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Tower Families of Ribbon Graphs

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1. Introduction

The notion of a *recursive family of graphs* was introduced by N. Biggs, R. Damerell, and D. Sands [BDS72], where the authors characterize an infinite sequence of graphs as a recursive family if the associated sequence of Tutte polynomials satisfies a linear recurrence relation with coefficients in $\mathbb{Z}[X, Y]$. Following this work, Noy and Ribó [NR04] defined a large class of infinite sequences of graphs, termed *recursively constructible families of graphs*, and showed all such families to be recursive.

In this paper we adapt those notions to infinite sequences of oriented ribbon graphs (hereafter, simply referred to as ribbon graphs) by means of the rank polynomial, a variant of the Bollobás-Riordan-Tutte polynomial [BR01]. We formalize the method of transfer matrices introduced by N. Biggs [Big10] (see also [CMNN03, NR04]) and show that a tower family of ribbon graphs, the analogue of a recursively constructible family of graphs, has associated rank polynomials satisfying a linear recurrence relation with coefficients in $\mathbb{Z}[X, Y, Z]$. As an application, we analyze the infinite sequence of ribbon graphs associated with a periodic link sequence. Utilizing a specialization of the Bollobás-Riordan-Tutte polynomial described in [DFK$^+$08], we obtain a recursive structure for the Jones polynomials of this link sequence.

The paper is structured as follows. Section 2 recalls the definition of a ribbon graph and that of the rank polynomial of a ribbon graph. In Section 3 we discuss amalgamation of ribbon graphs as a way of introducing a special class of infinite ribbon graph sequences, the tower families. In Section 4 we formalize the method of transfer matrices and establish our main result, that every tower family of ribbon graphs has a recursive associated polynomial sequence. To add concreteness, we devote Section 5 to two worked examples. The final section features an analysis of the family of dual ribbon graphs associated with the family of periodic links made from copies of $T_{3,4}$, this analysis yielding results for the Jones polynomial sequence according to the known specialization.

2. Ribbon Graphs

Before introducing ribbon graphs, we review several useful concepts from graph theory, as found in [GT01].

Definition 2.1. A graph G is a pair (V, E) consisting of a set V of vertices and a set E of edges. An edge $e \in E$ has an endpoint set $V(e)$ containing either one or two elements of V. Taken together, the $V(e)$ form the incidence structure of G, denoted I.

Remark. An edge e containing a vertex v as an endpoint (i.e., $v \in V(e)$) is said to be *incident* to v, while vertices sharing an edge are said to be *adjacent* or to be *neighbors*. If two vertices are separated by a sequence of edges, we say that they are *connected*. In this way we may decompose the vertex set V into maximal subsets of connected vertices, the *connected components* of G.

Definition 2.2. Let $G = (V_G, E_G)$ and $G' = (V_{G'}, E_{G'})$ be graphs. A *graph map* $f : G \to G'$ consists of a vertex map $f : V_G \to V_{G'}$ and an edge map $f : E_G \to E_{G'}$ such that the incidence strucutre is preserved. That is, for an edge $e \in E_G$, f maps the endpoints of e to the endpoints of $f(e)$. A *graph embedding* is simply an injective graph map.

Heuristically, a ribbon graph can be viewed as a multi-graph (i.e. loops and multiple edges are allowed) equipped with a cyclic order on the edges at every vertex. Formally, ribbon graphs can be viewed from a variety of perspectives, as seen in the recent monograph *Graphs on Surfaces and Their Applications* [LZ04]. We adopt the topological (*fat graph*) perspective in our definition.

Definition 2.3. Let $D = (V, E)$ be a graph with vertex set V, edge set E, and with each vertex equipped with a cyclic ordering on its incident edges. From D we form the ribbon graph $\mathbb{D}$ as a union of two disjoint sets of closed two-discs $V(\mathbb{D})$ (fat vertices) and $E(\mathbb{D})$ (ribbons), corresponding to vertices in V and edges in E respectively, and satisfying

a. Fat vertices intersect ribbons at disjoint line segments.
b. Each such line segment lies on the boundary of exactly one fat vertex and exactly one ribbon.
c. Each ribbon has exactly two such line segments.

Remark. By glueing two-cells along the boundary of a ribbon graph $\mathbb{D}$, we obtain a closed surface. We refer to these boundary components as boundary circles.

Definition 2.4. As with regular graphs, ribbon graphs have a rich combinatorial strucuture. Below we list a number of quantities needed in what follows.

a. The number of vertices: $v(\mathbb{D}) = |V(\mathbb{D})|$.
b. The number of edges: $e(\mathbb{D}) = |E(\mathbb{D})|$.
c. The number of boundary circles: $bc(\mathbb{D})$.
d. The number of connected components: $k(\mathbb{D})$
e. The rank: $r(\mathbb{D}) = v(\mathbb{D}) - k(\mathbb{D})$
f. The nullity: $n(\mathbb{D}) = e(\mathbb{D}) - r(\mathbb{D})$
g. The Euler characteristic: $\chi_{\mathbb{D}} = v(\mathbb{D}) - e(\mathbb{D}) + bc(\mathbb{D}) = 2k(\mathbb{D}) - 2g(\mathbb{D})$.

To conclude this section, we recall the definition of the rank polynomial, a modified version of the *Bollobás-Riordan-Tutte (BRT) polynomial* introduced in [BR01].

Definition 2.5. Given a ribbon graph $\mathbb{D}$, we define the rank polynomial of $\mathbb{D}$, $R(\mathbb{D}; X, Y, Z) \in \mathbb{Z}[X, Y, Z]$, as the sum

$$R(\mathbb{D}; X, Y, Z) = \sum_{\mathbb{H} \subset \mathbb{D}} X^{r(\mathbb{D}) - r(\mathbb{H})} Y^{n(\mathbb{H})} Z^{2g(\mathbb{H})},$$

over all spanning ribbon subgraphs $\mathbb{H}$ of $\mathbb{D}$.

3. Amalgamation of Ribbon Graphs

We adapt a notion defined for infinite sequences of graph to infinite sequences of ribbon graphs, namely that of *recursive constructibility* (see [NR04] for details on this construction for graph sequences). Essential to this adaptation is the notion of *amalgamation of ribbon graphs*, again built upon amalgamation of graphs.

Gross and Tucker define the graph amalgamation $G \sqcup_f G'$ of graphs G and G' with respect to the subgraph isomorphism $f : H \to H'$ by identifying H and H' according to f. See [GT01] for further details. Alternatively, given a graph A and embeddings $\iota_1 : A \hookrightarrow G$ and $\iota_2 : A \hookrightarrow G'$, we amalgamate by identifying $\iota_1(A)$ and $\iota_2(A)$ according to ι_1 and ι_2. Note that the amalgamation $G \sqcup_A G'$ depends upon ι_1 and ι_2, although we suppress the dependence for notational simplicity.

This type of amalgamation does not specify information about cyclic order, so it is insufficient for our purposes. To capture this information, we consider the *valence* associated with a vertex in $V(\mathbb{D})$.

Definition 3.1. Let $D = (V, E)$ be a graph, and let $v \in V(D)$. The *valence* of v in D is defined to be the number of edges in $E(D)$ incident to the vertex v, with loops counted twice.

As a last step before discussing ribbon graph amalgamation, we require the notion of a ribbon graph embedding.

Definition 3.2. Given ribbon graphs $\mathbb{A}$ and $\mathbb{D}$, we say that $\phi : \mathbb{A} \hookrightarrow \mathbb{D}$ is a ribbon graph embedding if the underlying graph map $\phi : A \hookrightarrow D$ is a graph embedding preserving the cyclic ordering of the vertices in A.

Given a ribbon graph $\mathbb{A}$ embeddable both in $\mathbb{D}$ and in $\mathbb{M}$ via ribbon graph embeddings ϕ and ψ respectively, we obtain the graph amalgamation $D \sqcup_A M$. To properly amalgamate ribbon graphs, we impose a condition on valence, namely that the valence of a vertex in $D \sqcup_A M$ be attained by one of its identified vertices in D or in M.

Definition 3.3. Conditions as outlined in the paragraph above, we define the *ribbon graph amalgamation* $\mathbb{D} \sqcup_{\mathbb{A}} \mathbb{M}$ by considering first the graph amalgamation $D \sqcup_A M$ and noting that each of the vertices in this graph has a well-defined cyclic ordering on its incident edges. In particular, those vertices in the image of A have well-defined cyclic orderings by our valence condition. Hence we are able to thicken $D \sqcup_A M$ to form the ribbon graph $\mathbb{D} \sqcup_{\mathbb{A}} \mathbb{M}$.

With this technical portion of the paper out of the way, we can define the main class of study, *tower families*.

Definition 3.4. Let $\{\mathbb{D}_n\}$ be an infinite sequence of ribbon graphs. The sequence forms a *tower family* if there is an associated triple ($\mathbb{M}$,$\mathbb{A}$,$\{\phi_n\}$), $\mathbb{M}$ and $\mathbb{A}$ ribbon graphs with $\mathbb{A} \subset \mathbb{M}$, $\{\phi_n\}$ a family of ribbon graph embeddings $\phi_n : \mathbb{A} \hookrightarrow \mathbb{D}_n$, such that $\mathbb{D}_{n+1} = \mathbb{D}_n \sqcup_{\mathbb{A}} \mathbb{M}$ by ϕ_n and the natural inclusion $\iota : \mathbb{A} \hookrightarrow \mathbb{M}$.

We refer to $\mathbb{M}$ as the pattern graph of $\{\mathbb{D}_n\}$ and to $\mathbb{A}$ as the amalgamation graph in $\mathbb{M}$. The point of the above definition is that the pattern graph contains all necessary information in a transition from $\mathbb{D}_n$ to $\mathbb{D}_{n+1}$, allowing the transition between members of the sequence to be described as attachment of pattern copies $\mathbb{M}$ along $\mathbb{A}$.

4. States and the Transfer Matrix

Here we extend the transfer matrix method of Noy and Ribó [NR04] by means of *states* assigned to the spanning ribbon subgraphs of members of a tower family $\{\mathbb{D}_n\}$. The motivation for our state definition is the idea that, given two distinct spanning ribbon subgraphs of $\mathbb{D}_n$ having certain shared properties, each spanning ribbon subgraph will experience the same variation in rank and in boundary circles when transitioning to a spanning subgraph of $\mathbb{D}_{n+1}$ by means of amalgamation with a spanning subgraph $\mathbb{M}_0 \subset \mathbb{M}$, amalgamation occuring along an appropriate spanning subgraph $\mathbb{A}_0 \subset \mathbb{A}$.

In what follows, let $\{\mathbb{D}_n\}$ be a tower family of ribbon graphs with pattern graph $\mathbb{M}$ and amalgamation graph $\mathbb{A}$. Note that we often speak of amalgamation of a

spanning ribbon subgraph $\mathbb{H}_n$ of $\mathbb{D}_n$ with a spanning ribbon subgraph $\mathbb{M}_0$ of the pattern graph. Implicit in this statement is the idea that common to $\mathbb{H}_{n-1}$ and $\mathbb{M}_0$ is a spanning subgraph $\mathbb{A}_0$ of $\mathbb{A}$ along which amalgamation is performed.

We begin by rewriting the expression for $R(\mathbb{D}_{n+1})$ as

$$R(\mathbb{D}_{n+1}) = \sum_{\mathbb{H}_{n+1}=\mathbb{H}_n \sqcup \mathbb{M}_0} X^{r(\mathbb{D}_{n+1})-r(\mathbb{H}_{n+1})} Y^{n(\mathbb{H}_{n+1})} Z^{2g(\mathbb{H}_{n+1})}$$

$$= \sum_{\mathbb{H}_n} X^{r(\mathbb{D}_{n+1})-r(\mathbb{H}_n)} Y^{n(\mathbb{H}_n)} Z^{2g(\mathbb{H}_n)} \sum_{\mathbb{M}_0} X^{-\Delta r(\mathbb{H}_n,\mathbb{M}_0)} Y^{\Delta n(\mathbb{H}_n,\mathbb{M}_0)} Z^{2\Delta g(\mathbb{H}_n,\mathbb{M}_0)},$$

$$= X^{an+b-1} Z^{an+b} \sum_{\mathbb{H}_n} X^{-r(\mathbb{H}_n)} Y^{n(\mathbb{H}_n)} Z^{-r(\mathbb{H}_n)+n(\mathbb{H}_n)-bc(\mathbb{H}_n)}$$

$$\sum_{\mathbb{M}_0} X^{a-\Delta r(\mathbb{H}_n,\mathbb{M}_0)} Y^{\Delta n(\mathbb{H}_n,\mathbb{M}_0)} Z^{a+\Delta n(\mathbb{H}_n,\mathbb{M}_0)-\Delta r(\mathbb{H}_n,\mathbb{M}_0)-\Delta bc(\mathbb{H}_n,\mathbb{M}_0)},$$

where $\Delta r, \Delta n, \Delta bc$ and Δg give the variation on rank, nullity, number of boundary circles, and genus in transitioning from $\mathbb{H}_n$ to $\mathbb{H}_n \sqcup \mathbb{M}_0$ and where $a = v(\mathbb{M}\backslash\mathbb{A})$ and $b = v(\mathbb{D}_0)$, so that $v(\mathbb{H}_{n+1}) = a(n+1) + b$. Note that we decompose $r(\mathbb{D}_{n+1})$ as $r(\mathbb{D}_{n+1}) = v(\mathbb{D}_{n+1}) - 1 = a(n+1) + b - 1$ in the sum above. The genus $g(\mathbb{H}_{n+1})$ is rewritten in the last sum according to the formula for the genus given in Section 1. It is crucial to understand that the variations for a particular $\mathbb{M}_0$ are completely determined by Δk and Δbc, Δk the analogous variation on the number of connected components.

Motivated by the rearrangement above, we define the state of a spanning subgraph $\mathbb{H}_n$ at each stage of a tower family $\{\mathbb{D}_n\}$. The state will be a pair, $s = (s_c, s_b)$, treating separately the variation in connected components in s_c and the variation in the number of boundary circles in s_b. Recall that $\mathbb{D}_{n+1}$ is the amalgamation $\mathbb{D}_{n+1} = \mathbb{D}_n \sqcup_{\mathbb{A}} \mathbb{M}$ by ϕ_n and ι.

Definition 4.1. Let V_0 be the set of vertices in $\mathbb{A}$ having neighboring vertices in $V(\mathbb{M})\backslash V(\mathbb{A})$. The component state s_c of a spanning ribbon subgraph $\mathbb{H}_n \subset \mathbb{D}_n$ is the partition of $\phi_n(V_0)$ induced by the connected components of $\mathbb{H}_n$. We call this the component partition.

The definiton of a boundary state is more involved. We begin by introducing the objects to be sorted, sectors.

Definition 4.2. The sectors of V_0 in $\mathbb{D}_n$ are components of the complement of the intervals in the boundary of the vertex discs in $\phi_n(V(\mathbb{A}))$ along which edges of $\mathbb{D}_n$ are attached, these componenets having the additional property that they are attachment sites for edges of $\mathbb{M}$ in the amalgamation forming $\mathbb{D}_{n+1}$. For a spanning ribbon subgraph $\mathbb{H}_n$ of $\mathbb{D}_n$, we simply take the same set of sectors.

Since the boundaries of ribbon graphs are oriented, there will be an ordering on sector partitions. We extend the concept of partition, to a partition in which the constituent sets have a cyclic order.

Definition 4.3. A cyclic partition of a set S is a partition into subsets, each equipped with a cyclic order.

To define a boundary state, we encode the partition of the sectors given by the boundary circles of $\mathbb{H}_n$ together with the cyclic order induced by the orientation of these boundary components.

Definition 4.4. The *boundary state* s_b of a spanning ribbon subgraph $\mathbb{H}_n$ of $\mathbb{D}_n$ is a cyclic partition of the sectors of V_0 induced by the boundary components of $\mathbb{H}_n$, with cyclic order induced by the orientation of $\mathbb{H}_n$.

In the example 1, if the large circle is V_0 (and the rest is $\mathbb{D}_n$), then there are three sectors (provided edges of $\mathbb{M}$ are attached there). At the end of the section, we give an example in which the cyclic orders on the same partition are distinct.

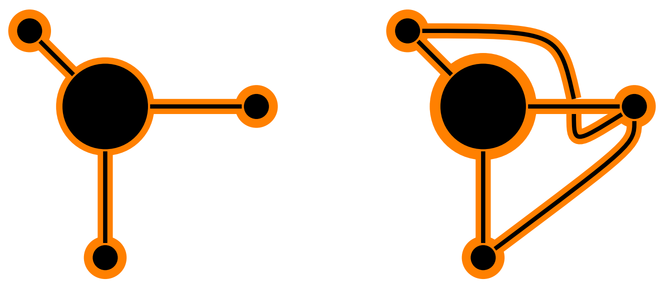


FIGURE 1. Boundary components.

Definition 4.5. The *state* of a ribbon subgraph $\mathbb{H}_n \subset \mathbb{D}_n$ is the ordered pair $s(\mathbb{H}_n) = (s_c(\mathbb{H}_n), s_b(\mathbb{H}_n)$.

Notation. Let $\mathbb{H}_n \subset \mathbb{D}_n$ be a spanning ribbon subgraph, with transfer state $s(\mathbb{H}_n)$. A spanning ribbon subgraph $\mathbb{M}_0 \subset \mathbb{M}$ *induces* a *state transition*, denoted $s(\mathbb{H}_n) \overset{\mathbb{M}_0}{\to} s(\mathbb{H}_n \sqcup \mathbb{M}_0)$.

Now $R(\mathbb{D}_{n+1})$ has the form

$$R(\mathbb{D}_{n+1}) = \sum_{\mathbb{H}_n} X^{r(\mathbb{D}_n)-r(\mathbb{H}_n)} Y^{n(\mathbb{H}_n)} Z^{2g(\mathbb{H}_n)}$$

$$\sum_{s(\mathbb{H}_n) \overset{\mathbb{M}_0}{\to} s(\mathbb{H}_n \sqcup \mathbb{M}_0)} \sum_{\mathbb{M}_0} X^{-\Delta r(\mathbb{H}_n,\mathbb{M}_0)} Y^{\Delta n(\mathbb{H}_n,\mathbb{M}_0)} Z^{2\Delta g(\mathbb{H}_n,\mathbb{M}_0)}.$$

Repeated decomposition of the sums above yields the matrix equation

$$R(\mathbb{D}_{n+1}) = \lambda_0 \cdot \Lambda^{n+1} \cdot \mathcal{I},$$

for $\mathcal{I}$ the all ones column vector, λ_0 the *initial state vector*, and Λ the *transfer matrix*, with constructions outlined below.

Begin by considering spanning subgraphs of $\mathbb{D}_0$, partitioned according to state. To a state s_i we associate a sum of monomials $X^{r(\mathbb{D}_0)-r(\mathbb{H})} Y^{n(\mathbb{H})} Z^{2g(\mathbb{H})}$ for every spanning subgraph $\mathbb{H}$ having transfer state s_i. Then the *initial state vector*, denoted λ_0, has $P(s_i)$ as its i^{th} entry, $P(s_i)$ being the monomial sum associated with the transfer state s_i.

The *transfer matrix*, denoted Λ, has the entries $\Lambda = [P(s_i, s_j)]_{ij}$, where $P(s_i, s_j)$ is the polynomial obtained by taking the sum of monomials $X^{a-\Delta r(\mathbb{H},\mathbb{M}_0)} Y^{\Delta n(\mathbb{H},\mathbb{M}_0)} Z^{2\Delta g(\mathbb{H},\mathbb{M}_0)}$. Here we fix a spanning subgraph $\mathbb{H} \subset \mathbb{D}_0$ with transfer state s_i and sum over all $\mathbb{M}_0$ inducing state transitions to transfer state s_j via amalgamation with $\mathbb{H}$.

The above constructions easily yield our main result, that tower families have recursive associated rank polynomial sequences.

Theorem 4.1. *Let $\{\mathbb{D}_n\}$ be a tower family of ribbon graphs. Then $\{R(\mathbb{D}_n)\}$ is a recursive sequence, satisfying a linear recurrence relation with coefficients in $\mathbb{Z}[X, Y, Z]$.*

Proof. Using the transfer matrix method, we are assured that

$$R(\mathbb{D}_n) = \lambda_0 \cdot \Lambda^n \cdot \mathcal{I}$$

holds for all $n \geq 0$. It remains to show that the ordinary generating function $\sum_{n\geq 0} R(\mathbb{D}_n; X, Y, Z)t^n$ is a rational function in X, Y, Z, and t. Simply rewrite the sum

$$\sum_{n\geq 0} R(\mathbb{D}_n; X, Y, Z)t^n = \sum_{n\geq 0} (\lambda_0 \cdot \Lambda^n \cdot \mathcal{I})t^n = \lambda_0 \cdot \sum_{n\geq 0} (t\Lambda)^n \cdot \mathcal{I} = \lambda_0 \cdot (I - t\Lambda)^{-1} \cdot \mathcal{I}.$$

Since the entries of Λ are rational functions in X, Y, and Z, the entries of the inverse matrix $(I - t\Lambda)^{-1}$ are rational in X, Y, Z, and t. The entries of λ_0 and $\mathcal{I}$ are also rational in X, Y, and Z, so that the generating function is rational in X, Y, Z, and t. □

Remark. For a closed tower family $\mathbb{D}_n$, we apply the above procedure to the underlying tower family $\mathbb{T}_n$ and apply a *closure vector* $\mathcal{E}$ to form $R(\mathbb{D}_n)$. This gives $R(\mathbb{D}_n)$ the form

$$R(\mathbb{D}_n) = \lambda_0 \cdot \Lambda^n \cdot \mathcal{E}.$$

Note that there is a need to enlarge the transfer state set needed for $\mathbb{T}_n$ in certain cases, as demonstrated in the next section.

Now we justify the use of transfer states in the result above by examining separately the component and boundary component states.

Begin by fixing some $n \in \mathbb{Z}_+$ and considering spanning ribbon subgraphs $\mathbb{H} \subset \mathbb{D}_n$ and $\mathbb{M}_0 \subset \mathbb{M}$, where $\mathbb{M}_0$ is chosen so that amalgamation with $\mathbb{H}$ can occur. To the pair $(\mathbb{H}, \mathbb{M}_0)$ we associate two graphs. First, we construct $G_{C_0}(\mathbb{H}, \mathbb{M}_0)$ with vertex set $\phi_n(V_0)$ and with an edge vw between v and w in $\phi_n(V_0)$ if and only if they lie in the same component within $\mathbb{H}$. Next we form $G_C(\mathbb{H}, \mathbb{M}_0)$ by means of amalgamation of $G_{C_0}(\mathbb{H}, \mathbb{M}_0)$ with M_0 along $\phi_n(V_0)$.

Lemma 4.1. *Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be spanning ribbon subgraphs of $\mathbb{D}_n$ which admit the same component partition. Then for any spanning ribbon subgraph $\mathbb{M}_0 \subset \mathbb{M}$ which permits amalgamation with each of the $\mathbb{H}_i$, we have*

$$\Delta k(\mathbb{H}_1, \mathbb{M}_0) = \Delta k(\mathbb{H}_2, \mathbb{M}_0).$$

Proof. We let K_i give the number of componenets of $\mathbb{H}_i$ lacking vertices in $\phi_n(V_0)$. Observe that $k(\mathbb{H}_i) = k(G_{C_0}(\mathbb{H}_i, \mathbb{M}_0)) + K_i$ and $k(\mathbb{H}_i \sqcup \mathbb{M}_0) = k(G_C(\mathbb{H}_i, \mathbb{M}_0)) + K_i$. Hence $\Delta k(\mathbb{H}_i, \mathbb{M}_0) = k(\mathbb{H}_i \sqcup \mathbb{M}_0) - k(\mathbb{H}_i) = k(G_C(\mathbb{H}_i, \mathbb{M}_0)) - k(G_{C_0}(\mathbb{H}_i, \mathbb{M}_0))$. Note

that the graphs $G_C(\mathbb{H}_i, \mathbb{M}_0))$ and $G_{C_0}(\mathbb{H}_i, \mathbb{M}_0))$ are entirely determined by the component state of $\mathbb{H}_i$, so that the graphs are the same for $\mathbb{H}_1$ and $\mathbb{H}_2$. Hence we have $\Delta k(\mathbb{H}_1, \mathbb{M}_0) = \Delta k(\mathbb{H}_2, \mathbb{M}_0)$.

□

Remark. Each of the $\mathbb{H}_i \sqcup \mathbb{M}_0$ will possess a component state based upon connectivity of the $\phi_{n+1}(V_0)$. We note that this connectivity is entirely determined by their connectivity in $G_C(\mathbb{H}_i, \mathbb{M}_0))$, so that $\mathbb{H}_i \sqcup \mathbb{M}_0$ give rise to the same component state.

The proof for the boundary component state proceeds in a similar fashion. To a pair of spanning ribbon subgraphs $(\mathbb{H}, \mathbb{M}_0)$, chosen as before, we associate a ribbon graph encoding all the essential information for the boundary state transition.

We construct $\mathbb{G}_B(\mathbb{H}, \mathbb{M}_0)$ as follows. Let $M = \{\{M_1\}, \{M_2\}, \dots, \{M_p\}\}$ be the boundary component state of $\mathbb{H}$. To each M_i we associate a fat vertex v_i having in its boundary disjoint intervals in correspondence with the sectors in M_i, with the intervals cyclically ordered according to the ordering of their associated sectors in M_i. Gathering these vertices into a set S, we have $V(\mathbb{G}_B(\mathbb{H}, \mathbb{M}_0)) = V(\mathbb{M}_0 \backslash \mathbb{A}) \coprod S$. The edge set of $\mathbb{G}_B(\mathbb{H}, \mathbb{M}_0)$ is constructed by examining the edge set of $\mathbb{M}_0$. Given an edge in $\mathbb{M}_0$ connecting a vertex in $\mathbb{A}$ with a vertex in $\mathbb{M}_0 \backslash \mathbb{A}$, this edge connects to a sector in M. We create an associated edge in $\mathbb{G}_B(\mathbb{H}, \mathbb{M}_0)$ connecting the same vertex in $\mathbb{M}_0 \backslash \mathbb{A}$ to the interval associated with the aforementioned sector (strictly speaking, we attach the edge to the vertex in S containing this interval). Given an edge in $\mathbb{M}_0$ connecting two vertices in $\mathbb{M}_0 \backslash \mathbb{A}$, we create an edge in $\mathbb{G}_B(\mathbb{H}, \mathbb{M}_0)$ connecting the very same vertices. Taken together, these two steps yield the edge set $E(\mathbb{G}_B(\mathbb{H}, \mathbb{M}_0))$.

Lemma 4.2. *Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be spanning ribbon subgraphs of $\mathbb{D}_n$ which admit the same boundary component state. Then for any spanning ribbon subgraph $\mathbb{M}_0 \subset \mathbb{M}$ which permits amalgamation with each of the $\mathbb{H}_i$, we have*

$$\Delta bc(\mathbb{H}_1, \mathbb{M}_0) = \Delta bc(\mathbb{H}_2, \mathbb{M}_0).$$

Proof. We let B_i give the number of boundary componenets of $\mathbb{H}_i$ lacking sectors used in the boundary component states. Observe that $bc(\mathbb{H}_i) = |V(S)| + B_i$ and $bc(\mathbb{H}_i \sqcup \mathbb{M}_0) = bc(\mathbb{G}_B(\mathbb{H}_i, \mathbb{M}_0)) + B_i$. Hence $\Delta bc(\mathbb{H}_i, \mathbb{M}_0) = bc(\mathbb{H}_i \sqcup \mathbb{M}_0) - bc(\mathbb{H}_i) = bc(\mathbb{G}_B(\mathbb{H}_i, \mathbb{M}_0)) - |V(S)|$. Again, the ribbon graphs are entirely determined by the boundary component state of $\mathbb{H}_i$, so that $\Delta bc(\mathbb{H}_1, \mathbb{M}_0) = \Delta bc(\mathbb{H}_2, \mathbb{M}_0)$. Likewise, we note that the $\mathbb{H}_i \sqcup \mathbb{M}_0$ will possess a common boundary component state as given by $\mathbb{G}_B(\mathbb{H}_i, \mathbb{M}_0))$

□

Taken together, these two results yield the theorem below.

Theorem 4.2. *Given a tower family $\{\mathbb{D}_n\}$, its state set partitions the set of spanning ribbon subgraphs of $\mathbb{D}_n$ such that, for any two spanning ribbon subgraphs $\mathbb{H}_1, \mathbb{H}_2 \subset \mathbb{D}_n$ having a common state and for $\mathbb{M}_0$ a spanning subgraph of $\mathbb{M}$ where amalgamation can occur, $\mathbb{H}_1 \sqcup \mathbb{M}_0, \mathbb{H}_2 \sqcup \mathbb{M}_0 \subset \mathbb{D}_{n+1}$ have the same transfer state and*

i. $\Delta k(\mathbb{H}_1, \mathbb{M}_0) = \Delta k(\mathbb{H}_2, \mathbb{M}_0)$
ii. $\Delta bc(\mathbb{H}_1, \mathbb{M}_0) = \Delta bc(\mathbb{H}_2, \mathbb{M}_0)$

Before moving on to the next section, we demonstrate explicitly that the cyclic boundary partition is necessary, and that a standard, non-cyclic, partition will not suffice, through the following simple example:

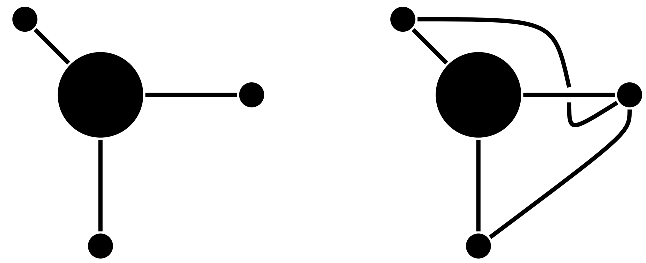


FIGURE 2. Two ribbon graphs, $\mathbb{H}_1$ (left) and $\mathbb{H}_2$ (right) with the same non-cyclic partition and different cyclic partitions.

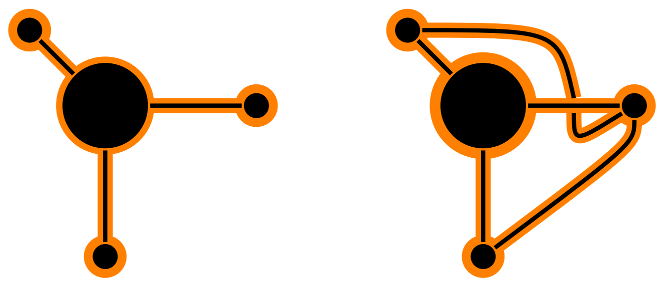


FIGURE 3. Labeling of the boundary components.

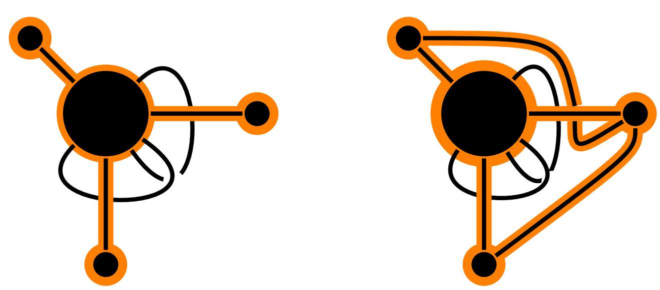


FIGURE 4. Insertion of two isomorphic edges.

5. EXAMPLES

5.1. **The Crossed Ladder Family.** Next we present the family of crossed ladders, $\{\mathbb{L}_n\}$, a tower family with nontrivial genus data. We begin with $\mathbb{L}_0$, a single crossed rung. Heuristically, the formation of $\mathbb{L}_n$ from $\mathbb{L}_{n-1}$ involves the addition of a crossed rung to $\mathbb{L}_{n-1}$. The process is easily comprehended through the following figure, where we show $\mathbb{M}$ and $\mathbb{A}$ to demonstrate the amalgamation.

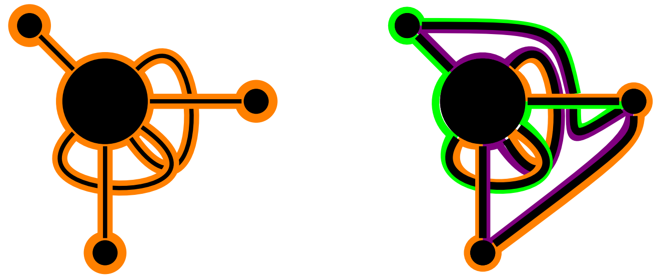
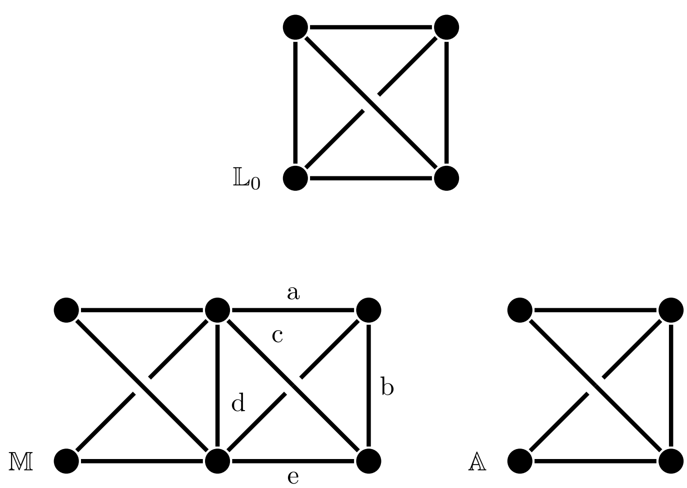


FIGURE 5. Demonstration of boundary components of $\mathbb{H}_1$ and $\mathbb{H}_2$. Note that $bc(\mathbb{H}_1) = 1$, while $bc(\mathbb{H}_2) = 3$.



In this case, $\mathbb{L}_n$ has a corresponding orientable surface with genus $n+1$. There are two amalgamating vertices, each having a single sector where attachment of edges occurs during amalgamation. The three states, s_1, s_2, s_3, are given as follows: s_1 gives connectedness of both pairs, s_2 gives connected vertices and disconnected sectors, and s_3 gives both pairs disconnected.

The computations are much more involved in this example; for the sake of brevity, we present the output only of the amalgamations with an initial state s_1. These appear in the table below, followed by λ_0 and Λ.

Initial state	E	$\Delta k(F,E)$	$\Delta bc(F,E)$	Final state	Contribution to Λ
s_1	ϕ	0	2	s_3	X^2
	{a}, {c}, {d}, {e}	1	1	s_3	$4X$
	{b}	1	1	s_1	X
	{a, b}	2	0	s_1	1
	{a, c}	2	0	s_1	1
	{a, d}	1	2	s_3	XY
	{a, e}	2	0	s_1	1
	{b, c}	2	0	s_1	1
	{b, d}	2	0	s_1	1
	{b, e}	2	0	s_1	1
	{c, d}	2	0	s_1	1
	{c, e}	1	2	s_3	XY
	{d, e}	2	0	s_1	1
	{a, b, c}	2	1	s_1	Y
	{a, b, d}	2	1	s_1	Y
	{a, b, e}	2	1	s_1	Y
	{a, c, d}	2	1	s_1	Y
	{a, c, e}	2	1	s_1	Y
	{a, d, e}	2	1	s_2	Y
	{b, c, d}	2	1	s_1	Y
	{b, c, e}	2	1	s_1	Y
	{b, d, e}	2	1	s_1	Y
	{c, d, e}	2	1	s_2	Y
	{a, b, c, d}	2	2	s_1	Y^2
	{a, b, c, e}	2	2	s_1	Y^2
	{a, b, d, e}	2	0	s_1	Y^2Z^2
	{a, c, d, e}	2	0	s_1	Y^2Z^2
	{b, c, d, e}	2	0	s_1	Y^2Z^2
	{a, b, c, d, e}	2	1	s_1	Y^3Z^2

$$\lambda_0 = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$$

with

$$\begin{aligned} x_1 &= 16 + 7X + X^2 + 2XY + 12Y \\ &\quad + 2Y^2 + 4Y^2Z^2 + Y^3Z^2; \\ x_2 &= 3Y; \\ x_3 &= 8X + 5X^2 + X^3 + 2XY. \end{aligned}$$

We write $\Lambda = [\Lambda_{(1)}, \Lambda_{(2)}]$, where $\Lambda_{(1)}$ is the first column of Λ and $\Lambda_{(2)}$ is the 3×2 remainder matrix of Λ.

$$\Lambda_{(1)} = \begin{pmatrix} 8 + X + 8Y + 2Y^2 + 3Y^2Z^2 + Y^3Z^2 \\ 6 + X + 2Y + 8YZ^2 + 4Y^2Z^2 + Y^3Z^4 \\ 6 + X + 2Y + 8X^{-1} + 4X^{-1}Y + X^{-1}Y^2Z^2 \end{pmatrix}$$

and

$$\Lambda_{(2)} = \begin{pmatrix} 2Y & 4X + X^2 + 2XY \\ 2 + Y^2Z^2 & 4X + X^2 + 2XYZ^2 \\ X^{-1}Y & 4 + 4X + X^2 \end{pmatrix}.$$

Computations grow complex very quickly in this family. For instance, the Bollobás-Riordan-Tutte polynomial for $\mathbb{L}_1$ is

$$\begin{aligned} R(\mathbb{L}_1) \quad = \quad & 192 + 256X + 157X^2 + 55X^3 + 11X^4 + X^5 + 336Y + 254XY + 74X^2Y + 8X^3Y \\ & + 184Y^2 + 72XY^2 + 8X^2Y^2 + 44Y^3 + 8XY^3 + 4Y^4 + 112Y^2Z^2 + 52XY^2Z^2 \\ & + 8X^2Y^2Z^2 + 117Y^3Z^2 + 26XY^3Z^2 + 2X^2Y^3Z^2 + 36Y^4Z^2 + 4XY^4Z^2 \\ & + 4Y^5Z^2 + 15Y^4Z^4 + 7Y^5Z^4 + Y^6Z^4. \end{aligned}$$

Using the rewritten expression for the generating function of the Bollobás-Riordan-Tutte polynomials $R(\mathbb{L}_n; X, Y, Z)$, the denominator of this rational function of X,Y,Z, and t gives the following recurrence relation:

$$R(\mathbb{L}_{n+3}) + p_1 R(\mathbb{L}_{n+2}) + p_2 R(\mathbb{L}_{n+1}) + p_3 R(\mathbb{L}_n) = 0,$$

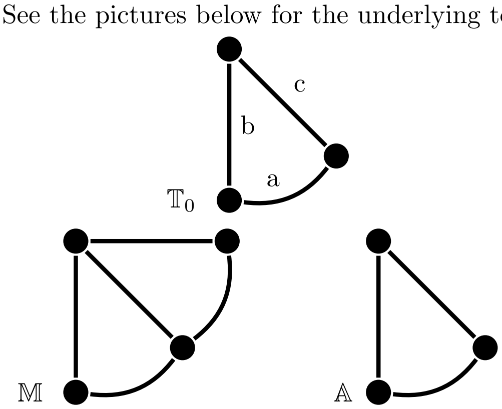
where

$p_1 = -14 - 5X - X^2 - 8Y - Y^3Z^2 - Y^2(2 + 4Z^2);$

$p_2 = 24 + 8YZ^2 + 4Y^3Z^2 + y^5Z^4 + Y^4Z^2(2 + Z^2) + X(14 + 5Y + 4Y^3Z^2 + 4Y^2(1 + 4Z^2))$
$\quad + X^2(4 + 4Y + Y^3Z^2 + Y^2(2 + 4Z^2));$

$p_3 = -(1 + Y)(2Y(4 - 2Y + Y^3Z^4) + X^2(4 + 4Y - 4Y^2Z^2 + 2Y^3Z^2 + Y^4Z^4)$
$\quad + X(8 + 14Y + 3Y^3Z^2 + 4Y^4Z^4 - 2Y(1 + 4Z^2))).$

5.2. **The Wheel Family.** To illustrate the hinted at extension to closed tower families, we begin with a well-understood family, the wheels $\{\mathbb{W}_n\}$. The underlying tower family here is denoted $\{\mathbb{T}_n\}$, consisting of triangle for $\mathbb{T}_0$ and with $\mathbb{T}_n$ formed by adjoining a triangle to $\mathbb{T}_{n-1}$. Given $\mathbb{T}_n$, we form $\mathbb{W}_n$ by closing up the wheel with a single edge. See the pictures below for the underlying tower family.



Notice that each member of this ribbon graph sequence has a planar underlying graph, so that the component state is the sole focus. In the tower family, the state set S has two elements s_1 and s_2 encoding connectivity of the vertices 0 and $n + 2$ in the picture above. However, we cannot simply rely upon this data. We also

need connectivity information of the vertices 1 and $n+2$ in forming $\mathbb{W}_n$. Hence we shall use an augmented state set S' consisting of states s_i as follows. For a spanning ribbon subgraph $\mathbb{H}$ of $\mathbb{T}_n$, $\mathbb{H}$ has state s_1 if vertices $0, 1,$ and $n+2$ are connected, state s_2 with vertices 0 and $n+2$ connected and vertex 1 not connected to either, state s_3 with vertices 1 and $n+2$ connected and vertex 0 not connected to either, state s_4 with vertices 0 and 1 connected and vertex $n+2$ not connected to either, and s_5 with all three vertices lying in separate components. Notice that contributions to states s_1 and s_2 ought to be the same during the formation of $\mathbb{T}_n$ from $\mathbb{T}_{n-1}$, and likewise for the states s_3, s_4 and s_5. Similarly, the pair s_1, s_3 and the triple s_2, s_4, s_5 agree at the formation of $\mathbb{W}_n$ from $\mathbb{T}_n$.

With this state set in mind, we begin by constructing the initial state vector and the transfer matrix for our tower family. Below we give a table of contributions to the initial state vector, edges corresponding to those in the picture above and identifying spanning ribbon subgraphs $\mathbb{H}_0$ of $\mathbb{T}_0$.

E	$r(\mathbb{H}_0)$	$n(\mathbb{H}_0)$	$s(\mathbb{H}_0)$	Contribution to λ_0
ϕ	0	0	s_5	X^2
{a}	1	0	s_3	X
{b}	1	0	s_4	X
{c}	1	0	s_2	X
{a, b}	2	0	s_1	1
{a, c}	2	0	s_1	1
{b, c}	2	0	s_1	1
{a, b, c}	2	1	s_1	Y

Hence we have our initial state vector

$$\lambda_0 = \begin{pmatrix} 3+Y & X & X & X & X^2 \end{pmatrix}.$$

Next we form the transfer matrix for the tower family. We present the contributions from initial states s_i below, with E identifying a spanning ribbon subgraph $\mathbb{M}_0$ of $\mathbb{M}$ to be amalgamated with a spanning ribbon subgraph $\mathbb{H}_0$ of $\mathbb{T}_0$ of the appropriate initial state.

$s(\mathbb{H}_0)$	E	$\Delta r(\mathbb{H}_0, \mathbb{M}_0)$	$\Delta n(\mathbb{H}_0, \mathbb{M}_0)$	$s(\mathbb{H}_0 \sqcup \mathbb{M}_0)$	Contribution to Λ
s_1	ϕ	0	0	s_4	X
s_1	{d}	1	0	s_1	1
s_1	{e}	1	0	s_1	1
s_1	{d,e}	1	1	s_1	Y
s_2	ϕ	0	0	s_5	X
s_2	{d}	1	0	s_2	1
s_2	{e}	1	0	s_2	1
s_2	{d,e}	1	1	s_2	Y
s_3	ϕ	0	0	s_5	X
s_3	{d}	1	1	s_3	1
s_3	{e}	1	1	s_2	1
s_3	{d,e}	2	2	s_1	X^{-1}
s_4	ϕ	0	0	s_4	X
s_4	{d}	1	1	s_1	1
s_4	{e}	1	1	s_4	1
s_4	{d,e}	2	2	s_1	X^{-1}
s_5	ϕ	0	0	s_5	X
s_5	{d}	1	1	s_2	1
s_5	{e}	1	1	s_5	1
s_5	{d,e}	2	2	s_2	X^{-1}

Hence the tower family admits a transfer matrix

$$\Lambda = \begin{pmatrix} 2+Y & 0 & 0 & X & 0 \\ 0 & 2+Y & 0 & 0 & X \\ X^{-1} & 1 & 1 & 0 & X \\ 1+X^{-1} & 0 & 0 & X+1 & 0 \\ 0 & 1+X^{-1} & 0 & 0 & X+1 \end{pmatrix}.$$

Finally, we construct a vector allowing for $\mathbb{W}_n$ to be built from $\mathbb{T}_n$ by means of a single edge addition. Again, we list a table of contributions, followed by the vector. Here we perform amalgamation with spanning ribbon subgraphs $\mathbb{M}_0^{'}$ of $\mathbb{M}^{'}$, a different graph altogether from the amalgamating graph used to form the initial state vector and the transfer matrix above.

$s(\mathbb{H}_0)$	E	$\Delta r(\mathbb{H}_0, \mathbb{M}_0^{'})$	$\Delta n(\mathbb{H}_0, \mathbb{M}_0^{'})$	Contribution to $\mathcal{E}$
s_1	ϕ	0	0	1
s_1	{f}	0	1	Y
s_2	ϕ	0	0	1
s_2	{f}	1	0	X^{-1}
s_3	ϕ	0	0	1
s_3	{f}	0	1	Y
s_4	ϕ	0	0	1
s_4	{f}	1	0	X^{-1}
s_5	ϕ	0	0	1
s_5	{f}	1	0	X^{-1}

$$\mathcal{E} = \begin{pmatrix} 1+Y & 1+X^{-1} & 1+Y & 1+X^{-1} & 1+X^{-1} \end{pmatrix}.$$

The final expression for $R(\mathbb{W}_n)$ is

$$R(\mathbb{W}_n) = \lambda_0 \cdot \Lambda^n \cdot \mathcal{E},$$

whence we can compute, for example,

$$R(\mathbb{W}_2) = 45+8X^3+X^4+52Y+28Y^2+8Y^3+Y^4+4X^2(7+Y)+X(52+25Y+4Y^2).$$

6. Application to Periodic Links

In [DFK$^+$08], the authors obtain a relation between the Jones polynomial of a link projection and the Bollobás-Riordan-Tutte polynomial of an associated ribbon graph, obtained by an *all-A smoothing* of the link projection. We record their main theorem below, advising the interested reader to consult [DFK$^+$08] for full details of the ribbon graph construction and the proof.

Theorem 6.1. *Let $\langle P \rangle \in Z[A, A^{-1}]$ be the Kauffman bracket of a connected link projection diagram P and $\mathbb{D}$ be the oriented ribbon graph of P associated to the all-A-splicing. Then the Bollobás-Riordan-Tutte polynomial $C(\mathbb{D}; X, Y, Z)$ and the Kauffman bracket are related by*

$$A^{-e(\mathbb{D})} \langle P \rangle = A^{2-2v(\mathbb{D})} C(\mathbb{D}; -A^4, A^{-2}\delta, \delta^{-2}),$$

where $\delta = (-A^2 - A^{-2})$.

Remark. The authors use a version of the Bollobás-Riordan-Tutte polynomial related to our rank polynomial by $C(\mathbb{D}; X, Y, Z^2) = R(\mathbb{D}; X-1, Y, Z)$.

A simple example is given by the family of torus knots $\{T_{2,2n+1}\}$, a periodic family formed from copies of the braid word σ_1. Taking the all-A-smoothing of the $T_{2,2n+1}$ yields a family of cycles with $2n+1$ vertices, $\{C_{2n+1}\}$. Applying the transfer matrix method outlined above, we find $C(C_{2n+1}) = \sum_{k=1}^{2n} X^k + (1+Y)$ and the derivative result $J(T_{2,2n+1}) = t^{-n}(1 + \sum_{k=1}^{2n} (-1)^{k+1} t^{-k})$, a consequence of the specialization above and the writhe computation $w(T_{2,2n+1}) = -(2n+1)$.

A less trivial example is the periodic link family formed by taking copies of the torus knot $T_{3,4}$, given by the braid word $(\sigma_1\sigma_2)^4$, so that the member L_n of the derived family is given by the braid word $(\sigma_1\sigma_2)^{4n}$.

The method applied to this family results in a large collection of eighteen states, giving equally complicated transfer vectors. Again, the periodic family of links displays recursiveness in its associated sequence of Jones polynomials. In general, the generated ribbon graph sequence forms a closed tower family to which the methods of this paper can be applied.

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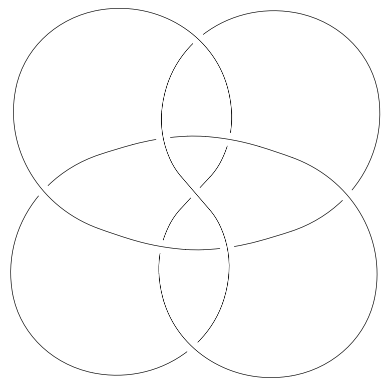

FIGURE 6. Link Diagram for Torus[3,4]

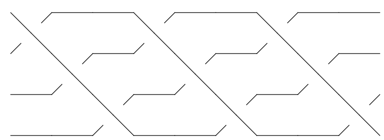

FIGURE 7. Braid Representation of TorusKnot[3,4]

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