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***Abstract*— We study continuous-time nonlinear systems with discrete measurements. We provide an estimate of the state variable that converges with a rate of convergence that can be made arbitrarily large by reducing the size of the largest sampling interval. Our proof of the convergence result is based on a recently developed trajectory based approach.**

***Index Terms*— Estimation, nonlinear systems, delay**

I. Introduction

The design of estimators plays an important role in current research in control systems; see for instance [3], [4], [5], [6], and [15]. One important consideration for observer design is the speed at which the estimation error converges to 0. For instance, [14] provided estimation results for a family of continuous-time systems which includes those of the type

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + \mu(y(t), u(t)) \\ y(t) & = & Cx(t) \end{array} \right. \tag{1}$$

where A and C are constant matrices, x is the state, y is the output, μ is a nonlinear function, and u is an input which can be a control. In fact, [14] proposed a finite time observer. Its expression incorporates delays and no dynamic extension is used. The instant of convergence is arbitrarily small and can be chosen by the user, which is clearly an advantage over traditional asymptotically converging observers. The design in [14] is only applicable when the continuous output measurement $y(t)$ is available. This is a limitation of the result because, in many engineering applications, only discrete output measurements of the form

$$y(t) = Cx(t_i) \text{ for all } t \in [t_i, t_{i+1}) \text{ and } i \geq 0 \tag{2}$$

are available, where t_i is an increasing sequence of nonnegative values. Then the observer of [14] cannot be constructed.

In the present work, our objective is to overcome this obstacle by proposing an adaptation of the technique of [14] to the case where the output is discrete and where the lengths $t_{i+1} - t_i$ of the sampling intervals are not required to be constant. This extension borrows key ideas from the celebrated paper [8], which, for nonlinear systems with discrete measurements, proposed a redesign of asymptotic observers that are available for systems with continuous measurements. The main ingredient in [8] is a dynamic extension which in some sense predicts the behavior of the measured variables between two measurements. We show that the key idea of the observer design of [8] used in combination with the approach of [14] produces an estimate of the state variable which converges with a rate of convergence that can be made arbitrarily large by reducing the size of the largest sampling interval. We study systems of the form

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + \varphi(Cx(t)) + \kappa(u(t)) \\ y(t) & = & Cx(t_i) \text{ for all } t \in [t_i, t_{i+1}) \text{ and } i \geq 0 \end{array} \right. \tag{3}$$

with the state x valued in $\mathbb{R}^n$ and the output y valued in $\mathbb{R}$, where u is an input, $C \in \mathbb{R}^{1\times n}$ is a nonzero constant matrix, φ and κ are nonlinear functions, $A \in \mathbb{R}^{n\times n}$ is a constant matrix, and t_i is an increasing sequence with $t_0 = 0$ and $\lim_{i\to+\infty} t_i = +\infty$. We allow nonperiodic sampling. We build a continuous-discrete observer which unlike the one of [14], will not converge in finite time but will converge with a rate of convergence of the form $c_1 \ln(c_2 \sup_i\{t_{i+1} - t_i\})$ for suitable constants $c_1 < 0$ and $c_2 > 0$, which can be made arbitrarily large by sampling frequently enough. We prove our result through a recent trajectory based approach from [1], [12], and [13]. Since the observer of [14] relies on a formula with delays, one cannot directly apply [8] to solve the problem we study. The observer from [14] exhibited good performance that may be beyond the scope of high gain or other earlier observer designs, and we believe that the analog that we present here provides similar advantages. We leave the comparison of the performance of our observer design with the performance of other methods for future papers.

The notation will be simplified whenever no confusion can arise from the context. The Euclidean norm is denoted by $|\cdot|$. We let I denote the identity matrix of any dimension. Given any constant $T > 0$, we let $C_{\rm in}$ denote the set of all continuous functions $\phi : [-T, 0] \to \mathbb{R}^a$. We define $\Xi_t \in C_{\rm in}$ by $\Xi_t(m) = \Xi(t + m)$ for all choices of Ξ, $m \leq 0$, and $t \geq 0$ such that $t + m$ is in the domain of Ξ.

II. Main result

A. Assumptions

Our main assumptions on (3) are as follows:

Assumption 1: The pair (A, C) is observable. □

Assumption 2: There are constants $\overline{T} > 0$ and $\underline{T} > 0$ such that for all integers $i \geq 0$, the inequalities

$$\underline{T} \leq t_{i+1} - t_i \leq \overline{T} \tag{4}$$

are satisfied and $t_0 = 0$. □

Assumption 3: The function κ is continuous. Also, φ is globally Lipschitz. □

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By Assumption 1, we can find a constant $\tau > 0$ (which can be chosen arbitrarily small) such that the matrix

$$\Omega(\tau) = \left(\begin{array}{c} C \\ Ce^{-A\tau} \\ \vdots \\ Ce^{-(n-1)A\tau} \end{array} \right) \tag{5}$$

is invertible. This can be done by choosing $\tau > 0$ such that $(-A^\top, C^\top)$ is τ-sampled controllable, using [16, Theorem 4 and Lemma 3.4.1] and the fact that the observability of (A, C) implies that $(-A^\top, C^\top)$ is controllable. We also fix a global Lipschitz constant $\overline{\varphi} \geq 0$ for φ, so $|\varphi(y_1) - \varphi(y_2)| \leq \overline{\varphi}|y_1 - y_2|$ holds for all $y_1 \in \mathbb{R}$ and $y_2 \in \mathbb{R}$. We set

$$\Psi(\tau) = \Omega(\tau)^{-1} , \tag{6}$$

$$\begin{array}{l} \lambda(\overline{T}, \tau) = \\ \mathcal{F}(\tau)\sqrt{n}\left[1 + |C|^2\overline{\varphi}e^{|A|(n-1)\tau}(\overline{T} + (n-1)\tau)\right] + |C|\overline{\varphi}, \end{array} \tag{7}$$

$$\text{and } \mathcal{F}(\tau) = |CA\Psi(\tau)|. \tag{8}$$

Our final assumption is then as follows:

Assumption 4: The inequality

$$\overline{T}\lambda(\overline{T}, \tau) < 1 \tag{9}$$

is satisfied, and $\lambda(\overline{T}, \tau)$ is positive. □

Remark 1: For any A, C, φ, and κ such that Assumptions 1-3 are satisfied, one can determine a constant $\overline{T} > 0$ such that (9) holds. For instance, this inequality is satisfied with

$$\overline{T} = \min\left\{\frac{3}{4}, \frac{1}{\lambda(0.75, \tau) + \epsilon_0}\right\} , \tag{10}$$

for any positive constant ϵ_0. □

B. Estimator Design

Let us introduce the continuous-discrete system

$$\left\{ \begin{array}{rcl} \dot{\omega}(t) & = & CA\hat{x}(t) + C\varphi(\omega(t)) + C\kappa(u(t)) \\ & & \text{for all } t \in [t_i, t_{i+1}) \text{ and } i \geq 0 \\ \omega(t_i) & = & Cx(t_i) \text{ for all } i \geq 0 \\ \hat{x}(t) & = & \Psi(\tau)\mathcal{U}_1(\omega_t) + \Psi(\tau)\mathcal{U}_2(\omega_t, u_t) \end{array} \right. \tag{11}$$

with $\hat{x}$ valued in $\mathbb{R}^n$, ω valued in $\mathbb{R}$, $\tau > 0$ being a constant such that $\Omega(\tau)$ is invertible,

$$\mathcal{U}_1(\omega_t) = \left(\begin{array}{c} \omega(t) \\ \omega(t-\tau) \\ \vdots \\ \omega(t-(n-1)\tau) \end{array} \right) \text{ and} \tag{12}$$

$$\mathcal{U}_2(\omega_t, u_t) = \left(\begin{array}{c} 0 \\ Ce^{-\tau A}\Delta H_1(t) \\ \vdots \\ Ce^{-(n-1)\tau A}\Delta H_{n-1}(t) \end{array} \right) \tag{13}$$

and where $\Delta H_j(t) = H_j(t) - e^{jA\tau}H_j(t - j\tau)$ and the H_j's solve

$$\dot{H}_j(t) \;=\; AH_j(t) + \varphi(\omega(t)) + \kappa(u(t)) \tag{14}$$

for $j = 1, \ldots, n-1$ and any constant initial functions that are defined over $[-(n-1)\tau, 0]$. The ω-subsystem of this observer is inspired by the one from [8] and $\hat{x}$ is reminiscent of the observer of [14]. Although we can apply variation of parameters on the intervals $[t - j\tau, t]$ to check that

$$\begin{array}{l} e^{-jA\tau}\left(H_j(t) - e^{jA\tau}H_j(t - j\tau)\right) = \\ \int_{t-j\tau}^{t} e^{A(t-m-j\tau)}[\varphi(\omega(m)) + \kappa(u(m))]\mathrm{d}m \end{array} \tag{15}$$

holds for all $t \geq (n-1)\tau$ and $j = 1, 2, \ldots, n-1$, we express (13) using solutions of (14) to obtain formulas for estimators that do not contain any integrations. Our main result is:

Theorem 1: Let (3) satisfy Assumptions 1-4. Then for all constant initial functions for (3) and (11), and with the choice $c_* = \sqrt{n}|\Psi(\tau)|\left(1 + |C|(n-1)\overline{\varphi}\tau e^{(n-1)|A|\tau}\right)$, we have

$$\begin{array}{l} |\omega(t) - Cx(t)| \leq \\ \sup\limits_{\ell \in [0, \overline{T} + (n-1)\tau]} |\omega(\ell) - Cx(\ell)|e^{\frac{\ln(\overline{T}\lambda(\overline{T},\tau))}{2(\overline{T}+(n-1)\tau)}(t - \overline{T} - (n-1)\tau)} \end{array} \tag{16}$$

for all $t \geq \overline{T} + (n-1)\tau$ and

$$\begin{array}{l} |x(t) - \hat{x}(t)| \leq \\ c_* \sup\limits_{\ell \in [0, \overline{T} + (n-1)\tau]} |\omega(\ell) - Cx(\ell)|e^{\frac{\ln(\overline{T}\lambda(\overline{T},\tau))}{2(\overline{T}+2(n-1)\tau)}(t - \overline{T} - (n-1)\tau)} \end{array} \tag{17}$$

for all $t \geq \overline{T} + 2(n-1)\tau$. □

Remark 2: Our observer (11)-(14) entails resetting the ω values at each time t_i when a new output value becomes available. Hence, at each time t, future sampling times can be uncertain. Theorem 1 provides the rate of convergence $r = c_1 \ln(c_2\bar{T})$ of $|x(t) - \hat{x}(t)|$ to 0 , where

$$c_1 = -\tfrac{1}{2(\overline{T}+2(n-1)\tau)} < 0 \text{ and } c_2 = \lambda(\bar{T}, \tau) > 0, \tag{18}$$

and r diverges to ∞ as $\overline{T} \to 0$. Thus, the continuous-time case, where an exact estimate of the state is obtained in finite time, can be seen as the limiting case as $\overline{T} \to 0$. □

Remark 3: An upper bound for $|\omega(t) - Cx(t)|$ over the interval $[0, \overline{T} + (n-1)\tau]$ depending on the initial conditions can be obtained. However since $[0, \overline{T} + (n-1)\tau]$ is in a sense arbitrary (because τ and $\overline{T}$ are chosen by the user), the behavior of the solutions over this interval has no significant interest from a practical point of view. □

Remark 4: Several observers in the literature can be applied to (3), notably those of [2], [7], and [11]. However, we believe that they either use high gain or do not achieve the arbitrarily fast convergence property from Theorem 1. □

III. Proof of Theorem 1

To simplify the proof, we consider the case where the function κ is not present. The extension to the general case is straightforward, by introducing $\kappa(u(m))$ in the integrals in (19)-(23).

First note that Assumptions 2-3 ensure that the system consisting of (3) and (11) is forward complete and that the so-called chattering phenomenon does not occur. By integrating (3) over any interval $[s, t]$ with $0 \leq s \leq t$, we obtain

$$x(t) = e^{A(t-s)}x(s) + \int_s^t e^{A(t-m)}\varphi(Cx(m))\mathrm{d}m . \tag{19}$$

Thus, for all $p \in \{0, ..., n-1\}$, we have

$$\begin{array}{rl} Ce^{-pA\tau}x(t) = & Cx(t-p\tau) \\ & + C\int_{t-p\tau}^{t} e^{A(t-m-p\tau)}\varphi(Cx(m))\mathrm{d}m \end{array} \tag{20}$$

for all $t \geq p\tau$. Setting

$$\Delta_1(x_t) = \Psi(\tau)\begin{pmatrix} Cx(t) \\ Cx(t-\tau) \\ \vdots \\ Cx(t-(n-1)\tau) \end{pmatrix} \text{ and} \tag{21}$$

$$\Delta_2(x_t) = \Psi(\tau)\begin{pmatrix} 0 \\ C\int_{t-\tau}^{t} e^{A(t-m-\tau)}\varphi(Cx(m))\mathrm{d}m \\ \vdots \\ C\int_{t-(n-1)\tau}^{t} e^{A(t-m-(n-1)\tau)}\varphi(Cx(m))\mathrm{d}m \end{pmatrix}, \tag{22}$$

we easily deduce from (20) that

$$x(t) = \Delta_1(x_t) + \Delta_2(x_t) \tag{23}$$

for all $t \geq (n-1)\tau$.

Let us introduce the variables

$$e_\omega(t) = \omega(t) - Cx(t) \ \text{ and } \ e_x(t) = \hat{x}(t) - x(t) \tag{24}$$

and let $\underline{i}$ be the integer such that $t_{\underline{i}-1} < (n-1)\tau$ and $t_{\underline{i}} \geq (n-1)\tau$. Then in terms of the functions

$$D_{3,i}(x_t, \omega_t) = \omega(t-i\tau) - Cx(t-i\tau) \tag{25}$$

for $i = 0, \dots, n-1$ and

$$\begin{array}{l} D_{4,i}(x_t, \omega_t) = \\ C\int_{t-i\tau}^{t} e^{A(t-m-i\tau)}[\varphi(\omega(m)) - \varphi(Cx(m))]\mathrm{d}m \end{array} \tag{26}$$

for $i = 1, 2, \dots, n-1$, one can use (11)-(13) and (15) to prove that

$$\left\{\begin{array}{rcl} \dot{e}_\omega(t) & = & CA\hat{x}(t) + C\varphi(\omega(t)) - CAx(t) \\ & & - C\varphi(Cx(t)) \text{ for all } t \in [t_i, t_{i+1}) \\ e_\omega(t_i) & = & 0 \\ e_x(t) & = & \Psi(\tau)\begin{pmatrix} D_{3,0}(x_t, \omega_t) \\ D_{3,1}(x_t, \omega_t) \\ \vdots \\ D_{3,n-1}(x_t, \omega_t) \end{pmatrix} \\ & & +\Psi(\tau)\begin{pmatrix} 0 \\ D_{4,1}(x_t, \omega_t) \\ \vdots \\ D_{4,n-1}(x_t, \omega_t) \end{pmatrix} \end{array}\right. \tag{27}$$

for all $i \in \mathbb{N}$ with $i \geq \underline{i}$ or, equivalently,

$$\left\{\begin{array}{l} \dot{e}_\omega(t) = CAe_x(t) + C[\varphi(\omega(t)) - \varphi(Cx(t))] \\ \qquad \text{for all } t \in [t_i, t_{i+1}) \\ e_\omega(t_i) = 0 \\ e_x(t) = \Psi(\tau)\mathcal{U}_1(e_{\omega,t}) + \Psi(\tau)\begin{pmatrix} 0 \\ D_{4,1}(x_t, \omega_t) \\ \vdots \\ D_{4,n-1}(x_t, \omega_t) \end{pmatrix} \end{array}\right. \tag{28}$$

for all $i \in \mathbb{N}$ with $i \geq \underline{i}$ and with $\mathcal{U}_1$ as defined in (12) (with ω_t replaced by $e_{\omega,t}$ in the $\mathcal{U}_1$ formula in (12)). Hence,

$$\left\{\begin{array}{rcl} \dot{e}_\omega(t) & = & CA\Psi(\tau)\mathcal{U}_1(e_{\omega,t}) \\ & & +CA\Psi(\tau)\begin{pmatrix} 0 \\ D_{4,1}(x_t, \omega_t) \\ \vdots \\ D_{4,n-1}(x_t, \omega_t) \end{pmatrix} \\ & & +C[\varphi(\omega(t)) - \varphi(Cx(t))] \\ & & \text{for all } t \in [t_i, t_{i+1}) \\ e_\omega(t_i) & = & 0 \end{array}\right. \tag{29}$$

for all $i \in \mathbb{N}$ with $i \geq \underline{i}$.

We deduce by integrating (29) over the interval $[t_i, t)$ that

$$\begin{array}{l} e_\omega(t) = C\int_{t_i}^{t} \left[\varphi(\omega(\ell)) - \varphi(Cx(\ell))\right]\mathrm{d}\ell \\ +CA\Psi(\tau)\mathcal{U}_3(t, x_t, \omega_t) + CA\Psi(\tau)\mathcal{U}_4(t, x_t, \omega_t) \end{array} \tag{30}$$

for all $t \geq 0$ and $i \in \mathbb{N}$ such that $t \in [t_i, t_{i+1})$, where

$$\mathcal{U}_3(t, x_t, \omega_t) = \begin{pmatrix} \int_{t_i}^{t} e_\omega(\ell)\mathrm{d}\ell \\ \int_{t_i}^{t} e_\omega(\ell-\tau)\mathrm{d}\ell \\ \vdots \\ \int_{t_i}^{t} e_\omega(\ell-(n-1)\tau)\mathrm{d}\ell \end{pmatrix} \tag{31}$$

and

$$\mathcal{U}_4(t, x_t, \omega_t) = \begin{pmatrix} 0 \\ C\mathcal{U}_{4,1}(t, x_t, \omega_t) \\ \vdots \\ C\mathcal{U}_{4,n-1}(t, x_t, \omega_t) \end{pmatrix} \tag{32}$$

and

$$\begin{array}{l} \mathcal{U}_{4,i}(t, x_t, \omega_t) = \\ \int_{t_i}^{t}\int_{\ell-i\tau}^{\ell} e^{A(\ell-m-i\tau)}[\varphi(\omega(m)) - \varphi(Cx(m))]\mathrm{d}m\mathrm{d}\ell \end{array} \tag{33}$$

for $i \geq \underline{i}$.

The equality (30) implies that

$$\begin{array}{l} |e_\omega(t)| \leq \mathcal{F}(\tau)\sqrt{\sum_{l=0}^{n-1}\left|\int_{t_i}^{t} e_\omega(\ell-l\tau)\mathrm{d}\ell\right|^2} \\ +\mathcal{F}(\tau)|C|\sqrt{\sum_{l=1}^{n-1}|\mathcal{U}_{4,l}(t, x_t, \omega_t)|^2} \\ +|C|\int_{t_i}^{t}|\varphi(\omega(\ell)) - \varphi(Cx(\ell))|\mathrm{d}\ell\,, \end{array} \tag{34}$$

where $\mathcal{F}$ is from (8). From Assumption 3, we deduce that

$$\begin{array}{l} |e_\omega(t)| \leq \mathcal{F}(\tau)\left\{\sqrt{\sum_{l=0}^{n-1}\left|\int_{t_i-l\tau}^{t-l\tau} e_\omega(\ell)\mathrm{d}\ell\right|^2}\right. \\ \left. + |C|^2\overline{\varphi}\sqrt{\sum_{l=1}^{n-1}\left(\int_{t_i}^{t}\int_{\ell-l\tau}^{\ell}|e^{A(\ell-m-l\tau)}||e_\omega(m)|\mathrm{d}m\mathrm{d}\ell\right)^2}\right\} \\ +|C|\overline{\varphi}\int_{t_i}^{t}|e_\omega(\ell)|\mathrm{d}\ell\,. \end{array}$$

By observing that

$$\begin{aligned}
&\int_{t_i}^{t}\int_{\ell-l\tau}^{\ell} |e^{A(\ell-m-l\tau)}||e_\omega(m)|\mathrm{d}m\mathrm{d}\ell \\
&\leq e^{|A|(n-1)\tau}\int_{t_i}^{t}\int_{\ell-l\tau}^{\ell} |e_\omega(m)|\mathrm{d}m\mathrm{d}\ell \\
&\leq e^{|A|(n-1)\tau}(t-t_i)\int_{t_i-(n-1)\tau}^{t} |e_\omega(m)|\mathrm{d}m \\
&\leq e^{|A|(n-1)\tau}\overline{T}\int_{t_i-(n-1)\tau}^{t} |e_\omega(m)|\mathrm{d}m
\end{aligned} \tag{35}$$

and

$$\begin{aligned}
&\left|\int_{t_i-l\tau}^{t-l\tau} e_\omega(\ell)\mathrm{d}\ell\right|^2 \leq (t-t_i)\int_{t_i-l\tau}^{t-l\tau} |e_\omega(\ell)|^2\,\mathrm{d}\ell \\
&\leq \overline{T}\int_{t_i-l\tau}^{t-l\tau} |e_\omega(\ell)|^2\,\mathrm{d}\ell
\end{aligned} \tag{36}$$

for all $l \in \{1,...,n-1\}$, where (36) is a consequence of Jensen's inequality, we deduce that

$$\begin{aligned}
|e_\omega(t)| \ \leq\ & \mathcal{F}(\tau)\sqrt{\overline{T}}\sqrt{\sum_{l=0}^{n-1}\int_{t_i-l\tau}^{t-l\tau} |e_\omega(\ell)|^2\,\mathrm{d}\ell} \\
&+\mathcal{G}(\tau)\overline{T}\int_{t_i-(n-1)\tau}^{t} |e_\omega(m)|\mathrm{d}m \\
&+|C|\overline{\varphi}\int_{t_i}^{t} |e_\omega(\ell)|\mathrm{d}\ell \\
\leq\ & \mathcal{F}(\tau)\sqrt{\overline{T}n(t-t_i)}\sup_{\ell\in[t_i-(n-1)\tau,t]} |e_\omega(\ell)| \\
&+\mathcal{L}(t)\sup_{\ell\in[t_i-(n-1)\tau,t]} |e_\omega(\ell)| \\
&+|C|\overline{\varphi}(t-t_i)\sup_{\ell\in[t_i,t]} |e_\omega(\ell)|
\end{aligned} \tag{37}$$

with

$$\mathcal{L}(t) = \mathcal{G}(\tau)\overline{T}[t-t_i+(n-1)\tau] \tag{38}$$

and

$$\mathcal{G}(\tau) = \mathcal{F}(\tau)|C|^2\overline{\varphi}e^{|A|(n-1)\tau}\sqrt{n}\ . \tag{39}$$

Since $t \in [t_i, t_{i+1})$, it follows that

$$\begin{aligned}
&|e_\omega(t)| \leq \overline{T}\mathcal{F}(\tau)\sqrt{n}\sup_{\ell\in[t_i-(n-1)\tau,t]} |e_\omega(\ell)| \\
&+\mathcal{G}(\tau)\overline{T}(\overline{T}+(n-1)\tau)\sup_{\ell\in[t_i-(n-1)\tau,t]} |e_\omega(\ell)| \\
&+|C|\overline{\varphi}\overline{T}\sup_{\ell\in[t_i,t]} |e_\omega(\ell)|\ .
\end{aligned} \tag{40}$$

Using the inequality

$$\sup_{\ell\in[t_i,t]} |e_\omega(\ell)| \leq \sup_{\ell\in[t_i-(n-1)\tau,t]} |e_\omega(\ell)|, \tag{41}$$

we obtain

$$|e_\omega(t)| \leq \overline{T}\lambda(\overline{T},\tau)\sup_{\ell\in[t-\overline{T}-(n-1)\tau,t]} |e_\omega(\ell)|\ , \tag{42}$$

where $\lambda(\overline{T},\tau)$ is the function defined in (7) for all $t \geq \overline{T}+(n-1)\tau$. Then (9) in Assumption 4 and [12, Lemma 1] (which we also include in Appendix A below) applied to the function $z(t) = e_\omega(t+\overline{T}+(n-1)\tau)$ and the constant $T^* = 2(\overline{T}+(n-1)\tau)$ ensure that

$$\begin{aligned}
&|e_\omega(t)| \leq \\
&\sup_{\ell\in[0,\overline{T}+(n-1)\tau]} |e_\omega(\ell)|e^{\frac{\ln(\overline{T}\lambda(\overline{T},\tau))}{2(\overline{T}+(n-1)\tau)}(t-\overline{T}-(n-1)\tau)}
\end{aligned} \tag{43}$$

for all $t \geq \overline{T}+(n-1)\tau$. This concludes the proof of the first conclusion of the theorem. The second conclusion now follows from the formula for e_x in (27).

IV. Illustration

A. Studied System: Special Case of (3)

We illustrate Theorem 1 in the particular case where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \tag{44}$$

$$\varphi(y) = \epsilon\sin(y)\begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \text{and}\ \kappa(u) = \begin{pmatrix} u^2 \\ -u \end{pmatrix} \tag{45}$$

for any constant $\epsilon \in \mathbb{R}$. Then Assumptions 1 and 3 are satisfied and $\overline{\varphi} = \epsilon$. Since

$$e^{-Am} = \begin{pmatrix} \cos(m) & -\sin(m) \\ \sin(m) & \cos(m) \end{pmatrix} \tag{46}$$

for all $m \in \mathbb{R}$, we deduce that

$$\Omega(\tau) = \begin{pmatrix} C \\ Ce^{-A\tau} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \cos(\tau) & -\sin(\tau) \end{pmatrix} \tag{47}$$

and for any $\tau \in \left(0, \frac{\pi}{2}\right]$, we have

$$\Psi(\tau) = \begin{pmatrix} 1 & 0 \\ \frac{\cos(\tau)}{\sin(\tau)} & -\frac{1}{\sin(\tau)} \end{pmatrix}. \tag{48}$$

Thus, in terms of the notation from Theorem 1, we have

$$\begin{aligned}
&\Psi\left(\tfrac{\pi}{2}\right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \mathcal{F}\left(\tfrac{\pi}{2}\right) = 1, \\
&\text{and}\ \lambda\left(\overline{T}, \tfrac{\pi}{2}\right) = \sqrt{2}\left[1+\epsilon e^{\frac{\pi}{2}}\left(\overline{T}+\tfrac{\pi}{2}\right)\right]+\epsilon\ .
\end{aligned} \tag{49}$$

With for instance $\epsilon = \frac{1}{6}$, we get

$$\lambda\left(\overline{T}, \frac{\pi}{2}\right) = \sqrt{2}\left[1+\frac{e^{\frac{\pi}{2}}}{6}\left(\overline{T}+\frac{\pi}{2}\right)\right]+\frac{1}{6}\ . \tag{50}$$

Then Assumption 4 is satisfied with

$$\overline{T} = \frac{1}{\sqrt{2}\left[1+\frac{e^{\frac{\pi}{2}}}{6}\left(0.3+\frac{\pi}{2}\right)\right]+1}\ . \tag{51}$$

It follows that with the choice $\tau = \pi/2$, Theorem 1 applies to the system we consider and it provides the observer

$$\left\{\begin{array}{rcl}
\dot{\omega}(t) &=& \hat{x}_2(t)+\frac{1}{6}\sin(\omega(t))+u^2(t) \\
&& \text{for all } t\in[t_i,t_{i+1}) \text{ and } i\geq 0 \\
\omega(t_i) &=& x_1(t_i) \text{ for all } i \geq 0 \\
\hat{x}(t) &=& \begin{pmatrix} \omega(t) \\ -\omega\left(t-\frac{\pi}{2}\right) \end{pmatrix} - \begin{pmatrix} 0 \\ \mathcal{R}_a(t) \end{pmatrix},
\end{array}\right. \tag{52}$$

where

$$\mathcal{R}_a(t) = \int_{t-\frac{\pi}{2}}^{t} \left(\cos\left(t-m-\frac{\pi}{2}\right)\left[\epsilon \sin(\omega(m)) + u^2(m)\right] - \sin\left(t-m-\frac{\pi}{2}\right) u(m)\right) \mathrm{d}m. \quad (53)$$

B. Simulation

In this part, we present numerical simulations, using Mathematica. We select the sampling instants $t_i = ij\overline{T}$ with $\overline{T}$ defined in (51) and the input $u(t) = \sin(t)$, with the choice $j = 1$, then with $j = 1/2$, and finally with $j = 0.25$. We choose the initial conditions $x_1(0) = x_2(0) = 1$ and $w(0) = 0$. In Fig. 1, we plot the resulting observation error values of $\hat{x}_2(t) - x_2(t)$ for the three different choices of the sampling rate. Since the figure shows rapid convergence of the errors to 0, with faster convergence as j decreases, it helps illustrate our theorem in the special case of (44)-(45).

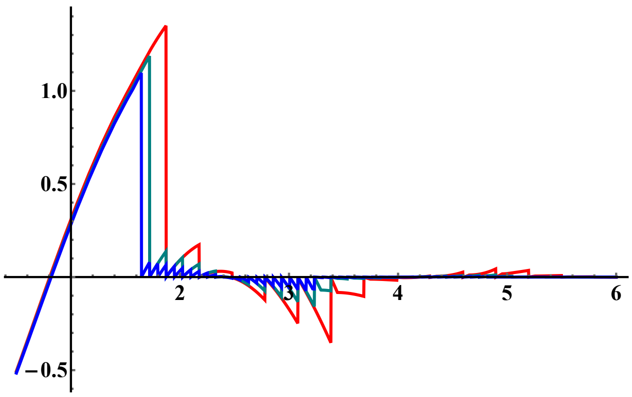

Fig. 1. Tracking Error $\hat{x}_2(t) - x_2(t)$ Converging to 0 over Time with Sampling Rates $\overline{T}$ (Red), $0.5\overline{T}$ (Green), and $0.25\overline{T}$ (Blue) with $\overline{T}$ as defined in (51).

V. Conclusions

We extended the observer design from [14] to cases where the measurements are discrete. A key novel feature was our use of a trajectory based approach, which enabled us to ensure arbitrarily fast convergence of the observation error to zero, provided the sampling in the output is frequent enough. Our trajectory based approach used a contractivity condition instead of a Lyapunov function, and the sampling rate is chosen so that the constant of contractivity ρ satisfies the requirement $\rho \in (0, 1)$. Many extensions can be expected. They pertain to time-varying systems, systems with delay in the measurements, robustness with respect to additive disturbances or the vector field of the system, local observer design of systems with general nonlinear terms, and multi-rate sampled-data observers in the spirit of what is proposed in [9] and [10] in the case where the dimension of the output is larger than 1.

Appendix A: Statement of [12, Lemma 1]

We provide a statement of a special case of [12, Lemma 1], which we used at the end of our proof of Theorem 1 above:

Lemma A.1: Let $T^* > 0$ be a constant. Let a piecewise continuous function $z : [-T^*, +\infty) \to [0, +\infty)$ admit a sequence of real numbers v_i and positive constants $\overline{v}_a$ and $\overline{v}_b$ such that $v_0 = 0$, $v_{i+i} - v_i \in [\overline{v}_a, \overline{v}_b]$ for all $i \geq 0$, z is continuous on each interval $[v_i, v_{i+1})$ for all $i \geq 0$, and $z(v_i^-)$ exists and is finite for each $i \in \mathbb{N}$. Assume that there is a constant $\rho \in (0, 1)$ such that

$$z(t) \leq \rho \sup_{\ell \in [t-T^*, t]} |z(\ell)| \quad (A.1)$$

holds for all $t \geq 0$. Then

$$z(t) \leq \sup_{\ell \in [-T^*, 0]} |z(\ell)| e^{\frac{\ln(\rho)}{T^*} t} \quad (A.2)$$

holds for all $t \geq 0$. □

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