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Measure Algebras Over a Compact Topological Semigroup.

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ABSTRACT

This paper is devoted to an investigation of the structure of a compact topological semigroup $S$, in terms of the measure algebra $\mathcal{M}(S)$ of complex valued countably additive regular Borel measures defined on the semigroup $S$. The investigation is motivated by the measure theoretic properties of a compact topological group.

If $G$ is a compact topological group it is known that $\mathcal{L}^1(m)$, the collection of all complex valued countably additive regular Borel measures absolutely continuous with respect to normalized Haar measure $m$ on $G$, is an ideal of $\mathcal{M}(G)$ with respect to convolution multiplication. Moreover, it is known that $\mathcal{L}^1(m)$ contains a subset consisting of non-negative measures having norm one which acts as an approximate identity for $\mathcal{L}^1(m)$. The ultimate aim of Chapter I is to find necessary and sufficient conditions, in terms of a non-negative idempotent measure having norm one and its $\mathcal{L}^1$-space, that the kernel (the minimal ideal) of the compact topological semigroup $S$ be a group. One of the main results of Chapter I is that the kernel is a group if and only if there exists a non-negative idempotent measure $\lambda$ having norm one whose $\mathcal{L}^1$-space is an ideal of $\mathcal{M}(S)$ and there exists a subset of $\mathcal{L}^1(\lambda)$ consisting of non-negative measures.
elements having norm one which acts as an approximate identity for $L^1(\lambda)$ in the weak-* topology (the measure algebra $\mathcal{M}(S)$ is identified with the adjoint space of the continuous complex valued functions, given the supremum norm topology, defined on $S$). A certain characterization of the convolution semigroup of non-negative measures having norm one gives rise to a number of necessary and sufficient conditions that the kernel be a group. A particular result of this characterization is that the kernel is a group if and only if there exists a unique idempotent non-negative measure having norm one whose $L^1$-space is an ideal of $\mathcal{M}(S)$ or equivalently, there exists a unique idempotent non-negative measure having norm one whose carrier is the kernel.

Chapter II is concerned with the study of a certain subset of the $L^1$-space of a non-negative idempotent measure $\lambda$. This subset, denoted by $L^2(\lambda)$, consists of those elements $\varphi$ of $L^1(\lambda)$ for which

$$\sup_{P_S} \sum_{E_i \in P_S} \frac{|\varphi(E_i)|^2}{\lambda(E_i)} < \infty,$$

where the supremum is taken over all finite partitionings $P_S$ of $S$ by Borel sets and the terms of the summation for which $\lambda(E_i) = 0$ are deleted (Note: $\lambda(E_i) = 0$ implies $\varphi(E_i) = 0$). One of the principle results of Chapter II v
is that the kernel is a group if and only if there exists a unique non-negative idempotent measure having norm one whose $L^2$-space is invariant relative to convolution multiplication by point measures. Moreover, it is shown that in the latter result the word "'invariant'" may be replaced by "'ideal'".

Necessary and sufficient conditions that the kernel be a group are given in terms of left and right regular representations of the $L^2$-space of a non-negative idempotent measure. Finally, a necessary and sufficient condition for $S$ to be a group is

(1) The kernel is a group, and

(2) $\gamma \in M(S)$ and $\gamma L^1(m) = \{0\}$

imply $\gamma = 0$, where $m$ is normalized Haar measure on the kernel (extended to $S$).

The latter result, in light of the theorems in Chapter I and II, give rise to many necessary and sufficient conditions for $S$ to be a group.
INTRODUCTION

In the text S will always denote a compact Hausdorff topological semigroup. To be more explicit, S will be a compact Hausdorff topological space enjoying a multiplication \( m: S \times S \rightarrow S \), denoted by \( m(x, y) = xy \), that is continuous and associative.

The sigma algebra of all Borel sets (the sigma algebra generated by the open sets of S) will be denoted by \( B(S) \) and \( M(S) \) will denote those countably additive complex-valued measures \( \mu \) defined on \( B(S) \) such that \( \mu \) is regular (i.e., \( \epsilon > 0 \) and \( E \in B(S) \) imply there exists an open set \( W \) containing \( E \) and a compact set \( K \) contained in \( E \) such that \( |\mu(W) - \mu(E)| < \epsilon \) and \( |\mu(K) - \mu(E)| < \epsilon \)).

For \( \mu \in M(S) \), the measure \( |\mu| \) defined at \( E \in B(S) \) by:

\[
|\mu|(E) = \sup_{P_E} \sum_{E_i \in P_E} |\mu(E_i)|
\]

is an element of \( M(S) \), where \( P_E \) now and hereafter will denote a partition, by Borel sets, of the set \( E \) and the supremum is taken over all such partitions of \( E \). \( M(S) \) is a Banach space under the norm given by \( \|\mu\| = |\mu|(S) \). If \( \mu \) and \( \lambda \) are elements of \( M(S) \) the convolution product of
\( \mu \) and \( \lambda \) is defined to be \( \mu \lambda(E) = \int \int \pi_E(xy) \, d\mu(x) \, d\lambda(y) \) for \( E \in B(S) \) \( \pi_E(z) \) will denote the characteristic function of the set \( E \). Under convolution multiplication \( M(S) \) is a Banach algebra.

A linear subspace \( I \) of \( M(S) \) will be called an ideal of \( M(S) \) if \( M(S)I \cup IM(S) \subseteq I \) where \( M(S)I = \{ \mu \eta : \mu \in M(S) \text{ and } \eta \in I \} \).

For each fixed point \( x \in S \) the measure \( x \) defined at \( E \in B(S) \) by:

\[
\chi(E) = \begin{cases} 
1 & \text{if } x \in E \\
0 & \text{if } x \notin E 
\end{cases}
\]

is a non-negative element of \( M(S) \) having norm one and will be called the point measure determined by \( x \). In an effort to simplify notation \( x \) may be replaced simply by \( x \); however, the distinction between the measure \( x = x \) and the point \( x \in S \) will be clear from context (e.g., the convolution product \( x\mu \) will usually be written \( x\mu \) for \( \mu \in M(S) \)). A set \( A \subseteq M(S) \) will be called invariant if \( A \) is closed under multiplication on the right and left by all point measures (i.e., \( xA \cup Ax \subseteq A \) for each \( x \in S \)).

If \( \alpha \) and \( \beta \) are elements of \( M(S) \) then \( \alpha \) will be called absolutely continuous with respect to \( \beta \) (written \( \alpha \ll \beta \)) if \( E \in B(S) \) and \( \beta(E) = 0 \) imply \( \alpha(E) = 0 \). A measure \( \lambda \in M(S) \) will be called non-negative (written \( \lambda \geq 0 \)) if for each \( E \in B(S) \), \( \lambda(E) \geq 0 \). For \( \mu \in M(S) \) and \( \mu \geq 0 \), \( L^1(\mu) \) will denote \( \{ \alpha \in M(S) : \alpha \ll \mu \} \) and \( L^2(\mu) \) will denote those elements \( \beta \) of \( L^1(\mu) \) for which:
\[
\sup_{P_S} \sum_{E_i \in P_S} \frac{|\beta(E_i)|^2}{\mu(E_i)} < \infty.
\]

The supremum is taken over all partitions \(P_S\) of \(S\) and the terms of the summation for which \(\mu(E_i) = 0\) are deleted (Note: \(\beta \in L^1(\mu)\) so that if \(\mu(E_i) = 0\), then \(\beta(E_i) = 0\)).

Since \(S\) is compact we may identify \(C(S)^*\), the space of bounded linear functionals on \(C(S)\), with the space \(M(S)\) where \(C(S)\) denotes the space of continuous complex valued functions on \(S\), endowed with the supremum norm topology. At times it will be convenient to use the same letter to denote the functional and the measure, writing \(\mu(f) = \int f(x)d\mu(x)\) for \(f \in C(S)\).

The set \(\mathcal{S} = \{ \lambda \in M(S) : \lambda \neq 0, \|\lambda\| = 1 \}\) is compact in the weak-\(*\) topology and is a topological semigroup relative to this topology and convolution multiplication \([5]\). For each \(f \in C(S)\) and \(x \in S\) the functions \(f^x\) and \(f_x\) defined by \(f^x(y) = f(yx)\) and \(f_x(y) = f(xy)\) respectively are elements of \(C(S)\) and \(x \longrightarrow f^x, x \longrightarrow f_x\) are continuous mappings of \(S\) into \(C(S)\), the latter being given the supremum norm topology \([5]\).

Now for each \(\lambda \in M(S)\) there exists a unique minimal

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1Numbers in brackets refer to correspondingly numbered bibliographical references. As above, \([5]\) refers to reference 5 in the Selected Bibliography.
closed set called carrier (λ) that is the complement of
the largest open set having |λ|-measure zero. If A ⊆ S,
then A will be called a right ideal of S (left ideal of S)
if A ≠ ∅ (∅ will denote the null set) and AS ⊆ A(SA ⊆ A)
where AS = {as : a∈A, s∈S}. A will simply be called an
ideal if it is both a left and right ideal. It is known
in the general theory of compact topological semigroups
that there exists, uniquely, a minimal closed ideal called
the kernel denoted by K [7] (the kernel of the convolution
semigroup S will be denoted by K(S)). Finally, the
idempotents of a set C will be denoted by E(C).

For the sake of completeness and ease of understanding the
general theorems of compact topological semigroups used in
following chapters will be stated here without proof.
However, credit will be given to the respective author or
authors and reference will be made to the Selected
Bibliography.

**Theorem I** [7]
Each e ∈ E(S) is contained in a unique maximal group H_e
and H_e is closed. Moreover, H_e ∩ H_f ≠ ∅ if and only if
e = f (and hence H_e = H_f).

**Theorem II** [7]
If G ⊆ S and G is a closed group, then G is a topological
group (i.e., multiplication in G is continuous and the
mapping $x \mapsto x^{-1}$ is continuous for $x \in G$).

**Definition I** Let $\mathcal{L}$ will denote the collection of all minimal left ideals of $S$ and $\mathcal{R}$ will denote the collection of all minimal right ideals of $S$ ($\mathcal{L} \neq \emptyset$ and $\mathcal{R} \neq \emptyset$ [7]).

**Theorem III** [2]
If $L \in \mathcal{L}$ and $R \in \mathcal{R}$ then $RL = R \cap L = eSe = H_e$ where $e^2 = e \in R \cap L$ and $H_e$ is the maximal group containing $e$.

**Theorem IV** [2]
The kernel $K$ of $S$ is the pairwise disjoint union of each of the sets $\mathcal{L}$, $\mathcal{R}$ and $\{eSe : e^2 = e \in K\}$.

**Definition II** A subsemigroup $T$ of $S$ is called simple if it does not properly contain any ideals of $T$.

**Theorem V** [5]
If $\mu$ and $\lambda$ are elements of $S$, then $\text{carrier}(\mu \lambda) = [\text{carrier}(\mu)][\text{carrier}(\lambda)]$.

**Theorem VI** [3]
If $\mu^2 = \mu \in S$ and $H = \text{carrier}(\mu)$, then $H$ is a compact simple subsemigroup of $S$ and $f \in C(S)$ implies that the mapping $x \mapsto \mu(f^x)$ is constant on each minimal left ideal of $S$ and the mapping $x \mapsto \mu(f_x^x)$ is constant on each minimal right ideal of $H$.

**Theorem VII** [3]
If $\mu \in S$ the following are equivalent:

1. $\mu \in K(S)$. 

(B) $\mu S \mu = \{\mu\}$.

(C) $\mu^2 = \mu$ and $H = \text{carrier}(\mu)$ is the union of maximal groups $H_\epsilon$ for $\epsilon^2 = \epsilon \in K$.

(D) $\mu^2 = \mu$ and $H \times H = H$ all $x \in S$.

**Theorem VIII [8]**

If $H$ is a subgroup of $S$, $\mu^2 = \mu \in S$, and carrier $(\mu) = H$, then the restriction of $\mu$ to the Borel sets of $H$ is normalized Haar measure on $H$.

**Theorem IX [3]**

The kernel $K$ of $S$ is the union of the sets carrier $(\mu)$ for $\mu \in K(S)$.

**Theorem X [3]**

The following are equivalent:

(A) $S$ is simple.

(B) $S$ consists of idempotents.

(C) $\mu S \mu = \{\mu\}$ all $\mu \in S$.

(D) $xS = \{x\}$ all $x \in S$ or $Sx = \{x\}$ all $x \in S$.

(E) $\mu S = \{\mu\}$ all $\mu \in S$ or $S \mu = \{\mu\}$ all $\mu \in S$.

**Theorem XI [6]**

If $\mu \in S$, then:

\[
\lim_{n \to \infty} \left( \frac{1}{n} \sum_{i=1}^{n} \mu_i^1 \right) = \lambda \text{ exists.}
\]

(B) $\lambda^2 = \lambda \in S$.

(C) $\mu \lambda = \lambda = \lambda \mu$. 
(Note: In the theorem $\mu^i$ denotes the convolution product of $\mu$ with itself $i$ times.)

**Theorem XII [1]**

The weak-* closed convex hull of the collection of all point measures of $\mathcal{M}(S)$ is $\mathcal{S}$. 
CHAPTER I

**Definition 1.1**
If $A$ is a subset of $S$ and $x \in S$, then $(x:A) = \{y \in S : xy \in A\}$ and $(A:x) = \{y \in S : yx \in A\}$.

**Theorem 1.1**
If $t \in S$ and $A$ is an open (closed) subset of $S$, then $(t:A)$ and $(A:t)$ are open (closed) subsets of $S$.

Proof: If $t \in S$ and $A$ is an open set, then if $(t:A) = \emptyset$, then $(t:A)$ is open. If $x \in (t:A)$, then $tx \in A$ and since multiplication is continuous there exists an open set $V$ containing $x$ such that $tV \subset A$. Hence $V \subset (t:A)$ and $(t:A)$ is open. A similar argument shows that $(A:t)$ is open.

Now if $A$ is closed we need, as before, only consider the case $(t:A) \neq \emptyset$. If $x \in \overline{(t:A)}$ (for $F \subset S$, $\overline{F}$ will denote the closure of $F$) and $x \notin (t:A)$ then $tx \in (S - A)$, an open set.

Again, by continuity of multiplication, there exists an open set $V$ about $x$ such that $V \cap (t:A) = \emptyset$ so that $x$ could not have been point of $(t:A)$. This contradiction and a similar argument for $(A:t)$ completes the proof.

**Theorem 1.2**
If $t \in S$ and $A$ is a Borel set, then $(t:A)$ and $(A:t)$ are Borel sets.

Proof: Fix $t \in S$ and consider the set $\mathcal{A}$ of those elements
A ∈ B(S) for which (t: A) ∈ B(S) and (A: t) ∈ B(S). Theorem 1.1 implies that $\mathcal{A}$ contains the open sets. Moreover, it is easily seen from the equalities $(t: \bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} (t: A_n)$ and $(t: [A_1 - A_2]) = [(t: A_1) - (t: A_2)]$ that $\mathcal{A}$ must be a sigma algebra. However, since $B(S)$ is the smallest sigma algebra containing the open sets we have that $\mathcal{A} = B(S)$. This completes the proof.

**Theorem 1.3**

If $\lambda \in M(S)$ and $\lambda \geq 0$, then carrier $(\lambda) = H$ is an ideal of $S$ if and only if $V$ open, $\lambda(V) = 0$ and $t \in S$ imply $\lambda(t: V) = \lambda(V: t) = 0$.

Proof: If carrier $(\lambda) = H$ is an ideal of $S$, $V$ is open, $\lambda(V) = 0$ and $t \in S$, then since $H$ is the complement of the largest open set having $\lambda$-measure zero we have that $V \cap H = \emptyset$. Now if $(t: V) \cap H \neq \emptyset$, then there exists $y \in H$ such that $ty \in V$ which is impossible since $H$ an ideal implies $ty \in H$ and $V \cap H = \emptyset$. A similar argument shows that $\lambda(V: t) = 0$.

Conversely, suppose $y \in H$ and $t \in S$. If $ty \in (S - H)$ then $y \in (t: [S - H])$. Since $(S - H)$ is an open set, Theorem 1.1 implies $(t: [S - H])$ is an open set. However, $\lambda(S - H) = 0$ so that $\lambda((t: [S - H])) = 0$. This means that $y \not\in H = \text{carrier } (\lambda)$, contrary to assumption. Thus, $ty \in H$ and a similar argument shows that $yt \in H$. This completes the proof.

**Theorem 1.4**

If $\mu \in M(S)$ and $p \in S$, then for $E \in B(S)$, $p \mu(E) = \mu(p: E)$ and $\mu p(E) = \mu(E: p)$. 
Proof: For $\mu \in \mathcal{M}(S)$, $E \in \mathcal{B}(S)$ and $p \in S$ we have that

$$p \mu(E) = \int \int \pi_{xy}(xy) dp(x) d\mu(y) = \int \pi_{E}(py) d\mu(y) = \int \pi_{(p;E)}(y) d\mu(y) = \mu(p;E).$$

A similar argument for $\mu(E;p) = \mu(p;E)$ completes the proof.

**Definition 1.2**

If $\mu \in \mathcal{M}(S)$, then $\mathcal{Z}(\mu)$ will denote the collection of all Borel sets $E$ for which $\mu(E) = 0$.

**Theorem 1.5**

If $\mu \in \mathcal{M}(S)$ and $\mu \not\geq 0$, then the following are equivalent:

(A) $L^1(\mu)$ is an ideal of $\mathcal{M}(S)$.

(B) $L^1(\mu)$ is invariant.

(C) $A \in \mathcal{Z}(\mu)$ and $t \in S$ imply $(t:A) \in \mathcal{Z}(\mu)$ and $(A:t) \in \mathcal{Z}(\mu)$.

(D) $x \mu \in L^1(\mu)$ and $\mu x \in L^1(\mu)$ for each $x \in S$.

(E) $L^1(\mu + x \mu) = L^1(\mu) = L^1(\mu + \mu x)$ for each $x \in S$.

Proof: (A) implies (B) is clear. To see that (B) implies (C) let $A \in \mathcal{Z}(\mu)$ and $t \in S$, then we have from Theorem 1.4 that $\mu(t:A) = t \mu(A) = 0$ and $\mu(A:t) = \mu t(A) = 0$ since $\mu(A) = 0$ and $L^1(\mu)$ is invariant. Again we have from Theorem 1.4 that if $\mu(E) = 0$ and $x \in S$ then $x \mu(E) = \mu(x;E) = 0$ and $\mu x(E) = \mu(E;x) = 0$ so that if (C) holds, then (D) must also hold. In order to see that (D) implies (E) we note that $\mu \not\geq 0$ and (D) imply that if $A \in \mathcal{B}(S)$, then $(\mu + x \mu)(A)$ is zero if and only if $(\mu + \mu x)(A)$ is zero if and only if $\mu(A)$ is zero so that these three measures
are mutually absolutely continuous with respect to one another and therefore generate the same $L^1$-spaces. To prove that (E) implies (A) let $\lambda \in L^1(\mu)$ and $\eta \in \mathcal{M}(S)$, then there exists a measurable $\mu$-integrable function $f$ such that for $E \in B(S)$, $\lambda(E) = \int f(z) d\mu(z)$ (Radon-Nikodym Theorem). Now suppose $A \in B(S)$ and $\mu(A) = 0$. Since by Theorem 1.4 $y \in S$ implies $\lambda(y(A)) = \lambda(A:y)$, we have that $\lambda \eta(A) = \int \int \eta(y) d\lambda(x) d\eta(y) = \int \eta(y) d\lambda(x) d\eta(y) = \int \lambda(A:y) d\eta(y) = \int (\int f(z) d\mu(z)) d\eta(y)$. However, (E) implies that for each $y \in S$, $\mu(A:y) = \mu_y(A) = 0$, since $\mu(A) = 0$ so that $\int f(z) d\mu(z) = 0$ for each $y$ and hence $\int (\int f(z) d\mu(z)) d\eta(y) = \int 0 d\eta(y) = 0$ and $\lambda \eta(A) = 0$. A similar argument together with the use of the Fubini theorem shows that $\eta(A) = 0$ finishing the argument that $L^1(\mu)$ is an ideal of $\mathcal{M}(S)$. This completes the proof.

**Theorem 1.6**

If $\mu^2 = \mu \in \mathcal{S}$, then $L^1(\mu) \otimes L^1(\mu) \subset L^1(\mu)$ (i.e., $L^1(\mu)$ is an algebra relative to convolution multiplication).

**Proof:** If $\eta, \lambda \in L^1(\mu)$, $E \in B(S)$ and $\mu(E) = 0$, then $\eta \lambda(E) = \int \int \eta(x) d\lambda(x) d\lambda(y) = \int \eta(E:y) d\lambda(y)$ and similarly $0 = \mu(E) = \mu \mu(E) = \int \mu(E:y) d\mu(y)$. Since $\mu > 0$ the latter equality yields that $\mu(E:y) = 0$, except on a set $F$ having $\mu$-measure zero. Now $\int \eta(E:y) d\lambda(y) = \int \eta(E:y) d\lambda(y) + \int F \eta(E:y) d\lambda(y)$ but $\eta(\langle \mu$ and
\[ \mu(E:y) = 0 \text{ except for } y \in F \text{ imply that } \eta(E:y) = 0 \text{ except possibly for } y \in F \text{ so that } \int_{(S-F)} \eta(E:y) \, d\lambda(y) = 0. \] Since \( \lambda \ll \mu \) and \( \mu(F) = 0 \) we have that \( \lambda(F) = 0 \) implying that \( \int \eta(E:y) \, d\lambda(y) = 0. \) Hence \( \eta \lambda(E) = 0 \) so that \( \eta \lambda \in L^1(\mu). \)

**Theorem 1.7**

If \( \mu \in \hat{S} \) and \( L^1(\mu) \) is an ideal of \( M(S) \) then carrier \( (\mu) \) is an ideal of \( S. \)

**Proof:** If \( x \in S, \) then the point measure \( x \in \hat{S} \) and since \( L^1(\mu) \) is an ideal of \( M(S) \) we have that \( x \mu \ll \mu \) and \( \mu x \ll \mu. \) Now carrier \( (x \mu) \subseteq \text{carrier} \ (\mu) \) for if \( V \) is an open set having \( \mu \)-measure zero then \( x \mu(V) = 0 \) and the definition of the carrier of a measure implies the desired inclusion. Similarly we have \( \text{carrier} \ (\mu x) \subseteq \text{carrier} \ (\mu). \)

Applying Theorem V (Wendel-Glicksberg) we have \( (x)\text{carrier}(\mu) = \text{carrier} \ (x \mu) \subseteq \text{carrier} \ (\mu) \) and \( \text{carrier} \ (\mu)(x) = \text{carrier} \ (\mu x) \subseteq \text{carrier} \ (\mu). \) This completes the proof of the theorem.

**Theorem 1.8**

If \( \mu^2 = \mu \in \hat{S} \) and carrier \( (\mu) = S \) then \( L^1(\mu) \) is an ideal of \( M(S) \) if and only if the mappings \( y \mapsto \mu(E:y) \) and \( y \mapsto \mu(y:E) \) are continuous for each \( E \in Z(\mu). \)

**Proof:** If \( L^1(\mu) \) is an ideal of \( M(S), \mu^2 = \mu \in \hat{S} \) and \( E \in Z(\mu), \) then Theorem 1.5 implies that \( \mu(E:y) = \mu(y:E) = 0 \) for each \( y \in S \) and thus the mappings are clearly continuous.
Conversely, if for $E \in Z(\mu)$ the mapping $y \rightarrow \mu(B:y)$ and $y \rightarrow \mu(y:E)$ are continuous then $0 = \mu(E) = \mu(E) = \int \int r_B(xy)d\mu(x)d\mu(y) = \int \mu(E:y)d\mu(y)$. Now if, for some $y \in S$, $\mu(\mu:y) \times 0$ then the continuity of the mapping $y \rightarrow \mu(E:y)$ implies that there is an open set $V$, containing $y$, such that $\mu(E:z) \times 0$ for $z \in V$. Since carrier $\mu = S$ we have that the $\mu$-measure of each non-void open set is positive so that $\mu(V) \times 0$. However, this is impossible for if this were the case, then $\mu(E) = \int \mu(E:y)d\mu(y)$ is larger than zero contrary to the assumption that $E \in Z(\mu)$. Therefore we see that $\mu(E:y) = 0$ and similarly $\mu(y:E) = 0$, for each $y \in S$. Invoking Theorem 1.5 once again, we have that $L^1(\mu)$ is an ideal of $M(S)$, completing the proof.

**Theorem 1.9**

If $\lambda^2 = \lambda \in S$ and $L^1(\lambda)$ is an ideal of $M(S)$ then:

(A) The kernel $K = \text{carrier } (\lambda)$.

(B) $\lambda \in K(S)$.

**Proof:** Since $L^1(\lambda)$ is an ideal of $M(S)$, Theorem 1.7 implies that carrier $(\lambda)$ is an ideal of $S$. Since the kernel $K$ is the minimal ideal we must have the $K \subset \text{carrier } (\lambda)$. On the other hand, Theorem VI implies that carrier $(\lambda)$ is a compact simple semigroup. Due to the fact that $K$ is an ideal of $S$ and $K \subset \text{carrier } (\lambda)$ we certainly have that $K$ is an ideal of carrier $(\lambda)$. The simplicity of carrier $(\lambda)$ yields $K = \text{carrier } (\lambda)$, completing the proof of (A).
In order to see that (B) holds we note that by (A), \( K = \text{carrier} (\lambda) \). Using Theorems III and IV the kernel \( K \) is the union of the maximal groups \( eS = H_e \) for \( e^2 = e \in K \). However, Theorem VII, together with these facts and the fact that \( \lambda^2 = \lambda \), implies that \( \lambda \in K(S) \). This completes the proof.

**Theorem 1.10**

If \( \mu^2 = \mu \in S \) and \( L^1(\mu) \) is an ideal of \( M(S) \), then \( L^1(\mu) \subseteq L^1(\lambda) \) if \( \lambda \in M(S), \lambda > 0, \lambda \neq 0 \) and \( L^1(\lambda) \) is an ideal of \( M(S) \).

Proof: If \( \mu^2 = \mu \in S \) and \( L^1(\mu) \) is an ideal of \( M(S) \), then Theorem 1.9 implies \( \mu \in K(S) \). For \( \lambda \in M(S), \lambda > 0 \), we have that \( \frac{\lambda}{\lambda} \) is in \( S \) so that using Theorem VII we have, \( \mu \frac{\lambda}{\lambda} \mu = \mu \). In view of the fact that \( L^1(\lambda) \) is an ideal of \( M(S) \) and \( L^1(\lambda) = L^1(\frac{\lambda}{\lambda}) \), we have that \( \mu \ll \lambda \) and hence that \( L^1(\mu) \subseteq L^1(\lambda) \). This completes the proof.

**Corollary 1.1**

If \( \mu^2 = \mu \in S, \lambda^2 = \lambda \in S \) and \( L^1(\mu) \) and \( L^1(\lambda) \) are ideals of \( M(S) \), then \( L^1(\mu) = L^1(\lambda) \).

Proof: The corollary follows immediately from Theorem 1.10.

**Theorem 1.11**

If \( \lambda \in S \), then the following are equivalent:

(A) \( \lambda \in K(S) \).

(B) \( \frac{\lambda + \alpha \lambda}{2} \in K(S), \) all \( \alpha \in S \).
(C) \( \frac{\lambda + \lambda \alpha}{2} \in K(\bar{S}) \), all \( \alpha \in \bar{S} \).

In particular, if \( \lambda \in K(\bar{S}) \), then \( \frac{\lambda + \lambda \alpha}{2} \) and \( \frac{\lambda + \lambda \alpha}{2} \) are idempotents in \( \bar{S} \) for each \( \alpha \in \bar{S} \).

**Proof:** If \( \lambda \in K(\bar{S}) \), then \( \lambda^2 = \lambda \in \bar{S} \) and \( \lambda \bar{S} \lambda = \{ \lambda \} \) (Theorem VII). Now if \( \beta \in \bar{S} \), then since \( \lambda \bar{S} \lambda = \{ \lambda \} \) and \( \bar{S} \) is a convex semigroup we have that \( \frac{\lambda + \alpha \lambda}{2} \in \bar{S} \) for all \( \alpha \in \bar{S} \) and \( \frac{\lambda + \alpha \lambda}{2} \) is equal to the measure

\[
\lambda \beta \lambda + \lambda (\beta \alpha) \lambda + \alpha \lambda \beta \lambda + \alpha \lambda (\beta \alpha) \lambda = \frac{\lambda + \alpha \lambda}{2} .
\]

This implies via Theorem VII that \( \frac{\lambda + \alpha \lambda}{2} \in K(\bar{S}) \).

Similarly, \( \frac{\lambda + \lambda \alpha}{2} \in K(\bar{S}) \) and Theorem VII implies both are idempotents.

Conversely, if \( \frac{\lambda + \alpha \lambda}{2} \in K(\bar{S}) \) for all \( \alpha \in \bar{S} \), then Theorem XI implies:

\[
\lim_{n \to \infty} \left( \frac{1}{n} \sum_{i=1}^{n} \lambda^i \right) = \eta \text{ exists,}
\]

\( \eta \lambda = \eta = \eta \lambda \) and \( \eta^2 = \eta \in \bar{S} \).

Moreover, we have:

\[
\frac{\lambda + \eta}{2} = \frac{\lambda + \eta \lambda}{2} = \frac{\lambda + \eta \lambda}{2} \eta^2 = \frac{\lambda + \eta \lambda}{2} .
\]

\[
\frac{\lambda + \eta}{2} = \frac{\lambda + \eta \lambda}{2} = \eta \lambda = \eta , \text{ so that}
\]

\( \frac{\lambda + \eta}{2} = \eta \) and hence \( \eta = \lambda \). Now note that:

\[
\lambda = \eta = \frac{\lambda + \eta \lambda}{2} \in K(\bar{S})
\]

so that (A) and (B) are equivalent. A dual argument for the equivalence of (A) and (C) completes the proof of the theorem.
Theorem 1.12

If $P = \{E_i\}_{i=1}^n$ is a partition of $S$ by Borel sets, then for each $x \in S$, $P' = \{(x:E_i)\}_{i=1}^n$ and $P'' = \{(E_i:x)\}$ are each partitions of $S$ by Borel sets.

Proof: If $P = \{E_i\}_{i=1}^n$ is a partition of $S$ by Borel sets, then in virtue of Theorem 1.2, $P' = \{(x:E_i)\}_{i=1}^n$ is a collection of Borel sets for each $x \in S$. Moreover, it is clear that $(x:S) = \{y: xy \in S\} = S$. From the latter and from the fact that $S = \bigcup_{i=1}^n E_i$ we have that $S = (x:S) = (x: \bigcup_{i=1}^n E_i) = \bigcup_{i=1}^n (x:E_i)$. Finally, since $E_i \cap E_j = \emptyset$ unless $i = j$ and $(x:E_i \cap E_j) = (x:E_i) \cap (x:E_j)$ we see that $(x:E_i) \cap (x:E_j) = \emptyset$ unless $i = j$. This, together with a similar argument for $P''$, completes the proof of the theorem.

Theorem 1.13

If $\mu^2 = \mu \in S$, $x \mu \ll \mu$ and $\mu x \ll \mu$ for each $x \in K$, then $\text{carrier } (\mu) = K$.

Proof: Since $x \mu \ll \mu$ and $\mu x \ll \mu$ for each $x \in K$, then $\text{carrier } (x \mu) \subseteq \text{carrier } (\mu)$ and $\text{carrier } (\mu x) \subseteq \text{carrier } (\mu)$. Now using Theorem V, we have that $(x) \text{carrier } (\mu) \subseteq \text{carrier } (\mu)$ and $\text{carrier } (\mu)(x) \subseteq \text{carrier } (\mu)$, for each $x \in K$ so that if $H = \text{carrier } (\mu)$, then $KHK \subseteq H$. We note now that Theorem VI implies that $H = \text{carrier } (\mu)$ is a compact simple semigroup. However, $KHK$ is clearly an ideal of $S$ and hence an ideal of $H$ so that, in view of the simplicity of $H$, $KHK = H$. On the other hand, $KHK \subseteq K$ and
the minimality of $K$ implies that $KHK = K$. Together, these facts imply that $K = H = \text{carrier } (\mu)$, completing the proof of the theorem.

**Definition 1.3**
If $\beta$ is a measure defined on the Borel sets of a closed subset $A$ of $S$, then the measure $\bar{\beta} \in \mathcal{M}(S)$, defined at a Borel set $E$ of $S$ by means of the equality $\bar{\beta}(E) = \beta(A \cap E)$, will be called the usual extension of $\beta$ to $B(S)$ (we naturally assume here that $\beta$ is a complex valued countably additive regular Borel measure defined on the Borel sets of $A$).

**Definition 1.4**
The set of all $\lambda \in \mathcal{S}$ such that $\text{carrier } (\lambda) \subseteq K$ will be denoted by $\mathcal{K}$.

**Theorem 1.15**
If the kernel $K$ is a group, $\mu$ is normalized Haar measure on $K$ and $\bar{\mu}$ denotes the usual extension of $\mu$ to the Borel sets of $S$ then:

(A) $\bar{\mu}^2 = \bar{\mu} \in \mathcal{S}$.

(B) $\bar{\mu}S = (\bar{\mu}) = \mathcal{S}\bar{\mu}$.

(C) $L^1(\bar{\mu})$ is an ideal of $\mathcal{M}(S)$.

**Proof:** In order to see that (A) holds we first note that the properties of normalized Haar measure $\mu$ and the definition of $\bar{\mu}$ immediately imply that $\bar{\mu} > 0$ and moreover that

$$||\bar{\mu}|| = ||\bar{\mu}||(S) = \bar{\mu}(S) = \mu(K \cap S) = \mu(K) = 1$$

so that $\bar{\mu} \in \mathcal{S}$. We have that $(\bar{\mu})$ is an invariant measure on
\( k \) so that if \( E \in B(S) \) we may write:

\[
\bar{m}(E) = \int \int \pi_E(xy) \bar{m}(x) d\bar{m}(y) = \int \int \pi_E(xy) d\bar{m}(x) d\bar{m}(y) = \int \int \pi_E(xy) d\bar{m}(x) d\bar{m}(y) = \int \pi_E(y) d\bar{m}(y) = \bar{m}(E).
\]

Thus, \( \bar{m}(E) = \bar{m} \in S \) and \( (A) \) holds.

To see that \( (B) \) holds, let \( \lambda \in \bar{S} \) and \( E \in B(S) \). We now write (denoting by \( e \) the identity element of \( K \))

\[
\bar{m}(E) \lambda(A) = \int \int \pi_E(xy) d\bar{m}(x) d\lambda(A) = \int \int \pi_E(xy) d\bar{m}(x) d\lambda(A) = \int \pi_E(x) d\bar{m}(x) d\lambda(A) = \bar{m}(E) \lambda(A) = \bar{m}(E) \bar{m} = \bar{m}(E).
\]

This implies \( \bar{m}(E) \bar{m} = \bar{m} \) and a similar argument, together with the use of the Fubini Theorem implies \( \bar{m} \bar{m} = \bar{m} \).

If \( x \in S \), then clearly \( x \in \bar{S} \) and \( (B) \) implies that \( x \bar{m} = \bar{m} = \bar{m} x \). This certainly implies that if \( E \in B(S) \) and \( \bar{m}(E) = 0 \), then \( x \bar{m}(E) = \bar{m} x(E) = 0 \). Now according to Theorem 1.5 this is equivalent to \( \bar{L}(\bar{m}) \) being an ideal of \( M(S) \) and thus \( (C) \) holds. This completes the proof of the theorem.

**Remark**

In general, a measure \( \beta \in M(S) \) will be called a left zero (right zero) of a subset \( A \) of \( M(S) \) if \( \beta A = (\beta)(A\beta = (\beta)). \)

The measure \( \beta \) will simply be called a zero of \( A \) if it is both a left and a right zero for \( A \).
Theorem 1.16
If $\mu^2 = \mu \in \mathcal{S}$ and $L^1(\mu)$ is an ideal of $\mathcal{M}(S)$ then:

(A) $K = \text{carrier } (\mu)$ is the countable union of maximal groups $H_e = eSe$ with $e^2 = e \in K$.

(B) The restriction of the measure $e\mu e$ to the Borel sets of $H_e$ is normalized Haar measure on $H_e$.

(C) The $\mu$-measure of each maximal group $H_e$ is positive.

Proof: Theorem 1.9 assures us that $K = \text{carrier } (\mu)$ and moreover, that $\mu \in K(S)$. Also, Theorem VII implies that $\mu^2 = \mu \in K(S)$. Now Theorem IV implies that $K$ is the disjoint union of maximal groups $H_e = eSe$ for $e^2 = e \in K$. Hence, since $\mu$ is countably additive and $\mu(K) < \infty$ there can be at most countably many maximal groups having positive $\mu$-measure. Since $\mu \in K(S)$ we have, for $e^2 = e \in K$, that $(e\mu e)(e\mu e) = e(\mu e\mu)e = e\mu e$. Thus, Wendel's results (Theorems V and VIII) imply that carrier $(e\mu e) = eKe = eSe$ and the restriction of the idempotent measure $e\mu e \in \mathcal{S}$ to the Borel sets of $eSe$ is normalized Haar measure on $eSe$.

Since $L^1(\mu)$ is an ideal of $\mathcal{M}(S)$ we have that $e\mu e \ll \mu$ for each $e^2 = e \in K$. If for some $e^2 = e \in K, \mu(eSe) = 0$, then by absolute continuity we would have $e\mu e(eSe) = 0$, which is absurd since $e\mu e(eSe) = 1$. Thus, there are only countably many maximal groups $H_e = eSe$ for $e^2 = e \in K$. This completes the proof of the theorem.
Theorem 1.17

The kernel $K$ is a group if and only if there exists $\mu^2 = \mu \in \mathcal{S}$ such that $L_1(\mu)$ is an ideal of $M(S)$ and a sequence $\{e_n \mu e_n\}_{n=1}^{\infty}$ converging in norm to $\mu$, where $e_n^2 = e_n \in K$.

Proof: If $K$ is a group and $\overline{m}$ is the usual extension of normalized Haar measure on $K$ to the Borel sets of $S$, then by Theorem 1.15 $L_1(\overline{m})$ is an ideal of $M(S)$ and $\overline{m}^2 = \overline{m} \in \mathcal{S}$. Moreover, $\overline{e} = \overline{m}$ where $e^2 = e \in K$.

Clearly, the constant sequence $\{e \overline{e}\}$ converges in norm to $\overline{m}$.

Conversely, if there exists $\mu^2 = \mu \in \mathcal{S}$ such that $L_1(\mu)$ is an ideal and a norm convergent sequence $\{e_n \mu e_n\}_{n=1}^{\infty}$ for $e_n^2 = e_n \in K$, then Theorem 1.16 implies that $K = \text{carrier } (\mu)$ and $K$ is the countable disjoint union of maximal groups of the form $H_e = eSe$, for $e^2 = e \in K$. If there are infinitely many distinct elements in $\{e_n \mu e_n\}_{n=1}^{\infty}$, then we may choose infinitely many distinct elements $\{e_n\}_{n=1}^{\infty}$ (for sake of notation the same index will be used) such that for each integer $n$, $\|e_n \mu e_n - \mu\| < \frac{1}{n}$. However, for $S_n = e_n Se_n$ we may write $|1 - \mu(S_n)| = |e_n \mu e_n(S_n) - \mu(S_n)| < \|e_n \mu e_n - \mu\| < \frac{1}{n}$. This means that the sequence $\{\mu(S_n)\}_{n=1}^{\infty}$ has limit 1, which is absurd since $\mu(S) < \infty$ implies that the series $\sum_{n=1}^{\infty} \mu(S_n)$ must converge. Hence there can be only finitely many distinct elements of the original sequence $\{e_n \mu e_n\}_{n=1}^{\infty}$ in which case $e_k \mu e_k = \mu$, for some integer $k$. Therefore, using
Theorem V, we see that carrier \((e_k \mu e_k) = e_k S e_k = \text{carrier} (\mu) = K\) so that the kernel is a group. This completes the proof of the theorem.

Theorem 1.18
The following conditions are equivalent:

(A) The kernel is a group.

(B) There exists a unique idempotent in \(S\) whose \(L^1\)-space is an ideal of \(M(S)\).

(C) There exists a unique idempotent in \(S\) whose carrier is the kernel.

Proof: To see that (A) implies (B) we note from Theorem 1.15 that if \(\overline{m}\) is the usual extension of Haar measure on \(K\) to the Borel sets of \(S\), then \(\overline{m}^2 = \overline{m} \in \mathcal{S}\) and \(L^1(\overline{m})\) is an ideal of \(M(S)\). Moreover, if there exists \(\lambda^2 = \lambda \in \mathcal{S}\) and \(\mathcal{L}^1(\lambda)\) is an ideal of \(M(S)\), then Theorem 1.9 implies that carrier \((\lambda) = K\) and hence that \(\lambda \in \mathcal{K}\). However, Theorem 1.15 also implies that \(\overline{m}\) is a zero for \(\mathcal{K}\) so that \(\overline{m} \lambda = \overline{m}\) and \(\lambda \overline{m} = \overline{m}\). These together imply that \(\overline{m} = (\lambda \overline{m})(\overline{m} \lambda) = \lambda \overline{m} \lambda\). Since by Theorem 1.9 \(\lambda \in \mathcal{K}(\mathcal{S})\), or equivalently, \(\lambda \mathcal{S} \lambda = \{\lambda\}\) we have that \(\overline{m} = \lambda \overline{m} \lambda = \lambda\) and that condition (B) holds.

In order to see that (B) implies (C) suppose that \(\lambda^2 = \lambda \in \mathcal{S}\) is the unique idempotent such that \(\mathcal{L}^1(\lambda)\) is an ideal of \(M(S)\). In this case we have immediately from Theorem 1.9 that carrier \((\lambda) = K\). Now if there exists \(\mu^2 = \mu \in \mathcal{S}\) such that carrier \((\mu) = K\), then Theorem VII and Theorem IV...
together imply that \( \mu S \mu = \{\mu\} \) and \( \lambda S \lambda = \{\lambda\} \). Hence, using Theorem 1.11, we have that \( \frac{\lambda + \lambda \mu}{2} \) and \( \frac{\lambda + \lambda \mu}{2} \) are idempotents in \( S \). Note now that since \( L^1(\lambda) \) is an ideal of \( \mathcal{M}(S) \) the measures \( \lambda, \frac{\lambda + \lambda \mu}{2} \) and \( \frac{\lambda + \lambda \mu}{2} \) are mutually absolutely continuous with respect to each other and hence generate the same \( L^1 \)-spaces. However, due to the uniqueness condition in (B) we have that

\[
\frac{\lambda + \lambda \mu}{2} = \lambda = \frac{\lambda + \lambda \mu}{2}
\]

and hence, \( \lambda = \mu \lambda = \lambda \mu \). The latter equality yields:

\[
\lambda = \lambda^2 = (\mu \lambda)(\lambda \mu) = \mu \lambda \lambda \mu = \mu \lambda \mu = \mu \text{ so that }
\]

\( \mu = \lambda \) and (B) implies (C).

In order to see that (C) implies (A), let \( \lambda^2 = \lambda \in S \) be the unique idempotent whose carrier is \( K \) and fix \( x \in K \). As before \( \frac{\lambda + x \lambda}{2} \) and \( \frac{\lambda + x \lambda}{2} \) are idempotents in \( S \) (Theorem 1.11). Now Theorem V implies that carrier \((x \lambda) = (x) \text{ carrier } \lambda = xK \subset K \) and carrier \((\lambda x) = \text{ carrier } \lambda x = K \subset K \), so that carrier \((\frac{\lambda + x \lambda}{2}) = \text{ carrier } (\frac{\lambda + x \lambda}{2}) = \text{ carrier } (\lambda) = K \). Thus by the uniqueness we have:

\[
\frac{\lambda + x \lambda}{2} = \lambda = \frac{\lambda + x \lambda}{2}
\]

so that, \( \lambda = x \lambda = \lambda x \). Now, using Theorem V once again, the latter equality yields:

\[
K = xK = Kx
\]

so that \( K \) is a group. This completes the proof of the theorem.
**Theorem 1.19**

If $\mu^2 = \mu \in S$, $L^1(\mu)$ is an ideal of $M(S)$ and the mappings $x \mapsto x\mu$, $x \mapsto \mu x$ are continuous in the norm topology, then carrier $(\mu') = K$ and $K$ is the finite disjoint union of maximal groups $eSe$, where $e^2 = e \in K$.

Proof: Theorem 1.9 assures that carrier $(\mu) = K$ and that $K$ is the countable disjoint union of maximal groups $eSe$ for $e \in K$. Now fix $e^2 = e \in K$ and $x \in eS$. Choose $\varepsilon = \frac{1}{2}$. In view of the continuity condition there exists an open set $U$ about $x$ such that $y \in U$ implies:

$$||y\mu - x\mu|| < \varepsilon.$$ 

Now since $x\mu \in S$ and carrier $(x\mu) = xK$:

\[|y\mu(xK) - 1| = |y\mu(xK) - x\mu(xK)| < ||y\mu - x\mu|| < \frac{1}{2}.\]

Therefore, the $K$ relatively open set $U \cap K$ must be contained in $eS$. If this were not the case, then we could choose $y' \in [(U \cap K) - eS]$ and $g^2 = g \in K$ such that $y' \in gS$ and $gS \cap eS = \emptyset$ (this follows since Theorem IV implies that $K$ is the disjoint union of the minimal right ideals $fS$ for $f^2 = f \in K$). However, we would then have that $y'\mu(xK) = 0$ since carrier $(y'\mu) = y'K \subset y'S \subset gS$ and $xK \subset xS \subset eS$. This, together with inequality (1), would imply that $1 < \frac{1}{2}$.

Hence, $U \cap K \subset eS$ and since $x$ and $e$ were arbitrary we have that each minimal right ideal $eS$, for $e^2 = e \in K$, is an open set in the relative topology on $K$. Moreover, since $K$ is compact there can be only finitely many minimal right ideals. A similar argument shows that there can be only a finite number of minimal left ideals. Together, these
facts imply that there can be only finitely many maximal groups in $K$ since a minimal left ideal and a minimal right ideal intersect in a maximal group (Theorem III). This completes the proof of the theorem.

**Definition 1.5**

If $Q \subseteq M(S)$, then the statement that $Q$ has a weak-* approximate identity means that there exists a net $\left\{ \lambda_a \right\}_{a \in A}$ contained in $Q \cap S$ such that for $\eta \in Q$, $\left\{ \eta \lambda_a \right\}_{a \in A}$ and $\left\{ \lambda_a \eta \right\}_{a \in A}$ converge in the weak-* topology to $\eta$. We make a similar definition for norm approximate identity replacing 'weak-*' by 'norm' in the definition.

**Theorem 1.20**

If $K$ is a group and $\overline{m}$ is normalized Haar measure on $K$ extended to the Borel sets of $S$, then $L^1(\overline{m})$ has a norm approximate identity.

**Proof:** In the argument we will work in the relative topology on $K$. Let $e^2 = e$ denote the identity of $K$ and let $U_a$, for $a$ in some index set $A$, be a neighborhood in $K$ containing $e$.

Since $K$ is a compact Hausdorff space there exists a continuous function $f'_a$ on $K$ such that $0 \leq f'_a \leq 1$, $f'_a(e) = 1$ and $f'_a(y) = 0$ for $y \in (K - U_a)$. Let $f_a$ be a continuous extension of $f'_a$ to all of $S$ and define $\lambda'_a$ at $E \in B(S)$ by:

$$\lambda'_a(E) = \int_E f_a(x)d\overline{m}(x).$$

Since $f_a(e) = 1$, $f_a \geq 0$ on $K$, and $f_a$ is continuous it is clear that $\lambda'_a(S) > 0$. Now define $\lambda_a$ at $E \in B(S)$ by:
\[ \lambda_a(E) = \frac{\lambda_a'(E)}{\lambda_a(S)}. \]

We now have that \( \lambda_a(S) = 1 \) and \( \lambda_a > 0 \) so that \( \lambda_a \in \mathcal{S} \). Moreover, \( \lambda_a \in L^1(\mu) \) by definition. If \( \eta \in L^1(\mu) \) then there exists a measurable \( \mu \)-integrable function \( g \) (Radon-Nikodym) such that for \( E \in \mathcal{B}(S) \), \( \gamma(E) = \int_E g(y) d\mu(y) \).

Moreover,
\[ \| \lambda_a \eta - \eta \| = \sup_{p \in C(S), \ p \neq 1} \left| \int p(z) d(\lambda_a \eta - \eta) \right| \]

Now for \( p \in C(S) \), \( p \neq 0 \) and \( g_a = \frac{f_a}{\lambda_a(S)} \) we have, using the invariance of \( \mu \) and the Fubini theorem,
\[ \left| \int p(z) d(\lambda_a \eta - \eta) \right| = \left| \int \int p(xy) g_a(x) g(y) d\mu(x) d\mu(y) - \int p(y) g(y) d\mu(y) \right| = \left| \int \int p(xy) g_a(x) g(y) d\mu(x) d\mu(y) - \int \int g_a(x) p(y) g(y) d\mu(x) d\mu(y) \right| = \left| \int \int g_a(x) g(y) d\mu(x) d\mu(y) - \int \int g_a(x) p(xy) g(xy) d\mu(x) d\mu(y) \right| = \left| \int \int g_a(x) (|g(y) - g(xy)| d\mu(y))^2 d\mu(x) \right|.

Now there exists an open set, say \( U_a \), about the identity such that for \( x \in U_a \), \( \int |g(y) - g(xy)| d\mu(y) < \varepsilon \) (Naimark [10]) so that in the former inequality the last item is less than:
\[ \varepsilon \int g_a(x) d\mu(x) = \varepsilon \int g_a(x) d\mu(x) = \varepsilon \frac{1}{\lambda_a(S)} \int f_a(x) d\mu(x) = \varepsilon, \]
so that \( \| \lambda_a \eta - \eta \| < \varepsilon \). A similar argument shows that \( \| \eta \lambda_a - \eta \| < \varepsilon \). Thus, the set \( A \) directed by inclusion of the open sets \( U_a \) together with the measures \( \lambda_a \) for \( a \in A \) form the net acting as a norm approximate identity for \( L^1(\mu) \).
(norm convergence implies weak-- convergence). Note, moreover, that \( \{ \lambda_a \}_{a \in A} \) converges weak-- to \( e \) where \( e^2 = e \) is the identity of \( K \). This completes the proof of the theorem.

**Theorem 1.21**

If \( \{ \lambda_a \}_{a \in A} \) is a net in \( M(S) \) converging weak-- to \( \lambda \in M(S) \), then for each \( \beta \in M(S) \), \( \{ \lambda_a \beta \}_{a \in A} \) and \( \{ \beta \lambda_a \}_{a \in A} \) converge weak-- to \( \lambda \beta \) and \( \beta \lambda \) respectively.

**Proof:** Suppose \( \{ \lambda_a \}_{a \in A} \) converges weak-- to \( \lambda \) and \( \beta \in M(S) \). Now for each \( f \in C(S) \) we have that the function \( g(x) = \int f(xy)d\beta(y) \) is an element of \( C(S) \) (the mapping \( x \rightarrow f_x \) is a continuous mapping of \( S \) into \( C(S) \), the latter being given the supremum norm topology [5]). For \( \epsilon \in A \) the Fubini theorem yields:

\[
\lambda_a \beta(f) = \int \int f(xy)d\lambda_a(x)d\beta(y) = \int (\int f(xy)d\beta(y))d\lambda_a(x) = \int g(x)d\lambda_a(x) = \lambda_a(g).
\]

Now since \( g \in C(S) \) we have that \( \lambda_a(g) \) converges to \( \lambda(g) = \int g(x)d\lambda(x) = \int (\int f(xy)d\beta(y))d\lambda(x) = \int \int f(xy)d\lambda(x)d\beta(y) = \lambda \beta(f) \). Therefore \( \{ \lambda_a \beta(f) \}_{a \in A} = \{ \lambda_a(g) \}_{a \in A} \) converges to \( \lambda \beta(f) \). Now since \( f \) was an arbitrary element of \( C(S) \), \( \{ \lambda_a \beta \}_{a \in A} \) converges weak-- to \( \lambda \beta \).

A similar argument shows that \( \{ \beta \lambda_a \}_{a \in A} \) converges weak-- to \( \beta \lambda \). This completes the proof of the theorem.

**Theorem 1.22**

The kernel, \( K \), is a group if and only if there exists \( \mu^2 = \mu \in S \) such that \( L^1(\mu) \) is an ideal of \( M(S) \) and \( L^1(\mu) \) has a weak-- approximate identity converging weak-- to an element of \( K \).
Proof: If there exists \( \mu^2 = \mu \in S \) such that \( L^1(\mu) \) is an ideal of \( M(S) \) and a weak-* approximate identity \( \{ \lambda_a \}_{a \in A} \) converging to \( \eta \in \mathbf{K} \), then by Theorem 1.16, \( \text{ carrier } (\mu) = \mathbf{K} \) and \( \mathbf{K} \) is the union of maximal groups \( eSe \) for \( e^2 = e \in \mathbf{K} \).

Fix an idempotent, say, \( e^2 = e \in \mathbf{K} \) and let \( \beta \) be normalized Haar measure on \( eSe \) extended to the Borel sets of \( S \). Theorem 1.16 implies that \( \beta \in L^1(\mu) \) so that, by virtue of Theorem 1.21, \( \{ \lambda_a \beta \}_{a \in A} \) and \( \{ \beta \lambda_a \}_{a \in A} \) converge weak-* to \( \beta \). On the other hand, \( \{ \lambda_a \}_{a \in A} \) converges weak-* to \( \eta \in \mathbf{K} \) so that \( \{ \lambda_a \beta \}_{a \in A} \) and \( \{ \beta \lambda_a \}_{a \in A} \) converge respectively to \( \lambda \beta \) and \( \beta \lambda \) (Theorem 1.21). Since weak-* limits are unique we now have that \( \eta \beta = \beta = \beta \eta \). Moreover, since \( \eta, \beta \in S \), Theorem V implies that:

\[
\text{ carrier } (\beta) = \text{ carrier } (\eta) = \text{ carrier } (\beta) \quad \text{and} \quad \text{ carrier } (\eta) = \text{ carrier } (\beta).
\]

Note that \( \text{ carrier } (\beta) = eSe \) is a group so that:

(1) \( \text{ carrier } (\eta) \subseteq \text{ carrier } (\beta) = \text{ carrier } (\beta) \). We will now show that \( \text{ carrier } (\eta) \subseteq eSe \). If this is not the case there exists \( y \in \text{ carrier } (\eta) \subseteq \mathbf{K} \) such that \( y \notin eSe \). However, since \( y \in \mathbf{K} \), \( y \) must be an element of some maximal group \( fSf \) where \( f^2 = f \in \mathbf{K} \) and \( eSe \cap fSf = \emptyset \). Moreover, for any \( x \in S \), we have \( yxy \in fSf \). The latter is clearly false since equation (1) implies that, in particular, \( yey \in eSe \). Hence, \( \text{ carrier } (\eta) \subseteq eSe \) and since \( e^2 = e \in \mathbf{K} \) was chosen arbitrarily we have that \( \text{ carrier } (\eta) \) is contained in each maximal group \( eSe \). Since the latter are disjoint, there can be only one maximal group, namely, \( \mathbf{K} \) itself.
The converse follows immediately from Theorems 1.15 and 1.20 together with the fact that the weak--* approximate identity \( \{ \lambda_a \}_{a \in A} \) for \( L^1(\mathfrak{m}) \) in Theorem 1.20 converges weak--* to \( e \), where \( e^2 = e \) is the identity of \( K \). This completes the proof.

**Theorem 1.23**
The set \( \overset{*}{K} \) is weak--* closed and hence weak--* compact.

**Proof:** If \( \{ \lambda_a \}_{a \in A} \) is a net in \( \overset{*}{K} \) converging weak--* to \( \lambda \), then since \( \overset{*}{S} \) is weak--* closed we have \( \lambda \in \overset{*}{S} \). We now show that carrier \( (\lambda) \subset K \). If this is not the case, there \( x \in \text{carrier} (\lambda) \) such that \( x \notin K \). Since \( S \) is regular there exists \( f \in C(S) \) such that \( 0 \leq f \leq 1 \) and:

\[
 f(y) = \begin{cases} 
 0 & \text{if } y \in K \\
 1 & \text{if } y = x 
\end{cases}
\]

Moreover, by virtue of the continuity of \( f \), there exists an open set \( V \) containing \( x \) such that for \( z \in V \), \( f(z) > 0 \). Now \( \lambda_a(f) = \int f(p) d\lambda_a(p) = \int K f(p) d\lambda_a(p) = 0 \) for each \( a \in A \), and since \( \lambda_a(f) \) converges to \( \lambda(f) \) we also have \( \lambda(f) = 0 \). Note now that \( \int_V f(y) d\lambda(y) > 0 \) since \( f \) is positive on \( V \) and \( x \in \text{carrier} (\lambda) \) implies \( \lambda(V) > 0 \). However, this would imply that:

\[
 0 = \lambda(f) = \int f(y) d\lambda(y) = \int_V f(y) d\lambda(y) + \int_{S-V} f(y) d\lambda(y) \geq \int_V f(y) d\lambda(y) > 0.
\]

This contradiction shows that carrier \( (\lambda) \subset K \), completing the proof of the theorem.
Theorem 1.24

The kernel, $K$, is a group if and only if there exists $\mu^2 = \mu \in \mathcal{S}$ such that $L^1(\mu)$ is an ideal of $M(S)$ and $L^1(\mu)$ has a weak-$*$ approximate identity.

Proof: Note first that if $\mu^2 = \mu \in \mathcal{S}$ and $L^1(\mu)$ is an ideal of $M(S)$ having a weak-$*$ approximate identity $\{a^a\}_{a \in A}$, then by virtue of Theorem 1.9 and Definition 1.5 we have that carrier $(\mu) = K$ and $a^a \in K$, all $a \in A$. Theorem 1.23 implies that $K$ is compact in the weak-$*$ topology so that net $\{a^a\}_{a \in A}$ has a weak-$*$ cluster point $\eta \in K$, and hence a subnet $\{b\}_{b \in B} \subset \{a^a\}_{a \in A}$ converging weak-$*$ to $\eta$. It is clear that $\{b\}_{b \in B}$ is also a weak-$*$ approximate identity for $L^1(\mu)$. Theorem 1.22 now implies that $K$ is a group.

The converse is an immediate consequence of Theorems 1.15, 1.20 and 1.22. This concludes the proof of the theorem.

Theorem 1.25

The following are equivalent:

(A) $S$ is a group

(B) $K$ is a group and $\gamma \in M(S)$, $\gamma L^1(\bar{m}) = \{0\}$ imply $\gamma = 0$, where $\bar{m}$ is the usual extension of normalized Haar measure on $K$ to the Borel sets of $S$.

Proof: If $S$ is a group, then clearly $S = K$ and $m = \bar{m}$.

Moreover, Theorem 1.20 assures the existence of a weak-$*$ approximate identity, $\{a\}_{a \in A}$, for $L^1(m)$. Now if $\gamma \in M(S)$ and $\gamma L^1(m) = \{0\}$ then since $a \in L^1(m)$ for each $a \in A$ we
have that $\gamma \lambda_a = 0$ for each $a \in A$. However, Theorem 1.20 implies that $(\lambda_a)_{a \in A}$ converges weak-$*$ to $e$ where $e^2 = e$ is the identity of $S$. Hence, by Theorem 1.21, $(\gamma \lambda_a)_{a \in A}$ converges weak-$*$ to $\gamma e = \gamma$. This implies that $\gamma = 0$.

Conversely, if (B) holds, then fix $y \in S$ and $k \in L^1(\overline{m})$. If $e^2 = e$ denotes the identity of $K$, then since carrier $(B)$ is contained in $K$ we have for $E \in B(S)$:

$$e_k(E) = \int_K \int_K \pi_E(pq)d\beta(q) = \int_K \pi_E(eq)d\beta(q) = \int_K \pi_E(q)d\beta(q) = \beta(E)$$

so that $e_k = \beta$. Now,

$$(ye - y)_k - (ye) - y_k = y(e_k) - y_k = y_k - y_k = 0$$

and hence $ye - y = 0$. This, together with Theorem V implies that $ye = y$. Therefore, since $e \in K$ and $K$ is an ideal of $S$, we have that $y = ey \in K$ so that $S \subset K$ and hence $S = K$ is a group. This completes the proof.

**Remark**

Note that condition (2) of (B) in the theorem could be replaced by:

$\gamma \in M(S)$ and $L^1(\overline{m}) \gamma = \{0\}$ imply $\gamma = 0$.

**Definition 1.6**

If $k \in M(S)$, the mapping $L_k(R_\gamma)$ is defined at $\gamma \in M(S)$ by $L_k(\gamma) = k \gamma$ ($R_\gamma(\gamma) = \gamma_k$) and maps $M(S)$ into $M(S)$. If $x \in S$, then $L_x(R_\gamma)$ will simply be denoted by $L_x(R_x)$. 
Theorem 1.26

The following are equivalent:

(A) $K$ is a group.

(B) There exists $\mu^2 = \mu \in \mathcal{S}$ such that:

1. $L\mu$ and $R\mu$ map $L^1(\mu)$ into $L^1(\mu)$ for each $\beta \in M(S)$.

2. $L\mu$ and $R\mu$ restricted respectively to the sets $K\mu$ and $\mu K$ are one to one maps.

Proof: If $K$ is a group and $m$ is the extension of normalized Haar measure on $K$ to the Borel sets of $S$, then by Theorem 1.15 $L^1(m)$ is an ideal of $M(S)$ so that the condition (1) of (B) clearly holds. Moreover, $m$ is an idempotent and a zero for $S$ so that $L_m$ and $R_m$ are clearly one to one on $mK \cup \overline{mK} = (m)$.

Conversely, if there exists $\mu^2 = \mu \in \mathcal{S}$ satisfying condition (B), then clearly from (1) $L^1(\mu)$ is an ideal of $M(S)$. Theorem 1.16 implies carrier $(\mu) = K$. Now fix $e^2 = e \in K$, $f^2 = f \in K$; then $L\mu(e\mu) = \mu e\mu$ and $L\mu(f\mu) = \mu f\mu$. By virtue of Theorem VII and Theorem 1.16 we have that $\mu e\mu = \mu$ so that $\mu e\mu = \mu f\mu = \mu$. Hence due to condition (2) we have $e\mu = f\mu$ and using Theorem V, $eK = fK$. However, $eK = eS$ and $fK = fS$ so that since the kernel is the disjoint union of minimal right ideals of the form $gS$ for $g^2 = g \in K$, we have that there must be exactly one minimal right ideal. Moreover, using a similar argument together with the fact that $R\mu$ is one to one on $\mu K$ we see that there is exactly one minimal left ideal so that the kernel is a group. This completes the proof of the theorem.
Remark
The following example, due to Collins [4], will demonstrate the need for the uniqueness conditions of Theorem 1.18. Let $S$ be the semigroup consisting of the elements $\{0,1,2,3\}$, with the multiplication given on the following page, and endowed with the discrete topology. In this case $S \cong K$ and $S$ has two minimal left ideals, $(0,1)$ and $(2,3)$ (the two minimal right ideals being, by necessity, $(0,2)$ and $(1,3)$). A measure $\mu \in M(S)$ will be represented in the form $\mu = (a,b,c,d)$ where $a$, $b$, $c$, and $d$ are complex numbers. That is, the measure $\mu$ is determined by attaching weights $a$, $b$, $c$, and $d$ to the points 0, 1, 2, and 3 respectively, and convolution multiplication is simply formal quadrational multiplication.

A measure $\mu$ will be an idempotent if and only if the following equations are satisfied:

1. $(a+b)(a+c) = a$
2. $(a+b)(b+d) = b$
3. $(c+d)(a+c) = c$
4. $(c+d)(b+d) = d$.

In particular let $\mu = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ and $\lambda = \left(\frac{1}{8}, \frac{1}{8}, \frac{3}{8}, \frac{3}{8}\right)$. Since equations (1) through (4) are satisfied by both $\mu$ and $\lambda$ we clearly have $\mu^2 = \mu \in S$ and $\lambda^2 = \lambda \in S$. Now note that $L^1(\mu) = L^1(\lambda) = M(S)$ and carrier ($\mu$) = carrier ($\lambda$) = $K$; however, the kernel $K = S$ is not a group. This example also shows that the conditions of Theorem 1.19 are not sufficient for the kernel to be a group.
**THEOREM 2.1**

The following are equivalent:

(A) The kernel is a group.

(B) There exists \( \mu^2 = \mu \in S \) such that \( L_x \) and \( R_x \) map \( L^2(\mu) \) onto \( L^2(\mu) \) for each \( x \in K \).

**Proof:** If \( K \) is a group, let \( \bar{m} \) be extended normalized Haar measure on \( K \). We will first show that \( L_x \) (and similarly \( R_x \)) map \( L^2(\bar{m}) \) into \( L^2(\bar{m}) \). Suppose \( \alpha \in L^2(\bar{m}) \), then \( L_x(\alpha) = x\alpha \). If \( E \in \mathcal{B}(S) \) and \( \bar{m}(E) = 0 \), then by Theorem 1.15, \( x\bar{m}(E) = \bar{m}(xE) = 0 \). On the other hand, \( x\alpha(E) = \alpha(xE) = 0 \) since \( \alpha \in L^1(\bar{m}) \). Since \( x\bar{m} = \bar{m} \), we have:

\[
\sup_{E_i \in \mathcal{P}_S} \sum_{E_i \in \mathcal{P}_S} \left| \frac{x\alpha(E_i)}{\bar{m}(E_i)} \right|^2 = \sup_{E_i \in \mathcal{P}_S} \sum_{E_i \in \mathcal{P}_S} \left| \frac{\alpha(xE_i)}{\bar{m}(xE_i)} \right|^2.
\]

Now Theorem 1.12 implies that \( \{(x:E_i) : E_i \in \mathcal{P}_S\} \) is a partition of \( S \) so that, since \( \alpha \in L^2(\bar{m}) \), the latter supremum is finite. Hence \( L_x \) maps \( L^2(\bar{m}) \) into \( L^2(\bar{m}) \). In order to show that \( L_x \) is onto, we choose \( \alpha \in L^2(\bar{m}) \) and let \( x' \) denote the inverse of \( x \) in \( K \). Clearly, \( x'\alpha \in L^2(\bar{m}) \) and \( L_x(x'\alpha) = xx'\alpha = e\alpha \) where \( e^2 = e \) denotes the identity of \( K \). However, since \( \alpha \in L^1(\bar{m}) \) we have for \( f \in C(S) \) that:

\[
e\alpha(f) = \int f(ex)d\alpha(x) = \int f(ex)d\alpha(x) = \int f(x)d\alpha(x) = \alpha(f)
\]
so that $\alpha \lambda = \lambda$ and similarly $\lambda e = \lambda$. Therefore, $\alpha = e\lambda$ and the preimage of $\lambda \in L^2(\mathbb{R})$ is $x\lambda$. A dual argument shows that $R_x$ has the desired property for $x \in K$. Conversely, if there exists $\mu^2 = \mu \in S$ satisfying condition (B) then by virtue of Theorem 1.13, carrier $(\mu) = K$ and Theorem VII implies $\mu \cdot \mu = \{\mu\}$. Now fix $e^2 = e \in K$.

Since $L_e$ and $R_e$ map $L^2(\mu)$ onto $L^2(\mu)$ we have that there exists $\gamma_1, \gamma_2 \in L^2(\mu)$ such that $L_e(\gamma_1) = \mu$ and $R_e(\gamma_2) = \mu$. Hence $e\gamma_1 = \gamma_2 e = \mu$ so that $\mu = \mu^2 = (e\gamma_1)(\gamma_2e) = e\gamma_1\gamma_2e = e(e\gamma_1)(\gamma_2e)e = e\mu e$. Using Theorem V, $K = \text{carrier}(\mu) = \text{carrier}(e\mu e) = eKe = eSe$. Since $e^2 = e \in K$ was arbitrarily chosen and $K$ is the disjoint union of maximal groups $fSf$ for $f^2 = f \in K$ we have that $K = eSe$ is a group. This completes the proof of the theorem.

The following result obtained in the proof of Theorem 2.1 will now be stated as a corollary.

**Corollary 2.1**

If $K$ is a group, $e^2 = e$ denotes the identity of $K$, $\alpha \in M(S)$ and carrier $(\alpha) \subset K$, then

$$e\alpha = \alpha = \alpha e.$$

**Lemma 2.1**

If $\mu^2 = \mu \in S$ and $e^2 = e \in K$ then:

(A) $e\mu(E)\mu(eS) \subseteq \mu(E)$ for $E \in B(S)$.

(B) $\mu(e(E))\mu(Se) \subseteq \mu(E)$ for $E \in B(S)$. 
Proof: If $\mu^2 = \mu \in \mathcal{S}$ and $e^2 = e \in K$, then for $E \in B(S)$:

\begin{align*}
(1) \mu(E) = \mu^2(E) &= \int x E(x)d\mu(x)d\mu(y) - \\
&= \int \left( \int x E(x)d\mu(y) \right) d\mu(x) = \int x \mu(E)d\mu(x) - \\
&= \int x \mu(E)d\mu(x) + \int \left( \int x \mu(E)d\mu(x) \right) E(x) d\mu(x).
\end{align*}

Now note that Theorem VI implies that the mapping $x \mapsto \mu(f(x))$ is constant on each minimal right ideal, for each $f \in C(S)$. In other words, if $p$ and $q$ are in the same minimal right ideal then:

\begin{align*}
q \mu(f) = \int f(xy)dq(x)dx - \int f(qy)d\mu(y) = \int f_q(y)d\mu(y) = \\
\mu(f_q) = \mu(f_p) = \int f_p(y)d\mu(y) = \int f(py)d\mu(y) = \\
\int f(xy)d(p(x)d\mu(y) = p \mu(f).
\end{align*}

Now in view of the identification of $\mathcal{M}(S)$ with $C(S)^*$ we have that $p \mu = q \mu$ if $p$ and $q$ are in the same minimal right ideal. Using inequality (1) and the latter statement applied to the minimal right ideal $eS$ we have

\begin{align*}
\mu(E) \leq \int x \mu(E)d\mu(x) = e \mu(E) \mu(eS).
\end{align*}

A similar argument shows that (B) holds, completing the proof of the lemma.

**Theorem 2.2**

The kernel is a group if and only if there exists a unique idempotent in $\mathcal{S}$ whose $L^2$-space is invariant.

Proof: If $K$ is a group and $\overline{\mu}$ is normalized Haar measure on $K$ extended to the Borel sets of $S$, then Theorem 1.15 implies that $L^1(\overline{\mu})$ is an ideal of $\mathcal{M}(S)$. Hence, if $x \in S$ and
\( \beta \in L^2(\overline{m}) \), then \( x \beta \) and \( \beta x \) are elements of \( L^1(\overline{m}) \).

Moreover, in the proof of Theorem 2.1 it was actually shown that \( L^2(\overline{m}) \) was closed under multiplication by point measures so that \( L^2(\overline{m}) \) is invariant. Now if there exists \( \lambda^2 = \lambda \in \mathcal{S} \) such that \( L^2(\lambda) \) is invariant, then using Theorem 1.5 we have that \( L^1(\lambda) \) is an ideal of \( M(S) \). On the other hand, since \( K \) is a group, Theorem 1.18 implies that \( \lambda = \overline{m} \) and hence that \( \overline{m} \) is unique relative to having an invariant \( L^2 \)-space.

Conversely, suppose there exists a unique idempotent \( \mu^2 = \mu \in \mathcal{S} \) whose \( L^2 \)-space is invariant. By virtue of Theorems 1.5 and 1.16, \( L^1(\mu) \) is an ideal of \( M(S) \) and carrier \( (\mu) = K \). If \( e^2 = e \in K \), then by Theorem 1.11 we have that \( \frac{e + e\mu}{2} \) is an idempotent measure in \( K(S) \). We will now show that 
\( L^2(\mu) = L^2(\frac{\mu + e\mu}{2}) \), noting first that, by Theorem 1.5, 
\( L^1(\mu) = L^1(\frac{\mu + e\mu}{2}) \). Now if \( \lambda \in L^2(\mu) \), then:

\[
\sup_{P_S} \sum_{E_1 \in P_S} \frac{2|\lambda(E_1)|^2}{\mu(E_1) + e\mu(E_1)} = 2 \sup_{P_S} \sum_{E_1 \in P_S} \frac{|\lambda(E_1)|^2}{\mu(E_1)}
\]

and the latter supremum is finite so that \( \lambda \in L^2(\frac{\mu + e\mu}{2}) \).

On the other hand, if \( \lambda \in L^2(\frac{\mu + e\mu}{2}) \) we then write the equality (for \( P_S \) a partition of \( S \), \( E_1 \in P_S \), and \( \mu(E_1) \neq 0 \))

\[
(1) \quad \frac{2|\lambda(E_1)|^2}{\mu(E_1)} = \frac{2|\lambda(E_1)|^2}{\mu(E_1) + e\mu(E_1)} + \frac{2|\lambda(E_1)|^2 e\mu(E_1)}{(\mu(E_1) + e\mu(E_1))\mu(E_1)}
\]

Now note, by Lemma 2.1, that for any \( E \in B(S) \) we have that
Moreover, since \( L^2(\mu) \) is an ideal of \( M(S) \) and carrier \( (\mu) = K \), Theorem 1.16 implies that \( \mu(es) > 0 \). Hence an application of Lemma 2.1 in this case yields for any \( E_i \in P_S \):

\[
\frac{e\mu(E_i)}{\mu(E_i)} < \frac{1}{\mu(es)}.
\]

Therefore, we have:

\[
2 \sup_{P_S} \sum_{E_i \in P_S} \frac{|\lambda(E_i)|^2}{\mu(E_i)} < 2 \left( 1 + \frac{1}{\mu(es)} \right) \sup_{P_S} \sum_{E_i \in P_S} \frac{|\lambda(E_i)|^2}{\mu(E_i) + e\mu(E_i)}.
\]

Since the latter supremum is finite we have that \( \lambda \in L^2(\mu) \) and finally that \( L^2(\mu) = L^2(\frac{\mu + e\mu}{2}) \). In view of the uniqueness condition we must have \( \mu = \frac{\mu + e\mu}{2} \), that is, \( \mu = e\mu \). A similar argument using the measure \( \frac{\mu + e\mu}{2} \) yields \( \mu = e\mu e \). Together, these facts imply:

\[
\mu = e\mu e = (e\mu)(\mu e) = e\mu \mu e = e\mu e.
\]

Using Theorem V, \( K = eKe = eSe \); that is, the kernel is a group. This completes the proof of the theorem.

**Theorem 2.3**

The kernel is a group if and only if there exists a unique idempotent in \( S \) whose \( L^2 \)-space is an ideal of \( M(S) \).

**Proof:** If \( K \) is a group and \( \tilde{m} \) is normalized Haar measure on \( K \) extended to the Borel sets of \( S \), then by Theorem 1.15, \( L^1(\tilde{m}) \) is an ideal of \( M(S) \). Fix \( \lambda \in L^2(\tilde{m}) \) and \( \gamma \in M(S) \), \( \gamma \neq 0 \). Clearly \( \lambda \in L^1(\tilde{m}) \) and there exists a Borel
measurable function \( f \) such that \( \int |f(y)|^2 \, d\bar{m}(y) \) exists and for \( B \in \mathcal{B}(S) \), \( \mathbf{A}(B) = \int_B f(y) \, d\bar{m}(y) \). Let \( e^2 = e \) denote the identity of \( K \), and note by Corollary 2.1 that \( e \mathbf{A} = \mathbf{A} = e \mathbf{A} \) so that \( \gamma \mathbf{A} = \gamma (e \mathbf{A}) = (\gamma e) \mathbf{A} \). Recall now that carrier \((\gamma e)\) is the complement of the largest open set having \( |\gamma e| \)-measure zero and that \( |\gamma e| \leq |\gamma|e \). Hence we have, using Theorem V, carrier \((\gamma e)\) = carrier \((|\gamma e|)\) which is contained in carrier \((|\gamma|e)\) = carrier \((\frac{|\gamma|}{||\gamma||}e)\) = carrier \((\frac{|\gamma|}{||\gamma||}e)\) ⊆ \( K \).

We will now show that there exists a function \( g \in L^2(\bar{m}) \) ( \( L^2(\bar{m}) \) will denote the Borel measurable and \( \bar{m} \)-square integrable functions defined on \( S \) for which \( \gamma \mathbf{A}(B) = \int_B g(y) \, d\bar{m}(y) \). We have, using the Fubini theorem and the invariance of \( \bar{m} \) on \( K \):

\[
\gamma \mathbf{A}(B) = (\gamma e) \mathbf{A}(B) = \int \int \pi_B(xy) \, d(\gamma e)(x) \, d\mathbf{A}(y) = \int \int \pi_B(xy) \, d\mathbf{A}(y) \, d(\gamma e)(x) = \int \int \pi_B(xy) f(y) \, d\bar{m}(y) \, d(\gamma e)(x) = \int \int \pi_B(y) f(x^{-1}y) \, d\bar{m}(y) \, d(\gamma e)(x) = \int \int \pi_B(y) f(x^{-1}y) \, d(\gamma e)(x) \, d\bar{m}(y) = \int \int \pi_B(y) \left( \int f(x^{-1}y) \, d(\gamma e)(x) \right) \, d\bar{m}(y) = \int \left( \int f(x^{-1}y) \, d(\gamma e)(x) \right) \, d\bar{m}(y).
\]

The Fubini theorem implies that \( g \) defined by:

\[
g(y) = \begin{cases} 
\int f(x^{-1}y) \, d(\gamma e)(x), & \text{if } y \in K \\
0, & \text{if } y \notin K
\end{cases}
\]

is a Borel measurable function. Moreover, using the Schwarz inequality and the invariance of \( \bar{m} \) on \( K \) we have:
so that $g \in L^2(\overline{m})$. This implies that $\gamma \alpha \in L^2(\overline{m})$ and similarly $\alpha \gamma \in L^2(\overline{m})$. It is clear that if $\gamma = 0$ then $\gamma \alpha$ and $\alpha \gamma$ are elements of $L^2(\overline{m})$. Moreover, if $\lambda^2 = \lambda \in S$ and $L^2(\lambda)$ is an ideal of $M(S)$, then as in Theorem 2.2, $L^1(\lambda)$ and $L^1(\overline{m})$ are ideals of $M(S)$ so that, since the kernel is a group, $\lambda = \overline{m}$.

Conversely, if there exists a unique idempotent $\mu \in S$ whose $L^2$-space is an ideal, then, as in the converse of Theorem 2.2, we have that for $e^2 = e \in K$:

$$L^2\left(\frac{\mu + e\mu}{2}\right) = L^2(\mu) = L^2\left(\frac{\mu + e\mu}{2}\right)$$

and hence that $\mu = e\mu e$ and $K = eSe$ is a group. This completes the proof of the theorem.

Remark

The four element example given in Chapter II also demonstrates the need for the uniqueness conditions in the latter theorems.
SELECTED BIBLIOGRAPHY


BIOGRAPHY

The author was born June 22, 1937, in Alexandria, Louisiana. He attended public schools in Marksville and Iowa, Louisiana. In 1959 he received a Bachelor of Science degree from McNeese State College and received a Master of Science degree from Louisiana State University in 1961. At this time, he is working toward the degree of Doctor of Philosophy in Mathematics.
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April 1, 1963