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On Two-Dimensional Continua Structured by Finite Families of Simple Closed Curves.

Arnold Raleigh Vobach

Louisiana State University and Agricultural & Mechanical College

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in

The Department of Mathematics

by
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ABSTRACT

The object of this dissertation is to generalize the concept of two-manifold to include certain spaces which triangulate like a compact two-manifold without boundary. Certain compact, locally connected, metric continua which partition into elements whose boundaries are simple closed curves which fit together like the boundaries of the two-simplices of a triangulation of a two-manifold are considered using results obtained by Anderson and Keisler.

If there is a sequence of such partitions, with mesh tending to zero, of such a space, M, and if successive collections of bounding simple closed curves can be mapped "nicely" onto preceding collections, then, for M homogeneous, easy characterizations, obtained by Anderson and Keisler, exist. These "nice" partitions and maps correspond, roughly, to successive subdivisions or refinements of a triangulation of a two-manifold. It is shown (Theorem 3.1) that a space in which a decreasing mesh sequence of partitions exists, but for which the maps of successive boundary collections are not given, i.e., a space for which the given partitions lack the
sequential or "subdividing" nature suggested above is still a space for which a sequential structure exists if the following condition is satisfied: If \( \{P_n\}_{n=1}^{\infty} \) is the sequence of partitions and \( C \in P_{n+1} \) is a simple closed curve of the \((n+1)\)st, then \( C \cap \bigcup_{i=1}^{n} P^*_i \) is a finite number of components.

On the basis of Theorem 3.1, by its homogeneity, and the Anderson-Keisler characterizations, the Universal Curve is excluded from the class of such spaces.
CHAPTER I
INTRODUCTION

The object of this dissertation is to generalize the concept of two-manifold to include certain spaces which triangulate like a compact two-manifold without boundary. Certain compact, locally connected, metric continua which partition into elements whose boundaries are simple closed curves which fit together like the boundaries of the two-simplices of a triangulation of a two-manifold are considered using results obtained by Anderson and Keisler.

If there is a sequence of such partitions, with mesh tending to zero, of such a space, $M$, and if successive collections of bounding simple closed curves can be mapped "nicely" onto preceding collections, then, for $M$ homogeneous, easy characterizations, obtained by Anderson and Keisler, exist. These "nice" partitions and maps correspond, roughly, to successive subdivisions or refinements of a triangulation of a two-manifold. It is shown (Theorem 3.1) that a space in which a decreasing mesh sequence of partitions exists, but for which the maps of successive boundary collections are not given, i.e., a space for which the given partitions lack the sequential or "subdividing"
nature suggested above is still a space for which a sequential structure exists if the following condition is satisfied: If \( \{P_n\}_{n=1}^\infty \) is the sequence of partitions and \( C \in P_{n+1}^* \) is a simple closed curve of the \((n+1)\)st, then \( C \cap \bigcup_{i=1}^nP_i^* \) is a finite number of components.

On the basis of Theorem 3.1, by its homogeneity, and the Anderson-Keisler characterizations, the Universal Curve is excluded from the class of such spaces.
CHAPTER II
PRELIMINARY DEVELOPMENTS

We begin by reviewing the definitions and results of Anderson and Keisler\textsuperscript{1}:

**Definition:** A triple of sequences \((\{F_i\}, \{\psi_i\}, \{\alpha_i\})\) is an inverse incidence system provided that for each \(i\):

1. \(F_i\) is a finite set,
2. \(\psi_i\) is a map of \(F_{i+1}\) onto \(F_i\),
3. \(\alpha_i\) is a reflexive and symmetric binary (incidence) relation on \(F_i\), and
4. If \((a, b), (b, c) \in \alpha_{i+1}\), then \((\psi_i(a), \psi_i(b)) \in \alpha_i\) (semi-transitivicy).

The pair \((\{F_i\}, \{\psi_i\})\) is an inverse incidence system whose inverse limit, \(L\), is a zero-dimensional compact metric space. If the sets \(\psi_i^{-1}(f)\), \(f \in F_i\), are non-degenerate, \(L\) is a Cantor set.

\textsuperscript{1}See Bibliography.
Let $R$ be a binary relation on $L$ defined by $(\{f_i\}, \{f_i^0\} \in R$ provided that, for each $i$, $(f_i, f_i^0) \in \alpha_i$. Using condition (4) of the definition above, it follows that the set of equivalence classes of $L$ defined by $R$ is an upper semicontinuous decomposition $\sim$ of $L$.

**Definition:** The collection $\sim L$ (topologized) is called the inverse incidence limit (I.I.L.) of $(\{F_i\}, \{I_i\}, \{\alpha_i\})$.

Whenever an I.I.L. is introduced, it will occur with respect to a particular inverse incidence system. In what follows, the inverse incidence system will be understood to be associated with the I.I.L.

**Definition:** A finite collection of simple closed curves, $G$, is called a $\kappa$-collection if:

1.) The intersection of any two is an arc or is null,
2.) The intersection of any three is a point or is null,
3.) $G^*$ (the union of the elements of $G$) is connected, and
4.) Except for a finite point set, each point of $G^*$ is in exactly two elements of $G$.

A subcollection, $G'$, of a $\kappa$-collection $G$ is called a $\lambda$-collection if $G \setminus G'$ is a non-null collection of disjoint elements of $G$. In such case, the union of the elements of $G \setminus G'$ is denoted by $B(G')$ and is called the boundary of $G'$. 
If $G'$ is a $\lambda$-collection, then $G'^* = G^*$, and hence $G'^*$ is connected. If $G'$ is a $\lambda$-collection then there is a canonical $\kappa$-collection containing it.

A non-degenerate $\lambda$-collection whose boundary is a single simple closed curve is called a $\mu$-collection.

Let $G'$ be a $\mu$-collection. If, for any two arcs $t_1$ and $t_2$ such that (1) $t_1 \cup t_2 = B(G')$ and (2) $t_1 \cap t_2$ is a set of two points each in two elements of $G'$, there exist two disjoint $\mu$-collections, $X_1$ and $X_2$, such that $X_1 \cup X_2 = G'$, $X_1 \supset t_1$, and $X_2 \supset t_2$, then $G'$ is said to be a $\psi$-collection.

Consider two propositions which may or may not hold in the canonical $\kappa$-extension, $G$, of a given $\mu$-collection:

I. There exists an arc $a \subset G^*$ such that (a) $a$ is not in any element of $G$, (b) for some $g \in G$, $a \cap g$ is the set of endpoints of $a$, and (c) $a$ does not separate $G^*$.

II. The elements of $G$ may be assigned orientations so that each arc which is the intersection of two elements of $G$ inherits opposite orientations from these two simple closed curves.

Definition: A $\mu$-collection whose $\kappa$-extension satisfies I and II is called a $T$-collection (toroidal collection).
A $\mathcal{K}$-collection whose $\mathcal{K}$-extension satisfies I but does not satisfy II is called a P-collection (projective collection).

Let $(\{F_i\}, \{\mathcal{U}_i\}, \{\alpha_i\})$ be an inverse incidence system where, for each $i$:

1. $F_i$ is a $\mathcal{K}$-collection,
2. For any $f \in F_i$, $\mathcal{U}_i^{-1}(f)$ is a $\nu$-collection whose boundary is canonically identified with $f$,
3. If $a, b \in F_i$, then $(a, b) \in \alpha_i$ if and only if $a \cap b \neq \emptyset$, and
4. If, for $f, f' \in F_i$, $f^* \cap f'^*$ is an arc $A$, then in each of $\mathcal{U}_i^{-1}(f)$ and $\mathcal{U}_i^{-1}(f')$ there are $g$ and $g'$, respectively, such that $g^* \cap [B(\mathcal{U}_i^{-1}(f))]* \subset A^0$, $g'^* \cap [B(\mathcal{U}_i^{-1}(f'))]* \subset A^0$ and each of these intersections is an arc.

Definition: A $\mathcal{K}$-inverse incidence limit is defined to be the I.I.L. of $\mathcal{K}$-collections of simple closed curves as described in conditions 1.) - 4.) above.

The inverse incidence limit of a sequence satisfying conditions 1.) - 3.) above can be identified as follows:
Definition: If for each $i$ and $f \in F_i$, $\varphi^1_i(f)$ is a $T$-collection, then the I.I.L. is called an orientable or non-orientable T-sphere according as $F_i$ satisfies or does not satisfy proposition II.

If, for each $i$ and $f \in F_i$, $\varphi^1_i(f)$ is a $P$-collection, then the I.I.L. is called a $P$-sphere.

The basic theorems are:

Theorem I: Every two $P$-spheres are homeomorphic to each other. A $P$-sphere is homogeneous and two-dimensional.

Theorem II: Every two orientable T-spheres are homeomorphic to each other. An orientable T-sphere is homogeneous and two-dimensional.

Theorem III: Every two non-orientable T-spheres are homeomorphic to each other. A non-orientable T-sphere is homogeneous and two-dimensional.

These results together with known results for two-manifolds almost classify continua with the properties that they are homogeneous and have bases for which every element has a simple closed curve boundary which separates (and separates locally) into two connected pieces. The one-dimensional Universal Curve also has these properties. The assumption of the sequential
structure is therefore needed.

Since our aim is to generalize, in a sense, the concept of two-manifold to cover objects which triangulate like two-manifolds, it is desirable to generalize the definitions of \( K', \lambda', \mu', \nu' \)-collections to allow our simple closed curves to fit together like the one-skeletons of the elements of the triangulation of a two-manifold:

Definition: A finite collection of simple closed curves, \( G \), is called a \( K' \)-collection if:

1'.) The intersection of any two is an arc or a point or is null,

2'.) The intersection of any three or more is a point or is null. If the point \( p \) is in exactly \( n(n \geq 3) \) of the elements of \( G \), then there is an ordering \( c_1, \ldots, c_n \) of these elements such that for \( i = 1, \ldots, n \):

\[
C_i \cap C_j = \begin{cases} 
\text{arc if } j = i \pmod{n} \\
\{p\} \text{ if } j \neq i \pmod{n}, 
\end{cases}
\]

3'.) \( G^* \) is connected, and

4'.) Except for a finite point set, each point of \( G^* \) is in exactly two elements of \( G \).

The definitions for \( \lambda', \mu' \) and \( \nu' \)-collections are analogous to those for the unprimed case.
The purpose of this generalization is to permit us to deal with collections in which the elements fit together like the one-skeletons of the two-simplexes of a triangulation of a two-manifold. Requirement (2'), the crucial one, says that at a "vertex" the simple closed curves of the collection intersect cyclically, as in a triangulation.

In the remainder of this chapter we shall show that a $\mathcal{K}$-I.I.L. of $\mathcal{K}'$-collections is still a $\mathcal{K}$-I.I.L. of $\mathcal{K}'$-collections. It is understood, of course, that a $\mathcal{K}$-I.I.L. of $\mathcal{K}'$-collections is defined just as a $\mathcal{K}$-I.I.L. of $\mathcal{K}$-collections except that primed collections replace unprimed collections throughout. We justify this seemingly pointless generalization by observing that, in the sequel, $\mathcal{K}'$-collections are vastly more convenient to work with.

As noted earlier, in the system $(\{F_i\}, \{\varphi_i\}, \{\alpha_i\})$ where the $F_i$'s are $\mathcal{H}$-collections of simple closed curves now, the pair $(\{F_i\}, \{\varphi_i\})$ is an inverse system with inverse limit $L$, a Cantor set. Using condition (4) of the I.I.L. definition and the definition of the binary relation $R$ given earlier, the set of equivalence classes is an upper semi-continuous decomposition, $\tilde{L}$, of the Cantor set.
Let us consider \( F \subseteq L \), where
\[
F = \{ x \in L | (x)_1 = f \in F_1 : \exists f^* \cap [B(\varphi_{1-1}^{-1}(f))]^* \neq \emptyset \}, f \in F_1.
\]
The map \( \varphi_{1-1}^{-1} \) is the composition map \( \varphi_{1-1} \cdots \varphi_2 \varphi_1 \). \( F \) is a subset of a Cantor set and we wish to show its decomposition, \( \widetilde{\sim} \) in \( \widetilde{L} \), is a simple closed curve which separates and separates locally into two connected pieces in \( \widetilde{L} \) and which intersects the simple closed curves corresponding to the other elements of \( F_1 \) in the same way that \( f \) does.

If, in Figure 1, we represent the large simple closed curve by \( f \), then we may order around it those simple closed curves of \( B(\varphi_2^{-1})^{-1}(f) \) in each of those of \( B(\varphi_1^{-1}(f)) \), etc. The fact that \( \varphi_1^{-1}(f) \) is a \( \mathcal{V} \)-collection assures us we may do this. In other words, we may coordinatize with an infinite sequence, chosen from a finite number for each term, the points of \( F \).

This coordinatization may be used to lay the points of \( F \) out on a Cantor set on an interval. The endpoints of deleted intervals correspond to points which are sequences of simple closed curves adjacent (in the ordering) at each stage and which therefore intersect and are identified in \( \widetilde{F} \).
We may, since these are $\mathcal{H}'$ instead of $\mathcal{H}$-collections, have to identify whole Cantor sets of points to get a copy of $F$; however these identifications will be of everything between endpoints of deleted intervals and cause no difficulty. Furthermore, the endpoints of the original interval itself get identified as part of a decomposition point, as suggested in Figure 1, from the fact that the last element in the ordering at each stage is adjacent to the first.

Figure 1
What we have done is to decompose the Cantor set on the interval into a simple closed curve.

Now suppose \( f \) and \( f' \in F \) and that \( f \cap f' \). We show that in \( \sim \) the associated simple closed curves, \( \sim \) and \( \sim' \), intersect. Consider a point of \( f \cap f' \). It is common to each simple closed curve of each of two sequences, \( \{ f_i \} \) and \( \{ f'_i \} \), where \( f_i \in (\varphi_1)_{i-1}^{-1}(f) \) and \( f'_i \cap [B((\varphi_1)_{i-1}^{-1}(f))]^* \neq \emptyset \) and \( f'_i \in (\varphi_1)_{i-1}^{-1}(f') \) and \( f_i^* \cap [B((\varphi_1)_{i-1}^{-1}(f'))]^* \neq \emptyset \). Since \( f_i \cap f'_i \neq \emptyset \), these two points, \( \{ f_i \} \) and \( \{ f'_i \} \) of \( \sim \) and \( \sim' \), respectively, are identified as a single point of \( \sim \). It is also true that if \( f \cap f' \) is an arc or point, then \( F \cap F' \) is also an arc or point, respectively. This follows from condition (4) of the \( \kappa \)-I.I.L. definition and semi-transitivity which say that intersection arcs stay "big" and do not get pinched to points in \( L \). To put it another way:

For \( f_i \in F_i \), \( (\varphi_1^{-1}(f_i)) \) is a \( \mathcal{Y} \)-collection whose boundary is identified with \( f_i \), and if \( f_{i+1} \in F_{i+1} \), then \( f_{i+1}^* \cap [B(\varphi_1^{-1}(f_i))]^* \) is homeomorphic to \( \sim f_{i+1} \cap \sim f_i \), where these are the corresponding simple closed curves of \( \sim \).

What we have said so far is that the \( \kappa \)-collection
F_i has an associated \( k' \)-collection, \( F_i \) in \( L \). Moreover, the same is true for each \( F_i \), and each simple closed curve of \( F_i \) in \( L \) bounds a \( \gamma \)-subcollection of \( F_{i+i} \).

We now show that each \( F \) of each \( F_i \) separates \( L \) and separates it locally into two connected pieces. We call this biseparation and local biseparation.

We define \( \text{Int} F \subset L \), for \( f \in F_i \), by first defining a set in the unde decomposed \( L \). \( I(f) = \{ x \in L | (x_j) = f_j, f_j \in (\psi_i^j)^{-1}(f_i) \} \) and, for some \( j \), \( f_j \in I(F(f)) \). Then \( \text{Int} F \) is the resulting set of equivalence classes of these points in \( L \). We define \( \text{Ext} F \) as \( L \setminus (F \cup \text{Int} F) \). It is clear that \( F \) separates \( \text{Int} F \) and \( \text{Ext} F \) from the way in which each set is defined. Furthermore, each of \( \text{Int} F \) and \( \text{Ext} F \) is connected, as may be inferred from the fact that \( F_i \) and later stages provide connecting "webbing" in each.

Before showing local biseparation, we show that mesh \( \frac{1}{F_i} \to 0 \). We may, in our constructions of \( F \) (unde decomposed) and of \( I(F) \), \( f \in F_i \), have required that the diameter of the set of elements in \( L \) (and under its metric) which have \( f \) as \( i \)-th coordinate have diameter < \( 1/i \). This includes \( F \) and \( I(F) \), the set which maps into \( \text{Int} F \). Next, for any \( \epsilon > 0 \),
we consider a finite $\varepsilon$-covering, in its metric, of $\sim L$. This induces, since $\sim L$ is a continuous image of $L$, a finite open cover of $L$. However, by the Lebesgue Covering Lemma, if $i$ is taken large enough, then every set of diameter $1/i$ maps into a set of diameter $< \varepsilon$ in $\sim L$. Hence for $i$ large enough, mesh $F_i < \varepsilon$. In other words, the sequence of $F_i$'s is a sequence of partitions of $\sim L$, of mesh tending to zero, in which each $\sim F \in F_i$ bounds a $\mathcal{V}'$-subcollection of $\sim F_{i+1}$.

We are able now to show local biseparation by $\sim F \in F_i$. Each point $p$ of $\sim F$ is interior to an arc of each of two simple closed curves, formed possibly by the union of two or more, in $\sim F_j$, $j > i$, such that each separates, their union is of suitably small diameter and their union is bounded by a simple closed curve which contains $p$ in a spanning separating arc of $\sim F$. See Figure 2.

Figure 2
We have now classified a $\kappa$-inverse incidence limit of $\kappa'$-collections as a metric space with a sequence of $\kappa'$-partitionings, of mesh tending to zero, such that each partitioning in the sequence $\nu'$-refines the preceding. It is clear that such a space may be written as an ordinary inverse limit space $\lim\left(\left\{ F_1^*\right\}, \left\{ \Theta_i^* \right\} \right)$ where:

1.) Each $F_1$ is a $\kappa'$-collection of simple closed curves,
2.) $\Theta_i : F_{i+1}^* \to F_i^*$ is a continuous onto map,
3.) For $f \in F_1$, $\Theta_i^{-1}(f^*) = [\Theta_i^{-1}(f)]^*$ and $[B(\Theta_i^{-1}(f))]*$ is carried homeomorphically onto $f^*$ (The $\left\{ F_1^* \right\}$ and $\left\{ \Theta_i^* \right\}$ sequences are of course still those of the $\kappa$-inverse incidence limit.), and,
4.) For each $\epsilon > 0$ and integer $n$, there exists an integer $m(n, \epsilon) > n$ such that $f^* \cap \Theta_n^* \leftarrow \Theta_m(n, \epsilon) \leftarrow m(n, \epsilon)+1$ is an $\epsilon$-net in $f^*$, $f \in F_n$. The finite subset $D$ is an $\epsilon$-net in $f^*$ if each point of $f^*$ is within $\epsilon$ of some point of $D$.

The set $D_{m(n, \epsilon)+1}$ is defined below.

The converse of all of the above may also be easily

---

2. The $\kappa'$-collection $\left\{ C_1, \ldots, C_m \right\}$ of simple closed curves in $M_\kappa'$-partitions $M$ if each $C_i$ biseparates and locally biseparates $M$.

3. The $\kappa'$-partitioning $P'$ $\nu'$-refines the $\kappa'$-partitioning $P$ if each simple closed curve of $P$ bounds a $\nu'$-subcollection of $P'$. 

established, i.e., any space which is an inverse limit in the above sense is also a $\mathcal{K}$-inverse incidence limit of $\mathcal{K}'$-collections. The equivalence of the two is now complete.

**Definition:** Let $F$ be a $\mathcal{K}'$-collection of simple closed curves. Let the finite set of points common to three or more of the simple closed curves be called the distinguished points of the collection, $D$ for short. Given a sequence, $\{F_n\}$, of $\mathcal{K}'$-collections, denote the distinguished points of $F_n$ by $D_n$.

**Definition:** Given an inverse incidence system $L'$, $(\{F_n\}, \{\varphi_n\}, \{a_n\})$, of $\mathcal{K}'$-collections, the interior of $f \in F_n$ is the set of all points of $L'$ which have as $n$-th coordinate $f$ and which have as $m$-th coordinate for some $m > n$, a simple closed curve $f \in (\varphi^m_n)^{-1}(f)$ which does not intersect $[B(\varphi^m_n)^{-1}(f)]^*$. Such simple closed curves in $F_m$ will be said to be in $\text{Int}_m(f)$, or the interior of $f$ at the $m$-th level.

**Lemma 2.1:** Let $L$ be a $\mathcal{K}$-inverse incidence limit, $(\{F_n\}, \{\varphi_n\}, \{a_n\})$, of $\mathcal{K}'$-collections. Then, if $C_1 \cap C_2 = A$, an arc, for $C_1, C_2 \in F_n$, for some integer $m(n) > n$, there exists $q \in A^0 \cap D_{m(n)}$ and $q \in f \in F_{m(n)}$ such that
\( f' \in (\mathcal{V}_m^{m(n)})^{-1}(C_1) \), or the subset of the inverse incidence limit corresponding to the intersection of the simple closed curves determined by \( C_1 \) and \( C_2 \) is a point. A similar statement holds for \( C_2 \).

Proof: If the first alternative of the conclusion were not true, there would be an infinite sequence of simple closed curves, one in each \( (\mathcal{V}_m^{m})^{-1}(C_1) \), \( m > n \), which contained \( A^0 \), but this says the subset of the \( \kappa \)-inverse incidence limit, as a particular decomposition of the Cantor set, corresponding to \( A \) is a point.

Theorem 2.1: Let \( L' \) be a \( \kappa \)-inverse incidence limit of \( \kappa' \)-collections, as in Lemma 2.1, then there is a \( \kappa \)-inverse incidence limit, \( L \), of \( \kappa \)-collections of simple closed curves such that \( L \) is homeomorphic to \( L' \).

Proof: Our proof will be in terms of inverse limit spaces rather than in terms of \( \kappa \)-inverse incidence limits. As mentioned earlier, for this we may assume that \( L' \) is homeomorphic to the inverse limit space generated by \( (F^*_n, \{ \Theta_n \}) \) where \( \Theta_n \) is any continuous map of \( F^*_{n+1} \) onto \( F^*_n \) which takes \( B(\mathcal{V}^*_n(f)) \) homeomorphically onto its copy, \( f \in F^*_n \), etc.
Given $L = \lim \left( \{ F_n^* \}, \{ \Theta_n \} \right)$, modify $F_n^*$ as follows:

Consider $D_n$. Replace each local cut point of order $> 3$ in $D_n$ by an "enlarged point," a simple closed curve. That is:

Let $G_n$ be a $\mathcal{K}$-collection obtained from $F_n$ by deleting in $F_n^*$ each of the local cut points of order $> 3$ and locally reconnecting with simple closed curves intersecting each of the simple closed curves containing the original point in an arc. Moreover, the cyclic ordering of the arcs, the point set union of which is to be the new simple closed curve is to be the same as the original cyclic ordering. The identification of the slightly modified simple closed curves with the old is easy to see, and the new simple closed curves replacing points are the only additions in the change from $\mathcal{F}'$ to $\mathcal{K}$-collection.

Now, given a $\mathcal{K}$-collection $G_n$ obtained from $F_n$, consider the map of $F^*_{m(n)}$, $m(n) > n$, onto $G_n$ determined (with the intermediate step of $H_n$) in the following way: Let $p$ be one of the replaced distinguished points of $F_n$ (to get $G_n$). Consider the arcs of the simple closed curves of $F_n$ containing $p$ which have as end points $p$ and other points of $D_n$ and with no interior points from $D_n$. By Lemma 2.1, there are four interior points,
two to each "side," of each of these arcs, which are points of \( D_{m(n)} \) for some sufficiently large \( m(n) > n \). Henceforth, we shall suppose \( m(n) \) to be sufficiently large to satisfy this condition for all the above such points \( p \) simultaneously. See Figure 3a). Thus, on the union of the arcs "into" \( p \) and "back out" for each of the simple closed curves containing \( p \), there is a pair of points of \( D_{m(n)} \) "bracketing" \( p \). Connect these with spanning arcs of these simple closed curves intersecting nothing else in \( F^*_n \). We connect those points of \( D_{m(n)} \) "nearest" -- in the order of the arc--to \( p \). These arcs exist in \( F_{m(n)} \) as we shall see.

Figure 3

(a) (b)
Now let $H_n$ be the collection of simple closed curves defined to be all of those in $F_n$ without a spanning arc plus the two created from each of these with a newly added spanning arc. See Figure 3b).

In $F_{m(n)}$ there exist two disjoint $\nu'$-collections, $X_1$, $X_2$, such that one has one of these two new curves as boundary and the other has the other curve as boundary. This follows from the definition of $\gamma'$-collection and the fact that, given a set of $\gamma'$-collections such that their boundaries form a $\gamma'$-collection, then the set of $\gamma'$-collections, plus the parts of their boundaries common to two or more, form a $\gamma'$-collection.

We define continuous onto maps, as indicated by Figure 4, carrying arcs shown in boldface onto arcs in boldface where this is not impossible. Graphically,
Figure 4
Note that $\Theta_{m(n)} : F^*_n \rightarrow F^*_n$. Also, so far, the inverse of the replacement simple closed curve in $G^*_n$ is not yet necessarily a $\gamma'$-collection. However, for a simple closed curve at some stage which bounds a $\mu'$-collection (not necessarily a $\gamma'$-collection) at a later stage, we may go out far enough in the sequence that each of the elements of the $\mu'$-collection bounds a $\gamma'$-collection. In this stage we may proceed from one distinguished point to another of the original $\mu'$-collection on the original bounding simple closed curve with an arc in the refining stage which stays "near enough" to one of the arcs of the original simple closed curve to break up the refinement collection into two $\mu'$-collections. We shall assume $m(n)$ is taken large enough to solve the problem for each of the finite number of simple closed curves involved simultaneously.

As the diagram indicates, canonical boundary identifications are possible for the maps $h_n \circ \Theta_{m(n)}^f \circ m(n)$, and thus $L$, the inverse limit for which a sample map and pair of terms is

$$G^*_m \rightarrow h_n \circ \Theta_{m(n)}^f \circ m(n) \rightarrow G^*_n$$

is a $\kappa$-inverse incidence limit of $\kappa$-collections. Since
they both correspond to subsequences of the same inverse
limit space, L is also homeomorphic to the $\kappa$-inverse incidence
limit of $\kappa'$-collections given by maps and terms of the form:

$$F^*_m(n) + 1 \quad \frac{f \cdot h \cdot g \cdot \Theta \cdot m(n)}{n \cdot n \cdot n \cdot n \cdot n \cdot m(n)} \quad F^*_n.$$ 

Note that the $f$, $h$, $g$ and $\Theta$-maps used may be taken
to be of the sort required in the earlier conversion from
inverse incidence limits to inverse limit spaces.

Hereafter, we may, in the light of this theorem, drop
the primes from our Greek-letter collections. It will be
understood that, in the strict sense, they should be primed.
In this chapter, we shall show (Theorem 3.1) that if a compact, locally connected metric continuum \( M \) has a sequence of \( \kappa \)-partitions with mesh tending to zero, then, even though each does not \( \psi \)-refine the preceding and the connecting maps required of a \( \kappa \)-inverse incidence limit are lacking, \( M \) is still a \( \kappa \)-inverse incidence limit if the partitions satisfy a finiteness condition with respect to their intersections. Theorem 3.1 is not the most desirable theorem here; however, to generalize it by removing this restriction appears to present grave technical difficulties.

Before proceeding, we recall that the simple closed curve \( S \) biseparates \( M \) if \( M \setminus S \) is the sum of two components. \( S \) locally biseparates \( M \) if for \( p \in S \) and \( \varepsilon > 0 \), there is an open set \( U \), containing \( p \), and contained in the \( \varepsilon \)-sphere about \( p \), such that \( S \) separates \( U \) into two components with an arc of \( S \) as common boundary. Although we used these terms in Chapter II, we have repeated their definitions here in view of their extensive use in the following theorems.

**Theorem 3.1:** Let \( M \) be a compact, locally connected metric
continuum with the following property: There exists a sequence of \( \kappa \)-partitionings, \( \{F_n\}_{n=1}^{\infty} \), of \( M \) such that:

1.) \( \text{Mesh } F_n \xrightarrow{n \to 0} 0 \), and

2.) \( C \in F_{n+1} \) implies \( C \cap \bigcup_{i=1}^{n} F_i^* \) has only a finite number of components. Then \( M \) is a \( \kappa \)-inverse incidence limit.

Before the proof of Theorem 3.1, we need a number of definitions and lemmas. Throughout the remainder of this chapter, \( M \) and the sequence \( \{F_i\} \) are to be as in the statement of Theorem 3.1.

**Definition:** The sequence \( \{F_i^*\}_{i=1}^{\infty} \) has Property I if for each \( i \), there is a \( \delta_i(1) > 0 \) such that if mesh \( F_j \leq \delta_i(1) \), then no element of \( F_j \) contains in the closure of its interior the closure of the interior of an element of \( F_k \), \( 1 \leq k \leq i \).

**Lemma 3.1:** The sequence \( \{F_i^*\}_{i=1}^{\infty} \) has Property I.

**Proof:** For each \( C \in F_k \), \( 1 \leq k \leq i \), pick a point in \( \text{Int } C \setminus \bigcup_{i=1}^{1} F_j^* \). About each point in this finite set we may put a sphere of small enough diameter that the closure of its interior is in \( \text{Int } C \setminus \bigcup_{i=1}^{1} F_j^* \). Let \( \delta_i(1) > 0 \) be less than one half the minimum of the sphere diameters. For \( F_n \) of mesh \( < \delta_i(1) \),
each element of $F_k$, $1 \leq k \leq i$, contains in its interior an element of $F_n$ plus its interior. Hence no element of $F_n$ can contain in the closure of its interior the closure of the interior of an element of $F_k$, $1 \leq k \leq i$. For it to do so would imply an element of $F_n$ contained in the closure of its interior the closure of the interior of another distinct, element of $F_n$.

**Note:** On the basis of Lemma 3.1, we may suppose, without loss of generality, that the original sequence, $\left\{F_i\right\}_{i=1}^{\infty}$, has the property that the closure of the interior of no element of $F_j$ contains the closure of the interior of an element of $F_i$, $i < j$.

**Lemma 3.2:** Each component of $\text{Int } C_1 \cap \ldots \cap \text{Int } C_n$, $C_i \in F_i$, is bounded by the union of a finite number of simple closed curves.

**Proof:** Suppose $n = 2$, then $p \in \text{Bdry } [\text{Int } C_1 \cap \text{Int } C_2]$ is a point of $C_1 \cup C_2$: Since $M$ is locally connected, there is a sequence of connected sets, each containing $p$, of diameter tending to zero such that each of these sets contains a point of $\text{Int } C_1 \cap \text{Int } C_2$ and a point not in $\text{Int } C_1 \cap \text{Int } C_2$, i.e., a point not in one of $\text{Int } C_1$ or $\text{Int } C_2$. Hence, each of these
sets must contain a point of one of the boundaries and thus of $C_1 \cup C_2$. Finally, then $p$ is a limit point of the closed set $C_1 \cup C_2$ and is in it.

Second, each point of $C_1 \cap \text{Int } C_2$ and of $C_2 \cap \text{Int } C_1$ is a boundary point of $\text{Int } C_1 \cap \text{Int } C_2$. Likewise, each point of $C_1 \cap C_2$ is a point of the boundary of $\text{Int } C_1 \cap \text{Int } C_2$ unless it is interior to an arc of $C_2$ in $M \setminus \text{Int } C_1$ or to an arc of $C_1$ in $M \setminus \text{Int } C_2$. All the above says, so far, is that the boundary of $\text{Int } C_1 \cap \text{Int } C_2$ is the union of a finite number of arcs from $C_2 \cap \text{Int } C_1$ and from $C_1 \cap \text{Int } C_2$ plus a finite number of arcs from $C_1 \cap C_2$.

We show now how these arcs may be expressed as the union of a finite number of simple closed curves. Consider in a three-face of a Hilbert Cube, a simple closed curve which we shall identify as $C_1$. We complete the configuration by adding arcs in the Hilbert Cube which are copies of each of the open arcs of $C_2 \cap \text{Int } C_1$. At the points corresponding to those at which $C_2$ crosses $C_1$ from $\text{Int } C_1$ to $\text{Ext } C_1$, we tie the ends to $C_1$. Where the endpoints of an arc of $C_2 \cap \text{Int } C_1$ are endpoints of arcs, possibly degenerate, shared by $C_1$ and $C_2$, we terminate them on $C_1$ and identify the arcs of $C_1$. 
(the copy) corresponding to these arcs of $C_1 \cap C_2$ in M.

Now, given an orientation on $C_1$ and starting from a point of $C_1$ in Int $C_2$ (We do not really have a problem if there are no such points.), we proceed to an intersection with an arc of the copied arc of $C_2$ if such an arc (and intersection) exists. Suppose such an arc does not exist. Then either Ext $C_1$ in M is contained in Int $C_2$ or the closure of the interior of $C_2$ is contained in the closure of the interior of $C_1$. We exclude the first case by requiring each of the original $F_1$'s to contain more than two elements ($F_1$ is more than a simple closed curve.). Then, if the first possibility held, the closure of the interior of an element of $F_1$ would be in $C_2 \cup$ Int $C_2$, contrary to the note following Lemma 3.1.

In the second case, Int $C_1 \cap$ Int $C_2$ is bounded by the simple closed curve $C_2$.

If intersections of $C_1$ with the closures of the copied arcs of $C_2$ do exist, then we proceed in the given orientation, along $C_1$ to such an intersection. This point of intersection may be a point at which $C_2$ crosses $C_1$ in M or an endpoint of a common arc of $C_1$ and $C_2$. See Figure 5.
If, as in Figure 5a), this is a crossing, turn off on the arc of \( C_2 \) leading into the interior of \( C_1 \). If the common arc, possibly degenerate, of \( C_1 \) and \( C_2 \) is bounded at both ends by arcs leading into \( \text{Int } C_1 \), turn off onto the first of these in the given orientation, as in Figure 5b). If the arc of \( C_2 \cap \text{Int } C_1 \) at which we have arrived leads into \( \text{Int } C_1 \), turn onto it from \( C_1 \), as in Figure 5c). To stay on \( C_1 \) past the endpoint of the common arc and into its interior would be to cover or traverse points with small neighborhoods not containing points of both \( \text{Int } C_1 \cap \text{Int } C_2 \) and \( M \setminus \text{Int } C_1 \cap \text{Int } C_2 \).
i.e., points which are not boundary points of $\text{Int } C_1 \cap \text{Int } C_2$. If the arc of $C_1 \cap C_2$ at which we have arrived is bounded at this end by an arc of $C_2$ coming from outside $C_1$, we stay on the common arc of $C_1 \cap C_2$ and turn off into $\text{Int } C_1$ along the arc of $C_2$ at the other end. See Figure 5d). At the other end of the copy of an arc of $C_2 \cap \text{Int } C_1$, we turn onto the arc of $C_1$ which is interior to $C_2$ or common to $C_1$ and $C_2$, etc. The set of points traversed in this way is one-dimensional, has no cut points and no local cut points of order greater than two. It is a simple closed curve. The same procedure for other arcs of $C_1 \cap \text{Int } C_2$, not already traversed, gives other simple closed curves. We cover, after finitely many circuits, all the boundary points (in our copy) of $\text{Int } C_1 \cap \text{Int } C_2$ in this way. The boundary of $\text{Int } C_1 \cap \text{Int } C_2$ is thus a finite collection of non-over-lapping (in the sense that no arc of one is shared with another) simple closed curves.

The proof of the assertion for boundaries of components of $\text{Int } C_1 \cap \ldots \cap \text{Int } C_n$, $n > 2$, is an easy generalization of the argument above for the boundary of the intersection of the interiors of two biseparating and locally biseparating simple closed curves which intersect in only finitely many components. Here we intersect $\text{Int } C_2$ with the components of
Int \( C_1 \cap \text{Int } C_2 \), each of which is bounded by the union of a finite number of simple closed curves - instead of just one- and so on.

**Definition:** Let, for \( i > 1 \), \( B(C_1, \ldots, C_i) \) denote the boundary, the union of a finite collection of simple closed curves, of \( \text{Int } C_1 \cap \ldots \cap \text{Int } C_n, C_j \in F_j, j = 1, \ldots, i \). It will also be convenient to denote by \( P(C_1, \ldots, C_i) \) the finite set of points (Lemma 3.2) common to two or more of the simple closed curves of \( B(C_1, \ldots, C_n) \).

**Lemma 3.3:** The sequence \( \left\{ F_i \right\}_{i=1}^{\infty} \) is such that, for each \( i \), each \( B(C_1, \ldots, C_i) \) and each finite subset \( Q(C_1, \ldots, C_i) \) of points of \( B(C_1, \ldots, C_i) \), there is a \( \mathcal{S}_2(i; C_1, \ldots, C_i; Q(C_1, \ldots, C_i)) > 0 \) such that if \( \text{mesh } F_j < \mathcal{S}_2(i; C_1, \ldots, C_i; Q(C_1, \ldots, C_i)) \), then there is, for each pair of maximal open arcs \( A_1^0 \) and \( A_2^0 \) (and each pair of simple closed curves or maximal open arc and simple closed curve) in \( B(C_1, \ldots, C_i) \backslash Q(C_1, \ldots, C_i) \) connected by a component of \( \text{Int } C_1 \cap \ldots \cap \text{Int } C_i \), an arc of \( F_j^* \) with endpoints in \( A_1^0 \) and \( A_2^0 \) (or in the pair of simple closed curves or in the arc and in the simple closed curve) and otherwise missing \( B(C_1, \ldots, C_i) \).

(This says that for small enough mesh \( \mathcal{A} \)-collections,
the boundary simple closed curves of a component of \( \text{Int } C_1 \cap \ldots \cap \text{Int } C_i \) are connected in the \( F_j \)-structure of \( M \).)  

**Proof:** Consider a component, \( K \), of some \( \text{Int } C_1 \cap \ldots \cap \text{Int } C_i \neq \emptyset \), fixed \( i \), bounded by the union, \( B_K \), of some subcollection of the simple closed curves determining \( B(C_1, \ldots, C_i) \). Consider a pair of maximal open arcs, \( A_1^o \) and \( A_2^o \), in \( B_K \setminus Q(C_1, \ldots, C_i) \), fixed collection \( Q(C_1, \ldots, C_i) \). Let \( C_1^o \) be an open arc in \( K \setminus B_K \) with endpoints in \( A_1^o \) and \( A_2^o \). Then there is a \( \delta(C_1^o, A_1^o, A_2^o) > 0 \) such that, for an \( F_j \) of mesh \( \delta(C_1^o, A_1^o, A_2^o) \), there is a (not necessarily) simple chain, \( D \), of closures of interiors of elements of \( F_j \) which contains \( C_1^o \) and such that there is an arc of \( \text{Bdry } D(C \cap F_j) \) from \( A_1^o \) to \( A_2^o \) in \( K \setminus B_K \). \( \delta(C_1^o, A_1^o, A_2^o) \) can be chosen as the \( \delta \) of a sufficiently small \( \delta \)-neighborhood of \( C_1^o \) in \( M \).

Since there are only finite numbers of pairings of open arcs like \( A_1^o \) and \( A_2^o \) in \( K \), and of components in \( \text{Int } C_1 \cap \ldots \cap \text{Int } C_i \), there is a \( \delta_2(1; C_1, \ldots, C_i; Q(C_1, \ldots, C_i)) \) small enough to serve all simultaneously. The argument for a pair of simple closed curves or for a pair consisting of a maximal open arc and a simple closed curve is obviously similar.
Note: Since there is only a finite number of non-empty intersections of interiors of elements of \( \{F_n\}_{n=1}^i \), and since the union of the sets \( P(C_1, ..., C_i) \) is a finite point set, there is a \( S_2(i) \) sufficiently small to insure that for \( F_j \) of mesh less than \( S_2(i) \), there is for each component \( K \) of each \( \cap \text{Int } C_1 \cap \ldots \cap \text{Int } C_1 \), \( C_k \in F_k, 1 \leq k \leq i \), an arc of \( F^*_j \) between each pair of the maximal open arcs (or simple closed curves or maximal open arc and simple closed curve) of \( \text{Bdry } K / P(C_1, ..., C_i) \) if \( \text{Bdry } K \cap P(C_1, ..., C_i) \neq \emptyset \). Hence, we may, without loss of generality, suppose that \( F_{i+1} \) is always of fine enough mesh to connect the boundary components of each component of each \( \cap \text{Int } C_1 \cap \ldots \cap \text{Int } C_1 \) in the above manner.

We now construct a manifold associated with each collection, \( \{F_{n_1}, ..., F_{n_i}\} \): Consider a copy of \( \cup_{k=1}^i F^*_k \). For each component \( K \) of \( \cap \text{Int } C_1 \cap \ldots \cap \text{Int } C_i \neq \emptyset \), \( C_k \in F_{n_k}, 1 \leq k \leq i \), in \( M \), consider a two-sphere with tubes leading off and "sewn in" along each of the boundary simple closed curves (Lemma 3.2) of \( K \) (in the copy of \( \cup_{k=1}^i F^*_k \)), the "end" of one tube for each of the simple closed curves. For a component bounded by a single simple closed curve, the
corresponding manifold is just a disk. In fact, in general we shall refer to the component-of-intersection "manifold" corresponding, in the copy, to \( k \) in \( M \) even though identifications of finite numbers of points of the bounding simple closed curves make this inaccurate. It is clear that these component-of-intersection manifolds - allowed to intersect only on \( \bigcup_{k=1}^{1} \frac{F^*}{n_k} \) - fill in all the simple closed curves identified with boundaries of intersections, \( \text{Int } C_1 \subset \ldots \subset \text{Int } C_1 \neq \emptyset \), in \( M \) and that, even along the arcs of the copy \( \bigcup_{k=1}^{1} \frac{F^*}{n_k} \), we get a space which is locally \( \mathbb{E}^2 \) since each side of an arc is used as a boundary for a "sewing" just once.

It must be mentioned here that the tubes leading to the bounding simple closed curves of a component-of-intersection manifold cannot be sewn on in a purely arbitrary fashion. We might, for example, fill in with a component-of-intersection manifold to yield a non-orientable manifold when \( M \) was an orientable manifold to start with.

To make sure the two-spheres with tubes filling in the boundary simple closed curves of a component of intersection do so "properly", we must examine an additional \( F_m \) -structure, \( m > n_1 \). Let \( K \) be a component of intersection of interiors
of elements of $F_{n_1}, \ldots, F_{n_1}$, in $M$. Let $K$ have as boundary
the collection of simple closed curves $B$, with point set
union $B^*$. By Lemma 3.3 we can choose an $F_m$, $m > n_1$, of
small enough mesh that the simple closed curves of $B$ are all
connected by arcs in $F_m^*$. Consider the manifold determined
by a copy of the union of $\bigcup \{ F_{n_1}, \ldots, F_{n_1}, F_m^* \}$ in which
the copies of boundaries of components of intersections of
interiors of elements of $F_m$ with interiors of elements of the
other collections are filled in with two-spheres and tubes
leading off to boundary simple closed curves in an arbitrary
sewing. Now the copy of $B$ bounds a "manifold" which is a
two-sphere with tubes leading to the simple closed curves of
$B$ and possibly added crosscaps and handles introduced by the
$F_m$-structure. If each of these extra features is inclosed
in a biseparating simple closed curve such that the simple
closed curves so obtained are pairwise disjoint, and if the
closure of the interior of each such simple closed curve is
identified to a point, the resulting "manifold" is a two-sphere
with tubes leading off to boundary simple closed curves. If
the simple closed curves are all disjoint, we would, of course,
have precisely a manifold. Making these identifications for
each of the components of intersection of interiors of elements of the collections $F_{n_1}, \ldots, F_{n_1}$ gives a manifold determined by $\{F_{n_1}, \ldots, F_{n_1}\}$ which is "consistent" with later structurings.

Two observations remain to be made regarding this process: First, there may be more than one way to decompose a component-of-intersection manifold in the manifold determined by $\{F_{n_1}, \ldots, F_{n_1}, F_m\}$ to get a two-sphere with tubes leading to the boundary simple closed curves. However, the "sewing" to the boundary curves are at least determined as they must be for finer future structures - orientability or non-orientability preserved, for example. Second, since we shrunk out the handle-producing ones in the decomposition manifold, it is clear that it does not matter how the two-spheres with tubes filling in interiors of elements of the copy of $F_m$ were sewn in.

If it is desired - and it will be - to construct such a manifold in a particular Hilbert Cube, one might start with a copy of $\bigcup_{k=1}^i F^*_{n_k}$ in a three-face and then, for each component-of-intersection manifold added, retreat into a higher dimensional face to avoid unwanted intersections.
It is also natural at this point to require that each component-of-intersection manifold, in each such imbedding, have diameter no more than some fixed $\delta > 1$ times the diameter of its boundary.

Note that there is a natural comparison between $M'$, determined by \( \{ F_{n_1}, \ldots, F_{n_j} \} \), and $M''$, determined by \( \{ F_{n_1}, \ldots, F_{n_j}, F_{n_{i+1}}, \ldots, F_{n_j} \} \), $j \geq 1$. Each is partitioned by a copy of $F_{n_k}$, $1 \leq k \leq j$. Also, corresponding to each component $K_i$ in $M$, of each $\text{Int } C_i \supset \ldots \supset \text{Int } C_i$, $C_k \in F_{n_k}$, $1 \leq k \leq j$, is a component-of-intersection manifold $K_j$ in $M'$ and one, $K_j$, in $M''$. By the note following Lemma 3.3, $F_{n_{i+1}}$ (and hence $F_{n_{i+1}}$) represents a "fine enough" structuring of $M$ that all the boundary simple closed curves of $K$ are connected in the $F_{n_{i+1}}$ "framework" of $M$ and hence $K_j$ is the two-sphere with tubes of $K_{i}$ with, possibly, additional cross-caps and handles. This says that the existence of a $\mathcal{K}$-partitioning collection, $P'$, of simple closed curves in $M'$ implies the existence of a $\mathcal{K}$-partitioning homeomorphic copy, $P''$, in $M''$ of $P'$ in $M'$. Further, $P''$ may be chosen so that the closure of the intersection of $P''$ with
the interior of an element of \( F_{n_1} \) in \( M \) is homeomorphic to
the closure of the intersection of \( F'* \) with the interior of
the "same" (corresponding) element of \( F_{n_1} \) in \( M \).

Strictly speaking, each time such a manifold is men­tioned, the determining collection of \( F_k \)'s should be indicated.
In what follows, it will be convenient to construct an
infinite sequence of such manifolds without indicating each
time, which of the \( F_k \)'s determine it. The context will, how­ever, make it clear which are involved.

Consider the manifold \( M_1 \) determined by the first \( N \) of
the \( F_i \)'s. Since it is a two-manifold, there is a partitioning,
\( P \), which \( V \)-refines each of the \( \kappa \)-partitionings of the
copies of the \( F_i \)'s. It will be convenient to locate the
distinguished points of \( P \) on \( \bigcup_{i=1}^{n} F_i^* \) for some sufficiently
large \( n \geq N \) in a manifold \( M_n \) structured by copies of each of
\( F_1, \ldots, F_N, \ldots, F_n \).

Let \( p \) be a distinguished point of \( P \) (if there is such
a point) which is not contained in \( \bigcup_{i=1}^{N} F_i^* \). A homeomorphic

[...]

...
to $p$ is in $\bigcap_{N+1} F_i \cap \bigcap C_1 \cap \ldots \cap \bigcap C_N$ where $C_i$ is the element of $F_i$ in whose interior $p$ lay in $M_i$. This may be accomplished by "sliding" the copy of $p$ in $P(l)$ in $M_2$ over to the $F_{N+1}$-structure. Likewise, for a second distinguished point $q$ of $P$ in $M_1$, we may require the copy of $g$ in $M_3$, determined by $\left\{ F_{N+2}^N \right\}_{i=1}$, to be contained in $F_{N+2}^*$. The net effect of all this is that from some $n = N + k$ on, we have a manifold $M_n$ which is $\kappa$-partitioned by a $\nu$-refinement of each of $F_1, \ldots, F_N$ and such that all the distinguished points of $P$ lie on $\bigcup_{i=1} F_i^*$. 

Now we need to say something about the "size" of crosscaps and handles in a manifold $M_k$, determined by $\left\{ F_i^k \right\}_{i=1}$, some $k$. Actually, all we really need to discuss is the "size" of handles and crosscaps in an $M$ determined by a single $F_i$ - as we shall see below.

In such an $M'$, consider a handle and denote by $U$ the union of the closures of the disk-interiors of a subcollection of the elements of $F_i$. $U$ may be thought of as bounded by the union, possibly empty, of a finite collection of simple closed curves, $\text{Bdry } U$. Let $U$ be such that it contains in
its interior a simple closed curve inscribed on the given handle which is not homotopic to a constant in \( M \). We say \( U \) contains the handle if each such simple closed curve is still not homotopic to a constant in the decomposition manifold obtained from \( U \) by identifying each of the boundary simple closed curves, if any, to a point. This says that, in some sense, \( U \) provides a "base" for the handle.

We define the diameter of the handle in \( M \) to be the minimum of the diameters, in the metric of \( M \), of the collections of elements of \( F_1 \) determining sets \( U \) which contain it. It is a measure, in terms of the structuring of \( M \) by \( F_1 \), of the size of the handle.

Similarly, for a given crosscap, define its diameter to be the minimum of the diameters of the collections of elements of \( F_1 \) determining, with their disk-interiors, the sets which contain it.

We shall wish in the sequel to be able to identify as the "same" handles and crosscaps in different manifolds determined by different collections \( \left\{ F_i \right\}_{i=1}^{k} \). To do this we make the following basic construction: Consider the manifolds \( M_{n_i} \), determined by \( \left\{ F_j \right\}_{j=1}^{n_i} \), imbedded successively in the
same Hilbert Cube. Let the imbeddings be such that each $M_{n_1}^n$ is contained in a finite-dimensional face of the Hilbert Cube and such that the imbedded manifolds intersect in exactly the copies of the $F_j$'s. This second condition may require retreating to higher-dimensional faces with each such successive imbedding to prevent component-of-intersection manifolds from intersecting except along boundaries. If the diameters of component-of-intersection manifolds are kept bounded by a common factor of $\$ $ times the diameters of their boundaries, as has been our practice, the limit set of the sequence is clearly $M$. We shall presume, henceforth, that such an imbedding has been made for a sequence $\left\{ M_{n_1}^n \right\}_{n_1=1}^{\infty}$ with $n_1 = 1$.

Now handles and crosscaps on some $M_{n_1}^n$, produced by the $F_j$-structures, can be identified with corresponding handles and crosscaps on an $M_{n_1}^{n_k}$, $n_k > n_1$, in terms of the common points of $\bigcup_{j=1}^{n_k} F_j^*$. Of course, the corresponding handles or crosscaps in the more finely structured $M_{n_k}^{n_k}$ may themselves be studded with handles and crosscaps.

We may choose an $n > k$ such that the $M$ determined by $\left\{ F_j \right\}_{j=1}^{n}$ reproduces at least those crosscaps and handles of $M'$ determined by $\left\{ F_j \right\}_{j=1}^{k}$, which are obtained by sewing
together the manifolds bounded by elements of $F_k$. Each such crosscap or handle may, as noted above, have additional crosscaps and handles sewn on it by the component-of-intersection structurings of the other $F_j$'s. If this were not possible, then there would be handles and crosscaps in $M'$-arising from the sewing together of the elements of $F_k$ - of arbitrarily small diameter (in the intuitive sense, not necessarily in the sense of our definition of handle and crosscap-diameter), else their structures would eventually "bulge out" of the interiors of elements of the $F_j$'s, $j > k$, in $M$. In the event of infinite intersections of elements from $F_i$ and $F_j$, i.e., without the finiteness condition of the theorem, crosscaps and handles in some $M$ might be arbitrarily small and the arguments here would not generalize - a measure of the magnitude of the general problem. Any "additional" crosscaps and handles in $M'$ will be contained in interiors of elements of $F_k$ and will hence be of diameter (our definition) $\leq \text{mesh } F_k$.

Note: Without loss of generality, we shall henceforth assume that the basic sequence $\left\{F_i\right\}_{i=1}^{\infty}$ of the theorem has the property that the manifold $M'$ obtained by filling in with disks the interiors of elements of a copy of $F_{i+1}$ is, except,
for possible additional crosscaps and handles, a homeomorph of \( M_i \) the corresponding manifold obtained for \( F_1 \). This is what we have indicated as possible above, relative to the given sequence of imbeddings in the Hilbert Cube and with each of the crosscaps and handles in which we were interested possibly carrying further crosscaps and handles produced by the other \( F_j \)-structurings. In fact, since \( F_{i+1} \) is already of small enough mesh to connect boundary components of a component of intersection, \( \text{Int} C_1 \cap \ldots \cap \text{Int} C_i, C_j \in F_j \), and thus to reproduce, with possible additional features, the component-o-intersection manifolds chosen for \( M_i \), determined by \( \{ F_j \}_{j=1}^{i+1} \), we may more generally require that \( M_{i+1} \), obtained from \( \{ F_j \}_{j=1}^{i+1} \), is, except for possible additional crosscaps and handles, a homeomorph of \( M_i \).

Let us now return to our earlier discussion, in which we assumed a \( \mu \)-partitioning \( P \) of \( M_n \), determined by \( \{ F_i \}_{i=1}^{n} \), which \( \nu \)-refined each of the first \( N \) of the \( F_i \)'s and such that the distinguished points of \( P \) all lie in \( \bigcup_{i=1}^{n} F_i^* \). Before the next lemma, we need to say what we mean by the statement that the arc \( A \), connecting distinguished points of \( P \), separates in the manifold \( M_k \) to within \( \varepsilon \). Suppose we have a not-necessarily-partitioning copy of \( P \) in \( M_k \), determined by \( \{ F_i \}_{i=1}^{k} \), \( k \geq n \),
such that all of $A$, except for $\epsilon$-small sets containing its endpoints, is contained in $\bigcup_{i=N+1}^{k} F_i^*$. This will be the situation in the sequel. Suppose, further, that the copy of $P$ would be a $\kappa$-partitioning of $M_k$, were it not for the possible existence of crosscaps and handles in the interiors of elements of $F_N$ whose diameter-determining collections of elements, in the collections $F_j, j > N$, are of diameter $\leq \epsilon$ and intersect $A \cap \bigcup_{i=N+1}^{k} F_i^*$ in their interiors. What this amounts to is that $P$ would be in $M_k$ a $\kappa$-partitioning except, possibly, for $\epsilon$-small crosscap and handle "leaks" in the neighborhood of $A$. With this definition and $M_n$ and $P$ as above, we have:

**Lemma 3.4**: Let $A$ be an arc of $P$ intersecting the distinguished points of $P$ only in its endpoints, $p$ and $q$. Let $A$ be contained in the closure of the interior of the element $C$ of $F_n$. For each $\epsilon > 0$ there is a $k > n$ such that in the interior of $C$ in $M_k$, determined by $\bigcup_{i=1}^{k} F_i$, there is an arc, $A(\epsilon)$, between $p$ and $q$ which separates to within $\epsilon$ ($A$ copy of $P$ may be inscribed in $M_k$ which agrees with $P$ in $M_n$ on $P^* \cap \bigcup_{i=1}^{n} F_i^*$.) and which is contained except for two mutually separated sets of diameters $< \epsilon$, each containing one of $p$ and $q$, in $F_k^*$. 
Proof: Let $k$ be large enough that $\delta/2^k < \varepsilon$ (the bound on the size of the component-of-intersection manifolds) and that $p$ and $q$ in $M_k$ are in the closures of the interiors of disjoint elements of $F_k$ in $M$. Further, let $k$ be large enough that $p$ and $q$ do not lie in the closures of the interiors of any elements of $F_k$ which are in minimal diameter collections containing handles or crosscaps of diameter $\geq \varepsilon$. Our imbeddings of the manifold in the Hilbert Cube, agreeing on the $F^*_k$'s, again permits us to identify those handles and crosscaps for which diameter-defining collections, when they appear, will always be as big as $\varepsilon$ in each manifold. Still further, let $k$ be large enough that there exists for each crosscap and handle of diameter $\geq \varepsilon$ a minimal diameter collection containing it, with closure missing $C$. Consider $M_k$ minus the interiors of those elements of $F_k$ which are elements of a minimal diameter collection determining the $F_k$-diameter of each crosscap or handle of diameter $\geq \varepsilon$, one such collection for each crosscap or handle. (This may require us to reduce the mesh of $F_k$ still further.) Subtract also those arcs of $F_k^*$ which in this new space have had the interiors of elements on "both sides" removed - but only if they were removed as interiors of the same crosscap or handle collection. We are, in effect,
leaving the "outer" boundary of each such collection intact.

We claim that what is left of $M_k$ is connected: Each of the sets - let us denote them by $U_1, \ldots, U_m$, some $m$ - which has been removed is the union of the interiors of elements of $F_k$ plus common boundary arcs. Hence, each of these sets, and each component of $U_j \cap \bigcup_{i=j}^{m} U_i$, $j = 1, \ldots, m$, is an open set bounded by the union of a finite number of simple closed curves in $F^*_k$. Removing these non-overlapping open sets (the components) does not disconnect the manifold $M_k$. It is now possible to pass an arc in the remainder of the $F^*_k$-structure interior to $C$ from near (< $\epsilon$) $p$ to near $q$. Any crosscaps or handles preventing separation by such an arc will necessarily be of diameter < $\epsilon$ (less than $S$ times mesh $F_k$, in the usual sense, for component-of-intersection manifolds in Int $C$). To complete our arc $A(\epsilon)$, and to get it to terminate at $p$ and $q$, it will, in general, be necessary to leave the $F^*_k$-structure - but only within $\epsilon$ of each of $p$ and $q$ - and finish the arc in what is left of $M_k$. The remaining arcs of a copy of $P$ may now be inscribed with the result that the arc $A(\epsilon)$ of $P$ separates to within $\epsilon$ in $M_k$.

**Lemma 3.5:** Let $A$ be an arc of $P$ connecting distinguished endpoints $p$ and $q \in \bigcup_{i=1}^{P} F^*_1$ in the closure of the interior of
C in $F_n$ in $M_n$. Let $A$ contain no other distinguished points of $P$. Then there is a sequence of arcs, $\{A(i)\}_{i=1}^{\infty}$, each $A(i)$ of which separates to within $1/2^i$ in $M_{n_i}$, determined by

$$\bigcup_{j=1}^i F_j, \quad n_i \geq n_{i-1} \geq n;$$

each is contained, except for two disjoint sets of diameters $< 1/2^i$, each containing one of $p$ and $q$ in $F^*$, and each is such that the limiting set of the sequence is an arc, $A$, between $p$ and $q$.

**Proof:** By Lemma 3.4, each $A(i)$ exists; the problem here is to show that they may be chosen so that the limit set is also an arc.

Since $A(i)$ separates to within $1/2^i$ in $M_{n_i}$ and $A(i+1)$ to within $1/2^{i+1}$ in $M_{n_{i+1}}$, we may choose $n_{i+1}$ sufficiently larger than $n_i$, and mesh $F_{n_i}$ sufficiently smaller than mesh $F_{n_{i+1}}$, that $A(i+1) \cap F^*$ need be "perturbed" by no more than $\delta/2^i$, the bound on the diameters of crosscaps and handles it must skirt but $A(i) \cap F^*$ need not, from the position of $F(i) \cap F^*$ in the Hilbert Cube in which all are imbedded.

Further, even the corresponding subsets (at each end) of $A(i+1) \setminus F^*$ and of $A(i) \setminus F^*$ in $M_{n_i}$ need not be more than $\delta/2^{i+1}$ apart. In short, $A(i+1)$ and $A(i)$ need be "crooked"
with respect to each other on sets of diameters no greater than \( \delta/2^i \).

This allows us to assert the existence of a homeomorphism \( h_1 : A(1) \to A(i+1) \) such that \( h_1(p) = p \) and \( h_1(q) = q \) and the distance in the Hilbert Cube (or in \( M \)) between \( x \in A(i) \cap F^* \) and \( h_1(x) \) is \( < \delta/2^i \). We are requiring, as we may, here that \( h_1(A(i) \cap F^*) \subset A(i+1) \cap F^* \). We wish now to show that the family, \( \{A(i)\}_{i=1}^{\infty} \), of arcs is equicontinuous\(^1\) and hence that the limiting set, \( A \), is also an arc.

In the Hilbert Cube, the limit set, \( A \), of the \( A(i)'s \) is a continuum containing \( p \) and \( q \). We may require of the \( h_1 \)'s that, in fact, for all \( x \in A(i) \), the distance in the Hilbert Cube between \( x \) and \( h(x) \) is less than \( \delta/2^i \). Now, given \( \epsilon > 0 \), first let \( i \) be large enough that \( 2 \delta \sum_{j=1}^{\infty} 1/2^j < \epsilon/2 \) and then let \( 0 < \delta' < \epsilon/2 \) be small enough that if \( \widehat{xy} \) is an interval of \( A(i) \) of diameter less than \( \delta' \); each of the intervals \( h_{i-1}^{-1} (\widehat{xy}) \) and \( h_k^{-1} h_{k+1}^{-1} \ldots h_{i-1}^{-1} (\widehat{xy}) \), \( i \leq k \leq i-2 \), is of diameter less than \( \epsilon \). This last takes care of the first \( i \) homeomorphs of \( \widehat{xy} \) and the diameters of the rest are less than \( \epsilon \).

\(^1\)A collection \( G \) of arcs is equicontinuous if, for every \( \epsilon > 0 \), there is a \( \delta > 0 \) such that if \( x \) and \( y \) are any two points of an arc \( g \in G \) at a distance apart less than \( \delta \), then the diameter of the interval \( \widehat{xy} \) of \( g \) is less than \( \epsilon \).
Thus, if we choose our $\delta'$ to be the $\delta$ of the definition of equicontinuity, we have shown \( \left\{ A(i) \right\}_{i=1}^{\infty} \) to be equicontinuous and then, as is well-known, $A$ is an arc.

**Lemma 3.6:** There exists in $M$ a copy of $P$ in which each of the arcs between distinguished points (and containing none in its interior) is the limit of a sequence of arcs of the $A(i)$-type described in Lemma 3.5.

**Proof:** We have already seen in Lemma 3.5 how to obtain in $M$ one such of the arcs connecting distinguished points of $P$. Let us suppose that, given an ordering of the finite number of such arcs composing $P$, we have constructed, in $M$, the limit arcs, \( \left\{ A_j \right\}_{j=1}^{k} \), of the first $k$ of them and wish to construct the $(k+1)$st.

Even though they may share endpoints, there is an open set $O_i$, $i=1, \ldots, k$, containing the interior of each $A_i$, $i=1, \ldots, k$, in $M$ such that the $O_i$'s are pairwise disjoint, and such, in fact, that their closures intersect only at the distinguished endpoints of the arcs they contain. We may require the $O_i$'s to contain only points sufficiently near their respective $A_i$'s that $M - \bigcup_{j=1}^{k} O_j$ is connected. In addition, we require the
$O_1$'s to be small enough in $M$ about their respective arcs that for some $n \geq N$, there is enough of $F^*_n \setminus \bigcup_{j=1}^{k} O_j$ left to provide in $M_n$, determined by $\{F_i\}_{i=1}^{n}$, homeomorphs of each of the crosscaps and handles in each of the elements of $F_N$ in $M_N$. In other words, the $O_1$'s are to be "unobtrusive" enough to permit, for some $n$, the $F_n$-structure remaining outside to carry a copy of the original $P$-structure except, possibly, near ($\leq$ mesh $F_n$) its distinguished points.

Now in a sequence of arcs, $\{A_{k+1}(i)\}_{i=1}^{\infty}$, $1/2^i$-separating approximations for $A_{k+1}$ in the $M_n$'s, from some $i_0$ on, corresponding to some sufficiently richly "veined" $F_{n_1}$-structure, the $A_{k+1}(i)$'s can be chosen, except for $1/2^i$ - small sets containing their endpoints, in $\bigcup_{j=1}^{n_1} F^*_j \setminus \bigcup_{j=1}^{k} O_j$ in $M_{n_1}$. Hence, the limiting set, $A_{k+1}$, will intersect $\bigcup_{j=1}^{k} A_j$, if at all, only in its endpoints - from which the conclusion of the lemma follows.

**Lemma 3.7:** Each simple closed curve $C$ of the $x$-collection $P$, constructed in Lemma 3.6, separates (and biseparates) in $M$.

**Proof:** Before we can claim $C$ separates in $M$, we must indicate the subset claimed to be the interior of $C$. We proceed to a definition of the "interior" of $C$: Our construction, one at a time, of the limit arcs $A_j$, $j = 1, \ldots, L$, which determine
P in M was undertaken in Lemma 3.6 so that approximations to the \((k+1)\)st arc were avoided, for large enough subscripts, those parts of \(F^*_i\)-structures contained in certain open sets (of \(M\)), \(\bigcup_{j=1}^{k} \{O_j \} \), containing the interiors of the first \(k\) limit arcs. This says that for some sufficiently large \(i_0\), all approximations \(A_j(i), i > i_0\), are disjoint except, possibly, in small open sets containing their distinguished endpoints. In each of these sets, in each of the \(M_{n_1}\)'s, we can alter, without affecting the limit arcs, the approximating arcs \(A_j(i)\)-which needn't be carried in \(\bigcup_{j=1}^{n} F^*_j\) here anyway-so that the \(A_j(i)'s\) intersect only at the distinguished points of\(P\). The result is a \(k\)-collection of simple closed curves with union homeomorphic to \(P^*\), each element of which separates to within \(1/2^1\) in \(M_{n_1}\) with the natural extension for simple closed curves of our definition of \(1/2^1\) -separation for arcs. Let us denote by \(C(i)\) the simple closed curve corresponding to \(C\) in the copy of \(P\) in \(M_{n_1}\), \(i > i_0\). Each \(C(i)\) has a naturally defined "interior", i.e., those points of \(M_{n_1}\) which would be interior to \(C(i)\) were it not for the possible existence of crosscap and handle "leaks" of diameter \(< 1/2^1\). To put it another way, each of the crosscaps and
handles of diameter $< 1/2^i$ in $M_{n_1}$ is contained in a set
made up of the interiors of elements of $F_{n_1}^*$ and of arcs of
$F_{n_1}^*$ adjoined on both sides by these interiors. If each of
these connected sets is decomposed to a point, then in the
decomposition space, what is left of $C(i)$, not necessarily
a simple closed curve anymore, separates. Those points of
$M_{n_1}$ which were in the interior of $C$ in the copy of $P$ in $M_{n_1}$
and which are separated from the rest of $M_{n_1}$, decomposed,
by $C(i)$, decomposed, we call the interior of $C(i)$.

We define the interior of $C$ in $M$ to be the limiting
set of the sequence of interiors of $C(i)$, $i = 1, \ldots$. It
is easy to see that $C$ separates the interior of $C$, so defined,
from the rest of $M$: Suppose $p$ is in the interior of $C$ and
$q$ is a point of $M$ in neither $C$ nor interior $C$. Then, if $C$
does not separate $p$ from $q$ in $M$, there is an arc $A$ in $M$, missing
$C$, with endpoints $p$ and $q$. $A$ is contained in a chain, not
necessarily simple, of closures of interiors of elements of
$F_i$, each $i$, in $M$. For large enough $i >$ some $i_0$, and small
enough mesh $F_i$, these chains will also miss $C$. Each such
chain of $F_k$-elements, $k > i_0$, then contains an arc $A_k$ in $F_k^*$
in $M$ (and in $M_{n_j}$, $n_j > k$) from the element of $F_k$ whose closure
contains $p$ to the element whose closure contains $q$. Consider
what must happen for a fixed $k > i$ with $A_k$ missing C.

Since $M$, some $n_j > k$, contains $A_k$ and since $M_n$ contains only crosscaps and handles of diameter $> 1/2^i$, for some $i$, the $1/2^i$-separating simple closed curves, $C(i)$, must intersect $A_k$ from some $i$ on. This implies the limiting set $C$ intersects $A_k$ as well, a contradiction.

It is clear from the increasingly rich "veining" or "webbing" of the interiors of the $C(i)$'s, as $i$ increases, that interior $C$ is connected, implying that $C$ not only separates but biseparates as well. Another way of seeing this is to observe that the interior of $C$, where "interior" now has the usual meaning, is the limiting set of the sequence of connected sets $\{\text{interior } C(i)\}_{i=1}^{\infty}$ and is thus connected. We shall prove biseparation in still another way in the sequel.

The next, and perhaps most natural, step would seem to be to show each element of $P$ locally biseparates. This, however, will be a very simple consequence of showing that $M$ is the inverse limit, in the ordinary sense, of a sequence of $\kappa$-collections each of which $\gamma$-refines the preceding (except, possibly, for local biseparation by each of the elements). Local biseparation will follow from this in the same manner as it did in Chapter II.
Lemma 3.8: There exists a \( \mathcal{K} \)-collection \( P \) in \( M \) which \( \mathcal{V} \)-refines (except, possibly, for the local biseparation required in the definition of \( \mathcal{V} \)-refining) each of \( F_{N+1} \) and the \( \mathcal{K} \)-collection \( P \) of the preceding lemmas.

Proof: Let \( P_0 \) be a \( \mathcal{K} \)-partitioning of \( M_n \), determined by some \( m > N + 1 \), which is homeomorphic to \( P \) in \( M \) and has all its distinguished points located at the corresponding points of \( P \) in \( M \). (They are also in \( M_n \)).

Now \( P \), in \( M \), and \( F_{N+1} \) may not, as collections of simple closed curves in \( M \), have the finite-number-of-components-of-intersection property of Theorem 3.1 and the sequence \( \{ F_i \}_{i=1}^\infty \). We may, however, choose \( P_0 \) in \( M_n \) to be such that each simple closed curve of \( P_0 \) intersects each of the curves of \( F_{N+1} \) in only a finite number of components. Let \( P_0' \) be a \( \mathcal{V} \)-refinement of each of \( P_0 \) and \( F_{N+1} \). Let \( n \geq m \) be chosen large enough so that the distinguished points of \( P_0' \) which are not in \( P_0^* \) are in \( \bigcup_{i=1}^n F_i^* \). This is a convenience we have justified before.

The strategy for the remainder of the proof will be to alter \( P_0' \) slightly so that the remaining distinguished points of \( P_0' \) (in \( P_0^* \)) are points of \( P \) in \( M \). Then, if each of the original arcs between distinguished points of \( P \) in \( M \) can be
rederived as unions of arcs between original distinguished points and the newly added distinguished points of the altered $P'_o$, we shall be able to fit in the remaining necessary limiting set arcs for $P'_o$, a homeomorph of $P'_o$ - just as we constructed $P$ in Lemma 3.6.

In $M$, each arc of $P'_o$, with distinguished points of $P'_o$ at each end and none in the interior, will, as an arc of $P'_o^*$, contain no more than some $k$ distinguished points of $P'_o$ in its interior. Our problem is to find $k$ "accessible" points on each such arc, $A$, of $P^*$ in $M$ (distinguished endpoints and no interior distinguished points) which are available for use as distinguished points of $P'_o$. Suppose, for example, that a "tail" of the closure of the interior, in $M$, of an element of $F_{N+1}^*$ spirals down around a point $p$, possibly distinguished, of $P^*$ as "vortex". Clearly, $p$, now also a point of $F_{N+1}^*$, need not be used as a distinguished point of the $\mathcal{V}$-subcollections of $P'_o$ $\mathcal{V}$-refining the elements of $F_{N+1}^*$ in $M$ which contain $p$. It must also not be used as a distinguished point of the $\mathcal{V}$-subcollections refining the elements of $P$ in which $p$ lies, and we must show that other points of $P^*$ are available for such use.

Each arc $A$, as above, of $P^*$ is either contained in $F_{N+1}^*$.
or contains a segment, \( A^0 \), disjoint from \( F_{n+1}^* \). Choose \( n' > n \) large enough that, in \( M \), each such arc \( A \) of \( F^* \), or arc \( A^0 \) if it exists, contains at least \( k \) different points of \( F_n^* \).

Now, back in \( M_{n'} \), determined by \( \{ F_i \}_{i=1}^{n'} \), we consider a copy of \( P_q \) with distinguished points of \( F_q \) in \( \bigcup_{i=1}^{n} F_i^* \) located as before. We modify \( P_o \) by "sliding" the arcs of \( P_q \) containing the remaining distinguished points of \( P_q \) (in \( P_o \)) over to the \( F_n^* \) structure at a finite number of points to make the remaining distinguished points coincide with points of \( P^* \cap F_n^* \) in \( M \). (Our choice of \( n' \) guaranteed the existence of enough such available points.) The arcs of \( P_o \) which terminated at these "transported" points may be made to "trail along", preserving \( P_o \) as a partition. Another way of describing the process above is to say that some or all of the arcs of \( P_o \) not contained in \( \bigcup_{i=1}^{n} F_i^* \) (The arcs of \( \bigcup_{i=1}^{n} F_i^* \cap F^* \) are fixed.) be required to contain finite numbers of points of \( F_n^* \cap F_{n+1}^* \) in \( M_{n'} \). These points are then to be used as the remaining (not already fixed in \( \bigcup_{i=1}^{n} F_i^* \)) distinguished points of \( P_o \).

One more comment needs to be made regarding our latest version of \( P_o \). Some of the points of \( P_o \), which may also have been points of \( P \), may have had to be abandoned as distinguished points of the refinement, \( P_o' \) of both \( F_{n+1}^* \) and \( P_o \) - for
example the "vortex" point \( p \) above if it were in \( P_0^* \). Such points, trailing their attendant arcs, get carried into "safe" open arcs, like the arc \( A^0 \) above, and we may find we have "stretched" or "squeezed" the original simple closed curves of \( P_0^* \) (in \( M_m \)) in our new manifold \( M_n \) to produce a not-necessarily-\( \kappa \) partition not-necessarily \( \mathcal{V} \)-refining each of \( P_0 \) and \( F_{N+1} \).

The trouble is that simple closed curves of \( P_0^* \) may now be pinched together or they may intersect one another in more than two components—contrary to the requirements for a \( \kappa \)-partition. It is possible, however, to re-establish from \( P_0^* \) a \( \kappa \)-partitioning \( \mathcal{V} \)-refinement of each of \( F_{N+1} \) and the copy (in \( M_n \)) of \( P_0^* \), determined by \( P_0^* \), by subdividing interiors of elements of \( P_0^* \) with spanning separating arcs finitely often. We shall call, for reasons of notational simplicity, this new collection of simple closed curves \( P_0^* \) again. It is important to note also that this readjustment requires the addition of no more distinguished points in \( P_0^* \). Thus, we shall presume that \( n \) and \( n' \) (possibly rechosen) are large enough that all the distinguished points of \( P_0^* \) not in \( P_0^* \) are in \( \bigcup_{1=1}^{N} F_1^* \) and that all the remaining distinguished points (in \( P_0^* \)) are in \( F_n^* \setminus F_{N+1}^* \).

Although this amounts to choosing two integers greater than or equal to each of the original \( n \) and \( n' \), we shall for subsequent
simplicity keep the same notation for the newly selected integers.

We are now in a position to apply Lemma 3.6, with $N+1$ replacing $N$, and $P'_0$ (or $P'$ in $M$) replacing $P$ in the statement. In fact, much of the construction of $P'$ in $M$ ($P'^* \cap P^*$) is already completed: If the arcs of $P'_0$ with distinguished points as endpoints and containing no distinguished points in their interiors are enumerated, $\bigcup_{i=1}^{K} A_i$, then since the distinguished points are all in $\bigcup_{i=1}^{n'} F_i^*$, we may proceed as in the proof of Lemma 3.6 - with the following convention. Whenever the arc $A_i$ of $P'_0$ is a subarc of an arc of $P'_0$, the limiting set arc $A_i$ of the sequence $\bigcup_{i=1}^{\infty} f_i$ has already been produced for us as a subarc of an arc of $P^*$ in $M$. The result of the construction is a homeomorphic copy, $P'$, of $P'_0$, each element of which biseparates in $M$, and which $y$-refines (except, possibly, for local biseparation) each of $P$ and $F_{N+1}$.

**Lemma 3.2**: $M$ can be represented as an inverse limit space, 

$$\lim_{\leftarrow} \left( \left\{ P_i \right\}_{i=1}^{\infty}, \left\{ f_i \right\}_{i=1}^{\infty} \right)$$

where each $P_i$ is a $\mathcal{K}$-collection of biseparating simple closed curves $y$-refining (except, possibly, for local biseparation) $P_{i-1}$ and $f_i : P_{i+1} \to P_i$ is the natural map taking the interior $y$-subcollections of $P_{i+1}$ into their boundary simple closed curves in $P_i$. 
Proof: We note that the mesh of \( P \) above is \( \leq \text{mesh } F_N \leq \frac{1}{2^N} \) and mesh \( P' \) above is \( \leq \text{mesh } F_{N+1} \leq \frac{1}{2^{N+1}} \). Lemma 3.8 is the inductive step in the construction of a sequence, \( \{P_i\}_{i=1}^{\infty} \), of \( \mathcal{K} \)-collections of biseparating (lemma 3.7) simple closed curves of mesh \( < \frac{1}{2} \) each of which \( \mathcal{U} \)-refines (except, possibly, for local biseparation) the preceding. Since the meshes of the \( P_i \)'s tend to zero, the interiors of the simple closed curves of the \( P_i \)'s and the interiors of simple closed curves bounded open sets with boundaries in the \( P_i \)'s (See Figure 2, Chapter II.) form a basis of open sets for \( M \), and the representation of \( M \) as an ordinary inverse limit of the \( P_i \)-sequence is immediate.

Finally:

Lemma 3.10: Each of the simple closed curves of each of the \( P_i \)'s above locally biseparates.

Proof: Let \( p \) be a point of a simple closed curve, \( C \), of \( P_i \), some \( i \). Then \( p \) is interior to an arc of each of two simple closed curves formed, possibly, by the union of two or more simple closed curves of some \( P_j \), \( j > i \), such that each simple closed curve separates \( M \) (Figure 2 again), their union is of suitably small diameter and their union is bounded by a single simple closed curve which contains \( p \) in a spanning separating arc of \( C \).
Now, finally, we can remove the nagging parenthetical restriction regarding \( \gamma \)-refinement by the \( P_1 \)'s above.

**Note:** Since \( M \) is connected and each \( C \in P_1 \) locally biseparates, it biseparates \( M \). This is another proof of an earlier observation. Since \( C \) locally biseparates, there is a connected open set in its interior, of which it is a boundary component, and also a similar connected open set in its exterior. These connected open "bands" on either "side", since \( M \) is connected, provide places for arcs to link pairs of points in the exterior and in the interior of \( C \)-biseparation.

**Proof of Theorem 3.1:** Since we now have in \( M \) a sequence of \( \kappa \)-collection \( \{ P_i \}^\infty_{i=1} \), each of which \( \kappa \)-partitions \( M \) (which requires biseparation and local biseparation) and each of which \( \gamma \)-refines the preceding, with mesh tending to zero with \( i \), we have shown \( M \) to be what we called a \( \kappa \)-inverse incidence limit - the conclusion of Theorem 3.1.
CHAPTER IV

CONCLUSIONS

Since the converse of Theorem 3.1 is obviously true for \( \kappa \)-inverse incidence limits, we have obtained a characterization of such spaces. Neither of these is, perhaps, surprising. It is, however, surprising that the Universal Curve should not have a "nice" (in the sense of Theorem 3.1) sequential, or \( \kappa \)-inverse incidence limit, structure.

The Universal Curve, as noted in Chapter II, has a neighborhood basis in which the boundary of each element is a simple closed curve which biseparates and biseparates locally. If, however, a given Universal Curve had a sequence \( \{P_n\}_{n=1}^\infty \) of \( \kappa \)-partitionings with mesh tending to zero, such that for \( C \in P_{n+1} \), \( C \cap \bigcup_{i=1}^{n} P_i^* \) was a finite number of components, and such that the elements of \( P_j \), \( j = 1, \ldots \), biseparated and biseparated locally, then it would be a \( \kappa \)-inverse incidence limit by Theorem 3.1. Hence, by its homogeneity and the Anderson-Keisler theorems of Chapter II, it would be a P or T-sphere and thus two-dimensional. In short, for a given Universal Curve, one or both of two things must happen: First, there is no decreasing mesh sequence of \( \kappa \)-partitions, nice
with respect to one another. Second, if there is such a sequence, there is a non-zero lower bound on the mesh of the partitions. The first possibility seems unlikely, but the natural generalization of Theorem 3.1, which would imply the second, is beyond the author.

While this is a negative sort of characteristic to ascribe to the Universal Curve, it does suggest how higher dimensional universal spaces ought not to be constructed. Further, since the techniques we have used depend on simple considerations of manifold theory, generalizations of our definitions and results to higher dimensional cases, with collections of bounding two-spheres, for example, are naturally suggested.
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