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Topological Analysis of Analytic Functions.

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ABSTRACT

In this dissertation we are concerned with three phases of topological analysis. The first phase, developed in Chapters 1 and 2, is devoted to the line of thought developed by Whyburn in his Topological Analysis. In Chapters 1 and 2 we develop a topological analogue of the line integral, and by using "local" inverses of the exponential function we obtain Whyburn's topological index. Making use of anti-derivatives of differentiable functions we obtain close analogues of the classical line integral and the Cauchy Integral Formula. Using these integral analogues and a modification of a standard proof of the Riemann Mapping Theorem, we obtain a new proof of the power series expansion of a differentiable function independent of the two proofs of Porcelli and Connell. We also use these techniques to obtain the Laurent expansion.

The second phase is concerned with the removable singularity problem. The setting of this problem consists of having a closed subset A of the closure U of the unit disk U and a continuous function f on U differentiable on U-A. The problem is to find conditions on f or A so that f is differentiable on U. Examples developed by Denjoy show that conditions need to be imposed. Our main results in this
direction consist of two sets of conditions which enable
us to conclude that \( f \) is differentiable on \( U \). One consequence
of our conditions is a new proof for the case where \( A \) is
a rectifiable arc dividing \( U \) into two disjoint regions.

The last phase of our work is involved with an extension
of the removable singularity problem. More precisely, suppose
\( A \) is a closed nowhere dense subset of \( U \) and \( f \) is a function
defined on \( U \) and differentiable on \( U-A \). What can be said
about \( f \) and \( A' \)? We are able to characterize \( A \) topologically
if we require \( f \) to be the limit of a pointwise convergent
sequence of analytic functions defined on an open set
containing \( U \) and \( A \) to be the set of all points of \( U \) for
which this sequence fails to locally converge uniformly.

Three necessary and sufficient conditions are given. Our
characterization is unchanged if we consider pointwise
convergent sequences of open maps. Sets such as the universal
plane curve are readily excluded from consideration because
of an excess of "fine" structure.
INTRODUCTION

In this dissertation we are concerned with three phases of topological analysis. The first phase, developed in Chapters 1 and 2, is devoted to the line of thought developed by Whyburn in his Topological Analysis. In Chapters 1 and 2 we develop a topological analogue of the line integral, and by using "local" inverses of the exponential function we obtain Whyburn's topological (circulation) index. Making use of antiderivatives of differentiable functions we obtain close analogues of the classical line integral and the Cauchy Integral Formula. Using these integral analogues and a modification of a standard proof of the Riemann Mapping Theorem, we obtain a new proof of the power series expansion of a differentiable function independent of the two proofs of Porcelli and Connell. We also use these techniques to obtain the Laurent expansion.

The second phase is concerned with the removable singularity problem. The setting of this problem consists of having a closed subset A of the closure U of the unit disk U and a continuous function f on U differentiable on U-A. The problem is to find conditions on f or A so that f is differentiable on U. Examples developed by Denjoy show that conditions need be imposed. Our main results in this
direction consists of two sets of conditions which enable us to conclude that $f$ is differentiable on $U$. One consequence of our conditions is a new proof for the case when $A$ is a rectifiable arc dividing $U$ into two disjoint regions.

The last phase of our work is involved with an extension of the removable singularity problem. More precisely, suppose $A$ is a closed nowhere dense subset of $\overline{U}$ and $f$ is a function defined on $U$ and differentiable on $U-A$. What can be said about $f$ and $A$. We are able to characterize $A$ topologically if we require $f$ to be the limit of a pointwise convergent sequence of analytic functions defined on an open set containing $\overline{U}$ and $A$ to be the set of all points of $\overline{U}$ for which this sequence fails to locally converge uniformly. Three necessary and sufficient conditions are given. Our characterization is unchanged if we consider pointwise convergent sequences of open maps. Sets such as the universal plane curve are readily excluded from consideration because of an excess of "fine" structure.

Let $K$ denote the complex numbers, and $\omega$ the positive integers. For $r > 0$, let $U_r$ denote the interior of the circle $C_r$ with center $0$ and radius $r$. We shall generally write $U$ for $U_1$ and $C$ for $C_1$. For $z \in K$, let $P_x(z)$ denote the real part of $z$, and $P_y(z)$ denote the imaginary part of $z$. For $r > 0$, and $z \in K$, let $P_r(z) = z$, if $|z| \leq r$, and $P_r(z) = \frac{r z}{|z|}$ if $|z| > r$. Let $I$ denote the interval $[0,1]$, and let $Q$ denote $I \times I$. If $M$ and $N$ are subsets of $K$, we
let \( \kappa(M,N) \) denote \( \sup_{x \in M, y \in N} |x-y| \).

Let \( K \) be a compact set. By the interior \( I(M) \) of \( M \), we shall mean the union of the bounded components of \( K-M \). By the exterior \( E(K) \) of \( K \) we shall mean the union of the unbounded components of \( K-M \). By the outer boundary \( B(M) \) of \( M \), we shall mean the set \( \overline{E(M)} - E(M) \). We shall call a set \( R \subseteq K \), a circular region, if there exists a circle \( T \), such that \( R = I(T) \). If \( H \) is a subset of \( K \), then \( m(K) \) shall denote the planar Lebesgue measure of \( H \). Unless otherwise stated point sets shall be considered as subsets of \( K \).

Unless otherwise stated the range of a function defined on a subset of \( K \) shall lie in \( K \). Let \( A \subseteq B \subseteq K \) and let \( f \) be a function defined on \( B \). Then \( f|_A \) shall denote the function \( g \) on \( A \), such that \( g(z) = f(z) \) for all \( z \in A \). Let \( f \) and \( g \) be functions on subsets of \( K \), such that the range of \( g \) lies in the domain of \( f \). Then \( fg \) shall denote the function \( h \), such that \( h(z) = f(g(z)) \) for all \( z \) in the domain of \( g \). For \( z \in K \), let \( I(z) = z \). Let \( f \) be a function defined on a set \( S \) in \( K \), into \( K \). If \( f \) is continuous we shall call \( f \) a map of \( S \); \( f \) is called an open map if \( f \) is a map, and \( f(V) \) is open for all open sets \( V \subseteq S \); \( f \) is called a light map if \( f \) is nonconstant on any nondegenerate continuum in domain \( f \) (cf. [9], p. 75).\(^1\)

\(^1\)Numbers in brackets refer to correspondingly numbered bibliographical references. As above, [9] refers to reference 9 in the Selected Bibliography.
CHAPTER I

We shall first derive a topological analogue to the line integral. The analogue is motivated by the notion of analytic continuation.

Definition 1.1. Let $S$ be an open set and $F$ a collection of functions on subsets of $K$. Then the statement that $F$ is a $C_S$ collection shall mean that for $f \in F$, $f$ is a map defined on an open set $S_f \subset S$, that $S = \cup_{f \in F} S_f$, and that $f, g \in F$, $S_f \cap S_g \neq \emptyset$ implies that there exists $c \in K$, such that $f(x) = g(x) + c$ for all $x \in S_f \cap S_g$. Let $h$ be a map of the interval $[a, b]$ into $S$, and $a = t_0 < t_1 < \ldots < t_{n+1} = b$ be a subdivision of $[a, b]$. Then we shall say that $t_0 < \ldots < t_{n+1}$ is a $C^p_{F, h}$ subdivision of $[a, b]$, if $h([t_i, t_{i+1}]) \subset S_f$ for some $f \in F$, for $i = 0, 1, \ldots, n$.

Theorem 1.1. Let $S$ be an open set, $F$ a $C_S$ collection, and $h$ a map of $I$ into $S$. Then there exists a $C^p_{F, h}$ subdivision of $I$, moreover, there exists a unique number $J$ denoted by $\int_0^I F \, dh$, such that $\int 0 = t_0 < t_1 < \ldots < t_{n+1} = 1$ is a $C^p_{F, h}$ subdivision of $I$, and $f_0, f_1, \ldots, f_n$ is a collection of elements of $F$ such that $h([t_i, t_{i+1}]) \subset S_{f_i}$ for $i = 0, 1, \ldots, n$, then $J = \sum_{i=0}^n f_i h(t_{i+1}) - f_i h(t_i)$.

Proof. Since $\{S_f\}_{f \in F}$ is a collection of open sets
covering the compact set $H = h(I)$, there exists a finite subcollection $G$ of $F$, such that $H \subseteq \bigcup_{g \in G} S_g$. $\{h^{-1}(S_g \cap H)\}_{g \in G}$ is a finite collection of open sets in $I$, and there exists a subdivision $T_1 = \{t_1\}_{0}^{n+1}$ of $I$, such that for $i = 0, 1, \ldots, n$, $h([t_i, t_{i+1}]) \subseteq S_g$ for some $g \in G$. Thus $T_1$ is a $C_{f, h}$ subdivision of $I$.

Let $T_2 = \{v_1\}_{0}^{q+1}$ be a $C_{f, h}$ subdivision of $I$, and $g_0, g_1, \ldots, g_q$ elements of $F$ such that $h([v_1, v_{i+1}]) \subseteq S_{g_i}$ for $i = 0, 1, \ldots, q$. Let $T_3 = \{w_1\}_{0}^{p+1}$ be the mesh of $T_1$ and $T_2$. Trivially $J_1 = \Sigma_o^p k_i h(w_{i+1}) - k_i h(w_i) = \Sigma_o^q g_i h(v_{i+1}) - g_i h(v_i)$, where for $i = 0, 1, \ldots, p$, $k_i = g_j$ and $j$ is the integer such that $[w_i, w_{i+1}] \subset [v_j, v_{j+1}]$. Also, $J_2 = \Sigma_0^p m_i h(w_{i+1}) - m_i h(w_i) = \Sigma_0^m f_i h(t_{i+1}) - f_i h(t_i)$, where for $i = 0, 1, \ldots, p$, $m_i = f_j$ and $j$ is the integer such that $[w_i, w_{i+1}] \subset [t_j, t_{j+1}]$. Let $i \in [0, 1, \ldots, p]$. Then from Definition 1.1, there exists $c \in K$, such that $m_i(x) = k_i(x) + c$ for all $x \in S_{m_1} \cap S_{k_1} h([w_i, w_{i+1}])$. Thus $m_i h(w_{i+1}) - m_i h(w_i) = k_i h(w_{i+1}) - k_i h(w_i)$, and consequently $J_1$ must be equal to $J_2$. Thus $I_0^1 F dh$ is uniquely determined.

The following theorem shows that $J$ is invariant under homotopic deformation.

**Theorem 1.2.** Let $h$ be a map of $Q$ into an open set $S$ such that $h$ satisfies one of the following: (1) $h(0, 0) = h(0, t)$ and $h(1, 0) = h(1, t)$ for $t \in I$, or (2) $h(0, t) = h(1, t)$ for $t \in I$. Let $h_t(x) = h(x, t)$ for all $(x, t) \in Q$, and let $F$ be a $C_S$ collection. Then $I_0^1 F dh_0 = I_0^1 F dh_1$. 
Proof. If \( t \in I \), then, by Theorem 1.1., there exists a subdivision \( 0 = x_0 < \ldots < x_{n+1} = 1 \) of \( I \) and \( f_0, f_1, \ldots, f_n \in F \), such that \( h_t([x_i, x_{i+1}]) \subset S_{f_i} \) for \( i = 0, 1, \ldots, n \). Thus \( \{h_t^{-1}(S_{f_i} \cap h(Q))\}_{0}^{n} \) is a collection of open sets in \( Q \) covering \( \{t\} \times I \), and hence there exists an open set \( S_t \subset I \) containing \( t \), such that \( S_t \times [x_i, x_{i+1}] \subset h^{-1}(S_{f_i} \cap h(Q)) \) for \( i = 0, 1, \ldots, n \).

Let \( u, v \in S_t \). Then for \( i = 0, 1, \ldots, n \), there exists \( c_i \in \mathbb{R} \) such that \( f_i(z) = f_{i+1}(z) + c_i \) for \( z \in S_{f_i} \cap S_{f_{i+1}} \). Hence we have

\[
\int_{0}^{1} F \, dh_u - \int_{0}^{1} F \, dh_v = [\sum_{0}^{n} f_h u(x_{i+1}) - f_h v(x_i)] - [\sum_{0}^{n} f_h v(x_{i+1}) - f_h v(x_i)]
\]

\[
= \sum_{0}^{n} [f_h u(x_i) - f_h v(x_i)]
\]

\[
= 0.
\]

Since \( I \) is compact and \( \bigcup_{t \in I} S_t = I \), there exists a subdivision \( t_0 < \ldots < t_{m+1} \) of \( I \) such that for \( i = 0, 1, \ldots, m \), \( [t_i, t_{i+1}] \subset S_t \) for some \( t \in I \). Thus clearly \( \int_{0}^{1} F \, dh_0 = \int_{0}^{1} F \, dh_1 \).

Theorem 1.3. Let \( S \) be an open set, \( F \) a \( C_S \) collection, \( T \) a simple closed curve, \( P \) a circle with center \( z_0 \) and radius \( r \) lying in \( I(T) \), and \( h \) a map of \( M = \overline{I(T)} \setminus I(P) \) into \( S \). Let \( a \) and \( b \) be distinct points of \( T \) and let \( A \) and \( B \) be distinct subarcs of \( T \) with endpoints \( a \) and \( b \). Then \( J = \int_{a}^{b} F \, dh_1 + \int_{a}^{b} F \, dh_2 = \int_{0}^{1} F \, dhk \) where \( h_1 = h|A \), \( h_2 = h|B \), and \( k(t) = \ldots \)
proof. Let $0 < r < 1$ and $g$ be a homeomorphism of $M$ onto $N = \mathbb{U} - \mathbb{U}r$ obtained in the following manner. Let $A_1$ and $A_2$ be line segments with endpoints $z_0 + ir$ and $a$, and $z_0 - ir$ and $b$ respectively, each intersecting $T$ in exactly one point. Then (cf. [9], p. 31) there exist distinct subarcs $A_3$ and $A_4$ of $P$ with endpoints $z_0 - ir$ and $z_0 + ir$, and distinct subarcs $A_5$ and $A_6$ of $T$ with endpoints $a$ and $b$, such that the simple closed curves $T_1 = A_1UA_3UA_2UA_5$ and $T_2 = A_1UA_4UA_2UA_6$ are such that $I(T_1) \cap I(T_2) = \emptyset$ and $I(T_1) \cap I(P) = \emptyset$ for $i = 1, 2$. Similar simple closed curves $V_1$ and $V_2$ are formed for $N$.

If $g_0$ is a suitable homeomorphism of $V_1 \cup V_2$ onto $T_1 \cup T_2$, then we extend $g_0$ to the interiors of $V_1$ and $V_2$, obtaining the desired homeomorphism (cf. [9], pp. 34-35).

For $(x, t) \in Q$, let $w(x, t) = hg[(t + r - rt)E(2\pi ix)]$. Then $w(x, 1) = hgE(2\pi ix)$ for $x \in I$, and $w(0, t) = w(1, t)$ for $t \in I$. Hence from Theorem 1.2, since $w_0$ and $w_1$ are homeomorphisms, $J = \pm I_0^1 F \, dw_1 = I_0^1 F \, dw_0 = \pm I_0^1 F \, dhk$.

To handle the case involving $v$, we take $w(x, t) = hg[t \cdot E(2\pi ix)]$ for $(x, t) \in Q$. In this case $J = \pm I_0^1 F \, dw_0 = 0$, where $w_0(x) = w(x, 0) = hg(0)$ for all $x \in I$.

In Theorem 1.4 we show to what extent $J$ is independent of the choice of path. In Theorem 1.5 we apply Theorem 1.4 to obtain an analogue to the Monodromy theorem.
Theorem 1.4. Let $S$ be a connected and simply connected open set, $F$ a $C_0$ collection, and $h$ and $k$ maps of $I$ into $S$, such that $h(0) = k(0)$ and $h(1) = k(1)$. Then $\int_0^1 F \, dh = \int_0^1 F \, dk$.

Proof. $M = h(I) \cup k(I)$ is a subcontinuum of $S$ and $\delta(M, K - S) = \epsilon > 0$. It follows from the Zoretti Theorem (cf. [9], p. 35), that there exists a simple closed curve $T$, such that $M \subset I(T)$, and $\delta(x, K - S) > \epsilon/2$ for $x \in T$. Thus $T \subset S$ and hence $I(T) \subset S$. There exists a homeomorphism $w$ of $I(T)$ onto $U$ (cf. [9], p. 38). For $(x, t) \in Q$, let $h(x, t) = w^{-1}[t \cdot w(k(x)) + (1-t)w(h(x))]$. Clearly $h$ is a map of $Q$ into $S$.

Now $h_0(x) = w^{-1}[0 \cdot w(k(x)) + (1-0)w(h(x))] = w^{-1}[w(h(x))] = h(x)$ for $x \in I$. Similarly $h_1(x) = k(x)$ for $x \in I$. Hence from Theorem 1.2, $\int_0^1 F \, dh = \int_0^1 F \, dh_0 = \int_0^1 F \, dh_1 = \int_0^1 F \, dk$.

Theorem 1.5. Let $S$ be a connected and simply connected open set and $F$ a $C_0$ collection. Then there exists a map $g$ of $S$ into $K$ such that if $f \in F$ and $R$ is a component of $S_f$, then there exists $c \in K$ such that $f(x) = g(x) + c$ for all $x \in R$.

Proof. Let $x_0 \in S$, $g(z_0) = 0$, and $z \in S - \{z_0\}$. Since $S$ is connected, there exists an arc $A_z \subset S$ with endpoints $z_0$ and $z$. Let $h_z$ be a homeomorphism of $I$ onto $A_z$ such that $h_z(0) = z_0$ and $h_z(1) = z$. Set $g(z) = \int_0^1 F \, dh_z$.

Let $x_0 \in S$. There exists $f \in F$ such that $x_0 \in S_f$. Let $y$ be a point of the component $R$ of $S_f$ containing $x_0$. There exists an arc $B \subset R$ with endpoints $x_0$ and $y$. If $k$ is a
homeomorphism of $I$ onto $B$ such that $k(0) = x_0$ and $k(1) = 1$, then from Theorem 1.4,
\[ g(y) = \int_0^1 F \, dh_y = \int_0^1 F \, dh_{x_0} + \int_0^1 F \, dk \]
\[ = \int_0^1 F \, dh_{x_0} + \{ f(y) - f(x_0) \}. \]
If $c = \int_0^1 F \, dh_{x_0} - f(x_0)$, then for all $y \in R$, we have $g(y) = f(y) + c$. Thus $g$ is the desired function.

So far strictly speaking we have been dealing with an analogue of the Stieltjes integral. We shall now show that an analogue to the integral along an arc falls out of our initial definition.

**Theorem 1.A.** Suppose $S$ is an open set, $F$ a $C^1$ collection, and $A$ an arc in $S$ with endpoints $a$ and $b$. Then there exists a unique number $\int_a^b F \, dz$ such that if $h$ is a map of $I$ onto $A$ satisfying $h(0) = a$ and $h(1) = b$, then $\int_a^b F \, dz = \int_0^1 F \, dh$.

**Proof.** If $n \in \omega$, then from the Zoretti Theorem, there exists a simple closed curve $C_n$ such that $A \subset (C_n)$ and $\delta(x,A) < 1/n$ for all $x \in C_n$. Suppose $x \notin K \cdot A$. Since $K \cdot A$ is connected (cf. [9], p. 29), there exists an arc $B$ with endpoints $x$ and $y$, such that $B \cap A = \emptyset$ and $|y| - \sup_{t \in A} |t| = 2$. Hence $y \notin H_n = C_n \cup I(C_n)$ for all $n \in \omega$. For $n > \Lambda(a,b)^{-1}$, $n \in \omega$, we have $\Lambda(z,A) < 1/n < \Lambda(A,B)$ for all $z \in C_n$, and hence $C_n \cap B = \emptyset$; consequently $x \notin H_n$. Thus $\cap_1^\infty H_n = A$. Now $\cap_1^\infty [H_n \cap (K-S)] = (K-S) \cap (\cap_1^\infty H_n) = (K-S) \cap A \subset (K-S) \cap S = \emptyset$; also $H_{n+1} \cap (K-S) \subset H_n \cap (K-S)$ for all $n \in \omega$. Hence there exists $n_0 \in \omega$ such that $H_{n_0} \cap (K-S) = \emptyset$, and consequently $H_{n_0} \subset S$. 
Let $h$ and $k$ be maps of $I$ onto $A$ such that $h(0) = k(0) = a$ and $h(1) = k(1) = b$. Then from Theorem 1.4, $\int_{a}^{b} F \, dh = \int_{0}^{1} F \, dk$, and hence $\int_{a}^{b} F \, dz$ is uniquely defined.

**Definition 1.2.** Let $S$ be an open set, $z_0 \in K$, and $F$ a collection of functions on subsets of $K$. Then the statement that $F$ is an $L_{S, z_0}$ collection means that $F$ is a $C_{S, \{z_0\}}$ collection such that $Ef(z) = z - z_0$ for $f \in F$ and $z \in S_f$.

**Theorem 1.7.** If $S$ is an open set in $K$ and $z_0 \in K$, then there exists an $L_{S, z_0}$ collection. Moreover if $h$ is a map of $I$ into $K$ and $z_0 \in K - h(I)$, then there exists a unique number $J$, such that if $S$ is an open set containing $h(I)$ and $F$ is an $L_{S, z_0}$ collection, then $J = \int_{0}^{1} F \, dh = 2\pi i \, \mu_{I}(h,z_0)$, where $\mu$ denotes Whyburn's topological index (cf. [9], p. 58). Finally, if $h(0) = h(1)$, then $J = 2\pi i n$ for some integer $n$.

**Proof.** For $z \in K$, let $F(z) = z - z_0$ and $y \in S - \{z_0\}$. Then $P(y) \neq 0$, and there exists $x \in K$, such that $E(x) = P(y)$. Now there exists an open set $V$ containing $x_0$ such that $E$ is a homeomorphism on $V$. Since $E$ is an open map, $E(V)$ is an open set containing $P(y)$, so there exists a circular region $R_y$ containing $y$ and lying in $S : P^{-1}(E(S))$. Hence there exists a map $f$ of $P(R_y)$ into $K$, such that $Ef(z) = z$ for all $z \in P(R_y)$. Let $f_y$ denote the function $fP$. Then $Ef_{y}(z) = EfP(z) = P(z) = z - z_0$ for all $z \in R_y$.

Let $u, v \in S$ and $z \in R_u \cap R_v$. Then $Ef_u(z) = Ef_v(z) = z - z_0$, and thus $E[f_u(z) - f_v(z)] = (z - z_0)/(z - z_0) = 1$, and
consequently \( f_u(z) = f_v(z) + 2\pi n(z)1 \), where \( n(z) \) is an integer. Then \( n = (\pi i)^{-1}(f_u - f_v) \) is a continuous function on the connected set \( R_u \cap R_v \) and hence is constant. Thus \( F = \{ f_y \mid y \in \mathbb{S} \} \) is an \( L_{C,z_0} \) collection.

Let \( F \) and \( G \) be \( L_{C,z_0} \) collections and for \( t \in I = \{ 1, \ldots, n \} \), set \( v(t) = \int_0^t F \, dh \) and \( w(t) = \int_0^t G \, dh \). Then clearly \( v \) and \( w \) are continuous. Now for \( t \in I \), there exist a collection \( f_0, f_1, \ldots, f_n \), and a subdivision \( 0 = x_0 < \ldots < x_{n+1} = t \) of \( [0,t] \), such that \( \int_0^t F \, dh = \sum_{1}^{n} f_1 h(x_{i+1}) - f_1 h(x_i) \), and hence \( E(v(t)) = E[\sum_{1}^{n} f_1 h(x_{i+1}) - f_1 h(x_i)] = \prod_{0}^{n} E f_1 h(x_{i+1}) / E f_1 h(x_i) = \prod_{1}^{n} (x_{i+1} - z_0)(x_i - z_0)^{-1} = [h(t) - z_0][h(0) - z_0]^{-1} \). Similarly for \( t \in I \), \( E(w(t)) = [h(t) - z_0][h(0) - z_0]^{-1} \) and hence \( E[v(t) - w(t)] = 1 \). Since \( v - w \) is continuous, there exists an integer \( n \) such that \( v(t) - w(t) = 2\pi ni \) for all \( t \in I \). Now \( v(0) = 0 \) and \( w(0) = 0 \), and hence \( n = 0 \). Thus \( v = w \) and \( J \) is uniquely defined.

Suppose \( h(0) = h(1) \). Then \( E(J) = E(v(1)) = [h(1) - z_0][h(0) - z_0]^{-1} = 1 \) and \( J = 2\pi ni \) for some integer \( n \).

Let \( c \) be a number such that \( E(c) = h(0) - z_0 \). Then for \( t \in I \), let \( Q(t) = v(t) + c \). Then \( E(Q(t)) = E[v(t) + c] = E[v(t)] \cdot E(c) = [h(t) - z_0][h(0) - z_0]^{-1} \cdot [h(0) - z_0] = h(t) - z_0 \). Thus \( J = m_1(h,z_0) \) (cf. \cite{9}, pp. 56-58).

**Theorem 1.8.** Let \( z_0 \in K \), and let \( S \) be a connected and simply connected open set excluding \( z_0 \). Then there exists a map \( g \) of \( S \) into \( K \), such that \( Eg(z) = z - z_0 \) for all \( z \in S \). Furthermore there exists a map \( k \) of \( S \) into \( K \), such that
\[ k(z)^2 = \bar{z}z_0 \text{ for all } z \in S. \]

**Proof.** From Theorem 1.7, there exists an \( L_{S,z_0} \) collection. Since \( z_0 \notin S \), the existence of the desired function \( g \) follows from Theorem 1.5.

Let \( k(z) = E[2^{-1}g(z)] \) for all \( z \in S \). Then \( k(z)^2 = E[2^{-1}g(z)]^2 = Eg(z) = \bar{z}z_0 \) for \( z \in S \).

**Theorem 1.9:** Suppose \( T \) is a simple closed curve, \( a \) and \( b \) distinct points of \( T \), and \( A \) and \( B \) distinct subarcs of \( T \) with endpoints \( a \) and \( b \). Then there exists \( n = \pm 1 \), such that for \( z_0 \in I(T) \), \( J = u_\alpha^b_A(z,z_0) + u_\beta^a_B(z,z_0) = u_\alpha^b_A(z,z_0) - u_\beta^b_B(z,z_0) = n \), where \( u_\alpha^b_A(z,z_0) \) denotes the unique number \( I_{a\alpha}^L \) \( dz \), where \( L \) is an \( L_{S,z_0} \) collection.

**Proof.** Let \( L \) be an \( L_{S,z_0} \) collection. Then from Theorem 1.3, \( J = I_{a\alpha}^L \) \( dz \) + \( I_{b\beta}^L \) \( dz \) = \( n \) \( I_{0\alpha}^L \) \( dk \) = \( n u_1(k,z_0) \), where \( k(t) = rE(2\pi it) + z_0 \) for some \( r > 0 \), \( n = \pm 1 \), for all \( t \in I \). Let \( p(t) = \log r + 2\pi it \) for \( t \in I \). Then \( Ep(t) = E(\log r + 2\pi it) = k(t) - z_0 \) for \( t \in I \). Thus \( J = n[p(1) - p(0)] = 2\pi n i \).

Let \( z_1 \in I(T) \), \( z_1 \notin z_0 \), and let \( W \) be an arc in \( I(T) \) with endpoints \( z_0 \) and \( z_1 \). Let \( h \) be a homeomorphism of \( I \) onto \( W \) such that \( w(0) = z_0 \) and \( w(1) = z_1 \), and \( v \) a continuous function on \( I \) onto \( T \) such that \( v(0) = a, \ v(1/4) \in A, \ v(1/2) = b \), and \( v \) is one-to-one on \([0,1]\). Set \( w(x,t) = v(x) - h(t) + z_0 \) for \((x,t) \in Q \). Now \( x_0 \notin w(Q) \), since \( W \cap T = \emptyset \), and \( w(x,0) = v(x) \) and \( w(x,1) = v(x) + z_0 \) for \( x \in I \). Then from
Theorem 1.2. $J = \int_0^1 L \, dw_0 = \int_0^1 L \, dw_1 = \mu_1(v+z_0, z_1, z_0) = \mu_1(v, z_1)$. Thus $J$ is uniquely determined.

**Theorem 1.10.** Let $S$ be an open set, $F$ a $C_2$ collection, $T$ a simple closed curve, and $h$ a map of $T$ into $S$. Then there exists a unique number $J = \int_T F \, dh$ such that if $a$ and $b$ are distinct points of $T$, and $A$ and $B$ are distinct subarcs of $T$ with endpoints $a$ and $b$, then $\int_a^b F \, dh_1 + \int_a^b F \, dh_2 = \int_a^b F \, dz + \int_a^b F \, L \, dz$ for all $L$, where $z_0 \in I(T)$, and $h_1 = h|A$ and $h_2 = h|B$.

**Proof.** The proof follows readily from Theorem 1.9.

**Remark.** Let $L$ be an $L_{K,0}$ collection and $h$ and $k$ maps of $I$ into $K - \{0\}$. Then for $t \in I$, $E(I_0^t L \, dh \cdot k) = [h(t)k(t)]$ $[h(o)k(o)]^{-1} = [h(t)/h(o)] \cdot [k(t)/k(o)] = E(I_0^t L \, dh) \cdot E(I_0^t L \, dk) = E(I_0^t L \, dh + I_0^t L \, dk)$, and thus $I_0^t L \, dh \cdot k = I_0^t L \, dh + I_0^t L \, dk + 2\pi m$ for some integer $m$. Now $I_0^t L dp = 0$, when $t = 0$ for $p = h, k, h \cdot k$ and hence $m$ must equal zero. With this observation we have completed the derivation of the machinery of Whyburn's topological index $\mu$.

**Theorem 1.11.** Let $A$ be a closed and nowhere dense subset of $U$, and $f$ a map of $U$, such that $f$ is differentiable on $U - A$ and $f(A)$ is nowhere dense in $K$. Then if $f$ is light, $f$ is an open map on $U$. Moreover if $f$ is light and $p \in f(U) - f(C)$, then $f^{-1}(p)$ has at most $m = \mu_C(f, p)$ elements. Finally if $f$ is light on $U - A$, then $\sup_{x \in U} |f(x)| \leq \sup_{t \in C} |f(t)|$, and if $f$ is light $|f(x)| < \sup_{t \in C} |f(t)|$ for all $x \in U$. 
Proof. Suppose $f$ is light on $U-A$ and $T$ is a simple closed curve in $\overline{U}$. Let $q \in f(H) - f(T)$, where $H = T \cup U \cap I(T)$, and let $S$ be the component of $K - f(T)$ containing $q$. We shall show that $S \subset f(H)$. Set $Q = f(H) \cap S$. Then $Q$ is open in the relative topology of $f(H)$, and hence $f^{-1}(Q)$ is open in $H$. Since $f^{-1}(Q) \cap T = \emptyset$, $f^{-1}(Q)$ is open in $K$. Now $Q_0 = f^{-1}(Q) - A$ is a non-empty open set in $U-A$, and hence $f(Q_0)$ is open in $K$. Let $G$ be the set of all points $x \in U-A$, such that $f'(x) = 0$. Then (cf. [9], pp. 72-73) $m(G) = 0$ and since $f(A)$ is nowhere dense in $K$, we have $Q_1 = f(Q_0) - G - f(A) \neq \emptyset$. Let $y \in Q_1$. Then paralleling Whyburn (cf. [9], pp. 67-68 and 72-74, and cf. Theorem 1.9) we see that $f^{-1}(y)$ has a finite number $n_0$ of elements, where $\mu_T(f,y) = n_0 \neq 0$. Hence $\mu_T(f,z) = n_0$ for all $z \in S$. It follows now from Theorem 1.9 that $S \subset f(H)$.

Let $S$ be the unbounded component of $K - f(C)$ and suppose $S \cap f(\overline{U}) \neq \emptyset$. Then from the above argument, $S \subset f(\overline{U})$. But $f(\overline{U})$ is compact. Thus $\sup_{x \in U} |f(x)| \leq \sup_{t \in C} |f(t)|$.

Suppose $f$ is light. We shall show that $f$ is an open map on $U$. Let $x_0 \in U$ and $R$ be a circular region with center $x_0$ lying in $U$. From the Zoretti Theorem, there exists a simple closed curve $T \subset R$, such that $x_0 \in I(T)$ and $T \cap f^{-1} f(x_0) = \emptyset$. Let $S$ denote the component of $K - f(T)$ containing $f(x_0)$ and lying in $f(\overline{I(T)}) \subset f(R)$. Thus if $V$ is an open set in $U$, $f(V)$ must be open. In particular $f(U)$ is open, and hence $|f(x)| < \sup_{t \in C} |f(t)|$ for all $x \in U$.

Suppose $f^{-1}(p) \cap I(T)$ contains more than $m$ elements.
Let $x_1, x_2, \ldots, x_{m+1}$ be distinct points of $f^{-1}(p) \cap I(T)$, and $R_1, R_2, \ldots, R_{m+1}$ be a collection of mutually disjoint circular regions in $U$, such that $R_i$ has center $x_i$ for $i = 1, 2, \ldots, m+1$. Then $f(R_i)$ is open in $K$ for $i = 1, 2, \ldots, m+1$, and hence $V = S \cap \bigcap_{i=1}^{m+1} f(R_i)$ is a non-empty open set in $K$ lying in $f(U)$ and containing $p$, where $S$ is the component of $K - f(C)$ containing $p$. Suppose $q \in V - f(A) - G$. Paralleling Whyburn, as above, we find that $u_C(f, q) \geq m+1$. But $u_C(f, q) = u_C(f, p) = m$. Thus $f^{-1}(p)$ has $m$ or fewer elements.

**Theorem 1.12.** Let $T$ be a simple closed curve, $S$ a set containing $R = I(T)$, $G$ a finite subset of $R$, $x_0 \in R - G$, and $f$ a map of $S$ such that $f$ is differentiable on $R - G$. For $x \in S$, let $Q_{f, x_0}(x) = (f(x) - f(x_0))(x - x_0)^{-1}$ for $x \neq x_0$, and $Q_{f, x_0}(x) = f'(x_0)$ for $x = x_0$. Set $\varphi = Q_{f, x_0}$. Then $\varphi$ is continuous on $R$, differentiable on $R - G - \{x_0\}$, and such that $|\varphi(x)| < \sup_{t \in T} |\varphi(t)|$ for all $x \in R$, if $\varphi$ is non-constant.

**Proof.** By definition of derivative $\varphi$ is continuous on $S$. Suppose $\varphi$ is non-constant. Then since $\varphi$ is differentiable on $R - G - \{x_0\}$, and $G$ is finite, $\varphi|R$ is light. Hence from Theorem 1.11, $\varphi$ is an open map on $R$, and $|\varphi(x)| < \sup_{t \in T} |\varphi(t)|$ for all $x \in R$.

**Theorem 1.13.** Let $A$ be a finite subset of $U$ and $0 < r < 1$. If $f$ is a map of $U$ such that $f$ is differentiable on $U - A$, then there exists $p > 0$, such that the function $g(z) =$
p \cdot f(z) + z \text{ for } z \in U \text{ is one-to-one on } \overline{U_r}.

\text{Proof.} \text{ Let } r < r_0 < 1, \ M = \sup_{z \in \overline{U_{r_0}}} |f(z)|, \text{ and } p = (r_0 - r)/2(M+1). \text{ For } t \in I, \text{ set } v(t) = r_0 E(2\pi t) \text{ and for } (s,t) \in \mathcal{U}, \text{ set } h(s,t) = sp \cdot f v(t) + v(t). \text{ Then } h \text{ is continuous, } h_0(t) = v(t), \text{ and } h_1(t) = p \cdot f v(t) \text{ for all } t \in I. \text{ Now for } (s,t) \in \mathcal{Q}, |h(s,t)| = |v(t) + sp \cdot f v(t)| \geq r_0 - sp|f v(t)| \geq r_0 - s(r_0 - r)/2 = (r_0 + r)/2. \text{ Thus } 0 \notin h(\mathcal{Q}), \text{ and hence from Theorems 1.2 and 1.9, } 1 = u(v,0) = \rho(h_0,0) = \rho(h_1,0) = \rho(p \cdot f v + v,0).

\text{Now } g \text{ is continuous and differentiable on } U - A. \text{ Since } g \text{ is not constant and } A \text{ is finite, } g \text{ must be light. Suppose } S \text{ is the component of } K - g(C_p) \text{ containing } C. \text{ Now for } z \in C_{r_0}, g(z) = |z + p \cdot f(z)| \geq r_0 - (r_0 - r)/2 = (r_0 + r)/2, \text{ and thus } U(r_0 + r)/2 \subset S. \text{ For } z \in \overline{U_r}, |g(z)| = |p \cdot f(z) + z| < (r_0 - r) 2^{-1} + r = (r_0 + r)/2, \text{ so that } f(\overline{U_r}) \subset S, \text{ and hence } \overline{U_r} \subset f^{-1}(S). \text{ For } x \in S, \rho(gv, x) = \rho(gv, 0) = 1, \text{ and hence from Theorem 1.11}, f^{-1}(x) \text{ has exactly one element. Consequently } g \text{ is one-to-one on } f^{-1}(S) \ni \overline{U_r}.

\text{Theorem 1.14. Let } A \text{ be an arc, } f \text{ a map of } K \text{ such that } f \text{ is non-constant and differentiable on } K - A, \text{ and } s = \lim_{x \to \infty} f(x) \text{ exists. Then } f(A) = f(K).

\text{Proof.} \text{ Suppose } p \in f(K) - f(A) \text{ and } p \neq s. \text{ Let } P \text{ be a circle with center } s \text{ such that } p \in E(P). \text{ There exists } r_0 > 0 \text{ such that } A \cup f^{-1}(p) \subset U_r \text{ for } r > r_0. \text{ Since } \lim_{x \to \infty} f(x) = s, \text{ there exists } r_1 > 0, \text{ such that } r > r_1 \text{ implies } f(C_r) \subset I(P). \text{ Let } r > \sup[r_0, r_1] \text{ and } B \text{ be an arc in } \overline{U_r} \text{ with endpoints}
a and b, such that \( B \cap C_r = \{a\} \cup \{b\} \), and \( A \subseteq B \). Let \( D_1 \) and \( D_2 \) be the components of \( U - B \), and \( T_i = B_i - D_i \) for \( i = 1, 2 \). Then from Theorem 1.11, \( \mu_{C_r}(f, p) = \mu_{T_1}(f, p) + \mu_{T_2}(f, p) \geq n_0 \), where \( n_0 \neq 0 \) is the number of elements of \( f^{-1}(p) \). But \( f(C_r) \subseteq I(P) \), and hence from Theorem 1.9, \( \mu_{C_r}(f, p) = 0 \). Thus \( f(K) = \{s\} \subseteq f(A) \) and \( f(A) = f(K) \).
CHAPTER II

This chapter shall center around the problem of obtaining polynomial approximations to differentiable functions; in particular power series expansions.

Lemma 2.1. Let \( f \) and \( g \) be polynomials such that \( g(z) \neq 0 \) for all \( z \in \bar{U} \), \( S \) an open set in \( U \), and \( h(z) = f(z)/g(z) \) for \( z \in \bar{U} \). Then if \( P_1, P_2, \ldots \) is a sequence of polynomials converging uniformly on compact subsets of \( h(S) \) to a limit function \( F \), there exists a sequence of polynomials \( Q_1, Q_2, \ldots \) converging uniformly on compact subsets of \( S \) to \( F h \).

Proof. Clearly for \( n \in \omega \), there exist polynomials \( f_n \) and \( g_n \) such that \( g_n(z) \neq 0 \) for all \( z \in \bar{U} \), and \( P_n h(z) = f_n(z)/g_n(z) \) for all \( z \in \bar{U} \). Let \( n \in \omega \). There exists a finite collection of numbers \( a_0, z_1, z_2, \ldots, z_p \) such that \( g_n(z) = a_0(z_1-z)(z_2-z) \ldots (z_p-z) \) for all \( z \in K \). [Note: for the Fundamental Theorem of Algebra cf. Whyburn [9], p. 77.] Since \( g(z) \neq 0 \) for all \( z \in \bar{U} \), we have \( |z_k| > 1 \) for \( k = 1, 2, \ldots, p \). Hence for \( k = 1, 2, \ldots, p \) the sequence of polynomials \( \{T_{km}\}_{m=1}^{\infty} \) converges uniformly on \( \bar{U} \) to \( (z_k-z)^{-1} = z_k^{-1}(1-z/z_k)^{-1} \), where \( T_{km}(z) = \sum_{j=0}^{m} (z/z_k)^j \) for \( k = 1, 2, \ldots, p \), \( m \in \omega \), and \( z \in K \). For \( m \in \omega \), let \( Q_{nm}(z) = a_0^{-1} f_n(z) T_{1m}(z) T_{2m}(z) \ldots T_{pm}(z) \) for all \( z \in K \). Then \( \{Q_{nm}\}_{m=1}^{\infty} \) is a sequence of
polynomials converging uniformly on $\overline{U}$ to $f_n/g_n$.

Suppose $p_1 < p_2 < \ldots$ is an increasing sequence in $\omega$ such that $|Q_{n^2}(z) - P_n h(z)| < 1/n$ for $n \in \omega$, $z \in \overline{U}$, $M$ a compact subset of $S$, and $\varepsilon > 0$. Since $h$ is continuous on $S$, $h(M)$ is compact, and there exists $N > 0$, such that $n > N$, $n \in \omega$, implies $1/n < \varepsilon/2$ and $|F(x) - P_n(x)| < \varepsilon/2$ for all $x \in h(M)$. Thus for $z \in M$ and $n > N$, $n \in \omega$, we have

$$|Fh(z) - Q_{n^2}(z)| \leq |Fh(z) - P_n h(z)| + |P_n h(z) - Q_{n^2}(z)| < 1/n + \varepsilon/2 < \varepsilon.$$ 

Lemma 2.2. Let $F$ be a uniformly bounded collection of differentiable functions on an open set $S$. Then $F$ is an equicontinuous family of functions.

Proof. Let $p \in S$ and $T$ be a circle with radius $r$ and center $p$ such that $H = T \cup I(T) \subset S$. There exists $M > 0$ such that $|f(z)| < M$ for all $z \in H$ and $f \in F$. If $f \in F$ and $z \in H$, then from Theorem 1.12, $|Q_{f,p}(z)| = \sup_{t \in T} |Q_{f,p}(t)| = \sup_{t \in T} |f(t) - f(p)| r^{-1} \leq 2M/r$ and thus $|f(z) - f(p)| \leq |z-p| 2Mr^{-1}$. Consequently $F$ must be an equicontinuous family of functions.

Remark. Lemma 2.2 immediately enables us to obtain the Vitali-Porter-Stieltjes Theorem. Hurwitz's Theorem and the standard Maximum Modulus Theorem follow immediately from the open mapping theorem for differentiable functions.

(Cf. Whyburn [9], pp. 72-76.)

Lemma 2.3. Suppose $f_1, f_2, \ldots$ is a sequence of
differentiable functions defined on an open set S, converging uniformly on compact subsets of S to a limit function F. Then F is differentiable and \( \{ f_p'(z) \}_{p=1}^{\infty} \) converges to \( F'(z) \) for all \( z \in S \).

**Proof.** The proof of this lemma is due to Porcelli and Connell (cf. [1] and [5]). Let \( p \in S \) and let \( T \) be a circle with center \( p \) and radius \( r \) such that \( D = T \cup I(T) \subset S \). Let \( Q = Q_{p,p} \) and for \( n \in \mathbb{N} \), let \( Q_n = Q_{f_n,p} \). Then for \( z \in D - \{ p \} \), \( \{ Q_n(z) \}_{n=1}^{\infty} \) converges to \( Q(z) \). From Theorem 1.12, we have

\[
|Q_n(z) - Q_m(z)| \leq \sup_{t \in T} |Q_n(t) - Q_m(t)| \leq \sup_{t \in T} |f_n(t) - f_m(t)| r^{-1}
\]

for \( z \in D \). Since \( \{ f_n \}_{n=1}^{\infty} \) converges uniformly on \( T \), we see that \( \{ Q_n \}_{n=1}^{\infty} \) converges uniformly on \( D \) to a limit function \( Q_0 \). Clearly \( Q_0(z) = Q(z) \) for all \( z \in D - \{ p \} \). Hence F is differentiable at \( p \) and \( F'(p) = Q_0(p) \). Moreover \( F'(p) \)

\[
= \lim_{n \to \infty} Q_n(p) = \lim_{n \to \infty} f_n'(p).
\]

**Lemma 2.4.** Suppose \( f_1, f_2, \ldots \) is a sequence of one-to-one differentiable functions on a simply connected bounded open set S into \( U \), converging uniformly on compact subsets of S to a limit function F non-constant on each component of S. Then F is a one-to-one differentiable function such that \( F(S) \subset U \). Moreover if \( M \) is a compact subset of \( f(S) \), there exists \( N > 0 \) such that \( n \geq N \), \( n \in \mathbb{N} \), implies \( M \subset f_n(S) \). Furthermore \( \{ f_n^{-1} \}_{n=1}^{\infty} \) converges uniformly to \( F^{-1} \) on \( M \).

**Proof.** By Lemma 2.3, F must be differentiable. Since F is non-constant on each component of S, by Hurwitz's
Theorem, $F$ must be one-to-one. From the Zoretti Theorem, there exists a simple closed curve $G \subseteq f(S)$ such that $M \subseteq I(G)$. Since $S$ is simply connected, $F(S)$ is simply connected, and hence $D = G \cup I(G) \subseteq F(S)$. Applying the Zoretti Theorem again, we obtain a simple closed curve $H \subseteq F(S)$, such that $D \subseteq I(H)$. Then $I(H) \subseteq F(S)$.

Let $J = F^{-1}(H)$. Now $F^{-1}[I(H)]$ is open in $K$ and hence $J - F^{-1}[I(H)] \neq \emptyset$. Let $W$ be a component of the complement of $J$ intersecting $F^{-1}[I(H)]$. Then $F^{-1}[I(H)] \cap W$ is open in $W$. Moreover $F^{-1}[I(H)] \cap W = F^{-1}[I(H)] \cap W$ is closed in $W$. Hence, since $W$ is connected, $W \subseteq F^{-1}[I(H)]$. Thus $F^{-1}[I(H)] = I(J)$.

There exists $N > 0$, such that $n > N$, $n \in w$, implies $|f_n(z) - F(z)| < \delta(H,D)/2$ for all $z \in J$, and hence $\delta(J_n,D) > \delta(H,D)$, where $J_n = f_n(J)$. Let $p \in F^{-1}(D)$. Then $p \in I(J)$. If for some $n > N$, $n \in w$, $D \cap E(J_n) \neq \emptyset$, then $D \subseteq E(J_n)$. If $L_n$ denotes the line segment with endpoints $f_n(p)$ and $F(p)$, then $F(p) \in D \subseteq E(J_n)$ and $f_n(p) \in I(J_n)$. Thus $L_n \cap I(J_n) \neq \emptyset$, and hence $\delta(J_n,D) < |F(p) - f_n(p)|$. There exists $M > N$, such that $n \geq M$, $n \in w$, implies $|F(p) - f_n(p)| < \delta(H,D)/2$. Hence for $n \geq M$, $n \in w$, we have $D \subseteq I(J_n)$ and consequently $D \subseteq F(S)$.

If $x_0 \in D$, then for $n \geq M$, $n \in w$, $x_0 \in I(J_n)$ and $f_n^{-1}(x_0) \in J$. Suppose that $\{f_n^{-1}(x_0)\}_{n=M}^{\infty}$ does not converge to $F^{-1}(x_0)$. Then there exist $p_1 < p_2 < \ldots$ in $w$ and a point $y_0 \in J \cup I(J)$ distinct from $F^{-1}(x_0)$, such that $\{f_{p_n}^{-1}(x_0)\}_{n=1}^{\infty}$ converges to $y_0$. Hence $\{f_{p_n}(y_0)\}_{n=1}^{\infty}$ converges to $x_0$ and $F(y_0) = x_0$. 
Since $F[F^{-1}(x_0)]$ is also equal to $x_0$ and $F^{-1}(x_0) \neq y_0$, we have that $F$ is not one-to-one, which is a contradiction.

Since $S$ is bounded, it follows from the Vitali-Porter-Stieltjes Theorem, that \( \{f_n^{-1}\}_{n=1}^{\infty} \) converges on compact subsets of $I(G)$; in particular \( \{f_n^{-1}\}_{n=1}^{\infty} \) converges uniformly on $D$. Let $n \in \omega$. The differentiability of $f_n^{-1}$ follows readily from the fact that $f_n'(z) \neq 0$ for all $z \in S$.

Let $z_0 \in S$ and $T$ be a circle with center $z_0$ lying in $S$.

Suppose $f_n'(z_0) = 0$ and set $w_0 = f(z_0)$ and $Q = Q_{f_n, z_0}$. Then $Q(z_0) = 0$, and hence from Theorem 1.11, $\mu_T(Q, 0) > 0$. Now $f_n(z) - w_0 = (z - z_0)Q(z)$ for $z \in S$, and hence from the remark preceding Theorem 1.11, $\mu_T(f_n, w_0) = \mu_T(z, z_0) + \mu_T(Q, 0) = 1 + \mu_T(Q, 0) > 1$. But since $f_n$ is a homeomorphism, $\mu_T(f_n, w_0) = 1$. (cf. [9], pp. 74-75 and 84-85.)

If $F(S) = U \neq \emptyset$, then since $F$ is an open map, there exists $x_0 \in S$ such that $|F(x_0)| > 1$. But then for some $n \in \omega$, we would have $|f_n(x_0)| > 1$, which is impossible. Thus $F(S) \subseteq U$.

**Lemma 2.5.** (cf. [9], pp. 225-230) Let $S$ be a connected and simply connected open set in $U$, such that $0 \in S$ and $S \subseteq U$. Then there exist polynomials $f$ and $g$, such that $g(z) \neq 0$ for all $z \in \bar{U}$, and a one-to-one differentiable function $h$ on $S$ into $U$ such that $h'(0) > 1$, $h(0) = 0$, and $fh(z)/gh(z) = z$ for all $z \in S$.

**Proof.** If $t \in U-S$, then for $z \in \bar{U}$, $|1 - \bar{t}z| > 1 - |ar{t}z| \geq 1 - |t| > 0$. For $z \in U$, let $A(z) = [t-z][1-\bar{t}z]^{-1}$. Then by
direct computation $A(z) = z$ for all $z \in \mathbb{U}$. Thus $A$ is one-to-one and $A(\mathbb{U}) = \mathbb{U}$. Trivially $A(0) = t$ and $A(t) = 0$, and hence $0 \notin A(S)$. Since $A$ is differentiable on $\mathbb{U}$, $A$ is an open map on $S$, and consequently $A(S)$ is a connected and simply connected open set in $\mathbb{U}$ not containing 0.

From Theorem 1.8, there exists a one-to-one differentiable function $H$ on $A(S)$ into $K$, such that $H(z)^2 = z$ for $z \in A(S)$. Then $|H(t)|^2 = |t| < 1$, and hence $|H(t)| < 1$. Then for $z \in \mathbb{U}$, $1 - H(t)z \neq 0$. Define $B(z) = [H(t) - z][1 - H(t)z]^{-1}$ for all $z \in \mathbb{U}$.

For $z \in S$, let $P(z) = BHA(z)$. $P$ is one-to-one and differentiable and $P(S) \subset \mathbb{U}$. Now $P(0) = BHA(0) = BH(t) = 0$.

For $z \in \mathbb{U}$, let $K(z) = z^2$. Then for $z \in \mathbb{U}$, let $Q(z) = AKB(z)$; $Q$ is differentiable and $QP(z) = AKBBHA(z) = AKHA(z) = AA(z)$ = $z$ for $z \in \mathbb{U}$. Then $P'(0)Q'(0) = 1$.

For $z \in \mathbb{U}$, set $Q_0(z) = Q(z)/z$ for $z \neq 0$ and $Q_0(z) = Q'(z)$ for $z = 0$. Then $Q_0$ is continuous on $\mathbb{U}$ and $Q$ is differentiable on $\mathbb{U} - \{0\}$. $Q$ is not one-to-one on $\mathbb{U}$ and hence $Q_0$ can not be constant on $\mathbb{U}$. From Theorem 1.11 (Cf. also Theorem 1.12), $|Q_0(z)| < \sup_{t \in \mathbb{C}} |Q_0(t)|$ for all $z \in \mathbb{U}$. In particular, $|Q'(0)| = |Q_0(0)| < 1$. Thus $|P'(0)| > 1$, and if $s = \bar{P'(0)}/|P'(0)|$, then $sP(z)$ is the desired function.

The proof of the following theorem is adapted from a proof of the Riemann Mapping Theorem given by Saks and Zygmund (Cf. [6], pp. 225-230).

Theorem 1.1. Let $S$ be a bounded connected and simply
connected open set and \( z_0 \in S \). Then there exists a one-to-one differentiable map \( F \) of \( S \) onto \( U \) such that \( F(z_0) = 0 \), \( F'(z_0) > 0 \), and a sequence of polynomials \( P_1, P_2, \ldots \) converging uniformly on compact subsets of \( U \) to \( F^{-1} \).

**Proof.** Let \( K \) be the set of all one-to-one differentiable maps \( f \) of \( S \) into \( U \) such that \( f(z_0) = 0 \), \( f'(z_0) > 0 \) and such that there exists a sequence of polynomials \( Q_1, Q_2, \ldots \) converging uniformly on compact subsets of \( f(S) \) to \( f^{-1} \). \( K \) is a non-empty set.

If \( s = \sup_{f \in K} f'(z_0) \), then there exists a sequence \( f_1, f_2, \ldots \) in \( K \) such that \( \lim_{n \to \infty} f_n'(z_0) = s \). From the Vitali-Porter-Stieltjes Theorem there exist \( p_1 < p_2 < \ldots \) in \( w \), such that \( \{f_{m_n}\}_{n=1}^{\infty} \) converges uniformly on compact subsets of \( S \) to a limit function \( F \). From Lemma 2.3, \( F \) is differentiable and \( F'(z_0) = s \), and hence \( F \) is non-constant. From the Hurwitz Theorem, \( F \) is one-to-one.

For \( n \in w \), let \( C_n = \{z \in F(S) \mid 0 = z - F(z) \geq 1/n \} \). \( C_n \) is closed and hence compact for \( n \in w \), and \( F(S) = \bigcup_{n=1}^{\infty} C_n \).

From Lemma 2.4, there exist \( p_1 < p_2 < \ldots \) in \( w \), such that if \( n \in w \), then \( m \geq p_n \), \( m \in w \), implies that \( C_n \subseteq f_m(S) \). If \( n \in w \), there exist a sequence of polynomials \( P_{n1}, P_{n2}, \ldots \) converging uniformly on compact subsets of \( f_{p_n}(S) \) to \( f_{p_n}^{-1} \).

There exist \( q_1 < q_2 < \ldots \) in \( w \), such that for \( n \in w \),

\[ |P_{nq_n}(z) - f_{p_n}(z)| < 1/n \text{ for } z \in C_n. \]

If \( D \) is a compact subset of \( F(S) \), then there exists an integer \( n_0 \), such that \( D \subseteq C_{n_0} \). Let \( \varepsilon > 0 \). Then from Lemma 2.4, there exists \( M > n_0 \),
such that \( m > M, m \in \mathbb{W} \), implies \(|f_{p_m}^{-1}(z) - F^{-1}(z)| < 1/m\) for all \( z \in C_{n_0} \). Hence for \( z \in D \subset C_{n_0} \) and \( m > M, m \in \mathbb{W} \), we have \(|F^{-1}(z) - P_{mq_m}(z)| \leq |F^{-1}(z) - f_{p_m}^{-1}(z)| + |f_{p_m}^{-1}(z) - P_{mq_m}(z)| < 1/m + 1/m = 2/m\). From Lemma 2.4, \( F(S) \subset U \) and we have \( F \in \mathbb{K} \).

Suppose \( F(S) \not\subset U \). Then clearly \( F(S) \) is a connected and simply connected open set. From Lemma 2.5, there exist polynomials \( f \) and \( g \) such that \( g(z) \not\equiv 0 \) for all \( z \in \bar{U} \), and a one-to-one differentiable function \( h \) on \( F(S) \) into \( U \), such that \( fh(z)/gh(z) = z \) for all \( z \in F(S) \), and such that \( h'(0) > 1 \). Then \( (hf)'(z_0) = h'(z_0) \cdot F'(z_0) = h'(0)s > s \). Now \( W = hF(S) \) is an open set in \( U \). Then \( F(S) = h^{-1}(W) \). Since \( F \in \mathbb{K} \), there exists a sequence of polynomials \( P_1, P_2, \ldots \) converging uniformly on compact subsets of \( h^{-1}(W) \) to \( F^{-1} \). From Lemma 2.1, there exists a sequence of polynomials \( Q_1, Q_2, \ldots \) converging uniformly on compact subsets of \( W \) to \( F^{-1}h^{-1} \). Thus \( hF \in \mathbb{K} \), which is a contradiction. Hence \( F(S) = U \).

**Theorem 2.2.** Let \( S \) be a bounded connected and simply connected open set in \( \mathbb{K} \), \( z_0 \in S \), and \( x_0 \in U - \{0\} \). Then there exists a unique differentiable one-to-one function \( f \) on \( U \) onto \( S \) such that (1) \( f^{-1} \) is differentiable, \( f(0) = z_0 \), \( f'(0) > 0 \), and there exists a sequence of polynomials \( P_1, P_2, \ldots \) converging uniformly on compact subsets of \( U \) to \( f \), and such that (2) if \( g \) is a one-to-one map of \( U \) onto \( S \), such that \( g \) is differentiable on \( U - \{x_0\} \), \( g(0) = z_0 \), and \( g'(0) > 0 \), then \( g = f \).
Proof. The existence of at least one function $f$ satisfying (1) is assured by Theorem 2.1. Let $g$ be a function satisfying (2) and set $Q(z) = f^{-1}g(z)$ for all $z \in U$. Then $Q$ is a one-to-one map of $U$ onto $U$, such that $Q$ is differentiable on $U - \{x_0\}$, $Q(0) = 0$, and $Q'(0) > 0$. Let $T = Q_{|0}$.

Then from Theorem 1.12, \( \sup_{z \in U} |T(z)| \leq \sup_{0 < r < 1} \sup_{t \in C_r} |T(t)| \leq \sup_{0 < r < 1} 1/r = 1 \). Thus $|T(z)| \leq 1$ for all $z \in U$.

Since $Q$ is one-to-one, $Q(z) \neq 0$ for $z \in U - \{0\}$ and, since $Q'(0) > 0$, we have $T(z) \neq 0$ for all $z \in U$. Thus we also have from Theorem 1.12, that $|T(z)| > 1$ for all $z \in U$. Consequently $|T(z)| = 1$ for all $z \in U$. From Theorem 1.11, since $Q'(0) > 0$, $T(z) = Q'(0) = 1$ for all $z \in U$, and thus $f^{-1}g(z) = z$ for all $z \in U$, and consequently $f = g$.

Theorem 2.3. Let $x_0 \in U$, $x_0 \neq 0$, and $f$ be a continuous function on $U$ such that $f$ is differentiable on $U - \{x_0\}$.

Then $f$ is differentiable and there exists a sequence of polynomials $P_1, P_2, \ldots$ which converge uniformly on compact subsets of $U$ to $f$.

Proof. If $\varepsilon > 0$ and $r = 1 - \varepsilon$, then, from Theorem 1.13, there exists $p > 0$ such that the function $g(z) = f(z) + pz$ for $z \in U$ is one-to-one on $U_r$. From Theorem 1.2, $g$ is differentiable and there exists a sequence of polynomials $P_1, P_2, \ldots$ which converges uniformly on compact subsets of $U_r$ to $g$. Hence $\{P_1 - P_i\}_{i=1}^\infty$ converges uniformly on compact subsets of $U_r$ and, in particular $U_{1-2\varepsilon}$, to $f$. By a diagonal process we obtain a sequence of polynomials $Q_1, Q_2, \ldots$, such
that \(|Q_n(z) - f(z)| < 1/2^n\) for \(z \in U_1 \cdot 1/n\) and \(n \in \omega\). Clearly \(Q_1, Q_2, \ldots\) is the desired sequence.

**Lemma 2.8.** If \(P(z) = \sum_0^n a_p z^p\) for \(z \in K\) and \(P(z) \leq 1\) for \(z \in \overline{U}\), then \(|a_j| \leq 1\) for \(j = 0, 1, \ldots, n\).

**Proof.** This theorem and proof are due to Porcelli and Connell [ ]. Trivially the theorem holds for polynomials of degree zero. Suppose for \(n \in \omega\), it holds for polynomials of degree \(n\) or less and \(P(z) = \sum_0^{n+1} a_p z^p\) is a polynomial of degree \(n+1\) such that \(|P(z)| \leq 1\) for \(z \in \overline{U}\). Let \(\theta \in [0, 2\pi]\) and \(Q(z) = 2^{-1}[P(z) - PE(1\theta)]\) for \(z \in K\). Then \(Q\) has no constant term and \(Q/I_0\) is a polynomial of degree \(n\) or less.

From Theorem 1.11. \(|Q(z)/z| \leq 1\) for \(z \in \overline{U} \setminus \{0\}\). By the inductive hypothesis, \(|2^{-1}a_{p+1}[1 - E((p+1)\theta)]| \leq 1\), for \(p = 0, 1, \ldots, n\). Upon setting \(\theta = \pi/p\) we have \(|a_p| \leq 1\), for \(p = 1, 2, \ldots, n+1\). Finally \(|a_0| = |P(0)| \leq 1\).

**Theorem 2.4.** If \(f\) is a differentiable function on \(U\), there exists a power series \(\sum_0^\infty a_n z^n\) which converges uniformly on compact subsets of \(U\) to \(f\).

**Proof.** From Theorem 2.3, there exists a sequence of polynomials \(P_1, P_2, \ldots\) such that for \(n \in \omega\), \(|P_i(z) - P_j(z)| < 1/2^n\), for \(z \in U_1 \cdot 1/n\) and \(i, j \geq n, 1, j \in \omega\). Let \(n \in \omega\).

Then from Lemma 0.6. \(|a_{1p} - a_{jp}| < [2^n(1-1/n)^p]^{-1}\), for \(i, j \geq n, 1, j \in \omega, p \in \omega\), where \(\{a_{1j}\}_{j=1}^\infty\) is a sequence in \(K\), such that \(P_j(z) = \sum_0^\infty a_{jp} z^p\) for \(j \in \omega\). Thus for \(p \in \omega\), there exists \(a_p \in K\) such that for \(n \in \omega\), \(|a_p - a_{1p}| <
2\left[2^n(1-1/n)^1\right]^1 \text{ for all } 1 > n, 1 \in w.

Let \( n \in w \) and let \( n_0 \) denote the degree of \( P_n \). Then for \( p > n_0, p \in w, a_{np} = 0 \) and hence \( |a_p| = |a_p - a_{np}| < 2\left[2^n(1-1/n)^p\right]^{-1}. \) Thus \( \limsup_{p \to \infty} |a_p|^{1/p} \leq (1-1/n)^{-1}. \) Since \( n \) is arbitrary, we have \( \limsup_{p \to \infty} |a_p|^{1/p} \leq \). Thus the power series \( T(z) = \sum_0^\infty a_p z^p \) converges uniformly on compact subsets of \( U \).

Let \( z \in U \). Then for \( n \in w, \) such that \( |z| < 1 - 2/n, \) we have

\[
|T(z) - P_n(z)| = |\sum_0^\infty (a_p - a_{np})z^p| \leq \sum_0^\infty |a_p - a_{np}| |z|^p \leq \sum_0^\infty 2|z|^p \left[2^n(1-1/n)^p\right]^{1} \leq 2^{n-1}(1-r)^{-1}, \text{ where } r = (1-2/n) \cdot (1-1/n)^{-1}. \] 

Thus \( \lim_{n \to \infty} P_n(z) = T(z). \) Since by hypothesis, \( \lim_{n \to \infty} P_n(z) = f(z), \) we have \( T(x) = f(x) \) for all \( x \in U. \)

**Theorem 2.5.** Let \( S \) be a connected open set, \( z_0 \in S, \)

\( f_1, f_2, \ldots \) a sequence of maps of \( S \) into \( K \) converging uniformly on compact subsets of \( S \) to a limit function \( f_0, \) and \( g_1, g_2, \ldots \)
a sequence of differentiable functions on \( S, \) such that for \( n \in w, \) \( g_n' = f_n \) and \( g_n(z_0) = 0. \) Then \( \{g_n\}_{n=1}^\infty \) converges uniformly on compact subsets of \( S \) to a limit function \( g_0, \) such that \( g_0 \) is differentiable and \( g_0' = f_0. \)

**Proof.** Let \( H \) be a compact subset of \( S. \) Since \( S \) is connected there exists a collection of squares \( Q_1, Q_2, \ldots, \)
such that \( H \subset Q \subset S, x_0 \in Q. \) and \( Q \) is connected, where \( Q = \bigcup_1^n T(Q). \) Then there exists \( M > 0, \) such that for \( x \in H, \) there exists a polygonal arc \( P \) with endpoints \( x \) and \( x_0, \)
having length \( L(P) \) and lying in \( Q. \) By the mean value theorem for real valued functions and for \( n,m \in w, \) \( |g_n(x) - g_m(x)| \leq 2M \sup_{t \in Q} |g_n'(t) - g_m'(t)| = 2M \epsilon_{nm}, \) where \( \epsilon_{nm} = \sup_{t \in Q} |f_n(t) - f_m(t)|. \) Since \( \{f_n\} \) converges
uniformly on \( Q \) to \( f_0 \), \( \lim_{n,m \to \infty} \epsilon_{nm} = 0 \), and thus \( \{g_n\}_{n=1}^{\infty} \) converges uniformly on \( Q \). Hence \( \{g_n\}_{n=1}^{\infty} \) converges uniformly on compact subsets of \( S \) to a limit function \( g_0 \). From Lemma 2.3, \( f_0(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} g_n'(x) = g_0'(x) \) for all \( x \in S \).

Definition 2.1. Let \( S \) be an open set, \( g \) a map of \( S \) into \( K \), \( F \) a collection of functions on subsets of \( K \). Then the statement that \( F \) is an \( I_S, g \) collection means that \( F \) is a \( C_S \) collection such that, for \( f \in F \), \( f'(x) = g(x) \) for all \( x \in S_f \).

Theorem 2.6. Let \( S \) be an open set \( h \) a map of \( I \) into \( S \). \( f \) and \( g \) differentiable functions on \( S \), and \( a \) and \( b \in K \). Then there exists an \( I_S, F \) collection. Moreover if \( S \) is connected and simply connected, then there exists a differentiable function \( k \) on \( S \) such that \( k' = f \), and a unique number \( I = \int_0^1 f \, dh \), such that if \( F \) is an \( I_S, f \) collection, then \( I = \int_0^1 F \, dh \). Finally \( \int_0^1 (af + bg) \, dh = a \int_0^1 f \, dh + b \int_0^1 g \, dh \).

Proof. Let \( x \in S \) and \( R_x \) be a circular region with center \( x \) lying in \( S \). From Theorem 2.3, there exists a sequence of polynomials \( P_1, P_2, \ldots \) converging uniformly on compact subsets of \( R_x \) to \( f \). Trivially there exists a sequence of polynomials \( Q_1, Q_2, \ldots \), such that, for \( n \in \omega \), \( Q_n(x) = 0 \), and \( Q_n' = P_n \). From Theorem 2.5, \( \{Q_n\}_{n=1}^{\infty} \) converges uniformly on compact subsets of \( R_x \) to a differentiable limit function \( f_x \), such that \( f_x'(y) = f(y) \) for all \( y \in R_x \).

Let \( x, y \in S \) and \( F(z) = f_x(z) - f_y(z) \) for all \( z \in R_x \cap R_y \).
Suppose $F(u) \neq F(v)$ for some $u, v \in \mathbb{R}^n \cap \mathbb{R}^y$. By the mean value theorem for real valued functions, there exists $w$ in the line segment with endpoints $u$ and $v$, such that $F'(w) \neq 0$. But $F'(w) = f'_x(w) - f'_y(w) = f(w) - f(w) = 0$. Thus $\{f_x\}_{x \in S}$ is an $I_S, f$ collection. Now if $S$ is connected and simply connected, then the existence of a differentiable function $h$, such that $h' = f$, follows from Theorem 1.5.

Let $F$ and $G$ be $I_S, f$ collections and $x \in S$. There exist $f_x \in F$ and $g_x \in G$, such that $x \in S_{f_x}$ and $x \in S_{g_x}$. Let $R_x$ be a circular region with center $x$ lying in $S_f \cap S_g$, and let $\tilde{f}_x = f_x|_{R_x}$ and $\tilde{g}_x = g_x|_{R_x}$. Then $F_0 = \{\tilde{f}_x\}_{x \in S}$ and $\{\tilde{g}_x\}_{x \in S}$ are $I_S, f$ collections, such that $\int_0^1 F dh = \int_0^1 F_0 dh$ and $\int_0^1 G dh = \int_0^1 G_0 dh$. Clearly $H = F_0 \cup G_0$ is an $I_S, f$ collection, and, from Theorem 1.1, $\int_0^1 F_0 dh = \int_0^1 G_0 dh$. Then $\int_0^1 F dh = \int_0^1 G dh$, and consequently $\int_0^1 f dh$ is uniquely defined.

We can obtain $I_S, f$ collections (vide supra) $M$ and $N$ such that $\{S_k\}_{k \in M} = \{S_k\}_{k \in N}$. It follows readily that $a \int_0^1 f dh + b \int_0^1 g dh = \int_0^1 (af + bg) dh$.

**Theorem 2.7.** Let $S$ be a simply connected open set. Let $C$ be a simple closed curve in $S$, $z_0 \in I(T)$, and $f$ a differentiable function on $S$. Then $f(z_0) = (2\pi i)^{-1} \int_T f(z)/(z-z_0) \, dz \tag{1}$

**Proof.** From Theorem 2.3, if $Q = Q_{f, z_0}$, then $f$ is differentiable on $S$. From Theorem 2.6, $\int_T f(z)/(z-z_0) \, dz = \int_T f(z_0)/(z-z_0) \, dz = \int_T \omega(z) \, dz = 0$, and thus $\int_T f(z)/(z-z_0) \, dz = f(z_0) \int_T (z-z_0)^{-1} \, dz$. 
If \( L \) is an \( L_{S, z_0} \) collection, then, for \( f \in L \), \( Ef(z) = z-z_0 \) for \( z \in S_f \), and hence \( 1 = Ef(z) \cdot f'(z) \) for \( z \in S_f \). Thus \( L \) is an \( I_{S, \{z_0\}, (z-z_0)^{-1}} \) collection and therefore from Theorem 1.10, \( \int_T (z-z_0)^{-1} = \int_T L \ dz = 2\pi i. \) Hence \( \int_T f(z)/(z-z_0) \ dz = 2\pi i f(z_0). \)

**Remark.** We can readily show that if \( T \) is rectifiable, then \( \int_T f(z)/(z-z_0) \ dz \) is defined and equal to \( \int_T f(z)/(z-z_0) \ dz \), and hence (1) reduces to the Cauchy Integral Formula.

**Theorem 2.8.** Let \( 0 < r_0 < 1 \) and \( f \) a differentiable function on \( S = U - \cup_{r_0}. \) Then there exists a power series \( \sum_{n=0}^{\infty} a_n z^n \) converging uniformly on compact subsets of \( S \) to \( f. \)

**Proof.** For \( n \in \omega, \) let \( \varepsilon_n = (1-r_0)/7n, \) \( V_n = \cup_{r_0+2\varepsilon_n} \cup_{r_0+3\varepsilon_n}, \) \( S_n = U_{1-3\varepsilon_n}, \) \( W_n = U_{1-\varepsilon_n} - U_{2-2\varepsilon_n}, \) \( P_n = C_{1-3\varepsilon_n/2}, \) and \( Q_n = C_{r_0+3\varepsilon_n/2}. \) Then from Theorem 2.7 and the proof of Theorem 1.9, we have, for \( n \in \omega \) and \( x \in S_n, \) \( 2\pi i f(x) = \int_{P_n} f(z)/(z-x) \ dz - \int_{Q_n} f(z)/(z-x) \ dz. \) For \( x \in U_{1-3\varepsilon_n}, \) set \( g_n(x) = \int_{P_n} f(z)/(z-z_0) \ dz, \) and for \( x \in R_n = \cup_{r_0+\varepsilon_n}, \) set \( h_n(x) = -\int_{Q_n} f(z)/(z-z_0) \ dz, \) for \( n \in \omega. \)

Let \( n \in \omega, x_0 \in R_n, \) and \( x_1, x_2, \ldots \) a sequence of points in \( R_n, \) distinct from \( x_0, \) converging to \( x_0. \) For \( z \in W_n \) and \( p \in \omega, \) set \( \nu_p(z) = f(z)\left[ (z-x_p)(z-x_0) \right]^{-1} \) and \( \nu_0(z) = f(z)/(z-x_0)^2. \) Then \( \{\nu_p\}_{p=1}^{\infty} \) is a sequence of differentiable functions converging uniformly on \( W_n \) to \( \nu_0. \) Expressing \( W_n \) as the union of two simply connected and connected open sets, and applying Theorems 2.5 and 2.6, we deduce that
\[ [g_n(x_p) - g_n(x_o)](x_p - x_o)^{-1} = \int p_n f(z)[(z \cdot x_p)(x_p - x_o)]^{-1} \, dz \]
\[ \int p_n f(z)[(z \cdot x_o)(x_p - x_o)]^{-1} \, dz = \int p_n v_p(z) \, dz \]
converges to \( \int p_n v_o(z) \, dz \) as \( p \to \infty \). Thus \( g_n \) is differentiable at \( x_o \), and \( g_n'(x_o) = \int p_n v_o(z) \, dz \). Similarly \( h_n \) is differentiable on \( R_n \). Clearly \( \lim_{x \to \infty} h_n(x) = 0 \).

For \( n > m, n, m \in \omega \), \( g_n(x) = g_m(x) \) for \( x \in U_1 - 3\varepsilon_m \) and \( h_n(x) = h_m(x) \) for all \( x \in R_m \). Thus there exist differentiable functions \( g \) defined on \( U \) and \( h \) defined on \( K - \overline{U}_{r_0} \), such that \( g(x) = g_n(x) \) and \( h(x) = h_n(x) \) for \( x \in S_n \), for all \( n \in \omega \). Then \( 2\pi i f(x) = g(x) + h(x) \) for \( x \in U - \overline{U}_{r_0} \). From Theorem 2.4, there exists a power series \( \sum_0^\infty a_n z^n \) converging uniformly on compact subsets of \( U \) to \( g \). Now for \( z \in U_1/r_0 \), let \( k(z) = h(1/z) \) if \( z \neq 0 \), and \( k(z) = 0 \) if \( z = 0 \). Then \( k \) is continuous on \( U_1/r_0 \) and differentiable on \( U_1/r_0 - \{0\} \). From Theorem 2.3, \( k \) is differentiable on \( U \) and from Theorem 2.4, there exists a power series \( \sum_0^\infty b_n z^n \) converging uniformly on compact subsets of \( U_1/r_0 \) to \( k \). Then for \( p \in \omega \), \( \sum_{-\infty}^\infty c_n z^n \) converges uniformly to \( f \) on \( S_p \), where for \( n \in \omega \), \( c_n = a_n \) if \( n > 0 \), \( c_n = b_n \) if \( n < 0 \), and \( c_0 = a_0 + b_0 \) if \( p = 0 \).
CHAPTER III

In this chapter we are concerned with the removable singularity problem. Our approach is motivated by the argument for the case when the singularity is a single point, given by Porcelli and Connell (Cf. [1] and [5]), using differences of difference quotients.

Definition 3.1. Let $A \subseteq K$, and $B = \{B_i\}_{i=1}^{\infty}$ be a sequence of subsets of $A$. Then $B$ is called a partition of $A$, if $A = \bigcup_{i=1}^{\infty} B_i$ and $B_i \cap B_j = \emptyset$, for $i \neq j$, $i, j \in \omega$. If $C = \{C_i\}$ is a partition of $A$, then $C$ is called a refinement of $B$, if for every $i \in \omega$, there exists $j \in \omega$, such that $C_i \subseteq B_j$. We shall call $A$ an $M$ set, if for every $\varepsilon > 0$, there exists a partition $T$ of $A$, such that if $V = \{V_i\}_{i=1}^{\infty}$ is a refinement of $T$, then $\sum_{i=1}^{\infty} \delta(V_i)^2 < \varepsilon$.

Let $f$ be a function on a subset of $K$ containing $A$. Then $f$ shall be called an $M_A$ function, if for every $\varepsilon > 0$, there exists a partition $T$ of $A$, such that if $V = \{V_i\}_{i=1}^{\infty}$ is a refinement of $T$, then $\sum_{i=1}^{\infty} \delta(f(V_i))^2 < \varepsilon$. We shall call $f$ a $P_A$ function, if for each $x \in A$, there exists $\mathcal{M}_{h_{x}} > 0$, such that $|f(y) - f(x)| \leq \mathcal{M}_{h_{x}}|y-x|$ for all $y \in A$.

Theorem 3.1. Let $A \subseteq B \subseteq K$. If $f$ and $g$ are $M_A$ functions on $B$, such that $f|A$ and $g|A$ are bounded, then $f \cdot g$ and $f + g$
are $M_A$ functions. If $h$ is a $P_A$ function on $B$, and $A$ is an $M$ set, then $h$ is an $M_A$ function.

Proof. There exists $M > 0$, such that $|f(x)| + |g(x)| < M$ for all $x \in A$. Let $\epsilon > 0$. There exists a partition $T$ of $A$, such that if $V = \{V_1\}_{i=1}^{\infty}$ is a refinement of $T$, then $\Sigma_1^\infty \delta[k(V_1)]^2 < \inf[\epsilon/4M^2, \epsilon/4]$ for $k = g$ and $k = h$. Let $W = \{W_1\}_{i=1}^{\infty}$ be a refinement of $T$, $i \in w$, and $x, y \in W_1$. Then

$$|\Sigma_1^\infty \delta[(f+g)(W_1)]^2 | < 2\delta[f(W_1)]^2 + 2\delta[g(W_1)]^2,$$

and hence

$$\Sigma_1^\infty \delta[(f+g)(W_1)]^2 < 2(\epsilon/4) + 2(\epsilon/4) = \epsilon.$$ Thus $f+g$ is an $M_A$ function.

Similarly for $i \in u$, and $x, y \in W_1$, \(|(f \cdot g)(y) - (f \cdot g)(x)|^2 = [(f(y)g(y) - f(y)g(x)) + (f(y)g(x) - f(x)g(x))]^2

\leq 2M^2 \Sigma_1^\infty \delta[f(W_1)]^2 + \delta[g(W_1)]^2.$$

Thus $\Sigma_1^\infty \delta[(f \cdot g)(W_1)]^2 < 2M^2 \Sigma_1^\infty \delta[f(W_1)]^2 + \delta[g(W_1)]^2 < 2M^2 \cdot (\epsilon/4M^2) = \epsilon$. Hence $f \cdot g$ is an $M_A$ function.

For $n \in w$, let $A_n = \{x \in A | n-1 < x \leq n\}$. Suppose $n \in w$, and $H \subseteq A_n$. Let $x, y \in A_n$. If $x \in A_n$ and $y \in A$, then $|h(x) - h(y)| \leq M_{h,x}|x-y| < n|x-y|$. Thus $\delta[h(H)] \leq n \Lambda(H)$. Let $\epsilon > 0$. Since $A$ is an $M$ set, for $n \in w$, there exists a partition $T_n = \{T_{ni}\}_{i=1}^{\infty}$ of $A$, such that if $V = \{V_1\}_{i=1}^{\infty}$ is a refinement of $T_n$, then $\Sigma_1^\infty \delta[h(V_1)]^2 < \epsilon/n2^n$. For $i \in w$, let $B_{ni} = T_{ni} \cap A_n$. Then $U_{n,i} = T_{ni} \cap A_n = U_{n,i} \in \infty T_{ni} \cap A_n = U_{n,i} \infty A_n = A$. Similarly $B_{ni} \cap B_{mj} = \emptyset$ for $i \neq j$ or $n \neq m$. Thus $\{B_{ni}\}_{i=1}^{\infty}$ is a partition of $A$. 
If $V = \{V_1\}_{i=1}^{\infty}$ is a refinement of $B$, then
\[ \sum_{i=1}^{\infty} \delta(h(V_1))^2 \leq \sum_{i=1}^{\infty} n \cdot \delta(h(V_{n1}))^2 \leq \sum_{n=1}^{\infty} n^2 \cdot \frac{\delta}{2^n} = \epsilon, \]
where for fixed $n \leq \omega$, $\{V_{n1}\}_{i=1}^{\infty}$ is the collection of all elements of $V$ lying in $A_n$, and hence $\{V_{n1}\}_{i=1}^{\infty}$ is a subcollection of a refinement of $T_n$. Thus $h$ is an $M_A$ function.

**Theorem 3.2.** If $A \subset B \subset K$ and $f$ and $g$ are $P_A$ functions on $B$ that are bounded on $A$, then $f+g$ and $f \cdot g$ are $P_A$ functions. If $h$ is a $P_A$ function on $B$ and $m(A) = 0$, then $m(h(A)) = 0$. If $S$ is an open set, $H$ a compact subset of $S$, and $f$ is a differentiable function on $S$, then $f$ is a $P_H$ function.

**Proof.** There exists $M > 0$, such that $|f(x)| + |g(x)| < M$ for all $x \in A$. Let $x \in A$. Then for all $y \in A$,
\[ |(f+g)(y) - (f+g)(x)| \leq |f(y) - f(x)| + |g(y) - g(x)| \leq M_{f,x} |x-y| + M_{g,x} |x-y| = [M_{f,x} + M_{g,x}] \cdot |x-y|. \]
also,
\[ |(f \cdot g)(y) - (f \cdot g)(x)| \leq |g(y)||f(y) - f(x)| + |f(x)||g(y) - g(x)| \leq [M_{f,x} + M_{g,x}] \cdot |x-y| \cdot M. \]
Thus $f+g$ and $f \cdot g$ are $P_A$ functions.

Suppose $m(A) = 0$ and $\epsilon > 0$. For $n \in \omega$, set $A_n = \{x \in A \mid M_{h,x} < n\}$. Then $\bigcup_{i=1}^{\infty} A_n = A$, and $m(A_n) = 0$. There exists a collection $\{R_{n1}\}_{n,i=1}^{\infty}$ of circular regions, such that for $n \in \omega$, $A_n \subset \bigcup_{i=1}^{\infty} R_{n1}$, and $\sum_{i=1}^{\infty} \delta(R_{n1})^2 < \frac{\epsilon}{n \pi 2^n}$. Then $f(A) \subset \bigcup_{i=1}^{\infty} f(R_{n1})$, and $\sum_{n,i=1}^{\infty} n \cdot \delta(f(R_{n1}))^2 \leq \sum_{n=1}^{\infty} n \cdot \sum_{i=1}^{\infty} \delta(f(R_{n1}))^2 \leq \sum_{n=1}^{\infty} n \cdot \sum_{i=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon$. Thus $m[f(A)] < \epsilon$. Since $\epsilon$
is arbitrary, \( m[f(A)] = 0 \).

Let \( x \in H \) and \( Q = Q_{h,x} \) (cf. Theorem 1.12 for the definition of \( Q_{h,x} \)). Then \( Q \) is continuous on \( H \) and there exists \( M > 0 \), such that \( |Q(z)| < M \) for all \( z \in H \). Thus for \( z \in H \), \( z \neq x \), \( |f(z) - f(x)| = |Q(z) \cdot (z-x)| < M|z-x| \).

**Theorem 3.3.** If \( A \) is a closed subset of \( U \) and \( f \) a map of \( U \) such that \( f \) is differentiable on \( U-A \), then \( f \) is differentiable on \( U \), if either:

1. \( A_0 = A-C \ (C = \bar{U}-U) \) is an \( M \) set and \( f \) is an \( M_{A_0} \) function, or
2. \( m(A_0) = 0 \), and \( f \) is a \( P_{A_0} \) function.

**Proof.** For \( x \in U-A \), let \( h_x = Qf,x \). For \( x,y \in U-A \), let \( h_{xy} = h_x - h_y \), and \( C \) be the set of all numbers \( c \), such that there exists a component \( S \) of \( U-B \), where \( B = A \cup \{x\} \cup \{y\} \), such that \( h_{xy}(z) = a + cz \) for \( z \in S \), for some \( a \in K \). Clearly \( C \) is empty or countable. Hence there exists a sequence of positive numbers \( c_1 > c_2 > \ldots \) converging to \( 0 \), each lying in \( K-C \). For \( z \in \bar{U}, \ i \in \omega \), set \( w_i(z) = h_{xy}(z) - c_iz \). From our construction \( w_i \) is non-constant on each component of \( U-B \), and thus \( w_i \) is light on \( U-B \), for \( i \in \omega \).

Suppose 1 holds. Then \( B_0 = B-C \) is an \( M \) set. For \( z \in \bar{U}-\{x\} \), let \( p(z) = (z-x)^{-1} \). Then \( p \) is differentiable, and hence from Theorem 3.2, \( p \) is a \( P_{B_0} \) function. Since \( B_0 \) is an \( M \) set, from Theorem 3.1, \( p \) must be an \( M_{B_0} \) function. Then since \( f \) is a bounded \( M_{B_0} \) function, and \( p \) is bounded on \( B_0 \), we have from Theorem 3.1, that \( f \cdot p \) is an \( M_{B_0} \) function,
where \( f_0 = f|_U-{x} \). Continuing in this manner, we deduce that \( W_1 \) is an \( M_{B_0} \) function for \( i \in w \). If \( i \in w \), then by Definition 3.1, \( m[w_1(B_0)] = 0 \), and thus \( w_1(B_0) \) is nowhere dense in \( K \). Then from Theorem 3.1, for \( z \in U \), \( |h_{xy}(z)| - |c_1| \leq |h_{xy}(z) + c_1z| = |w_1(z)| \leq \sup_{t \in C} |w_1(t)| \leq T_{xy} + c_1 \), where \( T_{xy} = \sup_{t \in C} |h_{xy}(t)| \), and hence \( |h_{xy}(z)| \leq T_{xy} + 2|c_1| \). Since \( \lim_{1 \to \infty} c_1 = 0 \), we have \( |h_{xy}(z)| \leq T_{xy} \) for \( z \in U \).

Suppose \( 2 \) holds. Then \( m(B_0) = 0 \). Since \( f \) and \( p \) are \( P_{B_0} \) functions, from Theorem 3.2, \( f \circ p \) is a \( P_{B_0} \) function. Continuing in this manner we deduce that \( w_1 \) is a \( P_{B_0} \) function for \( i \in w \). Then from Theorem 3.2, since \( m(B_0) = 0 \), we have \( m[w_1(B_0)] = 0 \). Then, as above, we have \( |h_{xy}(z)| \leq T_{xy} \) for \( z \in U \).

Hence in the case where \( 1 \) or \( 2 \) holds, letting \( x_0 \in A-C \), and \( x_1, x_2, \ldots \) be a sequence of points of \( U-A \) converging to \( x_0 \), we have \( |h_{x_1}(x_0) - h_{x_j}(x_0)| = |h_{x_1x_j}(x_0)| \leq T_{x_1x_j} \) for \( i, j \in w \). Clearly \( \lim_{i, j \to \infty} T_{x_1x_j} = 0 \) and hence \( \{h_{x_1}(x_0)\}_{i=1}^{\infty} \) is a Cauchy sequence. If \( y_1, y_2, \ldots \) is a sequence in \( U-A \) converging to \( x_0 \), then \( \{h_{y_1}(x_0)\}_{i=1}^{\infty} \) is a Cauchy sequence.

Now \( x_1, y_1, x_2, y_2, \ldots \) is a sequence in \( U-A \) converging to \( x_0 \), and hence \( h_{x_1}(x_0), h_{y_1}(x_0), h_{x_2}(x_0), h_{y_2}(x_0), \ldots \) is also a Cauchy sequence. Thus \( \lim_{1 \to \infty} h_{x_1}(x_0) \) must equal \( \lim_{1 \to \infty} h_{y_1}(x_0) \).

Since \( h_{x_1}(x_0) = [f(x_1) - f(x_0)]/[x_1-x_0]^{-1} \) for \( i \in w \), \( f \) must be differentiable at \( x_0 \). Thus \( f \) is differentiable on \( U \).

**Lemma 3.1.** If \( T \) is a rectifiable simple closed curve, \( H \) a compact subset of \( K-T \), \( A \) a subarc of \( T \) with endpoints \( x \) and \( y \), and \( s > 0 \), then there exists and arc \( B \) in \( T \cup I(T) \).
and an arc $T_0 \subset T$, with common endpoints $u$ and $v$, such that
$A \subset T_0$, $B \cap T = \{u\} \cup \{v\}$, $|u-x|, |v-y| < \varepsilon$, $L(B) < 8L(A)$,
where $L(A)$ denotes the length of $A$, and $H \subset E(B \cup T_0)$.

**Proof.** Let $W$ be a subarc of $T$ with endpoints $p$ and $q$
and $f(t) = L([p, t])$ for $t \in W$, where $[p, t]$ denotes the subarc of $W$ with endpoints $p$ and $t$. Then clearly $f$ is one-to-one and continuous, and hence $f$ is a homeomorphism.

Let $A_0$ be a subarc of $T$ with endpoints $a$ and $b$, such
that $A \subset A_0$, $x \notin [y, b]$, and such that $L([a, x]), L([y, b]) < \inf \{a, |x-y|/6\}$. Let $\varepsilon > 0$, and $x = x_0 < x_1 < \ldots < x_{n+1} = y$ be a subdivision of $A$, such that $L([x_i, x_{i+1}]) < 2^{-1} \inf \delta(A, T-(a, b))$, $\varepsilon$, $|x-y|/6}$. For $i = 0, 1, \ldots, n$, let $Q_1$ be
the square with side $2L([x_i, x_{i+1}])$ and center $x_i$ with sides parallel to the $x$ and $y$ axis. If $Q = U_0^n \overline{I(Q_1)}$, then $Q \cap T \subset A_0$. If $P = B(Q)$, then $P$ is a simple closed curve such that $A \subset I(P)$, and $\delta(p, A) < \varepsilon$, for all $p \in P$. From the proof of
Theorem 1.6, we may choose $\varepsilon$, so that $H \cup [T - (a, b)] \subset
E(P)$. Then since $x, y \in I(P)$, we have $(a, x) \cap P \neq \emptyset$ and $(y, b)$
$\cap P \neq \emptyset$.

There exists an arc $M \subset P$, with endpoints $x$ and $y$
such that $A_0 \cap M = \{x\} \cup \{y\}$, $x \in [a, x] \cap [y, b]$.
Thus $P_0 = P - [M-A_0]$ is a subarc of $P$, intersecting $[a, x]$
and $[y, b]$, and hence there exists an arc $N \subset P_0$, with end-
points $x$ and $y$, such that $N \cap A_0 = \{x\} \cup \{y\}$, $x \in [a, x]$, and $y \in [y, b]$.

Suppose both of $M-A_0$ and $N-A_0$ lie in the same component.
of $K-A$. Let $M_0 = M \cup \{x_1, y_1\}$ and $N_0 = N \cup \{x_2, y_2\}$. Since
$M \cap \bar{N} - A_0 = \emptyset$, $M - A_0 \subseteq E(M_0)$ (cf. [9], p. 31). $N - A_0 \subseteq E(M_0)$,
$|x_2 - x| < |x - y|/3$, $|y_2 - y| < |x - y|/3$, and hence there
exists $w \in N$, such that $|w - x| > |x - y|/3$ and $|w - y| > |x - y|/3$.
By our construction of $P$, there exists a polygonal arc $W \subseteq $ $Q$ with endpoints $w$ and $z$, such that $W \cap P = \{w\}$, and $W \cap$
$A_0 = \{z\}$, and such that $L(W) \leq |x - y|/6$. If $z \in [x, y]$, then
since $w \in E(M_0)$, $W - \{z\} \subseteq D$, and we must have $W \cap M \neq \emptyset$.
But then $[W - \{w\}] \cap P \neq \emptyset$. Thus $z \in [a, x]$ or $z \in [y, b]$, and
thus $|w - x| < |x - y|/3$ or $|w - y| < |x - y|/3$, which is impossible.
Hence $M \subseteq \overline{I(T)}$ or $N \subseteq \overline{I(T)}$, $|x_i - x| < \delta$, and $|y_i - x| < \delta$, for
$i = 1, 2$. Finally $L(M)$ and $L(N) \leq \sum_{1}^{n} L([x_i, x_{i+1}]) = 8 L(A)$.

We now give a new proof of a classical theorem (cf. Titus and Young [7]).

**Theorem 3.4.** If $A$ is a rectifiable arc in $U$ with end-
points $a$ and $b$ such that $A \cap C = \{a\} \cup \{b\}$, and $f$ a map of
$U$ such that $f$ is differentiable on $U - A$, then $f$ is differenti-
able on $U$.

**Proof.** Let $D_1$ and $D_2$ denote the components of $U - A$. Then
for $i = 1, 2$, there exist from Theorem 2.6, a differentiable
function $g_1$ on $D_1$, such that $g_1'(z) = f(z)$ for $z \in D_1$. For
$i = 1, 2$ we set $T_i = \overline{D_i} - D_i$. Suppose $p \in T_i$ and $\delta > 0$. Let $B$
be a subarc of $T_i$ with endpoints $a$ and $b$, such that $p \in (a, b)$
and $L(B) < \delta/24$, and $R$ be a circular region with center $p$
and radius less than $\delta/6$, such that $T_i \cap R = (a, b)$. Then
there exist line segments $P$ and $Q$ lying in $R$ with endpoints respectively $p_1$ and $p_2$, and $q_1$ and $q_2$, such that $P \cap T_1 = \{p_2\}$, $Q \cap T_1 = \{q_2\}$. Hence each of $L(P)$ and $L(Q)$ is less than $\delta/3$.

From Lemma 3.1, there exists a polygonal arc $W$, with endpoints $x$ and $y$, such that $[p_2, q_2] \sim (x, y)$ and $p_1, q_1 \notin I(W_0)$, where $W_0 = W \cup [x, y]$, and such that $L([x, y]) < 8L(B)$. Then $W \cap P \neq \emptyset$ and $W \cap Q \neq \emptyset$, and there exists an arc $W_1 \subset P \cup Q \cup W$ with endpoints $p_1$ and $q_1$. Hence $L(W_1) < \delta/3 + \delta/3 + 8L(B) < \delta$ and $|g_1(p_1) - g_1(q_1)| \leq 2L(W_1) \cdot \sup_{t \in W_1} |g_1'(t)| \leq 2\delta \sup_{t \in U} |f(t)|$. Thus $g_1$ may be continuous extended to $D_1$ for $1 = 1, 2$.

For $\varepsilon > 0$, there exists $\delta > 0$, such that $x, y \in U$ and $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Let $p$ and $q$ be points of $A$, such that $L([p, q]) < \delta/8$, and set $\varepsilon_0 = \varepsilon L([p, q])/8$. Let $l \in \{1, 2\}$. Then from Lemma 3.1, there exists a polygonal arc $P_l \subset D_l$ with endpoints $p_1, q_1 \in A$, such that $L(P_1) < 8L([p, q]) < \delta$, $P_1 \cap T_1 = \{p_1, q_1\}$, and $|f(p)| \cdot |p - p_1|, |f(p)| \cdot |q - q_1|, |g_1(p_1) - g_1(p)|, |g_1(q_1) - g_1(q)| < \varepsilon_0$. For $z \in K$, let $h_1(z) = g_1(z) - f(p)z$. Then $M = [g_1(p) - g_1(q)] - [g_2(p) - g_2(q)] = [h_1(p) - h_1(q)] \cdot [h_2(p) - h_2(q)]$, and $|h_1(p) - h_1(q)| \leq |g_1(p) - g_1(q)| + |f(p)| \cdot |p - p_1| < \varepsilon_0 + \varepsilon_0 = 3\varepsilon_0$. Similarly $|h_1(q_1) - h_1(q)| \leq 2\varepsilon_0$.

Let $N = [h_1(p_1) - h_1(q_1)] - [h_2(p_2) - h_2(q_2)]$. Then $|M - N| < 8\varepsilon_0 = \varepsilon L([p, q])$. Now for $1 = 1, 2$, $|h_1(p_1) - h_1(q_1)| \leq 2L(P_1) \cdot \sup_{t \in P_1} |h_1'(t)| \leq 16L([p, q]) \cdot \sup_{t \in P_1} |f(t) - f(p)| < 16\varepsilon L([p, q])$. Thus $|N| < 32\varepsilon L([p, q])$, and $|M| \leq \ldots$
$$|M-N| + |N| < \varepsilon L([p,q]) + 32 L([p,q]) = 33\varepsilon L([p,q]).$$

Let $u,v \in A$. Then taking a suitable subdivision of $[u,v]$, we see that $M_1 = [g_1(u) - g_1(v)] - [g_2(u) - g_2(v)] < 33\varepsilon L([u,v]) \leq 33 L(A)\varepsilon$. Since $\varepsilon$ is arbitrary, $M_1 = 0$. Thus there exists $c \in K$ such that $g_1(z) + c = g_2(z)$ for $z \in A$. Let $g(z) = g_1(z) + c$, for $z \in D_1$ and $g(z) = g_2(z)$ for $z \in D_2 - A$.

Using above methods, we deduce that for $u,v \in A$, we have $|g(u) - g(v)| < 8 L([u,v])$. Thus $g$ is of bounded variation on $A$ and $g$ is an $M_A$ function. Since $A$ is rectifiable, $A$ is an $M$ set. Then from Theorem 3.3, $g$ is differentiable on $U$. Since $g'(z) = f(z)$ for $z \in U-A$, and $g'$ is continuous, we have $f(z) = g'(z)$ for all $z \in U$. Thus $f$ is differentiable on $U$.

**Remark 3.1.** Theorem 3.4 gives a necessary but not sufficient condition for $f$ to be differentiable on $U$. The following example, due to Denjoy (cf. [3], p. 33), is analogous to functions met in potential theory. Let $H$ be a compact set such that $m(H) > 0$. For $x \in K$, let $F_x(z) = [z-x]^{-1}$ for $z \in K - \{x\}$, and let $F_x(z) = 0$ for $z = x$. Let $R$ be a circular region containing $H$. Then for $x \in R$,

$$\int_H |F_x| \, dm \leq \int_{R_x} |F_x| \, dm = \int_{R_x} \delta(x,z)^{-1} \, dm(z) = \int_0^{2\pi} r[\int_0^{2\pi} 1/r \, d\theta] \, dr = 2\pi \delta(R).$$

Thus $\int_H F_x \, dm$ exists for $x \in H$. For $x \in K-H$, the existence $\int_H F_x \, dm$ is obvious. For $x \in K$, let $f(x) = \int_H F_x \, dm$.

Suppose $x_0 \in K-H$, and $x_1, x_2, \ldots$ is a sequence of points distinct from $x_0$, converging to $x_0$. Then the sequence of
functions \( g_1, g_2, \ldots \), where \( g_n(z) = (x_n - z)/(x_0 - z) \) for \( n \in \omega \), \( z \in H \), converges uniformly to the function \( g_0 \), where \( g_0(z) = (x_0 - z)^{-2} \) for \( z \in H \). Then \( \lim_{n \to \infty} \int_H g_n \, dm = \int_H g_0 \, dm \). Now
\[
[f(x_n) - f(x_0)][x_n - x_0]^{-1} = (x_n - x_0)^{-1} \int_H [(x_n - z)^{-1} - (x_0 - z)^{-1}] \, dm
\]
Thus \( \lim_{n \to \infty} [f(x_n) - f(x_0)][x_n - x_0]^{-1} = \int_H g_0 \, dm \). Thus \( f \) is differentiable at \( x \), and hence \( f \) is differentiable on \( K - H \).

Let \( x \in H \), \( \epsilon > 0 \), \( R_x \) the circular region with center \( x \) and radius \( \epsilon /16\pi \), \( f_1(y) = \int_{R \cap H} F_x \, dm \), and \( f_2(y) = \int_{H - R} F_x \, dm \) for \( y \in K \). Then \( f = f_1 + f_2 \), and \( |f_1(y)| \leq 2\pi \delta(R) = \epsilon/4 \) for all \( y \in R \). Since \( f_2 \) is continuous on \( R \) and hence continuous on \( K \), there exists an open set \( S \subset R \), such that \( x \in S \) and \( |f(y) - f(x)| < \epsilon/2 \) for \( y \in S \). Then for \( y \in S \),
\[
|f(y) - f(x)| = |f_1(y) + f_2(y) - f_1(x) - f_2(x)| \leq |f_1(y)| + |f_1(x)| + |f_2(y) - f_2(x)| < \epsilon/4 + \epsilon/4 + \epsilon/2 = \epsilon.
\]
Thus \( f \) is continuous on \( K \).

Now for \( x \in K - H \), \( |f(x)| \leq m(H)/\delta(x, H) \), and hence \( \lim_{x \to \infty} f(x) = 0 \). For \( x \in K \), \( xf(x) = \int_H x/(x - z) \, dm = \int_H (1 - z/x)^{-1} \, dm \), and hence \( \lim_{x \to \infty} xf(x) = m(H) > 0 \). Thus \( f \) is non-constant on \( K - H \). Since \( A \) can be readily taken so that \( m(A) > 0 \), we have the desired example.

We note that Denjoy (cf. [3], p. 60) has also given an example of such a non-differentiable function, in the case when \( m(A) = 0 \).

**Theorem 3.5.** Let \( A \) be an arc in \( \overline{U} \) with endpoints \( a \) and \( b \) such that \( A \cap C = \{a, b\} \), and \( f \) a map of \( \overline{U} \) such that \( f \) is
differentiable on $U-A$. Then if $f$ is of bounded variation on $A$, $f$ is differentiable on $U$, and there exists a discrete set $G \subseteq U$, such that $A - G - C$ is the union of a countable collection of rectifiable open subarcs of $A$.

Proof. Since $f$ is of bounded variation on $A$, $f$ is an $M_A$ function. Hence $m[f(A)] = 0$, and $f(A)$ is nowhere dense in $K$. Omitting the trivial case when $f$ is constant, we have that $f$ is light (cf. [9], pp. 93-95) so that, from Theorem 1,11, $f$ is an open map on $U$. Let $p \in U$. Then from the proof of Theorem 1.11, there exists a simple closed curve $T \subseteq U$, such that $f^{-1}(p) \cap T = \emptyset$, and $f^{-1}(p) \cap I(T)$ is finite. Thus $f|U$ has the "scattered inverse property" (cf. [9], p. 83).

Making use of results of Stoilow (cf. [9], pp. 86-88) concerning light open maps, we see that $f$ is "locally equivalent to a power mapping" on $U$, and hence there exists a discrete set $G \subseteq U$, such that if $x \in U-G$, there exists a circular region $R_x \subseteq U$, with center $x$, such that $f|R_x$ is one-to-one.

Let $D_1$ and $D_2$ be the components of $U-A$. The lightness and openness of $f|U$ can also be deduced from a theorem of Titus and Young (cf. [7]) which makes use only of the fact that $f|D_i$ is light and open, for $i = 1,2$, and that $f(A)$ is nowhere dense in $K$.

Let $x \in A - C - G$. Then since $D_i$ is homeomorphic to $U$ for $i = 1,2$, there exists a simple closed curve $T \subseteq U$, such that setting $S = I(T)$, we have that $B = S \cap A$ is an arc in $S$ with endpoints $a$ and $b$, such that $B \cap T = \{a,b\}$,
and that \( f_0 = f|S \) is a homeomorphism. Then from Theorem 2.2, \( f_0^{-1} \) is differentiable on \( f(S) - f(B) \). Since clearly \( f(B) \) is a rectifiable arc, from Theorem 3.4, \( f_0^{-1} \) is differentiable on \( f(S) \), and hence from Theorem 2.2, \( f_0 \) is differentiable on \( S \). Thus \( f \) is differentiable on \( U - G \). Then from Theorem 2.3, \( f \) is differentiable on \( U \).

Let \( x \in A - C - G \), and define \( T, S, B \) as above. Then from Theorem 3.2, \( f_0^{-1} \) is a \( P \) function. Paralleling the arguments in the proof of Theorem 3.1, we see that each subarc of \( B - T \) is rectifiable. In particular, there exists an open subarc \( A_x \) containing \( x \) such that \( A_x \subseteq B \), and \( A_x \) is rectifiable. Thus \( \{A_x\}_{x \in A - C - G} \) is the desired collection of open subarcs. We note that if \( f \) is a continuous function on \( U \), and \( f \) is differentiable on \( U - A \), then from Theorem 3.4, \( f \) is differentiable on \( U - G \), and, from Theorem 2.3, \( f \) is differentiable on \( U \).

Remark 3.2. Let \( A \) be an arc and \( f \) a map of \( K \), such that \( f \) is nonconstant and differentiable on \( K - A \), and \( s = \lim_{x \to \infty} f(x) \) exists. Then from Theorem 1.14, \( f(A) = f(K) \). Now \( f|K - A \) is an open map and hence \( f(K - A) \) is open in \( K \). Thus \( f(A) \) is not nowhere dense in \( K \). If instead of the hypothesis of Theorem 3.5, we require that \( f(A) \) be nowhere dense, then as in the proof of Theorem 3.5, we are reduced to the case where \( f \) is a homeomorphism. It is not yet known whether this latter condition is sufficient to insure that \( f \) is differentiable on \( U \) (cf. [7]).

Remark 3.3. Functions of the form discussed in Remark
3.1 may be obtained in a natural manner from consideration of functions discussed in Theorems 3.4 and 3.5. Suppose $A$ is a compact subset of $U$, and $f$ is a map of $\overline{U}$, such that $f$ is differentiable on $U-A$. Now there exists $0 < r_0 < 1$, such that $A \subseteq U_{r_0}$. From the proof of Theorem 2.8, there exist differentiable functions $f$ on $U$ and $h$ on $K - \overline{U}_{r_0}$, such that $f(z) = g(z) + h(z)$ for $z \in U - \overline{U}_{r_0}$, and $\lim_{x \to \infty} h(x) = 0$. Let $f_0(z) = h(z)$ for $z \in K - \overline{U}_{r_0}$, and $f_0(z) = f(z) - g(z)$ for $z \in \overline{U}_{r_0}$. Then $f_0$ is continuous on $K$, $\lim_{x \to \infty} f_0(x) = 0$, $f_0$ is differentiable on $K-A$, and $f(z) = f_0(z) + g(z)$ for $z \in U$.

We now take $A$ to be an arc. If $f_0$ is differentiable on $K$, then, from the proof of Theorem 3.5, $f_0(A)$ is the union of a countable collection of arcs and points, and hence $f_0(A)$ is a first category set in $K$, and thus is nowhere dense in $K$. From Theorem 3.2, $f_0$ must be constant, and hence $f_0 \equiv 0$. Thus a necessary and sufficient condition that $f$ be differentiable on $U$, is that $f_0 \equiv 0$. 
CHAPTER IV

The primary concern of this chapter is the characterization of point sets in $K$ called "exceptional sets" (cf. Def. 4.1). The principle effort involved is the proceeding from an analytic characterization (using pointwise bounded collections of analytic functions) to a purely point-theoretic topological characterization. We make use of a polynomial approximation Theorem of Walsh (cf. Th. 4.1) to obtain polynomials having certain desired "contours" (cf. Th. 4.2). Using these polynomials we construct the desired sequences of analytic functions (cf. Th. 4.3).

**Definition 4.1.** Let $E \subset M \subset S$ be subsets of $K$ and $F$ a collection of functions defined on $S$. $E$ is called the exceptional set of $F$ with respect to $M$, if $E$ is the set of all points $x \in M$, such that $\bigcup_{f \in F} f(V \cap S)$ is unbounded for all open sets $V$ containing $x$. We shall write $E = E(F, M)$.

**Lemma 4.1.** Let $M$ be a compact set and $f$ a continuous function on $H = M \cup I(M)$ into $K$, such that $f$ is an open map on $I(M)$. Then for $x \in I(M)$, $|f(x)| < \sup_{t \in M} |f(t)|$.

**Proof.** We observe that since $H$ is compact, $f(H)$ is compact. Since $f[I(M)]$ is open in $K$, the lemma follows trivially.
If \( g \) is a non-constant differentiable function defined on a connected open set containing \( H \), then \( g \) is an open map on \( V \) and hence Lemma 4.1 applies to \( g \).

The following theorem is due to J. L. Walsh (cf. [8], p. 38) and shall be stated without proof.

**Theorem 4.1.** Let \( G \) be a simple closed curve in \( K \) containing the point \( 0 \) in its interior, and \( f \) a map of \( G \).
Then for every \( \varepsilon > 0 \), there exist polynomials \( P \) and \( Q \) such that \( |P(z) + Q(1/z) - f(z)| < \varepsilon \) for all \( z \in G \).

**Theorem 4.2.** Let \( A \) be an arc with endpoints \( 1 \) and \( y \), such that \( A - \{1\} \subset U \). Then if \( \varepsilon > 0 \), \( \delta > 0 \), \( N > 0 \), and \( v > 1 \), there exists a polynomial \( P \) and a point \( t \in U_y \), such that \( \delta[P_1(z), A] > \delta \) implies \( |P(z)| < \varepsilon \) for all \( z \in U_y \), \( |t-y| < \delta \), and \( |P(t)| > N \). (For definition of "\( P_1(z) \)" cf. Introduction.)

**Proof.** Let \( r > v, W = A \cup [1,1r], \) and \( h \) a homeomorphism of \( U \) onto \( U_r \) such that \( h([0,1]) = W \), where for \( v, w \in K \), \([v,w] \) denotes the line segment with endpoints \( v \) and \( w \). \( h \) may be obtained by extending \( W \) to an arc \( W_0 \) intersecting \( U_r \) in exactly two points, and by then paralleling arguments in the proof of Theorem 1.3. To simplify topological considerations one may consider all arcs mentioned to be polygonal.

Now \( h^{-1}(U_y) \) is a compact subset of \( U \), and hence there exists \( 0 < r_0 < 1 \), such that \( h^{-1}(U_y) \subset U_{r_0} \). Since \( h \) is uniformly continuous, there exists \( w > 0, w < r_0/2, \) such
that \(|u-v| \leq w| \) implies \(|h(u) - h(w)| < \delta \) for all \(u,v \in U\).

Let \(x_1 = -w + i\sqrt{r_0^2 - w^2}, \ x_2 = -w, \ x_3 = 1, \ x_4 = w - w^2 + iw, \ x_5 = w, \) and \(x_6 = w + i\sqrt{r_0^2 - w^2}. \) Let \(G_0\) be the subarc of \(C_{r_0}\)
with endpoints \(x_1\) and \(x_6\) not containing \(ir_0\) and \(G\) the simple closed curve \(G_0 \cup U^5 \left[x_1, x_1 + 1\right].\) Clearly \(x_0 = (1 + r_0)/2\) lies in \(I(G)\).

Let \(g\) be a map of \(h(G)\), such that \(g(z) = 0\) for all \(z \in h(G) - h([x_4, x_5])\), and \(|g(a)| > 2\) for some point \(a \in h([x_4, x_5])\).

From Theorem 4.1, there exists a differentiable function \(f\) on \(K - \{h(x_0)\}\) such that \(|f(x) - g(x)| < 1\) for all \(x \in h(G)\).

Then \(|f(x)| < 1\) for all \(x \in h(G) - h([x_4, x_5])\), and \(|f(a)| > 1\).

If \(m = -w + iw, \ n = w + iw, \ G_1 = G_0 \cup \left[x_1, x_2\right] \cup \left[x_2, x_5\right]
\cup \left[x_5, x_6\right], \) and \(G_2 = G_0 \cup \left[x_1, m\right] \cup \left[m, n\right] \cup \left[n, x_6\right], \) then \(G_1 \subset I(G_2)\) and \(x_0 \notin I(G_2). \)

If \(M_1 = \sup_{x \in h([I(G_1)])} |f(x)|\) and \(M_2 = \sup_{x \in h([I(G_2)])} |f(x)|\), then from Lemma 4.1, there exists \(b \in h(G_1)\) and \(t \in h(G_2),\) such that \(|f(b)| = M_1\) and \(|f(t)| = M_2. \) Now \(a \in I[h(G_2)]\) and hence \(|f(t)| > |f(a)| > 1.\) Consequently \(t \in h([m, n]).\)

Suppose \(M_1 \geq M_2.\) Then \(|f(b)| > 1,\) and hence \(b \in h([x_2 x_5])
\subset I[h(G_2)]. \) But then \(M_1 = |f(b)| < |f(t)| = M_2.\) Thus \(M_1 < M_2.\)

If for \(x \in U_r, \ f_0(x) = 2f(x)/(M_1 + M_2),\) then \(|f_0(x)| \leq 2M_1/(M_1 + M_2) < 1\) for all \(x \in h([I(G_1)]\) and \(|f_0(t)| > 1.\) There exists \(n \in w,\) such that \(|f_0(t)|^2 > N + \varepsilon/2\) and \(|f_0(x)|^n < \varepsilon/2\) for all \(x \in h([I(G_1)]\).

There exists a polynomial \(P\) such that \(|P(x) - f_0(x)^n| < \varepsilon/2\) for all \(x \in U_r.\) If \(x\) is a point of \(U_r\) such that \(\delta(P_1(x), \ A) > \delta,\) then \(\delta(x, w) > \delta\) and \(h^{-1}(x) \in U_r.\) Now if \(\delta[h^{-1}(x), \ A) \geq \delta,\) then \(\delta(x, w) > \delta\) and \(h^{-1}(x) \in U_r.\)
Then \( \delta(x,A) < \delta \). Thus \( x \in B[\{y \mid y \in A \}], \ |f_0(x)| < \varepsilon/2 \), and consequently \( |P(x)| < \varepsilon \). Since \( |P(t)| > N \), \( P \) is the desired polynomial.

**Theorem 4.3.** Let \( E \subseteq U \). Then the following statements are equivalent:

1. \( E = E(P,U) \) for some pointwise bounded collection \( P \) of open maps of a set \( S \) containing \( U \).
2. \( E \) is such that:
   a. \( E \) is compact.
   b. Each component of \( E \) intersects \( C \).
   c. There exists a sequence \( S_1 \supseteq S_2 \supseteq \ldots \) of open sets, such that \( \bigcap_{n=1}^{\infty} S_n = \emptyset \), and \( E - S_n \) is its own outer boundary for \( n \in \omega \).
3. There exists a sequence of polynomials \( P = \{P_n\}_{n=1}^{\infty} \), such that \( \{P_n(f(z))\}_{n=1}^{\infty} \) converges to 0 for all \( z \in U \), \( p = 0,1,\ldots \), and such that \( E = E(P,U) \).

**Proof.** We first prove that 1 implies 2. Since \( U \) is closed, \( E \) is closed. Suppose \( M \) is a component of \( E \) such that \( M \subseteq U \). Then, from the Zoretti Theorem, there exists a simple closed curve \( G \) such that \( M \subseteq I(G) \), \( \delta(x,M) < \delta(M,C) \) for all \( x \in G \), and \( G \cap E = \emptyset \). Hence \( G \subseteq U \). Since \( I(G) \) contains a point of \( E \), we have \( U_{f \in P} f[I(G)] \) is unbounded. But then by Lemma 4.1, \( U_{f \in P} f(G) \) is unbounded, and consequently \( G \) contains a point of \( E \). Thus all components of \( E \) intersect \( C \).

Let \( n \in \omega \), and let \( S_n' = \{x \in U \mid |f(x)| > n \text{ for some } f \in P\} \). Since all elements of \( P \) are continuous, \( S_n' \) is open
in the relative topology of $U$. Hence $S_n' = S_n'' \cap U$ for some open set $S_n''$ in $\mathbb{R}$. Let $S_n = S_n'' \cap \bigcup_{i=1}^{l-1/n}$. Trivially $S_{n+1} \subseteq S_n$ for $n \in \omega$.

If $x \in U$, there exists $M_x > 0$ such that $|f(x)| < M_x$ for all $f \in F$. Thus $x \notin S_n$ for all $n > M_x$, $n \in \omega$, and consequently $\bigcap_{n=1}^{\infty} S_n = \emptyset$. Let $n \in \omega$, and let $B = B(E-S_n)$. Suppose $B \notin E-S_n$ and let $x \in [E-S_n] - B$. Then $x \in I(B)$. Since $x \in E$, there exists $y \in I(B)$ and $f \in F$, such that $|f(y)| > n$. From Lemma 4.1, there exists $z \in B$, such that $|f(z)| > |f(y)| > n$. But then $z \in S_n$. Thus $B = E-S_n$.

We now show that 2 implies 3. Let $X$ be a dense countable subset of $E$, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $X$, such that every element of $X$ appears infinitely many times as a term of $\{x_n\}_{n=1}^{\infty}$. For $n \in \omega$, there exists a point $y_n$ of $E(E-S_n)$, such that $|x_n - y_n| < \varepsilon/4$, where $\varepsilon = 1/2^n$. There exists an arc $A$ with endpoints $x$ and $y_n$ such that $x \in C$, $A - \{x\} \subseteq U$, and $A \cap (E-S_n) = \emptyset$. From the Zoretti Theorem, there exists a simple closed curve $G$ such that the component $M$ of $E$ containing $x$ lies in $I(G)$, $G \cap E = \emptyset$, and $\delta(z, M) < \varepsilon/2$ for all $z \in G$. Since $M$ intersects $C$, $G$ must intersect $C$. Now $A \cap G = \emptyset$, or $A \cap G \neq \emptyset$ and there exists an arc $A_0$ in $A \cup G$ with endpoints $y$ and $z$, such that $z \in C$ and $A_0 - \{z\} \subseteq U$. Thus there exists an arc $A_n$ with endpoints $y_n$ and $z_n$ such that $z_n \in C$ and $A_n - \{z_n\} \subseteq U$, $\delta(x, E) < \varepsilon/2$ for all $x \in A_n$, and such that $A_n \cap (E-S_n) = \emptyset$.

Suppose $\varepsilon_0 > 0$ and $\varepsilon_0 < 2^{n-1} \inf \{1, \varepsilon, \delta(E-S_n, A_n)\}$.

From Theorem 4.1, there exists a polynomial $P_n$ and a point
tn ∈ U_2 - (E-S_n), such that |t_n - y_n| < ε_0, |P_n(t_n)| > n, and such that |P_n(z)| < ε_0^{n+1}/n! for all z ∈ U_2 for which δ[P_1(z), P_n] > ε_0.

Now for z ∈ E-S_n, we have δ(z, A_n) ≥ ε - ε/2 > ε_0. Let V_n = {x ∈ U_2 | x ∈ E-S_n or δ[P_1(x), E-S_n] > ε}. Then for z ∈ V_n, δ[P_1(z), A_n] > ε_0, and hence |P_n(z)| < ε_0^{n+1}/n!. Consequently

|f_{n}(p)(x)| = |p!(2πi)^{-1} \int_{D_x} f_{n}(z)(z-x)^{-p-1}dz| ≤ p!(2πi)^{-1}[2πε_0] \inf_{z \in D_x} |f_{n}(z)|/ε_0^{p+1} < ε_0^{-p}[ε_0^{n+1}/n!]

< ε_0 < ε = 1/2^n,

for x ∈ U ∩ V_n, and p = 0, 1, ..., n, where D_x is the circle with center x and radius ε_0.

Let x ∈ U-E and p = 0, 1, ... L_x denotes the interior of the circle with center x and radius δ(x,E)/3. There exists N > p such that n > N, n ∈ w, implies 2^{-n} < δ(x,E)/2, and thus δ[P_1(y), E-S_n] ≥ δ[P_1(y), E] > δ(x,E) - δ(x,E)/3 > 1/2^n, for all y ∈ L_x. Thus for n > N, n ∈ w, L_x ∩ U ⊂ V_n, and consequently |P_n^{(p)}(z)| < 1/2^n for all z ∈ L_x. Thus x ∉ E(P, U), where P denotes {P_n}_{n=1}^∞. Let x ∈ E, and p = 0, 1, ... Then there exists N > p, such that n > N, n ∈ w, implies x ∈ E-S_n, and hence x ∈ U ∩ V_n; and consequently |f_{n}(p)(x)| < 1/2^n.

Suppose again x ∈ E. There exist n_1 < n_2 < ... in w, such that [x_{n_1}]_{i=1}^∞ converges to x. Now [t_{n_1} - x_{n_1}]_{i=1}^∞ converges to zero, and |P_{n_1}(t_{n_1})| > n_1 for i ∈ w. Hence [t_{n_1}]_{i=1}^∞ converges to x, and x ∈ E(P, U). Thus E = E(P, U), and P is the desired sequence of polynomials.
Remark. Say $E$ is a compact subset of $U$ satisfying condition 2.c of Theorem 4.3. Then $E - S_n$ is the boundary of an open set in $K$ and hence is nowhere dense in $K$, for $n \in \omega$. Thus $E = \bigcup_{i=1}^{\infty} E - S_n$ is first category in $K$, and hence $K - E$ is dense in $K$. Since $E$ is closed, $K - E$ is open, and hence $E$ is nowhere dense in $K$.

Lemma 4.2. If $S_1 \supseteq S_2 \supseteq \ldots$ is a sequence of open sets in a compact metric space $M$ such that $\bigcap_{n=1}^{\infty} S_n = \emptyset$, then $T = \bigcap_{n=1}^{\infty} S_n$ is a compact nowhere dense subset of $M$.

Proof. A subset $V$ of $M$ is a spherical region if there exists $x \in M$ and $\varepsilon > 0$, such that $V = \{y \in M \mid d(x,y) < \varepsilon\}$. Since $M$ is a compact Hausdorff space, $T$ must be compact.

Suppose that there exists a non-empty open set $L$ such that $L \subset T$, and $n \in \omega$. Then $L \subset S_n$. Thus $L \cap S_n$ is a non-empty open set, and hence there exists a spherical region $V$, such that $V \subset L \cap S_n \subset T \cap S_n$. Thus if $T$ contains a non-empty open set $L_0$, we may define inductively a sequence $V_1, V_2, \ldots$ of spherical regions in $T$, such that $V_{n+1} \subset V_n \subset S_n$ for all $n \in \omega$. But then $\bigcap_{n=1}^{\infty} V_n \neq \emptyset$, since $M$ is a complete metric space, and $\bigcap_{n=1}^{\infty} S_n = \emptyset$. Hence $T$ is nowhere dense in $M$.

Theorem 4.4. Let $E$ be a compact subset of $K$, satisfying condition 2.c of Theorem 4.3. Then for each closed non-empty subset $M \subset E$, there exists a compact subset $T$ of $M$, such that $T$ is nowhere dense in the relative topology of $M$, and
such that if $S$ is an open set in $K$ containing $T$, then $M-S$ is its own outer boundary.

**Proof.** For $n \in \omega$, let $R_n = S_n \cap M$. Then $R_n$ is open in the relative topology of $M$, $\bigcap_{n=1}^{\infty} R_n = \emptyset$, and $R_{n+1} \subseteq R_n$ for $n \in \omega$. Let $T = \bigcap_{n=1}^{\infty} R_n$. Then from Lemma 4.2, $T$ is a compact nowhere dense subset of $M$.

Let $S$ be an open set in $K$ containing $T$. Then there exists $N > 0$, such that $n > N$, $n \in \omega$, implies $R_n \subseteq S$, and hence $M-S \subseteq M-R_n \subseteq M - (S_n \cap M) = M-S_n \subseteq E-S_n$. Since $E-S_n = W_n - W_n$, where $W_n = E(E-S_n)$, we have $M-S \subseteq E-S_n \subseteq W_n - W_n$, and hence $M-S = B(M-S)$ for $n > N$, $n \in \omega$.

**Remark.** Theorem 4.4 is useful in excluding from our consideration certain sets with too much "fine" structure. In particular we conclude from Theorem 4.3, that no exceptional set can contain the universal plane curve as a subset.

**Theorem 4.5.** Let $E_1$ and $E_2$ be subsets of $U$ satisfying condition 2 of Theorem 4.3. Then $E = E_1 \cup E_2$ also satisfies condition 2.

**Proof.** From Theorem 4.3, there exist pointwise bounded collections $F_1$ and $F_2$ of open maps of $U$ into $K$ such that $E_1 = E(F_1, U)$ and $E_2 = E(F_2, U)$. Let $x \notin E-U$. Then $x \notin E_1$ and $x \notin E_2$, and hence there exist open sets $V_1$ and $V_2$, and positive numbers $M_1$ and $M_2$, such that $x \in V_n$, and $|f(y)| < M_n$ for all $y \in V_n \cap U$, $f \in F_n$, and $n = 1, 2$. Then $|f(y)| < M_1 + M_2$ for $y \in (V_1 \cap V_2) \cap U$ and $f \in F_1 \cup F_2 = F$, and
consequently \( x \notin E(F, \overline{U}) \). Similarly \( E(F_1, \overline{U}) \cup E(F_2, \overline{U}) \subseteq E(F, \overline{U}) \).

Thus \( E = E(F, \overline{U}) \), and hence from Theorem 4.3, \( E \) satisfies condition 2.

**Theorem 4.6.** Let \( x_0 \in U \) and \( E \) a subcontinuum of \( \overline{U} \), irreducible between \( x_0 \) and \( C \). Then \( E \) satisfies condition 2 of Theorem 4.3.

**Proof.** Trivially \( E \) satisfies 2.a and 2.b. Let \( n \in \mathbb{N}, \ S_n = \{ z \in K \mid P_y(x_0) < P_y(z) < P_y(x_0) + 1/n \} \), and \( B_n = B(E - S_n) \). Suppose \( B_n \notin E - S_n \) and \( x \in (E - S_n) - B_n \). Let \( R \) be the component of \( I(B_n) \) containing \( x \) and \( D \) denote \( B(R) \). Then \( x \in I(D) \). From the Phragmén-Brouwer Theorem (cf. [9], p. 32) \( D \) is connected. Now by our construction \( x_0 \in E(E - S_n) \supseteq B_n \supseteq D \). If \( H \) and \( K \) are subcontinua of \( E \), such that \( H \) is irreducible between \( C \) and \( D \), and \( K \) is irreducible between \( D \) and \( x_0 \), then \( H \cup D \cup K \) is a subcontinuum of \( E \), intersecting \( C \), containing \( x_0 \), and excluding \( x \). From the irreducibility of \( E \), we have \( E = H \cup D \cup K \). But then \( x \notin E \). Thus \( B_n = E - S_n \), and consequently \( E \) satisfies condition 2.c.

**Theorem 4.7.** If \( E \subseteq U \), then the following statements are equivalent:

1. \( E = E(F, U) \) for some pointwise bounded collection \( F \) of open maps of \( U \) into \( K \).
2. \( E \) satisfies condition 2 of Theorem 4.3, and \( E \) is closed in the relative topology of \( U \).
3. There exists a sequence \( P = \{ P_n \}_{n=1}^{\infty} \) of polynomials,
such that $E = E(P,U)$, and $\{P_n^{(P)}(z)\}_{n=1}^{\infty}$ converges
to zero for all $x \in U$, and $p = 0,1,\ldots$

**Proof.** We first show that 1 implies 2. Let $x \in U$. Then
for $|x| < r < 1$, $x \in E(P, \overline{U_r}) \subset E$, and hence from Theorem
4.3, there exists a subcontinuum $M_r$ of $E(P, \overline{U_r})$ containing
$x$ and intersecting $\overline{C_r}$. Then $M = U_{|x|<r<1} M_r$ is a connected
set containing $x$, such that $M$ intersects $C$. Thus $E$ satisfies
condition 2.b of Theorem 4.3.

Let $x_0 \in \overline{E} \cap C$. For $n \in \mathbb{N}$, set $R_n = \{x \in U | |f(x)| > n
for some $f \in F\}$. Then $R_{n+1} \subset R_n$, $R_n$ is open, and $\bigcap_{n=1}^{\infty} R_n = \emptyset$. Let $S_n = R_n \cup (U - \overline{U_{1-1/n}}) \cup (L_n - \{x_0\})$, where $L_n$ is
the interior of the circle with center $x_0$ and radius $1/n$.
Now $S_n$ is open, $S_{n+1} \subset S_n$, and $\bigcap_{n=1}^{\infty} S_1 = \emptyset$. Now for $n \in \mathbb{N}$,
$E - S_n \cap C = \overline{E} \cap \overline{U_{1-1/n}} - R_n = E(P, \overline{U_{1-1/n}}) - R_n$, and hence
$E - S_n \cap C$ is its own outer boundary. Then clearly $E - S_n
= B(E-S_n)$, and thus $\overline{E}$ satisfies condition 2.c of Theorem
4.3. Trivially $E$ is closed in the relative topology of $U$.

The proof that 2 implies 3 follows readily from Theorem
4.3. The proof that 3 implies 1 is trivial.

**Theorem 4.8.** Let $E$ be a closed subset of $K$. Then the
following four statements are equivalent:

1. $E = E(F,K)$ for some pointwise bounded collection
   $F$ of open maps of $K$ into $K$.
2. $E$ is such that:
   a. Each component of $E$ is unbounded.
   b. There exists a sequence of open sets $S_1 \supseteq S_2 \supseteq \ldots$
such that for $n \in \mathbb{N}$, $E - S_n$ is its own outer boundary (we here define the outer boundary of a closed but not bounded set as the boundary of the union of the unbounded components of its complement), and such that $\cap_1^{\infty} S_1 = \emptyset$.

3. $E$ is such that:
   a. For $r > 0$, all components of $E \cap \overline{U_r}$ intersect $C_r$.
   b. There exists a sequence of open sets $S_1 \supset S_2 \ldots$
      such that for $r > 0$ $(E \cap \overline{U_r}) - S_n$ is its own outer boundary, and $\cap_1^{\infty} S_1 = \emptyset$.

4. There exists a sequence of polynomials $P = \{P_n\}_{n=1}^{\infty}$ such that $E = E(P, K)$, and $\{P_n(z)\}_{n=1}^{\infty}$ converges to 0, for $z \in K$, and $p = 0, 1, \ldots$

**Proof.** We first show that 1 implies 2. Let $x \in K$. Then for $r > |x|$, $x \in E(F, U) \subseteq E$, and hence from Theorem 4.3, there exists a subcontinuum $M_r$ of $E(F, U_r)$ containing $x$ and intersecting $C_r$. Then $M = U_r > |x| M_r$ is an unbounded connected subset of $E$ containing $x$. Thus $E$ satisfies 2.b.

For $n \in \mathbb{N}$, let $S_n = \{x \in K | |f(x)| > n \text{ for some } f \in F\}$. Then for $n \in \mathbb{N}$, $S_{n+1} \subseteq S_n$, $S_n$ is open, and $\cap_1^{\infty} S_1 = \emptyset$. Let $B = B(E - S_n)$. Suppose $B \nsubseteq E - S_n$ and $x \in (E - S_n) - B$. If $R$ is the component of $I(B)$ containing $x$, then $R$ is bounded. Now $U_{f \in F} f(R)$ must be unbounded. Hence by Lemma 4.1, there exists $y \in R - R \subseteq B$ and $f \in F$, such that $|f(y)| > n$. But then $y \in S_n$. Thus $B = E - S_n$ and $E$ satisfies 2.b.

The proof that 2 implies 3 is obvious. The sequence
We now show that 3 implies 4. Let \( X \) be a countable dense subset of \( E \), and let \( \{x_n\}_{n=1}^{\infty} \) be a sequence in \( X \), such that every element of \( X \) appears infinitely many times as a term of \( \{x_n\}_{n=1}^{\infty} \). Let \( 0 < r_1 < r_2 < \ldots \) be a sequence of numbers, such that \( x_n \in U_{r_n} \) for \( n \in \mathbb{N} \). Then \( \sup_n r_n = \infty \).

From the proof of Theorem 4.3, for \( n \in \mathbb{N} \), there exists a polynomial \( P_n \) and a point \( t_n \in K \), such that \( |f^{(p)}(z)| < 1/2^n \) for \( p = 0, 1, \ldots, n \), and \( x \in U_{r_{n+1}} \) such that \( x \in (E \cap U_{r_n}) - S_n \) or \( d(P_n(x), E \cup U_{r_n}) > 1/2^n \), and such that \( |P_n(t_n)| > n \) and \( |t_n - x_n| < 1/3^n \).

Let \( n \in \mathbb{N} \) and \( \{y_1\}_{i=1}^{\infty} \) be the subsequence of \( \{x_1\}_{i=1}^{\infty} \) consisting of all terms of \( \{x_1\}_{i=1}^{\infty} \) lying in \( U_n \). Then \( \{y_1\}_{i=1}^{\infty} \) is dense in \( E_n = E \cap U_n \). Using arguments identical to arguments in the proof of Theorem 4.3, it follows that \( E_n = E(P, U_n) \), and that \( \{P^{(p)}(z)\}_{i=1}^{\infty} \) converges to 0 for all \( x \in U_n \) and \( p = 0, 1, \ldots \). Now \( E = U_1^{\infty} E_n \) and hence \( E = E(P, K) \), and thus \( E \) satisfies 4.

The proof that 4 implies 1 is trivial.

Remark. Let \( F = \{f_n\}_{n=1}^{\infty} \) be a sequence of differentiable functions on \( U \), converging pointwise to 0 on \( U \), and let \( G = \{f_n'\}_{n=1}^{\infty} \). Then it may be readily shown that \( E(F, U) = E(G, U) \). However it is not necessarily true that \( G \) is pointwise bounded on \( U \). We shall now construct an example of such a sequence.

For \( n \in \mathbb{N} \), let \( A_n = [-3, 0] \cup \{0, 1/2^n\} \). Then from Theorem 4.2, there exists a polynomial \( P_n \) and a point \( t_n \in U_2 \), such
that \(|P_n(z)| < 1/2^{2n}\) for all \(z \in U_2\) such that \(\delta(z, A_n) > 1/2^{2n}\),
and such that \(|P_n(t_n)| > n\), and \(|t_n - 1/2^{2n}| < 1/3^{2n}\). Now
there exists \(x_n \in U_2\), such that \(|P_n(z)| < 1/2^n\) for all \(z \in U_2\)
such that \(P_y(z) > P_y(x_n)\), and such that \(|P_n(x_n)| = 1/2^n\).
Clearly \(P_y(x_n) \geq 2^{-n} - 2^{-2n}\), and hence \(\delta([-2,0], x_n) > 1/2^{2n}\).
Thus we must have \(|x_n - 1/3^n| < 1/2^{2n}\). Clearly \(P_y(x_n) < 2^{-n} + 2^{-2n}\).

Let \(y_n\) be a point of \(U_2\) such that \(P_x(y_n) = P_x(x_n)\), and \(P_y(y_n) = 2^{-n} + 2^{-2n}\). Then \(|P_n(y_n)| < 1/2^{2n}\). Now \(0 < |x_n - y_n| \leq 2/2^{2n}\), and \(|P_n(x_n) - P_n(y_n)| \geq |P_n(x_n)| - |P_n(y_n)| = 2^{-n} - \frac{|P_n(y_n)|}{2^{2n}} > 2^{-n} - 2^{-2n}.\) By the mean value theorem for real valued functions, there exists \(s_n \in U_2\), such that \(P_y(x_n) < P_y(s_n) < P_y(y_n)\), and \(|P_n'(s_n)| \geq 2^{-1} |P_n(x_n) - P_n(y_n)| \cdot |x_n - y_n|^{-1} > 2^{-n} - 2^{-2n} \cdot [2/2^{2n}]^{-1} = 2^n(1 - 2^{-n}) \geq 2^{n-1}\.\) Now \(|s_n - x_n| < 2/2^n\), and hence, since \(|x_n - 1/2^n| < 1/2^{2n}\), we have \(|s_n - 1/2^n| < 3/2^{2n}\). Since \(P_y(s_n) > P_y(x_n)\), we have \(|P_n(z)| < 1/2^n\) for all \(z \in U_2\), such that \(P_y(z) \geq P_y(s_n)\).

For \(n \in \mathbb{N}\), let \(Q_n(z) = P_n(z + s_n)\) for all \(z \in \mathbb{R}\). Then
for \(z \in \mathbb{U}\) we have \(|Q_n(z)| < 1/2^{2n}\) if \(P_y(z) < P_y(s_n)\), a
condition satisfied whenever \(P_y(z) < 0\) or \(P_y(z) > 2^{-n} + 3/2^{2n}\),
and we have \(|Q_n(z)| < 1/2^n\) whenever \(P_y(z) > P_y(x_n) - P_y(s_n)\),
where \(P_y(x_n) - P_y(s_n) < 0\). It follows that \(\{Q_n(z)\}_{n=1}^{\infty}\) con­
verges to 0 for all \(z \in \mathbb{U}\), and \(\lim_{n \to \infty} |Q_n'(0)| = \infty\).
SELECTED BIBLIOGRAPHY


BIOGRAPHY


Receiving a National Science Foundation graduate fellowship, he enrolled in the graduate school of Louisiana State University in Baton Rouge. He shall receive his Ph.D. in Mathematics from LSU in June, 1963.
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