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Almost Dedekind Domains.

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ABSTRACT

This paper is a study of commutative rings R with identity, such that each quotient ring R_P with respect to a proper prime P of R is a general ZPI ring, i.e., a ring in which each ideal is a product of finitely many prime ideals. The first section concerns such rings which are also integral domains and gives several characterizations of such domains. It is shown that a domain J has the above property if and only if each ideal of J which has prime radical, is a prime power. Such a domain J is shown to have no non-maximal proper prime ideals and to be strongly integrally closed. The second section concerns rings with the aforesaid property which are not domains. In such rings, it is shown that the condition that ideals with prime radical be prime powers is necessary and sufficient for each R_P to be a general ZPI ring.

The third section is a discussion of the properties of a commutative ring R with the property that the ideal system of each quotient ring R_P of R, with respect to a proper prime P of R, is linearly ordered. It is shown that 1) such a ring is integrally closed in its total quotient

ring, 2) each proper residue class ring modulo a proper prime of R is a Prüfer domain, and 3) any two primes of R, neither of which contains the other are relatively prime.

NOTATION

In this paper "ring" will mean a commutative ring with identity element. "Domain" will mean a ring in which the zero ideal is prime. "ZPI ring" will mean, [5;117][1], a ring in which each ideal is a product of finitely many prime ideals. "Prüfer domain" will mean a domain in which each finitely generated non-zero ideal is invertible. "Discrete valuation ring" will mean a Dedekind domain with at most one proper prime ideal, [11;278], where "proper ideal" means an ideal different from (0) and (1). A domain J, (ring R), will be called "integrally closed" if any element of its quotient field, (total quotient ring), which satisfies a monic polynomial with coefficients in J, (R), is already in J, (R). A domain J, (ring R), will be called "strongly integrally closed" if any element x of its quotient field, (total quotient ring), such that the polynomial ring J[x], (R[x]), is contained in a finite J-, (R-), module, is already in J, (R). The symbol "R_P" will indicate the quotient ring, [11;221], of R with respect to

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[1]In the symbol [A;B], A refers to the number of the bibliographical reference, B the page number in A.

the prime P. R_P will be called "proper" if P is proper; $A^{(n)}$ will indicate the n-th symbolic power of A, [11;232], (the overring with respect to which $A^{(n)}$ is formed will be clear from the context); rad(A) will indicate the radical of the ideal A, [11;147]. The symbol "$\subset$" will allow equality while "$<$" will indicate proper containment. For two sets A and B, A - B will denote the relative complement of B in A. Throughout this paper, J will be a domain with quotient field K, and R will be a ring with total quotient ring T.

ALMOST DEDEKIND DOMAINS

Definition 1.1

A domain in which each proper quotient ring is a discrete valuation ring will be called an "almost-Dedekind domain," (AD - domain).

Lemma 1.1

If, in J, each ideal with prime radical is a power of its radical, non-zero proper primes of J are maximal.

Proof: It clearly suffices to show that a minimal prime of a non-zero principal ideal is maximal. Let (a) be a non-zero principal ideal of J and P a minimal prime of (a). Since P is minimal for (a), $\mathrm{rad}(aJ_P) \cap J = PJ_P \cap J = P$. Therefore $aJ_P \cap J = P^n$ for some positive integer n. But then $aJ_P = (aJ_P \cap J)J_P = P^nJ_P = (PJ_P)^n$ and hence PJ_P is invertible since its n-th power is principal. This implies $PJ_P \neq (PJ_P)^2$ which implies $P \neq P^2$. Now $P^2 \subset P^{(2)} \subset P$ so that $\mathrm{rad}(P^{(2)}) = P$ and either $P^{(2)} = P$ or $P^{(2)} = P^2$. If $P^{(2)} = P$, $P^2J_P = PJ_P$ which cannot happen. Therefore, $P^{(2)} = P^2$ and P^2 is primary. Now let p be an element of $P - P^2$, and m an element of $J - P$. Since P^2 is primary, pm is not in P^2 and hence $P^2 + (pm) = P$. Then let $p = q + rpm$ where q is in P^2, r in J. We have $p(1 - rm)$ is in P^2 and p is not in P^2, primary, which implies 1 - rm is in P. But this implies 1 is in $P + (m)$ and since m was arbitrary outside P, P is maximal.

Theorem 1.1

J is an AD - domain if and only if each ideal of J, with prime radical, is a prime power.

Proof: If each proper J_P is a discrete valuation ring, proper primes of J are clearly maximal. Thus $\mathrm{rad}(A) = P$, a proper prime of J, implies A is primary, [11;153], so that $AJ_P \cap J = A$, [11;223], but since J_P is a discrete valuation ring, $AJ_P = (PJ_P)^n$ for some positive integer n. Now since A is primary and contained in P, $A = AJ_P \cap J$, and hence $A = P^n$. On the other hand, if each ideal with prime radical is a prime power, then according to lemma 1.1, each proper prime P of J is maximal so that J_P has exactly one proper prime ideal. Then since, (by the proof of lemma 1.1), PJ_P is invertible, J_P is a Dedekind domain, [8;234], thus a discrete valuation ring.

Remark 1.1

An AD - domain is strongly integrally closed in its quotient field.

Proof: It is easy to show that any domain J is the intersection of its quotient rings J_P for proper primes P of J. But each J_P of an AD - domain is strongly integrally closed.

Remark 1.2

In an AD - domain J, the powers of any proper ideal intersect in (0).

Proof: If A is a proper ideal of J, A is contained in P for some proper prime P of J. Thus $A \subset AJ_P \subset PJ_P$ and $\bigcap_{n=1}^{\infty} A^n \subset \bigcap_{n=1}^{\infty} (PJ_P)^n = (0)$ since J_P is a Dedekind domain.

Remark 1.3

A Noetherian AD - domain is a Dedekind domain.

Proof: We have already shown that an AD - domain is integrally closed and has no non-maximal proper prime ideals, [10;85,86].

We state here without proof a theorem of Krull, [3;554].

Theorem 1.2

In J, these are equivalent:

(a) J is a Prüfer domain,

(b) for each proper prime P of J, J_P is a Prüfer domain,

(c) for each proper prime P of J, J_P is a valuation ring.

Part (b) of theorem 1.2 is not in the statement of Krull's theorem, but is implied in the proof.

<u>Corollary 1.1</u>

An AD - domain is a Prüfer domain.

Since every Prüfer domain is integrally closed, [9;14], we have another proof of

<u>Corollary 1.2</u>

An AD - domain is integrally closed.

<u>Corollary 1.3</u>

J is a Prüfer domain if and only if, given a, b non-zero elements of J and P a proper prime of J, there exist elements c and d of J such that $a/b = c/d$ and (c,d) is not contained in P.

Proof: If (a,b) is contained in P, then $(a,b)J_P \subset PJ_P$ in J_P which is a valuation ring so that either $aJ_P \subset bJ_P$ or $bJ_P \subset aJ_P$, say $aJ_P \subset bJ_P$. Then $a = b(c/m)$ for some c in J, m in J - P, and $a/b = c/m$, (c,m) not contained in P.

<u>Theorem 1.3</u>

If J is a Prüfer domain and R is a ring such that $J \subset R \subset K$, then R is a Prüfer domain.

Proof: Let Q be a proper prime of R. Then $P = Q \cap J$ is a proper prime of J and J_P is a valuation ring. But any ring between a valuation ring and its quotient field is a valuation ring and $J_P \subset R_Q \subset K$ so that R_Q is a valuation ring. Therefore R is a Prüfer domain by theorem 1.3(c).

<u>Theorem 1.4</u>

If, in theorem 1.3, J has no proper non-maximal primes, R has no proper non-maximal primes.

Proof: Notice that in a Prüfer domain J, proper primes are maximal if and only if each proper J_P is a rank 1 valuation ring. We will show that if, (using the notation of theorem 1.3), J_P has rank 1, R_Q has rank 1. Suppose Q' is a proper prime of R_Q, then QR_Q contains Q' and since J_P has rank 1, $Q' \cap J_P = QR_Q \cap J_P$. Let m be an element of QR_Q. Then $m = a/b$ with a, b in J_P, b not in Q, hence $mb = a$ is an element of $QR_Q \cap J_P = Q' \cap J_P$ so that mb is in Q'. Since b is not in A', m is in A' and $QR_Q = Q'$ so that R_Q is of rank 1.

<u>Theorem 1.5</u>

If J is an AD - domain and $J \subset R \subset K$, then R is an AD - domain.

Proof: Since $J_P \subset R_Q \subset K$ and J_P is a Dedekind domain, R_Q is a Dedekind domain, [2;31].

<u>Theorem 1.6</u>

J is an AD - domain if and only if each proper primary ideal of J is a power of a maximal ideal.

Proof: It has already been shown that in an AD - domain, proper primes are maximal and each primary ideal is a power of its radical. On the other hand, each proper prime of J is a proper primary, hence is maximal. Thus each ideal with prime radical has maximal radical and is primary so that, by hypothesis, it is a maximal, (in particular prime), power. So J is an AD - domain by theorem 1.1.

<u>Theorem 1.7</u>

J is an AD - domain if and only if whenever A is an ideal of J with prime product radical, A is a prime power product.

Proof: "If" is clear from theorem 1.1. Then suppose J is an AD - domain and A an ideal of J with $\mathrm{rad}(A) = P_1 \ldots P_n$, with $P_1, \ldots, P_n$ different primes of J. Then since proper primes of J are maximal, A is the intersection of its isolated primary components, [4;22]. But since each J_P is a discrete valuation ring, the isolated primary components of A are powers of the proper primes containing A, namely $P_1, \ldots, P_n$. Then since each P_i is maximal, the isolated primary components of A are relatively prime and A is their product.

For completeness we state here without proof a result communicated by Robert Gilmer:

Theorem 1.8

J is an AD - domain if and only if for A, B, C non-zero ideals of J such that $AB = AC$, $B = C$.

Theorem 1.9

J is an AD - domain if and only if

(a) J is a Prüfer domain,

(b) proper primes of J are maximal, and

(c) J contains no proper idempotent prime.

Proof: We have already shown that an AD - domain has properties (a), (b) and (c). On the other hand, if J is a Prüfer domain, each J_P is a valuation ring. If J has no non-maximal proper primes, J_P has rank 1; but if J contains no non-zero proper idempotent prime, neither does J_P, since the prime powers of J are primary. Now a rank 1 valuation ring is a Dedekind domain if and only if its maximal ideal is not idempotent, [12;41].

Since only rank 1 valuation rings are strongly integrally closed, [11;255:12;45], we have

Corollary 1.4

J is an AD - domain if and only if (a), (c) of theorem 1.9 and

(b') Each proper J_P is strongly integrally closed.

Corollary 1.5

The union of a tower of AD - domains is an AD - domain if and only if it has no proper idempotent primes.

Proof: It is easily shown that the union of a tower of AD - domains is a Prüfer domain and each proper J_P is strongly integrally closed.

Theorem 1.10

If J is an AD - domain, F a finite algebraic extension of K and $J^\sim$ the integral closure of J in F, then $J^\sim$ is an AD - domain.

Proof: Let Q be a proper prime of $J^{\sim}$. Then $P = Q \cap J$ is a proper prime of J, hence J_P is a discrete valuation ring. Now we know that the integral closure L of J_P in F is a Dedekind domain, [11;281]. It can be easily shown that $L \subset J^{\sim}{}_Q \subset F$ and that F is the quotient field of L. Hence $J^{\sim}{}_Q$ is a Dedekind domain, (between a Dedekind domain and its quotient field, [2;31]).

Corollary 1.6

The ring of integral elements of an algebraic number field forms an AD - domain if and only if this ring has no proper idempotent primes.

Proof: This ring can be written as a union of a tower of rings each of which is the integral closure of the rational integers in a finite algebraic extension of the rational numbers. Hence corollary 1.6 follows from theorem 1.10 and corollary 1.5.

Example

Nakano, [7;426], gives the following example of an algebraic number field K, the integral elements of which form an AD - domain which is not a Dedekind domain. Let K be the field obtained by the adjunction of the p-th roots of unity for every rational prime p. Let J be the integral elements of K. Nakano showed that J has no idempotent proper primes, so that J is an AD - domain by corollary 1.6. He also showed that J has no finitely generated proper primes, so that J is not a Dedekind domain.

Theorem 1.11

The integral closure J is an AD - domain in an algebraic extension of its quotient field is an AD - domain if and only if J has no proper idempotent ideals.

Proof: Let J* be the union of a maximal tower of AD - domains in J, (such a tower exists by the Hausdorff maximality principle and the existence of one AD - domain in J). By corollary 1.5, J* is an AD - domain if and only if it has no idempotent proper primes. Now suppose J has no idempotent proper ideals. Then if P* is any proper prime of J*, P*J is a proper ideal of J so that P* is not idempotent since P*J is not. So J* is an AD - domain and if $J^* < J$, there exists an element x in J - J* and x is integral over J*. Then the domain J** which is the integral closure of J* in K*(x), (K* the quotient field of J*), is an AD - domain by theorem 1.10, contradicting the maximality of the tower which formed J*. Therefore $J^* = J$ and the theorem is proved.

ALMOST ZPI RINGS

Definition 2.1

For a prime ideal P of R, let M(P) be R - P and N(P) be the set of elements x of R such that 0 is an element of xM(P).

We will use the notation of Zariski and Samuel, [11;221], and let R_P be $(R/N(P))_{P/N(P)}$; A^e and A^c represent the extension and contraction of an ideal A of R and R_P respectively, [11;218]. Discrete valuation ring and special primary ring, (primärer zerlegbarer Ring, [4;84]), will be denoted by dvr and spr respectively.

Definition 2.2

R will be called an "almost ZPI ring, (AZPI-ring)," if for each proper prime P of R, R_P is a ZPI ring, [5;117], i.e., each ideal of R_P is factorable into a product of prime powers.

It has been shown by Asano, [1;83], that a ZPI ring with a unique maximal ideal is either a dvr or an spr. Therefore, R is an AZPI-ring if and only if each proper R_P is either a dvr or an spr.

For the proofs in this section it will be convenient to state here a theorem from Zariski and Samuel, [11;228], namely,

Theorem 2.1

Let P be a prime ideal of R. The mapping $A \longrightarrow A^e$ establishes a 1-1 correspondence between the set of prime (primary) ideals of R contained in P, and the set of all prime (primary) ideals of R_P.

Lemma 2.1

If R is an AZPI-ring and P a proper prime of R such that N(P) is not prime, then rad(N(P)) = P.

Proof: Since N(P) is not prime, R_P is not a domain and hence R_P is an spr. Therefore, there exists a positive integer n such that $(P^e)^n = (0)$ and thus for p in P, $((p)^e)^n = (0)$, i.e., p^n is in N(P). This implies P is contained in rad(N(P)), but the other containment always holds, so that rad(N(P)) = P.

Theorem 2.2

If R is an AZPI-ring, P a proper prime of R and N(P) is not prime, then P is minimal and maximal, R_P is an spr and rad(A) = P implies A is a power of P.

Proof: As in lemma 2.1, R_P is an spr and hence contains only one proper prime ideal, P^e; and by theorem 2.1, there are therefore no prime ideals of R properly contained in P, containing P. If $R_{P'}$ is an spr, P' is minimal and $P = P'$ is maximal. On the other hand, if $R_{P'}$ is a dvr, N(P') is prime in R. Again using theorem 2.1, P' and N(P') are the only primes of R contained in P' so that either $P = P'$ is maximal or $P = N(P') \subset N(P) \subset P$ which implies $P = N(P)$ and contradicts N(P) not being prime. Therefore P is maximal and $\mathrm{rad}(A) = P$ implies A is primary. So by theorem 2.1, since $A^e = (P^e)^n = (P^n)^e$, $A = P^n$.

<u>Theorem 2.3</u>

If R is an AZPI-ring, P a proper prime of R and N(P) is prime, then either (1) $P = N(P)$ or (2) P is maximal, N(P) is the only prime of R properly contained in P, R_P is a dvr, $\bigcap_{n=1}^{\infty} P^n = N(P)$ and $\mathrm{rad}(A) = P$ implies $A = P^n$ for some positive integer n.

Proof: If $P \neq N(P)$, $N(P) < P$. Since R_P is a domain, it is a dvr and hence by theorem 2.1, N(P) and P are the only primes of R contained in P. Let P' be a maximal ideal of R containing P. If N(P') were not prime, P' would be minimal by theorem 2.2 so that P' would be P and N(P) would not be prime. Therefore N(P') is prime and is the only prime properly contained in P', which implies either $P = N(P')$ or $P = P'$. If $P = N(P')$, $P = N(P)$; therefore $P = P'$ is maximal.

Then each P^n, for a positive integer n, is primary and $(P^n)^e = (P^e)^n$ so that N(P) is contained in each P^n and $N(P) \subset \bigcap_{n=1}^{\infty} P^n$. But since R_P is a dvr, $\bigcap_{n=1}^{\infty} (P^e)^n = (0)$ so that $\bigcap_{n=1}^{\infty} P^n \subset N(P)$, i.e., $N(P) = \bigcap_{n=1}^{\infty} P^n$. Now since P is maximal, rad(A) = P implies A is primary which implies by theorem 2.1 that $A = P^n$ for some positive integer n.

Lemma 2.2

If in R each ideal with prime radical is a prime power, this property also holds in each R_P.

Proof: Let A be an ideal of R_P and rad(A) = P*, a prime. Then, since $\mathrm{rad}(A^c) = (\mathrm{rad}(A))^c$ and since $(P^*)^c$ is prime, $\mathrm{rad}(A^c) = (P^*)^c$, prime so that $A^c = ((P^*)^c)^n$ and $A = A^{ce} = (((P^*)^c)^n)^e = ((P^*)^{ce})^n = (P^*)^n$ and the lemma is proved.

Theorem 2.4

If in R each ideal with prime radical is a prime power, then each proper R_P is a ZPI ring.

Proof: Let P be a proper prime of R. If P is minimal in R, P^e is the only proper prime of R_P and by lemma 2.2, each proper ideal is a power of P^e. In this case, R_P is either a field, (if P = N(P)), or an spr, $((0)^e = (P^e)^n)$. If P is not minimal, let P' be a minimal prime contained in P. It can be easily shown that the residue class ring of R modulo P' is an AD - domain so that proper primes of R/P' are maximal and P is maximal in R; and since P was any non-minimal prime, there are no primes properly between P' and P in R. We will now show that in R, $\bigcap_{n=1}^{\infty} P^n = P'$ is the only prime of R contained in P. Since in R/P' the intersection of the powers of P/P' is (0), we see that $\bigcap_{n=1}^{\infty} P^n \subset P'$. Now suppose there is a positive integer n such that $P' \subset P^n$ and $P' \not\subset P^{n+1}$. Then since $\mathrm{rad}(P' + P^{n+1}) = P$, $P' + P^{n+1} = P^n$ i.e., $(P/P')^{n+1} = (P/P')^n$ which cannot happen since R/P' is an AD - domain. Therefore $P' \subset \bigcap_{n=1}^{\infty} P^n$ so that $P' = \bigcap_{n=1}^{\infty} P^n$. But if P^* is any minimal prime of R contained in P, the same argument shows that $P^* = \bigcap_{n=1}^{\infty} P^n$ and P' is unique. Therefore, in R_P, P^e and $(P')^e$ are the only primes and every ideal has prime radical and is thus a prime power. This proves that R_P is a ZPI ring.

Summarizing theorems 2.2, 2.3 and 2.4 we see that

<u>Theorem 2.5</u>

R is an AZPI-ring if and only if each ideal of R with prime radical is a prime power.

Definition 2.3

We will call a ring a "multiplication ring" if whenever A and B are ideals of R with $A \subset B$, there is an ideal C of R such that $A = BC$, [6;2].

Lemma 2.3

A multiplication ring is an AZPI-ring.

Proof: Mori, [6], has shown that in a multiplication ring, primary ideals are prime powers and each ideal is the intersection of its isolated primary components. Therefore, any ideal with prime radical is primary, since it has only one isolated primary component; so any ideal with prime radical is a prime power and the lemma follows from theorem 2.5.

Theorem 2.6

R is an AZPI-ring if and only if each proper R_P is a multiplication ring.

Proof: "Only if" follows from the fact that every ZPI ring is a multiplication ring. Then if each R_P is a multiplication ring, each R_P is an AZPI-ring by lemma 2.3; but being its own quotient ring with respect to its maximal ideal, R_P is a ZPI ring.

<u>Theorem 2.7</u>

R is an AZPI-ring if and only if for each primary ideal Q of R there exists a maximal ideal M of R such that Q is either N(M) or a power of M.

Proof: "Only if" is clear from theorems 2.2 and 2.3. On the other hand, if each primary of R is either N(M) or a power of M, then each proper R_P is either a field, (in case $P = N(M)$), a dvr, (in case P is maximal but not minimal), or an spr, (in case P is maximal and minimal).

<u>Lemma 2.4</u>

Let R be a ring and A, B ideals of R. Then $A = B$ if and only if $A^e = B^e$ in every proper R_P.

Proof: It can be easily shown that any ideal A equals $\bigcap$ [A^{ec}: with respect to each proper R_P]. Then if $A^e = B^e$, $A^{ec} = B^{ec}$ and the lemma follows.

<u>Theorem 2.8</u>

If R is an AZPI-ring, then A, B and C are ideals of R with A regular and $AB = AC$ only if $B = C$.

Proof: For each proper prime P of R, $A^eB^e = A^eC^e$ and A^e is regular, and since R_P is a dvr or an spr, this implies $B^e = C^e$. Thus the theorem follows from lemma 2.4.

In concluding this section, we summarize from theorems 2.2, 2.3 and 2.4 the classification of the proper primes of an AZPI-ring.

Theorem 2.9

In an AZPI-ring, a proper prime P is either

(1) maximal and minimal in which case $N(P) = P^n$ and R_P is an spr,

(2) maximal and not minimal in which case $N(P)$ is the only prime below P, $N(P) = \bigcap_{n=1}^{\infty} P^n$, and R_P is a dvr, or

(3) minimal and not maximal in which case $P = N(P) = N(M) = \bigcap_{n=1}^{\infty} M^n$ for M the maximal ideal containing P, P is the only P-primary ideal of R and R_P is a field, the quotient field of R_M.

GENERALIZED PRÜFER DOMAINS

In this section we will discuss a generalization of the concept of "Prüfer domain" to rings with zero divisors. In particular, we will discuss rings with the property (*) that each proper quotient ring has a linearly ordered ideal system. Since a domain whose ideal system is linearly ordered is a valuation ring, we see that a domain with property (*) is, by theorem 1.2, a Prüfer domain. We will first prove some properties of the quotient rings of rings with property (*) in order to see the extent to which these quotient rings actually generalize the concept of "valuation ring."

Let R be a ring whose ideal system is linearly ordered. Then the nilpotent elements of R form a prime ideal P^* and the zero divisors of R form a prime ideal $P^{\sim}$.

Remark 3.1

Each ideal of R has prime radical.

Remark 3.2

Each finitely generated ideal of R is principal.

Remark 3.3

An ideal of R is regular if and only if it properly contains $P^{\sim}$.

Lemma 3.1

If x and y are non-zero elements of R such that $xy = 0$ and x is not in P^*, then x is in $P^{\sim}$ and y is in P^*.

Proof: x is in $P^{\sim}$ because it is a zero divisor. Then xy is in P^*, prime and x is not in P^*, so y must be in P^*.

Lemma 3.2

P^* is either idempotent or nilpotent.

Proof: If $(P^*)^2$ is not P^*, let p be in $P^* - (P^*)^2$. Then $(P^*)^2 \subset (p)$ since (p) is not in $(P^*)^2$; but p is nilpotent, hence P^* is nilpotent.

Lemma 3.3

Each ideal of R is the union of a tower of principal ideals.

Lemma 3.4

$P^* = P^{\sim}$ if and only if (0) is primary.

Proof: If $P^* = P^{\sim}$, (0) is primary by lemma 3.1. Now suppose (0) is primary. Let p be an element of $P^{\sim}$. Then there is a non-zero element m of R such that $pm = 0$. But $\mathrm{rad}(0) = P^*$ and $m \neq 0$ implies p is in P^*, i.e., $P^* = P^{\sim}$.

We will now give an example of a ring whose ideal system is linearly ordered and in which $P^* \neq P^{\sim}$.

Example:

Let K be the field of rational numbers and x, y, z indeterminants algebraically independent over K. Let I be the group of rational integers and define $F:K[x,y,z] \longrightarrow I \oplus I \oplus I$ by $F(f) = (r,s,t)$ where $f = \sum a_{ijk}x^i y^j z^k$

and $r = \min[i : a_{ijk} \neq 0]$

$s = \min[j : a_{ijk} \neq 0]$

$t = \min[k : a_{ijk} \neq 0]$.

Then the extension of F to $K(x,y,z)$ is a discrete rank 3 valuation. Let R be the valuation ring of F in $K(x,y,z)$ and $P < P'$ the two non-maximal proper primes of R. Then R/PP' is the desired ring with $P^* = P/PP'$, $P^{\sim} = P'/PP'$.

Lemma 3.5

If t is in T - R, 1/t is in R.

Proof: Let $t = a/b$ with a, b in R. Then either $(a) \subset (b)$ and t is in R or $(b) \subset (a)$ and 1/t is in R. We need only note that in case $(b) \subset (a)$, a is regular since b is.

Lemma 3.6

R is integrally closed in its total quotient ring T.

Proof: Let t be in T and r,...,s in R such that

$t^n + rt^{n-1} + \ldots + s = 0$ (I). Let $t = a/b$ with a, b in R. Then if t is not in R, $1/t = b/a$ is in R and a is regular by lemma 3.5. So we multiply the equation (I) by $(1/t)^{n-1}$ and we see that t is in R.

Now we will discuss the ring R with property (*).

<u>Theorem 3.1</u>

If R has property (*), then each R/P is a Prüfer domain for P prime in R.

Proof: Since the operations of quotient ring and residue class ring formation commute, [11;227], we see that any quotient ring of R/P, with respect to a proper prime, is a domain whose ideal system is linearly ordered; hence is a valuation ring. The theorem now follows from theorem 1.2.

<u>Theorem 3.2</u>

If R has property (*), R is integrally closed in its total quotient ring T.

Proof: Let t be an element of T, integral over R, $t = r/s$, with r, s in R. Let F be the natural map from R to R_P for a proper prime P of R. Then it is clear that F(s) is regular in R_P so that F(r)/F(s) is in the total quotient ring of R_P and is integral over R_P. Thus F(r)/F(s) is in

R_P by lemma 3.6. Then there is an element m in R - P such that rm is in (s). Therefore the ideal (s):(r), [11;147], is in no proper prime of R, hence equals R and (r) is in (s), showing that $t = r/s$ is in R.

Theorem 3.3

If R has property (*), any two primes of R are either relatively prime or one contains the other.

Proof: If two primes are not relatively prime, their sum is contained in a maximal ideal of R, with respect to which the quotient ring of R has a linearly ordered ideal system. But the primes of R which are contained in this maximal ideal correspond to the primes of this quotient ring in a 1-1 order preserving fashion.

Theorem 3.4

If R has property (*) and A, B and C are ideals of R with A finitely generated and regular and $AB = AC$, then $B = C$.

Proof: With respect to any proper prime of R, we have A^e is principal and regular, hence invertible, and $A^eB^e = A^eC^e$ so that $B^e = C^e$. Theorem 3.4 now follows from lemma 2.4.

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BIOGRAPHY

The author was born December 23, 1934, in Helena, Arkansas. He attended public schools in Helena and West Helena, Arkansas. In 1956 he received a Bachelor of Science degree from Ouachita Baptist College and received a Master of Science degree from Louisiana State University in 1958. He served two years in the U. S. Army Signal Corps at Fort Monmouth, New Jersey and is now working toward the degree of Doctor of Philosophy in Mathematics at Louisiana State University.

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