - kns_n^2\\ &\displaystyle\quad + \sum_{i=1}^{n} i|s_i|\sqrt{b_f W(\xi) + b_s|s_1+d(t)|^2}\\ &\displaystyle\le -\frac{k}{2}\left(\sum_{i=1}^{n-1} i s_i^2 - \sum_{i=2}^{n}(i-1)s_i^2\right) - kns_n^2\\ &\displaystyle\quad + \sum_{i=1}^{n} i|s_i|\sqrt{b_f W(\xi) + b_s|s_1+d(t)|^2}\\ &\displaystyle\le -\frac{k}{2}\sum_{i=1}^{n} s_i^2\\ &\displaystyle\quad + \sum_{i=1}^{n} i|s_i|\sqrt{b_f W(\xi) + b_s|s_1+d(t)|^2}\\ &\displaystyle\le -\frac{k}{2}\sum_{i=1}^{n} s_i^2 + \sqrt{b_f}\sum_{i=1}^{n} i|s_i|\sqrt{W(\xi)}\\ &\displaystyle\quad + \sqrt{b_s}\sum_{i=1}^{n} i|s_i||s_1| + \sum_{i=1}^{n} i|s_i|\sqrt{b_s}|d(t)|.\end{array} \tag{29}$$

Consider the candidate Lyapunov function

$$U(t,\xi,S) = \frac{n(n+1)}{2}b_f V(t,\xi) + Q(S) \tag{30}$$

and any trajectory of (25). Along the trajectories of (25),

$$\begin{array}{rl}\dot{U}(t) &\displaystyle\le -\frac{n(n+1)}{2}b_f W(\xi)\\ &\displaystyle\quad + \frac{n(n+1)}{2}b_f|s_1(t)+d(t)|^2\\ &\displaystyle\quad -\frac{k}{2}\sum_{i=1}^{n} s_i^2 + \sqrt{b_f}\sum_{i=1}^{n} i|s_i|\sqrt{W(\xi)}\\ &\displaystyle\quad + \sqrt{b_s}\sum_{i=1}^{n} i|s_i||s_1| + \sum_{i=1}^{n} i|s_i|\sqrt{b_s}|d(t)|\ .\end{array} \tag{31}$$

Using the triangle inequality to obtain

$$\sqrt{b_f}\, i|s_i|\sqrt{W(\xi)} \le \frac{1}{2}\left(b_f W(\xi) + i^2 s_i^2\right), \tag{32}$$

we deduce that

$$\begin{array}{l}\displaystyle\dot{U}(t) \le -\frac{nb_f}{2}W(\xi) - \frac{k}{2}\sum_{i=1}^{n} s_i^2 + \frac{n(n+1)}{2}b_f s_1^2\\ \displaystyle\quad + \frac{1}{2}\sum_{i=1}^{n} i^2 s_i^2 + \sqrt{b_s}\sum_{i=1}^{n} i|s_i||s_1| + \frac{n(n+1)}{2}b_f d^2(t)\\ \displaystyle\quad + n(n+1)b_f s_1 d(t) + \sum_{i=1}^{n} i|s_i|\sqrt{b_s}|d(t)|\ .\end{array} \tag{33}$$

Using $\sqrt{b_s}\, i|s_i||s_1| \le \frac{i}{2}s_i^2 + \frac{i}{2}b_s s_1^2$, we obtain

$$
\begin{aligned}
\dot{U}(t) \le & -\frac{nb_f}{2}W(\xi) - \frac{k}{2}\sum_{i=1}^{n} s_i^2 \\
& + n(n+1)\left(\frac{b_f}{2} + \frac{b_s}{4}\right) s_1^2 \\
& + \sum_{i=1}^{n} n^2 s_i^2 + \frac{n(n+1)}{2} b_f d^2(t) \\
& + n(n+1) b_f s_1 d(t) + \sum_{i=1}^{n} i|s_i|\sqrt{b_s}|d(t)| .
\end{aligned} \tag{34}
$$

From (24), we deduce that

$$
\begin{aligned}
\dot{U}(t) \le & -\frac{nb_f}{2}W(\xi) - \frac{k}{6}\sum_{i=1}^{n} s_i^2 + \frac{n(n+1)}{2} b_f d^2(t) \\
& + n(n+1) b_f s_1 d(t) + \sum_{i=1}^{n} i|s_i|\sqrt{b_s}|d(t)| .
\end{aligned} \tag{35}
$$

Since each λ_i (and therefore also each s_i) enters a fixed compact set after a finite time (by the boundedness of ψ and the structure of the system (19)), there are constants $\Lambda > 0$ and T_l such that for all $t \ge T_l$,

$$
\dot{U}(t) \le -\left[\frac{nb_f}{2}W(\xi) + \frac{k}{6}\sum_{i=1}^{n} s_i^2\right] + d_\sharp(t) \tag{36}
$$

with $d_\sharp(t) = n(n+1)b_f d^2(t) + \Lambda|d(t)|$. From this inequality, we can conclude from Barbalat's Lemma (applied to the function in squared brackets in (36), using the fact that $d_\sharp(t)$ converges exponentially to the origin) that all the solutions of (19) converge to the origin.

4. ILLUSTRATION

Consider a single-link direct-drive manipulator actuated by a permanent magnet DC brush motor, which produces the following model from Dawson et al. (1994) (after a change of coordinates which removes a constant):

$$
\left\{
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= g_1 - q_1 \sin(x_1) - q_2 x_2 \\
\dot{g}_1 &= u - q_3 \arctan(x_2) - a_{1,1} g_1
\end{aligned}
\right. \tag{37}
$$

with the output $Y = (x, g_1)$ and the $\mathbb{R}^2$-valued state $x = (x_1, x_2)$, where $a_{1,1} > 0$ and $q_i > 0$ for $i = 1$ to 3 are constants. Assumption 2 is satisfied with $\varpi = 0$ and $n = 1$ (because in this case, the system (5) with $\varpi = 0$ is a linear system that is exponentially stable to 0), and we choose the bounded function $\Omega(x, g_1) = -q_3 \arctan(x_2)$.

Let us check that Assumption 1 is satisfied with $n = 1$, under the added assumptions that

$$
\frac{q_1 q_2}{1 + q_1} < 1 \ \text{ and } \ \frac{q_1}{1 + q_1} < \frac{q_2}{2}. \tag{38}
$$

Consider

$$
\psi(\xi) = q_1 \sin(\xi_1) - \frac{\xi_1}{\sqrt{1 + \xi_1^2}} \tag{39}
$$

and any continuous function $d : [0, \infty) \to \mathbb{R}$ that exponentially converges to 0 and

$$
\left\{
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= -q_1 \sin(\xi_1) - q_2 \xi_2 + \lambda_1 + d(t) \\
\dot{\lambda}_1 &= k\left[-\lambda_1 + q_1 \sin(\xi_1) - \frac{\xi_1}{\sqrt{1 + \xi_1^2}}\right]
\end{aligned}
\right. \tag{40}
$$

with $k > 0$ being a constant to be specified. The choice

$$
s_1 = \lambda_1 - q_1 \sin(\xi_1) + \frac{\xi_1}{\sqrt{1 + \xi_1^2}}
$$

gives

$$
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= -q_2 \xi_2 - \frac{\xi_1}{\sqrt{1 + \xi_1^2}} + s_1 + d(t) \\
\dot{s}_1 &= -k s_1 + \left[-q_1 \cos(\xi_1) + \frac{1}{(1+\xi_1^2)\sqrt{1+\xi_1^2}}\right] \xi_2 .
\end{aligned} \tag{41}
$$

Let us choose

$$
V(\xi) = \sqrt{1 + \xi_1^2} - 1 + \tfrac{1}{2}\xi_2^2 + \tfrac{q_1}{4(1+q_1)} \tfrac{\xi_1}{\sqrt{1+\xi_1^2}} \xi_2 . \tag{42}
$$

Then the first inequality in (38) and the triangle inequality combine to give

$$
\begin{aligned}
& -q_2 \frac{q_1}{4(1 + q_1)} \frac{\xi_1 \xi_2}{\sqrt{1 + \xi_1^2}} \\
& \le \frac{q_1}{4(1 + q_1)} \left(\frac{1}{2}\frac{\xi_1^2}{1 + \xi_1^2} + \frac{1}{2} q_2^2 \xi_2^2\right) \\
& \le \frac{q_1}{8(1 + q_1)} \frac{\xi_1^2}{1 + \xi_1^2} + \frac{q_2}{8}\xi_2^2,
\end{aligned} \tag{43}
$$

and so also

$$
\begin{aligned}
\dot{V}(t) =\ & \tfrac{\xi_1 \xi_2}{\sqrt{1+\xi_1^2}} + \tfrac{q_1}{4(1+q_1)} \tfrac{\xi_2^2}{(1+\xi_1^2)\sqrt{1+\xi_1^2}} \\
& + \left(\xi_2 + \tfrac{q_1}{4(1+q_1)} \tfrac{\xi_1}{\sqrt{1+\xi_1^2}}\right) \\
& \times \left(-q_2 \xi_2 - \tfrac{\xi_1}{\sqrt{1+\xi_1^2}} + s_1 + d(t)\right) \\
=\ & -q_2 \left(\xi_2^2 + \tfrac{q_1}{4(1+q_1)} \tfrac{\xi_1 \xi_2}{\sqrt{1+\xi_1^2}}\right) \\
& - \tfrac{q_1}{4(1+q_1)} \tfrac{\xi_1^2}{1+\xi_1^2} + \tfrac{q_1}{4(1+q_1)} \tfrac{\xi_2^2}{(1+\xi_1^2)\sqrt{1+\xi_1^2}} \\
& + \left(\xi_2 + \tfrac{q_1}{4(1+q_1)} \tfrac{\xi_1}{\sqrt{1+\xi_1^2}}\right)(s_1 + d(t)) \\
\le\ & -\tfrac{3q_2}{4}\xi_2^2 - \tfrac{q_1}{8(1+q_1)} \tfrac{\xi_1^2}{1+\xi_1^2} + \xi_2(s_1 + d(t)) \\
& + \tfrac{q_1}{4(1+q_1)} \tfrac{\xi_1}{\sqrt{1+\xi_1^2}}(s_1 + d(t)) \\
\le\ & -\tfrac{q_2}{4}\xi_2^2 - \tfrac{q_1}{16(1+q_1)} \tfrac{\xi_1^2}{1+\xi_1^2} \\
& + \left(\tfrac{1}{2q_2} + \tfrac{q_1}{4(1+q_1)}\right)(s_1 + d(t))^2 \\
\le\ & -\tfrac{q_2}{4}\xi_2^2 - \tfrac{q_1}{16(1+q_1)} \tfrac{\xi_1^2}{1+\xi_1^2} \\
& + \left(\tfrac{1}{q_2} + \tfrac{q_1}{2(1+q_1)}\right) s_1^2 \\
& + \left(\tfrac{1}{q_2} + \tfrac{q_1}{2(1+q_1)}\right) d^2(t),
\end{aligned} \tag{44}
$$

along all solutions of (41) for all $t \ge 0$, where the first inequality in (44) used (38) and (43), the second equality in (44) used the relations

$$
\xi_2(s_1 + d(t)) \le \tfrac{1}{2} q_2 \xi_2^2 + \tfrac{1}{2q_2}(s_1 + d(t))^2 \ \text{ and }
$$
$$
\tfrac{\xi_1}{\sqrt{1+\xi_1^2}}(s_1 + d(t)) \le \tfrac{1}{4} \tfrac{\xi_1^2}{1+\xi_1^2} + (s_1 + d(t))^2,
$$

and the last inequality in (44) use the inequality $(a + b)^2 \le 2a^2 + 2b^2$ for suitable a and b. Also, the relations $\sqrt{1 + p} - 1 \ge p/(4(1 + p))$ for all $p \ge 0$, and (43) with q_2 replaced by 1, can be used to show that (42) is positive definite, so (42) is proper and positive definite.

Next let $U(\xi, s_1) = V(\xi) + \frac{1}{2}s_1^2$. Then (44) gives

$$
\begin{aligned}
\dot{U}(t) \le & -\tfrac{q_2}{4}\xi_2^2 - \tfrac{q_1}{16(1+q_1)}\tfrac{\xi_1^2}{1+\xi_1^2} \\
& + \left(\tfrac{1}{q_2} + \tfrac{q_1}{2(1+q_1)}\right) s_1^2 \\
& + \left(\tfrac{1}{q_2} + \tfrac{q_1}{2(1+q_1)}\right) d^2(t) - k s_1^2 \\
& + \left[-q_1\cos(\xi_1) + \tfrac{1}{(1+\xi_1^2)\sqrt{1+\xi_1^2}}\right] s_1\xi_2 \\
\le & -\tfrac{q_2}{4}\xi_2^2 - \tfrac{q_1}{16(1+q_1)}\tfrac{\xi_1^2}{1+\xi_1^2} + (q_1+1)|s_1\xi_2| \\
& + \left[\tfrac{1}{q_2} + \tfrac{q_1}{2(1+q_1)} - k\right] s_1^2 + c_* d^2,
\end{aligned}
\tag{45}
$$

where $c_* = \frac{1}{q_2} + \frac{1}{2}$. Since $(q_1+1)|s_1\xi_2| \le \frac{k}{4}s_1^2 + \frac{1}{k}(q_1+1)^2\xi_2^2$, it follows that if

$$
k \ge \max\left\{\tfrac{8(q_1+1)^2}{q_2}, 2\left(\tfrac{1}{q_2} + \tfrac{q_1}{2(1+q_1)}\right)\right\}, \tag{46}
$$

then

$$
\begin{aligned}
\dot{U}(t) \le & -\tfrac{q_2}{4}\xi_2^2 - \tfrac{q_1}{16(1+q_1)}\tfrac{\xi_1^2}{1+\xi_1^2} + \tfrac{1}{k}(q_1+1)^2\xi_2^2 \\
& + \left[\tfrac{1}{q_2} + \tfrac{q_1}{2(1+q_1)} - \tfrac{3}{4}k\right] s_1^2 + c_* d^2(t) \\
\le & -\left\{\tfrac{q_2}{8}\xi_2^2 + \tfrac{q_1}{16(1+q_1)}\tfrac{\xi_1^2}{1+\xi_1^2} + \tfrac{k}{4}s_1^2\right\} + c_* d^2(t)\ .
\end{aligned}
\tag{47}
$$

It follows from integrating (47) and applying Barbalat's Lemma to the function in curly braces in (47) that Assumption 1 is satisfied. Hence, Theorem 1 applies to (40) and provides the dynamic feedback $u(Z,x) = -\mathrm{sat}_{\overline{Z}}(-z_1 - k\psi(x_1)) + q_3\arctan(x_2)$ with

$$
\dot{z}_1 = k\left[-z_1 + q_1\sin(x_1) - \tfrac{x_1}{\sqrt{1+x_1^2}}\right]
$$

with $\overline{Z} = (1+\epsilon)(1+k)(q_1+1)$ and the choice (39) of ψ, since the proof of Theorem 1 gives $c_1 = -1$ and $c_2 = -k$.

5. CONCLUSIONS

We developed a new backstepping approach using a finite dimensional dynamic extension. Our work is motivated by the ubiquity of engineering applications that produce the required cascade forms. Our variant of backstepping offers possible advantages in terms of the design of output feedback control and boundedness, e.g., since it does not require artificial delays. We hope to develop local versions for systems that are only locally asymptotically stabilizable with prescribed input bounds, and for cases where the $a_{i,j}$'s in (1) depend on t, and to allow delays in the input. We will also investigate what family of systems can be stabilized by applying Theorem 1 repeatedly, and the possibility of chattering in the closed loop control.

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