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Partial Cosine-Funk Transforms at Poles of the Cosine-λ Transform on Grassmann Manifolds

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PARTIAL COSINE-FUNK TRANSFORMS AT POLES OF THE COSINE-λ TRANSFORM ON GRASSMANN MANIFOLDS

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Abstract

The cosine-λ transform, denoted $\mathcal{C}^\lambda$, is a family of integral transforms we can define on the sphere and on the Grassmannian manifolds of p-dimensional subspaces in $\mathbb{K}^n$ where $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$ or the skew field $\mathbb{H}$ of quaternions. We treat the Grassmannians as the symmetric spaces $\mathrm{SO}(n)/\mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(q))$, $\mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$ and $\mathrm{Sp}(n)/(\mathrm{Sp}(p) \times \mathrm{Sp}(q))$ and we work by analogy with the case of the cosine-λ transform on the sphere, which is also a symmetric space.

The family $\mathcal{C}^\lambda$ extends meromorphically in λ to the complex plane with poles at (among other values) $\lambda = -1, \ldots, -p$. In this dissertation we normalize $\mathcal{C}^\lambda$ and we use well known harmonic analysis tools to evaluate at those poles. The result is a series of integral transforms on the Grassmannians that we can view as *partial* cosine-Funk transforms. The transform that arises at $\lambda = -p$ is the natural Funk transform for the Grassmannians, which was introduced by B. Rubin.

1 Introduction

The cosine-λ transform is defined for functions on the sphere by

$$(\mathcal{C}^\lambda f)(u) = \int_{S^n} |u \cdot v|^\lambda f(v) dv.$$

Integral transforms of this kind have a rich history with connections to many diverse areas of mathematics. In the case of $\lambda = 1$ we have what Lutwak named the "cosine transform", noting that $|u \cdot v| = |\cos(\theta)|$ where θ is the angle between the vectors ([Lut90]). For a brief history of the cosine transform and a long list of references, see [ÓRP]. Here we offer just a few references to give a sense of it: there are connections to convex geometry ([RZ04], [Lut90], [GG99], [AA37]), harmonic analysis and singular integrals ([OR05], [OR06], [Rub98], [Rub02], [Str70]), integral geometry ([GGR84], [Rub98], [Rub99], [Rub03], [Sem63]), and others.

Of central importance to this dissertation is the observation that $\mathcal{C}^\lambda$ has a pole at $\lambda = -1$, and that if we normalize and then take the analytic continuation (a.c.) we get the Funk transform: that is,

$$\underset{\lambda=-1}{\text{a.c.}} \frac{\Gamma(-\lambda/2)}{\Gamma((1+\lambda)/2)} \mathcal{C}^\lambda f(u) = c \int_{u^\perp} f(v) dv$$

where c is computed by setting $f = 1$.

In the present paper we will explore similar relationships for a cosine-λ transform on the Grassmannian manifolds. We will see that an appropriate Funk transform on the Grassmannian similarly arises out of the cosine-λ transform there, and we will also note some important differences from the case on the sphere.

The Grassmannian manifolds $\mathcal{B} = \mathrm{Gr}(p, \mathbb{K}^n)$ are the manifolds of p-dimensional subspaces of $\mathbb{K}^n$ where $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, or the skew field of quaternions $\mathbb{H}$. We will often use the notation $q = n - p$, and throughout we assume $p \leq q$. Our methods here are largely based on the techniques and results of the paper [ÓP12], in which Ólafsson and Pasquale applied harmonic analysis and representation theory tools to the cosine-λ transform. The main result of that paper is to write down the spectrum for $\mathcal{C}^\lambda$ acting on $L^2(\mathcal{B})$. We will also use that result in this dissertation.

The definition of the cosine-λ transform on Grassmannian manifolds is analogous to the cosine transform on the sphere. There is a geometrical way to define $|\mathrm{Cos}(\sigma, \omega)|$ on two elements σ, ω. We follow [ÓP12] on this. Write d for the dimension of $\mathbb{K}$ as a real vector space. We view σ as a dp-dimensional real vector space and take a subset $E \subset \sigma$ containing the zero vector such that the volume of E is 1. Let $P_\omega : \sigma \to \omega$ denote orthogonal projection onto ω. Then we define $|\mathrm{Cos}(\sigma, \omega)| = \mathrm{Vol}_{\mathbb{R}}(P_\omega(E))^{1/d}$. For more details on this function, in particular to see that it is well-defined, see [ÓP12]. From now on we will use it as the appropriate generalization of the $|\cos(\theta)| = |u \cdot v|$ that we used on the sphere.

Having defined $|\mathrm{Cos}(\sigma, \omega)|$, it makes sense to define the $\mathcal{C}^\lambda$ transform on $L^2(\mathcal{B})$ by analogy with the sphere:

$$\mathcal{C}^\lambda f(\omega) = \int_{\mathcal{B}} |\mathrm{Cos}(\sigma, \omega)|^{d\lambda} f(\sigma) d\sigma$$

in the invariant measure. Our choice to put a $d\lambda$ power on the $|\mathrm{Cos}(\sigma, \omega)|$ (rather than λ or some other variant) suits the purposes of this dissertation. The reader will find variations on this in other papers. The choice is largely a matter of convenience to the work at hand.

This $\mathcal{C}^\lambda$ extends analytically to a meromorphic family of transforms. The first pole of $\mathcal{C}^\lambda$ occurs at $\lambda = -1$ and in this dissertation we will be interested in the poles $\lambda = -1, \ldots, -p$. We take an appropriate normalizing function $\gamma(\lambda)$ so that the analytic continuation of $\gamma(\lambda)\mathcal{C}^\lambda$ is entire. For a function $f \in C^\infty(B)$ and a fixed base point $\beta \in \mathcal{B}$ we compute

$$\underset{\lambda=-1}{\text{a.c.}}\, \gamma(\lambda)\mathcal{C}^\lambda f(\beta)$$

explicitly in coordinates using a familiar integral formula for compact symmetric spaces.

The striking result of this computation is an integral transform which is itself a certain cosine-λ transform on a lower-dimensional Grassmannian manifold $\mathcal{B}_1$ evaluated at $\lambda = 1$.

We consider $\mathcal{B}$ as a symmetric space K/L in the usual way: $K = \mathrm{SU}(n, \mathbb{K})$ and $L \cong \mathrm{S}(\mathrm{U}(\mathrm{p}, \mathbb{K}) \times \mathrm{U}(\mathrm{q}, \mathbb{K}))$, and there is an involution τ of K such that L is τ-fixed. In this picture, the base point β is L, but we will continue to use the notation β because we prefer to think of β as an element of a Grassmannian, in which case we think of L as the stabilizer of β.

We write $\mathfrak{k}$ for the Lie algebra of K, and throughout we will write lower case fraktur characters for Lie algebras. It is also customary to denote certain subspaces of Lie algebras by fraktur characters. Thus $\mathfrak{l}$ is the Lie algebra of L. Then $\mathfrak{l}$ is also the (+1)-eigenspace in $\mathfrak{k}$ of τ. Let $\mathfrak{q}$ denote the (-1)-eigenspace of τ. Then $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{q}$, and this is a Cartan decomposition.

We choose $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{q}$. The space $\mathfrak{a}$ has dimension p, the rank of $\mathcal{B}$. It is well known that we may write polar coordinates for K/L using the map $\Phi : L/M \times \mathfrak{a} \to K/L$ defined $(lM, X) \mapsto l\exp(X)L$ where M is the centralizer in L of $\mathfrak{a}$. We restrict the coordinates to a choice of positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ and then restrict further to a fundamental domain $D^+ \subset \mathfrak{a}^+$, which can be parameterized by coordinates $(t_1, \ldots, t_p)$ where $0 \le t_p \le \cdots t_1 \le \pi/2$. The integral $\mathcal{C}^\lambda f(\beta)$ can then be written as an integral over $L \times D^+$. The final analysis will not depend on our choice of $\mathfrak{a}^+$ because L permutes the Weyl chambers transitively.

The lower-rank Grassmannian $\mathcal{B}_1 \subset \mathcal{B}$ arises as follows. Let $\sigma(t_1, \ldots, t_p)$ denote $\Phi(e, (t_1, \ldots, t_p))$, where e is the identity. The fixed base element $\beta \in \mathcal{B}$ is $\sigma(0, \ldots, 0)$. The function

$$|\mathrm{Cos}(\sigma(t_1, \ldots, t_p), \beta)|^\lambda \Big|_{\lambda=-1}$$

becomes infinite at $t_1 = \pi/2$ where $|\mathrm{Cos}(\sigma, \beta)| = 0$. Since L is unitary and fixes β, for any $l \in L$ we have $|\mathrm{Cos}(l\sigma, \beta)| = |\mathrm{Cos}(\sigma, l^{-1}\beta)| = |\mathrm{Cos}(\sigma, \beta)| = 0$. This leads us to consider the set parameterized by $L \times (\pi/2, t_2, \ldots, t_p)$. In coordinate-free terms, this set is

$$\{\sigma \in \mathcal{B} \mid |\mathrm{Cos}(\sigma, \beta)| = 0\}$$

which we will denote by $Z(\beta)$. Clearly $Z(\beta)$ is of interest being the place where $|\mathrm{Cos}(\sigma(t_1, \ldots, t_p), \beta)|^\lambda$ blows up at the poles of $\mathcal{C}^\lambda$.

The set $Z(\beta)$ is not quite the embedded Grassmannian we mentioned. However, when we choose an appropriate subgroup $L_1 \subset L$, the coordinates $L_1 \times (\pi/2, t_2, \ldots, t_p)$ parameterize an embedded submanifold diffeomorphic to $\mathrm{Gr}(p-1, \mathbb{K}^{n-2})$. We call that manifold $\mathcal{B}_1$. Note that $\mathcal{B}_1$ lies in $Z(\beta)$ and we will see that

$$Z(\beta) \subset \bigcup_{l \in L} l\mathcal{B}_1.$$

This $\mathcal{B}_1$ has its own intrinsic cosine-λ transform, which we denote $\mathcal{C}_1^\lambda$. The main result of this analysis is that

$$\underset{\lambda=-1}{\text{a.c.}}\, \gamma(\lambda)\mathcal{C}^\lambda f(\beta) = c\ \mathcal{C}_1^\lambda f^L(\beta_1)|_{\lambda=1}. \tag{1.1}$$

Here c is a constant computed by putting 1 in for f. Throughout, $f^L(x) = \int_L f(lx)dl$, the integral in unit Haar measure. The element β_1 is a base point in $\mathcal{B}_1$ analogous to β.

B. Rubin defined a higher-rank Funk transform for Stiefel manifolds in his paper [Rub12]. His work applies to Grassmannian manifolds by assuming the function descends to the Grassmannian. In [ÓRP] the authors worked out a more specific relationship between Rubin's Funk transform and the cosine-λ transform. Rubin's results are restricted to the case of the field $\mathbb{R}$, but they are relevant to what we do here, so we explain how our results here fit together with his.

We restate his definition of the higher rank Funk transform in terms of Lie groups. For a function $f \in C^\infty(\mathcal{B})$ his definition can be stated

$$\mathscr{F} f(\beta) = \int_L f(l\sigma)dl \tag{1.2}$$

where $\sigma \subset \beta^\perp$ is arbitrarily chosen base point. The integral does not depend on this choice of σ.

He establishes that

$$\underset{\lambda=-p}{\text{a.c.}}\, \gamma(\lambda)\mathcal{C}^\lambda f(\beta) \propto \mathscr{F} f(\beta). \tag{1.3}$$

We agree that his notion of a Funk transform on $\mathcal{B}$ is the appropriate one. One may think of the classical Funk transform on the sphere as an integral

$$\mathscr{F} f(u) = \int_{G=\mathrm{Stab}(u)} f(gv)dg$$

where v is an arbitrarily chosen vector in $u^\perp$ and G is a subgroup of the special orthogonal group. Again, the integral is independent of this choice of v. The similarity to (1.2) is clear since $L = \mathrm{Stab}(\beta)$.

Observe that on the sphere, we might write $|\mathrm{Cos}(u, v)|$ for $|u \cdot v|$ since this is the natural cosine between two elements. However, on the sphere, the conditions $|\mathrm{Cos}(u, v)| = 0$ and $u \subset v^{\perp}$ are equivalent, but in a Grassmannian the condition $\sigma \in \beta^{\perp}$ is a stronger condition than $|\mathrm{Cos}(\sigma, \beta)| = 0$. The latter amounts to the statement that σ contains a vector orthogonal to β (and vice versa). That distinction is important in the present paper.

Using Rubin's result, we characterize $\mathcal{C}^{\lambda} f(\beta)$ at the poles $-1, -2, \ldots, -p$ for the real case, which ties together our results in this dissertation with his result at $\lambda = -p$. Having established that $\gamma(\lambda)\mathcal{C}^{\lambda} f(\beta)\Big|_{\lambda=-1}$ yields a cosine-λ transform of f over an embedded Grassmannian of rank $p - 1$, we further establish that $\gamma(\lambda)\mathcal{C}^{\lambda} f(\beta)\Big|_{\lambda=-2}$ yields a cosine-λ transform on a rank $p - 2$ embedded Grassmannian, and so on stepping down in rank at each pole until at $\lambda = -p$ we have Rubin's Funk transform.

This situation will be clearer to the reader once we have written $\mathcal{C}^{\lambda} f(\beta)$ in coordinates, but to give a general idea, the stepping down will look something like this. We start with an integral over $L \times D^{+}$. Then, at the first pole, we have an integral over $L \times D^{+}\Big|_{t_1=\pi/2}$: one vector in $\sigma(\pi/2, t_2, \ldots, t_p)$ is perpendicular to β. Then at $\lambda = -2$, we have $L \times D^{+}\Big|_{t_1=t_2=\pi/2}$: two independent vectors in σ are perpendicular to β, and we continue until we reach $L \times (\pi/2, \ldots, \pi/2)$. In the last expression, $\sigma(\pi/2, \ldots, \pi/2) \in \mathcal{B}$ is contained in $\beta^{\perp}$.

At each pole from -1 to -p we make a step down toward the Funk transform. By contrast, the sphere does not admit any division of its Funk transform into steps like this in quite so natural a way. Perhaps we may think of these intermediate cosine-λ transforms on embedded Grassmannians as some kind of *partial* cosine-Funk transforms. We leave that to the reader to consider.

Let us at the end mention the connection of those results to representation theory. Let $G = \mathrm{SL}(n, \mathbb{K})$. Then G acts on $\mathcal{B}$ in a natural way $g \cdot \beta = \{g(v) \mid v \in \beta\}$ and $\mathcal{B} = G/P$ where $P = MAN$ is a maximal parabolic subgroup in G. It was shown in [ÓP12] that $\mathcal{C}^{\lambda-\rho}$, where $\rho = d(n+1)/2$ is an G-intertwining operator between two representations π_λ and $\pi_{-\lambda} \circ \theta$. For a representation μ of K let $\eta_\mu(\lambda - \rho)$ be the eigenvalue of $\mathcal{C}^{\lambda-\rho}$ on $L^2_\mu(\mathcal{B})$, the space of L^2-functions of type μ. Then the zeros and poles of the sequence $\{\eta_\mu(\lambda - \rho)\}$ given information about the composition series for π_λ and $\pi_{-\lambda} \circ \rho$. This idea was used in [MS14] to determine the composition series explicitly. Our results then give extra information about intertwining operator onto the quotient and a geometric interpretation of the image, respectively the kernel.

1.1 Notation

The constant p is fixed throughout as the rank of the base Grassmannian $\mathcal{B}$. Since we consider embedded submanifolds that are Grassmannians of lower rank, in several instances we will use a subscript k to denotes that we are considering an element in the rank $p-k$ setting. For the coordinates, we write $\mathbf{t}_k = (t_{k+1}, \ldots, t_p)$.

1.2 Outline

In Section 2 we recall some basic results from harmonic analysis including an integral formula for compact symmetric spaces in polar coordinates. We also summarize some of the results from the paper [ÓP12] and establish a few elementary corrolaries.

In Section 3 we compute the transform that arises from $\mathcal{C}^\lambda$ at its first pole $\lambda = -1$. We use the integral formula from Section 2 to write $\mathcal{C}^\lambda$ in coordinates. This yields an integral transform we have called $\mathcal{F}_1$, and we call this a "partial cosine-Funk transform" on the Grassmann manifold.

Since the result of taking this limit is an integral in coordinates, we spend some time in Subsection 3.4 examining the geometric interpretation of $\mathcal{F}_1$.

Next we observe that $\mathcal{F}_1$ is an intertwining operator for the left regular representation on $\mathcal{C}^\infty(\mathcal{B})$ and we compute the image and kernel of $\mathcal{F}_1$.

In Section 4 we consider the poles $\lambda = -1, \ldots, -p$ of C^λ. Unlike the work in previous sections, in this section we rely on a result proved by B. Rubin. The argument presented here may appear to subsume our work on the first pole, but in fact it is quite different because in our analysis of the first pole we did not use Rubin's result, and his methods are quite different from ours.

2 Background

In this section we will first recall some of the basic notation and results we will use in the present paper. We use a well known integral formula for compact symmetric spaces which can be found in S. Helgason's *Groups and Geometric Analysis.* After that, we specialize to the cosine-λ transform on Grassmannian manifold where we establish the notation we will use and review some of the results from [ÓP12]. Here we have also included some small corollaries concerning poles of the cosine-λ transform that follow quickly from [ÓP12].

2.1 Grassmannian Manifolds and Symmetric Spaces

We fix $\mathbb{K}$ as the field $\mathbb{R}$ or $\mathbb{C}$ or the skew field $\mathbb{H}$ of quaternions. Then the Grassmannian manifolds $\mathrm{Gr}(p, \mathbb{K}^n)$ are the manifolds of p-dimensional subspaces of $\mathbb{K}^n$.

In this dissertation, we will deal with these spaces using matrix groups. Assume we have fixed some orthonormal basis $\{b_1, \ldots, b_n\}$ for the underlying vector space $\mathbb{K}^n$. We pick a base point in $\mathrm{Gr}(p, \mathbb{K}^n)$, say β, the span of $\{b_1, \ldots, b_p\}$

The appropriate matrix groups for this work are the special unitary groups. For the fields $\mathbb{R}$, $\mathbb{C}$ and the skew field $\mathbb{H}$ these matrix groups are customarily denoted $\mathrm{SO}(n)$ (the special orthogonal group), $\mathrm{SU}(n)$ (the special unitary group) and $\mathrm{Sp}(n)$ (the unitary group over the quaternions).

Refer to Knapp, page 112, for a very brief discussion on the "determinant" for matrices of quaternions. Knapp refers to Dieudonné's notion of determinant and concludes that for members of $\mathrm{Sp}(n)$ the determinant is automatically 1.

To unify notation, we will write $\mathrm{SU}(n)$ for all of these, considering the underlying field to be fixed and understood. Note however, that over $\mathbb{H}$, $\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q)) = \mathrm{Sp}(p) \times \mathrm{Sp}(q)$ because of the above discussion about the determinant.

The space of $n \times n$ special orthogonal matrices $\mathrm{SU}(n)$ acts transitively on $\mathrm{Gr}(p, \mathbb{K}^n)$. Let $q = n - p$ here and throughout. The stabilizer of β is diffeomorphic to $\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$. We will simply write $\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$ for the stabilizer of the base point, and we will often refer to this subgroup as L. Therefore, the the quotient space $\mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$ is diffeomorphic to $\mathrm{Gr}(p, \mathbb{K}^n)$. This situation is very common, and one can work in this way any time we have a Lie group acting transitively on a manifold. This is called a homogeneous space.

This places the study of Grassmannian manifolds firmly within the realm of Lie groups.

We may now define an involutive automorphism on $K = \mathrm{SU}(n)$ by

$$\tau(x) = \left(\begin{smallmatrix} I_p & 0 \\ 0 & -I_q \end{smallmatrix}\right) x \left(\begin{smallmatrix} I_p & 0 \\ 0 & -I_q \end{smallmatrix}\right),$$

and we observe that L is exactly the subgroup of K that is fixed by τ.

In this dissertation, I will often refer to *symmetric spaces*. For our purposes here, one could simply think of this situation as the definition. Namely:

Definition 1. A symmetric space is a homogeneous space G/H, where G is a semisimple Lie group and H is a compact subgroup, together with an involutive automorphism (also simply called an *involution*) so that H is an open subgroup of the set fixed by τ.

Technically and historically speaking, that is not the most correct definition of a symmetric space. Rather, symmetric spaces are smooth manifolds having at every point an inversion symmetry. One may think of the inversion symmetry at a point as an automorphism that reverses the direction of any geodesic curve passing through the point.

It is highly non-obvious that such spaces are encompassed by homogeneous spaces with an involution as described above. This is due to the work of Élie Cartan, who studied symmetric spaces extensively and classified them. Cartan's work was later clarified in Helgason's book *Differential Geometry, Lie Groups, and Symmetric Spaces* ([Hel01]). There one can also find many pages of bibliographic references, including an entire page of references to original papers by É. Cartan (in French).

Thanks to Cartan and Helgason, we will not need to delve into the details of Riemannian symmetric spaces. We will use the Lie theory tools.

We also note that the sphere S^n is a symmetric space. One of the primary motivations of this dissertation was to generalize a result from the sphere to a different symmetric space, the Grassmannian manifolds, and to characterize the similarities and differences.

2.2 Roots and Weyl Chambers

Here we recall some well known facts from semisimple Lie algebra structure theory. We referred primarily to Helgason's *Differential Geometry, Lie Groups, and Symmetric Spaces* for this material. We intend to be as brief and basic as possible on these matters of notation and well-known results while at least pointing in the direction of a correct understanding of the ideas.

Definition 2. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{R}$ with Cartan involution θ and Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ is the (+1)-eigenspace of θ and $\mathfrak{p}$ is the (-1)-eigenspace. Also, let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Then for each λ in the dual of $\mathfrak{a}$, we define the notation

$$\mathfrak{g}_\lambda := \{X \in \mathfrak{g} \mid [H, X] = \lambda(H)X \text{ for all } H \in \mathfrak{a}\}.$$

Definition 3. With notation as in Definition 2, the element λ is called a *root* of $(\mathfrak{g}, \mathfrak{a})$ (or a restricted root) if $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq 0$. When λ is a root, the spaces $\mathfrak{g}_\lambda$ are called *root spaces*.

Also, we define the *multiplicity* of a root λ denoted here m_λ to equal the real vector space dimension of $\mathfrak{g}_\lambda$.

We will use the notation Σ to denote the set of restricted roots.

Definition 4. A point $H \in \mathfrak{a}$ is called *regular* if $\lambda(H)$ is nonzero for all $\lambda \in \Sigma$. Otherwise, it is *singular*. Let $\mathfrak{a}'$ denote the set of regular elements.

Proposition 1. The subset $\mathfrak{a}' \subset \mathfrak{a}$ of regular elements consists of the complement of finitely many hyperplanes.

Definition 5. Each connected component of $\mathfrak{a}'$ is called a *Weyl chamber*.

Definition 6. For any fixed choice of Weyl chamber $\mathfrak{a}^+$, we define the set of *positive roots* by $\Sigma^+ = \{\lambda \in \Sigma \ : \ \lambda \restriction \mathfrak{a}^+ > 0\}$.

In this dissertation we will be interested in the case where the Lie algebra is compact and real. For our purposes, it suffices to think of a *compact* Lie algebra as the Lie algebra of a compact Lie group (See Helgason, page 132, [Hel01]). In this setting, we need to consider the complexification of the Lie algebra to get a handle on the roots.

Note that in this section for convenience we will use a star notation $\mathfrak{p}^*$ to refer to $i\mathfrak{p}$ to emphasize the duality between a compact real form and a non-compact real form in a complex Lie algebra. In particular, we do not write $\mathfrak{p}^*$ for the dual space—the space of linear functionals on $\mathfrak{p}$. We will just write "the dual of $\mathfrak{p}$" for that space.

Let $\mathfrak{u}$ be a compact real Lie algebra and assume we have a involution θ on $\mathfrak{u}$ and we have a direct sum decomposition $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{p}^*$ into the +1 and -1 eigenspaces.

Let $\mathfrak{u}^{\mathbb{C}}$ be the complexification of $\mathfrak{u}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a non-compact real form of $\mathfrak{u}^{\mathbb{C}}$ where $\mathfrak{p}$ and $\mathfrak{p}^*$ are related by $\mathfrak{p}^* = i\mathfrak{p}$, and this is a Cartan decomposition of $\mathfrak{g}$ with involution θ.

Now that we have a non-compact real Lie algebra $\mathfrak{g}$, we find a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ as we did above, and we can write down the roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$ as above.

We can extend $\mathfrak{a}$ to a maximal abelian subalgebra $\mathfrak{h}$ in $\mathfrak{g}^{\mathbb{C}}$. Then for any linear form α in the dual space $\mathfrak{h}^*$ we define, similar to the above, the space

$$\mathfrak{g}_\alpha^{\mathbb{C}} = \{X \in g^{\mathbb{C}} \ : \ [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}.$$

If $\alpha \neq 0$ and $g_\alpha^{\mathbb{C}} \neq 0$, then α is called a *root of* $\mathfrak{g}^{\mathbb{C}}$ *with respect to* $\mathfrak{h}$. We denote the set of these roots by Δ.

One can show that the roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$ are the restrictions to $\mathfrak{a}$ of roots in Δ that do not vanish on $\mathfrak{a}$. For this reason the roots in Σ are called *restricted roots.*

Suppose now that λ is a (restricted) root of $\mathfrak{g}$ with respect to $\mathfrak{a}$. It is real-valued on $\mathfrak{a}$ because it is contained in the dual of $\mathfrak{a}$. In the compact real form $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{p}^*$, one can show that $i\mathfrak{a}$ is maximal abelian in $\mathfrak{p}^*$, and because those elements have the form iH for some $H \in \mathfrak{a}$, the roots are pure imaginary on $i\mathfrak{a}$.

Therefore, the appropriate notion of a "root" of $\mathfrak{u}$ with respect to $\mathfrak{a}^* = i\mathfrak{a}$ is really the same as the restricted roots Σ of $\mathfrak{g}$ with respect to $\mathfrak{a}$ except that they are defined, by linear extension, on $\mathfrak{a}^*$ and then restricted to $\mathfrak{a}^*$.

This is one way of thinking about the reason why we needed to go into the complexification. For compact lie algebras and a decomposition based on an involutive automorphism like we started with, the roots are complex-valued—in fact pure imaginary. Therefore, they do not lie in the dual of $\mathfrak{a}^*$.

Note that the above notions of regular points and Weyl chambers go through for $\mathfrak{u}$ and $\mathfrak{a}^*$ just as well.

Let G be a semisimple Lie group with an involution θ (always assumed nontrivial). Let K be an open subgroup of $G^\theta = \{g \mid \theta g = g\}$ and assume K is compact. Then the homogeneous space G/K is a Riemannian symmetric space. Conversely, any Riemannian symmetric space can be written as such a homogeneous space.

We will assume now that $K = G^\theta$ since that will be true in the cases that interest us.

Let $\mathfrak{g}$ denote the Lie algebra of G. Then θ yields an involution on $\mathfrak{g}$ defined

$$\theta X = \left.\frac{d}{dt}\right|_{t=0} \theta \exp(Xt).$$

The Lie algebra of K, denoted $\mathfrak{k}$, is the $(+1)$-eigenspace of θ. Let $\mathfrak{p}$ denote the (-1)-eigenspace in $\mathfrak{g}$. Then θ is a Cartan involution on $\mathfrak{g}$ and

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

is a Cartan decomposition.

Fix some maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ as above.

Definition 7. Let $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$—the normalizer in K of $\mathfrak{a}$ modulo the centralizer in K of $\mathfrak{a}$. This group is called the *Weyl group.*

Proposition 2. The Weyl group W acts simply transitively on the Weyl chambers of $\mathfrak{a}$

In the following work, we will make a choice of positive Weyl chamber and then we will see that our results did not depend on the choice because K contains $N_K(\mathfrak{a})$. A permutation of the Weyl chambers is given by an element $kZ_K(\mathfrak{a})$ of the Weyl group, and the element $k \in K$ has the same action on $\mathfrak{a}$.

2.3 Symmetric Space Integral Formula

In this section we will recall an integral formula for symmetric spaces that may be found in Helgason's *Groups and Geometric Analysis* starting on page 187 ([Hel00]). It is basically a kind of polar decomposition. We build upon the structure developed in the previous section.

Our main interest in this work is in Grassmannian manifolds, which are Riemannian symmetric spaces of the *compact* type. Therefore, let K be a compact Lie group with an involutive automorphism τ ($\tau \neq Id$) such that L is the τ-fixed subgroup of K.

Let $\mathfrak{k}$ be the Lie algebra of K, $\mathfrak{l}$ the Lie algebra of L. Then $\mathfrak{l}$ is the (+1)-eigenspace of the derived involution $\tau : \mathfrak{k} \to \mathfrak{k}$. Let $\mathfrak{q}$ be the (-1)-eigenspace. Then $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{q}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{q}$, and let M denote the centralizer of $\mathfrak{a}$ in L.

Let Σ denote the set of restricted roots of $\mathfrak{k}$ with respect to $\mathfrak{a}$ as described above for the case of a compact Lie algebra. Note that roots take pure imaginary values on $\mathfrak{a}$. Given some choice of positive Weyl chamber, we let Σ^+ denote the set of positive roots. Also let A denote the set $\exp \mathfrak{a}$. This is a closed subgroup of K.

The integral formula arises from the following polar coordinate map Φ defined

$$\Phi : L/M \times A \to K/L$$

$$(kM, a) \mapsto kaL.$$

Theorem 1. The map Φ is onto and $|\det(d\Phi_{(kM,a)})| = \prod_{\alpha\in\Sigma^+} |\sin\alpha(iH)|^{m_\alpha}$, where $a = \exp H$ $(H \in \mathfrak{a})$ and m_α denotes the multiplicity of α.

For proof, see Helgason, 2000, page 188. ([Hel00])

Let $\delta(b) := \prod_{\alpha\in\Sigma^+} |\sin\alpha(i\log(b))|^{m_\alpha}$. We now exclude certain singular points from the map. Let A' be the set $\{x \in A \mid \delta(x) \neq 0\}$. There is a notion of *singular* points in K/L, and the complement of this set is the nonsingular set $(K/L)_r$. It would take us rather far afield to explain the meaning in detail, but see Helgason's book, where he will also refer you to his earlier book ([Hel01], [Hel00]). The essential idea is that $(K/L)_r$ is precisely the set so that the restriction of Φ to $L/M \times A'$ maps regularly onto $(K/L)_r$.

Theorem 2. The restriction $\Phi : (L/M) \times A'$ maps onto $(K/L)_r$ and this map is regular. Furthermore, $(K/L)_r$ is open and of full measure in K/L. Therefore,

$$\int_{K/L} f(kL)dk = c\int_L\int_{A'} f(laL)\delta(a)dadl$$

for a constant c.

Integrals are always written in this dissertation with respect to Haar measure (for Lie groups) or invariant measure (for symmetric spaces).

Proposition 3. Fix a particular Weyl chamber $\mathfrak{a}^+$ and let $A^+ = \exp\mathfrak{a}^+$. Then

$$\int_{K/L} f(kL)dk = c\int_L\int_{A^+} f(lbL)\delta(a)dadl$$

for a constant c.

In each case, the constant c is determined by letting $f = 1$ so that in Prop. 3, $c = 1/\int_{A^+} \delta(b)db$.

Proof. Recall that Weyl chambers are the connected components of A' and that the Weyl group can be represented in L and it acts simply transitively on the Weyl chambers. $\square$

Throughout, let us write f^L for the function given by

$$f^L(x) = \int_L f(lx)dl$$

with the integral taken with respect to unit Haar measure.

Later, when dealing with specific coordinates, we will be able to restrict A^+ to a subset which is relatively compact (has compact closure) by looking at a fundamental domain, and we will pull the integral back to a fundamental domain in the Weyl chamber $\mathfrak{a}^+$

2.4 Cosine-λ Transform on $\mathrm{Gr}(p, \mathbb{K}^n)$

Here we establish some background and notation, and mention some of the essential results from [ÓP12] we will be using.

We specialize to the setting $\mathcal{B} = \mathrm{Gr}(p, \mathbb{K}^n)$, the Grassmannian manifold of p-dimensional subspaces in $\mathbb{K}^{p+q}$ where $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$ or the skew field $\mathbb{H}$ of quaternions. We will assume $q \geq p \geq 2$. Let $n = p+q$. Let $\{b_1, \ldots, b_n\}$ be an ordered orthonormal basis for the underlying space $\mathbb{K}^n$. We set $K = \mathrm{SU}(p+q, \mathbb{K})$ and $L = \mathrm{S}(\mathrm{U}(p, \mathbb{K}) \times \mathrm{U}(q, \mathbb{K}))$. Then $\mathcal{B} \cong K/L$, which is a compact symmetric space with involution

$$\tau(x) = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} x \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$$

and $K^\tau = L$.

2.4.1 Lie Algebra Decomposition and Simple Roots

We take the decomposition $\mathfrak{k} = \mathfrak{l} + \mathfrak{q}$ with $\mathfrak{l}$ and $\mathfrak{q}$ the +1 and -1 eigenspaces of $\mathfrak{k}$ with respect to the infinitesimal involution τ. Then

$$\mathfrak{q} = \left\{ Q(X) = \begin{pmatrix} 0_{pp} & X \\ -X^* & 0_{qq} \end{pmatrix} | X \in M(p \times q, \mathbb{K}) \right\}.$$

We also write down a maximal abelian subspace of $\mathfrak{q}$ and coordinates for it. Here our choice will differ from the one in [ÓP12] by a conjugation. Let $E^{(r,s)}_{\nu,\mu} = (\delta_{i\nu}\delta_{j\mu})_{i,j}$, the matrix in $M(r \times s, \mathbb{K})$ with all entries equal to 0 but the (ν, μ)th, which equals 1. Let $\mathbf{t} = (t_1, \ldots, t_p)$ and

$$X(\mathbf{t}) = -\sum_{j=1}^{p} t_j E^{(p,q)}_{j,p+q+1-j}$$

and

$$Y(\mathbf{t}) = Q(X(\mathbf{t})) \in \mathfrak{q}.$$

Then $\mathfrak{a} = \{Y(\mathbf{t}) | \mathbf{t} \in \mathbb{R}^p\}$ is a maximal abelian subspace of $\mathfrak{q}$ and

$$\exp Y(\mathbf{t}) = \begin{pmatrix} \cos(t_1) & & & & & & -\sin(t_1) \\ & \ddots & & 0 & & ⋰ & \\ & & \cos(t_p) & & -\sin(t_p) & & \\ & 0 & & I_{q-p} & & 0 & \\ & & \sin(t_p) & & \cos(t_p) & & \\ & ⋰ & & 0 & & \ddots & \\ \sin(t_1) & & & & & & \cos(t_1) \end{pmatrix}. \tag{2.1}$$

Denote by Σ the set of roots of $\mathfrak{k}$ with respect to $\mathfrak{a}$ and let Σ^+ denote the positive roots with respect to some choice of ordering. Below we will make this choice explicit.

Let us define $\{\epsilon_j\}$ as the basis of the dual of $\mathfrak{a}$ which is dual to $\{Y^j = Y(\sum_{m=1}^{p} \delta_{j,m} t_m)\}$ so that $\epsilon_j(Y(\mathbf{t})) = it_j$.

The following proposition is Lemma 5.2 of [ÓP12].

Proposition 4. The roots are

$$\Sigma = \{\pm\epsilon_i \pm \epsilon_j\ (1 \leq i \neq j \leq p, \pm \text{ independent}),\ \pm \epsilon_i\ (1 \leq i \leq p), \pm 2\epsilon_i\ (1 \leq i \leq p)\}$$

with multiplicities, respectively, d (and not there in case $p = 1$), $d(q-p)$ (and not there in case $p = q$) and $d - 1$ (and not there if $d = 1$).

Let us pick a simple system of roots to work with in each case. We indicate this choice and the corresponding positive Weyl chamber for each case in Table 2.1.

TABLE 2.1: Systems of Simple Roots

Case	Simple System of Roots	Positive Weyl chamber
$p = q, d = 1$	$\{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{p-1} - \epsilon_p, \epsilon_{p-1} + \epsilon_p\}$	$\lvert t_p\rvert < t_{p-1} < \cdots < t_1$
$p \neq q, d = 1$	$\{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{p-1} - \epsilon_p, \epsilon_p\}$	$0 < t_p < t_{p-1} < \cdots < t_1$
$p \neq q,\ d > 1$	$\{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{p-1} - \epsilon_p, \epsilon_p\}$	$0 < t_p < t_{p-1} < \cdots < t_1$
$p = q,\ d > 1$	$\{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{p-1} - \epsilon_p, 2\epsilon_p\}$	$0 < t_p < t_{p-1} < \cdots < t_1$

2.4.2 Highest Weights and Spherical Representations

In this subsection we discuss the irreducible spherical representations of K and their highest weights. Ultimately, we will use this discussion in the next subsection to decompose $L^2(\mathcal{B})$ into irreducible subspaces and thereby study the spectrum of C^λ.

Let $\widehat{K}$ denote the set of equivalence classes of irreducible representations of K, and let $\widehat{K}_L$ denote the subset of $\widehat{K}$ of L-spherical representations (representations having a nonzero L-fixed vector). Define the notation

$$\Lambda^+ := \left\{ \mu \in i\mathrm{L}(\mathfrak{a}) \text{ (the dual) } | (\forall \alpha \in \Delta_{\mathfrak{k}}^+) \frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{N}_0 \right\}.$$

The map $\pi \mapsto$ (highest weight of π) maps $\widehat{K}_L$ injectively into Λ^+. This map is bijective if $\mathcal{B}$ is simply connected ([Hel00], p. 535). In general, $\widehat{K}_L$ maps to a subset $\Lambda^+(\mathcal{B})$ of Λ^+ ([ÓS11]).

Proposition 5. If $\mathbb{K} = \mathbb{R}$ and $p = q$, then

$$\Lambda^+ = \left\{ \mu = \sum_{j=1}^{p} m_j \epsilon_j \middle| m_j \in \mathbb{Z}, m_i - m_{i+1} \in 2\mathbb{N}_0 \text{ for } i = 1, \ldots, p, m_{p-1} > |m_p| \right\}.$$

Observe that as a consequence, $m_1 \geq m_2 \geq \cdots m_{p-1} \geq |m_p|$. If $\mathbb{K} = \mathbb{R}$ and $p \neq q$, then

$$\Lambda^+ = \left\{ \mu = \sum_{j=1}^{p} m_j \epsilon_j \middle| m_j \in \mathbb{N}_0, m_i - m_{i+1} \in 2\mathbb{N}_0 \text{ for } i = 1, \ldots, p \right\}.$$

In the other cases,

$$\Lambda^+ = \left\{ \mu = \sum_{j=1}^{p} m_j \epsilon_j \middle| m_j \in 2\mathbb{N}_0 \text{ and } m_1 \geq m_2 \geq \cdots m_{p-1} \geq m_p \right\}.$$

Proof. First consider the case $\mathbb{K} = \mathbb{R}$ and $p = q$. Let $\mu \in \Lambda^+$. The set $\{\epsilon_i | i = 1, \ldots, p\}$ is an orthogonal basis for the dual of $\mathfrak{a}$, so we can write $\mu = m_1\epsilon_1 + \cdots + m_p\epsilon_p$. The positive roots in this case are given by $\epsilon_i \pm \epsilon_j$ where $1 \leq i < j \leq p$. Then the condition

$$\frac{\langle \mu, \epsilon_i - \epsilon_j \rangle}{\langle \epsilon_i - \epsilon_j, \epsilon_i - \epsilon_j \rangle} \in \mathbb{N}_0$$

simplifies to

$$\frac{m_i|\epsilon_i|^2 - m_j|\epsilon_j|^2}{|\epsilon_i|^2 + |\epsilon_j|^2} \in \mathbb{N}_0.$$

Since the ϵ_i all have the same norm, we have $m_i - m_j \in 2\mathbb{N}_0$ for $i < j$. Similarly, the roots $\epsilon_i + \epsilon_j$, $i < j$, yield $m_i + m_j \in 2\mathbb{N}_0$. It follows that the m_i are all integers. These conditions are all summarized by the equivalent conditions $m_i \in \mathbb{Z}$, $m_1 \geq m_2 \geq \cdots m_{p-1} \geq |m_p|$, and $m_{i+1} - m_i \in 2\mathbb{N}_0$ for $i = 1, \ldots, p$.

In the case $\mathbb{K} = \mathbb{R}$, and $p \neq q$, all the conditions of the previous case apply and, in addition, we have

$$\frac{\langle \mu, \epsilon_i \rangle}{\langle \epsilon_i, \epsilon_i \rangle} \in \mathbb{N}_0, i = 1, \ldots, p.$$

This yields $m_i \in \mathbb{N}_0$ for $i = 1, \ldots, p$

Supposing $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$, so that $d = 2$ or $d = 4$, we have the condition

$$\frac{\langle \mu, 2\epsilon_i \rangle}{\langle 2\epsilon_i, 2\epsilon_i \rangle} \in \mathbb{N}_0, i = 1, \ldots, p$$

which yields $m_i \in 2\mathbb{N}_0$ for $i = 1, \ldots, p$. This is in addition to the condition that $m_i \pm m_j \in 2\mathbb{N}_0$ for $i < j$.

In this case, whether $p = q$ or $p \neq q$ makes no difference in computing Λ^+.

□

For the following computation, we will need to recall some basic topology of Lie groups. The groups $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$ are simply connected ([Kna02], p. 112). Also $\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$ and $\mathrm{Sp}(p) \times \mathrm{Sp}(q)$ are connected. Therefore $\mathrm{SU}(p+q)/(\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$ and $\mathrm{Sp}(p+q)/(\mathrm{Sp}(p) \times \mathrm{Sp}(q))$ are simply connected ([Kna02], p. 85). However, the real Grassmannian $\mathrm{SO}(p+q)/\mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(q))$ is not simply connected.

The space $\mathrm{SO}(p+q)/(\mathrm{SO}(p) \times \mathrm{SO}(q))$ is itself a symmetric space ([Hel01], p. 518). It can be interpreted as the oriented real Grassmannian. It is simply connected ([FR84], p. 443) and there is a natural two-cover of $\mathrm{SO}(p+q)/(\mathrm{SO}(p) \times \mathrm{SO}(q))$ onto $\mathrm{SO}(p+q)/\mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(q))$.

Proposition 6. In the cases $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{H}$, the subset $\Lambda^+(\mathcal{B})$ of Λ^+ given by highest weights of irreducible spherical representations is the full set Λ^+.

For the case $\mathbb{K} = \mathbb{R}$, an element $\mu = \sum m_i \epsilon_i \in \Lambda^+$ is in $\Lambda^+(\mathcal{B})$ if and only if $m_i \in 2\mathbb{Z}$ for $i = 1, \ldots, p$.

Proof. For $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$, [Hel00], p. 535, Theorem 4.1 and Corollary 4.2 following that theorem show that every $\mu \in \Lambda^+$ is the highest weight of an irreducible spherical representation.

Now consider the case $\mathbb{K} = \mathbb{R}$. Let $\mu = \sum m_i\epsilon_i \in \Lambda^+$. The space $\mathrm{SO}(p+q)/\mathrm{S}(\mathrm{O}(p)\times\mathrm{O}(q))$ is not simply connected. The space $\mathrm{SO}(p+q)/(\mathrm{SO}(p)\times\mathrm{SO}(q))$ is a double cover

Let us temporarily set the notation $\tilde{L} = \mathrm{SO}(p)\times\mathrm{SO}(q)$ and $L = \mathrm{S}(\mathrm{O}(p)\times\mathrm{O}(q))$. Let J be the element $J = \exp Y(0,\dots,0,\pi) \in L$. Then $L/\tilde{L} = \{\tilde{L}, J\tilde{L}\}$.

Let (π, V) be an irreducible spherical representation of $SO(p+q)$ with respect to L with highest weight $\mu = \sum m_i\epsilon_i$. Let $v \in V$ be a highest weight vector and let e_L be an L-fixed vector in V. Since K is compact, we can assume π is unitary with respect to the inner product on V. Also, $\langle v, e_L\rangle \neq 0$ ([Kna02], p. 545). Therefore

$$\langle v, e_l\rangle = \langle v, \pi(J^{-1})e_L\rangle = \langle \pi(J)v, e_L\rangle = e^{\mu\log J}\langle v, e_L\rangle \tag{2.2}$$

and so $e^{\mu\log J} = 1$. Therefore $e^{m_p\pi} = 1$, so $m_p \in 2\mathbb{Z}$. It follows, since $\mu \in \Lambda^+$ that $m_i \in 2\mathbb{Z}$ for $i = 1,\dots,p$.

Let us suppose toward the converse that $\mu \in \Lambda^+$ is given and μ satisfies $m_i \in 2\mathbb{Z}$. Then there is a representation (π_μ, V) of $\mathrm{SO}(p+q)$ that is $\tilde{L}$-spherical and has highest weight μ. Since $\mathrm{SO}(p)\times\mathrm{SO}(q)$ is normal in $\mathrm{S}(\mathrm{O}(p)\times\mathrm{O}(q))$, $\pi(J)e_L$ is also $\tilde{L}$-fixed. The pair $(K, \tilde{L})$ is a Gelfand pair ([Wol07], 265), and so for an irreducible representation of K the space of $\tilde{L}$-fixed vectors is at most 1-dimensional ([vD09], p. 82). Now, since $e^{\mu\log J} = 1$ and $\pi(J)e_{\tilde{L}} = ce_{\tilde{L}}$ for some constant c, it follows by (2.2) that $\langle v, \frac{1}{c}e_{\tilde{L}}\rangle = \langle v, e_{\tilde{L}}\rangle$, so $c = 1$ and therefore $e_{\tilde{L}}$ is in fact L-fixed. Therefore π is an L-spherical representation having highest weight μ.

□

2.4.3 $\mathcal{C}^\lambda$ as an Intertwining Operator and its Spectrum

The following definition follows [ÓP12] except that we use a different exponent on the $|\mathrm{Cos}(x,\omega)|$ factor. This is a matter of convenience for the work at hand. See [ÓRP] for further remarks.

Definition 8. Let d be the dimension of $\mathbb{K}$ over $\mathbb{R}$. On the space $\mathcal{B} = \mathrm{Gr}(p,\mathbb{K}^n)$ the Cosine-λ transform is defined for $\mathfrak{Re}\ (d\lambda) > -1$ and $f \in L^2(\mathcal{B})$ by

$$\mathcal{C}^\lambda f(\omega) = \int_{\mathcal{B}} |\mathrm{Cos}(x,\omega)|^{d\lambda} f(x)dx.$$

See the introduction for a definition of $|\mathrm{Cos}(x,\omega)|$ and refer to [ÓP12] for a detailed explanation.

Theorem 3. The $\mathcal{C}^\lambda$ transform extends to a meromorphic family of intertwining operators $\mathcal{C}^\lambda : \mathcal{C}^\infty(\mathcal{B}) \to \mathcal{C}^\infty(\mathcal{B})$.

This is Theorem 4.5 (1) of [ÓP12].

The space $L^2(\mathcal{B})$ decomposes into

$$L^2(\mathcal{B}) \cong_K \bigoplus_{\mu\in\Lambda^+(\mathcal{B})} L^2_\mu(\mathcal{B}) \tag{2.3}$$

where $L^2_\mu(\mathcal{B})$ is an irreducible subrepresentation of K with highest weight μ, and $\mathcal{C}^\lambda$ acts by scalar on each of these spaces $L^2_\mu(\mathcal{B})$. We therefore speak of the K-spectrum of $\mathcal{C}^\lambda$, meaning the set $\{\eta_\mu(\lambda)|\mu \in \Lambda^+(\mathcal{B})\}$ where $\eta_\mu(\lambda)$ is the scalar such that $\mathcal{C}^\lambda \restriction_{L^2_\mu(\mathcal{B})} = \eta_\mu(\lambda)\mathrm{id}_{L^2_\mu(\mathcal{B})}$. We will identify μ with the p-tuple $(m_1,\ldots,m_p)$ where $\mu = \sum_1^p m_i\epsilon_i$.

The next theorem is Theorem 5.11 of [ÓP12] with notation changes to suit this dissertation. It is one of the primary results of that paper. We will use it in this dissertation to investigate the poles of the $\mathcal{C}^\lambda$ transform.

Theorem 4. The K-spectrum of the Cosine-λ transform is

$$\eta_\mu(\lambda) = (-1)^{|\mu|/2} \frac{\Gamma_{p,d}(\frac{1}{2}dn)}{\Gamma_{p,d}(\frac{1}{2}dp)} \frac{\Gamma_{p,d}(\frac{1}{2}(d\lambda + dp))\Gamma_{p,d}(\frac{1}{2}(-d\lambda + \mu))}{\Gamma_{p,d}(-\frac{1}{2}d\lambda)\Gamma_{p,d}(\frac{1}{2}(d\lambda + dn + \mu))}.$$

The function $\Gamma_{p,d}(\lambda)$ is defined for a p-tuple $\lambda \in \mathbb{C}^p$ by

$$\Gamma_{p,d}(\lambda) = \prod_{j=1}^{p} \Gamma(\lambda_j - \frac{d}{2}(j-1)).$$

When $\lambda \in \mathbb{C}$, the notation $\Gamma_{p,d}(\lambda)$ means $\Gamma_{p,d}(\lambda, \ldots, \lambda)$.

Corollary 1. If $d = 1$ or 2, the cosine-λ transform has poles at the negative integers -1, -2, -3, ...except in the case $p = 1$, $d = 1$ where the poles occur only at the odd negative integers. If $d = 4$, then the poles are $\lambda = -1, -3/2, -2, -5/2, -3, \ldots$

Proof. First consider the factor

$$\frac{\Gamma_{p,d}(\frac{1}{2}(d\lambda + dp))}{\Gamma_{p,d}(\frac{1}{2}(d\lambda + dn + \mu))}.$$

For any fixed λ, we can take μ large so that the denominator is finite. The numerator expands to

$$\Gamma(\tfrac{1}{2}d(\lambda + p))\Gamma(\tfrac{1}{2}d(\lambda + p - 1)) \cdots \Gamma(\tfrac{1}{2}d(\lambda + 1)).$$

For $d = 1$ the poles $-1, -3, -5, \ldots$ come from the factor $\Gamma(\frac{1}{2}d(\lambda+1))$ and the poles $-2, -4, -6, \ldots$ come from $\Gamma(\frac{1}{2}d(\lambda + 2))$. If $p = 1$, these latter poles don't exist. If $d = 2$ then all $\lambda = -1, -2, -3, \ldots$ are poles of $\Gamma(\frac{1}{2}d(\lambda + 1))$. In the case $d = 4$, the list includes fractional values: $\lambda = -1, -3/2, -2, -5/2, -3, \ldots$

Now if we expand the other factor

$$\frac{\Gamma_{p,d}(\frac{1}{2}(-d\lambda + \mu))}{\Gamma_{p,d}(-\frac{1}{2}d\lambda)}$$

and take each m_i to be positive, we see that this is a polynomial. (Recall each m_i is even.) $\square$

Corollary 2. In the case $d = 1$, the poles $\lambda = -i$ for $i = 1, \ldots, -p$ have orders $i/2$ for i even, $(i+1)/2$ for i odd.

Proof. Assume $p \neq 1$. The case $p = q$ is potentially weirder than the others because m_p can be negative. However, we will see that this causes no problems.

Of the factor

$$\frac{\Gamma_{p,1}(\frac{1}{2}(-\lambda+\mu))}{\Gamma_{p,1}(\frac{1}{2}(-\lambda))} = \frac{\Gamma(\frac{1}{2}(-\lambda+m_1))}{\Gamma(\frac{1}{2}(-\lambda))}\frac{\Gamma(\frac{1}{2}(-\lambda-1+m_2))}{\Gamma(\frac{1}{2}(-\lambda-1))}\cdots\frac{\Gamma(\frac{1}{2}(-\lambda-p+1+m_p))}{\Gamma(\frac{1}{2}(-\lambda-p+1))}$$

only the last element

$$\frac{\Gamma(\frac{1}{2}(-\lambda-p+1+m_p))}{\Gamma(\frac{1}{2}(-\lambda-p+1))}$$

can fail to be holomorphic, and then only if m_p is negative. We will see how to deal with that in a moment.

We expand the factor

$$\frac{\Gamma_{p,1}(\frac{1}{2}(\lambda+p))}{\Gamma_{p,1}(\frac{1}{2}(\lambda+p+q+\mu))}$$

which yields

$$\frac{\Gamma(\frac{1}{2}(\lambda+p))}{\Gamma(\frac{1}{2}(\lambda+p+q+m_1))}\frac{\Gamma(\frac{1}{2}(\lambda+p-1))}{\Gamma(\frac{1}{2}(\lambda+p+q+m_2-1))}\cdots\frac{\Gamma(\frac{1}{2}(\lambda+1)}{\Gamma(\frac{1}{2}(\lambda+q+1+m_p))}$$

Of the numerators in this expansion, the number of them having poles at $\lambda = -i$ for $i = 1, \ldots, p$ is exactly $i/2$ for i even, $(i+1)/2$ for i odd. The denominators are all finite except possibly the last one, $\Gamma(\frac{1}{2}(\lambda+q+1+m_p))$. However, at any fixed λ, if this factor is infinite, then

$$\frac{\Gamma(\frac{1}{2}(-\lambda-p+1+m_p))}{\Gamma(\frac{1}{2}(\lambda+q+1+m_p))}$$

is finite. $\square$

Corollary 3. In the cases $d = 1$, $d = 2$ and $d = 4$, the Cosine-λ transform has a simple pole at $\lambda = -1$.

Throughout, we can use the function

$$\gamma(\lambda) = \frac{1}{\Gamma_{p,d}(\frac{d}{2}(\lambda + p))}$$

to normalize C^λ so that $\gamma(\lambda)C^\lambda$ is holomorphic.

3 The First Pole of $\mathcal{C}^\lambda$

In this section we write down the C^λ transform in coordinates using well known harmonic analysis tools. Then we compute the limit of the normalized trasnform

$$\lim_{\lambda\to -1}\gamma(\lambda)C^\lambda(f)$$

where C^λ has its first pole in our notation. This limit yields an integral transform $\mathcal{F}_1$ which we also write explicitly in coordinates. We explore the geometric interpretation of this transform in detail and show that we can view $\mathcal{F}_1$ as a cosine-λ transform on an embedded submanifold that is diffeomorphic to a rank-$(k-1)$ Grassmannian. To minimize clutter in the notation, we have used a subscript 1 rather than $k-1$.

This transform is also an intertwining operator for the left regular representation of K on $\mathcal{C}^\infty(\mathcal{B})$, and we explicitly compute its image and kernel.

We have been thinking of this transform $\mathcal{F}_1$ as a kind of *p*artial cosine-Funk transform which makes sense on the higher rank Grassmanians but does not exist on the sphere.

3.1 Weyl Chambers and Fundamental Domains

We let β denote $\{(x_1,\ldots,x_p,0,\ldots,0)|x_1,\ldots x_p\in\mathbb{K}\}$ and let $\{b_1,\ldots,b_p\}$ be the standard basis. Then for any $\omega=k\beta$ where $k\in K$, because k is an element of the orthogonal we have

$$|\mathrm{Cos}(\sigma,\omega)|=|\mathrm{Cos}(k^{-1}\sigma,\beta)|.$$

Then

$$\int_{\mathcal{B}}|\mathrm{Cos}(\sigma,\omega)|^{d\lambda}f(\sigma)d\sigma=\int_{\mathcal{B}}|\mathrm{Cos}(\sigma,\beta)|^{d\lambda}(L_{k^{-1}}f)(\sigma)d\sigma$$

so it suffices to consider the cosine transform evaluated at β.

Let $\alpha(b) = |\mathrm{Cos}(b \cdot \beta, \beta)|$ (see Theorem 4.2 from [ÓP12]) and note that $\alpha(b)$ is L-invariant.

The following is Lemma 5.8 from [ÓP12]:

Lemma 1. Let $\mathbf{t} \in \mathbb{R}^p$ and $\lambda \in \mathbb{C}$. Then $\alpha(\exp Y(\mathbf{t}))^\lambda = \prod_{j=1}^p |\cos(t_j)|^\lambda$.

Because of this, we write $\alpha(\mathbf{t}) = \prod_{j=1}^p |\cos(t_j)|^\lambda$ and $\alpha_k(\mathbf{t}_k) = \prod_{j=k+1}^p |\cos(t_j)|^\lambda$

Proposition 7. As before, take $A^+ = \exp \mathfrak{a}^+$ where $\mathfrak{a}^+$ is a positive Weyl chamber. Then

$$\mathcal{C}^\lambda f(\beta) = c \int_L \int_{A^+} \alpha(a)^{d\lambda} f(la\beta)\delta(a)da\, dl \tag{3.1}$$

$$= c \int_{A^+} \alpha(a)^{d\lambda} f^L(a\beta)\delta(a)da \tag{3.2}$$

where $c = \mathcal{C}^\lambda 1(\beta) / \int_{A^+} \alpha(a)\delta(a)da$. This c can be calculated.

Proof. Apply Proposition 3. □

We will now play somewhat loose with the constant c, which may vary from line to line. In each case, it can be evaluated explicity by setting $f = 1$.

Proposition 8. Fix a fundamental domain $U \subset \mathfrak{a}^+$ for the map $\exp : \mathfrak{a}^+ \to A^+$. Then for an L-invariant f,

$$\mathcal{C}^\lambda f(\beta) = c \int_U \alpha(\exp X)^{d\lambda} f(\exp X\beta)\delta(\exp X)dX. \tag{3.3}$$

Using the map Y as a coordinate chart, we have

$$\mathcal{C}^\lambda f(\beta) = c \int_{Y^{-1}(U)} \alpha(\exp Y(\mathbf{t}))^{d\lambda} f(\exp Y(\mathbf{t})\beta)\delta(\exp Y(\mathbf{t}))d\mathbf{t}. \tag{3.4}$$

Proposition 9. Given a fundamental domain U for the map $\exp \circ Y : \mathbb{R}^p \mapsto A^+$ and the same f as above, we have

$$\mathcal{C}^\lambda f(\beta) = c \int_U \prod_{j=1}^{p} |\cos(t_j)|^{d\lambda} f(\exp Y(\mathbf{t})\beta)\delta(\exp Y(\mathbf{t}))d\mathbf{t}. \tag{3.5}$$

Lemma 2. The map $\Psi : (t_1, t_2, \ldots, t_p) \mapsto \exp Y(t_1, \ldots, t_p)\beta$ is π-periodic in each t_i and if $\Psi(t) = \Psi(s)$ then for each i, $s_i = t_i + k_i\pi$ where k_i is an integer depending on i.

Proof. Suppose $\exp Y(\mathbf{t})\beta = \beta$ and $e_i(\mathbf{t})$ is the ith column of $\exp Y(\mathbf{t})$. Then $e_i(\mathbf{t})$ is linearly independent of $\{b_j | j \neq i\}$, so $e_i(\mathbf{t}) = b_i$, which forces $t_i = k_i\pi$ for some integer k_i.

□

Now let us deal with the action of the Weyl group. Consider a p-cube centered at the origin:

$$D = [-\pi/2, \pi/2] \times \cdots \times [-\pi/2, \pi/2].$$

Let D^+ denote the intersection of D with the positive Weyl chamber, and let D' denote the subset of D containing just the regular elements.

Lemma 3. Up to a set of measure zero

$$D' = \bigcup_{g \in W} gD^+,$$

and W acts simply transitively on the connected components of D'.

Proof. The Weyl group acts simply transitively on the Weyl chambers and D' is W-invariant. □

Proposition 10. The set $D^+ = \mathfrak{a}^+ \cap D$ is a fundamental domain as in Proposition 9.

3.2 Cosine-λ Transform in Coordinates

We will now write the $\mathcal{C}^\lambda$ transform concretely in coordinates. The basic form we use comes from (3.5) in Proposition 9. Two parts of (3.5) need to be specified: δ and U. Both depend on the particular case—that is, they depend on the value of d and whether $p = q$.

Recall that for brevity we defined $\mathbf{t}_k = (t_{k+1}, \ldots, t_p)$. Then we will write $f(\mathbf{t}) = f(\exp Y(t_1, \ldots, t_p)\beta)$ and $f_k(\mathbf{t}_k) = f(\pi/2, \ldots, \pi/2, t_{k+1}, \ldots, t_p)$ (the first k arguments are $\pi/2$).

Recall that f is π-periodic in all variables and $\delta(b) := \prod_{\alpha\in\Sigma^+} |\sin\alpha(i\log(b))|^{m_\alpha}$. Applying Proposition 4 for the roots, we have

$$\begin{aligned}\delta(\mathbf{t}) &= \delta(\exp Y(\mathbf{t}))\\ &= \prod_{i=1}^{p}[|\sin t_i|^{d(q-p)}|\sin(2t_i)|^{d-1}] \prod_{1\le i<j\le p} |\sin(t_i - t_j)\sin(t_i + t_j)|^d.\end{aligned} \tag{3.6}$$

Since $\sin(2u) = 2\cos u\sin u$ and $|\sin(u - v)\sin(u + v)| = |\cos^2 u - \cos^2 v|$, this simplifies. We set the notation

$$\delta_k(\mathbf{t}_k) = \prod_{i=k+1}^{p}[|2\cos t_i|^{d-1}|\sin t_i|^{d-1+d(q-p)}] \prod_{k+1\le i<j\le p} |\cos^2 t_i - \cos^2 t_j|^d$$

because we will need to consider the lower rank δs later.

To write down the region U in each case, we refer to Table 2.1 and apply Proposition 10. The case $p = q, d = 1$ is different from the other three cases because the positive Weyl chamber has a different shape.

For the case $p = q, d = 1$, one can see that $U = \{(t_1, \ldots, t_p) | |t_p| < t_{p-1} < \cdots < t_1 < \pi/2\}$ is a fundamental domain. The integral over this region can therefore be written

$$\mathcal{C}^\lambda f(\beta) = c \int_0^{\pi/2} \int_0^{t_1} \cdots \int_{-t_{p-1}}^{t_{p-1}} \alpha(\mathbf{t})^{d\lambda} f(\mathbf{t}) \delta(\mathbf{t}) dt_p \cdots dt_1. \tag{3.7}$$

Observe that $\mathbf{t} \mapsto \alpha(\mathbf{t})^{d\lambda}\delta(\mathbf{t})$ is even in each variable, so if f is odd in t_p then the integral is zero. We will therefore assume f is even in t_p, so

$$\mathcal{C}^\lambda f(\beta) = c \int_0^{\pi/2} \int_0^{t_1} \cdots \int_0^{t_{p-1}} \alpha(\mathbf{t})^{d\lambda} f(\mathbf{t}) \delta(\mathbf{t}) dt_p \cdots dt_1. \tag{3.8}$$

In the other three cases, the fundamental domains are all

$$U = \{(t_1, \dots, t_p) | 0 < t_p < t_{p-1} < \cdots < t_1 < \pi/2\},$$

and the integral takes the same form (3.8).

We perform a change of variable $u_i = \cos(t_i)^2$ and define $\mathbf{u} = (u_1, \dots, u_p)$. Then we have

$$c \int_0^1 \int_{u_1}^1 \cdots \int_{u_{p-1}}^1 \prod_{i=1}^p u_i^{(d(\lambda+1)-2)/2} f^\vee(\mathbf{u}) \nu_p(\mathbf{u}) du_p \cdots du_1 \tag{3.9}$$

where $f^\vee(\mathbf{u}) = f(\cos^{-1}\sqrt{u_1}, \dots, \cos^{-1}\sqrt{u_p})$ and $\nu_k^m(u_{k+1}, \dots, u_m) = \prod_{i=k+1}^m (1 - u_i)^{(d-2+d(q-p))/2} \prod_{k+1 \le i < j \le m} |u_i - u_j|^d$ (and $\nu_k = \nu_k^p$). Note that ν_k is not just δ_k after a change of variable. Rather, we have collected all the factors of u_i, so none appear in ν_k. Also, the quantity $q - p$ which appears will always be the same in our work even as k and m change in ν_k^m.

The integral (3.8) is independent of our choice of Weyl chamber and ordering. All such choices are conjugate under L, and we assume f is bi-L-invariant. More explicitly, had we chosen some other maximal abelian $\mathfrak{a}' \subset \mathfrak{q}$ and positive Weyl chamber $(\mathfrak{a}')^+$, it is well known that there is an element $l \in L$ giving an automorphism $\mathrm{Ad}(l)$ so that $\mathfrak{a} \mapsto \mathfrak{a}'$ and $\mathfrak{a}^+ \mapsto (\mathfrak{a}')^+$. All our work throughout this text is independent of this choice.

3.3 $\mathcal{F}_1$ in Coordinates

We now take the limit of the normalized $\gamma(\lambda)\mathcal{C}^\lambda(f)(\beta)$ as λ goes to -1. That is the location of the first pole of $\mathcal{C}^\lambda(f)(\beta)$, and this computation yields $\mathcal{F}_1$.

Let

$$F(\lambda, u_1) = \int_{u_1}^{1} \cdots \int_{u_{p-1}}^{1} \prod_{i=2}^{p} u_i^{(d(\lambda+1)-2)/2} f^\vee(\mathbf{u})\nu_1(\mathbf{u}) \prod_{1<j\leq p} |u_1 - u_j|^d du_p \cdots du_2. \tag{3.10}$$

Then up to a constant factor c, the limit $\lim_{\lambda\to-1} \gamma(\lambda)\mathcal{C}^\lambda f(\beta)$ equals

$$\lim_{\lambda\to-1} \gamma(\lambda) \int_0^1 u_1^{(d(\lambda+1)-2)/2}(1-u_1)^{(d-2+d(q-p))/2} F(\lambda, u_1) du_1.$$

Recall that the order of the pole at $\lambda = -1$ is 1. Therefore, for this computation we may replace $\gamma(\lambda)$ with $\Gamma(\lambda)$ for simplicity.

The difficulty we must overcome now is the dependence of F on λ. Limits of the form

$$\lim_{\lambda\to-1} \Gamma(\lambda) \int_0^1 u^\lambda f(u) du$$

are comparatively trivial, but we must carefully deal with the interior λs in F. This difficulty is the purpose of the following lemma.

Lemma 4. Let $f(\lambda, s, t)$ be a function that is bounded on $[-1, 0] \times [0, 1] \times [0, 1]$ and assume that $f(\lambda, s, t)$ converges to $f(\lambda, 0, t)$ uniformly in λ as $s \to 0$. Define

$$F(\lambda, s) = \int_s^1 t^\lambda |s-t|^d f(\lambda, s, t)\mu(t) dt$$

where $d \geq 1$ and μ is integrable on $[0, 1]$ and bounded on some $[0, \eta]$, $\eta > 0$. Then $F(\lambda, s)$ converges to $F(\lambda, 0)$ uniformly in λ as $s \to 0$ and F is bounded.

Proof. The essential observation is that $u_p^d - |u_p - u_{p-1}|^d$ is $O(u_{p-1})$ as $u_{p-1} \to 0$ and $\int_{u_{p-1}} u_p^{-1} du_p \mu(u_p)$ is $O(\ln(u_{p-1}))$.

First we show that F is bounded. Observe that

$$|s-t|^d \leq |s-t| \leq s+t.$$

Also, because $t \in [0,1]$ and $-1 \leq \lambda$, we have

$$t^\lambda \leq t^{-1}.$$

Therefore $\int_s^1 t^\lambda |s-t|^d \mu(t) dt \leq \int_s^1 t^{-1} s \mu(t) dt + \int_s^1 t^{-1} t \mu(t) dt$ and since $t^{-1}s \leq s^{-1}s = 1$, it follows that $F(\lambda, s) \leq 2M \int_0^1 |\mu(t)| dt$ where M is a bound on f.

Now let us show the uniform convergence part. For any s,

$$|F(\lambda,s) - F(\lambda,0)| \tag{3.11}$$

$$\leq \left| \int_s^1 t^\lambda |s-t|^d f(\lambda,s,t)\mu(t)dt - \int_s^1 t^{\lambda+d} f(\lambda,s,t)\mu(t)dt \right| \tag{3.12}$$

$$+ \int_0^s t^{\lambda+d} |f(\lambda,0,t)| \mu(t) dt \tag{3.13}$$

As $s \to 0$, the integral in (3.13) goes to 0 uniformly in λ. To see this, observe that $t^{\lambda+d} \leq t^{d-1}$ and $d-1 \geq 0$. Thus $t^{d-1}|f(\lambda,0,t)|\mu(t)$ is bounded on $[0,s]$ for s small.

Now consider the integral in (3.12). We add and subtract $t^\lambda |s-t|^d f(\lambda,0,t)\mu(t)$ to get the inequality

$$(3.12) \leq \int_s^1 t^\lambda |s-t|^d |f(\lambda,s,t) - f(\lambda,0,t)|\, |\mu(t)| dt \tag{3.14}$$

$$+ \int_s^1 |f(\lambda,0,t)| |t^\lambda |s-t|^d - t^{\lambda+d}| |\mu(t)| dt \tag{3.15}$$

Since f converges uniformly, as $s \to 0$, and $\int_s^1 u^\lambda |s-t||\mu(t)|dt$ is bounded, we can see that the integral in (3.14)$\to 0$ uniformly.

In (3.15), consider the $||s-t|^d - t^d|$ factor. Using binomial expansion, $||s-t|^d - t^d| = s\, m(t,d)$ where m is a polynomial. Then

$$(3.15) \leq s \int_s^1 t^{-1} |f(\lambda, 0, t)| m(t,s) |\mu(t)| dt.$$

Since f is bounded and μ is bounded near 0, there are constants a and b so that

$$\int_s^1 t^{-1} |f(\lambda, 0, t)| m(t,s) |\mu(t)| dt \leq a + b \ln(s).$$

Then we can see that $s(a+\ln(s)$ goes to zero uniformly in λ as $s \to 0$. This concludes the proof of uniform convergence. □

Lemma 5. For $\lambda \in [-1, 0]$, as $u \to 0$ the function $F(\lambda, u)$ from (3.10) converges to $F(\lambda, 0)$ uniformly in λ and F is bounded.

Proof. Apply Lemma 4 repeatedly to $f^\vee(u_1, \ldots, u_p)$ multiplying by $|u_i - u_j|^d$ factors as needed to build F.

□

Theorem 5. With the notation as above,

$$\lim_{\lambda \to -1} \Gamma(\lambda) \int_0^1 u_1^{(d(\lambda+1)-2)/2} (1-u_1)^{(d-2+d(q-p))/2} F(\lambda, u_1) du_1 = cF(-1, 0)$$

for a nonzero constant c.

Lemma 6. We perform a change of variable to make the notation nicer:

$$\lim_{\lambda \to -1} \Gamma(\lambda) \int_0^1 u_1^{(d(\lambda+1)-2)/2} (1-u_1)^{(d-2+d(q-p))/2} F(\lambda, u_1) du_1$$

$$= \lim_{\lambda\to -1} \frac{2}{d}\Gamma(\lambda) \int\limits_0^1 u_1^\lambda (1-u_1)^{(d-2+d(q-p))/2} F\left(\frac{2}{d}(\lambda+1)-1, u_1\right) du_1.$$

We may ignore the $\mu(u_1) = (1-u_1)^{(d-2+d(q-p))/2}$ factor in the limit.

Lemma 7. With notation as above,

$$\lim_{\lambda\to -1} \Gamma(\lambda) \int\limits_0^1 u^\lambda \mu(u) F(2/d(\lambda+1)-1, u) du$$
$$= \lim_{\lambda\to -1} \Gamma(\lambda) \int\limits_0^1 u^\lambda F(2/d(\lambda+1)-1, u) du.$$

Proof. The idea of the proof is to break the integral over two subintervals $[0,\eta]$ and $[\eta,1]$ so that on $[0,\eta]$ we have $|1-\mu(u)|$ small. On $[\eta,1]$, μ is integrable and the u^λ term drives the integral to 0.

Let $\epsilon > 0$ be given. We may assume $\epsilon < 1$ without loss of generality. Let M be a bound on $|F|$. Fix some $\eta > 0$ such that $|1-(1-u_1)^{(d-2+d(q-p))/2}| < \frac{\epsilon}{(4M)}$ for $|u_1| < \eta$. Then

$$\left|\int_0^1 u^\lambda(\lambda+1)F(\lambda,u)\mu(u)du - \lim_{\lambda\to -1}\int_0^1 u^\lambda(\lambda+1)F(\lambda,u)du\right| \tag{3.16}$$
$$\leq M\int_0^1 u^\lambda(\lambda+1)|1-\mu(u)|du \tag{3.17}$$
$$\leq \frac{\epsilon}{4}\int_0^\eta u^\lambda(\lambda+1)du + M\int_\eta^1 u^\lambda(\lambda+1)|1-\mu(u)|du \tag{3.18}$$
$$\leq \frac{\epsilon}{4}\eta^{\lambda+1} + \eta^\lambda(\lambda+1)M\int_\eta^1 |1-\mu(u)|du \tag{3.19}$$

Observe that $|1-\mu(u)|$ is integrable on $(\eta,1)$. Since $\lim_{\lambda\to -1}\eta^{\lambda+1} = 1$ and $\lim_{\lambda\to -1}\eta^\lambda(\lambda+1) = 0$, there exists $\eta' > 0$ such that for $|\lambda+1| < \eta'$ we have

$|\eta^{\lambda+1}| < 1 + \epsilon$ and $\eta^{\lambda}(\lambda+1)M\int_{\eta}^{1}|1-\mu(u)|du < \epsilon/2$. In this case

$$(3.19) < \epsilon/4(1+\epsilon) + \epsilon/2 \leq \epsilon.$$

$\square$

Proof of the Theorem. Now we will see that

$$\lim_{\lambda\to-1}(\lambda+1)\int_{0}^{1} u^{\lambda}F(2/d(\lambda+1)-1,u)du = F(-1,0).$$

As before, the proof is just a matter of breaking up the integral into subintervals $[0,\eta]$ and $[\eta,1]$. We choose η so that $|F(\lambda,u)-F(\lambda,0)|$ is uniformly small on $[0,\eta]$. Then, as $\lambda \to -1$, the integral over $[\eta,1]$ is driven to 0 by the u^{λ} factor.

Let $\epsilon > 0$ be given. Assume without loss of generality that $\epsilon < 1$. Let M be a bound on F. Choose η so that $|F(\lambda,u)-F(\lambda,0)| < \epsilon/8$ when $|u| < \eta$.

$$\left|\int_{0}^{1} u_1^{\lambda}(\lambda+1)F(\lambda,u_1)du_1 - F(-1,0)\right| \tag{3.20}$$

$$= \left|\int_{0}^{1} u_1^{\lambda}(\lambda+1)F(\lambda,u_1)du_1 - \int_{0}^{1} u_1^{\lambda}(\lambda+1)F(-1,0)du\right| \tag{3.21}$$

$$= \int_{0}^{\eta} u_1^{\lambda}(\lambda+1)|F(\lambda,u_1)-F(-1,0)|du_1 + \int_{\eta}^{1} u_1^{\lambda}(\lambda+1)\left|F(\lambda,u_1)-F(-1,0)\right|du_1 \tag{3.22}$$

$$\leq \int_{0}^{\eta} u_1^{\lambda}(\lambda+1)\left(|F(\lambda,u_1)-F(\lambda,0)|+|F(\lambda,0)-F(-1,0)|\right)du_1 + 2M\int_{\eta}^{1} u_1^{\lambda}du_1 \tag{3.23}$$

Observe that in the integral, $|F(\lambda, u_1) - F(\lambda, 0)| \leq \epsilon/8$. Let us choose η' so that for $|\lambda + 1| < \eta'$ we have

$$|F(\lambda, 0) - F(-1, 0)| < \epsilon/8$$

and

$$|\eta^{\lambda+1} - 1| < \min\{\epsilon, \epsilon/(4M)\}.$$

Then

$$(3.23) \leq \frac{\epsilon}{4}\eta^{\lambda+1} + 2M(\eta^{\lambda+1} - 1) \tag{3.24}$$

$$\leq \frac{\epsilon}{4}(\epsilon + 1) + \epsilon/2 \leq \epsilon \tag{3.25}$$

□

Corollary 4. In coordinates $\lim_{\lambda \to -1} \Gamma(\lambda)\mathcal{C}^\lambda f(\beta)$ equals

$$c \int_0^1 \cdots \int_{u_{p-1}}^1 \prod_{i=2}^p u_i^{d-1} f^\vee(0, u_2, \ldots, u_p)\nu_{p-1}(u_2, \ldots, u_p) du_p \cdots du_2 \tag{3.26}$$

If we reverse the change of variable $u_i = \cos^2(t_i)$ for $i = 2, \ldots, p$, we get

$$c \int_0^{\pi/2} \cdots \int_0^{t_{p-1}} \alpha_1(\mathbf{t}_1)^d f_1(\mathbf{t}_1)\delta_1(\mathbf{t}_1) dt_p \cdots dt_2. \tag{3.27}$$

Definition 9. For $f \in C^\infty(\mathrm{Gr}(p, \mathbb{K}^n))$, the transform $\mathcal{F}_1$ is defined by

$$\mathcal{F}_1 f = \lim_{\lambda \to -1} \gamma(\lambda)\mathcal{C}^\lambda f.$$

3.4 Geometric Interpretation of $\mathcal{F}_1$

Our goal in this section is to arrive at a geometrically meaningful interpretation of the transform $\mathcal{F}_1$. We will see that this transform can be interpreted as a cosine transform on a lower-dimensional Grassmannian $\mathcal{B}_1 \cong \mathrm{Gr}(p-1, \mathbb{K}^{n-2})$ embedded in $\mathcal{B} \cong \mathrm{Gr}(p, \mathbb{K}^n)$.

3.4.1 Cosine-λ Transform on a Lower Rank Grassmannian

Consider the integral (3.27). Recall $\mathbf{t}_1 = (t_2, \ldots, t_p)$ and $Y_1(\mathbf{t}_1) = Y(0, t_2, \ldots, t_p)$. Let $k_1 = \exp Y(\pi/2, 0, \ldots, 0)$. With this notation, for (3.27) we can write

$$c \int_0^{\pi/2} \cdots \int_0^{t_{p-1}} \alpha_1(\mathbf{t}_1)^d f(\exp Y_1(\mathbf{t}_1) k_1 \beta) \delta_1(\mathbf{t}_1) dt_p \cdots dt_2 \tag{3.28}$$

Let

$$K_1 = \{x \in K | xe_1 = \gamma_1 e_1 \text{ and } xe_n = \gamma_n e_n; \gamma_1 \text{ and } \gamma_n \text{ scalars}\}$$

and $L_1 = K_1 \cap L$. Then $K_1/L_1 \cong \mathrm{SU}(n-2, \mathbb{K})/(\mathrm{SU}(n-1, \mathbb{K}) \times \mathrm{SU}(n-1, \mathbb{K}))$.

Lemma 8. Let $\pi : K_1 \to K/L$ denote the quotent map $x \mapsto xL$. Then $\pi(K_1)$ is a closed, embedded submanifold of K/L and it is diffeomorphic to K_1/L_1.

Proof. The group K_1 acts on K/L and K_1 is compact, so the action is proper (see [Lee13] on proper group actions). It follows that orbits in K/L are closed embedded submanifolds. Observe that $\pi(K_1)$ is the orbit of L. Thus $\pi(K_1)$ is a closed manifold on which K_1 acts transitively, and it is clear that the stabilizer of L under this action is L_1, so $\pi(K_1)$ is diffeomorphic to K_1/L_1. □

The following proposition is very straightforward and proof is omitted.

Proposition 11. Let $\mathfrak{a}_1 = \{Y_1(\mathbf{t}_1) | t_i \in \mathbb{R}\}$.

1. The involution τ of K restricts to an involution on K_1 and $K_1^\tau = L_1$. We use the notation $\mathfrak{k}_1 = \mathfrak{l}_1 \oplus \mathfrak{q}_1$ for the eigenspace decomposition of the Lie algebra. Then $\mathfrak{l}_1 \subset \mathfrak{l}$ and $\mathfrak{q}_1 \subset \mathfrak{q}$.

2. The subspace $\mathfrak{a}_1$ is a maximal abelian subspace of $\mathfrak{q}_1$.

3. The submanifold $\pi(K_1)$ is a symmetric space under the action of the group K_1. Its involution is the restriction of τ.

4. The translate $\mathcal{B}_1 := \pi(K_1)k_1$ is an embedded submanifold and is diffeomorphic to $\mathrm{Gr}(p-1, \mathbb{K}^n)$.

Since $\mathcal{B}_1 = \pi(K_1)k_1$ is a Grassmannian in its own right, there is a cosine-λ transform defined on it which we denote $\mathcal{C}_1^\lambda$.

In line with our development above and applying Proposition 11 we can write this transform in coordinates. Observe that in dropping down from $\mathrm{Gr}(p, \mathbb{K}^n)$ to $\mathrm{Gr}(p-1, \mathbb{K}^{n-2})$, the important value $q - p = (q-1) - (p-1)$ is preserved, so the root system falls into the same category. We take the positive Weyl chamber on $\mathfrak{a}_1$ induced by our choice on $\mathfrak{a}$. We let $\mathfrak{a}_1^+$ denote the positive Weyl chamber and $A_1^+ = \exp \mathfrak{a}_1^+$. We take a fundamental domain U_1 for the map $\exp : \mathfrak{a}_1^+ \to A_1^+$ in the same way as before. We will also evaluate the $\mathcal{C}_1^\lambda$ transform on a particular $\beta_1 = k_1\beta$. Then we have

$$\mathcal{C}_1^\lambda f(\beta_1) = \int\limits_{U_1} \prod_{i=2}^{p} |\cos \epsilon_i(Y)|^{d\lambda} f(\exp Y \beta_1)\delta_1(\exp Y)dY \tag{3.29}$$

$$= \int\limits_0^{\pi/2} \cdots \int\limits_0^{t_{p-1}} \alpha_1(\mathbf{t}_1)^{d\lambda} f(\exp Y_1(\mathbf{t}_1)\beta_1)\delta_1(\exp Y_1(\mathbf{t}_1))dt_p \ldots dt_2 \tag{3.30}$$

Here $\delta_1 = \prod\limits_{\alpha \in \Sigma_1^+} |\sin \alpha(i \log(b))|^{m_\alpha}$, where Σ_1^+ denotes the positive restricted roots of $\mathfrak{k}_1$ with respect to $\mathfrak{a}_1^+$. Proposition 4 applies with the modification that the indices range between 2 and p. Therefore $\delta_1(\exp Y_1(\mathbf{t}_1)) = \delta_1(\mathbf{t}_1)$.

We observe now that

$$(3.30) = \mathcal{C}_1^1(f)(\beta_1).$$

Note that this is a $\mathcal{C}_1^\lambda$ transform with $\lambda = 1$. This is the essential geometric observation. The normalized cosine-λ family of transforms on a Grassmannian yields at $\lambda = -1$ a transform that is itself a cosine-λ transform on a Grassmannian of lower rank.

3.4.2 *L*-Orbits of Lower Rank Grassmannians

We make a geometric observation about the way these lower rank Grassmannians sit inside $\mathcal{B}$.

Observe that $\mathcal{B}_1 = \{\sigma \in \mathrm{Gr}(p, \mathbb{K}^n) \mid e_n \in \sigma \text{ and } \sigma \subset e_1^\perp\}$. Let us set the notation

$$Z_u^v = \{\eta \in \mathrm{Gr}(p, \mathbb{K}^n) \mid u \in \eta \text{ and } \eta \subset v^\perp\}.$$

Proposition 12. Given $\xi, \eta \in \mathrm{Gr}(p, \mathbb{K}^n)$, ξ contains a vector orthogonal to η if and only if $|\mathrm{Cos}(\xi, \eta)| = 0$

Proof. Assume $|\mathrm{Cos}(\xi, \eta)| = 0$. We consider ξ and η dp-dimensional real vector spaces. Given an orthonormal basis $\{\xi_1, \ldots, \xi_{pd}\}$ for ξ, let E denote the unit-volume parallelepiped formed with these vectors at its edges. Let v_i' denote $P_\eta v_i$, the orthogonal projection onto η. Since $|\mathrm{Cos}(\xi, \eta)| = 0$, we have $\mathrm{Vol}(P_\eta(E)) = 0$, so the set $\{v_1', \ldots, v_{dp}'\}$ is linearly dependent. Therefore the span of $\{v_1, \ldots, v_{dp}\}$ contains some element v contained in the kernel of P_η, which is $\eta^\perp$. This gives us an element v orthogonal to η in the real dot product. Since $\eta = i\eta = j\eta = k\eta$ over $\mathbb{H}$ and $\eta = i\eta$ over $\mathbb{C}$, for $d > 1$ this implies that v is orthogonal to η in the hermitian form $\langle \cdot, \cdot \rangle_{\mathbb{K}}$ also.

For the converse, if ξ contains an element in the kernel of P_η then the volume of $P_\eta(E)$ is clearly 0. $\square$

Proposition 13. The action of K on $\mathrm{Gr}(p, \mathbb{K}^n)$ induces an action on the family $\{Z_\nu^\omega | \nu, \omega$ perpedicular unit vectors in $\mathbb{K}^n\}$ and this action is given by

$$k \cdot Z_\nu^\omega = Z_{k\nu}^{k\omega}$$

For $\eta \in \mathrm{Gr}(p, \mathbb{K}^n)$, let us set $Z(\eta) = \{\xi \in \mathrm{Gr}(p, \mathbb{K}^n) \mid |\mathrm{Cos}(\xi, \eta)| = 0\}$. Then $Z(\beta)$ is the set where $\eta \mapsto |\mathrm{Cos}(\eta, \beta)|^{-1}$ blows up. One might think of this set $Z(\beta)$ as the appropriate notion of $\beta^\perp$ in the Grassmannians by analogy with $v^\perp$ for v on the sphere. On the sphere, $v^\perp$ is the lower dimensional sphere where $u \mapsto |\mathrm{Cos}(u, v)|$ takes zero values.

Observe that

$$Z(\beta) = \bigcup_{v \in \beta, \omega \in \beta^\perp} Z_v^\omega = L Z_{e_1}^{e_n} = L \mathcal{B}_1.$$

Therefore, in this sense, $Z(\beta)$ decomposes into copies of $\mathrm{Gr}(p-1, \mathbb{K}^{n-1})$, and L acts on this family of lower-dimensional Grassmannians transitively. Since we assume f is L-invariant, nothing is lost by restricting attention to $\mathcal{B}_1$.

3.5 Image and Kernel of $\mathcal{F}_1$

Now we turn to some representation-theoretic considerations regarding the integral transform $\mathcal{F}_1$.

Proposition 14. The transform $\mathcal{F}_1$ is an intertwining operator of the left regular representation of K on $\mathcal{C}^\infty(\mathcal{B})$.

Proof. Let L_k denote left translation by k. The cosine-λ transform $\mathcal{C}^\lambda$ is a mermomorphic family of intertwining operators (see [ÓP12], Theorem 4.5). Thus for any $k \in K$ and $\lambda > -1$, we have $\mathcal{C}^\lambda \circ L_k - L_k \circ \mathcal{C}^\lambda = 0$. It follows by the analytic continuation that the equality $\gamma(\lambda)\mathcal{C}^\lambda \circ L_k - L_k \circ \gamma(\lambda)\mathcal{C}^\lambda = 0$ extends to the limit at $\lambda = -1$. □

Therefore, the image and kernel of $\mathcal{F}_1$ are invariant subspaces and we will characterize them in terms of μ, the highest weights in the decomposition in (2.3).

Proposition 15. The image of $\mathcal{F}_1$ is composed of those subspaces $L^2_\mu(\mathcal{B})$ with highest weight $\mu = (m_1, \ldots, m_p)$ where $m_2 = \cdots m_p = 0$.

Recall the spectrum of $\mathcal{C}^\lambda$ and consider the function

$$\frac{\eta_\mu(\lambda)}{\eta_0(\lambda)} = \left(\frac{\Gamma_{p,d}(\frac{1}{2}(-d\lambda) + \mu))}{\Gamma_{p,d}(\frac{1}{2}(-d\lambda))}\right)\left(\frac{\Gamma_{p,d}(\frac{1}{2}(d\lambda + dn))}{\Gamma_{p,d}(\frac{1}{2}(d\lambda + dn + \mu))}\right). \tag{3.31}$$

We examine for which values of μ this function is 0 at $\lambda = -1$ and for which values it is nonzero.

Here, recall that $\mu = (m_1, \ldots, m_p)$ and in all cases we have that the m_i are all even integers and $m_1 \geq \cdots \geq |m_p|$ where m_p can only be negative in the case $p = q$ and $d = 1$.

Suppose $p > 2$. We expand the factor on the left in (3.31):

$$\frac{\Gamma(\frac{d}{2}(-\lambda) + m_1/2)}{\Gamma(\frac{d}{2}(-\lambda))}\frac{\Gamma(\frac{d}{2}(-\lambda - 1) + m_2/2)}{\Gamma(\frac{d}{2}(-\lambda - 1))}\cdots\frac{\Gamma(\frac{d}{2}(-\lambda - p + 1) + m_p/2)}{\Gamma(\frac{d}{2}(-\lambda - p + 1))} \tag{3.32}$$

The factor on the right in (3.31) expands to

$$\frac{\Gamma(\frac{d}{2}(\lambda + n))}{\Gamma(\frac{d}{2}(\lambda + n) + m_1/2)}\frac{\Gamma(\frac{d}{2}(\lambda + n - 1))}{\Gamma(\frac{d}{2}(\lambda + n - 1) + m_2/2)}\cdots\frac{\Gamma(\frac{d}{2}(\lambda + q))}{\Gamma(\frac{d}{2}(\lambda + q) + m_p/2)}. \tag{3.33}$$

Note that (3.33) cannot be infinite since $q > 1$, so for the product in (3.31) to be finite, the other factor (3.32) must be nonzero.

Considering (3.32), each factor $\Gamma(\frac{d}{2}(-\lambda - j))$ in the denominator for $j > 0$ is infinite at $\lambda = -1$ if $\frac{d}{2}(-\lambda - j)$ is an integer. For (3.32) to be nonzero, then, each of these factors must be matched by an infinity in the numerator. In particular, the factor $\Gamma(\frac{d}{2}(-\lambda - 1))$ in the denominator must be matched by $\Gamma(\frac{d}{2}(-\lambda - 1) + m_2/2)$

in the numerator, which requires that $m_2 = 0$. This forces $m_2 = \cdots = m_p = 0$. However, $\Gamma(\frac{d}{2}(-\lambda))$ is finite and m_1 is free.

Suppose $p = 2$. Then we have expansions

$$\frac{\Gamma(\frac{1}{2}(-\lambda) + m_1/2)}{\Gamma(\frac{1}{2}(-\lambda))} \frac{\Gamma(\frac{1}{2}(-\lambda - 1) + m_2/2)}{\Gamma(\frac{1}{2}(-\lambda - 1))}$$

and

$$\frac{\Gamma(\frac{d}{2}(\lambda + n))}{\Gamma(\frac{d}{2}(\lambda + n) + m_1/2)} \frac{\Gamma(\frac{d}{2}(\lambda + n - 1))}{\Gamma(\frac{d}{2}(\lambda + n - 1) + m_2/2)}$$

By the same reasoning as above, $m_2 = 0$ and m_1 is free unless it's possible for m_2 to be negative. That can only happen when $p = q$ and $d = 1$. In that case, the factor $\frac{\Gamma(\frac{d}{2}(-\lambda-1)+m_2/2)}{\Gamma(\frac{d}{2}(-\lambda-1))}$ is nonzero for $m_2 \leq 0$. However, noting that $n = 4$, the factor

$$\frac{\Gamma(\frac{1}{2}(\lambda + 4 - 1))}{\Gamma(\frac{1}{2}(\lambda + 4 - 1) + m_2/2)}$$

is zero for $m_2 < 0$ (and m_2 an even integer).

Thus, in all cases, the function $\frac{\eta_\mu(\lambda)}{\eta_0(\lambda)}$ is nonzero at $\lambda = -1$ precisely for those values of μ where $m_2 = \cdots = m_p = 0$.

4 Higher Poles of $\mathcal{C}^\lambda$

4.1 Overview

We now turn our attention to the higher poles of the cosine-λ transform—those on the negative integers -2,..., -p. Here we restrict attention to the Grassmannians over $\mathbb{R}$. In this case, B. Rubin has eplored the analytic continuation of a normalized cosine-λ transform to $\lambda = -p$. Recall (1.3) from the introduction.

In this section we will assume $p \geq 2$ because the $p = 1$ case is well understood and because in that case there are no "higher poles" above $\lambda = -1$ to consider.

At first glance our work here would seem to render our previous analysis of the first pole unnecessary because it applies to that pole also. There are two reasons for presenting both that argument and this argument separately. Our analysis in previous sections applied whether the field was $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{K}$, but here we use Rubin's result, which was only proved in a setting over $\mathbb{R}$. The second reason is that our analysis in previous sections is quite different from Rubin's methods.

We translate Rubin's work into the language and notations of this dissertation. Rubin works in terms of Stiefel manifolds, but as he points out we may apply his theorems to the Grassmannian picture by viewing the functions on the Stiefel manifold as right-O(p)-invariant functions so they lift to the Grassmannian. He proves that

$$\underset{\lambda=-p}{a.c.}\,\gamma(\lambda)\mathcal{C}^\lambda f(\beta) = c\int_{\sigma\subset\beta^\perp} f(\sigma)d\sigma$$

in the invariant measure, where c is a nonzero constant. Note that in our view in this dissertation, we have fixed β and L is the stabilizer of β in K. Thus, fix any $\eta \in \beta^\perp$ and

$$\int_{\sigma\subset\beta^{\perp}} f(\sigma)d\sigma = \int_{L} f(l\eta)dl = f(\eta)$$

where we assume f is L-invariant as before. The coordinates used above provide a convenient choice of η given by $(t_1,\ldots,t_p) = (\pi/2,\ldots,\pi/2)$. Then, in our view of things, this result can be stated as such: the analytic continuation of $\gamma(\lambda)\mathcal{C}^{\lambda}f(\beta)$ to $\lambda = -p$ is $f(\pi/2,\ldots,\pi/2)$ up to a non-zero factor.

The intuitive idea of the following results is as follows. The analytic continuation of $\gamma(\lambda)\mathcal{C}^{\lambda}f(\beta)$ to the pole at -1 yields an integral transform on an embedded submanifold which is itself a Grassmannian of rank $p-1$ Further, this transform has the form of a cosine-λ transform evaluated at $\lambda = 1$. At $\lambda = -p$, we have a simple evaluation at a point, which we may view as the rank 0 case. At the poles in between, we step down in rank at each iteration from -1 to -p. That is, at -2 we will have a cosine-λ transform over an embedded submanifold which is a Grassmannian of rank p-2, evaluated at a particular λ which comes out of the analysis, and it continues in this way.

4.2 Partial Cosine-Funk Transforms

Recall that in coordinates

$$\mathcal{C}^{\lambda}f(\beta) = \int_{0}^{1}\int_{u_1}^{1}\cdots\int_{u_{p-1}}^{1}\prod_{i=1}^{p} u_i^{(d(\lambda+1)-2)/2} f^{\vee}(\mathbf{u})\nu_p(\mathbf{u})du_p\cdots du_1 \tag{4.1}$$

Lemma 9. Fix $k = 1,\ldots,p-1$ and let

$$F(\lambda,u_1,\ldots,u_{p-k}) = \int_{u_{p-k}}^{1}\cdots\int_{u_{p-1}}^{1}\prod_{i=1}^{p} u_i^{(d(\lambda+1)-2)/2} f^{\vee}(\mathbf{u})$$

$$\prod_{i\le p-k<j}|u_i-u_j|\nu_1(\mathbf{u})du_p\cdots du_{p-k+1} \tag{4.2}$$

which is the inner k integrals in (4.1). Then F is uniformly continuous at $\lambda = -p+k$.

Proof. The innermost integral is

$$F_1(\lambda, u_{p-1}) = \int_{u_{p-1}}^{1} u_p^{\frac{\lambda-1}{2}} \prod_{i<p} |u_i - u_p| f^\vee(\mathbf{u}) \nu_1(\mathbf{u}) du_p.$$

We have suppressed the dependence on the other variables. Observe that $u_p^{\frac{-p+k-1}{2}} \prod_{i<p} |u_i - u_p|$ is bounded for $u_p \in [0,1]$ since $u_1 \leq \cdots \leq u_p$. This is because

$$u_p^{\frac{-p+k-1}{2}} \prod_{i<p} |u_i - u_p| \leq u_p^{\frac{-p+k-1}{2}} u_p^{p-1}$$

which is bounded when $0 \leq p-2$, and we have assumed $p \geq 2$.

Then $|F_1(-p+k, u_{p-1}) - F_1(-p+k+h, u_{p-1})| \leq \int_{u_{p-1}}^{1} u_p^{-p+k}(1-u_p^h) \prod_{i<j} |u_i - u_p|\, |f(u_{p-1}, u_p)| du_p \leq M \int_{u_{p-1}}^{1} (1-u_p^h) du_p$, some M.

We iterate outward like this observing that at each stage $u_j^{\frac{-p+k-1}{2}} \prod_{i<j} |u_i - u_j|$ is bounded.

$\square$

Theorem 6. With k, F and γ as above,

$$\underset{\lambda=-p+k}{\text{a.c.}} \gamma(\lambda) C^\lambda f(\beta) = F(-p+k, 0, \ldots, 0)$$

which equals

$$\int_0^{\pi/2} \cdots \int_0^{t_{p-1}} \prod_{i=p-k}^{p} |\cos t_i|^{p-k} f_k(\mathbf{t}_k) \delta_k(\mathbf{t}_k) dt_p \cdots dt_{p-k}. \tag{4.3}$$

Proof. We view $\mathcal{C}^\lambda f(\beta)$ as a cosine-λ transform $\mathcal{C}_k^\lambda$ of a function F defined on a Grassmannian manifold of rank $p-k$. In this case, it is evaluated at β_k, the element spanned by $\{b_1, \ldots, b_{p-k}\}$. Rubin proved that

$$\underset{\lambda=-p+k}{\text{a.c.}} \gamma(\lambda) \mathcal{C}_k^\lambda [F(-p+k, u_1, \ldots, u_{p-k})](\beta_k) = cF(-p+k, 0, \ldots, 0)$$

where c is a nonzero constant.

For λ close enough to $-p+k$, we have

$$|F(\lambda, u_1, \ldots, u_{p-k}) - F(-p+k, u_1, \ldots, u_{p-k})| < \epsilon$$

uniformly. By linearity, and supressing dependence except on λ,

$$\mathcal{C}_k^\lambda[F(\lambda)] = \mathcal{C}_k^\lambda[F(-p+k)] + \mathcal{C}_k^\lambda[F(\lambda) - F(-p+k)].$$

Then as $\lambda \to -p+k$, it follows that

$$\gamma(\lambda)\mathcal{C}_k^\lambda[F(\lambda) - F(-p+k)](\beta_k) \to 0.$$

The integral (4.3) is just an evaluation of $F(-p+k, 0, \ldots, 0)$ followed by a change of variable. $\square$

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Vita

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