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Enculturational practices in the teaching of proof in mathematics

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ENCULTURATIONAL PRACTICES IN THE TEACHING OF PROOF IN MATHEMATICS

A Dissertation

Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy

in

The Department of Educational Theory, Policy, and Practice

by

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This thesis is dedicated to the lotus feet of Lord Venkateswara, the all-pervading almighty God. This study had been possible only with divine intervention. God has helped me in each and every step of this incredible journey and the credit for the successful completion of this endeavor goes to him.
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ABSTRACT

Mathematics education reform is informed by constructivist theories that forefront student learning of concepts, and by sociocultural theories whose focus is on students’ mastery of mathematical practices. As Cobb (1994) pointed out, these theorizations are inconsistent with one another, leading to conflict as some theorists seek to promote their approach as the correct one. Alternatively, Cobb, and many others in the social constructivism or the situated cognition camps, seek some sort of integration or balancing of these priorities in pedagogical theorizing. Kirshner (2002, 2004, 2008) argued that instead of either selecting one theory or balancing/coordinating the two theories, we should regard each theory as an independent basis for pedagogical practice, and articulate a separate genre of teaching for each. In that spirit, the current study sought to explore pedagogical methods directed exclusively to enculturating students into mathematical practices, particularly, practices of argumentation characteristic of mathematical proof. The researcher worked with a group of 11 average-ability students in the 11-12 age range, over 24, half-hour sessions. At first, students were called upon to discuss various basic geometric terms, and then to present arguments establishing the truth of 10 basic geometric theorems. Students worked together in groups to discuss the problems, and presented their proofs. All sessions were videotaped and transcribed, and each student’s arguments were coded for sophistication on a 4-level system based on the work of Lolli (2005) and Douek (2009). The results indicated that all students advanced in their level of sophistication, most moving from level 1 in which one understands that an explanation is required, but one does not understand the obligation for the explanation to be logically persuasive to level 3 in which one coordinates the elements of the argument in a way that is consistent with logically sound deductive reasoning. The qualitative analysis of interactional processes illustrates the influence of the group’s level of discourse on individual development.
CHAPTER 1
INTRODUCTION

The general trend in mathematics teaching and learning is for students to memorize procedures and practice them without much conceptual understanding of the various mathematical principles underlying the material. The current culture of the math classroom is one in which students repeat what the teacher models. The goal always seems to be the mastery of routine skills. These problems are not new. According to Courant and Robbins (1941), “The teaching of mathematics has sometimes degenerated into empty drill in problem solving, which may develop formal ability but does not lead to real understanding or to greater intellectual independence” (p. 1). Though reformers decry such practices and several approaches to teaching mathematics have been tried, still the current trend is to deliver required procedures to a group of passive learners. Although some teachers try to make students understand the underlying principles behind mathematical concepts, for the most part, learning mathematics still involves a focus on procedural skills that makes math difficult to understand and boring to pursue. Teachers are often blamed for the lack of motivation on part of their learners and their low achievement in mathematics. The teachers in turn blame the curriculum and the restrictions placed on them for course completion and student achievement on standardized tests. This becomes a vicious circle in which people blame one another rather than seeking a solution. This is not just the problem of a particular country; it is the same around the globe as teachers teach their students in the same old ways that they were taught or exposed to when they did their student teaching.
This study intended to shed light on pedagogical practices that can ensure that the student’s role is not that of a passive absorber of knowledge, but an active participant in the process of acquiring it. In the words of Bishop (1988), “It is not enough merely to teach them mathematics, we need also to educate them about mathematics, to educate them through mathematics and to educate them with mathematics” (p. 3). This quote deepens the relevance of the discipline and should inspire math educators to adopt new ways of teaching mathematics. For example, the area of proofs has the inherent capacity to establish the connection between the mathematics being taught to students and the actual discipline of mathematics. Fawcett (1938) clearly states:

The concept of proof is one which not only pervades work in mathematics but is also involved in all situations where conclusions are to be reached and decisions to be made. Mathematics has a unique contribution to make in the development of this concept, and … this concept may well serve to unify the mathematical experiences of the pupil. (p. 120)

Proof is an important aspect of mathematical education. Learning to use proof involves developing higher order thinking capabilities which students must master before attempting theoretical math at the college level. Higher levels of geometrical thinking include the ways in which students apply the inductive process to find, justify, and prove generalizations. Though proof was considered to be an integral part of the geometry curriculum alone, it has now expanded into the other core areas like algebra and calculus, and math educators are striving to incorporate proof into daily mathematical activities. This study aimed to address the area of proof, particularly how it is taught. Though proof is an integral part of the mathematics curriculum, it is often the most neglected part. In order to suggest how proof should be taught, it is necessary to understand the limitations of the traditional approach used to teach proof in the regular classroom.
The traditional classroom approach relies heavily on the two-column format of proving. This format is the method most frequently used to introduce students to formal proof—writing in mathematics. Consider the sequence of a two-column proof. In the left column goes a list of statements, each one a consequence of the ones above it in the list. Adjacent to each statement (in the right column) is the reason why this statement does indeed follow from the previous steps. This format is seen in almost every geometry textbook. The problem is not with the two-column format, because this is a representation of the proof, the problem is with how it is being taught. In the traditional scenario, the teacher explains what a two-column format is and then writes the statement of the theorem on the board and asks the students to prove the theorem through a sequence of steps. Once this is taught, students are given similar postulates or theorems to prove. Though most likely a justification is provided for each argument, the students frequently do not comprehend the essence of why they should be following these steps. Research suggests that this traditional teaching method is the main reason why students do not have any conceptual understanding when writing proofs. As Hershcowitz (1990) puts it, “In the traditional approach to teaching geometry, the process of inductive discovery, formulated as conjectures, was almost [completely] neglected” (p. 88).

Understanding that proof is both a product and a process is lacking in current educational strategies. Alibert and Thomas (1991) point out that, students lose sight of the process aspect of proof because the traditional teaching method always emphasizes the final product. The possibility is not considered that students themselves should be coming up with propositions and proving them, so creativity is lacking. Professional mathematicians’ methods include working backwards, mapping a strategy, or structuring
a proof. However, in the classroom little relevance is given to these aspects. Instead, achievement of the final product is stressed. Reid and Knipping (2010) aptly say, “The results of teaching proof in an axiomatic context in the New Math and with a focus on form through two column proofs did not lead to students learning. This may be because excessive formality ignores the semi-formal nature of mathematicians’ proofs and the usefulness of pre-formal proofs in schools” (p. 220).

Recent research demonstrates other methods of teaching proof rather than following the traditional two-column format, which is the norm in a regular classroom. One such method that needs to be used more often to teach proofs is the socio-constructivist approach. Scholars in math education view proof as an “activity with social character” (Alibert & Thomas, 1991, p. 216). They have examined how mathematical meaning is socially negotiated. What they observed in their studies regarding the learning of proofs from elementary through college levels was that there was a social lens attached to learning them. Reid (1995) observed students in the process of verifying, explaining, and exploring in social contexts and concluded that “the organization of class activities should accommodate the development of a ‘culture of proving,’ in which students feel that deduction is an appropriate way to reason about mathematics” (p. vii). Keeping in mind the limitations of the traditional approach to teaching proof and drawing inspiration from the socio-constructivist approaches to teaching proofs, the current study intends to employ mathematical enculturation as a pedagogical tool to draw attention to the aspect of proof as a process without totally disregarding the aspect of proof as a product.
The advantages of using proof in mathematical learning are touted by many math educators. For example, Yackel and Hanna (2003) point out that proof is a tool for “verification, explanation, systematization, discovery, communication, construction of empirical theory, exploration of definition and of the consequences of assumptions and incorporation of a well-known fact into a new framework” (p. 228). The general opinion is also that proof can be an invaluable tool to enhance logical thinking. In a classic study, Fawcett (1938) conducted an experiment in his geometry course using proof as a means to develop logical thinking. The study involved teaching a geometry course to students in grades 9 through 11 for a period of two years. The students’ skills were measured based on six forms of data, ranging from tests to observations by others and the students themselves. Fawcett found that the students did develop logical thinking and critical thinking skills.

Reformers in mathematics education are striving to bring a more constructivist approach to the teaching and learning of proof in order to get students involved in the culture of proving. An overall view as to why proof is needed can be obtained by the comprehensive definition provided by Stylianides (2007) of a “mathematical argument,” a connected sequence of assertions for or against a mathematical claim, with the following characteristics:

1. It uses statements accepted by the classroom community (set of accepted statements) that are true and available without further justification;
2. It employs forms of reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of, the classroom community; and
3. It is communicated with forms of expression (modes of argument representation) that are appropriate and known to, or within the conceptual reach of, the classroom community. (p. 291)
The time has come for a new paradigm for teaching proof. Educators need to give more academic status to activities which help students transfer and connect everyday practices of argumentation with mathematical practices of argument. Activities should also focus on the aspects of inductive and deductive reasoning and highlight their role in the proving process. Polya (1957) writes:

having verified the theorem in several particular cases, we gathered strong inductive evidence for it. The inductive phase overcame our initial suspicion and gave us a strong confidence in the theorem. Without such confidence we would have scarcely found the courage to undertake the proof which did not look at all a routine job. When you have satisfied yourself that the theorem is true, you start proving it. (p. 83–84)

For Schoenfeld (1986), “the foundation, on which geometrical performance is based, includes both inductive and deductive competencies” (p. 226).

Researchers also assert that empirical observations in geometry are necessary because they introduce a discovery aspect in which the learner uncovers the necessity to prove what has been conjectured to be true. These inductive experiences are the intuitive base upon which the understanding and generation of a deductive proof can be built, making the act of proving a dynamic interplay between induction and deduction. The problem in incorporating proof is the formal way in which it is taught and learned. Harel and Sowder (1998) worry that “students do not learn that proofs are first and foremost convincing arguments, that proofs (and theorems) are a product of human activity, in which they can and should participate” (p. 297). Reasoning is another aspect that goes hand in hand with proof. It is imperative to develop both inductive and deductive reasoning skills in order to proceed to proofs. Reasoning and proofs are the effective media through which mathematical ideas can be constructed in a classroom.
The National Council for Teachers of Mathematics (2000) has recognized the following standards for reasoning and proof to be developed across the grades from K-12:

- recognize reasoning and proof as fundamental aspects of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs;
- select and use various types of reasoning and methods of proof. (p. 56)

**Purpose of the Study**

Students should learn mathematics through the cultural practices of exploring, arguing and thinking. According to Bishop (1988), this is the crux of the concept of mathematical enculturation. He defines mathematical enculturation as an intentional process of shaping ideas, “where concepts, meanings, process and values are shaped according to a certain criteria” (p. 124). He prioritizes knowledge over action when he says, “Teaching children to do mathematics emphasizes knowledge as a ‘way of doing.’ A mathematical education seems to me, in contrast, to be essentially concerned with ‘a way of knowing.’ That then speaks to me of a cultural perspective on mathematical knowledge” (Bishop, 1988, p. 3). This can be achieved through mathematical enculturation. He further adds, “The goal of this process is the development in each individual child of a *modus sciendi* – a way of knowing” (Bishop, 1988, p. 124). Other researchers concur. As Lee and Smith (2009) put it, “More inquiries are needed to investigate the relation between students’ interpretations and proof constructions and also to see how instructional practices can attend to the cognitive challenges faced by students who are new to the rules of the proving game” (p. 25). Reid (1995) is also of the opinion that:
if we can teach in a way that acknowledges that importance of explaining and exploring as motivations to prove, and that creates social contexts that allow the development of a culture of proving, then we may find that our students prove and understand proving well enough to understand that other ways of thinking are sometimes better. (p. 123)

This study is an exploration of the effects of an enculturationist approach to students’ learning to think about and write proofs by employing Bishop’s idea that an investigative approach should be “the modus operandi in nurturing an increasing sophistication of mathematical thought” (p. 115). Holding to his idea that “an investigation is an extended piece of work . . . intended to imitate some of the activities of the mathematician” (Bishop, 1988, p. 115), the teacher gives opportunities that specifically shape students’ creative behaviors which can be paralleled to “acquiring dispositions through enmeshment in a cultural community” (Kirshner, 2008, p. 16).

**Goals Associated with the Study**

In this study, the researcher intended to implement instructional procedures meant to encourage students to think and act like mathematicians (their ways of knowing) and utilize their knowledge in meaningful and culturally (mathematically) appropriate ways. That is to say, the researcher intended to involve the students in making sense of mathematical ideas through argumentation and justification. The researcher as a teacher intended to provide the students with the “pressurizing, encouraging, restricting or freeing social environment” (Bishop, 1988, p. 127) to make the students engage in mathematical practices. Through observation and documentation, modes of thinking and motivations for participation with which they come into the classroom should be perceived to undergo a gradual change.
Dispositions are tendencies to engage in mathematical practices. Kirshner (2004) interprets dispositions broadly as:

Inclinations to engage with people, problems, artifacts, or oneself in culturally particular ways. Thus establishing an enculturationist teaching agenda requires identifying a reference culture and target dispositions within it. In mathematics education, the reference culture usually is presumed to be mathematical culture, wherein a wide range of distinctive dispositional characteristics has been identified as instructional objectives. These include mathematical proof, the characteristic mode of argumentation by which new knowledge is established for the community through logical (rather than empirical) considerations. (p. 5)

Reasoning is a cognitive disposition that the study intended to address.

The Pedagogy

The instruction to be used for this study was intended to facilitate significant classroom discussion leading students to reason about different possibilities and outcomes. The teacher intended to position herself as a facilitator/catalyst in order to encourage the students to engage in different ways of reasoning and proof throughout the class sessions. The teacher even planned to participate in the discourse as a collaborator sometimes. The goal was to establish a collaborative learning community, which in turn was expected to facilitate students learning from one another. The students as a group were expected to come up with some defining solutions to the tasks at hand. The teacher planned to give them opportunities to construct their own body of mathematical knowledge to make learning more meaningful. The aim was to encourage their ability to do mathematics throughout the course of instruction. Confidence in dealing with tasks and utilizing their capabilities to the fullest extent was to be encouraged throughout each session. Tasks were planned and assigned with the goal of initiating into the different ways of mathematical thought. This reflects an enculturationist pedagogy wherein
the instructional focus is on the classroom microculture. The enculturationist teacher works to shape the microculture so that it comes to more closely resemble the reference culture [of disciplinary mathematics] with respect to the target dispositions. Students, thus, come to acquire approximations of the target dispositions of the reference culture through their enmeshment in the surrogate culture of the classroom (Kirshner, 2004, p. 7).

The study also focused on how the individual students’ mathematical activity was influenced by or influenced the classroom microculture. Cobb (1995) suggests that individual students’ mathematical activity and the classroom microculture are reflexively related. This suggests that the route to progress is a gradual one in which minor variations in the current forms of argumentation are worked by the teacher so that there is incremental progress toward normative mathematical forms of argumentation. The teacher intended to plan the phases of her instruction in such a way that individual students who start with no/little idea about proofs, would improve their reasoning through group action and come to a level of proving independently. It would be left to her discretion to decide at what juncture in the course of the tasks the students should start working independently. The study intended to cover at least 10 major theorems in the secondary school geometry curriculum.

**Research Questions**

1. How do the reasoning processes of individual students evolve in the context of enculturational instruction?

2. How do the reasoning processes of the group evolve in the context of enculturational instruction?

3. Given the reflexive relationship between the group and individuals who comprise it, how do the reasoning processes of individual students evolve in relation to the group?
Analyzing the sophistication in reasoning and argumentation is the main aim of the study. Lolli (2005) developed a framework to analyze the four modes of activity in proof production. Later, Douek (2009) adapted this framework to understand the construction of proof and the role of argumentation and used it as a tool to analyze proving as a cognitive, culturally situated activity involving four modes of reasoning. For the purpose of analyzing the students’ sophistication in reasoning and argumentation, a new framework was developed based on Lolli’s and Douek’s frameworks. A detailed description of the frameworks is presented in the methodology section. The analysis also makes use of more general notions of data, claim, and warrants (Toulmin, 1958), often referred to as elements of an argumentation structure.
CHAPTER 2
REVIEW OF LITERATURE

The review of literature deals exclusively with enculturational approaches to teaching and learning mathematics and proofs in particular. The theoretical perspectives of mathematical enculturation with supporting evidence from various studies are dealt within this review.

In the context of enculturation, learning is viewed as a process acquired through social interaction. Vygotsky (1978) emphasized that students construct knowledge through social interactions with others. Tappan (1998) captures this perspective and asserts that effective learning is more likely to take place when individuals are afforded opportunities to grow into the culture that surrounds them. Cognitive skills are mediated by words, language, and forms of discourse and have their origins rooted in social relations and within a socio-cultural setting.

Scholars in the field of mathematics education like Schoenfeld (1999), Tall (1998) and many more are of the opinion that a proof can be viewed as more of a communication tool in the classroom that reflects the socio-cultural perspective of learning. Social interaction is the key for enculturation as stressed by Vygotsky and others. One of Bishop’s (1988) approaches to mathematical enculturation is the use of an investigative environment, in which the main aim for the teacher is to work with the learners’ own creative potential, emphasizing aspects such as:

- choosing of symbols,
- the exploration of possibilities,
- hypothetical thinking,
- the representation of relationships,
- the development of conjectures,
- conviction, argumentation, and proof. (p. 147)

All these aspects in some way or other echo the sentiments of other proponents like Maher and Martino (1996), who are of the opinion that mathematical enculturation involves approaches like classifying, organizing and reorganizing data, constructing personal relationships, predicting, and formulating generalizations. In their longitudinal study on a six-year-old girl named Stephanie, they claim that a proper environment will empower the child to build the idea of mathematical proof over a period of time. The idea of proof, which is one of the assets that mathematicians possess, can be literally adopted in the classroom setting with the help of mathematical enculturation tool as Maher and Martino’s study shows us. In enculturationist teaching, the teacher begins by identifying a target culture (in this case, the culture of mathematicians) and target dispositions (in this case, attributes that mathematicians possess) within that culture and focuses on activities that develop those dispositions. Healy (1993) established an enculturational environment in his classroom through which he made his students develop a book on geometry entirely on their own. The focus is on the development of a mathematical culture in which students assume authority and responsibility for their learning. Healy’s class was an all-discovery geometry class. In the long run there was evidence that students showed increased sophistication in the writing of proofs.

Enculturation is an interactive, interpersonal process involving humanistic approaches. It focuses on shaping ideas and meanings rather than behaviors and techniques. Lakatos (1976) proposed an account of mathematical knowledge based on the
idea of heuristics. He explored the processes of mathematical proof and discovery through counterexamples to conjectures. Polya’s (1957) work was mostly concerned with the challenge of finding a proof; his heuristics can also be interpreted for more general a problem solution. He encourages teachers to inculcate many mathematical habits of mind including the habit of guessing. In his own words, “At the high school level, when the teacher proposes a problem for class discussion, he should begin with letting the students guess the result: The students being impatient to know whether their guess will turn out to be right or not will afterwards work with much more interest” (p. 258). Yackel and Cobb (1996) focus on socio-mathematical norms as a part of mathematical enculturation to guide the discussions in a classroom and articulate that:

Our observations consistently indicate that teachers capitalize on the learning opportunities that arise for them as they begin to listen to their students' explanations. The increasingly sophisticated way they select tasks and respond to children's solutions, shows their own developing understanding of the students' mathematical activity and conceptual development. These learning opportunities for the teachers are directly influenced by the sociomathematical norms negotiated in the classrooms. In particular, children continue to give a variety of explanations when different solutions are emphasized and developmentally sophisticated solutions are legitimized. These inform the teachers about the students' conceptual possibilities and their current understandings. The latter, in turn, contribute to the teachers evolving notions of what is sophisticated and efficient for the children. (p. 466, 467)

In this approach, the teacher plays an important role in the mathematical quality of the classroom. Yackel and Cobb emphasize the teacher’s own personal mathematical beliefs, values, knowledge, and understanding in creating the classroom microculture that enhances student mathematical thinking. In the context of proofs, unless the teachers themselves are confident and knowledgeable enough in dealing with them, they cannot expect to see those characteristics in their students. Bartolini (1996) came up with the term “mathematical discussion” to refer to a combination of different opinions on a
particular mathematical concept. Mathematical discussion works as a lever to address two major issues: the need for proof and the distinction between argumentation and proof. Mariotti (2000) on similar grounds has the opinion that “meanings are rooted in the phenomenological experience but their evolution is achieved by means of social construction in the classroom under the guidance of the teacher” (p. 278).

Maher (2009) discussed the role of mathematical argumentation in a cross-sectional study of a third grade class, where the activity involved students manipulating towers made of colored plastic cubes and articulated that even though “proof learning for students is difficult, our research shows that, in the process of convincing oneself and others the validity of a solution, arguments are presented by young children that take the form of mathematical proof” (p. 131).

Reid and Zack (2009) analyzed a multiyear teaching experiment in a fifth grade classroom and concluded that three characteristics that were central in her (Vicki, one of the researchers) class activities were “conjecturing, leaving the criteria for a correct solution up to the students and expectations for communication” (p. 145). They add that “These three characteristics put an emphasis on reasoning and arguing.” (p. 146) finally leads to proofs.

Boero (2006) recognizes three aspects to be considered while dealing with proofs:

– the epistemic aspect, inherent in the conscious validation of statements according to shared premises and legitimate ways of reasoning (“shared” and “legitimate” in the context of a given mathematical culture: secondary school mathematics, present or past mathematicians’ mathematics, etc.);

– the teleological aspect, inherent in the tension towards the products to be attained, and in the conscious choices to be made in that direction (i.e. in the problem solving character of conjecturing and proving processes);
the communicative aspect, inherent in the conscious adhering to rules that ensure both the possibility of communicating steps of reasoning, and the conformity of the products (statements and proofs) to mathematical standards in a given mathematical culture. (p. 2)

Nickerson and Rasmussen (2009) used a pedagogical intervention to characterize the enculturation process of students involved in the process of proving and concluded that “the criteria one brings to bear on proof are related to the socio–mathematical norms negotiated within the classroom community. The manner in which they engage in the activity of creating and critiquing proofs is related to classroom math practices” (p. 123).

Bauersfeld (1995) explored and analyzed language games and discussed participating in a culture of mathematizing. He talks about language games in mathematics classrooms as taken-as-shared activities that promote mathematical thinking. Cobb (1995) makes a similar point regarding the culture of mathematizing:

Participating in the process of a mathematics classroom is participating in a culture of mathematizing. The many skills, which an observer can identify and will take as the main performance of the culture, form the procedural surface only. These are the bricks of the building, but the design of the house of mathematizing is processed in another level. As it is with culture, the core of what is learned through participation is when to do what and how to do it….The core part of school mathematics enculturation comes into effect on the meta–level and is “learned” indirectly. (p. 9)

Bauersfeld (1995) also stated that “negotiation of meaning in social interaction functions as a starting point for the development of an effective language game” (p. 282). In a set of longitudinal studies, Cobb (1995) analyzed small–group activities and concluded that “the view that emerged in the course of analysis was that of a reflective relationship between students’ mathematical activity and the social relationships that they established” (p. 10). All the ideas presented above clearly reflect the enculturation aspects of mathematical learning. As Reid (2010) asserts:
Teaching experiments involving students solving problems, conjecturing and explaining, and verifying their conjectures seem to provide good contexts for proving. Careful examination and description of students’ patterns of reasoning and argumentation structures in these and other classrooms should permit researchers to explore in detail how different meanings and roles of proof and different levels of formality, along with other factors, influence teaching (p. 220).
CHAPTER 3
RESEARCH METHODOLOGY

This study used enculturationist pedagogy to engage students in the practice of proving in the context of geometry. The aim was to create a culture of mathematical learning in the classroom that nurtured the development of sophistication in the students’ discussions about mathematics. Proof is a form of investigation the students used to explore geometric figures and relationships. It is a complex activity involving various kinds of thinking including inductive and deductive reasoning and argumentation structures. It is a process as well as a product. The study focused on the Euclidean proofs taught in the Geometry curriculum.

The Subjects

The subjects for this study were 11 sixth grade students selected from two classes in a rural Title I school in Southern Louisiana. The sample was a convenience sample because the selection of the students was done based on accessibility. Formal proofs are introduced to students of geometry in the high school grades, so these students had never been exposed to proof and ways of proving. This is the main reason for selecting this group of children. The students were of different racial backgrounds. Twelve students began the study, but two wanted to drop out after the first few sessions. Two others joined but one of these dropped out after one session, leaving a total of eleven participants. The school principal gave permission for the researcher to conduct the sessions during the students’ enrichment time. The duration for each class in this study was half an hour, two to three days a week for a period of two months. There was strict adherence to making use of the instructional time to the fullest extent. The experiment
covered 10 major proofs that the geometry curriculum offers. The researcher allocated a pre-session to make the students acquainted with the goals and objectives of the study. Participation during the sessions, punctuality, commitment to the study, and regular attendance were stressed, but there were some absences which were beyond the control of the researcher as the students sometimes had other school-related obligations to fulfill. The students were allowed to take part in the study only after the required consent forms were signed by the parents and students.

**Design of the Study**

The study design is based on Ball’s (2000) research on and into practice from the perspective of the teacher-researcher which is research on teaching and learning where the principal investigator of the research is also the teacher and where at least one central goal is to contribute to scholarly discourse communities and to the development of theory... It strives to illuminate a broader point, probe a theoretical issue, develop an argument or framework. (p. 374)

The current study was an attempt to put theory into practice. The teaching method had its theoretical basis in mathematical enculturation, by which the teacher creates a mathematical culture that nurtures the cognitive capabilities of math proof. Enculturation pedagogy was used to support students’ acquisition of sophisticated forms of reasoning through a shaping of the mathematical culture of the classroom to foster the intended dispositions. Although the teaching of proof was the focus of this study, the students also made gains in their understanding of the particular concepts dealt with in the geometry curriculum.

The exposition of students’ geometric understanding was documented in the course of this study. This served to demonstrate that an enculturational focus in
instruction does not have to come at the expense of conceptual learning. Although students’ conceptual development was not formally assessed in the study (the researcher focused instead on the forms of students’ argumentation), this aspect of the instructional outcome will be explored in subsequent publications related to this study. Geometry proofs have the potential to make students think and get motivated to learn mathematics. It is the most feared venue by students and has been neglected for a long time. The aim of the classroom interactions was to develop intuitions about geometry and to make their geometric knowledge explicit.

The pedagogy was inspired by Fawcett’s (1938) and Healy’s (1993) approaches to dealing with geometry. Fawcett conducted an experimental study emphasizing reasoning processes and reflective thinking while accommodating to content knowledge. The study was conducted on high school students and consisted of an experimental group and a control group. Fawcett himself taught the students in the experimental group for two years, and the students from both groups were tested after two years. Comparisons were drawn between the achievements of both the groups.

Healy (1993) worked with high school students who developed a book on geometry entirely without content supervision by the instructor. Healy’s focus was on the development of a mathematical culture in which students assume authority and responsibility for their learning. Healy’s role was that of a facilitator. In the long run there was evidence that students showed increased sophistication even in the writing of proofs. No text was used, and the theorems were all discovered by the students. Though Healy’s focus was not on formal proofs, his students included in their notebooks mathematical arguments ranging from simple restatement of theorems to two-column
proofs. However, these were not the two-column proofs of traditional instruction that are cut off from authentic mathematical explorations. The description of Fawcett’s (1938) method gives a general idea of what the study looked like in action:

1. No formal text is used. Each pupil writes his own text as the work develops and is able to express his own individuality in organization, in arrangement, in clarity or presentation and in the kind and number of implications established.
2. The statement of what is to be proved is not given the pupil. Certain properties of a figure are assumed and the pupil is given an opportunity to discover the implications of these assumed properties.
3. No generalized statement is made before the pupil has had an opportunity to think about the particular properties assumed. This generalization is made by the pupil after he (she) has discovered it.
4. Through the assumptions made the attention of all pupils is directed toward the discovery of a few theorems, which seem important to the teacher.
5. Assumptions leading to theorems that are relatively unimportant are suggested in mimeographed material, which is available to all pupils but not required of any.
6. The major emphasis is not on the statement proved, but rather on the method of proof.
7. The extent to which pupils profit from the guidance of the teacher varies with the pupil and the supervised study periods are particularly helpful in making it possible to care for these variations. In addition individual conferences are planned when advisable. (p. 62)

Though the current study was modeled based on these studies done decades ago, the processes of learning were examined through more sophisticated lenses.

**Data Sources**

The main sources of data were:

- video tapes of the classroom discussions,
- transcription of the videos,
- notes made by the students,
- student work done on the board, and
- day to day written reflections of the researcher.
All the classroom discussions were video recorded without omission and with much clarity.

**Analysis of Argument Sophistication**

This study aimed to look at the development of sophistication of reasoning in the process of proof production. Lolli’s (2005) and Douek’s (2009) frameworks to analyze proof as a cognitive and socio–culturally situated activity engaging four modes of reasoning was utilized to develop a new framework to analyze the argumentation structures of the students. Douek’s framework initially was adapted from Lolli’s analysis of proof production.

In Lolli’s framework, production of reasons for validity is attributed to Level 1. It seems that this is too advanced for the initial level of proof production. Similarly, reasoning into a cogent argumentation is also too sophisticated for Level 2. Usually a cogent argument is considered to be logically sound. In the new framework developed for this study, these first two levels were aligned with Douek’s first two modes of reasoning in her framework.

Level 3 of Douek’s framework includes production of a deductive text following mathematicians’ norms. Mathematical norms extend deductive reasoning in specialized ways beyond ordinary notions of sound reasoning. For this reason, Level 3 of the new framework was aligned with Lolli’s framework. Level 4 of the new framework was aligned with both the frameworks except that the criterion here is more explicit.
Lolli’s and Douek’s Frameworks

Lolli:

MODE 1: exploration and production of reasons for validity of the statement;

MODE 2: organisation of reasoning into a cogent argumentation;

MODE 3: production of a deductive text according to specific cultural constraints concerning the nature of propositions and their enchaining;

MODE 4: formal structuring of the text according to shared rules of communication. (Douek, 2009, p. 142, 143)

Douek:

1) **Heuristic exploration** occurs when one tries to interpret a proposition or to produce a proposition or an example. One has in mind a target but the main focus is not on attaining the target through an acceptable mathematical reasoning. Any accidental event, writing, metaphor, may move the exploration activity. This type of reasoning is typically open to divergent paths.

2) **Organization of reasoning**, making explicit the threads of reasoning holding propositions together. When a proposition seems pertinent, a calculation promising, a writing efficient, one searches for a convincing coherent link to a local goal or to the global one. The links may be theoretical reasons of validity. The intentional and planning characters are typical of this mode, and abduction is a good example of it. Deductive reasoning is not yet a priority. Such organizational intention may concern partial arguments or the whole of the argumentation aimed at proof construction.

3) **Production of a deductive text following mathematicians’ norms.** Once ideas of proof are brought to light, they must be organized in a deductive reasoning.

4) **Formal structuring of the text**, to approach a formal derivation. To be able to produce a verbal or textual organization organized in a deductive way. (Douek, 2009, p. 333)
The word “Level” was used in the new framework instead of the word “mode” used by Lolli and Douek. Modes of thinking do not imply a hierarchical structure. One may adopt a mode at one point and then a different one at a different point, with no structure or order. On the other hand, “Level” suggests a hierarchical structure and at the same time does not prohibit regression to a prior level (for instance, “mode” would be excluded by the use of “phase”—another term considered for this taxonomy). Considering this, “Level” was considered to be the best term for this framework.

The New Framework

Level 1

One understands that an explanation is required, but one does not understand the obligation for the explanation to be logically persuasive. Perhaps one just restates the conditions, thinking that constitutes a sufficient explanation, or perhaps one makes associations to the contents of the theorem, but without any logical structure. Or one may cite authority to validate the claim, not recognizing the need for arguments to be explicitly made.

Specifics:

Whether one seeks to apply previous theorems as part of one’s argument, or to marshal data, does not, in itself, determine the level. What matters is if one is applying this information into something that is recognizable as an argument structure that attempts to be logically persuasive. If one does so, then the level is 2 or 3. If one simply incorporates these elements (previous theorems or data) into one’s explanation, but without asserting their persuasive character, then one is at Level 1.
Level 2

At this level, the obligation to be logically persuasive is understood, but the forms of deductive logic that compel agreement (Modus Ponens and Modus Tollens) have not yet been appropriated. Elements of correct arguments may be assembled, but the appropriate structures for coordinating those elements into a logical argument are not evidenced.

Specifics:
Applying previous theorems, and marshaling data, typically occur at this level. In general, one recognizes these elements need to be part of a persuasive argument, but the appropriate structures for coordinating those elements into a logical argument are not evidenced. Justifications for the premises emerging from an understanding of argumentation structure are not apparent.

Level 3

At this level, one coordinates the elements of the argument in a way that is consistent with logically sound deductive reasoning. Of course, this does not imply that one successfully completes a proof. However, if one is unsuccessful, one recognizes that fact; one does not change the rules of reasoning so that the argument one has managed to assemble is considered adequate – that is, one does not regress to the initial level.

Specifics:
In applying a previous theorem, one may deal with it as a condition/implication sequence that one has memorized, or one may understand and be able to
reproduce the reasoning that led to that prior result. One provides data with reasons of support and also provides data claim links with appropriate warrants.

**Level 4**

At this level, one has progressed beyond the general principles of sound deductive reasoning (Modus Ponens and Modus Tollens) that operate in literate society to embrace specific technical variations employed by mathematicians, including reductio ad absurdum, proof by induction, and the like. One also has a sense of the formal relation of given conditions to the structure of a proof; if a given condition is not utilized in the proof, one recognizes one is proving, not the originally stated theorem, but a more powerful theorem.

**Specifics:**

In applying a previous theorem, one recognizes one’s obligation to understand and be able to reproduce the reasoning that led to that prior result.

In working with children in this study, the stated expectation was that students should provide a justification for each step. This requirement was imposed for two reasons. First, even though mathematicians do not state a justification for each step of a proof, they keenly understand the obligation that each step be justifiable. Requiring explicit justifications ensured this obligation was realized in the classroom setting. Second, the students in the study were not all expected to be at the same level of sophistication. When mathematicians omit simple justifications in a proof, it is in the context of peer publication in which all readers are equally sophisticated in their level of reasoning. We needed to include all justification to make sure proofs were maximally accessible to the entire class.
For this reason, in evaluating a student’s level, when a justification was omitted the researcher counted this as indicating that the structure of sound argumentation had not been fully appropriated. This risks the possibility that in some cases a student may have fully comprehended the structure of argument and the necessity that all steps be justifiable but omitted to provide the justification for a step in the same way as a professional mathematician might. However, the researcher had no empirical way to separate those more mature cases from other cases in which students really did not think through how to justify the step in question. In general, the transcripts show that the expectation of providing explicit justification for each step became normative in the classroom microculture. Initially, students failed to provide justifications fairly often, but this almost never happened toward the end of the instructional sessions.
CHAPTER 4
DATA ANALYSIS

This section presents an overview of the general flow of instruction including the onset of tasks followed by the task-wise analyses of individual work and group work.

Overview of the Class Dynamic

The teacher-researcher started the first session by asking the students to express their ideas about a point, line, and plane and then coming to a common agreement on the definitions of these concepts. They put forth their ideas, listening closely to what others were saying, expressing their opinions about others’ ideas, and coming to a few definitions that everybody agreed upon. Then the discussion turned to the various possibilities that would arise in representing two lines. The teacher asked the students: “If I give you two lines, what would you do on your paper? How would you represent them?” The students came up with representations of two lines in different forms as shown below:

Delbert

Jeremy

Ricky

Julia

Marcy

Laila

Cathy

Joey

Darren

Tommy

Figure 4.1: Representation of Two Lines
The students started by representing the two lines as intersecting lines, parallel lines, etc., and the discussion progressed to the students’ inferences about intersecting lines. The teacher asked them to explain what each of their representations meant. The students stumbled over more geometrical terminology like angles, point of intersection, intersecting lines, parallel lines, non-parallel lines, and straight angles in this activity. However, the teacher made sure that the students defined these terms and the students came up with comprehensive definitions for these terms as well. This is different from the way mathematicians approach definitions. For a mathematician, the mathematics itself initiates the necessity of formulating the definition depending on when meanings become problematic (Lakatos, 1974). However, at this point, the students would not be able to make their interpretations precise enough to recognize the necessity of defining terms. The teacher’s goal in dealing with the definitions at this point of time was to help the students develop more shared meanings of the definitions and also go to a deeper level of analyzing statements and figures while operating in the mathematical culture that was being established.

Following up on the representations given above, the teacher-researcher directed the students’ attention to the representation of intersecting lines and asked the students to draw inferences about the different angles. The students then made observations and conjectures about the figure. The teacher researcher here was not only concerned with the students’ interactions but was also hoping that the students would notice a specific geometry principle underlying the figure that the vertical angles were congruent. In this activity the teacher tried to make the students make those observational intuitions and arrive at generalizations, which the students did do eventually. The instructor also asked
the students to draw a construction and make conjectures about their construction. The discussion led to the conjecture of the vertical angle theorem. This theorem was considered the first task for analysis.

For a mathematician, the proof of the first pair is completely general and encompasses any pair of vertical angles for any pair of intersecting straight lines. As a result, there is no need to address the other pair in the diagram. However, these students are not at that point of sophistication in their reasoning. Furthermore, considering the fact that there are 11 students working in these sessions, one or two students who are very articulate may start the discussion and prove it while others do not get a chance. To give the others a fair chance to present their own thoughts to prove the theorem, the teacher–researcher employed the students to prove the other angle pair in the hope that if some students did not get a chance to prove the first pair, they would have the chance to prove the other pair.

The instruction in this study is enculturationist, and multiple dispositions are targeted, including providing students with a deep sense of how a proof establishes generality and having students learn to participate in the basic practices of logical argumentation that constitute proving. These practices are not logically opposed to one another. Teachers often come to points where the priorities they hold are in conflict with one another, and they must make hard choices. This is an inherent part of the complexity of teaching. Initially, the teacher prioritized the second goal as she felt that the students first needed to learn to participate in the basic practices of logical argumentation that constitute proving. Later on she changed her practices to emphasize the first goal of attaining generality. The next phase of sessions related to parallel line theorems started
with giving the students three lines and asking them to do a representation with them. The students came up with the following representations:

Carl:

![Carl's representation]

Cathy:

![Cathy's representation]

Delbert:

![Delbert's representation]

Jeremy:

![Jeremy's representation]

Ricky:

![Ricky's representation]

Marcy:

![Marcy's representation]

Julia:

![Julia's representation]

Darren:

![Darren's representation]

Figure 4.2: Representation of Three Lines
The discussion ended on the topic of two lines intersected by a transversal. Here, they were introduced to the new vocabulary word “transversal.” These sessions involved observations about the figures drawn on the board by the teacher. The students observed two parallel lines cut by a transversal and arrived at conjectures. The teacher drew a set of parallel lines with a transversal and a set of skew lines with a transversal, as shown below:

![Figure 4.3: Parallel and Non-parallel Lines Intercepted by a Transversal](image)

The discussion then led to the number of angles formed by a transversal, and the students were again introduced to new vocabulary including “corresponding angle pairs,” “alternate interior angles,” “alternate exterior angles,” and so forth. The students then were asked to observe what would happen if the lines were translated up and down or the transversal was translated across. The students came up with different ideas and the discussion led them to point out that the corresponding angle pairs seemed to be congruent. The discussion then proceeded to conjectures that the alternate interior and alternate exterior angles were congruent.

At this point, the teacher divided the students into two groups of five and seven. There were two sixth-grade sections in the school where the study was conducted, and the sample for the study was a convenient sample from those two sections. When she divided the groups for these parallel line tasks, the teacher allotted all the students from
one section to same group for convenience. The discussions for these tasks were video-recorded separately. The purpose of making the two groups was to increase the participation and talk time of each student because a small group provides more opportunity for individual input. The students worked in two groups on all the theorems related to the parallel lines.

After the conjectures had been set up, the teacher introduced the concepts of postulates and theorems. The need for proving a theorem as opposed to taking a postulate “for granted” was clearly identified at this point. The proofs of the four theorems related to parallel lines followed the conjecturing. Even in these tasks, the teacher–researcher asked the students to prove the other pairs in order to overcome the problems discussed earlier.

A construction activity was introduced at that juncture to break the monotony. The students were asked to construct a parallel line through a given point on one side of a line already given. In the sessions that followed, the groups were combined and the students discussed the properties of triangles and proved the triangle sum theorem and exterior angle theorem.

A brief discussion on congruence and similarity of geometrical figures took place after proving those theorems. The class first discussed the fixedness of one triangle in order to proceed to the congruence of two triangles. The teacher-researcher started by saying, “I have a triangle on my piece of paper. How many pieces of information do you need for me to give you to draw a triangle congruent to mine?” The students then tried to arrive at the conditions that fix the triangle by exploring the possibilities. Individuals identified too few conditions, minimal conditions, and surplus conditions in this process.
The students in the course of this line of thinking came up with the SSS, SAS, and ASA congruence postulates and theorems. The discussion then led to the congruences of right triangles, namely HL, LL, and HA congruences. The students did not actually prove these congruence theorems.

After the discussion about congruence of triangles, the teacher researcher asked the students to do an application of the parallel line postulates and theorems. This problem had the potential to make the students engage in different types of reasoning and argumentation while utilizing the knowledge gained through proving the parallel line postulates and theorems. At this juncture, the teacher made the students work independently on the remaining tasks in order to determine the extent to which students had become autonomous thinkers while also giving them an opportunity to experience the methods of proof employed by others.

Next the discussion moved to circles, and the students recognized the various terms related to circle including center, radius, diameter, chord, and so on. The concept of locus was introduced with specific reference to the circle as a set of loci equidistant from a point. The locus of points equidistant from the endpoints of a line segment forming a perpendicular bisector and the locus of points equidistant from the sides of an angle forming an angular bisector were also discussed. The teacher then asked the students an open-ended question related to the chord of a circle: “What observations can you make when we have a chord in a circle and we draw a perpendicular bisector for it?” The discussion led to the following conjectures:

- The perpendicular bisector of a chord contains the center of the circle.
- A diameter that bisects a chord is perpendicular to the chord.
The teacher then asked them to prove the two theorems, “The diameter that is perpendicular to the chord bisects the chord” and “If two chords are equal in measure, then they are equidistant from the center.

The tasks dealt with during these teaching episodes are presented in Table 4.1

Table 4.1: Tasks Considered for Analysis

<table>
<thead>
<tr>
<th>Task number</th>
<th>Description of the Task</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Proof of the vertical angle theorem.</td>
</tr>
<tr>
<td>2.</td>
<td>Proof involving Alternate Interior Angles (first pair)</td>
</tr>
<tr>
<td>3.</td>
<td>Proof involving Alternate Interior Angles (second pair)</td>
</tr>
<tr>
<td>4.</td>
<td>Proof involving Alternate Exterior Angles (first pair)</td>
</tr>
<tr>
<td>5.</td>
<td>Proof involving Alternate Exterior Angles (second pair)</td>
</tr>
<tr>
<td>6.</td>
<td>Proof involving Consecutive Interior Angles (first pair)</td>
</tr>
<tr>
<td>7.</td>
<td>Proof involving Consecutive Interior Angles (second pair)</td>
</tr>
<tr>
<td>8.</td>
<td>Proof involving Consecutive Exterior Angles (first pair)</td>
</tr>
<tr>
<td>9.</td>
<td>Proof involving Consecutive Exterior Angles (second pair)</td>
</tr>
<tr>
<td>10.</td>
<td>Proof of the triangle sum theorem</td>
</tr>
<tr>
<td>11.</td>
<td>Proof of the exterior angle theorem</td>
</tr>
<tr>
<td>12.</td>
<td>Task on parallel lines</td>
</tr>
<tr>
<td>13.</td>
<td>Theorem–1 on circles</td>
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<td>14.</td>
<td>Theorem–2 on circles</td>
</tr>
</tbody>
</table>

The analyses of the students’ work are presented in two sections. In the first section, the evolution of reasoning capabilities and proof competencies of each individual student is analyzed based on the framework that was developed for the study to address the research question “How do the reasoning processes of individual students evolve in the context of enculturational instruction?” In the second section, the evolution of group competencies as a whole is analyzed task by task to answer the research question “How
do the reasoning processes of the group evolve in the context of enculturational instruction?"

**Analysis of Individual Work**

The purpose of doing the analysis of individual work is to look into the evolution of sophistication in reasoning and proof competencies of each participant during the tasks. The analysis helps us to understand the performance of the individual during the course of time. A detailed analysis on how students’ reason, and the ways in which students present, explain and justify arguments are presented in the individual analysis. Instances of how students navigated different levels of reasoning are also presented. The analysis provides us with an insight into the various facets of individual work.

**Carl’s Work**

**Analysis of Task 1**

In this task, students had to prove the vertical angle theorem. The discussion started with students putting some of their thoughts on paper before they started to discuss. Once they had some ideas to talk about, they started to explain what they had written. Carl also presented his argument:

1.25 Carl: I said it’s congruent because it looks like a reflection and all intersecting lines have at least two pairs. [He was referring to angle pairs]

1.26 Teacher: Oh…that’s a new thing. It’s a reflection.

1.27 Julia: I agree if you have a mirror, we can see that. I agree.

1.28 Teacher: Like…I see something unique about what he said. Carl can you come here and just show what you were meaning…on the board. These were your angles, right? You said something unique. We want to see what you said. Okay…Just say it like…show us what you were trying to mean.

1.29 Carl: I meant this, if you took a mirror or if you just see like how far these are…and draw a circle like Darren said. That’s what I did. I took my marker and lined it like that and then I noticed that they are same. That’s when I thought of mirror and looked
at it, then when I noticed that you make an ‘X’ and I kept
drawing an ‘X’ and it always… to see that it looks the same
thing.

There were two things that Carl talked about in his explanation. First, he referred
to a reflection in proving that the vertical angles are congruent (line 1.25). The other one
he said was that if he kept on going drawing intersecting lines which he referred to as
‘X’, he would see that the vertical angle pairs are congruent in all cases (line 1.29). The
analysis can be done based on these two different approaches to the proof. One is related
to the ‘reflection’ which he referred to in his explanation. Visualization is the ability to
look at a figure or an image in a different way other than in its existing form which also
includes the thought of what might happen if it is moved, turned or flipped, geometrically
are termed as the translation, rotation and reflection. Research shows that students tend to
present arguments in terms of these transformations when they have to provide an
explanation to why the geometrical property works. As Tall (2009) points out, students
“Give meaning to the definition by using images or diagrams or dynamic change,
building on met– befores to construct a natural route to formal proof (p.224).Tall (2004)
defines a met– befor to be a current conceptual structure in the mind that is linked to a
previous experience. But in the case of proof production, this is considered to be the very
basic level as it relates to just interpreting a proposition. So in this case, Carl is
considered to be operating on Level 1.

Now considering the other approach that he took in continuation of earlier one, he
referred to different cases of intersecting lines having the same vertical angle measures.
This points to the fact that he went for a proof based on multiple representations which
relates to the inductive approach to a proof. He made a generalization of the vertical
angle pair congruence and provided that as the proof itself. Even in this case, he operated on Level 1 as he provided only different examples to prove his case.

**Analysis of Task 5**

In Task 5, the students were required to prove that a pair of alternate exterior angle pairs was congruent. They had the figure below:

![Diagram of alternate exterior angles](image)

Figure 4.4: Alternate Exterior Angles (Second Pair)

The task was to prove that \( \angle 3 \cong \angle 8 \).

Carl worked with the group while Tommy represented the group in writing the proceedings of the proof. The following discussion shows how they approached the theorem:

5.36 Tommy: Okay. What did you say?
5.37 All: \( \angle 3 \) equals \( \angle 5 \) and \( \angle 5 \) equals \( \angle 8 \).
5.38 Teacher: \( \angle 3 \) is
5.39 Tommy: Equal to
5.40 Teacher: Congruent to
5.41 Carl: \( \angle 5 \).
5.42 Teacher: Guys, you should say it a bit more louder, he has to write it, right?
5.43 Carl: And \( \angle 5 \).

The whole group stated off with; corresponding angles \( \angle 3, \angle 5 \) (line 5.37) and vertical \( \angle 5, \angle 8 \) (line 5.37). However, Tommy considered vertical angles \( \angle 3 \) and \( \angle 1 \) (line
5.58) and was looking for the corresponding angle to associate with $\angle 1$. At that time, Carl helped him by pointing to $\angle 8$, (lines 5.62, 5.65) and this is what they conclude:

5.58 Tommy: $\angle 3$ is vertical… $\angle 3$ is congruent to $\angle 1$.
5.59 Teacher: Aha. Why?
5.60 Tommy: Because they are vertical.
5.61 Teacher: Okay. They are vertical.
5.62 Tommy: $\angle 1$ mean no… yeah $\angle 1$ is…
5.63 Teacher: Aha.
5.64 Tommy: is yeah.
5.65 Carl: To $\angle 8$.
5.66 Tommy: Yeah… it’s corresponding.
5.67 Teacher: Carl, can you say it louder? $\angle 1$ is congruent to what?
5.68 Carl: $\angle 8$, $\angle 8$.
5.69 Teacher: $\angle 8$. Why are they congruent?
5.70 Tommy: Because they are corresponding angles.
5.71 Teacher: Aha. They are corresponding?
[Tommy claps]
5.72 Teacher: So what can you conclude from that?
5.73 Tommy: That $\angle 3$.
5.74 Teacher: Aha.
5.75 Tommy: $\angle 3$ is congruent to…
5.76 Teacher: To what angle?
5.77 Tommy: $\angle 8$.
5.78 Teacher: Perfect. You… you proved this right?

Even though Tommy was writing, the contribution from Carl had its own significance. He was able to recognize the relevant data along with others. He is considered to be a part of the group operating on Level 2. Here one can see the effect of students’ efforts to adopt the normative practices of the classroom which in turn influenced their rate of learning.

**Analysis of Task 6**

In Task 6, the students were asked to prove that a pair of consecutive interior angles are supplementary.
In this task the students were required to prove that $\angle 5 + \angle 3 = 180^\circ$. Carl volunteered to prove the theorem. The discussion ensured in the following manner:

6.88 Carl: Can I solve this problem.
6.89 Teacher: You want to solve that?
6.90 Carl: Yeah... like the $\angle 1$ plus $\angle 2$ equals $180^\circ$. [He was referring to the pair that was proved earlier.]
6.91 Teacher: Okay. Can you like... can you show us.
6.92 Carl: So... you have to know like... linear pairs and linear pairs is like... something like this.
6.93 Teacher: Aha.
6.94 Carl: You see... this equals to $90^\circ$ and this $90^\circ$. $90^\circ$ plus $90^\circ$ equals $180^\circ$. So you find something linear.
6.95 Teacher: Aha.
6.96 Carl: And so... you see the $\angle 3$ and $\angle 4$, add, equals to $180^\circ$.
6.97 Teacher: Aha.
6.98 Tommy: They are linear pairs.
6.99 Teacher: They are linear pairs too? Tommy?
6.100 Tommy: Aha.
6.101 Carl: $\angle 5$ and $\angle 6$ equals to $180^\circ$.
6.102 Teacher: Aha.
6.103 Carl: So, all you have to do is...
6.104 Teacher: So... this time you are approaching it as like linear pairs?
6.105 Tommy: Aha...
6.106 Teacher: All of you are on the same page? All of you think that.
6.107 Carl: So, this is pretty easy, so all you have to do is, put the 5 where the 4 is.
6.108 Teacher: Aha. [Carl replaces the $\angle 4$ with the $\angle 5$.]
6.109 Carl: And that’s how you get $\angle 3$ equals $\angle 3$ plus $\angle 5$ equals $180^\circ$. 

Figure 4.5: Consecutive Interior Angle (First Pair)
Carl started with the linear pairs $\angle 3 + \angle 4 = 180^\circ$ (line 6.96) and $\angle 5 + \angle 6 = 180^\circ$ (line 6.101) and then replaced the $\angle 4$ with $\angle 5$ (line 6.107) in the first one to prove the theorem.

Carl began to understand the shape of a proof and presented it with reference to relevant data, linked to the claim in a logical way. However he did not provide justification for the replacement of $\angle 4$ with $\angle 5$ which were a corresponding angle pair. He also considered additional data which was not required. He was operating on Level 2 as he did not present the whole of the argumentation required to make the proof complete.

**Analysis of Task 8**

In Task 8, the students were required to prove that the consecutive exterior angles $\angle 7$ and $\angle 8$ are supplementary. Each student worked independently and presented their proof on the board. Carl presented the proof below with the note that shows the justification for his steps:

Note!!!

Corresponding angles and they are allowed
to be replaced as long as they are congruent.

Corresponding angles always supposed to be congruent.

Figure 4.6: Consecutive Exterior Angles (First Pair)
Carl explained the proof in the figure above. He considered the linear pair \( \angle 1 + \angle 8 = 180^\circ \) and replaced \( \angle 1 \) with \( \angle 7 \) as they were corresponding angles to arrive at the conclusion that \( \angle 7 + \angle 8 = 180^\circ \).

Carl in this task presented a complete proof. He presented the data and the warrant that supports the logical link between it and the claim. He was operating on Level 3.

**Analysis of Task 11**

In Task 11, it was required to present a proof to the exterior angle theorem. The students’ first worked independently and presented their proofs on the board. Carl presented the proof below:

![Figure 4.7: Exterior Angle](image)

Carl presented the following proof on the board:

\[
\angle 1 + \angle 2 + \angle 3 = 180^\circ \\
\angle 3 + \angle 4 = 180^\circ \\
180^\circ - \angle 3 = \angle 4 \\
\angle 4 = 180^\circ - \angle 3 \\
\angle 1 + \angle 2 = 180^\circ - \angle 3 \\
\angle 4 = \angle 1 + \angle 2
\]
The proof Carl presented was very sophisticated in that it was algebraic in form. But, as can be seen below, these algebraic steps were based on Carl’s conceptualizing of the quantitative relations depicted in the problem, not on formal algebraic skills.

When Carl came to the board to explain, he found it difficult to explain his reasoning. So the teacher helped him in the following way:

11.105 Carl: It’s all in the inside. Since ∠1 and ∠3 are in the interior and ∠1 and ∠2 are interior angles. So that’s how I wrote this… [Referring to ∠1 + ∠2 + ∠3 = 180°.]

11.106 Teacher: Aha… Aha.

11.107 Carl: Then ∠1 + ∠2 = 180° – ∠3 because they are the same…

11.108 Teacher: Okay. Let me help you. I think he wrote something nice here. Look, he wrote, ∠1 + ∠2 = 180° – ∠3. Look at this. Guys… guys look. He has a good explanation here. First he says ∠1 + ∠2 + ∠3 is 180°, right. So ∠1 + ∠2 will be 180° – ∠3.

11.109 Student: ∠3… ∠3… Yeah.

11.110 Teacher: ∠3 + ∠4. Where? Did you write that? Yeah. ∠3 + ∠4 is also 180°. Why?

11.111 Student: Because they are on the same line. Linear pair…

11.112 Teacher: Linear pair. Very good. So ∠3 + ∠4 = 180°. So ∠4 is also 180° – ∠3.

11.113 Tommy: Yeah.

11.114 Teacher: Got it?

11.115 Student: Yeah.

11.116 Teacher: So both are 180° – ∠3, so ∠4 = ∠1 + ∠2. The thing is, he wrote it very clearly, the only thing is, and he couldn’t explain it properly.

Carl was able to recognize the application of the triangle sum theorem and linear pair postulates to deduce the exterior angle theorem with the help of algebraic transformations. He was operating on Level 3.

Analysis of Task 12

In Task 12 on parallel lines, the teacher presented the following problem.
Figure 4.8: Parallel Line Task

Prove that: \( \angle 3 = \angle 1 + \angle 2 \) when the two horizontal lines are parallel to each other.

The students worked individually and presented their proofs on the board. Carl presented the following proof:

Since \( \angle 1 \) and \( \angle 3 \) are alternate exterior angles and \( \angle 2 \) and \( \angle 3 \) are alternate interior angles, \( \angle 3 \) is replaced.

So, \( \angle 3 = \angle 1 + \angle 2 \)

In this task, Carl was not even able to present data relevant to the task at hand. He had target in mind but did not reach it through proper mathematical reasoning. The difficulty in this case seemed to stem out from his inability to recognize the angles. He made a fallacy in associating the given angles to the ones formed when two parallel lines intersected by a transversal. Carl was operating on Level 2. As observed in the case of other students as well, there is a possibility that the diagram itself presented a complexity which was beyond the comprehension to most of the students.

**Analysis of Task 13**

In Task 13, the students all presented their proofs to show that “The diameter that is perpendicular to the chord bisects the chord.” They had to prove that \( AC = BC \) from the figure presented below:
The students worked individually and presented their proofs on the board. Carl
drew the figure below, on the board and wrote his proof in the following manner:

Carl’s proof:

\[ AC = BC \text{ because they are [the triangles] congruent [and] also since all angles are} \]
\[ 90^\circ \text{ [Referring to the angles at the intersection of the two perpendicular lines],} \]
\[ \text{they are the same.} \]

When his turn came to explain the proof, he explained it in the following manner:

13.77 Teacher: Okay. Chris, explain yours baby.
13.78 Carl: So on mine, I said AC and BC are congruent because, they are
congruent also, no they are congruent because like they both…
all that the angles are 90° and I showed you that.[He points to
the right angles formed at the intersection of the two line
segments]
13.79 Teacher: Aha.
13.80 Carl: So they are the same [Points to the two pairs of right triangles,
formed on either side of the vertical line] and I showed that
each side of it... [Points to the sides of those two triangles] they are all congruent. So if you put them right next to each other, then they are the same.

Earlier, the students discussed about the right triangle congruences and Carl used that knowledge to present his proof. He explained by his actions that the two triangles had $90^\circ$ at the common vertex (point of intersection) and the hypotenuses of those two triangles (radii of circle) were same. He referred to the fact that the two triangle pairs which were formed as result of the two perpendicular lines, were congruent right triangles with same side measures and hence $AC = BC$. He was operating on Level 3 as he organized the ideas of proof in a deductive reasoning.

**Summary of Carl’s Work**

Carl initially started with presenting his proofs referring to visual transformations like reflection and producing multiple examples to make generalizations. He was operating on Level 1 of reasoning. This can be seen in the Task 1, related to vertical angles. In the subsequent tasks, he observed other students presenting deductive logical arguments and presented his own arguments in Task 5 related to exterior angles and in Task 7, related to consecutive interior angles.

In the exterior angle theorem he presented a proof that is worth mentioning in terms of the structure and logic. He provided the proof clearly showing all the steps leading to a valid conclusion while dealing with the various transformations as a mature prover would approach. His approach showed a level of sophistication and understanding about the nature of proof.

He then produced complete proofs in Tasks 8, 11 and 13, related to consecutive exterior angles theorem, triangle exterior angle sum theorem and theorem on circles.
respectively where he operated on Level 3 except in Task 12 on parallel lines where he might have been deceived by the complexity of figure.

**Dalton’s Work**

**Analysis of Task 1**

The task was related to the proof of the vertical angle theorem and the students had to prove that \( \angle PRQ = \angle SRT \) in reference to the figure presented on the board. The students discussed as a group and added their input. Dalton used an idea presented by Darren during the task. Earlier, Darren referred to using an angle measure of 30° for one of the angles, calculated the adjacent angle to be 150°, and repeated the calculation to prove that the other vertical angle would also be 30°.

![Figure 4.11: Vertical angles](image)

Dalton used the same concept and used the angles given in the figure. The proof unfolded in the following way:

1. Teacher: \( \angle PRQ = 180° \) minus… Wow, looks good. So he is going for the angles instead of the measures. So \( \angle PRQ \) equals…If somebody wants to help him you can go.
2. Darren walks on to the board and writes \( \angle SRT \)
3. Darren: \( \angle SRT \)… \( \angle SRT \)
4. Teacher: Okay…Okay. Somebody should have a question here. He says… \( \angle PRQ \) is 180° minus this one [The teacher points to \( \angle SRT \) in the figure.]
1.114 Ricky: No
1.115 Teacher: Ricky… Ricky…Ricky wants to add something. [Ricky comes to the board]
1.116 Teacher: PRQ is
1.117 Ricky: This is ∠SRQ
1.118 Teacher: Good. ∠SRQ. You are going…getting at something ∠SRQ, okay…Darren sit down. Ricky… I think Dalton can do the remaining because we want to know what you wanted to do. So ∠PRQ is 180° − ∠SRQ
1.119 Dalton: ∠SRT equals to 180° minus ∠SRQ.
1.120 Teacher: Okay. You tell your friends because I didn’t understand what you wrote.
1.121 Dalton: These are congruent, (referring to 180° − ∠SRQ) so take them out. We are left with ∠PRQ and ∠SRT. [Dalton writes]

Dalton applied the same rule to the angles presented in the figure and made a general argument. He started with ∠PRQ = 180° − ∠SRQ with some help from Ricky, proceeded to take ∠SRT = 180° − ∠SRQ (line 1.119), and concluded as follows:

1.123 Dalton: Because these are here there [circling ∠SRQ] congruent. So it’s just these two...
1.124 Teacher: Okay.
1.125 Dalton: And these are congruent [pointing to ∠SRQ] so this one [pointing to ∠PRQ] is congruent to [pointing to ∠SRT] without using the measurement.

He referred to 180° − ∠SRQ being the same, as the ∠SRQ is common in both expressions, and hence ∠PRQ = ∠SRT. In this task, Dalton was able to distinguish general arguments from discussions of a specific case.

When a proposition seemed pertinent, and a calculation promising, Dalton searched for a convincing coherent link from the local one to a global one. Dalton in this task was operating on Level 2. The links he used were theoretical reasons of validity.

In the same task, Laila suggested that she would prove the theorem in a different way. Laila approached the proof with linear pairs, ∠PRQ + ∠QRS = 180° and ∠SRT + ∠TRP = 180°. Once she presented this, Dalton had a suggestion to offer:
Dalton: It could be ∠TRP and… yah… it could be ∠TRP, but this is what I say. Since you have ∠TRP here and ∠SRT here [Referring to the second equation] ∠QRS here [Referring to the first equation] … you should have ∠SRQ in… [Referring to the second equation]

The reason he suggested to have ∠SRQ in both equations was to ensure that both of them had a common angle. Laila immediately understood what he was referring to, and changed the second angle pair to ∠SRT + ∠SRQ = 180°.

Later in the same task, when Laila and Delbert presented the proof to the other pair, he made a similar contribution. Both the students discussed and wrote ∠QRS + ∠SRT = 180° and ∠PRT + ∠PRQ = 180°. Dalton again intervened:

Dalton: He made the same mistake. We are trying to prove that they are congruent. So it would be better if ∠SRT is there on the top and bottom.

These suggestions played a crucial role in the process of the production of this proof. He was considered to be operating on Level 2 along with Laila and Delbert. Dalton’s interventions to their arguments suggest that he was able to identify the conditions that actually produce the result in a theorem. In this case, he was able to point out that for arriving at ∠QRS equals ∠PRT, both the linear pairs that Laila and Delbert took should have ∠SRT in common. So he himself would have known how the proof would work out eventually.

**Analysis of Task 2**

In this task the students were required to prove that the alternate interior angles are congruent. The figure was generated to help them in the process of proving, and the task was to prove that ∠6 ≅ ∠4.
Figure 4.12: Alternate Interior Angles (First Pair)

Laila, Ricky, and Marcy started discussing together. Dalton, Jeremy, and Ryan also discussed together at the same time. In the groups Laila and Dalton started to refer to corresponding angles.

The teacher asked the groups to join together so that she could hear why they were both referring to the corresponding angles. Laila responded:

2.54 Laila: Look at the information in the corresponding angles.

Automatically, Laila and Dalton started to think of what information was already at hand that might be relevant to their goal. The students considered $\angle 4 \cong \angle 8$ as the first data set.

2.56 Students: $\angle 4 \cong \angle 8$
2.57 Teacher: $\angle 4 \cong \angle 8$, so you want to look at this one? $\angle 4 \cong \angle 8$. You wanted to take that one? Okay. These are congruent? Why? They are corresponding angles, right?
2.58 Students: Yes.
2.59 Teacher: Okay, What else do you want me to write? [Darren says something and the teacher responds.]

The discussion resumed:

2.64 Students: $\angle 1$ and $\angle 6$.
2.65 Teacher: $\angle 1$ and $\angle 6$, Okay, those are also, what? What kind of angles?
2.66 Students: Corresponding.
The discussion continued after a brief interruption by Darren. Dalton said that he noticed something:

2.83  Dalton:  Ms. Indira, I just noticed this…
2.84  Teacher:  Come show me what you noticed. Ssh… give them a… give them time to think. Aha.
2.85  Dalton:  \( \angle 6 \) and \( \angle 8 \).
2.86  Teacher:  Oh, you wanted \( \angle 6 \) and \( \angle 8 \). Guys, he says \( \angle 6 \) and \( \angle 8 \). What are those?
2.87  Dalton:  They are… I think they are…
2.88  Laila:  Interior…
2.89  Dalton:  I really don’t know…
2.90  Teacher:  \( \angle 6 \) and \( \angle 8 \)
2.91  Laila:  Interior? Vertical…

Both Dalton and Laila came up to that point and started pondering their next step.

It was at that juncture that Ryan provided them with the logical link that connected the data \( \angle 4 \cong \angle 8 \) and \( \angle 6 \cong \angle 8 \) to the claim \( \angle 6 \cong \angle 4 \) (lines 2.95, 2.96) using the transitive property of relations. He concluded as follows:

2.95  Ryan:  \( \angle 4 \) and \( \angle 8 \) is congruent, and \( \angle 6 \) and \( \angle 8 \) is congruent.
2.96  Ryan:  So, they both equal, so \( \angle 6 \cong \angle 4 \).

So Dalton, in this task, was considered to be operating on the Level 2, as the appropriate structures for coordinating those elements into a logical argument are not evidenced.

**Analysis of Task 5**

The teacher asked Jeremy to prove the theorem as a way of encouraging all of the students. He used the figure given on the board to prove that the pair of alternate exterior angles \( \angle 1 \) and \( \angle 8 \) was congruent.
Jeremy started with $\angle 6 \cong \angle 8$ and $\angle 4 \cong \angle 8$. Dalton added that Jeremy needed one more pair of angles to prove the theorem. The teacher directed Dalton to help him:

5.11 Teacher: He needs one more? Dalton help him what he needs.
5.12 Dalton: $\angle 1$ and $\angle 4$
5.13 Teacher: He needs $\angle 1$ and $\angle 4$? He says you need $\angle 1$ and $\angle 4$.
5.14 Students: Ooh…
5.15 Teacher: Now, can you, can you summarize that for us? To prove $\angle 1 = \angle 8$?
5.16 Jeremy: $\angle 1 = \angle 4$; $\angle 4 = \angle 8$; so…
5.17 Teacher: Therefore…
5.18 Jeremy: Therefore $\angle 1 \cong \angle 8$.

The extra pair; $\angle 1$ and $\angle 4$ (line 5.12) helped Jeremy to complete the proof.

Dalton’s contribution in this way, as also seen in Task 1, proved to be an invaluable tool in helping his peers to arrive at complete proofs. His intervention at the right moment was crucial in the process of proof production. He was operating with others on Level 2. Though, Dalton did not get an opportunity to prove this theorem, it is obvious that he understood how the conditions $\angle 4 \cong \angle 8$ and $\angle 1 \cong \angle 4$ actually produce $\angle 8 \cong \angle 1$, the result in a theorem.
Analysis of Task 6

In Task 6, the students were required to prove that the consecutive angles on the same side of the transversal are supplementary. They were given the figure below for reference:

![Figure 4.14: Consecutive Interior Angles (First Pair)](image)

The students were required to prove that $\angle 3 + \angle 5 = 180^\circ$. Dalton came forward and presented what he had on his paper. He set out with $\angle 4$ and $\angle 3$ as the linear pair and $\angle 4$ and $\angle 5$ as the congruent corresponding angle pair. He explained it in this way:

6.20 Dalton: This is what I was thinking. If $\angle 3 + \angle 5 = 180^\circ$ then $\angle 4$, $\angle 3$ is a linear pair and $\angle 4$ is corresponding angle to $\angle 5$. They are congruent and that’s all I got.
6.21 Teacher: Baby can you like, show me on your piece of paper. Move near Laila, listen to what he said.
6.22 Dalton: $\angle 3 + \angle 5 = 180^\circ$ and…
6.23 Teacher: You have to show me that, right, $\angle 3 + \angle 5 = 180^\circ$, you have to show me that. Show me, okay?
6.24 Dalton: And $\angle 4$, $\angle 3$. They are $180^\circ$ so they are all linear, linear pairs.
6.25 Teacher: Oh… They are a linear pair? Okay.
6.26 Dalton: So the $\angle 4$ and $\angle 4$ and $\angle 5$. They are Corresponding angles.
6.27 Teacher: Aha.
6.28 Dalton: And that which makes $\angle 4$ and $\angle 5$ congruent. So they are equal…
The teacher then asked him to present it on the board so that everyone could see.

He said:

6.30  Dalton:  $\angle 3 + \angle 5$. Okay, we are trying to show that $\angle 3 + \angle 5 = 180^\circ$, which already $\angle 4$, $\angle 4$ and $\angle 3$. These two are a linear pair [Referring to $\angle 4$ and $\angle 3$] which makes $180^\circ$, and $\angle 4$ and $\angle 5$. $\angle 4$ and $\angle 5$ are corresponding angles, which makes them congruent and so we have that part…

The teacher directed him to write everything on the board. At that point, Laila wanted to present her solution and the teacher allowed her. Laila could not arrive at the proof, with the data; $\angle 4 + \angle 3 = 180^\circ$, $\angle 5 + \angle 6 = 180^\circ$, $\angle 4 \cong \angle 5$ and $\angle 5 \cong \angle 6$. Dalton came and tried the proof with the same data and presented the argument below:

6.39  Dalton:  Just like 4 and 5 are corresponding angles, 3 and 6 are also corresponding angles, and… and what they have both… have in common is that both $\angle 3$ and $\angle 5$, they share something with either $\angle 4$ or $\angle 6$. They both share. Since they both share that and simple… that little information… it can help prove that $\angle 3 + \angle 5 = 180^\circ$.

The teacher asked him to put it in writing and the whole discussion started again. Jeremy came to the board and completed what Dalton was trying to say. Jeremy concluded:

6.67  Jeremy:  Place the $\angle 4$ with $\angle 5$ and $\angle 3 + \angle 5 = 180^\circ$.

Dalton was very close to arriving at the final conclusion but somewhere in the interruptions he lost track of what he wanted to say. He was operating on Level 2 as he was able to bring to light the ideas of proof but was not able to organize his reasoning in achieving the target.
Analysis of Task 7

In a similar task to prove that the other pair of consecutive interior angles on the same side of the transversal, Ricky started with $\angle 1 + \angle 8 = 180^\circ$ and $\angle 2 = \angle 8$, but he wanted some time to think about it. Dalton wanted to give it a try and proceeded in the following manner:

7.33  Dalton: And $\angle 2$ and $\angle 8$ are congruent, corresponding.
7.34  Teacher: Corresponding right? Let me put the words there. These are corresponding. This is a linear pair right? Okay where are you going from these?
7.35  Dalton: $\angle 7$ and…
7.36  Students: $\angle 7$?
7.37  Dalton: $\angle 7$ and $\angle 2$ are linear pairs.
7.38  Teacher: Okay that’s also a linear pair. Okay…
7.39  Dalton: And $\angle 7$ and $\angle 1$ form a linear pair… no congruent.
7.40  Teacher: $\angle 7$ and $\angle 1$, okay $\angle 7$ and $\angle 1$. Let him try. He already did the last one right, $\angle 7$ and what?
7.41  Dalton: $\angle 1$
7.42  Teacher: $\angle 7$ and $\angle 1$. Okay.
7.43  Dalton: And so…
7.44  Teacher: You have to show $\angle 2 + \angle 1 = 180^\circ$ so show us from that.
7.45  Dalton: $\angle 7 + \angle 2 = 180^\circ$ and replace $\angle 7$ with … $\angle 1$ …

Dalton first started with Ricky’s corresponding angle pair, $\angle 2$ and $\angle 8$ (line 7.33) but presented a new linear pair, $\angle 7 + \angle 2 = 180^\circ$ (line 7.37). He thought for a few seconds and listed another set of corresponding angles, $\angle 7$ and $\angle 1$ (line 7.42). He replaced the $\angle 7$ with $\angle 1$ (line 7.45) to complete the proof. Dalton understood that there could be multiple ways of arriving at a proof. Even though he started with the data that Ricky presented, he chose to find a coherent link with data of his own to arrive at the conclusion.

Dalton operated on Level 3, as he organized his reasoning into a cogent argumentation. Once the ideas of proof were brought to light, he organized them in a deductive form.
Analysis of Task 8

The task was to prove that $\angle 7 + \angle 5 = 180^\circ$, using the given figure as reference.

The students worked together again in proving the theorem.

![Figure 4.15: Consecutive Exterior Angles (First Pair)](image)

Ryan came to the board and started the proof with the help of Laila and Dalton.

The conversation continued in the following way:

8.13 Ryan: $\angle 7$ and $\angle 6$ is corresponding.
8.14 Laila: $\angle 7$ and $\angle 6$.
8.15 Student: Exterior.
8.16 Laila: I think you should put $\angle 7$ and $\angle 9$.
8.17 Teacher: She said you should put $\angle 7$ and $\angle 9$.
8.18 Laila: I also see $\angle 4$ and $\angle 5$.
8.19 Teacher: So… what are those?
8.20 Laila: Those are exterior.
8.21 Teacher: Exterior angles? They are congruent? Okay. So where are you going from there? Guys you should be helping her helping him. Sorry.
8.22 Dalton: I think $\angle 7$ and $\angle 9$, $\angle 7$ and $\angle 2$ equal to $180^\circ$, because it’s a linear pair.
8.23 Teacher: So… Dalton, can you repeat. What you said again baby.
8.24 Dalton: I told Ryan to use $\angle 7$ plus $\angle 2$ because they were linear pairs and they equal $180^\circ$.

After a few minutes, Laila and Dalton both added:
8.30 Laila & Dalton: So you should… replace the $\angle 2$ with $\angle 5$ and… $\angle 7$ plus $\angle 5$ equals 180°.

The teacher then asked Laila to write what they inferred. As she wrote, the teacher questioned why she replaced the $\angle 2$ with $\angle 5$. Laila failed to explain to which Dalton answered:

8.56 Dalton: I say… replace the $\angle 2$ with the $\angle 5$ because they are… corresponding angles.

One can see that Dalton clearly understood how to present relevant data, like the linear pair; $\angle 7, \angle 2$ (line 8.22) and corresponding angle pair; $\angle 2, \angle 5$ (line 8.56). He was also able to link the data to the claim $\angle 7 + \angle 5 = 180°$ using deductive reasoning.

Dalton was operating on Level 3, as he produced a valid deductive proof. He also provided data with reasons of support.

**Analysis of Task 9**

The students had a similar task to show that the exterior angles on the same side of the transversal are supplementary. They had to prove that $\angle 4 + \angle 9 = 180°$ from the same figure given in the previous task. The teacher directed the students to work individually and present their proofs on the board. Dalton presented the following proof on the board. He wrote:

$$\angle 1 + \angle 9 = 180°$$
$$\angle 4 \text{ and } \angle 1 \text{ are corresponding angles.}$$
$$\angle 1 + \angle 9 = 180°$$
$$\angle 4 + \angle 3 = 180°$$
So you replace the $\angle 1$ with $\angle 4$ because they are corresponding angles.

$$\angle 4 + \angle 9 = 180°$$

He continued to operate on Level 3 as he presented data, and arrived at a proof backing each step with justifications related to linear pair properties and corresponding
angle postulate. He coordinated elements of the arguments that is consistent with logically sound deductive reasoning.

**Analysis of Task 11**

In Task 11, the students were required to present a proof to the exterior angle theorem. The students first worked independently, then presented their proofs on the board. Dalton modified the figure presented initially to prove that \( \angle 4 = \angle 1 + \angle 2 \).

![Figure 4.16: Exterior Angle](image1)

![Figure 4.17: Dalton’s Exterior Angle](image2)

He presented the following proof on the board:

\[
\begin{align*}
\angle 3 + \angle 4 &= 180^\circ \\
\angle 1 + \angle 2 &= 180^\circ \\
\angle 1 + \angle 2 + \angle 3 &= 180^\circ \\
\text{So } \angle 4 &= \angle 1 + \angle 2
\end{align*}
\]

When his chance came to explain his proof, he did it in this way:

11.52 Dalton: I said, that \( \angle 3 + \angle 4 = 180^\circ \).
11.53 Teacher: Aha
11.54 Dalton: \( \angle 5 + \angle 2 = 180^\circ \).
11.55 Teacher: Aha.
11.56 Dalton: And \( \angle 1 + \angle 2 + \angle 3 = 180^\circ \).
11.57 Teacher: And where did you get the \( \angle 1 + \angle 2 + \angle 3 \)? From your sum of the angles, you said that, right? Okay.
11.58 Dalton: And then I classified that \( \angle 4 = \angle 1 + \angle 2 \)
11.59 Teacher: \( \angle 1 + \angle 2 \). So, he… he was similar to what… uh… Joey was saying.
Dalton presented a valid proof when he claimed that $\angle 4 = \angle 1 + \angle 2$ from $\angle 3 + \angle 4 = 180^\circ$ and $\angle 1 + \angle 2 + \angle 3 = 180^\circ$. The students first tended to present the data that they saw on the figure based on the geometrical relationships that they perceive and those they think that would be helpful to solve the problem. In this case Dalton presented $\angle 5 + \angle 2 = 180^\circ$ along with $\angle 3 + \angle 4 = 180^\circ$ in a similar way. However his proof shows that he used only $\angle 3 + \angle 4 = 180^\circ$. Dalton was operating on Level 3.

**Analysis of Task 13**

In Task 13, all the students presented their proofs to show that “The diameter that is perpendicular to the chord bisects the chord.” They had to prove that $AC = BC$ from the figure presented below:

![Figure 4.18: Circles Theorem-1](image1)

![Figure 4.19: Dalton’s Circles Theorem-1](image2)

Dalton’s proof as presented on the board:

I drew two radiuses [ii] that form, two right triangles which are congruent.

He explained it in the following way:

13.68  Dalton: I said that there are two radiuses that which form two right triangles which are congruent. [Pointing to the radii and triangles formed]

13.69  Dalton: The reason why I said that they were congruent also, because when the bisector, intersects the… [He points to the diameter and says it is a bisector]

13.70  Teacher: The chord.

13.71  Dalton: The chord.
Teacher: Aha.

Dalton: And the bisector that makes 90° angle which is 90° on each side, which proves AC = BC. [Here he specifically mentions that the bisector is a perpendicular bisector]

Dalton joined the radii OA and OB to form right triangles OCA and OCB (line 13.68 and figure). He also pointed out that, because of this construction; the two right triangles formed were congruent (line 13.68). Dalton was operating on Level 3 as he produced a deductive argument according to specific cultural constraints concerning the nature of propositions and their enchaining.

Summary of Dalton’s Work:

The uniqueness of Dalton’s argumentation is reflected in his critique and interventions to the arguments that other students presented during the episodes. From the discussions in most of the earlier Tasks 1 through 8, it can be seen that he was attentive to the ideas presented by other students, and provided them with timely suggestions in taking relevant data required to solve the problem at hand. He also took the initiative to question and critique the arguments presented by his peers.

Regarding Dalton’s own proof construction, Dalton started presenting his argument in Task 1 with reference to Darren’s use of a particular angle measure to prove that the vertical angles were congruent. Dalton used the same logic but proved it for a general case of vertical angles. He started to operate on Level 2 as he tried to search for a coherent link between a local goal and a general one. In the same task when Laila wanted to present the proof for the vertical angles theorem through a different approach, he helped Laila in finishing it by making her see the importance of taking the data that was relevant to the task at hand. He was able to understand the need to present relevant data to prove the theorem at that point. He continued to apply his understanding of the process of
producing a proof consisting of data, linked to the claim through logical means. He
continued to operate on Level 2 in Tasks, 2, 5 and 6 related to the vertical angles and
some parallel line properties. He understood the form of a proof, presented relevant data
and linked them to the claim at hand through appropriate rules of inference. However by
that juncture, he did not understand how to present a complete proof in terms of
providing justifications for each step of the proof. Dalton came to understand the
complete shape of a proof through observations of arguments provided with justifications
by others. From Task 7 onwards, he started operating on Level 3. In Tasks 7, 8, 9, 11, and
13 on theorems related to consecutive interior angles, consecutive exterior angles, the
exterior angle theorem and the theorem on chords of circles, he started to justify each of
his steps to produce complete proofs.

Darren’s Work

Analysis of Task 1

The students previously conjectured that the vertical angles were congruent and
the task at hand was to prove that theorem. The episode began with various students
expressing their approaches to the task. Darren also presented his argument:

1.14 Darren: I said you could just use a pencil or something and make a
circle around the lines and you should like measure it in your
head whether it is a right angle or obtuse, how you can get that
this is a right angle or obtuse.

Darren came to the board to explain his work in more detail. Darren started
engaging visually with the statement by generating an image and then inspecting the
image. He drew the following figures to explain his reasoning.

He first drew two intersecting lines and then drew an arc as shown below:
Darren: It’s in the mind . . . I would see how far they are apart and I would draw a circle like this.

When the teacher questioned him about that, he completed the circle as shown below:

![Figure 4.21: Darren’s Vertical Angles-2](image)

Then this exchange followed:

1.34 Darren: No, I will go like that and see how far they are apart and they would be the same.
1.35 Teacher: How do you? We are not still able to visualize. I understand your circle part. You drew a circle, that’s right. But how do you?
1.36 Darren: You see how these two are like that. I would see how far they are apart on each side and like that.

It was not clear from Darren’s words if he was referring to the fact that congruent sectors on a circle subtend equal central angles or that equal arc lengths subtend equal central angles. On close observation of the video, the second inference seems to be more compatible with his action. As he pointed to the intersection of the circle and the two rays on both sides with two fingers, it seemed that he was referring to the arc lengths. But in
either case, these were merely perceptual elements Darren was linking together. He had no argument to offer either to establish that the arcs were of equal length or to justify his claim that therefore the angles must be equal. He was operating on Level 1. He not only did a heuristic exploration of the figure at hand but also presented an informal analysis of the attributes of the components of the diagram.

In a later part of the discussion related to the same task, Laila initiated an argument using her knowledge of the symbol for a right angle to form a square, as shown below:

![Figure 4.22: Darren’s Vertical Angles-3](image)

Darren picked up on this idea and explained the congruence of vertical angles using right angles. He presented the following argument:

1.56 Darren: Because . . . can you see this and this [referring to a pair of vertical angles], the same amount? This is a right angle also when we look at that side that has the same thing then you get the idea that they are both the same.

1.57 Teacher: Okay. One more time repeat the one [sentence] which you said right now . . . Repeat.

1.58 Darren: You could see that they are right angles. When you draw that measurement line, you see that that is right angles all together.

Darren continued to operate on Level 1 in this episode. An accidental reference to the symbol of a right angle by Laila led him to explore the possibility of explaining the congruence of vertical angles using right angles. The initial thread of Darren’s
argumentation had to do with the exploration of the concept of an angle in terms of arc lengths. In the second frame, he gave the example of right angles to prove that the vertical angles are congruent. Here, he was using a particular example to make a general case. In this instance, Darren was not aiming for a generality of vertical angle congruence but merely using an example to justify the case at hand. He thought that it sufficed to provide one example to prove a general case. According to the framework, justifying the proposition by means of an example is a characteristic of Level 1 reasoning whereas attaining generality is not. The assertion that he had not attained that generality at this point is supported by his subsequent claims in the immediate argumentation as well as in Task 12. In both these frames he tried to justify his reasoning with reference to particular angle measures in proving a general case.

The argumentation that followed was also about the issue of using a particular example to make a general case:

1.84 Darren: I have to prove that they are congruent, so . . . I would take this.
1.85 Teacher: Aha . . .
1.86 Darren: As 30°.
1.87 Teacher: Aha . . .
1.88 Darren: So, this would be 150° because 180° minus ∠QRS
1.89 Teacher: Aha . . .
1.90 Darren: 150°.
1.91 Teacher: So you got 150° out of that? Okay? So what else?
1.92 Darren: If that is 150°. . .
1.93 Teacher: Aha . . .
1.94 Darren: You have to do the same thing.
1.95 Teacher: So that is how it would be 30°.

Darren said that he would use a measure of 30° for one of the angles, calculates the adjacent angle to be 150°, and repeats the calculation to arrive at the fact that the other vertical angle would also be 30°. Darren continued the same thread of reasoning as in the previous frame. He was still operating on Level 1. This is reflected in the above
discussion when he referred to taking some measurements for the angles and gave a concrete case to prove his assertion. He had not yet learned to distinguish general arguments from discussions of a specific case.

Interestingly, at the conclusion of this episode, Laila interrupted him at the point where he used particular angle measures and noted that measurements cannot be used anymore to prove an argument, saying, “But we don’t want to have measures.” Her criticism of Darren’s approach was based on her understanding that it was not enough to provide an example in proving a general case. Instead, something more abstract and theoretical was necessary for proving the argument. It is worth noting that Darren’s efforts at proof were subject to critique from his peers.

**Analysis of Task 2**

In Task 2, the students were required to prove that pairs of alternate interior angles are congruent. The following figure was generated to help them in the process. Specifically, the task was to prove that \( \angle 4 \cong \angle 6 \).

![Figure 4.23: Alternate Interior Angles (First Pair)](image)
The students started off by thinking about corresponding angle pairs because the discussion that preceded this task supplied them with the information about the corresponding angle pair postulate. They automatically started to think of what information was already at hand that might be relevant to their goal. The students started off by considering \( \angle 4 \cong \angle 8 \). This was a reasonable starting point as \( \angle 4 \) and \( \angle 8 \) were a corresponding angle pair, and one of the pairs was required in the current task. Darren interjected and asked the group to consider looking at the pairs of interior angles for information. This was how the discussion unfolded:

2.54 Laila: Look at the information in the corresponding angles.
2.55 Teacher: Look for information in the corresponding angles? Okay, what information do you see in the corresponding angles; you want me to put here.
2.56 Students: \( \angle 4 \cong \angle 8 \).
2.57 Teacher: \( \angle 4 \cong \angle 8 \), so you look at this one. \( \angle 4 \cong \angle 8 \). You wanted to take that one. Okay, these are congruent why? They are corresponding angles, right?
2.58 Students: Yes.
2.59 Teacher: Okay, what else do you want me to write?
2.60 Darren: Look at the interior angles.
2.61 Teacher: Look at the exterior angles?
2.62 Darren: No… No… Look at the interior angles.
2.63 Teacher: We are looking at the interior, right? We are going to prove this [Referring to the pair of alternate interior angles] [The teacher was pointing to the fact that the alternate interior angles are the ones to be proved instead of taking them as a part of the dataset]

The objective of the task was to prove that the alternate interior angles are congruent. So at this point, he was clearly not able to recognize what needed to be proved in the first place. In comparison, the other students had shown advancement in their approach because clearly they recognized that the interior angle pair could not be a part of the data as it was the one which needed to be proved, so they objected to Darren’s considering the interior angles as a part of the data gathering.
As the discussion resumed, another objection was raised against Darren’s next strategy. He asked them to consider ∠3 and ∠7 along with ∠1 and ∠6, the other corresponding angle pair that the students wanted to take next. The following discussion brought to light the objection raised against Darren’s considering ∠3 and ∠7.

2.68 Darren: This is like they say ∠1 and ∠6; would you do ∠3 and ∠7?
2.69 Teacher: Okay, ∠3 and ∠7 Okay.
2.70 Dalton: I don’t get it.
2.71 Teacher: So, he wants ∠3 and ∠7.
2.72 Darren: ∠1 and ∠6, ∠3 and ∠7, it’s just the opposite.
2.73 Teacher: These are corresponding too right? Okay, No…like, like.
2.74 Laila: We are, like trying to find out how ∠4 and ∠6 are congruent, right?
2.75 Teacher: ∠4 and ∠6 are congruent, right.
2.76 Laila: Why do you pull out, what do you get? What do you get from ∠3 and ∠7?

As the discussion later unfolded, the students were trying to produce a coherent link to the corresponding angle pair and vertical angle pair in an attempt to prove that the alternate interior angles are congruent. They were picking out those specific vertical and corresponding pairs that would lead them to the alternate angle pair to be proved. Since it had to be proved that ∠4 ≅ ∠6, the students were trying to take the angle pairs which involved either of those angles. However, Darren overlooked the reason why the others were considering ∠4 and ∠8 as well as ∠1 and ∠6 and asked them to consider ∠3 and ∠7 along with ∠1 and ∠6. Dalton objected to this. Laila supported Dalton and also interjected and stated that ∠3 and ∠7 were not required to show that ∠4 and ∠6 were congruent.

From these engagements, we can see that Darren was still operating on Level 1 as he was unable to formulate a strategic plan for applying a previous theorem in a way that helps to solve the problem at hand. The other students were trying to gather relevant data.
required to present a valid explanation for the claim that the pair of alternate interior angles is congruent. Although Darren knew how to match up the corresponding angles to the diagram at hand and recognized the need to apply previous theorems as part of an argument, he did not yet know how to formulate an overall plan for applying a prior theorem that leads to the desired result.

**Analysis of Task 7**

Task 7 required the students to prove that the consecutive angles on the same side of the transversal are supplementary. They were given the following figure for reference:

![Figure 4.24: Consecutive Interior Angles (Second Pair)](image)

The students were required to prove that $\angle 1 + \angle 2 = 180^\circ$. Darren participated in the task of proving that the consecutive angles are supplementary, but his contribution was very limited in terms of substance. Laila and Dalton started off the argument by taking into consideration that $\angle 7 + \angle 2 = 180^\circ$. Darren diverted their attention to $\angle 1 + \angle 8 = 180^\circ$ and said that they add up to $180^\circ$ as they were corresponding angles [which was not the case]. Laila then rectified his mistake of referring to $\angle 1$ and $\angle 8$ as corresponding angles.
7.2 Laila: [Pointing to the \( \angle 7 \) and \( \angle 2 \)] These equal to 180°. [Pointing to \( \angle 1 \) and \( \angle 8 \)] These equal to 180°.

7.3 Teacher: Aha.

7.4 Laila: Okay, \( \angle 7 \) and \( \angle 1 \) are corresponding angles. [The teacher asks Darren to come and help Laila in writing on the board]

7.10 Darren: \( \angle 7 \) and \( \angle 2 \) equal 180°.

7.11 Teacher: Aha.

7.12 Darren: \( \angle 1 + \angle 8 = 180° \).

7.13 Teacher: \( \angle 1 + \angle 8 = 180° \) too, write that too if you want write that. \( \angle 1 + \angle 8 \). So where are you going from there? 180°.

7.14 Darren: They are equal to 180° because they are corresponding.

7.15 Teacher: Corresponding?

7.16 Laila: \( \angle 7 \) and \( \angle 1 \) is corresponding. [Darren interjects]

7.17 Teacher: Wait… wait, listen to her.

The students proceeded from there and proved the theorem:

7.18 Laila: If you change \( \angle 2 \) with \( \angle 7 \), they would be…

7.19 Dalton: \( \angle 1 \) with the \( \angle 7 \)

7.20 Laila: Yeah… we change \( \angle 1 \) with the \( \angle 7 \).

Laila and Dalton had already grasped the fact that the theorems can be proved by linking pre-established facts to arrive at something that needs to be proved. Darren was not at the same level but was trying to make the connection after listening to the arguments that the others presented. Darren was still operating on Level 1 as in the earlier task. Even at this stage, he was only able to see the need to refer to previous theorems; he still did not know how to make a logical argument from them.

**Analysis of Task 9**

Task 9 required the students to prove that \( \angle 4 + \angle 9 = 180° \). All the students were required to prove the theorem involving the exterior angles on their own and present their proofs. They had the figure below for reference.
In this task, Darren started to shift his focus from just making connections based on what other students were asserting to making a valid argument on his own. He presented the following proof on the board:

\[ \angle 4 + \angle 3 = 180^\circ \]
\[ \angle 1 + \angle 9 = 180^\circ \]

He explained the proof in the following manner:

9.55 Darren: I observed that they … like \( \angle 9 \) and switched it with \( \angle 3 \). So yeah \( \angle 4 + \angle 9 \) equals \( 180^\circ \).

In this case, Darren approached the proof with linear pairs \( \angle 4 \) and \( \angle 3 \) and \( \angle 1 \) and \( \angle 9 \) as the data and switched \( \angle 3 \) with \( \angle 9 \) to arrive at \( \angle 4 + \angle 9 = 180^\circ \). He did not explain why he was switching \( \angle 3 \) and \( \angle 9 \).

In this task, Darren began to operate on Level 2 of reasoning. He was organizing his proof, but only offering partial arguments, characteristic of Level 2. He was able to produce the required data which would eventually lead him to the theorem but missed the important aspect of providing the detail of the switch between \( \angle 3 \) and \( \angle 9 \) based on the corresponding angle pair congruence postulate. This was the crucial link to the data and to the claim that \( \angle 4 + \angle 9 = 180^\circ \). The corresponding angle pair congruence which
justified his switch was not mentioned at all. At this stage, he did not understand that the normative way of proof production was to offer warrants for the various data-claim links to make his argument more justifiable and reasonable.

**Analysis of Task 11**

Task 11, was proving the exterior angle theorem.

![Figure 4.26: Exterior Angle](image)

Darren first modified the figure presented initially to a figure that he was comfortable with, in this case a right triangle. He said:

11.60 Darren: Instead of using a regular triangle, I used a right triangle instead.

![Figure 4.27: Darren’s Exterior Angle](image)

$\angle 4 + \angle 1 + \angle 5 = 180^\circ$

$\angle 1 + \angle 2 + \angle 3 = 180^\circ$

When asked to present his proof, he articulated the following argument:
Darren: And like Joey said, he has the $\angle 4$ over here and I have $\angle 4 + \angle 1 + \angle 5$.

Teacher: Okay.

Darren: And that equals 180° [referring to $\angle 4 + \angle 1 + \angle 5$].

Teacher: How did you say that was 180°?

Darren: Because to me, it was…

Teacher: Aha…

Darren: I said this could be 90°.

Teacher: Are you guys listening to what he is saying?

Student: Yeah.

Darren: That could be a 90° and that could be a 180°. $\angle 2 = 90°$, $\angle 1 = 90°$ and that is 30°. That would be 20°.

Darren tried to search for a convincing coherent link to a familiar shape (i.e., a right triangle) to the task at hand. It is possible that Darren wanted to use a specialized version of the theorem—instead of using “the exterior angle is equal to the sum of the two opposite interior angles,” he could be limiting it to the case where one of those interior angles is 90°. This would show a lack of awareness of the need to maintain the generality of the conditions in formulating a proof. It was also possible that earlier he saw the teacher and other students introducing constructions as part of the proof process. He could be emulating this step, but without the benefit of an overall plan for how the argument should progress in order to establish the desired conclusion. Both of these possibilities were reflected in the figure that he drew in which he referred to the $\angle 2$ as a right angle and drew the line at the vertex of $\angle 2$ to make two more angles ($\angle 4$ and $\angle 5$). The $\angle 4$ that was formed by this transformation was not the exterior angle for the triangle anymore, but since it was outside the triangle he assumed it to be exterior to the triangle; this shows his limited understanding of the concept of an exterior angle. As we shall see, this misunderstanding of exterior angle also surfaced in his performance in Task 13. In this task, once he tried to link the data he presented to the claim at hand, he himself felt
not so convinced and finally reverted to some measurements to fill in the gap of reasoning.

Darren was operating on Level 2 in this task. He understood the need to apply prior theorems and tried to search for a convincing coherent link. He was not able to recognize the newly formed angles due to the construction added to the figure, perhaps because of his limited exposure to the concept of multiple angles at a vertex.

**Analysis of Task 12**

In Task 12, the students were given the figure below by the teacher and required to prove that \( \angle 3 = \angle 1 + \angle 2 \) given that the horizontal lines are parallel to each other.

![Figure 4.28: Parallel Line Task](image)

In this task, Darren was still navigating the network of relationships but was not able to provide a justification for his assertions. He was still operating on Level 2 of reasoning. He was not putting much thought into reading and analyzing the figure presented. The students had proved the exterior angle theorem in an earlier task, but it seemed that Darren did not have a grasp of what an exterior angle to a triangle means because in this task he mistakenly identified the \( \angle 3 \) to be an exterior angle based on its
position in the figure and applies the exterior angle theorem to conclude that $\angle 3 = \angle 1 + \angle 2$. This error had arisen from his misunderstanding of the concept of an exterior angle, which can be seen in the earlier task as well. Though he now understood that he could rely on past theorems to make an assertion, he was unable to apply the exterior angle theorem to the problem at hand as there seemed to be a gap in his understanding of the exterior angle theorem itself.

**Analysis of Task 13**

In Task 13, the students all presented their proofs to show that “The diameter that is perpendicular to the chord bisects the chord.” They had to prove that $AC = BC$ from the figure presented below:

Earlier, the students had discussed the various congruence properties of triangles in general, including right triangle congruence. Darren’s argument was based on right triangle congruence, but he did not make it explicit in his argument:

13.52 Darren: I drew perimeter lines [radii] or radius lines on both sides to show that they [the triangles] are congruent to each other.

13.53 Teacher: Aha.

13.54 Darren: Because this side is 90° triangle, they make each side equal and the same. So you see they are congruent.

13.55 Teacher: Aha.
Darren: If each side is the same, they are equal and AC = BC.

[He was applying the concept of right triangle congruence]

Darren joined the radii OA and OB to form right triangles OCA and OCB. He pointed out that because of this construction; the two triangles that are formed are congruent.

Darren was operating on Level 2. He understood the need to break down the prior theorems to check conditions and apply the results, but he was still not good at providing a robust justification either verbally or in writing. He joined the radii to form two congruent right triangles as shown above. This was a step more advanced than just transforming the figure without considering the consequence of transformation. He then tried to explain his reasoning using the congruence of triangles, particularly right triangle congruence, but he did not indicate in his writing how the two right triangles were congruent in terms of particular corresponding measures that make them congruent and in turn made AC and BC congruent. This kind of clarity was needed when presenting a complete proof and to be operating on Level 3, but it was absent in his presentation. The two combinations of right triangle congruence (HS/HL congruence and SS congruence) had been discussed earlier in the previous session, so it was expected that he would provide the basis to say why the two right triangles were congruent in order to conclude that the remaining corresponding measures were congruent. In his writing, he just mentioned that the radii were same and since the angles are 90°, the two triangles are congruent. When he came to the board he just showed with his fingers that OA = OB. He was still operating on Level 2.
Analysis of Task 14

In Task 14, the students were required to prove that OX = OY when the given chords are of equal length and a perpendicular line is drawn through the center as shown below:

Figure 4.31: Circles Theorem-2

The students worked individually and presented their proofs on the board. Darren presented the following proof on the board:

Darren’s proof:

The radius makes them the same. Also because they are right triangles each one is even [Pointing to all of the right triangles formed]. Being the same means “congruent.” So OX = OY.

He joined OA, OB, OC and OD, which he referred to as the radii, resulting in four right triangles OXA, OXB, OYC, and OYD. He then justified his claim that OX = OY by stating that all the right triangles that were formed were congruent.
Darren continued to operate on the Level 2 of reasoning. Like in Task 13, he made a correct choice in terms of proving the current theorem in relation to right triangle congruence and felt the need to justify his thinking with reference to the congruence of right triangles while proving his claim that \( OX = OY \), but he missed the important aspect of clearly identifying the conditions or measures leading to the congruence of those two triangles in the first place, which was crucial in determining the congruence of the remaining corresponding measures. He was working from a sound conceptualization of the proof in relation to theoretical reasons of validity, but he did not know what elements of that reasoning he needed to explicitly state in order to be seen as an effective prover.

**Summary of Darren’s Work**

Considering the different types of arguments presented by Darren in the various tasks, there was progress in his approaches to proofs. In the beginning, he could not sense that the shape of a proof had a series of claims supported by *data* and backed up by justifications and arriving at the claim through logical means. This can be observed in the discussions leading to the proofs of the vertical angle theorem, alternate interior angle theorem and consecutive interior angle theorem in Tasks 1, 2 and 7, respectively, where he was clearly not able to even formulate arguments on his own and mostly commented on the proofs of others.

In the later tasks, however, he could sense the shape of a proof but did not approach it with sufficient detail and rigor to assemble all of its pieces in a correct sequence. This could be seen in Tasks 9, 11, 12, 13 and 14 related to the exterior angle theorem, the task on parallel lines and in the theorems related to the circles.
Delbert’s Work

Analysis of Task 1

The task at hand was to prove the vertical angle theorem. The students had to prove that \( \angle PRT \cong \angle QRS \) in reference to this figure presented on the board:

![Figure 4.33: Vertical Angles](image)

This episode began with students expressing their approaches to the task. Delbert read out what he wrote on his paper:

1.2 Delbert: I think they are congruent because they are on opposite sides of one another.

He was just reiterating the conjecture with a twist of his own. He was operating on Level 1 as he felt at least an obligation to create an argument in support of something that needs to be proved. As the discussion continued, Laila presented a proof to show the congruence of \( \angle PRQ \) and \( \angle SRT \). The teacher asked Delbert to prove the other pair and told Laila to provide support. Laila started to direct him to prove the theorem in the way that she had approached it in the earlier pair but discussed with him regarding what needed to be done:

1.178 Laila: So Delbert, the first thing… [Laila and Delbert consult each other and write]
1.179 Teacher: You always look at your figure so that it makes it easier for you.
1.180 Laila: Now \( \angle QRS + \angle SRT \). What are you going to do? [Delbert writes] And since they are a linear pair they equal…
1.181 Delbert: 180°
Laila: Okay and write the other one. [Delbert writes] and write the other linear pair that one. [Delbert writes $\angle PRT$ plus $\angle PRQ$ equals…]

Both students discussed and wrote $\angle QRS + \angle SRT = 180^\circ$ (line 1.180) and $\angle PRT + \angle PRQ = 180^\circ$ (line 1.182). Dalton interrupted them at this point and the following discussion ensued:

Teacher: Dalton has a question there.
Dalton: He made the same mistake. We are trying to prove that they are congruent. So it would be better if $\angle SRT$ is there on the top and bottom.

Teacher: Wow. [Laila and Delbert correct the equation]
Laila: To have something in common. [Delbert writes]

Teacher: What do you conclude from that? Tell us what you conclude Delbert? So you have two equations here, right? Laila you should be helping him because you did that. So help him and make him understand what you wrote.

Laila: This is what… this is what I did. They both equal to $180^\circ$, so they are both… and they share $\angle SRT$.

Teacher: $\angle SRT$? Therefore…
Laila: Therefore these two must be equal. [Pointing to $\angle QRS$ and $\angle PRT$]

Laila helped Delbert to set up the data $\angle QRS + \angle SRT = 180^\circ$ (line 1.180) and let him write the other pair (line 1.182), $\angle PRT + \angle SRT = 180^\circ$. They then discussed and claimed that $\angle PRT \equiv \angle QRS$ (line 1.190) by eliminating $\angle SRT$ from both linear pairs (line 1.188). Though it seemed that Laila was directing him in the proof, she was involving him in presenting it. He was surely involved in the process as he seems to understand the shape of the proof from his discussion with Laila. Like Laila, he was also missing some parts of the argumentation which was filled in by others (line 1.184). However, he was able to quickly grasp what was lacking in their argument and rectified it accordingly. He was operating on Level 1 as he recognized the need to apply previous
theorems as part of one’s argument, but did not know the normative fashion for doing so himself.

**Analysis of Task 2**

In Task 2, the students were required to prove the alternate interior angles theorem involving angles $\angle 6$ and $\angle 9$. They already have been taught that corresponding angles are congruent. The figure below was given to help them in the process:

![Figure 4.34: Alternate Interior Angles (First Pair)](image)

Julia started to present a proof:

2.163 Julia: Since we have to prove $\angle 6$ equals $\angle 9$, $\angle 6$ equals $\angle 8$ right here and $\angle 8$ equals $\angle 9$.

As Julia pondered how to give a plausible justification to link the corresponding angles $\angle 6$ and $\angle 8$ and the vertical angles $\angle 8$ and $\angle 9$, Delbert picked up the thread and arrived at the conclusion that $\angle 6$ and $\angle 9$ are congruent:

2.181 Delbert: $\angle 6$ equals $\angle 8$ and $\angle 8$ equals $\angle 9$. So, these two [pointing to $\angle 6$ and $\angle 9$] should be the same.

Delbert provided Julia with a link between the angles using the transitive property (which had not been explicitly taught) Delbert started to operate on Level 2 as he was able to organize the reasoning to produce the desired result.
Analysis of Task 3

In Task 3, the students were required to prove the alternate interior angle theorem. The following figure was given to help them in the process. The task was to prove that

\[ \angle 4 \cong \angle 7. \]

Figure 4.35: Alternate Interior Angles (Second Pair)

Tommy approached the theorem first and he used \( \angle 5 \) equals \( \angle 3 \) to link \( \angle 4 \), \( \angle 5 \) and \( \angle 3 \), \( \angle 7 \) and hence to deduce \( \angle 4 \cong \angle 7 \). But, by this time, the students had not proved the exterior angle theorem which deduces \( \angle 5 \cong \angle 3 \).

After thinking for some more time, Tommy came back again to present a different approach and Delbert had a share in the proving the theorem.

3.43 Tommy: I think.
3.44 Teacher: I think… Tommy got it.
3.45 Tommy: \( \angle 4 \) equals \( \angle 3 \) and \( \angle 3 \) equals…
3.46 Delbert: \( \angle 7 \).
3.47 Teacher: Tell us that… do you want to tell us that as a group or one person go there and tell me… Write that for me baby. Tommy, write it for me, or show it to me here first.
3.48 Tommy: \( \angle 4 \) is vertical to \( \angle 3 \) and \( \angle 3 \) is corresponding angle to \( \angle 7 \).
3.49 Teacher: Perfect. Put it over, put it over here. Let me see, how it goes on the board. [Tommy writes] \( \angle 4 \) is congruent to \( \angle 3 \); \( \angle 3 \) is congruent to \( \angle 7 \).
3.50 Teacher: So why \( \angle 4 \) and \( \angle 3 \) are congruent? Can you tell me why they are congruent?
3.51 Joey,
   Julia: Vertical.
3.52 Teacher: They are vertical. Good. Put vertical for me in the bracket, there across the $\angle 4$ and $\angle 3$. Just put vertical, V, just put ‘V’ there in the bracket. So they are vertical. The ‘v’ stands for vertical. Why $\angle 3$ and $\angle 7$ are congruent?

3.53 Tommy: Corresponding…

3.54 Teacher: So put ‘C’ for me in the bracket. Like put a bracket for me. Put a parenthesis over there.

3.55 Student: Parenthesis.

3.56 Teacher: Just like I kept here and put one for the ‘V’ too. So what can you say about $\angle 4$ and $\angle 7$ now?

3.57 Tommy: They are congruent.

3.58 Teacher: Can you write that for, also that.

3.59 Delbert: $\angle 4$ equals $\angle 7$ [Tommy writes]. They are congruent.

3.60 Teacher: Okay. So, do you guys realize what you did now, you have actually proved that the alternate interior angles are congruent.

Delbert clearly understood the argument that was presented but he himself could not present the whole argument. Delbert was operating on Level 2 as he presented only partial arguments to the proof.

**Analysis of Task 4**

In Task 4, the students were required to prove that a pair of alternate exterior angles are congruent. They had the figure below:

![Figure 4.36: Alternate Exterior Angles (First Pair)](image_url)
The task was to prove that $\angle 7 \cong \angle 6$. The teacher directed Delbert to be the writer for the group. The discussion among the whole group unfolded in the following way:

4.78 Delbert: $\angle 2$ equals $\angle 7$.
4.79 Julia: Yeah.
4.80 Delbert: $\angle 7$ equals $\angle 2$ and $\angle 7$ equals $\angle 6$.
4.81 Tommy: $\angle 7$ equals $\angle 2$?
4.82 Julia: $\angle 7$ equals $\angle 4$ and $\angle 4$ equals $\angle 6$.
4.83 Tommy: Yeah.
4.84 Delbert: $\angle 7$ equals $\angle 4$ and $\angle 4$ equals $\angle 6$.
4.85 Julia: Which one?
4.86 Joey: $\angle 7$ is vertical to $\angle 4$ and $\angle 4$…
4.87 Delbert: You want me to write that down?
4.88 Tommy: Okay. $\angle 7$ equals $\angle 4$ and $\angle 4$ equals $\angle 6$.
4.89 Teacher: So you are the representative? You are just recording what you all thought of?
4.90 Tommy: Aha.
[Delbert writes]
4.92 Teacher: Guys, when he is writing, you have to look whether he is writing what you want him to write. He wrote $\angle 7$ is congruent to $\angle 4$. Why is $\angle 7$ congruent to $\angle 4$?
4.93 Students: Because they are vertical.
4.94 Teacher: They are vertical? Can you ask him to put that… in a bracket like in a parenthesis somewhere? Yeah. Vertical angles. Okay, and then he wrote $\angle 4$ is congruent to $\angle 6$. Guys, $\angle 4$ is congruent to $\angle 6$. Why?
4.95 Julia: [Inaudible] they look like each other.
4.96 Teacher: Julia, he wrote $\angle 4$ is congruent to $\angle 6$. So what was the reason behind writing that?
4.97 Julia: Because $\angle 6$ is equal to $\angle 4$ and $\angle 4$ is equal to $\angle 7$. So $\angle 6$ is equal to $\angle 7$. Which means $\angle 7$ is equal to $\angle 6$.
4.98 Teacher: Yeah… we got that, what are $\angle 4$ and $\angle 6$? That’s what I am asking. Tommy?
4.99 Tommy: Hah…?
4.100 Teacher: Why did you all put $\angle 4$ and $\angle 6$? What are they?
4.101 Tommy: Because they are corresponding angles.
4.102 Teacher: They are corresponding angles? Okay. Can you put corresponding angles for me there?
4.103 Delbert: Write across it?
4.104 Teacher: Yeah, put it in the parentheses, yeah… because they are corresponding angles.
[Delbert writes]
The students started with the data $\angle 7 \cong \angle 4$ and $\angle 4 \cong \angle 6$ (line 4.82). They presented relevant data and produced the logical link between the data and the claim that $\angle 7 \cong \angle 6$. The group was able to present theoretical reasons of validity to prove a theorem. However, they did not provide the justifications that made that reasoning plausible until they were questioned by the teacher (see #31, #39). In this case, they needed to justify that $\angle 7 \cong \angle 4$ and $\angle 4 \cong \angle 6$ with reference to the vertical angle theorem and the corresponding angle postulate respectively. Delbert was a part of the group that was operating on Level 2. He was able to acknowledge that $\angle 7$ equals $\angle 4$ and $\angle 4$ equals $\angle 6$ (line 4.84), which would lead to the result $\angle 7$ equals $\angle 6$ (line 4.110). He concluded along with others that $\angle 7 \cong \angle 6$ from the data set. Even though he was recording what others were telling him to write on the board, he was a part of the discussion leading to the result and he himself understood the working of the proof.

**Analysis of Task 5**

A similar task followed in which the students were required to prove the congruence of the other pair of alternate interior angles.
The students had to prove that $\angle 3 \cong \angle 8$, and the teacher asked them to present the proof as a group again. The students chose Tommy to write on the board, and the discussion flowed as follows:

5.29 Tommy: It’s easy. $\angle 3$ is equal to…
5.30 Delbert: $\angle 5$.
5.31 Tommy: $\angle 3$ and $\angle 1$.
5.32 Joey: $\angle 3$ is what?
5.33 Del, Joey: $\angle 3$ equals $\angle 5$ and $\angle 5$ equals $\angle 8$.
5.34 Tommy: There are two ways to put this.

Tommy wanted the others to start with $\angle 3$ and $\angle 1$ (line 5.31) as the vertical angle pair. Delbert and Joey wanted him to start with $\angle 3$ and $\angle 5$ (line 5.33). The discussion continued:

5.35 Teacher: Good. Yeah. There are… there might be many ways right? Tommy, first write what they told you and then you write what you thought, okay?
5.36 Tommy: Okay. What did you say?
5.37 Students: $\angle 3$ equals $\angle 5$ and $\angle 5$ equals $\angle 8$.

Tommy with the help of others took a few minutes to write the data on the board and Delbert augmented the proof:
5.44 Delbert: $\angle 3$ equals $\angle 5$ and $\angle 5$ equals $\angle 8$.
5.45 Teacher: So what… what are you guys concluding from that. Guys you should try helping him write. [Addressing Tommy]
5.46 Delbert: $\angle 3$ is congruent to $\angle 8$.

Dynamic interaction between the students can be seen in this episode. They assigned one of their friends to present their ideas and took full responsibility for providing the valid information to arrive at the target of proving the above mentioned alternate exterior angle pair. Delbert in particular, recognized the main logical link connecting the data $\angle 3 \cong \angle 5$ and $\angle 5 \cong \angle 8$ (lines 5.33, 5.44) to the claim that $\angle 3 \cong \angle 8$.

Delbert was still operating on Level 2. Even though he was providing the relevant data, he did not feel the necessity to justify his steps, in this case with the vertical angle theorem and corresponding angle postulate to support his data. Without these, the proof is considered to consist of only partial arguments.

**Analysis of Task 7**

In Task 7 the students were asked to prove that consecutive interior angles are supplementary when two parallel lines are intersected by a transversal. The students did not conjecture about this; the teacher presented the task to them. The diagram below was given for reference:

![Diagram](image)

Figure 4.38: Consecutive Interior Angles (Second Pair)
The students were required to prove that $\angle 1 + \angle 2 = 180^\circ$. The teacher asked Julia, Delbert, and Joey to come to the board and present a proof together. The following conversation shows the development of the argumentation for the proof of this theorem:

7.80 Delbert: $\angle 7$ and $\angle 2$ are linear pairs and $\angle 1$ and $\angle 8$ are linear pairs.
7.81 Teacher: Okay. Write them for me. Let me write them for you. You are saying $\angle 7$ and…?
7.82 Delbert, Julia, Joey: $\angle 2$.
7.83 Teacher: $\angle 2$ are linear pairs.
7.84 Julia: And $\angle 8$ and $\angle 1$
7.85 Teacher: $\angle 8$ and $\angle 1$ are linear pairs too? Okay. Where are you going from there?
7.86 Julia: Because we thought…
7.87 Delbert: Because they are corresponding angles [pointing to $\angle 7$ and $\angle 1$.] [Delbert draws an arrow from $\angle 7$ to $\angle 1$.]
7.88 Teacher: Aha.
7.89 Julia: Put the equal thing.
7.90 Delbert: Corresponding angles are congruent.
7.91 Teacher: Okay. So you are saying, because $\angle 7$ and $\angle 1$ are corresponding which one do you want to replace, in this one [$\angle 7$ plus $\angle 2$ equals $180^\circ$] or this one? [$\angle 8$ plus $\angle 1$ equals $180^\circ$]
[The students point to $\angle 7$ plus $\angle 2$ equals $180^\circ$.] Teacher: In the top one? The top one becomes what? Joey, can you write that for me? What will the top one become now? You are replacing…
7.92 Joey: $\angle 1$ and the $\angle 7$.
7.93 Teacher: Okay. $\angle 1$…
[Joey writes $\angle 1$ plus $\angle 2$ equals.]
7.94 Teacher: Equals.
7.95 Joey: It’s [inaudible]… $180^\circ$.

As the argument frame shows, the students slowly began to produce a deductive proof by making their arguments explicit with data like $\angle 7 + \angle 2 = 180^\circ$ (line 7.80) and $\angle 7 \cong \angle 1$ (line 7.87). They backed up the arguments with the use of linear pair properties and corresponding angle pair congruence (lines 7.80, 7.87). Delbert’s contribution is worth noting. He provided the basis for the proof by presenting both the linear pairs $\angle 7 + \angle 2 = 180^\circ$ (line 7.80) and $\angle 1 + \angle 8 = 180^\circ$ (line 7.80) as possible starting points to
arrive at the proof of the theorem. However, the group considered taking $\angle 7 + \angle 2 = 180^\circ$ (line 7.91) and then they used what Delbert suggested, taking $\angle 7$ and $\angle 1$ as the congruent corresponding angle pair so that $\angle 7$ could be replaced with $\angle 1$ and they all concluded that $\angle 1 + \angle 2 = 180^\circ$.

Delbert began to operate on Level 3. He was bringing the ideas of the proof to light, organizing reasoning aimed at deduction, and providing the justifications that made those logical links possible.

**Analysis of Task 8**

In Task 8, the students were asked to prove that the consecutive exterior angles are supplementary when two parallel lines are intersected by a transversal. They had the same figure for reference from the preceding task. Here, each student presented his or her proof on the board. Delbert presented the following proof along with required warrants in the parentheses to show that $\angle 7 + \angle 8 = 180^\circ$ with reference to the figure below.

![Figure 4.39: Consecutive Exterior Angles (First Pair)](image)

Delbert’s written proof on the board:

$\angle 7 + \angle 2 = 180^\circ$

$\angle 1 + \angle 8 = 180^\circ$ [They are corresponding angles.]
So $\angle 7 + \angle 8 = 180^\circ$ [They switch because they are corresponding angles and they are congruent.]

Delbert presented a deductive proof by first listing the linear pairs that he thought could solve the problem. Because the proof involved $\angle 7$ and $\angle 8$, he might have taken the linear pairs that involved both the angles: $\angle 7 + \angle 2 = 180^\circ$ and $\angle 1 + \angle 8 = 180^\circ$. The subsequent steps involve the elimination of unnecessary data. In this case, Delbert was supposed to eliminate $\angle 1 + \angle 8 = 180^\circ$ as it was not needed to prove the theorem. A possible reason for not striking out $\angle 1 + \angle 8 = 180^\circ$ can be explained as follows:

Since he put an arrow between $\angle 2$ and $\angle 8$ to show that $\angle 2$ in the first linear pair can be switched with corresponding angle $\angle 8$, he could not strike off $\angle 1 + \angle 8 = 180^\circ$. His statement, “They are corresponding angles” shows that he knew $\angle 8$ was congruent to $\angle 2$. This indicates that he is considering only the linear pair $\angle 7 + \angle 2 = 180^\circ$ to arrive at $\angle 7 + \angle 8 = 180^\circ$.

Delbert was clearly operating on Level 3 as he was able to recognize that $\angle 7 + \angle 2 = 180^\circ$ is the relevant data that could lead him to the conclusion that $\angle 7 + \angle 8 = 180^\circ$, by using the corresponding angle postulate.

**Analysis of Task 11**

In Task 11, proving the exterior angle theorem, Delbert first modified the figure presented initially:

![Figure 4.40: Exterior Angle](image1)

![Figure 4.41: Delbert’s Exterior Angle](image2)
Delbert’s proof on the board:

\[ \angle 4 = \angle 5 + \angle 6 \]

\[ \angle 4 = \angle 1 + \angle 2 \] [Replace \( \angle 5 \) with \( \angle 1 \) and \( \angle 6 \) with \( \angle 2 \)]

He provided the following explanation for what he wrote on the board:

11.123 Delbert: Okay. \( \angle 4 \) and \( \angle 1 \) are like interior. So I draw a line up here to make an angle 5. And like I wrote this like \( \angle 6 \) as a substitute.

Delbert did not understand that once he drew the parallel line, \( \angle 4 \) will be the sum of 90\(^\circ\) and \( \angle 1 \), not just \( \angle 1 \). He continued:

11.124 Teacher: Where is the \( \angle 6 \), baby? [He shows \( \angle 6 \) in the figure.]
11.125 Teacher: Oh… the \( \angle 6 \) is there. Okay. \( \angle 4 = \angle 5 + \angle 6 \). [The teacher reads what he wrote on the board.]
11.126 Delbert: I replaced them with \( \angle 1 \) and \( \angle 2 \).
11.127 Teacher: Oh…
11.128 Tommy: I don’t see what he did.
11.129 Julia: Even though you are my cousin, I cannot understand you.

The other students also said that they did not understand what he implied (lines 11.128, 11.129). After the teacher encouraged them to question his proof, the following conversation took place:

11.130 Teacher: You guys have to ask when you don’t understand, because I also didn’t understand that… or can you guys like… I have a question, Delbert. How did you replace the \( \angle 5 \) with the \( \angle 6 \) here?
11.131 Julia: Yeah.
11.132 Teacher: Julia, did you have the same question? Julia? Or ask your question.
11.133 Julia: I was kind of confused because I saw they were two right angles.
11.134 Teacher: Guys, he replaced the \( \angle 5 \) with the \( \angle 1 \) and \( \angle 6 \) with the \( \angle 2 \).
11.135 Ryan: How could you replace the \( \angle 1 \) and \( \angle 5 \) with \( \angle 6 \) and \( \angle 2 \), \( \angle 6 \) and \( \angle 2 \) equals 180 \(^\circ\) and \( \angle 5 \) and \( \angle 1 \), doesn’t it?
11.136 Teacher: So think about what they are asking, okay? We are not complete yet. Okay. We still have to go through everybody. It was good. Ryan asked how he could replace the \( \angle 5 \) with the \( \angle 1 \) and the \( \angle 6 \) with the \( \angle 2 \). There should be a logical
explanation for it. So think about it. [Delbert goes back to his seat to think about it.]

It is possible that Delbert earlier has seen the teacher or other students using a construction as part of a proof and might have assumed that he would benefit from by drawing one on his own. So he drew the right triangle with the extended sides and a parallel line as shown in the above figure. He then tried to find meaning to the given task through an association with the parallel line properties. Now, the figure itself became a complex construction with multiple angles at each vertex and he made an association fallacy of locating the angles in the first place.

This confusion could have arisen due to the fact that he had not yet been exposed to such a complex situation, and it threw him off track as to what he was supposed to take as the data to present his case.

Even though Delbert understood how the process of a proof works and was even capable of producing a deductive text, in this particular task, he was operating on Level 2. He understood the obligation to be logically persuasive, but his inability to make a valid argument can be attributed to his misinterpretation of the angles caused by the complexity of the figure that he presented.

**Analysis of Task 12**

In Task 12 on parallel lines, the teacher presented the following problem:

![Figure 4.42 Parallel Line Task](image-url)
Prove that $\angle 3 = \angle 1 + \angle 2$ when the two horizontal lines are parallel to each other.

The students worked individually on this task then presented their proofs on the board. Delbert drew the figures below and presented this proof:

Figure 4.43: Delbert’s Parallel Line 1

![Figure 4.43: Delbert’s Parallel Line 1](image)

Delbert’s proof: $\angle 3$ is an alternate interior angle. The problem is like the one we already did. $\angle 3 = \angle 1 + \angle 2$.

He explained it in this way:

12.79 Delbert: I put the same angle because they were the same.
12.80 Teacher: Oh… you put the same angle because they were the same? Good.
12.81 Delbert: Because they were alternate interior.
12.82 Teacher: Aha.
12.83 Delbert: And the problem is like this [showing a part of the proof]
12.84 Teacher: Wonderful.
12.85 Delbert: That’s how I got.

Delbert in this task first placed another angle 3 in the figure and referred to the exterior angle theorem that he showed in another representation (see figure). He then concluded that since $\angle 3$ is the exterior angle to the triangle in the given figure,

$\angle 3 = \angle 1 + \angle 2$.

Technically, it is not correct to give the same label for alternate angles. One may say that Delbert gave the same label to alternate angles, since he was shifting between
working with angles and angle measures. He should have given a different label to the newly formed angle and then should have showed that they were congruent. Even though he did not do so, in his arguments he said the angles were the same as they were alternate interior (line 12.79 & 12.81). This shows that he understood the concept of alternate interior angles and congruence. He might have used the same label “3” to keep the problem simple since in the back of his mind he knew that the angles were congruent.

Nobody else in this study used the same labeling for two congruent angles which shows that Delbert’s concept of using same label did not influence the other students. Delbert was again operating on Level 3, as he showed how the conditions actually produce the result in a theorem even though a first look at his labeling of angles produces a sense of ambiguity of his understanding between angles and angle measures.

**Analysis of Task 13**

In Task 13, the students all presented their proofs to show that the diameter that is perpendicular to the chord bisects the chord. Specifically, they had to prove that $AC = BC$ from the figure presented below:

![Figure 4.45: Circles Theorem-1](image1)

![Delbert’s figure:](image2)

Figure 4.45: Circles Theorem-1  
Figure 4.46: Delbert’s Circles Theorem-1
Delbert joined the radii OA and OB to form right triangles OCA and OCB (see figure). He pointed out in his explanation (see # 94) that because of this construction; the two triangles that were formed are congruent. Delbert’s argument was based on triangle congruence and he made it explicit in the written text of his proof:

Delbert’s proof: It looks like it will bisect in the middle. It [will] also have the same length for everything.

He was referring to the fact that both triangles have corresponding sides of the same length, making them congruent triangles. He pointed to the triangles being right triangles when he later said:

13.118 Delbert: Mine is kind of like Darren’s and Dalton’s. I had the 90° triangles and the radius.

This statement indicates that he was going for right triangle congruence. Delbert was operating on Level 3 as he produced a verbal explanation of the proof using valid reasons of inference by saying that the two triangles are right triangles.

Analysis of Task 14

In Task 14, the students were required to prove that OX = OY when the given chords are of equal length and a line is drawn through the center as shown below:

Figure 4.47: Circles Theorem-2
The students worked individually and presented their proofs on the board. Delbert presented the following diagrammatic proof on the board:

![Diagram](image)

Figure 4.48: Circles Theorem-2

and explained it in the following way:

14.95 Delbert: I just drew like lines like this [shows the radii]. If these two are the same size and those two are same size [Referring to the two sets of triangles on the top and then on the bottom] then these two are the same. [Points to OX and OY]

He joined OA, OB, OC and OD, refers to them as sides of equal lengths forming 4 right triangles. He then concluded that OX = OY because the triangles are congruent. The right angle symbols that he inserted in his construction clearly indicate that the triangles he was considering are right triangles. Delbert is considered to be creative enough to present a proof diagrammatically in addition to being able to verbalize it. He made a clear and concise argument so that everyone could understand the hidden implications and assertions that he was presenting as a part of the proof. He was still operating on Level 3 as he was able to produce a verbal deductive explanation for his proof.
Summary of Delbert’s Work

Delbert started at Level 1, feeling an obligation to create an argument in support of a conclusion but not quite knowing how to produce an argument in Task 1 related to vertical angles. In the same episode, he continued to operate on Level 1 but received guidance from Laila in proving the theorem. Here he understood that he was expected to make interpretations of the task with logical arguments rather than just inferences based on observation alone and he also began to understand the shape of a proof. In the subsequent Tasks 2, 4 and 5 related to the alternate interior and alternate exterior angle theorems, he worked at Level 2. He put this understanding of the shape of the proof to use by drawing logical conclusions from the data presented by others in the group and he himself produced a deductive argument in proving the alternate exterior angle theorem. However he did not feel the need to present justifications for his assertions at that juncture. Through observation, he learned that he needed to present those in order to produce a wholesome argumentation which he implemented in Tasks 6 and 8, related to consecutive interior angles and consecutive exterior angles theorems. It shows that he started working at Level 3 from Task 7 onwards. However in Task 11, he complicated the given figure and found it difficult to produce the data. The confusion might have arisen due to his inexperience in dealing with multiple angles at the different vertices that his new construction produced. Though he was not efficient in producing that particular proof, he still exhibited his understanding of the process of proof production in the remaining Tasks 12, 13 and 14, in which he continued to work on Level 3 by showing the same logical consistency in arriving at justifiable conclusions to his claims as he did in the earlier sessions.
Jeremy’s Work

Analysis of Task 1

The task was to prove the vertical angles theorem. First, the students presented their thoughts regarding the proof of the theorem. Jeremy had an explanation:

![Diagram of vertical angles]

Figure 4.49: Vertical angles

1.18 Jeremy: They are congruent because it matters what angle it is.
1.19 Teacher: It matters what angle it is? Can you be more...can you be more clear about what you are trying to say?
1.20 Jeremy: It matters what angle it is because if they are not the same angle they are not the same measure.

In this explanation, Jeremy was just referring to the fact that the vertical angles can be congruent only if their angles measures are the same. He was operating on the Level 1 as he at least felt an obligation to create an argument in support of a conjecture though not yet capable of making a substantial one.

In the later part of the task, Delbert and Laila proved the theorem together. They came up with the data; \( \angle QRS + \angle SRT = 180° \) and \( \angle PRT + \angle SRT = 180° \) and concluded that, \( \angle PRT \cong \angle QRS \) by excluding the angles which were there in both the linear pairs.

Jeremy then volunteered to prove the same thing in a different way. The following conversation ensued:

1.135 Teacher: So go back and we will give a chance to Jeremy. Jeremy, do the second one to show that you can make them true. [Students talking.]
1.136 Teacher: You have reasoning? I will ask [addressing the students] them too. [Jeremy writes.]
Jeremy: [Inaudible] $\angle QRS$ equals $180^\circ$ minus $\angle PRQ$

Teacher: Good. Guys, your reasoning is really good. $180^\circ$ minus $\angle PRQ$, wow... See $\angle QRS$. I think this is really wonderful $\angle QRS$. Appreciate how they did it $\angle QRS$. Let us see here $\angle QRS$ equal $180^\circ$ minus $\angle PRQ$. Perfect. Right and then what about $\angle PRT$? [Jeremy writes $\angle PRT = 180^\circ – PRQ$] $\angle PRT$ is also $180^\circ$ minus...

Students: $\angle PRQ$

Teacher: $\angle PRQ$. So, what can you say?

Jeremy: Same, equal.

Teacher: Very good. Tell us why are they equal?

Jeremy: Common. [Pointing to both $180^\circ – \angle PRQ$]

Teacher: These are common, right... $180^\circ$ minus $\angle PRQ$ and $180^\circ$ minus $\angle PRQ$ are common, so you can say the same, right, therefore what can you write? You proved this... [$\angle QRS$] is congruent to...

Students: P...R...T...

Jeremy started with the linear pair $\angle QRS + \angle PRQ = 180^\circ$, but arranged it in the form of $\angle QRS = 180^\circ – \angle PRQ$ (line 1.137) and then he took $\angle PRT = 180^\circ – \angle PRQ$ (1.138) and concluded that $\angle QRS = \angle PRT$. Jeremy had been observing Laila and Delbert while they were proving the theorem. Before Laila and Delbert presented their proofs, Dalton proved that $\angle PRQ = \angle SRT$. When Jeremy came to present the proof for the other vertical pair $\angle QRS = \angle PRT$, he used Dalton’s reasoning and arrived at a valid conclusion. This shows that he clearly understood the reasoning and was able to apply it appropriately.

Jeremy began to operate on Level 2 of reasoning as he searched for a coherent link between the propositions based on theoretical reasons of validity though he was using the approach of another student.

**Analysis of Task 5**

Jeremy was given a chance to prove the alternate exterior angle theorem. The task was to prove that $\angle 1 \cong \angle 8$ using the figure below:
Jeremy approached the proof in the following way:

5.1 Teacher: You have to show me $\angle 1 \cong \angle 8$. Can you show it now? Come here. Jeremy wants to… because I want everybody to be involved.

[Jeremy writes]

5.2 Teacher: Show us; how you show us $\angle 1 \cong \angle 8$ Ssh… look at what he is doing.

5.3 Jeremy: $\angle 6$ equals to $\angle 8$.

5.4 Teacher: Stand… Stand next to him if you want to help him okay? $\angle 6 \cong \angle 8$. Why is $\angle 6 \cong \angle 8$? Jeremy?

5.5 Jeremy: Because they are vertical.

5.6 Teacher: Vertical angles? Okay next.

[Jeremy writes $\angle 4 \cong \angle 8$]

5.7 Teacher: Let him do it. Let him do it, Dalton. $\angle 4 \cong \angle 8$. Why are $\angle 4$ and $\angle 8$ congruent?

5.8 Jeremy: They are corresponding.

5.9 Teacher: They are corresponding. Perfect. So what can you say now?

[Jeremy thinks]

5.10 Students: Pooh…

5.11 Teacher: He needs one more? Dalton help him what he needs.

5.12 Dalton: $\angle 1$ and $\angle 4$

5.13 Teacher: He needs $\angle 1$ and $\angle 4$? He says you need $\angle 1$ and $\angle 4$.

5.14 Students: Ooh…

5.15 Teacher: Now, can you, can you summarize that for us? To prove $\angle 1 = \angle 8$?

5.16 Jeremy: $\angle 1 = \angle 4; \angle 4 = \angle 8$; so…

5.17 Teacher: Therefore…

5.18 Jeremy: Therefore $\angle 1 \cong \angle 8$. 

Figure 4.50: Alternate Exterior Angles (Second Pair)
Jeremy started with the vertical angle pair $\angle 6 \cong \angle 8$ (line 5.3) and the corresponding angle pair $\angle 4 \cong \angle 8$ (line 5.6). He stopped to think about next step in the process. Dalton suggested him to take $\angle 1$ and $\angle 4$, the vertical angle pair. Jeremy immediately picked up $\angle 4 \cong \angle 8$ in combination with $\angle 1 \cong \angle 4$ to conclude that $\angle 1 \cong \angle 8$ (line 5.16).

Jeremy was operating on Level 2 as he was able to break down the application of a prior theorem to check conditions and apply results. However, he was not quite good at it as a part of the argument was supplemented by another student.

**Analysis of Task 6**

The students had to prove that $\angle 3 + \angle 5 = 180^\circ$, the consecutive angles on the same side of the transversal are supplementary. They used the figure below for their proof:

![Figure 4.51: Consecutive Interior Angles (First Pair)](image)

First Dalton started the proof and then Laila continued. But she was not able to come to a valid conclusion. Laila presented $\angle 4 + \angle 3 = 180^\circ; \angle 5 + \angle 6 = 180^\circ$ as the linear pairs and $\angle 4 \cong \angle 5$ and $\angle 5 \cong \angle 6$. From there, she was not able to arrive at $\angle 3 + \angle 5 = 180^\circ$. Jeremy at this juncture, raised his hand and the teacher allowed him to try the proof. He explained his proof in this way:
6.59 Teacher: Okay, Jeremy wants to say something. All of you guys should try. Okay, he is… he is helping Dalton, in what he is trying to do okay.

6.60 Jeremy: \( \angle 4 + \angle 3 = 180^\circ, \angle 4 \cong \angle 5. \)

6.61 Teacher: Why is \( \angle 4 \cong \angle 5 \)?

[Jeremy thinks]

6.62 Students: They are corresponding, Jeremy.

6.63 Jeremy: Corresponding?

6.64 Laila: Yeah, they are corresponding angles.

6.65 Teacher: The corresponding angles are congruent, right?

6.66 Jeremy: Place the \( \angle 4 \) with \( \angle 5 \) and \( \angle 3 + \angle 5 = 180^\circ \).

Jeremy arrived at \( \angle 3 + \angle 5 = 180^\circ \) from the data \( \angle 4 + \angle 3 = 180^\circ; \angle 4 \cong \angle 5 \) that Dalton initially took. Jeremy was able to bring ideas of proof to light and link them in a coherent way. However he was not able to back up his data with particular reference to the corresponding angle postulate which made the logical link possible in this case. He was still operating on Level 2.

**Analysis of Task 9**

The students had to prove that \( \angle 4 + \angle 9 = 180^\circ \), the other pair of exterior angles on the same side of the transversal. The students worked on their papers individually and presented their proofs on the board. Jeremy presented the proof below on the board:

![Figure 4.52: Consecutive Exterior Angles (Second Pair)](image)

Figure 4.52: Consecutive Exterior Angles (Second Pair)
Jeremy’s proof:

\[ \angle 4 + \angle 3 = 180^\circ \]
\[ \downarrow \]
\[ \angle 1 + \angle 9 = 180^\circ \]

So \( \angle 4 + \angle 9 = 180^\circ \)

Jeremy presented the linear pairs; \( \angle 4 + \angle 3 = 180^\circ \) and \( \angle 1 + \angle 9 = 180^\circ \) though the second one was irrelevant to the task. Then, he placed an arrow to show that \( \angle 3 \) and \( \angle 9 \) were switched to arrive at \( \angle 4 + \angle 9 = 180^\circ \). However, he did not write or explain explicitly why he replaced them. He might have been influenced by the classroom taken-as-shared meaning of representing congruent angles with arrows and assumed that the arrow itself explains that they were congruent. He was still operating on Level 2 as the proof contained only partial arguments rather than a complete one.

**Analysis of Task 10**

The students had to prove triangle sum theorem with reference to the figure below:

![Triangle Sum](image)

**Figure 4.53: Triangle Sum**

Ryan first proved the theorem and a few students were absent on the day that he presented the proof. The next day, the teacher asked Jeremy, Tommy and Ricky to explain the proof to the others. The presentation of the proof was as follows:
The students used the above figure while explaining the proof:

10.135 Ricky: $\angle 4 + \angle 1 + \angle 5 = 180^\circ$.
10.136 Jeremy: Because it is a straight line.
10.137 Tommy: Yeah, we said it was 180 degrees because…
10.138 Jeremy: That is a straight line, that’s why we said it was 180 degrees.
[The students discuss what to write and they write $\angle 4$, $\angle 2$; $\angle 3$, $\angle 5$ on the board and add the word alternate interior beneath those two pairs.]
10.139 Teacher: This is… This is a phase where you are learning from one another. Okay so… you… I will… so…. What did you do there? Let me ask you a question. Why did you put? What is the $\angle 4$ and $\angle 2$ and what is the $\angle 3$ and the $\angle 5$?
10.140 Ricky: They are interior, alternate interior.
10.141 Tommy: Yeah.
10.142 Teacher: So, what are you doing… where are you going from there?
10.143 Ricky, Jeremy,
Tommy: We would replace the interior. $\angle 4$ and $\angle 2$.
They approached the theorem by referring to $\angle 4 + \angle 1 + \angle 5 = 180^\circ$ (line 10.135) as the angles on a straight line and then added the alternate interior angle pairs; $\angle 4$, $\angle 2$; $\angle 3$, $\angle 5$ (line 10.138 ). Then, they went for the replacement of the alternate interior angles to prove that $\angle 2 + \angle 1 + \angle 3 = 180^\circ$. The students completed the theorem in the following manner:

10.162 Teacher: What they did was… instead of the $\angle 4$, they put $\angle 2$.
10.163 Tommy: Yeah.
10.164 Teacher: Instead of the $\angle 5$…
10.165 Jeremy: We put $\angle 3$.
10.166 Teacher: They put…
10.167 Students: $\angle 3$. 
Jeremy and the others replicated the proof that Ryan did the earlier day with the details that he referred to in his proof. This presentation shows that they clearly understood the working of that proof.

Jeremy was working with a group that was operating on Level 3. The group not only understood how the conditions actually produce the result in a theorem but also saw the necessity of backing up their warrants with justifications while producing a proof.

**Analysis of Task 11**

Task 11 was to prove the exterior angle theorem. The students worked independently and presented their proofs on the board. They had to prove that, \( \angle 4 = \angle 1 + \angle 2 \) from the figure below:

![Figure 4.55: Exterior Angle](image)

Jeremy’s proof on the board:

\( \angle 4 \) and \( \angle 1 \) are interior pair and you can switch them

\( \angle 4 = \angle 1 + \angle 2 \)

Jeremy did not write anything on his paper or the board of substance. He came to the board and said:

11.72 Teacher: Jeremy, go and explain what your reasoning was… Ssh… guys, you have to see.
He started with considering $\angle 1$ and $\angle 4$ and addressed them as alternate interior angles as he pointed them out. However, Jeremy felt that he could not continue further and went back to his seat.

It seems that he wanted to apply the alternate interior angle theorem, but did not draw a parallel line to support his idea. He might have assumed it (a parallel line) at the vertex opposite to the base to conclude that $\angle 4$ and $\angle 1$ were an alternate interior pair. However, he was not able to comprehend that they cannot be an alternate interior angle pair. The reason for his inability to deal with this situation can be attributed to the fact that he had not yet been involved in sophisticated practices of dealing with multiple angles at a single vertex. Imagining the parallel line rather than drawing it might have complicated the issue further. He was at a disadvantage to be able to use this approach. However, this does not mean that he had not understood how the process of proof works. It is just that he could not proceed due to the complication that arose from his imagination of the parallel line and formation of multiple angles at vertex. In this task, his participation was not sufficient to reveal his level of understanding.

**Analysis of Task 12**

A problem on parallel lines was presented to the students in Task 12. The students were required to prove that $\angle 3 = \angle 1 + \angle 2$ with respect to the two parallel lines:
The students worked individually and presented their proofs on the board. Jeremy presented the following proof with the figures below:

$\angle 3$ and $\angle 4$ are alternate interior angles.
$\angle 4 = \angle 2 + \angle 1$ because we proved
Switch $\angle 4$ with the $\angle 3$
$\angle 3 = \angle 1 + \angle 2$

Jeremy first assigned $\angle 4$ to the one of the angles in the figure and started the proof with $\angle 4$ and $\angle 3$ as alternate interior angles. He then applied the exterior angle theorem to the angles $\angle 4$, $\angle 2$, and $\angle 1$ and replaced $\angle 4$ with $\angle 3$ to arrive at $\angle 3 = \angle 1 + \angle 2$.
12.93 Teacher: So… yours looks similar to whose in the…
12.94 Jeremy: This guy [Delbert]… That guy [Ryan]

Jeremy was operating on Level 3 as he produced a deductive text with explicit reference to the warrants about the alternate interior angle theorem and the exterior angle theorem to back up his reasoning.

**Summary of Jeremy’s Work:**

Jeremy started on Level 1, in the beginning of Task 1, related to the vertical angle theorem. He felt the need to present an argument but did not quite know how to present one. In the same task at a later stage, he began to understand the nuances of proof production as he observed others presenting valid arguments. He himself presented an alternate proof for the theorem, by rearranging the data presented by two others in a way that was easier for the remaining students to comprehend. He started to operate on Level 2 and continued to do so in the next few tasks. In Task 5, even though he was able to find the logical link between the data once it was set, he seemed not confident in setting up the data by himself. However by this time, he understood how the proof process worked. In Task 6 and 9, related to the consecutive interior angle and consecutive exterior angle theorems, he presented a valid proof but did not make the warrants explicit. He did not quite understand the need to provide justifications for his assertions. In Tasks 10 and 12, he was operating on Level 3 as he was able to produce deductive texts for the proofs. However in Task 11, he made a mistake in identifying the angles, a complication which might have arisen due to his imagination of a parallel line and inability to identify multiple angles at a single vertex.
Joey’s Work

Analysis of Task 1

The task was to give a proof for the vertical angle theorem. Earlier the students conjectured about the congruence of the vertical angle theorem and Joey’s idea about the nature of the vertical angles formed by two intersecting lines formed the basis for the conjecture. He laid the foundation for the conjecture of the vertical angle theorem by referring to specific measures and helped others to arrive at the conjecture that the vertical angles are congruent. At this stage, students resorted to measurement to provide a justification as they have not yet been channelized to give deductive arguments. So, it is natural that he started using measurements to support his claim that the vertical angle measures make two congruent acute and obtuse angle pairs. This provided the basis for others to generalize that the vertical angle pairs are congruent. But in this task, when the students were asked to prove the theorem, he did not add anything in terms of the content to the discussions that followed it. He was merely observing the ways in which others were presenting their ideas of proof. He was operating on Level 1 as he did not make an argument of his own. Either he was not sure how to present a logical argument or was not confident enough to do it on his own.

Analysis of Task 2

In this task, the students had to prove that pairs of alternate interior angles are congruent. They were required to prove that $\angle 6 \cong \angle 9$. 

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The students started to discuss and the discussion ensued in this manner:

2.113 Tommy: Okay. [Students start discussing. Julia and Delbert start discussing together, Tommy and Joey start discussing together.]
2.114 Joey: So the corresponding angles…
2.115 Teacher: And you can talk louder if you want, it doesn’t matter, because we can’t hear what you are talking. [Students come together]
2.116 Julia: Let’s all get in a group. It’s all in the…did you guys get it? [Asking the other two students]
2.117 Tommy: Are they vertical angles?
2.118 Julia: Yeah, they are vertical, but I meant the numbers and how they are equal to each other. \(\angle 6\) equals to \(\angle 8\); \(\angle 8\) equals \(\angle 1\); and \(\angle 1\) equals to \(\angle 9\).
2.119 Tommy: Hmm… [Nods]
2.120 Teacher: So what… did you guys find something?
2.121 Julia: Yeah. 2 ways.
2.122 Teacher: 2 ways. Okay. Can you come here, Julia? Can you come here and show us, like what you were… but you explained to them too right. Just say it loud, when you look at them, okay?

After that Tommy started to prove the theorem and the discussion continued from there in this way:

2.138 Tommy: I said they are vertical angles.
2.139 Teacher: Which ones are vertical angles? \(\angle 6\)…
2.140 Tommy: \(\angle 6\) and \(\angle 9\).
2.141 Teacher: \(\angle 6\) and \(\angle 9\)?
2.142 Joey: Well its \(\angle 9\) and \(\angle 8\). Or \(\angle 1\) and \(\angle 6\).
2.143 Teacher: Okay, you want to take \(\angle 9\) and \(\angle 8\)? Let us see \(\angle 9\) what is that? Is congruent to? Which one?
2.144 Joey: \(\angle 8\).
2.145 Teacher: \(\angle 8\)? Okay. What else do you have for me to write?
Tommy started with incorrect identification of the vertical angles. He started with angles, $\angle 6$ and $\angle 9$ which were required to be proven to be congruent. Joey corrected him and asked him to consider $\angle 9$ and $\angle 8$ or $\angle 1$ and $\angle 6$ (line 2.142). Even though Joey was able to recognize the different angles correctly, he found it difficult to link the data that he had; that $\angle 1$ and $\angle 6$ is a congruent vertical angle pair and $\angle 1$ and $\angle 9$ is a congruent corresponding angle pair. He could not provide the link from there to the claim that $\angle 6 \cong \angle 9$.

He was operating on Level 1 as he understood that an explanation is required, but did not understand the obligation for the explanation to be logically persuasive.

**Analysis of Task 5**

In Task 5, the students were required to prove the congruence of a pair of alternate exterior angles.
Figure 4.60: Alternate Exterior Angles (Second Pair)

The students as a group had to present the proof for $\angle 3 \equiv \angle 8$. The students chose Tommy to write on the board and the discussion flowed as follows:

5.29 Tommy: It’s easy. $\angle 3$ is equal to…
5.30 Delbert: $\angle 5$.
5.31 Tommy: $\angle 3$ and $\angle 1$.
5.32 Joey: $\angle 3$ is what?
5.33 Delbert & Joey: $\angle 3$ equals $\angle 5$ and $\angle 5$ equals $\angle 8$.
5.34 Tommy & others: There are two ways to put this.

Tommy wanted others to start with $\angle 3$ and $\angle 1$ (line 5.31) as the vertical angle pair. Delbert and Joey wanted him to start with $\angle 3$ and $\angle 5$ (line 5.33). The discussion continued:

5.35 Teacher: Good. Yeah. There are… there might be many ways right? Tommy, first write what they told you and then you write what you thought, okay?
5.36 Tommy. Okay. What did you say?
5.37 Students: $\angle 3$ equals $\angle 5$ and $\angle 5$ equals $\angle 8$.

Tommy with the help of others took few minutes to write the data on the board and it was Delbert who augmented the proof:

5.44 Delbert: $\angle 3$ equals $\angle 5$ and $\angle 5$ equals $\angle 8$.
5.45 Teacher: So what… what are you guys concluding from that. Guys you should try helping him write. [Addressing Tommy]
5.46 Delbert: $\angle 3$ is congruent to $\angle 8$.

Joey again was able to present the relevant data; $\angle 3 \cong \angle 5$ and $\angle 5 \cong \angle 8$ (line 5.33) that would lead to the final conclusion, $\angle 3 \cong \angle 8$. However, it was Delbert who completed the theorem. Tommy is still considered to be operating on Level 1 as he knew what conditions lead to the theorem but did not know the normative way to apply those conditions to get the desired result.

**Analysis of Task 7**

In Task 7, the students proved the consecutive interior angle sum theorem. They had to prove that $\angle 1 + \angle 2 = 180^\circ$ with reference to the diagram below:

![Diagram of consecutive interior angles](image)

Figure 4.61: Consecutive Interior Angles (First Pair)

Julia, Delbert, and Joey presented the proof together. The following conversation shows how they developed the proof to the theorem:

7.80 Delbert: $\angle 7$ and $\angle 2$ are linear pairs and $\angle 1$ and $\angle 8$ are linear pairs.
7.81 Teacher: Okay. Write them for me. Let me write them for you. You are saying $\angle 7$ and…?
7.82 Delbert, Julia, Joey: $\angle 2$.
7.83 Teacher: $\angle 2$ are linear pairs.
7.84 Julia: and $\angle 8$ and $\angle 1$
7.85 Teacher: $\angle 8$ and $\angle 1$ are linear pairs too? Okay. Where are you going from there?
7.86 Julia: Because we thought…
7.87 Delbert: Because they are corresponding angles [pointing to $\angle 7$ and $\angle 1$.] [Delbert draws an arrow from $\angle 7$ to $\angle 1$.]
Teacher: Aha.
Julia: Put the equal thing.
Delbert: Corresponding angles are congruent.
Teacher: Okay. So you are saying, because \( \angle 7 \) and \( \angle 1 \) are corresponding which one do you want to replace, in this one [\( \angle 7 + \angle 2 \) equals \( 180^\circ \)] or this one [\( \angle 8 + \angle 1 \) equals \( 180^\circ \)]? [The students point to \( \angle 7 + \angle 2 \) equals \( 180^\circ \)] In the top one? The top one becomes what? Joey, can you write that for me? What will the top one become now? You are replacing...

Joey: \( \angle 1 \) and the \( \angle 7 \).
Teacher: Okay. \( \angle 1 \)… [Joey writes \( \angle 1 \) plus \( \angle 2 \) equals.]
Teacher: Equals.
Joey: It’s [inaudible]… \( 180^\circ \).

Delbert, Julia and Joey started with the data; \( \angle 7 + \angle 2 = 180^\circ \) and \( \angle 7 \cong \angle 1 \). They backed them up with reference to the corresponding angle postulate and linear pair properties. Joey had a share in presenting the proof though he himself did not put out arguments of his own. He was acting as the writer for the group and he understood how the proof worked out. He was operating on Level 2 by this time, as he understood how the process of proof production works.

**Analysis of Task 8**

In continuation of the above task, the students were asked to prove that \( \angle 7 + \angle 8 = 180^\circ \) which was the other pair of consecutive exterior angles. Each student first worked out the proof on their papers and then presented their proof on the board. The students used the same figure. Joey presented the proof below:

Joey’s proof on the board:

\[
\begin{align*}
\angle 7 + \angle 2 &= 180^\circ \\
\angle 8 + \angle 1 &= 180^\circ \\
\angle 7 + \angle 1 &= 180^\circ
\end{align*}
\]
Joey presented the linear pairs; $\angle 7 + \angle 2 = 180^\circ$ and $\angle 8 + \angle 1 = 180^\circ$ and replaced $\angle 2$ with $\angle 8$ but he did not explain why he replaced them. He did not present the justification for the move either verbally or in written form. Also, it seems that the taken-as-shared meaning of representing congruent angles with arrows seemed to be influencing his representation. The warrant that was required in this case was the corresponding angle theorem which he did not make explicit.

Joey started to operate on Level 2 as he was able to set up the data and then link it to the claim. However, he presented only partial arguments aimed at the proof construction.

**Analysis of Task 11**

In Task 11, the students were required to present a proof to the exterior angle theorem. The students first worked independently, then presented their proofs on the board. Joey presented the proof below to prove that $\angle 4 = \angle 1 + \angle 2$.

![Diagram of Exterior Angle](image)

FIGURE 4.62: Exterior Angle

$\angle 3 + \angle 4 = 180^\circ$

$\angle 1 + \angle 2 + \angle 3 = 180^\circ$

So $\angle 4 = \angle 1 + \angle 2$

The linguistic complexity is the challenge in these cases. The student cannot say that $\angle 4$ is the supplement of $\angle 3$, and $\angle 1 + \angle 2$ is the supplement of $\angle 3$; therefore, $\angle 4$ must
be equal to $\angle 1 + \angle 2$. But one can see the sense of reasoning that Joey is exhibiting. He began operating on Level 3 as he produced a deductive text. Also he was able to understand the application of the triangle sum theorem in the context of a different proof.

**Analysis of Task 12**

In Task 12, the students were presented with the following problem.

![Figure 4.63: Parallel Lines Task](image)

They had to prove that, $\angle 3 = \angle 1 + \angle 2$ when the two horizontal lines are parallel to each other. The students presented their proofs on the board. Joey had the proof below:

Joeys’s diagram:

![Figure 4.64: Joey’s Parallel Lines Task](image)

Joeys’s proof on the board:

$\angle 4 + \angle 3 = 180^\circ$

$\angle 1 + \angle 2 + \angle 5 = 180^\circ$

$\angle 4 = \angle 5$

$\angle 1 + \angle 2 = \angle 3$
Later he explained it in the following way:

12.99  Teacher:  Let us give a chance to Joey to talk.
12.100  Joey:  $\angle 3 + \angle 4$ equal $180^\circ$ because they are on the same side. [He refers to the consecutive interior angles sum theorem.]
12.101  Teacher:  Wow.
12.102  Joey:  $\angle 1 + \angle 2 + \angle 5$ equal $180^\circ$ and these two are vertical angles [Pointing to $\angle 4$ and $\angle 5$] and all these add up to $180^\circ$ [Pointing to $\angle 1 + \angle 2 + \angle 5$]
12.103  Teacher:  Aha.
12.104  Joey:  [Pointing to $\angle 4$ equals $\angle 5$] these are vertical angles.
12.105  Teacher:  So… so you conclude what. You conclude that $\angle 3$ equals $\angle 1 + \angle 2$.

Joey started off with the linear pair, $\angle 4 + \angle 3 = 180^\circ$ (line 12.100) because earlier they proved that the consecutive interior angles on the same side of the transversal are supplementary. He then applied the triangle sum theorem to the angles, $\angle 1, \angle 2$ and $\angle 5$ (line 1.102). He also pointed out the $\angle 4$ and $\angle 5$ are vertical angles and since they are congruent in both, $\angle 4 + \angle 3 = 180^\circ$ and $\angle 1 + \angle 2 + \angle 5 = 180^\circ$. From that, he concluded that $\angle 1 + \angle 2 = \angle 3$. This was a unique proof. He thought of so many logical steps, arranged them sequentially in order to arrive at this proof.

He began to operate on Level 3 as he coordinated the elements of the argument in a way that is consistent with mathematically sound argumentation.

**Summary of Joey’s Work**

In Task 1, Joey started off on Level 1. He did not make an argument on his own as he might not have known how to do it. However, it seems that he felt an obligation to present it. In Tasks 2 and 5, related to the alternate interior angle and alternate exterior angle theorems, he succeeded in presenting relevant data that can be used to arrive at a valid conclusion, but was not quite capable of providing the logical link between the data and the claims. He seemed to recognize the need to apply previous theorems as part of
one’s argument, but did not know the normative fashion for doing so. However, he seemed to understand the shape of a proof. In Task 8, Joey started to operate on Level 2 as he moved beyond from just presenting data to provide the logical link between the data and claim. However, he presented only partial arguments aimed at the proof construction. He did not provide the justification explicitly for making that logical link possible. In Task 12, related to the parallel lines, he moved into Level 3 by presenting a complete proof to the theorem, by taking relevant data sets and then linking them to the final conclusion through a series of logical steps. He also justified his reasoning using specific reference to the application of earlier theorems.

Julia’s Work

Analysis of Task 1

Before Task 1, the students conjectured that the vertical angles were congruent. They then had to prove it. The lesson began with all of the students expressing their approaches to the task. The conversation turned to Carl’s idea of reflecting the figure. He observed the figure and said that each vertical angle pair was a reflection of the other angle.

1.25 Carl: I said it’s congruent because it looks like a reflection and all intersecting lines have at-least two pairs. [Angles which are equal]

Julia agreed with what Carl said. She added that if a mirror was used, one could see that the vertical angles were indeed congruent.

1.27 Julia: I agree… if you have a mirror, we can see that. I agree.

Both of the students in this task used spatial reasoning and visualization skills. They generated a mental representation and examined the properties of the figure. A mental representation plays an important role in the development of geometric ideas, and
is an important tool in understanding and interpreting geometrical concepts in relation to physical objects. In the early stages of proof production, it is common for children to refer to visual representations which enable them to give convincing explanations for their assertions. They rely on the intuitions they have of geometrical shapes that they’ve encountered previously, and tend to provide valid explanations from the thoughts rising from those intuitions. Also, geometrical transformations, like reflections, help them visualize the attributes of the representation without an actual measurement. They help students form a connection between physical objects and objects of reasoning.

Julia was operating on Level 1 of reasoning by exploring physical and mental representations. The use of mental imagery to justify an assertion can be considered one of the initial levels of reasoning.

**Analysis of Task 2**

In Task 2, the students were required to prove that pairs of alternate interior angles were congruent. The following figure was generated to help them in the process. The task was to prove that $\angle 6 \cong \angle 9$.

![Figure 4.65: Alternate Interior Angles (First Pair)](image)
The students started discussing in pairs until the teacher asked them to discuss as a group. Julia initiated the conversation by asking the others to join the group, and then explained her reasoning. She said:

2.116 Julia: Let’s all get in a group. It’s all in the…did you guys get it? [Asking the other two students]

She gave an explanation of her claim that the interior angles were congruent with reference to exterior angles. An excerpt from the transcript is presented to get a glimpse of her approach to the proof:

2.123 Julia: We are proving that $\angle 6$ equals $\angle 9$. This is what I say.
2.124 Teacher: Aha.
2.125 Julia: $\angle 6$ is congruent to $\angle 8$.
2.126 Teacher: Show us on the… show us on that picture, like the diagram.
2.127 Julia: We are proving that $\angle 6$ equals $\angle 9$. $\angle 6$ equals $\angle 8$ which is equal to $\angle 1$ and $\angle 1$ equals $\angle 9$.
2.128 Teacher: Wow. So can you tell us what are $\angle 6$ and $\angle 8$? So you said $\angle 6$ equals $\angle 8$ right? So what is $\angle 6$… what are $\angle 6$ and $\angle 8$? What are they called?
2.129 Julia: Corresponding angles.
2.130 Teacher: Oh… They are corresponding angles? So let me write them here. So angles… Okay. That makes sense.
2.131 Julia: And $\angle 8$ and $\angle 1$ are alternate exterior angles.
2.132 Teacher: But we didn’t prove them yet right? We have to prove them $\angle 6$ and $\angle 8$.
2.133 Julia: $\angle 6$ and $\angle 8$ are corresponding angles.
2.134 Teacher: Aha.
2.135 Julia: $\angle 8$ equals $\angle 1$ vertical…

Julia approached the proof using exterior angles which had not yet been proved to be congruent. The proof presented by Julia, using the alternate exterior angles, gave the teacher an opportunity to point out that the students could only use already proved axioms, theorems, and postulates. She also pointed out the distinction between a postulate and theorem, which was discussed earlier.
In this task, Julia began to organize her reasoning, but was not making explicit the threads of reasoning holding propositions together to get the desired outcome through the proper channel. She was operating on Level 1. The thing that was keeping her from functioning on Level 2 in this task was the fact that she was missing the importance of using proven axioms and theorems.

For example, in accordance with the flow of the tasks, the exterior angles were conjectured to be true, but not yet proven. Had the exterior angle theorem already been proved, Julia would be considered as operating on Level 2, because she started presenting a logical link between different statements using the rules of inference.

In continuation of the same task, when she was asked to prove the theorem without using alternate exterior angle pairs, she started off with corresponding pairs again, but then used vertical angle pairs which were already proved to be congruent. She said:

2.163 Julia: Since we have to prove \( \angle 6 \) equals \( \angle 9 \), \( \angle 6 \) equals \( \angle 8 \) right here and \( \angle 8 \) equals \( \angle 9 \).

As Julia paused to see the connection between the corresponding angles \( \angle 6 \) and \( \angle 8 \) and the vertical angles \( \angle 8 \) and \( \angle 9 \), Delbert picked up the thread and arrives at the conclusion that \( \angle 6 \) and \( \angle 9 \) were congruent.

2.181 Delbert: \( \angle 6 \) equals \( \angle 8 \) and \( \angle 8 \) equals \( \angle 9 \). So these two, [pointing to \( \angle 6 \) and \( \angle 9 \)] should be the same.

Here one can see the delightful classroom dynamic that went on between the students in the acquisition of knowledge that led to a collaborative construction of proofs.
Julia started to operate on Level 2 in the second frame of the task. She put up a logical chain of reasoning by linking the propositions together, but did not present the whole argumentation. She also missed on providing the justifications for her steps.

**Analysis of Task 4**

In Task 4, the students were required to prove that a pair of alternate exterior angles are congruent. They had the figure below:

![Figure 4.66: Alternate Exterior Angles (First Pair)](image)

The task was to prove that $\angle 7 \cong \angle 6$. The teacher asked one of the students in the group to be the writer. Julia asked the teacher if they could all work together on the task, and then present the task as a group. She said:

4.69 Julia: Can we do it together?
4.70 Teacher: You want to…?
4.71 Delbert: That’s what she showed me.
4.72 Teacher: Yeah… you are all talking together. Guys remember you are a group right?
4.73 Teacher: So you are talking together, and then once you talk, you can go there and show us how you proved it.
   [Students talking to each other]
4.74 Julia: $\angle 7$ equals $\angle 2$ and $\angle 2$ equals $\angle 4$ and $\angle 4$ equals $\angle 6$.
4.75 Tommy: Sounds [inaudible].
4.76 Julia: Yeah?
Julia first showed Delbert what she had done on her paper. She then showed it to the group when they sat together. Julia started with the corresponding angle pair, \( \angle 7 \) equals \( \angle 2 \) and alternate angle pair \( \angle 2 \) equals \( \angle 4 \) and concludes with the corresponding angle pair, \( \angle 4 \) equals \( \angle 6 \). She is approaching the proof using transitivity rule twice when she is linking; \( \angle 7 \equiv \angle 2, \angle 2 \equiv \angle 4, \angle 2 \equiv \angle 4, \angle 4 \equiv \angle 6 \) to arrive at \( \angle 7 \equiv \angle 6 \). Julia, Tommy, Joey and Delbert discussed near the board further and proved the theorem in a different way. The conversation resumes:

4.78 Delbert: \( \angle 2 \) equals \( \angle 7 \).
4.79 Julia: Yeah.
4.80 Delbert: \( \angle 7 \) equals \( \angle 2 \) and \( \angle 7 \) equals \( \angle 6 \).
4.81 Tommy: \( \angle 7 \) equals \( \angle 2 \)?

Delbert started with \( \angle 7 \) equals \( \angle 2 \) the corresponding angle pair and concluded that \( \angle 7 \) equals \( \angle 6 \). Julia asked the group to consider the combination of the angles, \( \angle 7 \) equals \( \angle 4 \) and \( \angle 4 \) equals \( \angle 6 \). Julia’s contribution to the proof was very significant as it is her selection of angles that lead the group to the proof. In one frame she presented two different ways of presenting the proof. The discussion is as follows:

4.82 Julia: \( \angle 7 \) equals \( \angle 4 \) and \( \angle 4 \) equals \( \angle 6 \)?
4.83 Tommy: Yeah.
4.84 Delbert: \( \angle 7 \) equals \( \angle 4 \) and \( \angle 4 \) equals \( \angle 6 \)?
4.85 Julia: Which one?
4.86 Joey: \( \angle 7 \) is vertical to \( \angle 4 \) and \( \angle 4 \)...
4.87 Delbert: You want me to write that down?
4.88 Tommy: Yeah.
4.89 Delbert: Okay. \( \angle 7 \) equals \( \angle 4 \) and \( \angle 4 \) equals \( \angle 6 \).
4.90 Teacher: So you are the representative right? You are just recording what you all thought of?
4.91 Tommy: Aha.
[Delbert writes]
4.92 Teacher: Guys when he is writing, you have to look whether he is writing what you want him to write. He wrote \( \angle 7 \) is congruent to \( \angle 4 \). Why is \( \angle 7 \) congruent to \( \angle 4 \)?
4.93 Students: Because they are vertical.
4.94 Teacher: They are vertical? Can you ask him to put that… in a bracket like in a parenthesis somewhere? Yeah. Vertical angles. Okay, and then he wrote $\angle 4$ is congruent to $\angle 6$. Guys, $\angle 4$ is congruent to $\angle 6$. Why?

4.95 Julia: [Inaudible] they look like each other.

4.96 Teacher: Julia, he wrote $\angle 4$ is congruent to $\angle 6$. So what was the reason behind writing that?

4.97 Julia: Because $\angle 6$ is equal to $\angle 4$ and $\angle 6$ is equal to $\angle 7$. So $\angle 6$ is equal to $\angle 7$. Which means $\angle 7$ is equal to $\angle 6$.

4.98 Teacher: Yeah… we got that, what are $\angle 4$ and $\angle 6$? That’s what I am asking. Tommy?

4.99 Tommy: Hah…?

4.100 Teacher: Why did you all put $\angle 4$ and $\angle 6$? What are they?

4.101 Tommy: Because they are corresponding angles.

4.102 Teacher: They are corresponding angles? Okay. Can you put corresponding angles for me there?

4.103 Delbert: Write across it?

4.104 Teacher: Yeah, put it in the parenthesis, yeah… because they are corresponding angles.

[Delbert writes]

4.105 Teacher: So what did you all conclude from that? He can be your speaker too, if you want. Can you tell us why… what you finally came to? The conclusion… can you tell us?

4.106 Delbert: Umm…

4.107 Teacher: He wants you to help him. If anybody wants to come there, they can come too. This is your group work, right?

[Students discuss again]

4.108 Delbert: We need the conclusion, right?

4.109 Teacher: Aha. Conclusion is? You said $\angle 7$ is congruent to $\angle 4$ and $\angle 4$ is congruent to $\angle 6$ right? So what did you conclude from that? Why?

4.110 Students: $\angle 7$ equal $\angle 6$ and $\angle 6$ equals $\angle 7$.

Here one can see the classroom dynamic between the students in the acquisition of knowledge. The dynamic involved a culture of doing mathematics in which students are geared toward appreciating each other’s ideas, being open to different approaches, and making sense of new ideas, in this case leading to a collaborative construction of proofs.
In this task, Julia continued to operate on the Level 2 still missing the important aspect of providing the required justifications for each step that make the argumentation more holistic.

**Analysis of Task 7**

In Task 7, the students were asked to prove that the consecutive interior angles were supplementary when two parallel lines are intersected by a transversal. The students did not conjecture about this, but the teacher presented the task to them. The below diagram was given for reference:

![Diagram](image)

**Figure 4.67: Consecutive Interior Angles (Second Pair)**

The students were required to prove that $\angle 1 + \angle 2 = 180^\circ$. The teacher asked Julia, Delbert, and Joey to come to the board and present a proof together. The following conversation shows the development of the argumentation for the proof of this theorem:

7.80 Delbert: $\angle 7$ and $\angle 2$ are linear pairs and $\angle 1$ and $\angle 8$ are linear pairs.

7.81 Teacher: Okay. Write them for me. Let me write them for you. You are saying $\angle 7$ and...?

7.82 Delbert, Julia, Joey: $\angle 2$.

7.83 Teacher: $\angle 2$ are linear pairs.

7.84 Julia: and $\angle 8$ and $\angle 1$

7.85 Teacher: $\angle 8$ and $\angle 1$ are linear pairs too? Okay. Where are you going from there?

7.86 Julia: Because we thought...

7.87 Delbert: Because they are corresponding angles [pointing to $\angle 7$ and $\angle 1$.] [Delbert draws an arrow from $\angle 7$ to $\angle 1$.]

7.88 Teacher: Aha.

7.89 Julia: Put the equal thing.
7.90 Delbert: Corresponding angles are congruent.
7.91 Teacher: Okay. So you are saying, because $\angle 7$ and $\angle 1$ are corresponding which one do you want to replace, in this one [$\angle 7$ plus $\angle 2$ equals 180°.] or this one [$\angle 8$ plus $\angle 1$ equals 180°.]? [The students point to [$\angle 7$ plus $\angle 2$ equals 180°.].] In the top one? The top one becomes what? Joey, can you write that for me? What will the top one become now? You are replacing…
7.92 Joey: $\angle 1$ and the $\angle 7$.
7.93 Teacher: Okay. $\angle 1$…
[Joey writes $\angle 1$ plus $\angle 2$ equals.]
7.94 Teacher: Equals.
7.95 Joey: It’s [inaudible]… 180°.

As the argument frame shows, the students slowly began to produce a deductive proof by making their arguments explicit with data like $\angle 7 + \angle 2 = 180°$ and $\angle 7 \cong \angle 1$.

They backed them up with the linear pair properties and corresponding angle pair congruence. Julia, being a part of that shared communication, is also considered to be approaching proof in a more advanced way. Julia was a part of the group that is approaching the proof at Level 3.

**Analysis of Task 8**

In Task 8, the students were asked to prove that the consecutive exterior angles were supplementary when two parallel lines were intersected by a transversal. They had the same figure for reference from the preceding task. Here, each student presented their proof on the board. Julia presented the below proof, including the warrants for her arguments in square brackets:

**Problem:** $\angle 7 + \angle 8 = 180°$

**Julia:**

<table>
<thead>
<tr>
<th>C:</th>
<th>$\angle 7 \cong \angle 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear:</td>
<td>$\angle 7$ and $\angle 2$</td>
</tr>
<tr>
<td>$\angle 2 \cong \angle 8$</td>
<td>$\angle 8$ and $\angle 1$</td>
</tr>
</tbody>
</table>

$\angle 7 + \angle 2 = 180°$

$\uparrow$

$\angle 1 + \angle 8 = 180°$ [switch.]

$\angle 7 + \angle 8 = 180°$

[C: Corresponding angles]

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Julia started to present a structured proof as the content on the board suggests. She started with what needed to be proved, \( \angle 7 + \angle 8 = 180^\circ \), listed the linear pairs and corresponding angle pairs that she thought could solve the problem and presented the logical link between what was already known to what needed to be proved by replacing \( \angle 2 \) with \( \angle 8 \).

The presentation of the proof clearly reflected that she was then operating on Level 4. She produced a written text organized in a deductive way.

**Analysis of Task 10**

The task was to prove that \( \angle 1 + \angle 2 + \angle 3 = 180^\circ \).

![Triangle Sum](image)

Figure 4.68: Triangle Sum

The theorem was discussed as a group with individual contributions to reach the final proof. In this task, after a discussion about the various properties of triangles, the teacher presented them with a triangle and an auxiliary parallel line passing to help them prove the theorem. Julia and Laila came to the board and discussed how they would prove that the sum of the angles in a triangle was 180°. They referred to the below figure as they proved. Their discussion was as follows:
Figure 4.69: Rylan’s Triangle Sum

10.81 Julia: Well… \( \angle 3 \) and \( \angle 7 \) equal 180°.
10.82 Teacher: Okay. He also wrote that, right?
10.83 Julia: Yes, but \( \angle 4 \) plus \( \angle 1 \) plus \( \angle 5 \) will be equal to 180 degrees too.
10.84 Teacher: Okay. Yeah write it \( \angle 4 \) plus \( \angle 1 \) plus \( \angle 5 \).
              [Julia writes][Laila and Julia whisper]
10.85 Teacher: You got it?
10.86 Laila: \( \angle 6 \) and \( \angle 2 \) would be 180 degrees too.
10.87 Teacher: Okay.
10.88 Laila: So that you can…
10.89 Tommy: Oh yeah.
10.90 Laila: Then \( \angle 1 \) plus \( \angle 2 \) plus \( \angle 3 \) equals 180°.
10.91 Julia: Well, you can swap around the numbers. The ones which are
              inside you can swap around them. [She points out to: \( \angle 2 \) and
              \( \angle 4 \)]
10.92 Teacher: Aha.
10.93 Julia: Like \( \angle 2 \) plus \( \angle 1 \) plus \( \angle 5 \) equals 180 degrees.
10.94 Laila: \( \angle 2 \) plus \( \angle 1 \) plus \( \angle 3 \).

Julia and Laila were very close to arriving at the proof, but somehow failed to
explain the switch of \( \angle 2 \) and \( \angle 4 \); \( \angle 5 \) and \( \angle 3 \) in clear terms. Ryan came to their aid at the
end. He helped them explain the proof in the following manner:

10.107 Ryan: Well… and \( \angle 4 \) and \( \angle 2 \) are exterior… alternate interior
            angles…
10.108 Teacher: Good. So you are replacing \( \angle 4 \) with \( \angle 2 \)?
10.109 Ryan: Yes.
10.110 Teacher: Okay.
10.111 Ryan: And \( \angle 6 \) and \( \angle 3 \) are also alternate interior angles.
10.112 Teacher: Aha.
10.113 Ryan: Replace the \( \angle 6 \) with the \( \angle 3 \), so… and it would be \( \angle 2 \) plus \( \angle 1 \)
              plus \( \angle 3 \)…
In the process of producing the proof for this theorem, Julia and Laila provided only partial arguments to link their data: \( \angle 3 + \angle 7 = 180^\circ \); \( \angle 1 + \angle 4 + \angle 5 = 180^\circ \); \( \angle 6 + \angle 2 = 180^\circ \) to the claim \( \angle 1 + \angle 2 + \angle 3 = 180^\circ \) in a systematic manner to produce a valid proof. They were not able to justify the switch between \( \angle 2 \) and \( \angle 4; \angle 5 \) and \( \angle 3 \), which were the alternate interior angle congruence pairs. In this task, both Julia and Laila were operating on Level 2 of reasoning.

**Analysis of Task 11**

In the task that involved the proof of the exterior angle theorem, each student again worked independently and provided an explanation on the board. Julia wrote the following data on the board as part of her proof with reference to the figure that she drew:

\[
\begin{align*}
\angle 1 + \angle 2 + \angle 3 &= 180^\circ \\
\angle 1 &\cong \angle 2 \angle 1 \cong \angle 3 \\
\angle 2 &\cong \angle 1 \angle 2 \cong \angle 3 \\
\angle 3 &\cong \angle 1 \angle 3 \cong \angle 2 \\
\angle 4 &\cong \angle 5 \angle 4 \cong \angle 6 \angle 4 + \angle 3 = 180^\circ \\
\angle 5 &\cong \angle 4 \angle 5 \cong \angle 6 \angle 2 + \angle 5 = 180^\circ \\
\angle 6 &\cong \angle 4 \angle 6 \cong \angle 5 \angle 6 + \angle 1 = 180^\circ
\end{align*}
\]

Figure 4.70: Julia’s Exterior Angle

As soon as she began to explain what she had written, Dalton and Ryan raised an objection to some of the data presented by her on the board. The discussion was as follows:

11.140 Dalton: I disagree.
11.141 Teacher: You disagree. Why do you disagree?
11.142 Dalton: Because…
11.143 Teacher: Okay. Alright go… He has a doubt about the congruence.
11.144 Dalton: I say that all these congruences aren’t right.
11.145 Julia: Don’t they look similar?
11.146 Ryan: I think what Dalton is trying to say is that you know how you said \( \angle 1 \) and \( \angle 2 \) equal each other.
11.147 Dalton: How she has, $\angle 3$ is congruent to $\angle 4$. $\angle 2 \ldots \angle 2 \ldots$ I say $\angle 3$ can’t be congruent to $\angle 4$. $\angle 2$ can’t be congruent to $\angle 3$ and $\angle 3$ can’t be congruent to $\angle 4$.

11.148 Ryan: Because they are off that side.

11.149 Julia: I know that, but $\angle 3 + \angle 4 = 180^\circ$ and $\angle 6 + \angle 1 = 180^\circ$ too, but, $180^\circ$ except that they are all… except that they are all equal.

11.150 Ryan: The first step…

[Teacher addressing a student]

11.151 Teacher: You are confused about that too? So let’s just go and sit down. But…

11.152 Julia: Well…

11.153 Teacher: But the thing that they asking, Julia is. They are all… guys… guys… you should listen to me, as I am trying to ask the question that you are asking me. Okay. They are saying. All of them are interior inside the triangle. They add up to $180^\circ$. But they cannot be individually congruent, right?

11.154 Julia: Oh.

11.155 Teacher: So, go and think about it.

Julia went back and thought for some time. She came back again after a few minutes and started the following conversation:

11.176 Julia: I see why you guys are confused. Let’s see $180^\circ$, right here. $180^\circ = \angle 1 + \angle 2 + \angle 3$.

11.177 Teacher: Okay.

11.178 Julia: Am I supposed to put an equal sign here?

11.179 Teacher: You already… yeah… you can put, it doesn’t matter.

11.180 Julia: So I am going to take $\angle 6$ and $\angle 1$.

[She writes $\angle 6 + \angle 1 = 180^\circ = \angle 1 + \angle 2 + \angle 3$]

11.181 Julia: Okay. They are both $\angle 1$s.

11.182 Teacher: So you mean, they are both $\angle 1$?

11.183 Julia: So both of these are equal together.

11.184 Teacher: Aha.

11.185 Julia: Look… these are… I got rid of this [referring to $\angle 1$], I got rid of this [referring to the other $\angle 1$] they would be equal.

11.186 Teacher: Guys. Oh… oh… guys, guys… [Students clap]

Initially, Julia laid down data that were not only irrelevant, but wrong as well. She is just getting her bearing with respect to the problem. This seems to suggest that she is not yet sufficiently grounded in the mature processes of argumentation to be able to coordinate exploring open-ended possibilities while maintaining the logical structure of
The objection from her peers brought back her focus to arrive at a justifiable proof. Once she got her ideas together, the correct flow of argumentation reasserted itself.

The reason for her poor reasoning in the beginning of the task is due to the lack of her understanding to recognize the relevant data. She knew that she had to put forth some data to present a proof. She did this as she knew the shape of a proof by now but she presented that data without giving much thought to the consequences of taking such a data set. The critique by others at that juncture in the proof production helped her to reflect on what she had presented. Then, she eliminated the extraneous irrelevant data and arrived at the proof. Her proof on the board is given below:

\[ \angle 6 + \angle 1 = 180^\circ = \angle 1 + \angle 2 + \angle 3 \]

[If we get rid of \( \angle 1 \)]

\[ \angle 6 = \angle 2 + \angle 3 \]

The students got excited that Julia was able to prove the theorem. At the end of this task, the students were asked to take a vote of the proof that they thought was more logical. Julia’s proof was one of those proofs.

Julia was operating on Level 3 of the framework. She produced a deductive text once the ideas of proof were brought to light. She understood how the conditions actually produced the result in a theorem.

In Task 8, where she presented a structured proof, she started with what was needed to be proved, and then proceeded to what was already known. She listed the statements that were true with reasons and presented the logical steps leading to the conclusion. Her proof presentation in that task nearly resembles the two column proof that is generally used in geometry. She was able to produce a textual organization, organized in deductive way as the framework describes. The same kind of written
organization cannot be seen in her current proof. The reason for this kind of shift was maybe due to the fact that in the earlier part of dealing with the task she made an association fallacy where she was relating the values looking at them superficially and making a judgment about the nature of the angle congruencies which led to the discord in her demonstration. As it shows, she had not yet developed the open-minded skepticism in seeking out the valid information sources. Once she realized that error through the critique offered by her friends, she strived to maintain the goal of arriving at the target through justifiable means as she did in previous tasks.

**Analysis of Task 12**

In Task 12 on parallel lines the teacher presented the following problem.

![Figure 4.71: Parallel Line Task](image)

Prove that: $\angle 3 = \angle 1 + \angle 2$ when the two horizontal lines are parallel to each other.

The students worked individually on this task then presented their proofs on the board. She drew the figure and presented this proof:

Julia’s proof:

$\angle 1 + \angle 2 + \angle 3 = 180$

$\angle 1 + \angle 2 = 180$

![Figure 4.72: Julia’s Parallel Line Task](image)
Julia in this task first modified the figure by adding a parallel line and a line segment to complete the triangle, as shown above. This manipulation of the figure created a dilemma for her regarding what she had at hand and what she had to prove. At one point during the task, she said that she was confused.

12.70 Julia: These are all triangles, the sum equals to 180 °.
12.71 Teacher: Aha.
12.72 Julia: ∠1 + ∠2 + ∠3 equals 180 °.
12.73 Teacher: Guys…
12.74 Julia: So… Umm… equal to 180 °. So both will be equal. ∠1 + ∠2 = ∠3.
12.75 Teacher: I have a question baby. You said first like ∠1 + ∠2 + ∠3 = 180° right? But in the next step, ∠1+∠2 = ∠3. How is it possible? Did you guys understand what I asked? She said ∠1 + ∠2 + ∠3 = 180° and then she says ∠1 + ∠2 = ∠3.
12.76 Julia: I think my mind is confused.

An observation that was made in this case, and in a few other cases, was that the students tended to lose focus on the proof production when they were trying to manipulate the original figure to create a figure of their own. The students tried to find new meaning from the figure that they developed, and it had a tendency to throw them off guard and confuse them. In Julia’s case the difficulty in understanding the new constructions may have been caused by the misreading of the angle labels. The convention of using a single letter to name an angle is valid when the vertex has only a single angle; however, when she constructed additional lines, this resulted in multiple angles at a vertex. Her error was consistent with having misread which angle is indicated at these more complex vertices.

This can be inferred from Julia’s figure. She formed a triangle by joining the parallel lines. She then considered ∠1 to be the whole angle at the newly formed vertex.
even though it was a portion of that angle. This can be seen in the data that she presented as a part of her proof; ∠1 + ∠2 + ∠3 = 180°.

This was one of the tasks that some students found difficult, like the interior angle sum theorem. Julia was operating on the Level 2, as she was trying to apply previous theorems, but the appropriate structures for coordinating those elements into a logical argument are not evidenced.

**Analysis of Task 13**

In Task 13 the students all presented their proofs to show that “the diameter that is perpendicular to the chord bisects the chord”. They had to prove that AC = BC from the figure presented below:

![Figure 4.73: Circles Theorem-1](image)

Figure 4.73: Circles Theorem-1

She joined the radii OA and OB and used symbols to show that the corresponding sides and angles were congruent. Given below is Julia’s proof and figure:

Also it’s a bisector (note: bisectors split shapes in half). Equally…

AC = BC.

![Figure 4.74: Julia’s Circles Theorem-1](image)

Figure 4.74: Julia’s Circles Theorem-1
She explained the proof referring to the SAS congruence.

13.90 Julia: When I came up to add these.
[She was referring to the radii that she drew]
13.91 Teacher: Aha.
13.92 Julia: I thought everybody had like triangles and right angles and Ms. Indira showed us about congruence. I used SAS.
13.93 Teacher: Aha. Wow. You used SAS there?
13.94 Julia: Because they are congruent, right?
13.95 Teacher: Aha. So you used that property to show that these two are congruent?
13.96 Julia: Aha.
13.97 Teacher: That looks reasonable.
13.98 Julia: Also [since] they are 90° angles, the diameter cuts half equally and is a bisector.
13.99 Teacher: Aha... It becomes a bisector too.
13.100 Julia: It is also is.

She continued to operate on Level 2 of reasoning. Proof-wise, she did not write much on the board, but the figure and the geometrical symbols used by her in the figure reflect the ideas and connections that she presented and even though she referred to the SAS congruence, she actually pointed out to the sides and the right angle, which describes the SS90° congruence.

**Analysis of Task 14**

In Task 14, the students were required to prove that $OX = OY$ based on the below figure, when the given chords are of equal length and a perpendicular line is drawn through the center:

![Figure 4.75: Circles Theorem-2](image)

Figure 4.75: Circles Theorem-2
All the students again worked individually and presented their proofs on the board. Julia made an additional diagrammatic representation to support her assertions.

![Diagram of Julia's Circles Theorem](image)

**Figure 4.76: Julia’s Circles Theorem-2a**

**Figure 4.77: Julia’s Circles Theorem-2b**

OXB and COY are congruent triangles.

She joined the points C and B, and drew a line through ‘O’ which bisected the two chords perpendicularly. She referred to the line BC as the transversal to the two parallel chords. She formed two congruent triangles OXB and COY. She explains her proof in the following manner:

14.38 Teacher: Julia seems to be very excited. Let me see what she says.
14.39 Julia: Umm. Okay. I was thinking to draw a line through it.
14.40 Teacher: Aha.
14.41 Julia: And since these two are parallel…
14.42 Teacher: Aha.
14.43 Julia: You see this is outside the circle over here the diameter, here it’s a transversal.
14.44 Teacher: Wow.
14.45 Julia: These angles over here are equal. [Referring to the alternate interior angles formed when she considered the chord to be parallel and vertical angles]
14.46 Teacher: Aha.
14.47 Julia: And… that’s it.

She referred to the diameter as the transversal to the two parallel lines and pointed out to the radii being the same in both triangles. She also pointed out to the congruence of the vertical angles; COY, BOX and the alternate interior angles; OCY, OBX formed when
the diameter intersected the parallel lines referring back and forth to the two figures that she drew on the board. However, she did not make the ASA congruence explicit. Julia was operating on Level 2 of reasoning.

**Summary of Julia’s Work:**

Julia began approaching proof through mental imagery in the episodes where the students proved the vertical angle theorem. She referred to reflection as a means of proving that the vertical angles were congruent. This reference to transformations was considered to be Level 1 of reasoning as the argumentation was just based on perception alone. After she began to break away from this kind of thinking as a means of presenting a proof, she slowly began to understand the shape of a proof but did not completely comprehend of what a proof constitutes in the tasks related to the alternate interior and exterior angle theorems. It slowly dawned on her that she should be using previously proved results in setting up the data. She then proceeded to tasks related to consecutive interior angles on the same side of the transversal where she along with others, presented a proof with relevant data linked to the claim through deductive reasoning. They also provided justification to their assertions. Being now able to see what a proof constitutes of, in Task 8, related to the consecutive exterior angle theorem, she produced a structured proof which was in close resemblance to any kind of formal proof used generally in the mathematics community. In tasks related to parallel lines, she altered the given figure and failed to present the proof from it. In the tasks related to the theorems on circles, she explained her proofs diagrammatically, incorporating symbols that represent congruent sides and angles that showed the creative potential in her. However she did not make the
warrants explicit in these tasks that can be attributed to Cobb’s, “taken-as-shared” ways of doing things which permit a certain degree of inexplicitness.

Also through Julia’s words and actions, during the tasks, she tried to bring out a dynamic in classroom interaction by encouraging others to join the discussions, and making them feel that they were a part of the learning community where discoveries were made through shared meaning. The dynamic interaction mentioned above also involves a culture of doing mathematics in which students are geared toward appreciating each other’s ideas and being open to different approaches. There is no doubt that Julia benefitted from the constant support and critique of her arguments during the tasks.

**Laila’s Work**

**Analysis of Task 1**

The task was to prove that the vertical angles are congruent. All the students started to discuss as a group and presented their ideas on the board. When it is Laila’s turn, she explained her approach:

1.40 Laila: Okay. This is what I say. I drew a box… because in my 4th grade, my teacher said that we can draw a perfect square… its 90°.

The teacher then drew the figure for Laila:

![Figure 4.78 Laila’s Vertical Angles](image)

1.41 Teacher: Like this?
1.42 Laila: Yes… that’s not a square.
She questioned the teacher that it did not look like a square.

1.43 Teacher: It’s okay. This is 90°. Okay.
1.44 Laila: That’s 90° and this one isn’t, right?

What she referred to was the symbol to represent a right angle ‘\( \perp \)' as shown in books. The teacher in this instance actually misunderstood what Laila was intending to say. Laila was trying to use the symbol for the right angle rather than a square at the intersection. So the teacher questioned her:

1.45 Teacher: You said… you wanted me to draw a square.
1.46 Laila: Yes.
1.47 Teacher: Everything is a 90°. Yah…I drew a square for you.

The teacher pointed out to the edges of the square which are right angles.

1.48 Laila: Well… then they are both equal. They are both equal. Isn’t it?

Even at this time, the teacher did not clearly understand Laila’s underlying assumption that the symbol for the square itself was the 90° angle. So the teacher continued:

1.49 Teacher: We are measuring [Pointing to the center of intersection of the two lines] Not this one. [Pointing to the edges of the square drawn]
1.50 Laila: Oh… [Stomps her foot]

In this task, Laila intended to use her knowledge of the symbol for a right angle to form a square (when the lines are perpendicular to each other) and she actually was talking about the case when all the angles at the intersection were right angles represented by four squares to show that the vertical angles were congruent. This was according to what her teacher in an earlier grade said about a square being used as a symbol for a right angle. Even though the teacher misunderstood Laila’s thinking in this instance, Laila was exploring the theorem at a very basic level as she was just using a symbol and a single
measure to prove a theorem. She also cited an authority to justify her argument. Clearly she was operating on Level 1 of reasoning.

In the later part of the discussion about the proof of the vertical angles theorem, after different students presented their ideas, Dalton and Jeremy with the help of their friends provided the class with a valid logical argument to prove two different pairs of vertical angles to be congruent. Laila said that she had a different idea to prove the theorem. The discussion unfolded in the following manner as the teacher gave Laila a chance to present her proof. She was referring to the figure below as she gave her explanation to prove $\angle PRQ \cong \angle SRT$:

![Figure 4.79: Vertical Angles](image)

1.147 Laila: $\angle PRQ$
1.148 Teacher: $\angle PRQ$. Guys, you should be seeing your friend, what she is doing.
1.149 Laila: $\angle PRQ$
1.150 Teacher: Aha… [Laila writes $\angle PRQ$ plus $\angle SRQ$ equals to $180^\circ$].
1.151 Teacher: Okay guys. Did you see the difference? This is not much difference, but still there is a difference. Okay. Laila, why did you put that?
1.152 Laila: Because it’s a linear pair.
1.153 Teacher: It is a linear pair. Very good. So what else? What else are you going to give us?
1.154 Laila: $RT$
1.155 Teacher: Are you doing the first one or the second one?
1.156 Laila: Both…
1.157 Teacher: Are you going to this one [pointing to the first pair, $\angle PRQ \cong \angle SRT$] or this one [pointing to the second pair $\angle QRS \cong \angle PRT$]
[There were actually two vertical angle pairs on the board: \( \angle PRQ \cong \angle SRT \) and \( \angle QRS \cong \angle PRT \) – Laila points to the first pair]

1.158 Teacher: This one. Okay. Show them.
1.159 Laila: Equals to [Inaudible]
1.160 Teacher: Aha…
1.161 Laila: \( \angle SRT + \angle TRP \ldots \)

After Laila wrote \( \angle PRQ + \angle QRS = 180^\circ \) and \( \angle SRT + \angle TRP = 180^\circ \) on the board, Dalton made a suggestion. Laila moved away from the board to give him room to explain what he wanted to say. He proceeded in the following manner:

1.165 Dalton: It could be \( \angle TRP \) and… yeah… it could be \( \angle TRP \), but this is what I say. Since you have \( \angle TRP \) here and \( \angle SRT \) here [Referring to the second equation] \( \angle QRS \) here [Referring to the first equation] … you should have \( \angle SRQ \) in… [Referring to the second equation].

The reason he was suggesting \( \angle SRQ \) in both equations was to ensure that both of them had a common angle. Laila immediately understood what he was referring to and changed the second angle pair to \( \angle SRT + \angle SRQ = 180^\circ \). This suggestion played a crucial role in the process of the production of this proof. This interjection showed that the other students were listening intently to what their friends were saying and in turn giving valuable suggestions at the right juncture to help the prover arrive at a justifiable conclusion. The discussion resumed as Laila completed her proof:

1.169 Teacher: Oh… Good. It’s \( \angle SRQ \). Wow… Okay, both are linear pairs. But I don’t know where she is going to go to. Tell us where you are going to go to. Tell us clearly.
1.170 Laila: So… like he said, these two are common. [Circling both 180°s]
1.171 Teacher: Okay.
1.172 Laila: These two are common [Circling both \( \angle SRQ \)s]. So, both are equal.
1.173 Teacher: Wow… So let me put in other words what she said. She says these two are common in this equation, in these two equations and 180° is common to both of them this is the same way that they said but she put it in different way. That’s the only thing.
What happens? What is the final thing Laila? Wow …this is very good reasoning. So, what can you say?

Laila: \( \angle PRQ \) is congruent to \( \angle SRT \).

Laila finally proved the theorem using linear pairs. She concluded that since the linear pairs had two angles in common, \( \angle SRQ \) and 180°, the other two angles namely, \( \angle PRQ \) and \( \angle SRT \) were congruent.

At this point, Laila was operating on Level 2 as she organized her reasoning and made explicit the threads of reasoning, holding propositions together with a little help from her peers. She was also able to check conditions and apply the results to make a valid argument.

After Laila presented her proof for the first pair of vertical angles, the teacher asked Delbert to prove the other pair and told Laila to stand near him to see what he was doing. In this episode, Delbert came to the board to prove the other pair and Laila guided him in the process:

Laila: So Delbert, the first thing… [Laila and Delbert consult each other and write]

Teacher: You always look at your figure so that it makes it easier for you.

Laila: Now \( \angle QRS \) plus \( \angle SRT \). What are you going to do? [Delbert writes] And since they are a linear pair they equal…

Delbert: 180°

Laila: Okay and write the other one. [Delbert writes] and write the other linear pair that one. [Delbert writes \( \angle PRT \) plus \( \angle PRQ \) equals…]

Both the students discussed and wrote \( \angle QRS + \angle SRT = 180° \) and \( \angle PRT + \angle PRQ = 180° \). Dalton again intervened:

Teacher: Dalton has a question there.

Dalton: He made the same mistake. We are trying to prove that they are congruent. So it would be better if \( \angle SRT \) is there on the top and bottom.

Teacher: Wow. [Laila and Delbert correct the equation]
Laila: To have something in common. [Delbert writes]  
Teacher: What do you conclude from that? Tell us what you conclude Delbert? So you have two equations here, right? Laila you should be helping him because you did that. So help him and make him understand what you wrote.  
Laila: This is what… this is what I did. They both equal to 180°, so they are both… and they share ∠SRT.  
Teacher: ∠SRT? Therefore…  
Laila: Therefore these two must be equal. [Pointing to ∠QRS and ∠PRT]

Delbert and Laila came up with the data ∠QRS + ∠SRT = 180° and ∠PRT + ∠SRT = 180° and concluded that ∠PRT ≅ ∠QRS by excluding the angles which were in both the linear pairs. Laila operated on Level 2 as she was searching for a coherent link between promising propositions and the result but was missing some parts of the argumentation, which were filled in by others. However, she was able to recognize her mistake as soon as it was pointed out and rectified it to arrive at a justifiable conclusion. This negotiation of meanings played a major role in the process of learning.

**Analysis of Task 2**

In this task, the students were required to prove that alternate interior angles are congruent. The following figure was generated to help them in the process of proving.

The specific task was to prove that ∠6 ≅ ∠4.

![Figure 4.80: Alternate Interior Angles (First Pair)](image-url)
Laila and the group started by approaching the proof using the concept of corresponding angle pairs because they thought it would be useful to start with information already at hand. Laila expressed this when the teacher questioned why she wanted to use corresponding angles:

2.45  Laila: [to another student] Now look at the corresponding angles. See the corresponding angles are always congruent. See if we could use that information.

2.46  Teacher: Talk a bit louder.

Dalton, Jeremy and Ryan were discussing together at the same time:

2.47  Dalton: Corresponding angles are $\angle 4$, $\angle 4$ is congruent.

The teacher asked the two groups to join together and the discussion resumed:

2.49  Teacher: You want to look at the corresponding angles?
2.50  Laila: To gather some information. […]
2.54  Laila: Look at the information in the corresponding angles.
2.55  Teacher: Look for information in the corresponding angles? Okay, what information do you see in the corresponding angles you want me to put here?

Automatically Laila and Dalton started to think of what information was already at hand that might be relevant to their goal. The students considered $\angle 4 \cong \angle 8$ as the first data set.

2.56  Students: $\angle 4 \cong \angle 8$
2.57  Teacher: $\angle 4 \cong \angle 8$, so you want to look at this one? $\angle 4 \cong \angle 8$. You wanted to take that one? Okay. These are congruent? Why? They are corresponding angles, right?
2.58  Students: Yes.
2.59  Teacher: Okay, What else do you want me to write? [Darren interrupts about something]
2.64  Students: $\angle 1$ and $\angle 6$.
2.65  Teacher: $\angle 1$ and $\angle 6$, Okay, those are also, what? What kind of angles?
2.66  Students: Corresponding.

Here, Darren asked them to take $\angle 3$ and $\angle 7$ along with $\angle 1$ and $\angle 6$, to which Laila and Dalton objected:
Laila: We are, like, trying to find out how $\angle 4$ and $\angle 6$ are congruent, right?

Teacher: $\angle 4$ and $\angle 6$ are congruent, right.

Laila: Why do you pull out, what do you get? What do you get from $\angle 3$ and $\angle 7$?

Teacher: Think... think some more.

The discussion resumed and Dalton said he noticed something:

Dalton: Ms. Indira, I just noticed this...

Teacher: Come show me what you noticed. Ssh... give them a... give them time to think. Aha.

Dalton: $\angle 6$ and $\angle 8$.

Teacher: Oh, you wanted $\angle 6$ and $\angle 8$. Guys, he says $\angle 6$ and $\angle 8$. What are those?

Dalton: They are... I think they are...

Laila: Interior...

Dalton: I really don’t know...

Teacher: $\angle 6$ and $\angle 8$

Laila: Interior? Vertical...

Both Dalton and Laila came up to this point and started pondering their next step.

Ryan provided them with the logical link that connected the data $\angle 4 \cong \angle 8$ and $\angle 6 \cong \angle 8$ to the claim $\angle 6 \cong \angle 4$ using the transitive property of relations. He concluded:

Ryan: $\angle 4$ and $\angle 8$ is congruent, and $\angle 6$ and $\angle 8$ is congruent.

So, they both equal, so $\angle 6 \cong \angle 4$.

The proof production was a group effort in this case as the input to the argumentation came from different students other than the ones who came forward to prove the theorem. While they were pondering the next step some other students thought of the next step ahead of them and completed the argument to prove the case.

Laila was a part of the group operating on Level 2 as she was able break down the application of prior theorems and postulates to check conditions and was able to arrive at the result based on those conditions with group effort.
Analysis of Task 3

The students used the same figure from the above task to prove that the other pair of alternate interior angles was congruent (i.e., $\angle 5 \cong \angle 7$). The students together presented the following data as part of the proof. They had $\angle 3 \cong \angle 7$, $\angle 5 \cong \angle 9$, and $\angle 7 \cong \angle 9$. Laila wanted to present the remaining argument and did it in the following manner:

3.29 Laila: If these are the same, like these two [Pointing to $\angle 7 \cong \angle 9$ and $\angle 5 \cong \angle 9$] congruent… and this has… [Showing $\angle 9$] in both of them… [Pauses for a second]
3.30 Students: Ooh… Yeah…
3.31 Teacher: She is doing it, right?
3.32 Laila: And $\angle 7$, $\angle 7$ equals $\angle 9$ and $\angle 9$ equals that… [Pointing to $\angle 5$]. So both of them are congruent [Pointing to $\angle 5 \cong \angle 7$]

Here she was able to make the logical link between the data and the claim and made a cogent argument. Laila was operating on Level 2 as she was organizing her reasoning using laws of inference, in this case using the law of transitivity, in explaining that $\angle 7 \cong \angle 9$ and $\angle 5 \cong \angle 9$ and pointing to the fact that since $\angle 9$ was common to both the pairs of angles, she was concluding that $\angle 7 \cong \angle 5$ which was a valid conclusion.

Analysis of Task 6

In Task 6, it was required of the students to prove that the consecutive angles on the same side of the transversal are supplementary. They were given the figure below for reference:

![Figure 4.81: Consecutive Interior Angles (First Pair)](image-url)

Figure 4.81: Consecutive Interior Angles (First Pair)
The students were required to prove that \( \angle 3 + \angle 5 = 180^\circ \). Dalton came forward and tried to present his proof. He started with \( \angle 4 \) and \( \angle 3 \) as the linear pair and \( \angle 4 \) and \( \angle 5 \) as the congruent corresponding angle pair. When the teacher asked him to write that down, Laila wanted to continue the proof from there. The teacher gave her a chance and her explanation followed in this way:

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At this point, Jeremy also raised his hand and arrived at \( \angle 3 + \angle 5 = 180^\circ \) from the data presented by Dalton. Laila on the other hand could not arrive at that conclusion because she presented too much data in terms of \( \angle 4 + \angle 3 = 180^\circ \) and \( \angle 5 + \angle 6 = 180^\circ \) as linear pairs and \( \angle 4 \cong \angle 5 \) and \( \angle 5 \cong \angle 6 \) and got confused after that. Had she just concentrated on the data provided by Dalton, she would have seen the link between the angles in the linear pair and the corresponding angle pair.
Laila was operating on Level 1 as she failed to eliminate extraneous data that inhibited her from arriving at a logical conclusion. She had a target in mind to prove that $\angle 3 + \angle 5 = 180^\circ$ but lost focus on attaining the target through acceptable mathematical reasoning.

**Analysis of Task 7**

Task 7 was similar to Task 6. Using the same figure, the students were required to prove that $\angle 1 + \angle 2 = 180^\circ$. Laila gave it a try and proved the theorem. Her explanation was as follows:

7.1 Teacher: Laila wants to try, okay.
7.2 Laila: [Pointing to the $\angle 7$ and $\angle 2$] These equal to 180°. [Pointing to $\angle 1$ and $\angle 8$] These equal to 180°.
7.3 Teacher: Aha.
7.4 Laila: Okay, $\angle 7$ and $\angle 1$ are corresponding angles. [Darren interjects and after that the discussion resumes.]
7.17 Teacher: Wait… wait, listen to her.
7.18 Laila: If you change $\angle 2$ with $\angle 7$, they would be…
7.19 Dalton: $\angle 1$ with the $\angle 7$
7.20 Laila: Yeah… we change $\angle 1$ with the $\angle 7$.

Laila started to operate on Level 3. She slowly brought out relevant data like $\angle 7 + \angle 2 = 180^\circ$ and $\angle 7 \cong \angle 1$ to prove the theorem at hand and linked them in a logical and deductive way to arrive at the result that $\angle 1 + \angle 2 = 180^\circ$. At the same time, she backed her assertions with warrants like linear pair properties and corresponding angle congruence. She also understood how the conditions actually produce the result in a theorem.

**Analysis of Task 8**

The task was to prove that $\angle 7 + \angle 5 = 180^\circ$ using the figure below as reference:
The students worked together again in proving the theorem. Ryan came to the board and started the proof with the help of Laila and Dalton, but at one point, Laila came and presented her proof. Laila started with two linear pairs: $\angle 7 + \angle 2 = 180^\circ$ and $\angle 6 + \angle 5 = 180^\circ$. The discussion flowed in this way:

8.33 Laila: Okay. I will write this one over here. [She writes $\angle 7 + \angle 2 = 180^\circ$]
8.34 Teacher: Ssh…
8.35 Dalton, Ryan: That’s 180°.
8.36 Laila: Oh my god… $\angle 6 + \angle 5$ equals 180°. [She writes $\angle 6 + \angle 5 = 180^\circ$]
8.37 Teacher: Ssh…
8.38 Laila: Okay. So now, you swap the $\angle 2$ and $\angle 5$ up.
8.39 Teacher: $\angle 2$ and the $\angle 5$. Show, show us the diagram. Can you… can you just move so that we can see the diagram. Swap the $\angle 2$ and the $\angle 5$. Okay. Why are swapping the $\angle 2$ and the $\angle 5$?
8.40 Laila: Because, these are linear pairs, they are equal to 180°.
8.41 Teacher: Aha.
8.42 Laila: So swap these two.
8.43 Teacher: Okay, you are swapping… we are… we are… guys, you ask them to be more clear or ask them to explain so can you…
8.44 Darren: Can you explain a little better?
8.45 Teacher: Yeah…
8.46 Laila: These two are linear pairs, like 180°. Okay.
8.47 Teacher: Okay.
8.48 Laila: And they both equal 180°. You can swap $\angle 5$ and the $\angle 2$.
8.49 Teacher: How can you swap them like… if they are equal to 180°? [Darren asks something.]
8.50 Teacher: Darren, ask her once again.
8.51 Darren: When you say switch. Like you don’t mean the other way around or?

8.52 Laila: Change the \( \angle 2 \) to the \( \angle 5 \).

8.53 Teacher: Oh… change the \( \angle 2 \) to the \( \angle 5 \). We… we understood that, but why are you changing the \( \angle 2 \) and the \( \angle 5 \)? How can you switch the \( \angle 2 \) and \( \angle 5 \)? That’s the question.

8.54 Laila: Because these are both linear pairs.

8.55 Teacher: Okay. Somebody… we have to be… you want to try? Okay. [Dalton raises his hand] Dalton try.

8.56 Dalton: I say… replace the \( \angle 2 \) with the \( \angle 5 \) because they are… corresponding angles.

Laila explained almost everything but failed to answer the teacher’s question about swapping \( \angle 2 \) and \( \angle 5 \). She actually produced the proof of the theorem, but what the teacher was trying to do in this episode was to make the other students see why she was switching those angles. She knew that they were corresponding angles but lost track of that fact. Dalton came to her aid and produced the warrant to finish off the theorem with a justifiable conclusion.

Laila was again operating on Level 2. She was able to break down the application of a prior theorem to check conditions and apply results but was presenting only partial arguments and leaving some arguments for the listeners to infer.

**Analysis of Task 9**

Task 9 was similar to Task 8 in that the students had to show that the other pair of exterior angles on the same side of the transversal is supplementary. Specifically, they had to prove that \( \angle 4 + \angle 9 = 180^\circ \). The teacher directed the students to work individually and present their proofs on the board. Laila presented the following proof on the board:

\[
\angle 4 + \angle 3 = 180^\circ \\
\angle 1 + \angle 9 = 180^\circ \text{ (Corresponding angles)} \\
\text{So } \angle 4 + \angle 9 = 180^\circ
\]
Her intention was that ∠3 and ∠9 were the corresponding angles, but she did not make this explicit in her written proof. She was still operating on Level 2, as the proof contained only partial arguments rather than a robust justification.

However, in the same task she said that she could prove the theorem in a different way. This is how the second proof unfolded:

9.78 Laila:  Okay. ∠7 is congruent to ∠9 and ∠4 is congruent to ∠5.
9.79 Teacher:  ∠4 and ∠5. Where are ∠4 and ∠5? Okay. ∠4 and ∠5. Good. Let me write for you. So that you just, just tell me, I will write… She said… ∠7 is congruent to ∠9; ∠5 is congruent to ∠4. Okay. Okay. Let her tell and then…
9.80 Laila:  Okay. Dalton you can probably help because…
9.81 Teacher:  These are exterior.
9.82 Laila:  Angles.
9.83 Teacher:  Alternate exterior angles are congruent Okay. I got it. Guys observe, she is thinking deeply. Yeah.
9.84 Laila:  When they are congruent, you can write the ∠4 plus ∠7.
9.85 Teacher:  ∠4 and the ∠7. Oh… you want to write ∠4 plus ∠7.
9.86 Teacher:  Oh… my goodness. Guys look… there are lots of ways of proving all this, right… so she thought of one way… Very good. She said… ∠4 plus ∠7 equals 180° and then she replaced ∠7 with the…
9.87 Student:  Oh… Okay.
9.89 Student:  ∠9…
9.90 Student:  Oh… Yeah…
9.91 Teacher:  Very good Laila… very good thinking. Give her a clap. [Students clap.]

Here, Laila started by taking the alternate exterior angle pairs ∠7 ≅ ∠9 and ∠4 ≅ ∠5 along with the linear pair ∠4 and ∠7 and switched ∠7 with ∠9 to arrive at the valid conclusion that ∠4 + ∠9 = 180°.

In this task, Laila was operating on Level 3. Once the ideas of the proof were brought to light, she presented them in a deductive manner and provided warrants to back up the data and the assertions that she was making along the way.
Analysis of Task 10

Before this task, the teacher had a discussion about the properties of a triangle in general. One of the properties the students articulated was about the sum of the interior angles of a triangle. The teacher then presented them with a triangle and an auxiliary parallel line passing through the vertex opposite to the base of the triangle to help them in proving the theorem. The students did this task as a group with each individual student making a contribution to the argumentation. Ryan presented his ideas first. He drew the figure below:

Figure 4.83: Ryan’s Triangle Sum

and wrote: \( \angle 6 + \angle 2 = 180^\circ \), \( \angle 3 + \angle 7 = 180^\circ \) and \( \angle 4 + \angle 1 + \angle 5 = 180^\circ \) and then added that since \( \angle 4 \) and \( \angle 3 \) and also \( \angle 5 \) and \( \angle 2 \) were alternate interior angles, he would replace \( \angle 4 \) with \( \angle 3 \) and \( \angle 5 \) with \( \angle 2 \) to get \( \angle 3 + \angle 1 + \angle 2 = 180^\circ \). The teacher raised an objection that \( \angle 3 \) and \( \angle 4 \) were not alternate interior angles. At that point, Laila interjected and said:

10.63 Laila: Oh… I knew something. \( \angle 4 \) is not interior… it’s \( \angle 4 \) and \( \angle 2 \).

Ryan admitted what she said was true and corrected his argument. Laila then said she had a similar type of argument. After a brief discussion with others, she drew the figure below, but did not know how to arrive at the proof from that figure:
Figure 4.84: Laila’s Triangle Sum

Julia came to help her but again referred to Ryan’s figure and they both started discussing the problem with reference to the earlier figure, as follows:

10.81 Julia: Well… ∠3 and ∠7 equal 180°.
10.82 Teacher: Okay. He also wrote that, right?
10.83 Julia: Yes, but ∠4 plus ∠1 plus ∠5 will be equal to 180 degrees too.
[She writes ∠1 + ∠4 + ∠5 = 180°]
10.84 Teacher: Okay. Yeah, write it ∠4 plus ∠1 plus ∠5.
[Julia writes ∠1 + ∠4 + ∠5 = 180°]
[She and Julia talk in low tones]
10.85 Teacher: You got it.
10.86 Laila: ∠6 and ∠2 would be 180° too.
10.87 Teacher: Okay.
10.88 Laila: So that you can…
10.89 Tommy: Oh yeah.
10.90 Laila: Then ∠1 plus ∠2 plus ∠3 equals 180°.
10.91 Julia: Well, you can swap around the numbers. The ones which are inside you can swap around them.

She was referring to swapping ∠4 and ∠2 and ∠5 and ∠3 but was not clear about it. Ryan came to help them and completed the proof.

10.103 Ryan: Well… like what Julia said, how ∠5 and ∠2 and ∠4; ∠1 and ∠6; ∠3 and ∠7 are 180°. We can write that down.
10.104 Teacher: Yeah, put that down. 4… He is just applying the same rule too over there, right. Look at those, look at what he is doing. So all are 180°.
10.105 Laila: What is that?
10.106 Ryan: ∠1.
10.107 Ryan: Well and ∠4 and ∠2 are exterior… alternate interior angles…
10.108 Teacher: Good. So you are replacing \( \angle 4 \) with \( \angle 2 \)?
10.109 Ryan: Yes.
10.110 Teacher: Okay.
10.111 Ryan: And \( \angle 6 \) and \( \angle 3 \) are also alternate interior angles.
10.112 Teacher: Aha.
10.113 Ryan: Replace the \( \angle 6 \) with the \( \angle 3 \), so… and it would be \( \angle 2 \) plus \( \angle 1 \) plus \( \angle 3 \)…
10.114 Teacher: Anything \( \angle 2 \) plus \( \angle 1 \) plus \( \angle 3 \) is 180°.
[Students clap.]
10.117 Laila: Good job.

Laila and Julia were very close to arriving at the proof but could not effectively explain the logical link between replacing the alternate interior angles with the angles in \( \angle 1 + \angle 4 + \angle 5 = 180° \) to get \( \angle 1 + \angle 2 + \angle 3 = 180° \).

Laila was operating on Level 2 as she was able to present only partial arguments aimed at proof production.

**Summary of Laila’s Work**

Laila worked through the process of proof production enthusiastically as she progressed through the tasks. She started initially on Level 1 in the task related to vertical angles, citing an authority and referring only to symbols in order to prove a theorem in Task 1 related to vertical angles. With the help of her peers, she slowly began to understand the workings of the proof process and approached the proof by organizing her reasoning and holding propositions together in the same task and in Tasks 3 and 4 related to the proof of alternate interior angles where she operated on Level 2. In Task 7 related to consecutive interior angles on the same side of the transversal, she started to operate on Level 3 as she produced a deductive text and understood how the conditions actually produce the result in a theorem. She exercised her understanding of the process of proving, by presenting alternate approaches to arrive at the same result using different
data sets in Task 8 related to proving that the consecutive exterior angles are supplementary. Though she now clearly understood the process of proof, in Task 10 which dealt with the triangle sum, she was not able to successfully search for links which provided the basis for validity of her assertions on her own, though she was able to do it with the support of others. Through her classroom interactions with her fellow students, her ideas were refined and shaped as Bishop (1999) points out: “Concepts, meanings, processes and values are what are being shaped, and these belong to the learner […] they are shaped in response to certain messages received not just from the teacher but from the whole environment, both physical and social” (p. 126).

Ricky’s Work

Analysis of Task 1

The task was related to the proof of vertical angle theorem and the students had to prove that $\angle PRQ = \angle SRT$ with reference to the figure presented on the board. The students discussed as a group and added their input. Ricky was involved in the discussion in which Dalton presented the proof for the theorem on board. The discussion ensued in the following manner:

Figure 4.85: Vertical Angles

[Dalton writes on the board $\angle PRQ = 180^\circ$ – and stops to look at the figure]
Teacher: $\angle PRQ$ equals $180^\circ$ minus… Wow, looks good. So he is going for the angles instead of the measures. So $\angle PRQ$ equals… If somebody wants to help him you can go. [Darren walks on to the board and writes $\angle SRT$]

Darren: $\angle SRT \ldots \angle SRT$

Teacher: Okay…Okay. Somebody should have a question here. He says… $\angle PRQ$ is $180^\circ$ minus this one [The teacher points to $\angle SRT$ in the figure.]

Ricky: No

Teacher: Ricky… Ricky… Ricky wants to add something. [Ricky comes to the board]

Teacher: PRQ is

Teacher: This is $\angle SRQ$

Teacher: Good. $\angle SRQ$. You are going…getting at something $\angle SRQ$, okay… Darren sit down. Ricky… I think Dalton can do the remaining because we want to know what you wanted to do. So $\angle PRQ$ is $180^\circ - \angle SRQ$.

Though Ricky played a very small part in the discussion, he was able to recognize the geometrical relationship that Dalton tried to portray. However, he was not able to create an argument of his own in this task. Ricky was operating on Level 1 as he appeared to understand the need to present an argument but was not quite capable of doing so himself.

**Analysis of Task 3**

The Task 3 required the students to prove that the pair of alternate interior angles are congruent. The conversation started as a group and each student gave their own input to the argumentation. The figure that Ricky referred to is given below:

![Alternate Interior Angles (Second Pair)](image)

Figure 4.86: Alternate Interior Angles (Second Pair)
He expressed that:

3.19 Ricky: $\angle 3 \cong \angle 7; \angle 7 \cong \angle 9$. So $\angle 9$ must be congruent to $\angle 3$.

3.20 Teacher: But you have to prove that $\angle 5 \cong \angle 7$ right?

He provided an argument to prove that the alternate exterior angles are congruent instead of the alternate interior angles. There was a logical consistency in his approach but that did not justify the task at hand.

In this task, Ricky was operating on Level 1. He recognized the need to apply a prior theorem but did not pay attention to check the conditions that necessarily led to the result. One can see that he lost focus on attaining the target at hand.

**Analysis of Task 6**

In Task 6, the students were given the figure drawn below and asked to prove that $\angle 3 + \angle 5 = 180^\circ$. The students started the discussion as a group and added their own arguments.

![Figure 4.87: Alternate Interior Angles (Second Pair)](image)

Ricky started his argument with a pair of corresponding angles:

6.41 Ricky: $\angle 4$ Equals to... $\angle 4 \cong \angle 5$  
[Starts off with corresponding angle pairs]

6.42 Teacher: Aha.

6.43 Ricky: So like $\angle 3 \cong \angle 4$ and $\angle 4 \cong \angle 5$ and $\angle 4 \cong \angle 3$,  
So $\angle 3 \cong \angle 4$. I mean $\angle 3 \cong \angle 5$... Wait... [Ricky goes back to think.]
Ricky first started the argument with presenting a pair of corresponding angles (line 6.41). In the succeeding argument, he seemed to find it difficult even to recognize the congruent angle pairs (line 6.45). While presenting the data for his claim, he lost track of the conditions that could lead him to the result and gave up. He himself felt not convinced with his argument (line 6.45). At this point, he seemed to be trying to replicate the type of argumentations that were presented in the earlier tasks of proving the congruence of alternate interior and exterior angles, i.e. using pairs of congruent angle pairs either vertical or corresponding to arrive at the conclusions. In doing so, he missed the important aspect of modifying the conditions to suit the required Task.

Like in the earlier task, Ricky was still operating on Level 1 where-in he recognized the need to apply a previous theorem as a part of his argument but did not get a grasp of the way how to arrive at a justifiable argument to prove the task at hand.

**Analysis of Tasks 7**

A similar task followed the Task 6, in which the students were required to prove that the other pair of consecutive interior angles are supplementary. They were required to prove that \(\angle 1 + \angle 2 = 180^\circ\).

Ricky came to the board and explained his approach:

7.25  Ricky: \(\angle 2 \text{ and } \angle 8 \text{ are corresponding. So let me put it like this, } \angle 2 \cong \angle 8\).
7.26  Teacher: Okay, \(\angle 2 \cong \angle 8\). So where are you...? Where are you getting from these?
7.27  Ricky: \(\angle 1 + \angle 8 = 180^\circ\)
7.28  Teacher: You are taking this one? [Pointing to \(\angle 1 \text{ and } \angle 8\)] \(\angle 1 + \angle 8 = 180^\circ\). So what are you doing now?
7.29  Students: Oh...
7.30  Teacher: Let him try.
7.31  Ricky: We select... these are congruent [Pointing to \(\angle 2 \cong \angle 8\)]... same thing \(\angle 1 \text{ and } \angle 2\), just like here [pointing to \(\angle 1 + \angle 8 = 180^\circ\)]... I cannot explain.
Even though Ricky found it difficult to explain his proof, one can see that he was close to make a valid argumentation for his claim. He went back to his seat to re–think. After some time he came back and said:

7.56 Ricky: \( \angle 1 + \angle 8 = 180^\circ, \angle 8 \cong \angle 2 \). So \( \angle 8 \) and \( \angle 2 \) are congruent. \( \angle 2 \) wait… \( \angle 8 + \angle 1 = 180^\circ \) also… So if 8 equals… \( \angle 8 \cong \angle 2 \ldots \angle 8 \) should be congruent to…

Ricky was still not able to organize his reasoning into a cogent argumentation. Laila helped him in finishing the proof. He was operating on Level 2 as he was able to present the data; \( \angle 1 + \angle 8 = 180^\circ \), \( \angle 8 \cong \angle 2 \), but was not able to make a coherent link to the data and the claim, \( \angle 1 + \angle 2 = 180^\circ \).

From a psychological point of view, the difficulty comes from the implicit reference to \( \angle 8 \) in the statement \( \angle 1 + \angle 8 = 180^\circ \). If he had a way to make \( \angle 8 \) the subject of the sentence, as in \( \angle 8 \) is the supplement of \( \angle 1 \), then it would become much easier to construct the argument: \( \angle 8 \) is the supplement of \( \angle 1 \), but \( \angle 2 \) is congruent to \( \angle 8 \), so \( \angle 2 \) is the supplement of \( \angle 1 \). Obviously he is not able to make the angle as a subject of reference which would have made his work easier while giving an explanation for those transformations.

**Analysis of Task 9**

In Task 9, the students were asked to prove that \( \angle 4 + \angle 9 = 180^\circ \) with respect the figure below:

![Figure 4.88: Consecutive Exterior Angles (Second Pair)](image_url)

Figure 4.88: Consecutive Exterior Angles (Second Pair)
Each student was required to work individually and present their proof to the class. By this time, Ricky seemed to get a hold of the process of proof production. He linked the data that he took to the claim in a logical way as seen in his proof on the board:

\[ \angle 4 + \angle 3 = 180^\circ \]
\[ \angle 1 + \angle 9 = 180^\circ \]

\( \angle 3 \) and \( \angle 9 \) are corresponding angles

So \( \angle 4 + \angle 9 = 180^\circ \)

Ricky presented a deductive proof by first listing the linear pairs that he thought could solve the problem. Because the proof involved \( \angle 4 \) and \( \angle 9 \), he might have taken the linear pairs that involved both the angles; \( \angle 4 + \angle 3 = 180^\circ \) and \( \angle 1 + \angle 9 = 180^\circ \).

Subsequently, Ricky was supposed to eliminate \( \angle 1 + \angle 9 = 180^\circ \) as it was not needed to prove the theorem. A possible reason for not striking out \( \angle 1 + \angle 9 = 180^\circ \) can be explained as follows:

Since he put an arrow between \( \angle 3 \) and \( \angle 9 \) to show that \( \angle 3 \) in the first linear pair can be switched with corresponding \( \angle 9 \), he could not strike off \( \angle 1 + \angle 9 = 180^\circ \). He clearly stated that \( \angle 3 \) and \( \angle 9 \) were corresponding angles. This indicates that he was considering only the linear pair \( \angle 4 + \angle 3 = 180^\circ \) to arrive at \( \angle 4 + \angle 9 = 180^\circ \). This kind of representation seems to have become a group practice and Ricky was influenced by this taken-as-shared practice of representing congruent angles with arrows while presenting his own proof. He produced a deductive text, checked the necessary and sufficient conditions leading to the result and provided a wholesome proof.

Ricky started to operate on Level 3 as he understood the shape of the proof as well as searching for conditions that actually produce the result in the theorem.
Analysis of Task 10

The task was to prove that the sum of the interior angles of a triangle is 180°. The students were given the figure below:

![Triangle Sum](image)

Figure 4.89: Triangle Sum

Earlier, the students discussed about the properties of triangles. When the task was presented by the teacher to prove the triangle sum theorem Ryan’s ideas led the group to present a valid proof for the theorem. A few students were absent on that day. So in the next session, the teacher asked Ricky, Jeremy and Tommy to present the proof that Ryan presented the day before. The discussion ensued as follows:

![Rylan’s Triangle Sum](image)

Figure 4.90: Rylan’s Triangle Sum

10.135 Ricky: $\angle 4 + \angle 1 + \angle 5 = 180^\circ$.
10.136 Jeremy: Because it is a straight line.
10.137 Tommy: Yeah, we said it was 180 degrees because…
10.138 Jeremy: That is a straight line, that’s why we said it was 180 degrees.
   [The students discuss what to write and they write $\angle 4$, $\angle 2$; $\angle 3$, $\angle 5$ on the board and add the word alternate interior beneath those two pairs.]
10.139 Teacher: This is… This is a phase where you are learning from one another. Okay. So… you… I will… so… What did you do
there? Let me ask you a question. Why did you put? What is the \( \angle 4 \) and \( \angle 2 \) and what are the \( \angle 3 \) and the \( \angle 5 \)?

10.140 Ricky: They are interior, alternate interior.
10.141 Tommy: Yeah.
10.142 Teacher: So, what are you going… where are you going from there.
10.143 Ricky, Jeremy, Tommy: We would replace the interior, \( \angle 4 \) and \( \angle 2 \).

The three students approached the theorem by considering the sum of the angles on the line opposite to the base. They referred to \( \angle 4 + \angle 1 + \angle 5 = 180^\circ \) (line 10.135) as the angles on a straight line. They then presented the alternate interior angle pairs; \( \angle 4, \angle 2; \angle 3, \angle 5 \) (line 10.138) and replaced the respective alternate interior angles in \( \angle 4 + \angle 1 + \angle 5 = 180^\circ \) to prove that \( \angle 2 + \angle 1 + \angle 3 = 180^\circ \). The students finished the proof as follows:

10.162 Teacher: What they did was… instead of the \( \angle 4 \), they put \( \angle 2 \).
10.163 Tommy: Yeah.
10.164 Teacher: Instead of the \( \angle 5 \)…
10.165 Jeremy: We put \( \angle 3 \).
10.166 Teacher: They put…
10.167 Students: \( \angle 3 \).
10.168 Teacher: \( \angle 3 \). So the answer is \( \angle 1 \) and…
10.169 Students: \( \angle 2 \) plus \( \angle 3 \).
10.170 Teacher: \( \angle 2 \) plus \( \angle 3 \) is
10.171 Tommy: 180 degrees.

Ricky is considered to be a part of the group which was operating on Level 3 as he understood how to arrive at \( \angle 2 + \angle 1 + \angle 3 = 180^\circ \) in the theorem by using the conditions; \( \angle 4 + \angle 1 + \angle 5 = 180^\circ \), \( \angle 4 \cong \angle 2 \), \( \angle 3 \cong \angle 5 \) and was able to reproduce the reasoning that was presented earlier by Ryan.

**Analysis of Task 11**

In Task 11, the proof of the exterior angle theorem in a triangle was discussed as a group and the students individually presented their ideas.
Ricky started with modifying the figure to something similar to the triangle sum theorem discussed earlier and assigned labels to the newly formed angles. He presented the following argument in relation to the figure that he drew:

Figure 4.92: Ricky’s Exterior Angle

11.91 Ricky: $\angle 6$ and $\angle 2$ are interior.
11.92 Teacher: Ssh… Everybody should be listening. What is it? $\angle 6$ and $\angle 2$? Guys, he is saying $\angle 6$ and $\angle 2$.
11.93 Darren: Exterior… No…
11.94 Teacher: Ryan… Ryan is saying something. $\angle 6$ and $\angle 3$ are interior? Okay. What else Ricky?
11.95 Ricky: $\angle 5$ and $\angle 1$ are linear.
11.96 Teacher: Only $\angle 5$ and $\angle 1$ are linear?
11.97 Ricky: $\angle 5$, $\angle 1$ and $\angle 6$.
11.98 Teacher: $\angle 5$, $\angle 1$ and $\angle 6$. Okay. Let me write for you, because sometimes not all kids can explain, right? You said $\angle 5$ and $\angle 1$ and $\angle 6$ they become…
11.99 Ricky: $180^\circ$.
11.100 Teacher: $180^\circ$. So where are you going from there?
11.101 Ricky: So… so… I would switch the $\angle 1$ with the $\angle 2$. I will switch $\angle 5$ with the $\angle 1$ and the $\angle 6$ with the $\angle 2$.
11.102 Teacher: $\angle 5$ with the $\angle 1$? Show me… show me what you want to switch?
11.103 Ricky: $\angle 5$ and $\angle 1$; $\angle 6$ and $\angle 2$. 

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Ricky tried to make sense of the current task by reverting to the figure that was used in the triangle sum theorem. In the triangle sum theorem, the students used a parallel line to the base in the process of its proof production. Ricky modified the current figure by adding a parallel line, in footsteps of the previous theorem proof related to the triangle sum and tried to make the connections in relation to this figure. He then tried to replicate the argument presented in the triangle sum theorem. This diverted his focus and he ended up recognizing the alternate interior angles as; $\angle 6$, $\angle 2$ and $\angle 5$, $\angle 1$ instead of; $\angle 6$, $\angle 3$ and $\angle 5$, $\angle 2$. Ricky might have tried to use a similar approach used in triangle some theorem as this current theorem also involved a triangle as the previous one. In this process, he ended up with a complete replication of the previous proof which might have completely thrown him off the track. Though by this time, he understood the shape of a proof and how to structure the arguments to arrive at the proof, his idea of relying on the same type of arguments presented in earlier theorems hindered his ability to think of his own and present arguments relevant to the task at hand. He was operating on Level 2. It seems that in many occasions, he was looking at a task through the last one, instead of approaching it directly, on its own terms.

**Analysis of Task 12**

This task was on parallel lines and the students were required to prove that, $\angle 3 = \angle 1 + \angle 2$ from the figure given by the teacher. All the students worked independently and presented their proofs on the board. Ricky added an extra angle 4 as shown below and wrote the following proof on the board:
\[ \angle 3 = \angle 4 \]
\[ \angle 4 = \angle 1 + \angle 2 \]
\[ \angle 3 = \angle 1 + \angle 2 \]

Figure 4.93: Parallel Line Task

Figure 4.94: Ricky’s Parallel Line Task

When presenting to the class, he went into more detail:

12.38 Ricky: I made another angle. [He refers to the \( \angle 4 \) that he adds in the figure] Okay. \( \angle 3 \) and \( \angle 4 \) are a linear pair right?
12.39 Teacher: \( \angle 3 \) and what?
12.40 Ricky: They are alternate interior …
12.41 Student: Yeah…
12.42 Teacher: Oh…\( \angle 3 \) and \( \angle 4 \) are alternate interior? Okay.
12.43 Ricky: And how previously we had this. [Referring to a part of his diagram depicting the exterior angle sum theorem which was discussed in an earlier session]
12.44 Ricky: I just switched them around and this like that. Outside equals all inside, but not this one.

Here Ricky linked his arguments to theorems that have already been proved. In contrast to the previous task in which he tried to replicate previous arguments, in this task he used his own reasoning and ended on a successful note. The proof that he presented, really marked out the progression from his initial ways of thinking.

Though the task on parallel lines proved to be difficult for several students, Ricky produced a complete proof by making the necessary links to earlier theorems. He was operating on Level 3 as he produced a deductive text following mathematicians' norms and understood how the conditions actually produce the result in a theorem.
Analysis of Task 14

In Task 14, a theorem on equal length chords was given to the students to prove. Each student worked individually and presented their argumentation. Ricky drew the diagram below:

![Diagram of circles and chords]

Figure 4.95: Ricky’s Circles Theorem-2

14.54 Ricky: So, I drew this line right over it, to make a perpendicular line. Since it is perpendicular, all of them are 90°. So… when all these are equal isn’t $OX = OY$?
14.55 Teacher: Like how can you say that?
14.56 Ricky: Because you see how it is 90° right here, wouldn’t they be?
14.57 Teacher: You are saying this is 90°, this is 90°.

Ricky was referring to the fact that if he drew a perpendicular line to $XY$ through the center, all the angles at the center are 90° and then $OX$ should be equal to $OY$. He might be assuming that the line that he drew will be a perpendicular bisector drawn to $XY$ and that it would bisect the line. Though he could not explain it in clear terms the reasoning that he exhibited was correct. When he drew the perpendicular line through the center, it will bisect the given line connecting the chords of equal length. Ricky was operating on Level 3. Even though he did not succeed at this task, one can observe that he had advanced to thinking about the theorem on its own terms, rather than to just replicate a previous proof and try to apply it.
Summary of Ricky’s Work

Ricky’s progress in understanding the shape of a proof gradually improved with the progression of tasks. He started off by providing support to an argument presented by another student in Task 1, related to the vertical angle theorem. He had trouble keeping focus on attaining the target through the next few tasks as he started to present partial arguments of his own and began to operate on Level 1, in the Tasks 2, 6 and 7, related to the alternate interior angles and consecutive interior angles on the same side of the transversal with reference to parallel lines. In those tasks, he recognized the need to apply previous theorems as part of one’s argument, but did not quite know the way for doing so. Slowly his competence developed from the outside in, replicating the forms of participation, and gradually working toward the intentionality (Vygotsky, 1972) as he progressed into the remaining tasks.

He started working on Level 3 and proceeded to maintain logical consistency to arrive at deductive proofs in Tasks 9, 10 and 12 related to the consecutive exterior angle theorem, triangle sum theorem and in the task related to parallel lines. Though the task on parallel lines proved to be difficult for some students, Ricky gave a creative proof for that task which showed that there was a transformation in his approach towards proof.

As seen in Tasks 6 and 11, the main hindrance in Ricky’s proofs seems to be that, whenever he saw some similarity in successive tasks, he tried to approach the proof in a similar way and ended up with replication of the previous proof. This threw him off the track and left him unsuccessful. Whenever there was a change in the nature of task, he found it difficult to supply the required conditions that would lead him to the goal but one can notice that he understood the process of proof production.
Ryan’s Work

Analysis of Task 2

Ryan joined the sessions after the vertical angle theorem had been proved. He was explained by his friends, how the discussions progressed till then and how the vertical angle theorem was proved. Task 2 was related to the proof of the alternate interior angle theorem and the students had to prove that $\angle 6 \cong \angle 4$ in reference to the figure presented on the board. The students discussed as a group and presented their thoughts.

![Figure 4.96: Alternate Interior Angles (First Pair)](image)

In this episode, initially the students started to discuss in two groups. Dalton, Jeremy and Ryan started discussing together. Dalton initiated the discussion and made others to pay attention to corresponding angle pairs. Laila, who was working with other group till then, joined the conversation and said that they should consider corresponding angle pairs. She expressed that they were considering those angles as they were trying to gather information already known. Ryan and the other students agreed with her.

Both Dalton and Laila came up to the stage of picking up the data that seemed relevant in proving the theorem. They settled for $\angle 4 \cong \angle 8$, the corresponding angle pair and $\angle 6 \cong \angle 8$, the vertical angle pair. Then, they started to ponder on their next step of
connecting the data to the claim. At this juncture, Ryan provided them with the logical link that connected the data $\angle 4 \cong \angle 8$ and $\angle 6 \cong \angle 8$ to the claim $\angle 6 \cong \angle 4$. He concluded:

2.95 Ryan: $\angle 4$ and $\angle 8$ is congruent, and $\angle 6$ and $\angle 8$ is congruent.
2.96 Ryan: So, they both equal, so $\angle 6 \cong \angle 4$.

While others were thinking how to link the data to the claim, Ryan perceived the association between the vertical angle pair and the corresponding angle pair in relation to the alternate interior angle pair to be proved. He was able to identify that logical connection just by observation.

Ryan was operating on Level 2 of reasoning. Even though he did not formulate the data, he came up with a logical interpretation of the data into a valid conclusion. His contribution was crucial in arriving at a definitive conclusion and completing the proof.

**Analysis of Task 4**

In Task 4, the students were required to prove that the alternate exterior angles are congruent. The same figure was used in this task too. The students started to prove this theorem as a group and everybody were giving their input to the proof. The task was to show that $\angle 3$ and $\angle 9$ are congruent. Dalton started to write on the board on behalf of his group in which Ryan was a member. He and the group ended up in showing that $\angle 5$ and $\angle 7$ were congruent instead of $\angle 3$ and $\angle 9$. When the teacher pointed out the mistake done by the group, Ryan volunteered to provide the proof. He started with a different data set and proved the theorem. Below is an excerpt from the transcript that shows his reasoning:

4.33 Teacher: Give him a chance, give him a chance first. So you want to take $\angle 3$ and $\angle 7$, Ryan? Okay, but what are these two angles? $\angle 3$ and $\angle 7$?
4.34 Laila: $\angle 3$ and $\angle 7$, I can’t see?
4.35 Ryan: They are corresponding.
4.36 Teacher: They are corresponding? [Students talking]
Teacher: Don’t do that. [Addressing another student]
Teacher: So what can you say about? Now Ryan wants to say something. What do you want to say?
Ryan: \( \angle 7 \cong \angle 9, \angle 3 \cong \angle 9 \ldots \angle 7 \cong \angle 9, \angle 3 \cong \angle 7 \). Therefore \( \angle 7 \) is common and \( \angle 3 \cong \angle 9 \).

Ryan started off by taking a different data set \( \angle 7 \cong \angle 9 \) and \( \angle 3 \cong \angle 7 \). He concluded that since \( \angle 7 \) is common in the data, \( \angle 3 \cong \angle 9 \) (line 4.39). However he did not justify why \( \angle 3 \cong \angle 7 \) and \( \angle 7 \cong \angle 9 \), till the teacher questioned. He continued to present data without proper justifications in the next few tasks as well.

He continued to operate on Level 2 as he made logical data-claim links without providing the justifications unless questioned by the teacher or others.

**Analysis of Task 8**

In Task 8, the students had to prove that \( \angle 7 + \angle 5 = 180^\circ \) with the figure below as reference:

![Figure 4.97: Alternate Interior Angles (First Pair)](image)

In this task, everybody started giving their input to the discussion. The teacher asked Ryan if he wanted to come to board and try. After a slight hesitation, he started the discussion with a corresponding angle pair and the discussion ensued in the following manner:

Teacher: You want to try Ryan? [Ryan first hesitates and then says that he wants to do it.]
Teacher: You want to… yeah… everything here is trying, so you have to show me \( \angle 7 \) plus \( \angle 5 \) equals \( 180^\circ \). Okay? So here’s your chalk
and you guys need to be helping. Yeah… something, it’s your class.

8.12 Laila: Okay. So…
8.13 Ryan: $\angle 7$ and $\angle 6$ is corresponding.
8.14 Laila: $\angle 7$ and $\angle 6$.
8.15 Student: Exterior.
8.16 Laila: I think you should put $\angle 7$ and $\angle 9$.
8.17 Teacher: She said you should put $\angle 7$ and $\angle 9$.
8.18 Laila: I also see $\angle 4$ and $\angle 5$.
8.19 Teacher: So… what are those?
8.20 Laila: Those are exterior.
8.21 Teacher: Exterior angles? They are congruent? Okay. So where are you going from there?
8.22 Dalton: I think $\angle 7$ and $\angle 9$, $\angle 7$ and $\angle 2$ equal to 180°, because it’s a linear pair.
8.23 Teacher: So… Dalton, can you repeat what you said again baby?
8.24 Dalton: I told Ryan to use $\angle 7$ plus $\angle 2$ because they were linear pairs and they equal 180°.
8.25 Teacher: Oh… Okay. So, Ryan he is giving you a suggestion to start with $\angle 7$ plus $\angle 2$.

After a few minutes, Laila and Dalton both added:

8.30 Laila & Dalton: So you should… replace the $\angle 2$ with $\angle 5$ and… $\angle 7$ plus $\angle 5$ equals 180°.
8.31 Ryan & others: Agreed.

Even though Ryan started on the right track, he did not get a chance to complete the proof on his own as the others propagated their ideas and proved the theorem. He was operating on Level 2 as he seemed to understand how the proof was approached.

**Analysis of Task 9**

The students proved a similar task to show that the other pair of exterior angles on the same side of the transversal are supplementary. They had to prove that $\angle 4 + \angle 9 = 180°$. The teacher directed the students to work individually and present their proofs on the board. Ryan presented the following proof on the board:
Ryan’s proof: \( \angle 4 + \angle 3 = 180^\circ \)

\[ \therefore \angle 1 + \angle 9 = 180^\circ \]

So \( \angle 4 + \angle 9 = 180^\circ \)

Figure 4.98: Consecutive Exterior Angles (Second Pair)

Ryan first drew the above figure and took the linear pairs; \( \angle 4 + \angle 3 = 180^\circ \) and \( \angle 1 + \angle 9 = 180^\circ \). Though it was not required to take \( \angle 1 + \angle 9 = 180^\circ \) to prove the theorem, he took that linear pair along with \( \angle 4 + \angle 3 = 180^\circ \). He introduced an arrow between \( \angle 9 \) in this pair and \( \angle 3 \) in other pair to illustrate that \( \angle 3 \) can be switched with \( \angle 9 \) to arrive at \( \angle 4 + \angle 9 = 180^\circ \). This may be a possible reason for not eliminating the additional data in subsequent steps. Even though he was able to link the available data to come up with a valid conclusion, he did not provide the justification for why he was replacing \( \angle 3 \) with \( \angle 9 \). He was operating on Level 2 as he presented partial arguments and presented the data-claim links without producing the warrants.
Analysis of Task 10

The task was to prove that $\angle 1 + \angle 2 + \angle 3 = 180^\circ$ using the figure below:

![Figure 4.100: Triangle Sum](#)

In this task, the teacher gave the students a figure with a triangle and a parallel line passing through one of the vertices as shown above to prove the theorem. The teacher asked the students to prove that sum of the angles is $180^\circ$ in this task. First the teacher asked the students to sit in a group and work together. Ryan had some idea about the proof and he wrote it on his paper. Since the other students could not see what he wrote, the teacher asked him to come to the board to present it. The proof that Ryan presented was very logical and beyond the understanding of other students. The thinking that lead to the proof is presented below:

10.33 Teacher: Ryan wants to say something, let me see.
10.34 Ryan: Well... when we draw line ‘l’, on top, I thought of parallel lines, and drew one at the bottom and I saw how they made angles... So I made the angles, $\angle 4, \angle 5, \angle 6$ and $\angle 7$, then...

![Figure 4.101: Rylan’s Triangle Sum](#)
Teacher: Let...let me see yours. Okay, I think it’s …do you guys think it is better if I ask you to write it on the board?

Students: Yes.

Teacher: Because everybody can see it right?

Students: Yah….

Teacher: It’s not clear here. Ryan wants to say. I will give a chance to everybody, Okay, and then we will agree on one thing, which everybody agrees on, okay?

Julia: [inaudible]

Teacher: Okay. Let him explain what he thinks.

Ryan: And…I saw them make angle and I named it, \(\angle 4\), \(\angle 5\), \(\angle 6\) and \(\angle 7\).

Teacher: Oh… he named them…Okay.

Ryan: And I saw that…\(\angle 6\) and \(\angle 2\) are linear.

Teacher: Linear?

Ryan: Linear angles.

Teacher: You mean linear pair?

Ryan: Linear pairs and therefore 180°.

[He writes \(6 + 2 = 180°\)]

Teacher: These are not number 6 and number 2 right? They are angles. Even though he didn’t put that he meant angles. It’s not 6 and 2. I just wanted to make sure.

Ryan: And \(\angle 3\& \angle 7\)…

[He writes \(\angle 3 + \angle 7 = 180°\)]

Teacher: Is 180°?

Ryan: 180° and \(\angle 4\), \(\angle 1\) and \(\angle 5\) is also 180°.

Teacher: \(\angle 4\) and \(\angle 1\) and what is the other number?

Ryan: \(\angle 5\).

Teacher: \(\angle 5\). [Addressing other students] Okay. Somebody…when you see somebody thinking about that, you can get some more ideas too right? So think about what he is trying to do.

Ryan: Then I saw that \(\angle 4\) and \(\angle 3\) are interior…

Teacher: \(\angle 4\) and \(\angle 3\)?

Laila: I can’t see. \(\angle 4\) and \(\angle 3\). Will they be corresponding?

Teacher: Okay, you were trying to say interior angles? [Addressing Ryan]

Ryan: Yeah.

Ryan: \(\angle 4\) and \(\angle 3\) are interior angles. \(\angle 5\) and \(\angle 2\) are interior angles and the I saw that \(\angle 4\) and \(\angle 3\) are interior angles, I can replace the \(\angle 4\) and the \(\angle 3\); \(\angle 5\) and \(\angle 2\) are interior, so I replace the \(\angle 5\) with the \(\angle 2\) and it would be \(\angle 3 + \angle 1 + \angle 2 = 180°\).

First he presented the linear pairs \(\angle 6 + \angle 2 = 180°\) (line 10.44), \(\angle 3 + \angle 7 = 180°\) (line 10.50) and then stated that he would take \(\angle 4 + \angle 1 + \angle 5 = 180°\) (line 10.52) formed
by the parallel line. It was not clear whether he was considering them as alternate interior angle pair or interior angles on same side of transverse. In continuation, he argued that he could replace \( \angle 4 \) with \( \angle 3 \) and \( \angle 5 \) with \( \angle 2 \) (line 10.59). He might have been assuming that \( \angle 4 \) and \( \angle 3 \) were alternate interior angle pair and hence congruent. Actually they were neither alternate interior angles nor interior angles on same side of transverse. He might have been deceived by the formation of multiple angles at the vertex. At that juncture the teacher raised a question about the replacement to which Laila added her input:

10.62 Teacher: I have one question Ryan. Does anybody have a question or can I go? He says \( \angle 4 \) and \( \angle 3 \) are interior; I have a question about that.
10.63 Laila: Oh...I know something. \( \angle 4 \) is not interior... \( \angle 4 \) and \( \angle 2 \).
10.64 Teacher: It’s \( \angle 4 \) and \( \angle 2 \). What was yours? [Addressing Dalton]
10.65 Dalton: \( \angle 4 \) and \( \angle 3 \) would be exterior?
10.66 Teacher: Anybody has still a question about it? Okay, look again. He says \( \angle 4 \) and \( \angle 3 \).

Laila actually pointed out the error that he made in referring to; \( \angle 4 \) and \( \angle 3 \) as the alternate interior angle pairs and corrected that \( \angle 4 \) and \( \angle 2 \) were the alternate interior angles (line 10.63). Ryan realized his mistake and said:

10.67 Ryan: I think I will just agree with Laila now.
10.68 Teacher: Oh... you are agreeing with Laila?
10.69 Ryan: Yes.
10.70 Teacher: Because what Laila said was right. What did she say? \( \angle 4 \) and \( \angle 2 \) are...
10.71 Students: interior.
10.72 Teacher: Alternate interior angles. He thought he replaced them right, but the only thing is he put ...
10.73 Student: The wrong numbers.
10.74 Teacher: The wrong angles, but still he...
10.75 Laila: He proved it.
Ryan gave a deductive proof. Even though he made a small error in identifying the alternate interior angles that he replaced, he quickly realized his mistake and rectified it. He then coordinated the elements of the argument in a way that is consistent with mathematically sound argumentation. Ryan not only knew the alternate interior angle theorem in the sense of memorizing the conditions needed to apply it and the result that can then be claimed, he understood the logic of how the result follows from those conditions. He was operating on the Level 3 of reasoning.

**Analysis of Task 11**

In Task 11, it was required to present the proof for the exterior angle theorem. The students worked independently and presented their proofs on the board. The students had to prove $\angle 4 = \angle 1 + \angle 2$ from the figure below:

![Figure 4.102: Exterior Angle](image)

Ryan did not write anything on the board but explained his proof in the following way:

11.191 Ryan: Like others, I thought $\angle 1 + \angle 2 + \angle 3 = 180^\circ$ and $\angle 3$ and $\angle 4$ is a linear pair. Therefore equals $180^\circ$.

11.192 Teacher: Aha.

11.193 Ryan: I replaced the $\angle 3$ with the $\angle 4$.

11.194 Teacher: Then it becomes $\angle 1 + \angle 2 + \angle 4 = 180^\circ$. If you replace the $\angle 3$ with the $\angle 4$, it becomes $\angle 1 + \angle 2 + \angle 4$ equal $180^\circ$. But we need to know $\angle 4 = \angle 1 + \angle 2$ right? [Ryan goes back to think about the proof but is not able to finish it.]

When the teacher questioned him about the replacement of $\angle 3$ with the $\angle 4$ in $\angle 1 + \angle 2 + \angle 3 = 180^\circ$. Ryan went back to think further (line 11.194). The reason for him
not being able to present a valid argument can be construed as follows. Initially when the teacher gave the task and asked the students to present their proofs, Ryan had nothing to offer. But as he saw the proofs presented by other students, he began picking up partial threads of those arguments to formulate a proof of his own which ended on an unsuccessful note. Even though by this time, he knew how to present a proof and understood the shape of the proof, he could not achieve the goal because he might have not put much thought to it. In this task, his participation was not sufficient to reveal his level of understanding.

**Analysis of Task 12**

The task was on parallel lines and the students were required to prove that, \( \angle 3 = \angle 1 + \angle 2 \) from the figure given by the teacher. All the students worked independently and presented their proofs on the board. Ryan added an extra angle 4 as shown below and wrote the following proof on the board:

Ryan’s diagram:

![Diagram of parallel lines with angles 1, 2, 3, and 4]

Figure 4.103: Ryan’s Parallel Line Task

Ryan’s proof:

\[
\begin{align*}
\angle 4 &= \angle 1 + \angle 2 \\
\angle 3 &= \angle 1 + \angle 2
\end{align*}
\]

[Replace the \( \angle 4 \) with \( \angle 3 \)]

\( \angle 3, \angle 4 \) are alternate [exterior] interior angles.
Even before Ryan came to explain his proof, he realized the mistake that he had done on the board i.e. referring to $\angle 3$, $\angle 4$ as alternate exterior angles. He made the change to alternate interior angles. He explained his proof as follows:

12.58 Teacher: Next is Ryan. Ryan explain yours.
12.59 Ryan: I messed up. This is supposed to be interior [changes what he wrote as exterior]
12.60 Teacher: Okay. Write it. Change it. Ssh…
12.61 Ryan: Okay. I kind of like said that the angle that is right here $\angle 3$ and $\angle 4$ are interior angles.
12.62 Teacher: Aha.
12.63 Ryan: A while ago we reviewed… This would… this [Points to $\angle 4$] equals this one [Points to $\angle 1$] + this [Points 2].
12.64 Teacher: You mean the exterior angle theorem? Exterior angle is equal to the sum of the opposite interior angles? Good.
12.65 Ryan: So then I wrote $\angle 4$ equals $\angle 1 + \angle 2$ and since $\angle 4$ and $\angle 3$ are interior angles, then we replace $\angle 4$ with $\angle 3$.

Ryan first took $\angle 4 = \angle 1 + \angle 2$ which he referred to as something that was reviewed earlier (12.63). He was referring to the exterior angle theorem that was proved earlier and applied it to the current figure. He then added that he replaced $\angle 4$ with $\angle 3$ as they were alternate interior angles (12.65).

Ryan was operating on Level 3 as he made up an argument of his own by putting some thought to it and by making his reasoning explicit with reference to the alternate interior angles and the exterior angle theorem to prove the task at hand.

**Summary of Ryan’s Work**

Ryan from the day he joined the sessions, seemed to understand the dynamics of proof production. Although he joined the group in the second task, he was able to understand how the process of proving worked out right from that task. He was able to recognize the shape of proof consisting of claims based on the data presented in Tasks 2, 4, 8 and 9 related to the alternate interior angle theorem, alternate exterior angle theorem
and consecutive exterior angle theorem respectively. However, he took some time to realize the importance of explicating the warrants that link the data to the claim which he did at the end. He exhibited a logical consistency all through the tasks except in the exterior angle theorem where he tried to replicate the arguments that others presented and ended up being unsuccessful in producing a valid proof.

**Tommy’s Work**

**Analysis of Task 1**

The task at hand was to prove vertical angles theorem. The students had to prove that $\angle PRT \cong \angle QRS$, in reference to the figure presented on the board.

![Figure 4.104: Vertical Angles](image)

Delbert started to read what he wrote on his paper. He said:

1.2 Delbert: I think they are congruent because they are on opposite sides of one another.

Tommy asked him to repeat again:

1.3 Teacher: Okay…Tommy? Yah repeat that again because he didn’t listen.
1.4 Delbert: They are congruent because they are on opposite sides of one another.
1.5 Teacher: Okay…So, my question is, is that proof enough to tell that they are congruent?

The teacher then asked the class whether it was proof enough to say that, to which Tommy responded as follows:

1.6 Tommy: I think…kind of not.
1.7 Teacher: Kind of not?
1.8 Tommy: Kind of not.
In this episode, Tommy seemed to be able to critique a claim made by Delbert, though he did not present an argument of his own. Delbert at the beginning of the task, just rephrased the conjecture that the students made earlier about the vertical angles and presented it as the proof. Though it was the teacher who initially questioned about the sufficiency of the statement to be the proof in itself, Tommy too seemed to recognize that it was not a valid proof or he was just assuming from the teacher’s question that it must not be a proper proof.

Either way, at that point he himself could not present an argument of his own. He was operating on Level 1 as he recognized the need to create an argument though he could not do it himself.

**Analysis of Task 3**

In Task 3, the students were required to prove the alternate interior angle theorem. The following figure was generated to help them in the process. The task was to prove that $\angle 4 \cong \angle 7$.

![Figure 4.105: Alternate Interior Angles (Second Pair)](image)

Tommy started his proof with reference to alternate exterior angles. He said:

3.35 Tommy: $\angle 5$ equals $\angle 3$ and $\angle 3$ equals $\angle 7$.
3.36 Teacher: Tommy, can you repeat that baby?
3.37 Tommy: $\angle 4$ equals $\angle 5$ and $\angle 5$ equals $\angle 3$ and $\angle 3$ equals $\angle 7$ and implies that $\angle 4 \cong \angle 7$. 

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Teacher: But \( \angle 5 \) equals \( \angle 3 \) is needed to be proved right? Just get me the eraser for a minute because I want to erase this because you have to prove that. So you cannot take it be granted right?

Tommy used the corresponding angle pairs; \( \angle 4 \), \( \angle 5 \), \( \angle 3 \), and \( \angle 7 \), the alternate exterior angles, \( \angle 5 \) and \( \angle 3 \) to arrive at the claim that \( \angle 4 \cong \angle 7 \) (line 3.37). It was similar to the approach that Julia presented in Task 2, where she started off with data related to alternate exterior angles and could not complete the proof at that time. As the teacher objected to Tommy’s approach of considering those angles, and he went back to his seat to think for some more time.

Tommy used \( \angle 5 \) equals \( \angle 3 \) to link \( \angle 4 \), \( \angle 5 \), \( \angle 3 \), \( \angle 7 \) and hence to deduct \( \angle 4 \cong \angle 7 \) by using theory of transitivity. But, by this time, the students had not proved the exterior angle theorem which deduces \( \angle 5 \cong \angle 3 \). It shows that at this juncture Tommy did not understand the structure of proof and did not recognize that the proof consists of data–sets recognized from already established truths linked to the claim using valid reasoning. He was operating on Level 1 of reasoning.

After thinking for some more time, Tommy came back again to present different approach as below:

3.43 Tommy: I think.
3.44 Teacher: I think… Tommy got it.
3.45 Tommy: \( \angle 4 \) equals \( \angle 3 \) and \( \angle 3 \) equals…
3.46 Delbert: \( \angle 7 \).
3.47 Teacher: Tell us that… do you want to tell us that as a group or one person go there and tell me… Write that for me baby. Tommy, write it for me, or show it to me here first.
3.48 Tommy: \( \angle 4 \) is vertical to \( \angle 3 \) and \( \angle 3 \) is corresponding angle to \( \angle 7 \).
3.49 Teacher: Perfect. Put it over, put it over here. Let me see, how it goes on the board. [Tommy writes] \( \angle 4 \) is congruent to \( \angle 3 \); \( \angle 3 \) is congruent to \( \angle 7 \).
3.50 Teacher: So why \( \angle 4 \) and \( \angle 3 \) are congruent? Can you tell me why they are congruent?
3.51 Joey, Julia: Vertical.
3.52 Teacher: They are vertical. Good. Put vertical for me in the bracket, there across the $\angle 4$ and $\angle 3$. Just put vertical, v, just put ‘v’ there in the bracket. So they are vertical. The ‘v’ stands for vertical. Why $\angle 3$ and $\angle 7$ are congruent?

3.53 Tommy: Corresponding...
3.54 Teacher: So put ‘C’ for me in the bracket. Like put a bracket for me. Put a parenthesis over there.
3.55 Student: Parenthesis.
3.56 Teacher: Just like I kept here and put one for the ‘V’ too. So what can you say about $\angle 4$ and $\angle 7$ now?
3.57 Tommy: They are congruent.
3.58 Teacher: Can you write that for, also that.
3.59 Delbert: $\angle 4$ equals $\angle 7$ [Tommy writes] they are congruent.
3.60 Teacher: Okay. So, do you guys realize what you did now, you have actually proved that the alternate interior angles are congruent.

After the teacher’s comment about considering the exterior angles, Tommy pondered for a few minutes. He then started with $\angle 4$ equals $\angle 3$ (line 3.45), and specified that he was considering them as vertical angles (line 3.48). He also pointed to $\angle 3$ and $\angle 7$ (line 3.48) as the related corresponding angle pair that he would consider to prove that $\angle 4$ equals $\angle 7$. He began to operate on Level 3 in the later part of the argumentation in the same task. He produced a deductive connection with data; $\angle 4 \cong \angle 3$ and $\angle 3 \cong \angle 7$ linked to the claim $\angle 4 \cong \angle 7$ in a logical sequence and backed them with warrants related to the corresponding angle postulate and the vertical angle theorem.

**Analysis of Task 5**

In Task 5, the students were required to prove the alternate exterior angle theorem. They had the figure below to prove that $\angle 3 \cong \angle 8$. 
Delbert first presented a valid proof for the theorem, and Tommy said that he would present the proof in an alternate way. It is worth looking at Delbert’s proof to be able to understand how Tommy’s approach was different. Delbert started with $\angle 3 \cong \angle 5$ and $\angle 5 \cong \angle 8$ and concluded that $\angle 3 \cong \angle 8$. The alternate approach was presented by Tommy in the following way:

5.47 Tommy: $\angle 3$ is vertical… $\angle 3$ is congruent to $\angle 1$.
5.48 Teacher: Aha. Why?
5.49 Tommy: Because they are vertical.
5.50 Teacher: Okay. They are vertical.
5.51 Tommy: $\angle 1$ mean no… yeah $\angle 1$ is…
5.52 Teacher: Aha.
5.53 Tommy: Is… yeah.
5.54 Chris: To $\angle 8$.
5.55 Tommy: Yeah… it’s corresponding.
5.56 Teacher: Chris, can you say it louder? $\angle 1$ is congruent to what?
5.57 Chris: $\angle 8$, $\angle 8$.
5.58 Teacher: $\angle 8$. Why are they congruent?
5.59 Tommy: Because they are corresponding angles.
5.60 Teacher: Aha. They are corresponding? [Tommy claps]
5.61 Teacher: So what can you conclude from that?
5.62 Tommy: That $\angle 3$.
5.63 Teacher: Aha.
5.64 Tommy: $\angle 3$ is congruent to…
5.65 Teacher: To what angle?
5.66 Tommy: $\angle 8$.
5.67 Teacher: Perfect. You… you proved this right?

Figure 4.106: Alternate Exterior Angles (Second Pair)
Tommy started with $\angle 3 \cong \angle 1$ (line 5.47) which he referred to as vertical angles. As he was looking for another angle that he could associate $\angle 1$, Carl suggested $\angle 8$ (line 5.57). So Tommy considered that angle and inferred that $\angle 1 \cong \angle 8$, as they were corresponding angles. He finally arrived at the valid conclusion that $\angle 3 \cong \angle 8$ (line 5.66). He was able to understand that the proof could be approached in multiple ways with proper backing to support the data.

He was operating on Level 3, as he memorized the theorem as condition implication sequence, and understood how the conditions; $\angle 3 \cong \angle 1$, $\angle 1 \cong \angle 8$ actually produced the result, $\angle 3 \cong \angle 8$ in the theorem.

**Analysis of Task 6**

In Task 6, the students were asked to prove that a pair of consecutive interior angles are supplementary.

![Figure 4.107: Consecutive Interior Angles (First Pair)](image)

In this task the students were required to prove that $\angle 5 + \angle 3 = 180^\circ$. Carl volunteered to prove the theorem. The discussion ensured in the following manner:

6.88  Carl: Can I solve this problem?
6.89  Teacher: You want to solve that?
6.90  Carl: Yeah... like the $\angle 1$ plus $\angle 2$ equals $180^\circ$. [He was referring to the pair that was proved earlier.]
6.91  Teacher: Okay. Can you like... can you show us.
6.92 Carl: So... you have to know like... linear pairs and linear pairs is like... something like this.
6.93 Teacher: Aha.
6.94 Carl: You see... this equals to 90° and this 90°. 90° plus 90° equals 180°. So you find something linear.
6.95 Teacher: Aha.
6.96 Carl: And so... you see the ∠3 and ∠4, add, equals to 180°.
6.97 Teacher: Aha.
6.98 Tommy: They are linear pairs.
6.99 Teacher: They are linear pairs too? Tommy?
6.100 Tommy: Aha.
6.101 Carl: ∠5 and ∠6 equals to 180°.
6.102 Teacher: Aha.
6.103 Carl: So, all you have to do is...
6.104 Teacher: So... this time you are approaching it as like linear pairs?
6.105 Tommy: Aha...
6.106 Teacher: All of you are on the same page? All of you think that.
6.107 Carl: So, this is pretty easy, so all you have to do is, put the 5 where the 4 is.
6.108 Teacher: Aha. [Carl replaces the ∠4 with the ∠5.]
6.109 Carl: And that’s how you get ∠3 equals ∠3 plus ∠5 equals 180°.

Tommy understood the approach of the argument that Carl tried to employ to arrive at the proof. He was operating on Level 2 as he provided partial arguments to the proof and could not present the whole of the argumentation by himself.

**Analysis of Task 8**

In Task 8, the students were asked to prove that ∠7 + ∠8 = 180°, the angles being the consecutive alternate interior angles on the same side of the transversal. Each student presented their proof on the board. Tommy presented his proof on the board to show that

∠7 + ∠8 = 180° with reference to the figure below:

![Figure 4.108: Consecutive Exterior Angles (First Pair)](image-url)
Tommy’s proof on the board with warrants:

\[ \angle 7 + \angle 2 = 180^\circ \]

\[ \angle 1 + \angle 8 = 180^\circ \]

“So then \( \angle 2 \) and \( \angle 8 \) were corresponding and \( \angle 1 \) and \( \angle 7 \) were corresponding, and then you could switch the numbers so that \( \angle 7 + \angle 8 = 180^\circ \).”

Tommy started with the linear pairs on the same side of the transversal; \( \angle 7 + \angle 2 = 180^\circ \), \( \angle 1 + \angle 8 = 180^\circ \) and made explicit the condition that \( \angle 2 \) and \( \angle 8 \) were corresponding angles, as a justification to replace the \( \angle 2 \) with \( \angle 8 \) in arriving at the valid conclusion \( \angle 7 + \angle 8 = 180^\circ \).

One can notice that he started with two linear pairs even though one pair is enough to come up with the proof. Because the proof involved \( \angle 7 \) and \( \angle 8 \), he might have taken the linear pairs that involved both the angles; \( \angle 7 + \angle 2 = 180^\circ \) and \( \angle 1 + \angle 8 = 180^\circ \). Then, he might have realized that he can prove it by using either of the linear pairs by switching \( \angle 2 \) with \( \angle 8 \) in the first linear pair as \( \angle 2 \) and \( \angle 8 \) are corresponding angles or by switching \( \angle 1 \) with \( \angle 7 \) in the second linear pair as \( \angle 1 \) and \( \angle 7 \) are corresponding angles. It showed his ability to utilize the additional data to provide the proof in another way also.

Tommy continued to operate on the Level 3, as he was able to produce a deductive text once the ideas of proof were brought to light.

**Analysis of Task 10**

The task was to prove that the sum of the interior angles of a triangle was \( 180^\circ \). The students were given the figure below:
The properties of triangles were discussed earlier and the teacher asked the students to prove the triangle sum theorem in particular. Ryan’s argumentation was significant and he presented a valid proof for the theorem. As the class was missing a few students when Ryan presented the proof, the teacher asked Tommy and two other students in the next to present the proof that Ryan presented earlier. The following discussion ensued:

![Figure 4.109: Triangle Sum](image)

Figure 4.109: Triangle Sum

Ricky: \( \angle 4 + \angle 1 + \angle 5 = 180^\circ \).

Jeremy: Because it is a straight line.

Tommy: Yeah, we said it was 180 degrees because…

Jeremy: That is a straight line, that’s why we said it was 180 degrees.

[The students discuss what to write and they write \( \angle 4, \angle 2; \angle 3, \angle 5 \) on the board and add the word alternate interior beneath those two pairs.]

Teacher: This is… This is a phase where you are learning from one another. Okay. So… you… I will… so… What did you do there? Let me ask you a question. Why did you put? What is the \( \angle 4 \) and \( \angle 2 \) and what are the \( \angle 3 \) and the \( \angle 5 \)?

Ricky: They are interior, alternate interior.

Tommy: Yeah.

Teacher: So, what are you going… where are you going from there.

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Ricky, Jeremy, Tommy: We would replace the interior, ∠4 and ∠2.

The students started with particular reference to ∠4 + ∠1 + ∠5 = 180° (line 10.135) the sum of the angles on a straight line and then presented the alternate interior angle pairs; ∠4, ∠2; ∠3, ∠5 (line 10.138). They then replaced the congruent alternate interior angles in ∠4 + ∠1 + ∠5 = 180° to prove that ∠2 + ∠1 + ∠3 = 180°. The students concluded in the following way:

10.162 Teacher: What they did was… instead of the ∠4, they put ∠2.
10.163 Tommy: Yeah.
10.164 Teacher: Instead of the ∠5…
10.165 Jeremy: We put ∠3.
10.166 Teacher: They put…
10.167 Students: ∠3.
10.168 Teacher: ∠3. So the answer is ∠1 and…
10.169 Students: ∠2 plus ∠3.
10.170 Teacher: ∠2 plus ∠3 is
10.171 Tommy: 180 degrees.

Tommy, Jeremy and Joey replicated the proof that Ryan presented before, but they clearly understood the working of the proof as they understood how the data; ∠4 + ∠1 + ∠5 = 180°, ∠4 ≅ ∠2, ∠3 ≅ ∠5 produced the result, ∠2 + ∠1 + ∠3 = 180°. Tommy was considered to be a part of the group operating on Level 3.

**Analysis of Task 11**

In Task 11 the students had to prove the exterior angle theorem. The students produced their proofs independently and presented them on the board. The task was to prove that ∠4 = ∠1 + ∠2 from the below figure:
Tommy first modified the figure to the figure below:

Figure 4.112: Tommy’s Exterior Angle

He then explained the proof in the following manner:

11.86 Tommy: I said that $\angle 4 + \angle 1$ ... I mean $\angle 4 = \angle 1 + \angle 2$ because... well... first I put a 5 here and then a 6. [Pointing to where he added the angles.]

11.87 Teacher: Aha.

11.88 Tommy: Then, I said that $\angle 6$ and $\angle 2$ are interior [Interior opposite angles] and I need my paper.

11.89 Teacher: Okay. We have to give them time. This is a thinking process right? We cannot rush. Let them think, why they think $\angle 4 = \angle 1 + \angle 2$, okay?

He came back again and finished explaining his proof in this way:

11.158 Tommy: I figured it out.

11.159 Teacher: He figured it out. Guys... guys listen.

11.160 Tommy: I said that $\angle 1 + \angle 6 = 4$. So, if that can happen... then...

11.161 Teacher: Let me write that for you. Okay. You want to write? Okay. Guys you should be listening.

11.162 Tommy: $\angle 4 = \angle 1 + \angle 6$. So if $\angle 1 + \angle 6 = \angle 2$ and $\angle 6$ are interior. So...

11.163 Teacher: You mean... alternate interior, right?

11.164 Tommy: Yeah.

11.165 Teacher: Okay.
Tommy: So… then we switch $\angle 6$ with $\angle 2$.

Teacher: Oh… Hey, hey guys. He gave a very good reasoning. Listen to what he said.

Tommy: And then… after that $[\angle 1]$… that $[\angle 2]$ equals that $[\angle 4]$.

Tommy first changed the triangle to a right triangle, added a parallel line to that at the vertex opposite to the base to prove the theorem. He started with right triangle because he might have thought that it may easier to work with a right triangle as on angle already known. But his proof did not contain any thing that is specifically applicable to only right triangle.

There was uniqueness in his approach to this theorem. It showed the creative aspect of his approach. If one observes the figure that he drew, one can notice that he was capable of locating angles in the right way even though the construction caused multiple angles at the vertex opposite to the base of the triangle. For instance, he correctly recognized $\angle 4$ and $\angle 1 + \angle 6$ (line 11.162) respectively as the alternate interior angles and then replaced the $\angle 6$ with $\angle 2$ (line 11.166) while referring them as alternate interior angles.

It is worth noting this accomplishment, as in the case of some students, the multiple angles formed at the vertices with added constructions to the given figure, threw them off guard. They found it difficult even to present their data sets whereas in this particular case it seems that Tommy’s thinking might be more mature in that aspect. But he faltered with multiple angles in the next task on parallel lines when he made the figure more complex. However in the current task he did an excellent job in terms of presentation of the proof.
Tommy continued operating on Level 3 as he presented a complete deductive proof with explicit warrants linking the data; \( \angle 4 = \angle 1 + \angle 6, \angle 6 \cong \angle 2 \) to the claim, \( \angle 4 = \angle 1 + \angle 2 \).

**Analysis of Task 12**

The teacher in this task presented the figure below excluding the top horizontal line and asked them to prove that \( \angle 3 = \angle 1 + \angle 2 \) from it. The students worked independently and presented their proofs on the board. Tommy added all the angles shown below and explained his proof in the following way:

![Figure 4.113: Tommy’s Parallel Line Task](image)

12.49 Tommy: I drew a line at the top first.
12.50 Teacher: You did a parallel line or just the line.
12.51 Tommy: Uh... a parallel line.
12.52 Teacher: Okay.
12.53 Tommy: And then I was thinking that \( \angle 4 \) and \( \angle 6 \) were interior and exterior but I noticed that I was messed up and then I noticed that \( \angle 3 \) and \( \angle 6 \) were neither and so I messed up too.

[Students laugh]

He added an additional parallel line on the top of the figure and formed additional angles, \( \angle 6 \) and \( \angle 5 \). As a result of that construction, multiple parallel lines formed in the figure, and the alteration presented a breach in his understanding about the different angles formed near the intersection of the transversal and the three parallel lines. This can be seen in his statement that \( \angle 4 \) and \( \angle 6 \) were interior and exterior (line 12.53). However,
he accepted responsibility for not being able to recognize the related pairs of congruent angles to proceed with the proof (line 12.53). Tommy’s inability to arrive at the data could be related to identifying the relationships between the components of the diagram. This must have arisen due to the complexity of the diagram itself. Even though he was able to deal with multiple angles formed at vertex in conjunction with two parallel lines in the previous task, he might have been confused with the multiple angles in combination with three parallel lines. As the figure changed, the students had to deal with more complex systems of relationships than those that existed in the original one.

Even though in this task, Tommy was operating on Level 2 and was not able to provide a valid proof in this task, he had already exhibited his understanding of the process of proof production in all the preceding tasks. Managing to interpret the features of the figure seemed to be the difficulty in this task which in turn inhibited him from proving the theorem.

**Summary of Tommy’s Work**

Tommy started off on Level 1, in Task 1, with an understanding that rephrasing a conjecture again in a different way is not the proof of the theorem in itself. Delbert in that task, rephrased the conjecture that the students made earlier about the congruence of vertical angles and presented it as the proof. Tommy did not think it was argument that could be presented as a proof. In this task, he felt the obligation to present an argument even though he could not present one, himself. At the beginning of Task 2, he again started operating on Level 1 as he approached the proof by using theorems that were not proven. Due to the intervention of the teacher in making him understand the importance
of using proven axioms and theorems, he approached the proof differently and presented a complete valid proof.

The uniqueness of Tommy’s work lay in the fact that he started gaining knowledge about the nuances of proof production and about the shape of a complete proof earlier than others. By Task 2, he realized the importance of linking relevant data to the claim and backing up reasoning with exclusive warrants related to the logical links.

He understood the importance of presenting warrants to justify his assertions which could be tracked in his arguments during later episodes. He operated in the third phase in all the remaining tasks, except in the parallel line task, where he might have lost focus on the target at hand by making the figure more complex with third parallel line.

Once he understood the proof structure, he consistently presented a complete proof all through the remaining tasks except for the task on parallel lines, by providing justifications to his assertions in every step towards achieving that goal. However the regression that we see in these students to less sophisticated proof practices under the stress of a difficult problem illustrates their immaturity as provers.

**Analysis of Group Work**

The purpose of doing the task wise group analysis is to look into the evolution of thinking of the group as well as the individual’s thinking in relation to the group over the course of the task. Instances of discussions that enable to look at the sophistication in reasoning are presented. The analysis not only helps us to understand the performance of the group as a whole but also provides us an insight into the ways in which a group promotes an individual’s progress. The ways in which students explain and justify their arguments, argue and counter argue statements, modify their own arguments in relation to
the demands of the group can be observed in the group analysis. It is a general assumption that the group often will be able to operate on a higher level than all of the students. That’s because a prior conversation provides some elements for thought, and then a student picks up on some of those elements to produce a higher level argument which means that student did not produce that higher level argument alone. The group, as a whole, may be able to do better than all of the individuals who make it up and the group analysis. In this way, the group, collectively, provides resources that enable all students to able to progress in their level of reasoning. The group analysis provides us with evidence of these aspects of group work.

**Task 1**

In this task, students had to prove the vertical angle theorem. Students first started with putting some of their thoughts on paper before starting the discussion. Once they had some ideas to talk about, they started to explain what they had written. One can see that some students started off with feeling an obligation to create an argument but were not quite sure how to do it. Instances of which are presented below:

1.4 Delbert: They are congruent because they are on opposite sides of one another.

Even though Delbert was just rephrasing the conjecture, he felt an obligation to present an argument. Another student, Jeremy, said:

1.18 Jeremy: They are congruent because it matters what angle it is.
1.19 Teacher: It matters what angle it is? Can you be more…can you be more clear about what you are trying to say?
1.20 Jeremy: It matters what angle it is because if they are not the same angle they are not the same measure.

Jeremy was just referring to the fact that the vertical angles can be congruent, only if their angles measures are the same. Both these students were operating on the
Level 1 of reasoning where the cultural practice of feeling an obligation to create an argument was present, but they were not yet able to present one on their own. They were just restating the conjecture, thinking that itself constitutes a sufficient explanation.

Some students started to engage visually with the statement by associating oneself with *met– befores*. Tall (2007) who coined the word *met– before* defines it as “Part of the individual’s concept image in the form of a mental construct that an individual uses at a given time based on experiences they have met before” (p. 1). Two instances of students referring to the symbols used in geometry and making an argument for the proof from those symbols is below presented:

Darren started his argument as follows:

1.14 Darren: I said you could just use a pencil or something and make a circle around the lines and you should like measure it in your head whether it is a right angle or obtuse, how you can get that this is a right angle or obtuse.

Darren started engaging visually with the statement by creating an image. He drew the following figures to explain his reasoning.

He first drew two intersecting lines and then drew an arc as shown below:

![Figure 4.114: Darren’s Vertical Angles 1](image)

1.32 Darren: It’s in the mind . . . I would see how far they are apart and I would draw a circle like this.
When the teacher questioned him about that, he completed the circle as shown below:

![Figure 4.115: Darren’s Vertical Angles 2](image)

He presented the following argument:

1.34 Darren: No, I will go like that and see how far they are apart and they would be the same.
1.35 Teacher: How do you? We are not still able to visualize. I understand your circle part. You drew a circle, that’s right. But how do you?
1.36 Darren: You see how these two are like that. I would see how far they are apart on each side and like that.

By his actions in the video, it seems that he was making reference to the fact that equal arc lengths subtend equal central angles. Darren was merely linking perceptual elements. He had no argument to offer either to establish that the arcs were of equal length or to justify his claim that therefore the angles must be equal. Using the symbol of the arc as a symbol for an angle is conjuring up for Darren something about circles, making him to construct a circle and then proceed from there. Tall’s notion of met–befores theorizes this kind of reference as guided not by the internal logic of the current problem, but the free association to prior experiences that is in accordance with Level 1 of reasoning. Also, it seems that Darren is misconstruing a symbol (the arc) that is providing information about the diagram in question for a geometric figure, part of the diagram, itself. Actually the arc is an “iconic symbol” or just an “icon” to use Bruner’s (1966) terminology, because its physical structure is related to its meaning.
Laila explained her proof as below:

1.40  Laila:  Okay. This is what I say. I drew a box… because in my 4th grade, my teacher said that we can draw a perfect square… its 90°.

Figure 4.116: Laila’s Vertical Angles

In fact she was referring to the symbol to represent a right angle ‘\[\text{\L}\]’ at the intersection of the lines. Laila was trying to use the symbol for the right angle rather than a square at the intersection. So the teacher questioned her:

1.45  Teacher:  You said… you wanted me to draw a square.
1.46  Laila:  Yes.
1.47  Teacher:  Everything is a 90°. Yah…I drew a square for you.

The teacher pointed out to the edges of the square which are right angles.

1.48  Laila:  Well… then they are both equal. They are both equal. Isn’t it?

Even at this time, the teacher did not clearly understand Laila’s underlying assumption that the symbol for the square itself was, itself, the 90° angle. So the teacher continued:

1.49  Teacher:  We are measuring [Pointing to the center of intersection of the two lines] Not this one. [Pointing to the edges of the square drawn]
1.50  Laila:  Oh… [Stomps her foot]

In this task, Laila used her knowledge of the symbol for a right angle and that the four right symbols drawn, form a square. She was referring to the case where all the angles are right angles. Laila was exploring the theorem at a very basic level as she was
just using a symbol and a single measure to prove a theorem. She also cited an authority to justify her argument. As one can see, she was operating on Level 1 of reasoning.

Some students started off with reference to geometric transformations. For instance:

1.25  Carl:  I said it’s congruent because it looks like a reflection and all intersecting lines have at least two pairs. [He was referring to angle pairs]
1.26  Teacher:  Oh…that’s a new thing. It’s a reflection.
1.27  Julia:  I agree if you have a mirror, we can see that. I agree.

Both Carl and in turn Julia, presented a mental imagery in order to produce an argument for the proof.

Carl went on to add:

1.29  Carl:  I meant this, if you took a mirror or if you just see like how far these are…and draw a circle like Darren said. That’s what I did. I took my marker and lined it like that and then I noticed that they are same. That’s when I thought of mirror and looked at it, then when I noticed that you make an ‘X’ and I kept drawing an ‘X’ and it always… to see that it looks the same thing.

Carl talked two things when he presented his argument. First, he referred to reflection in proving that the vertical angles are congruent (line 1.25). The other was that if he had multiple cases of intersecting lines, the vertical angle pairs would be congruent in all cases (line 1.29). Here, he went for a proof based on multiple representations to make a generalization of the vertical angle pair congruence and provided that as the proof itself. These approaches are considered to be the very basic levels in context of proof production. He was operating on Level 1.

Darren moved beyond his first explanation and provided another proof, this time, with reference to specific measures:
He gave a concrete case to prove his assertion. All these students started to operate on Level 1 where, they created arguments with reference to measurements, geometric transformations, and justifications based on particular examples to arrive at a general case.

As the discussion unfolded, some students started to gather information known to them with reference to the figure given and presented their proofs. Instances of which are presented below.

![Figure 4.117: Vertical Angles](image)

Dalton started with $\angle PRQ = 180^\circ - \angle SRQ$. With Ricky’s help he came up with $\angle SRT = 180^\circ - \angle SRQ$ and concluded that:

1.123 Dalton: Because these are here there [circling $\angle SRQ$] congruent. So it’s just these two...
1.124 Teacher: Okay.
Dalton: And these are congruent [pointing to $\angle SRQ$] so this one [pointing to $\angle PRQ$] is congruent to [pointing to $\angle SRT$] without using the measurement.

He referred to $180^\circ - \angle SRQ$ being the same, as the $\angle SRQ$ is common in both expressions, and hence $\angle PRQ = \angle SRT$. In this task, Dalton was able to distinguish general arguments from discussions of a specific case. Dalton started to operate on the Level 2 of reasoning.

Jeremy proved the other pair using the linear pair $\angle QRS + \angle PRQ = 180^\circ$, and rearranged it in the form of $\angle QRS = 180^\circ - \angle PRQ$. He then took $\angle PRT = 180^\circ - \angle PRQ$ and concluded that $\angle QRS = \angle PRT$.

In the same episode Laila said that she would prove, $\angle PRQ \cong \angle SRT$, that Dalton proved earlier in a different way. After Laila started writing $\angle PRQ + \angle QRS = 180^\circ$ and $\angle SRT + \angle TRP = 180^\circ$ on the board, Dalton made a suggestion.

Dalton: It could be $\angle TRP$ and... yeah... it could be $\angle TRP$, but this is what I say. Since you have $\angle TRP$ here and $\angle SRT$ here [Referring to the second equation] $\angle QRS$ here [Referring to the first equation] ... you should have $\angle SRQ$ in... [Referring to the second equation].

He suggested to her that it would be easy if both linear pairs share a common angle. Laila understood his implication and changed the second angle pair to $\angle SRT + \angle SRQ = 180^\circ$. This suggestion played a crucial role in the process of the production of this proof. This instance shows that Laila had moved from her initial level of reasoning and started to operate on Level 2.

These are some instances where one can see a progress in achieving the target with guidance from the students who are able to think in a level beyond their peers. This is in accordance with the Vygotskian perspective that, “Children’s participation in
cultural activities with the guidance of more skilled partners allows children to internalize the tools for thinking” (Rogoff, 1990, p.13) and in this context the students are enculturated into the ways of proving through constant interactions with more knowledgeable others.

All the students started in Level 1 of reasoning in this task. Delbert and Jeremy started with feeling an obligation to create an argument but they were not yet able to create arguments of their own. Darren, Carl, Laila started creating arguments with reference to measurements, geometric transformations, and justifications based on particular examples to arrive at a general case. Justifying ones argument with reference to an example is typical to students operating on the Level 1 of reasoning. As the discussion unfolded, some students started to gather information known to them. For example the idea of Dashawn to make use of linear pair postulates in proving the theorem, was a step more advanced than doing a heuristic exploration. The students slowly started to gather data in pieces as a group by helping one another with supplement missing pieces of information. There was critique to the arguments presented by one another which served as a feedback and students modified their arguments to the demand of logicality. The group proceeded to a stage, where they were able to present relevant data. Most of the time the arguments provided was a result of group work. Even though arguments were initiated by Dashawn and Laila, the students were able to present complete argument through collective effort. The group as a whole operated on Level 2, as the students started providing the logical links between the data and claim though not fully aware of the argumentation structure of proof leading form premises followed by reasons of support.
**Task 2**

While proving the theorems related to parallel lines, the teacher divided the students into two groups and the discussions were recorded separately. This was adopted to increase the talk time of individuals in the group. The participants remained in the same group till the theorems on parallel lines were proved. In Task 2, the students proved that a pair of alternate interior angles are congruent.

**First Group Work**

The following figure was generated to help them in the process of proving. The specific task was to prove that $\angle 6 \cong \angle 4$.

![Figure 4.118: Alternate Interior Angles (First Pair)](image)

Laila and Dalton initiated the discussion by starting off with corresponding angles for proving the theorem. The students considered $\angle 4 \cong \angle 8$ as the first data set. They told others that they were collecting known information as an initial step in the proof process. The group also thought of taking, $\angle 1$ and $\angle 6$ as it was one more corresponding angle pair that can be inferred from the figure. The motive, as one can understand is that they were trying to take the possible corresponding angle pairs involving $\angle 4$ and $\angle 6$. Darren asked
them to consider $\angle 3$ and $\angle 7$ along with $\angle 1$ and $\angle 6$. Laila said to him there was no point in taking $\angle 3$ and $\angle 7$ as they were proving $\angle 4 \cong \angle 6$.

2.74 Laila: We are, like, trying to find out how $\angle 4$ and $\angle 6$ are congruent, right?
2.75 Teacher: $\angle 4$ and $\angle 6$ are congruent, right.
2.76 Laila: Why do you pull out, what do you get? What do you get from $\angle 3$ and $\angle 7$?
2.77 Teacher: Think… think some more.

After a few minutes, Dalton said he noticed something:

2.83 Dalton: Ms. Indira, I just noticed this…
2.84 Teacher: Come show me what you noticed. Ssh… give them a… give them time to think. Aha.
2.85 Dalton: $\angle 6$ and $\angle 8$.
2.86 Teacher: Oh, you wanted $\angle 6$ and $\angle 8$. Guys, he says $\angle 6$ and $\angle 8$. What are those?
2.87 Dalton: They are… I think they are…
2.88 Laila: Interior…
2.89 Dalton: I really don’t know…
2.90 Teacher: $\angle 6$ and $\angle 8$
2.91 Laila: Interior? Vertical…

It seemed, at that point students were finding it difficulty with the vocabulary of the different angles associated with the parallel lines. Both Dalton and Laila came up to that point and started pondering their next step. Ryan who was looking at the data that was provided them perceived a logical link that connected the data $\angle 4 \cong \angle 8$ and $\angle 6 \cong \angle 8$ to the claim $\angle 6 \cong \angle 4$. He concluded as follows:

2.95 Ryan: $\angle 4$ and $\angle 8$ is congruent, and $\angle 6$ and $\angle 8$ is congruent.
2.96 Ryan: So, they both equal, so $\angle 6 \cong \angle 4$.

The group continued to operate on Level 2 while one or two students were still not in that level of reasoning. Even though the group arrived at a valid conclusion, the
individuals who constitute the group are presenting only partial arguments which are filled in by others.

**Second Group Work**

The following figure was provided by the teacher to help them in the process of proving. The specific task was to prove that $\angle 6 \cong \angle 9$.

![Diagram of Alternate Interior Angles (First Pair)](image)

Figure 4.119: Alternate Interior Angles (First Pair)

The students started to discuss in pairs. Julia initiated the conversation by asking the others to join the group, and she explained her approach to the proof. This group started off with Julia presenting a data set. But the data she started with was based on theorems that have not been proved earlier.

2.127 Julia: We are proving that $\angle 6$ equals $\angle 9$. $\angle 6$ equals $\angle 8$ which is equal to $\angle 1$ and $\angle 1$ equals $\angle 9$.

$\angle 8$ and $\angle 1$ formed a pair of alternate exterior angles. The alternate exterior angle theorem was not proved by this task. The teacher reminded them that they have to consider taking theorems that were proved earlier. Tommy and Joey also added their input to the discussion but had difficulty in presenting the data. They started off with $\angle 6 \cong \angle 8$; $\angle 9 \cong \angle 8$ and $\angle 1 \cong \angle 6$.
2.150 Joey: Since $\angle 1$ is vertical to $\angle 6$; $\angle 6$ is vertical… wait no… $\angle 6$ is congruent to corresponding to $\angle 8$.

2.151 Teacher: Aha.

2.152 Joey: And… $\angle 1$ and $\angle 6$ and $\angle 9$; $\angle 6$ and $\angle 1$ are vertical to each other… $\angle 1$ and $\angle 9$ are congruent.

2.153 Teacher: You are saying $\angle 6$ and $\angle 8$; $\angle 1$ and $\angle 6$ right? So what are you saying from $\angle 6$ and $\angle 8$ and $\angle 1$ and $\angle 6$? What are you saying from $\angle 6$ and $\angle 9$ and $\angle 1$ and $\angle 6$?

2.154 Joey: Oh man… I forgot what I wanted to say.

Since they listed three congruent pairs, they were confused as to which two pairs would lead them to their goal. Joey recognized the need to apply previous theorems as part of one’s argument, but could not co-ordinate the elements of the proof at this juncture. The discussion continued and Julia again presented another data set:

2.163 Julia: Since we have to prove $\angle 6$ equals $\angle 9$, $\angle 6$ equals $\angle 8$ right here and $\angle 8$ equals $\angle 9$.

As she pondered to see the connection between the corresponding angles $\angle 6$ and $\angle 8$ and the vertical angles $\angle 8$ and $\angle 9$, Delbert who was watching her, make the coherent link between the data and claim by concluding that:

2.181 Delbert: $\angle 6$ equals $\angle 8$ and $\angle 8$ equals $\angle 9$. So these two, [pointing to $\angle 6$ and $\angle 9$] should be the same.

Delbert was able to see relationship between the data and the claim and guided others to approach in that way to complete the proof. The group as a whole assembled the relevant information, and reached a sound deductive conclusion. The group was operating on Level 3.

In both the episodes, the negotiation of meaning played a major role. As Bauersfeld (1988) points out, “Learning is characterized by the subjective reconstruction of societal means and models through negotiation of meaning in social interaction” (p.39). A classroom microculture is a venue for such a social interaction, where meanings
are negotiated among the students. Newman et.al, defined the negotiation as the process of *mutual appropriation*, where students gain insight into the utterances of each other to mutually build up the body of mathematical knowledge.

The success in proving the task also was based on the negotiation of meaning. As these negotiations played around, one can see a spiraling of success of the group which in turn acted as suction for individual growth. Voigt’s (1995) view point, “Through mutual accommodations, the participants form the impression that they know what mathematics teaching and learning is” (p.176) supports the idea of success.

**Task 3**

**First Group Work**

The students had to prove that \( \angle 5 \cong \angle 7 \). They had the figure below as a frame of reference:

![figure](image.png)

*Figure 4.120: Alternate Interior Angles (Second Pair)*

Marcy started with a data set; \( \angle 3 \) and \( \angle 7 \) a pair of corresponding angles. However, all the other students wanted her to consider a different pair; \( \angle 5 \) and \( \angle 9 \). The group settled for the data \( \angle 5 \) and \( \angle 9 \) that all the students agreed upon. Then, Ryan
directed the group’s attention to $\angle 7$ and $\angle 9$, a pair of vertical angles which contained one of the angles in the corresponding angle pair that the group initially decided to put forth.

But when the teacher asked what the outcome of taking that data was, Ricky started off with an entire new set of data; $\angle 3 \cong \angle 7$; $\angle 7 \cong \angle 9$ and proved that $\angle 9 \cong \angle 3$. This was in fact an alternate exterior angle pair. He proved the alternate exterior angle theorem instead of the alternate interior angle theorem. One can see that some students were trying to think of proofs of their own, rather than depending on the group’s input though unsuccessful sometimes in their attempts to attain the target. When the teacher reminded Ricky of what needed to be proved Laila came up and presented a valid argument leading to the result. The group cheered with her.

3.29  Laila: If these are the same, like these two [Pointing to $\angle 7 \cong \angle 9$ and $\angle 5 \cong \angle 9$] congruent… and this has… [Showing $\angle 9$] in both of them… [Pauses for a second]

3.30  Students: Ooh… Yeah…
3.31  Teacher: She is doing it, right?
3.32  Laila: And $\angle 7$, $\angle 7$ equals $\angle 9$ and $\angle 9$ equals that… [Pointing to $\angle 5$]. So both of them are congruent [Pointing to $\angle 5 \cong \angle 7$]

The group as a whole came up with the data and Laila found the data claim link. Here also one can see that the group was operating on Level 2 while one or two students were still trying to understand the nuances of proof production and attempting to understand the approaches of others.

**Second Group Work**

The group had to prove that $\angle 4 \cong \angle 7$. They had the figure below:
In this group, Tommy started with a data set consisting of a pair of corresponding angles, a pair of vertical angles and a pair of alternate exterior angles; $\angle 4$ equals $\angle 5$, $\angle 5$ equals $\angle 3$ and $\angle 3$ equals $\angle 7$ and proves that, $\angle 4 \cong \angle 7$.

3.35 Tommy: $\angle 5$ equals $\angle 3$ and $\angle 3$ equals $\angle 7$.
3.36 Teacher: Tommy, can you repeat that baby?
3.37 Tommy: $\angle 4$ equals $\angle 5$ and $\angle 5$ equals $\angle 3$ and $\angle 3$ equals $\angle 7$ and implies that $\angle 4 \cong \angle 7$.
3.38 Teacher: But $\angle 5$ equals $\angle 3$ is needed to be proved right? Just get me the eraser for a minute because I want to erase this because you have to prove that. So you cannot take it be granted right?

Here, the teacher reminded him that he could not use the theorems which have not been proved yet. The reference was to the use of the alternate exterior angle pair. He was operating on Level 1 at that point. After thinking for some more time, Tommy came back again to present a different approach as shown below:

3.43 Tommy: I think.
3.44 Teacher: I think… Tommy got it.
3.45 Tommy: $\angle 4$ equals $\angle 3$ and $\angle 3$ equals…
3.46 Delbert: $\angle 7$.
3.47 Teacher: Tell us that… do you want to tell us that as a group or one person go there and tell me… Write that for me baby. Tommy, write it for me, or show it to me here first.
3.48 Tommy: $\angle 4$ is vertical to $\angle 3$ and $\angle 3$ is corresponding angle to $\angle 7$. 
3.49 Teacher: Perfect. Put it over, put it over here. Let me see, how it goes on the board. [Tommy writes] \( \angle 4 \) is congruent to \( \angle 3 \); \( \angle 3 \) is congruent to \( \angle 7 \).

3.50 Teacher: So why \( \angle 4 \) and \( \angle 3 \) are congruent? Can you tell me why they are congruent?

3.51 Joey, Julia: Vertical.

3.52 Teacher: They are vertical. Good. Put vertical for me in the bracket, there across the \( \angle 4 \) and \( \angle 3 \). Just put vertical, v, just put ‘v’ there in the bracket. So they are vertical. The ‘v’ stands for vertical. Why \( \angle 3 \) and \( \angle 7 \) are congruent?

3.53 Tommy: Corresponding…

3.54 Teacher: So put ‘C’ for me in the bracket. Like put a bracket for me. Put a parenthesis over there.

3.55 Student: Parenthesis.

3.56 Teacher: Just like I kept here and put one for the ‘V’ too. So what can you say about \( \angle 4 \) and \( \angle 7 \) now?

3.57 Tommy: They are congruent.

3.58 Teacher: Can you write that for, also that.

3.59 Delbert: \( \angle 4 \) equals \( \angle 7 \) [Tommy writes] they are congruent.

3.60 Teacher: Okay. So, do you guys realize what you did now, you have actually proved that the alternate interior angles are congruent.

Tommy arrived at the valid conclusion, \( \angle 4 \cong \angle 7 \). The group as a whole was operating on Level 3 as the students understood the argument presented by Tommy.

In the case of both the groups, though it seems that only two students finally proved the theorem, the videos show that the others were paying attention to what was going on and how the arguments led to the proof of the theorem. The case was of internalizing the dynamics of the process on the part of the other members of the group. In both the groups, there was a progress in achieving the target with guidance from the students who were able to think in a level beyond their peers. This is in accordance with the Vygotskian perspective, expressed by Cobb (1995) as, “It is reasonable to treat learning as primarily a process of enculturation, and to emphasize the crucial role played by both children’s interactions with more knowledgeable others and their mastery of tools that are specific to the culture” (p.123).
Task 4

Task 4, was related to the proof of the alternate exterior angle theorem.

First Group Work

The students had to prove that $\angle 3 \cong \angle 9$.

![Diagram of Alternate Exterior Angles](image)

Figure 4.122: Alternate Exterior Angles (First Pair)

For a change, the teacher asked the group members to write on the board instead of her writing it on the board. Dalton was the writer. The discussion was initiated by Dalton and they started with a pair of congruent vertical angles $\angle 7 \cong \angle 9$ and a corresponding angle pair $\angle 5 \cong \angle 9$. Dalton asked one of his friends to think of what was the possible outcome from the data. Jeremy gave them a logical conclusion but the data provided them with the proof of the alternate interior angle theorem instead of the alternate exterior angle theorem. The group ended up in showing that $\angle 5$ and $\angle 7$ were congruent instead of $\angle 3$ and $\angle 9$ which was the required task. When the teacher pointed out the mistake done by the group, Ryan wanted to try and prove the theorem. He started with a different data set and proved the theorem.

4.33 Teacher: Give him a chance, give him a chance first. So you want to take $\angle 3$ and $\angle 7$, Ryan? Okay, but what are these two angles? $\angle 3$ and $\angle 7$?
4.34 Laila: \( \angle 3 \) and \( \angle 7 \), I can’t see?
4.35 Ryan: They are corresponding.
4.36 Teacher: They are corresponding?
        [Students talking]
4.37 Teacher: Don’t do that. [Addressing another student]
4.38 Teacher: So what can you say about? Now Ryan wants to say something. What do you want to say?
4.39 Ryan: \( \angle 7 \cong \angle 9, \angle 3 \cong \angle 9 \ldots \angle 7 \cong \angle 9, \angle 3 \cong \angle 7 \), Therefore \( \angle 7 \) is common and \( \angle 3 \cong \angle 9 \).

The group was initially operating on Level 1 of reasoning as the group lost focus on attaining the target at hand. At a later stage, one student was able to coordinate the elements of the argument in a way that was consistent with a mathematically sound argumentation. The other students of the group also understood the approach to the proof and acknowledged the proof. The group was operating on Level 2 by the end of the task. However, they were still not providing justifications for the arguments presented.

**Second Group Work**

The second group had to prove that \( \angle 7 \cong \angle 6 \).

![Figure 4.123: Alternate Exterior Angles (First Pair)](image)

Figure 4.123: Alternate Exterior Angles (First Pair)
Even in this group, the teacher asked one the students to be the writer on the board. Julia requested the teacher if they could do the proof together to which the teacher agreed. She then provided an explanation of the proof by presenting relevant data. She had; \( \angle 7 \cong \angle 2 \), \( \angle 2 \cong \angle 4 \) and \( \angle 4 \cong \angle 6 \) as data and approached the proof using the transitivity rule twice when she is linked; \( \angle 7 \cong \angle 2 \), \( \angle 2 \cong \angle 4 \) and \( \angle 4 \cong \angle 6 \) to arrive at \( \angle 7 \cong \angle 6 \). After Julia presented her proof, the group presented a different data set. The students began to question among themselves the other possibilities for proof.

4.82 Julia: \( \angle 7 \) equals \( \angle 4 \) and \( \angle 4 \) equals \( \angle 6 \)?
4.83 Tommy: Yeah.
4.84 Delbert: \( \angle 7 \) equals \( \angle 4 \) and \( \angle 4 \) equals \( \angle 6 \)?
4.85 Julia: Which one?
4.86 Joey: \( \angle 7 \) is vertical to \( \angle 4 \) and \( \angle 4 \)…
4.87 Delbert: You want me to write that down?
4.88 Tommy: Yeah.
4.89 Delbert: Okay. \( \angle 7 \) equals \( \angle 4 \) and \( \angle 4 \) equals \( \angle 6 \).
4.90 Teacher: So you are the representative right? You are just recording what you all thought of?
4.91 Tommy: Aha.
[Delbert writes]
4.92 Teacher: Guys when he is writing, you have to look whether he is writing what you want him to write. He wrote \( \angle 7 \) is congruent to \( \angle 4 \). Why is \( \angle 7 \) congruent to \( \angle 4 \)?
4.93 Students: Because they are vertical.
4.94 Teacher: They are vertical? Can you ask him to put that… in a bracket like in a parenthesis somewhere? Yeah. Vertical angles. Okay, and then he wrote \( \angle 4 \) is congruent to \( \angle 6 \). Guys, \( \angle 4 \) is congruent to \( \angle 6 \). Why?
4.95 Julia: [Inaudible] they look like each other.
4.96 Teacher: Julia, he wrote \( \angle 4 \) is congruent to \( \angle 6 \). So what was the reason behind writing that?
4.97 Julia: Because \( \angle 6 \) is equal to \( \angle 4 \) and \( \angle 4 \) is equal to \( \angle 7 \). So \( \angle 6 \) is equal to \( \angle 7 \). Which means \( \angle 7 \) is equal to \( \angle 6 \).
4.98 Teacher: Yeah… we got that, what are \( \angle 4 \) and \( \angle 6 \)? That’s what I am asking, Tommy?
4.99 Tommy: Hah…?
4.100 Teacher: Why did you all put \( \angle 4 \) and \( \angle 6 \)? What are they?
4.101 Tommy: Because they are corresponding angles.
4.102 Teacher: They are corresponding angles? Okay. Can you put corresponding angles for me there?
4.103 Delbert: Write across it?
4.104 Teacher: Yeah, put it in the parenthesis, yeah… because they are corresponding angles.
    [Delbert writes]
4.105 Teacher: So what did you all conclude from that? He can be your speaker too, if you want. Can you tell us why… what you finally came to? The conclusion… can you tell us?
4.106 Delbert: Umm…
4.107 Teacher: He wants you to help him. If anybody wants to come there, they can come too. This is your group work, right?
    [Students discuss again]
4.108 Delbert: We need the conclusion, right?
4.109 Teacher: Aha. Conclusion is? You said $\angle 7$ is congruent to $\angle 4$ and $\angle 4$ is congruent to $\angle 6$ right? So what did you conclude from that? Why?
4.110 Students: $\angle 7$ equal $\angle 6$ and $\angle 6$ equals $\angle 7$.

The group as a whole came up with a valid conclusion from the data presented by their peers. The group as well as some individuals was operating on Level 2 as they were organizing the elements into a cogent argumentation, but still were not providing justifications for their steps.

In these episodes one can see that the microculture of involving in the practices of explaining and proving was being strengthened and as Voigt (1995), “The microculture can be constituted without the participants talking about it explicitly, and it can be accomplished indirectly, by way of mathematical activities” (p.178).

**Task 5**

The students proved the second pair of alternate exterior angles in this task.

**First Group Work**

The students had to prove, $\angle 1 \equiv \angle 8$. 

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The teacher in this task directed Jeremy to prove the theorem. The teacher employed this strategy as a way of providing an opportunity to the students who were not so outspoken but had their own ideas to present if given a chance. Jeremy willingly attempted to prove the theorem and when he seemed to pause, others helped him to prove the theorem.

5.1 Teacher: You have to show me $\angle 1 \cong \angle 8$. Can you show it now? Come here. Jeremy wants to… because I want everybody to be involved. [Jeremy writes]

5.2 Teacher: Show us; how you show us $\angle 1 \cong \angle 8$ Ssh… look at what he is doing.

5.3 Jeremy: $\angle 6$ equals to $\angle 8$.

5.4 Teacher: Stand… Stand next to him if you want to help him okay? $\angle 6 \cong \angle 8$. Why is $\angle 6 \cong \angle 8$? Jeremy?

5.5 Jeremy: Because they are vertical.

5.6 Teacher: Vertical angles? Okay next. [Jeremy writes $\angle 4 \cong \angle 8$]

5.7 Teacher: Let him do it. Let him do it, Dalton. $\angle 4 \cong \angle 8$. Why are $\angle 4$ and $\angle 8$ congruent?

5.8 Jeremy: They are corresponding.

5.9 Teacher: They are corresponding. Perfect. So what can you say now? [Jeremy thinks]

5.10 Students: Pooh…

5.11 Teacher: He needs one more? Dalton help him what he needs.

5.12 Dalton: $\angle 1$ and $\angle 4$

5.13 Teacher: He needs $\angle 1$ and $\angle 4$? He says you need $\angle 1$ and $\angle 4$.

5.14 Students: Ooh…
5.15 Teacher: Now, can you, can you summarize that for us? To prove \( \angle 1 = \angle 8 \)?

5.16 Jeremy: \( \angle 1 = \angle 4; \angle 4 = \angle 8 \); So…

5.17 Teacher: Therefore…

5.18 Jeremy: Therefore \( \angle 1 \cong \angle 8 \).

Here the group as well as the individual who presented was operating on Level 2 again as they were presenting data in support of the argument structure, but were not giving the reasons of support for their premises.

**Second Group Work:**

The group had to prove that \( \angle 3 \cong \angle 8 \).

![Figure 4.125: Alternate Exterior Angles (Second Pair)](image)

When the group started to present the proof, they chose one of the group members to write it on the board on their behalf. Delbert started with \( \angle 3 \cong \angle 5 \) and \( \angle 5 \cong \angle 8 \) and concluded that \( \angle 3 \cong \angle 8 \). The input for the arguments came from different students and the individual contributions also increased. Tommy started with \( \angle 3 \cong \angle 1 \) which he referred to as vertical angles. As he was looking for another angle that he could associate \( \angle 1 \), Carl suggested \( \angle 8 \). So Tommy considered that angle and added that \( \angle 1 \cong \angle 8 \), as they were corresponding angles. He finally arrived at the valid conclusion that \( \angle 3 \cong \angle 8 \).
5.47 Tommy: ∠3 is vertical… ∠3 is congruent to ∠1.
5.48 Teacher: Aha. Why?
5.49 Tommy: Because they are vertical.
5.50 Teacher: Okay. They are vertical.
5.51 Tommy: ∠1 mean no… yeah ∠1 is…
5.52 Teacher: Aha.
5.53 Tommy: Is… yeah.
5.54 Carl: To ∠8.
5.55 Tommy: Yeah… it’s corresponding.
5.56 Teacher: Chris, can you say it louder? ∠1 is congruent to what?
5.57 Carl: ∠8, ∠8.
5.58 Teacher: ∠8. Why are they congruent?
5.59 Tommy: Because they are corresponding angles.
5.60 Teacher: Aha. They are corresponding?[Tommy claps]
5.61 Teacher: So what can you conclude from that?
5.62 Tommy: That ∠3.
5.63 Teacher: Aha.
5.64 Tommy: ∠3 is congruent to…
5.65 Teacher: To what angle?
5.66 Tommy: ∠8.
5.67 Teacher: Perfect. You… you proved this right?

Tommy finally arrived at the valid conclusion that ∠3 \cong ∠8. The group as well as individual participants were operating on Level 2 as the justifications were not made explicit on their own without which the argumentation is not whole.

Both the groups as whole were operating on Level 2 consistently. The interaction that the students had in these episodes as well as previous ones seems to be more than a sequence of actions and reactions as Voigt (1995) puts it. He explains it in detail as below:

One participant of an interaction monitors his or her action in accordance with what he or she assumes to be the other participants’ background, understandings, expectations and so forth. At the same time, the other participants make sense of the action by adopting what they believe to be the actor’s background, understandings, intentions, and so forth. The subsequent actions of the other participants are interpreted by the former actor with regard to his or her expectations and can prompt a reconsideration, and so on (p. 169).
Task 6

In this task, the teacher presented the theorem to the groups and they were asked to prove it. They did not conjecture about this theorem earlier. The task was to prove the consecutive interior angle sum theorem.

First Group Work

The students had to prove that, $\angle 3 + \angle 5 = 180^\circ$.

The discussion was initiated by Dalton. Dalton came forward and presented what he had on his paper. He set out with $\angle 4$ and $\angle 3$ as the linear pair and $\angle 4$ and $\angle 5$ as the congruent corresponding angle pair.

6.20 Dalton: This is what I was thinking. If $\angle 3 + \angle 5 = 180^\circ$ then $\angle 4$, $\angle 3$ is a linear pair and $\angle 4$ is corresponding angle to $\angle 5$. They are congruent and that’s all I got.

The teacher wanted him to do the same on the board so that everybody could see what he achieved. But when he came to explain it on the board, he had some difficulty in explaining his proof.

6.30 Dalton: $\angle 3 + \angle 5$. Okay, we are trying to show that $\angle 3 + \angle 5 = 180^\circ$, which already $\angle 4$, $\angle 4$ and $\angle 3$. These two are a linear pair [Referring to $\angle 4$ and $\angle 3$] which makes $180^\circ$, and $\angle 4$ and $\angle 5$. $\angle 4$ and $\angle 5$ are corresponding angles, which makes them congruent and so we have that part…
Some group members started to pick up the threads of those arguments and tried to complete the proof.

6.44 Teacher: Okay, Laila wants to say something.
6.45 Laila: Okay, if \( \angle 3 \cong \angle 4 \), \( \angle 5 \cong \angle 6 \). No?
6.46 Teacher: Linear pair, right? You are confused with the word linear pair.
6.47 Laila: Linear pair.
6.48 Teacher: Aha.
6.49 Laila: \( \angle 4 \) and \( \angle 6 \), and \( \angle 3 \) and \( \angle 5 \)
6.50 Teacher: Tell us again.
6.51 Laila: \( \angle 4 \) and \( \angle 5 \).
6.52 Teacher: Aha.
6.53 Laila: \( \angle 3 \) and \( \angle 6 \).
6.54 Teacher: You have to prove \( \angle 3 + \angle 5 \) is…
6.55 Laila: \( \angle 4 \) and \( \angle 5 \) are congruent. These two are… add up to equal to 180° [pointing to \( \angle 4 \) and \( \angle 3 \)] Oh man… I forgot what to say…
6.56 Teacher: Guys, you should be helping her. Talk something. Help her in something.
6.57 Laila: If these are like 180° and this and that… [Pointing to \( \angle 4 \) and \( \angle 5 \); \( \angle 3 \) and \( \angle 6 \)] They are all congruent… Uggh…
6.58 Laila: I was going to say that these are 180° [marking \( \angle 4 \) and \( \angle 3 \), marking \( \angle 5 \) and \( \angle 6 \)] and \( \angle 4 \cong \angle 5 \) and \( \angle 3 \cong \angle 6 \). They are all congruent to each other.

Laila came up with \( \angle 4 + \angle 3 = 180° \), \( \angle 5 + \angle 6 = 180° \), \( \angle 4 \cong \angle 5 \) and \( \angle 3 \cong \angle 6 \) but she could not arrive at \( \angle 3 + \angle 5 = 180° \) by linking the relevant data. Jeremy, who was observing the ongoing discussion, geared up and helped his friends in proving the theorem (line 6.67). He used the data that Dalton had initially and concluded:

6.67 Jeremy: Place the \( \angle 4 \) with \( \angle 5 \) and \( \angle 3 + \angle 5 = 180° \).

In this task, the students started to present warrants for their arguments. The group also started to operate on Level 3 as they collectively understood how the conditions lead to the result in the theorem.

**Second Group Work**

The students had to prove that \( \angle 3 + \angle 5 = 180° \).
Carl and Tommy discussed the theorem on their own for a few minutes. Carl volunteered to prove the theorem and produced a valid deductive argument. The discussion ensured in the following manner:

6.88 Carl: I solve this problem.
6.89 Teacher: You want to solve that?
6.90 Carl: Yeah... like the $\angle 1$ plus $\angle 2$ equals $180^\circ$. [He was referring to the pair that was proved earlier.]
6.91 Teacher: Okay. Can you like... can you show us.
6.92 Carl: So... you have to know like... linear pairs and linear pairs is like... something like this.
6.93 Teacher: Aha.
6.94 Carl: You see... this equals to $90^\circ$ and this $90^\circ$. $90^\circ$ plus $90^\circ$ equals $180^\circ$. So you find something linear.
6.95 Teacher: Aha.
6.96 Carl: And so... you see the $\angle 3$ and $\angle 4$, add, equals to $180^\circ$.
6.97 Teacher: Aha.
6.98 Tommy: They are linear pairs.
6.99 Teacher: They are linear pairs too? Tommy?
6.100 Tommy: Aha.
6.101 Carl: $\angle 5$ and $\angle 6$ equals to $180^\circ$.
6.102 Teacher: Aha.
6.103 Carl: So, all you have to do is...
6.104 Teacher: So... this time you are approaching it as like linear pairs?
6.105 Tommy: Aha...
6.106 Teacher: All of you are on the same page? All of you think that.
6.107 Carl: So, this is pretty easy, so all you have to do is, put the 5 where the 4 is.
6.108 Teacher: Aha.[Carl replaces the $\angle 4$ with the $\angle 5$.]
6.109 Carl: And that’s how you get $\angle 3$ equals $\angle 3$ plus $\angle 5$ equals $180^\circ$.
6.110 Teacher: Can... can you really write what you are saying?
6.111 Tommy
& others: I get it.

Carl started with the linear pairs \( \angle 3 + \angle 4 = 180^\circ \) (line 6.96) and \( \angle 5 + \angle 6 = 180^\circ \) (line 6.101) and then replaced the \( \angle 4 \) with \( \angle 5 \) (line 6.107) in \( \angle 3 + \angle 4 = 180^\circ \) to prove the theorem.

Though the argumentation was provided by Carl, still the group was in agreement with the proceedings of the approach. However the group was operating on Level 2 as the elements of the argument were organized but the warrant required in support of the link, namely the corresponding angle postulate was not made explicit to make the argumentation more wholesome.

Task 7

The Task 7 was related to prove that the consecutive interior angles on the same side of a transversal are supplementary, given the lines are parallel.

First Group Work

The task was to prove that \( \angle 2 + \angle 1 = 180^\circ \).

![Diagram](image)

Figure 4.128: Consecutive Interior Angles (Second Pair)

Instances of individual argumentations are presented below. Laila was the first one to present her proof and she did it in this manner.
Laila was operating on Level 3 as she presented the complete proof. She set out with relevant data like $\angle 7 + \angle 2 = 180^\circ$ and $\angle 7 \cong \angle 1$ to prove the theorem at hand, linked them in a logical and deductive way to arrive at the result that $\angle 1 + \angle 2 = 180^\circ$. She also provided the warrant, the corresponding angle postulate to justify her reasoning.

Then, Ricky came to the board to give a try at the proof. He proceeded in the following way:

7.25 Ricky: $\angle 2$ and $\angle 8$ are corresponding. So let me put it like this, $\angle 2 \cong \angle 8$.
7.26 Teacher: Okay, $\angle 2 \cong \angle 8$. So where are you...? Where are you getting from these?
7.27 Ricky: $\angle 1 + \angle 8 = 180^\circ$
7.28 Teacher: You are taking this one? [Pointing to $\angle 1$ and $\angle 8$, $\angle 1 + \angle 8 = 180^\circ$. So what are you doing now?
7.29 Students: Oh...
7.30 Teacher: Let him try.
7.31 Ricky: We select... these are congruent [Pointing to $\angle 2 \cong \angle 8$, same thing $\angle 1$ and $\angle 2$, just like here [pointing to $\angle 1 + \angle 8 = 180^\circ$]... I cannot explain.

Ricky was very close to the conclusion but found it difficult to explain it. He went back to his seat to re-think. After some time he came back and said:

7.56 Ricky: $\angle 1 + \angle 8 = 180^\circ$, $\angle 8 \cong \angle 2$. So $\angle 8$ and $\angle 2$ are congruent. $\angle 2$ wait... $\angle 8 + \angle 1 = 180^\circ$ also... So if $8$ equals... $\angle 8 \cong \angle 2$... $\angle 8$ should be congruent to...
Ricky was still not able to organize his reasoning. Ricky was operating on Level 2. At that point, Dalton came to the board and proved the theorem in the following manner:

7.33 Dalton: And \( \angle 2 \) and \( \angle 8 \) are congruent, corresponding.
7.34 Teacher: Corresponding right? Let me put the words there. These are corresponding. This is a linear pair right? Okay where are you going from these?
7.35 Dalton: \( \angle 7 \) and…
7.36 Students: \( \angle 7 \)?
7.37 Dalton: \( \angle 7 \) and \( \angle 2 \) are linear pairs.
7.38 Teacher: Okay that’s also a linear pair. Okay…
7.39 Dalton: And \( \angle 7 \) and \( \angle 1 \) form a linear pair… no congruent.
7.40 Teacher: \( \angle 7 \) and \( \angle 1 \), okay \( \angle 7 \) and \( \angle 1 \). Let him try. He already did the last one right, \( \angle 7 \) and what?
7.41 Dalton: \( \angle 1 \)
7.42 Teacher: \( \angle 7 \) and \( \angle 1 \).Okay.
7.43 Dalton: And so…
7.44 Teacher: You have to show \( \angle 2 + \angle 1 = 180^\circ \) so show us from that.
7.45 Dalton: \( \angle 7 + \angle 2 = 180^\circ \) and replace \( \angle 7 \) with … \( \angle 1 \) …

Dalton was operating on Level 3. The group was also operating on Level 3 as a result of the effort of individual students operating at that stage though a few were operating on Level 1. It is because there is a certain group dynamic that operates at the level of the flow of events, even if those events are just a result of one or two individuals, or even of no individual, but just the pieces assembling themselves into proofs. What makes that fast moving current have integrity as a social process is taken– as– shared practices. It is the fact that certain ways of operating cause no dissension, cause no comment or question, that enables them to operate as the social nexus for the whole group. Even in cases where some students are not able to keep up with the flow, and they may make some incorrect comments, or operate in a way that is not helpful to the progress of the group, they do not stop the progress of the group.
Second Group Work

The task was to prove that $\angle 1 + \angle 2 = 180^\circ$.

Figure 4.129: Consecutive Interior Angles (Second Pair)

Julia, Delbert, and Joey came to the board and presented a proof together. The following discussion brings to light the development of proof of this theorem:

7.80 Delbert: $\angle 7$ and $\angle 2$ are linear pairs and $\angle 1$ and $\angle 8$ are linear pairs.
7.81 Teacher: Okay. Write them for me. Let me write them for you. You are saying $\angle 7$ and…?
7.82 Delbert, Julia, Joey: $\angle 2$.
7.83 Teacher: $\angle 2$ are linear pairs.
7.84 Julia: And $\angle 8$ and $\angle 1$
7.85 Teacher: $\angle 8$ and $\angle 1$ are linear pairs too? Okay. Where are you going from there?
7.86 Julia: Because we thought…
7.87 Delbert: Because they are corresponding angles [pointing to $\angle 7$ and $\angle 1$.] [Delbert draws an arrow from $\angle 7$ to $\angle 1$.]
7.88 Teacher: Aha.
7.89 Julia: Put the equal thing.
7.90 Delbert: Corresponding angles are congruent.
7.91 Teacher: Okay. So you are saying, because $\angle 7$ and $\angle 1$ are corresponding which one do you want to replace, in this one [$\angle 7$ plus $\angle 2$ equals $180^\circ$] or this one? [$\angle 8$ plus $\angle 1$ equals $180^\circ$]
[The students point to $\angle 7$ plus $\angle 2$ equals $180^\circ$.] In the top one? The top one becomes what? Joey, can you write that for me? What will the top one become now? You are replacing…
7.92 Joey: $\angle 1$ and the $\angle 7$.
7.93 Teacher: Okay. $\angle 1$…
[Joey writes $\angle 1$ plus $\angle 2$ equals.]
7.94 Teacher: Equals.
7.95 Joey: It’s [inaudible]… $180^\circ$. 
The students produced a deductive proof by making their arguments explicit with data like \( \angle 7 + \angle 2 = 180^\circ \) (line 7.80) and \( \angle 7 \cong \angle 1 \) (line 7.87). They backed the arguments with the use of linear pair properties and corresponding angle pair congruence (lines 7.80, 7.87). In this task one can see that a prior conversation provided the students with some elements for thought, and then one student picked up on some of those elements to produce a higher level argument. The group, as a whole, is able to do better than all of the individuals who make it up.

This episode also had a particular aspect that is worth mentioning. Delbert in this task introduced a notation for representing congruent angles (line 7.87). When he presented his argument, he used an arrow between them for showing that the corresponding angles \( \angle 7 \) and \( \angle 1 \) are congruent. This influenced this particular group in using this kind of representation, which one can see in the presentations leading to the proof in the next task. This is a classic example of classroom taken as shared meanings.

The second group continued to operate on Level 3 as they understood how the conditions lead to the result in the theorem.

The argumentation of both groups bring to notice, that these students were slowly progressing towards intellectual autonomy in the sense that they “are aware of, and draw on, their own intellectual capabilities when making mathematical decisions and judgments as they participate in (these) classroom practices” (Yackel & Cobb, 1996, p.473).
Task 8

The students had to prove that the consecutive interior angles are supplementary.

First Group Work

The task was to prove that $\angle 7 + \angle 5 = 180^\circ$.

Figure 4.130: Consecutive Exterior Angles (First Pair)

The students started to work together and Ryan came to the board to prove the theorem. Laila and Dalton started to help him initially, but Laila took over the proving at one point. Laila started with two linear pairs: $\angle 7 + \angle 2 = 180^\circ$ and $\angle 6 + \angle 5 = 180^\circ$. She and Dalton then proceeded to switch $\angle 5$ and the $\angle 2$.

8.30 Laila & Dalton: So you should... replace the $\angle 2$ with $\angle 5$ and... $\angle 7$ plus $\angle 5$ equals $180^\circ$.

When the teacher questioned about the reason how she switched the angles, Laila could not explain. Dalton filled in the answer for her by saying that:

8.56 Dalton: I say... replace the $\angle 2$ with the $\angle 5$ because they are... corresponding angles.

The first group as a whole still was operating on Level 3 as they were bringing the ideas of proof to light and organizing them into deductive texts.
Second Group Work

The students had to prove that $\angle 7 + \angle 8 = 180^\circ$.

![Figure 4.131: Consecutive Exterior Angles (First Pair)](image)

The teacher wanted the second group to present their proofs individually on the board. They first wrote their proofs on the board and explained them later as their chance came. Three students in the group started to use a kind of representation of arrows, to specify congruent angle pairs. As mentioned in the earlier task, Delbert first used this kind of representation for showing congruent angle pairs and in this task, these students did the same. The proofs that they presented on the board are shown below:

Joey’s proof:

\[
\begin{align*}
\angle 7 + \angle 2 &= 180^\circ \\
\angle 8 + \angle 1 &= 180^\circ
\end{align*}
\]

Julia’s proof:

Problem: $\angle 7 + \angle 8 = 180^\circ$

C: $\angle 7 \cong \angle 1$  Linear: $\angle 7$ and $\angle 2$

$\angle 2 \cong \angle 8$  $\angle 8$ and $\angle 1$

$\angle 7 + \angle 2 = 180^\circ$

$\angle 1 + \angle 8 = 180^\circ$ [switch.]

$\angle 7 + \angle 8 = 180^\circ$

Delbert’s proof:

$\angle 7 + \angle 2 = 180^\circ$

$\angle 1 + \angle 8 = 180^\circ$ [They are corresponding angles.]

Tommy’s proof:

$\angle 7 + \angle 2 = 180^\circ$

$\angle 1 + \angle 8 = 180^\circ$

So then $\angle 2$ and $\angle 8$ are corresponding and $\angle 1$ and $\angle 7$ are corresponding and then you can...
They switch because they are corresponding angles and they are congruent.

Carl’s proof:

\[ \angle 7 + \angle 8 = 180^\circ \]

Note!!!

That corresponding angles always supposed to be congruent.

Corresponding angles and they are allowed to be replaced as long as they are congruent.

Figure 4.131: Carl’s Consecutive Exterior Angles (First Pair)

Most of the students were autonomously operating on Level 3 as each student produced a deductive text and organized their reasoning deductively. The group itself is considered to be operating on Level 3. Carl’s proof stands out as Carl explained the proof in the figure above. He considered the linear pair \( \angle 1 + \angle 8 = 180^\circ \) which he showed with an angle simple encompassing \( \angle 1 \) and \( \angle 8 \) and then replaced \( \angle 1 \) with \( \angle 7 \), shown with the arrow from \( \angle 7 \) with \( \angle 1 \) as they were corresponding angles to arrive at the conclusion that \( \angle 7 + \angle 8 = 180^\circ \).

It seems that students were using this representation to isolate the congruent angles. In the simplest language, we would say \( \angle 4 \) is the supplement of \( \angle 3 \), but since \( \angle 3 \) is equal to \( \angle 9 \). So, mentally though they were enacting the language of substitution, the children, when faced with the task of presenting it were adopting this representation to
show the substitution. Students were using this as a ready means to express their understanding.

This episode also brings to light, “Explanations and justifications that individual students give in specific instances and the classroom mathematical practices that become taken-as-shared” (Yackel, 2001, p.5). In this context, the students took the representation of congruent angles with an arrow first started by Delbert in a previous task as a taken-as-shared meaning and applied it when they presented their proofs.

**Task 9**

Task 9 was related to the proof of the second pair of consecutive exterior angle theorem.

**First Group Work**

The task was to prove that $\angle 4 + \angle 9 = 180^\circ$. This group started to work independently in this task.

![Figure 4.132: Consecutive Exterior Angles (Second Pair)](image)

Ricky’s proof:

$\angle 4 + \angle 3 = 180^\circ$

$\angle 1 + \angle 9 = 180^\circ$

$\angle 3$ and $\angle 9$ are corresponding angles

So $\angle 4 + \angle 9 = 180^\circ$

Ryan’s proof:

$\angle 4 + \angle 3 = 180^\circ$

$\angle 1 + \angle 9 = 180^\circ$

So $\angle 4 + \angle 9 = 180^\circ$
Dalton’s proof:
\[ \angle 1 + \angle 9 = 180^\circ \]
\[ \angle 4 \text{ and } \angle 1 \text{ are corresponding angles.} \]
\[ \angle 1 + \angle 9 = 180^\circ \]
\[ \angle 4 + \angle 3 = 180^\circ \]

So you replace the \( \angle 1 \) with \( \angle 4 \) because they are corresponding angles.

\[ \angle 4 + \angle 9 = 180^\circ \]

Laila’s proof:
\[ \angle 4 + \angle 3 = 180^\circ \]
\[ \angle 1 + \angle 9 = 180^\circ \text{ (Corresponding angles)} \]

So \( \angle 4 + \angle 9 = 180^\circ \)

Darren’s proof:
\[ \angle 4 + \angle 3 = 180^\circ \]

Jeremy’s proof:
\[ \angle 4 + \angle 3 = 180^\circ \]

Only the first group proved the second pair of the consecutive exterior angle theorem. The students presented their proofs on the board individually and explained them. After the students presented their proofs, the teacher asked them to evaluate the proofs of other students and to provide their opinion on the proofs presented. The students were unanimous in saying that they all approached the theorem in a similar way.

The teacher then asked them if there were other ways to prove the theorem to which Laila responded by taking a different data set and arrived at a valid conclusion.

9.78 Laila: Okay. \( \angle 7 \) is congruent to \( \angle 9 \) and \( \angle 4 \) is congruent to \( \angle 5 \).
9.79 Teacher: \( \angle 4 \) and \( \angle 5 \). Where are \( \angle 4 \) and \( \angle 5 \)? Okay. \( \angle 4 \) and \( \angle 5 \). Good. Let me write for you. So that you just, just tell me, I will write… She said… \( \angle 7 \) is congruent to \( \angle 9 \); \( \angle 5 \) is congruent to \( \angle 4 \). Okay. Okay. Let her tell and then…
9.80 Laila: Okay. Dalton you can probably help because…
9.81 Teacher: These are exterior.
9.82 Laila: Angles.
Alternate exterior angles are congruent.
Okay. I got it. Guys observe, she is thinking deeply. Yeah.

When they are congruent, you can write the \( \angle 4 \) plus \( \angle 7 \).

\( \angle 4 \) and the \( \angle 7 \). Oh… you want to write \( \angle 4 \) plus \( \angle 7 \).

Oh… my goodness. Guys look… there are lots of ways of proving all this, right… so she thought of one way… Very good. She said… \( \angle 4 \) plus \( \angle 7 \) equals 180° and then she replaced \( \angle 7 \) with the…

Oh… Okay.

\( \angle 9 \).

\( \angle 9 \)…

Oh… Yeah…

Very good Laila… very good thinking. Give her a clap.

Laila approached the proof by taking the linear pair \( \angle 4 + \angle 7 = 180° \) and alternate exterior angle pairs \( \angle 7 \cong \angle 9 \) and \( \angle 4 \cong \angle 5 \). She then switched \( \angle 7 \) with \( \angle 9 \) to arrive at the valid conclusion that \( \angle 4 + \angle 9 = 180° \). She was operating on Level 3.

The group was operating on Level 3 as the students organized the proofs into deductive texts and also produced data claim links followed by explicit warrants. Even in this group, four students adopted a system of representation for showing congruent angles.

Also as discussed in the earlier task, “Classroom mathematical practices, in contrast, focus on the taken-as-shared ways of reasoning, arguing, and symbolizing established while discussing particular mathematical ideas” (Cobb et.al, 2001, p.126).

Task 10

From this task onwards, both the groups were combined together. The students had to prove the triangle sum theorem. The task was to prove that \( \angle 1 + \angle 2 + \angle 3 = 180° \) using the figure below:
In this task, the teacher first gave the students the above figure with a triangle and a parallel line passing through one of the vertices. The teacher asked the students to prove that sum of the angles $\angle 1 + \angle 2 + \angle 3$ was $180^\circ$. The teacher asked the students to sit in as group and work together. Ryan wrote something on his paper that he wanted to present. Since the other students were not able to see what had written, the teacher asked him to come and explain it on the board. Laila and Julia also said that they had some ideas. The proof that Ryan presented was very systematic and logical. The discussion that led to the proof is presented below:

10.33 Teacher: Ryan wants to say something, let me see.
10.34 Ryan: Well… when we draw line ‘l’, on top, I thought of parallel lines, and drew one at the bottom and I saw how they made angles….So I made the angles, $\angle 4, \angle 5, \angle 6$ and $\angle 7$, then…
10.35 Teacher: Let...let me see yours. Okay, I think it’s ...do you guys think it
is better if I ask you to write it on the board?
10.36 Students: Yes.
10.37 Teacher: Because everybody can see it right?
10.38 Students: Yah....
10.39 Teacher: It’s not clear here. Ryan wants to say. I will give a chance to
everybody, Okay, and then we will agree on one thing, which
everybody agrees on, okay?
10.40 Julia: [inaudible]
10.41 Teacher: Okay. Let him explain what he thinks.
10.42 Ryan: And...I saw them make angle and I named it, ∠4, ∠5, ∠6 and
∠7.
10.43 Teacher: Oh... he named them...Okay.
10.44 Ryan: And I saw that...∠6 and ∠2 are linear.
10.45 Teacher: Linear?
10.46 Ryan: Linear angles.
10.47 Teacher: You mean linear pair?
10.48 Ryan: Linear pairs and therefore 180°.
[He writes 6+ 2 = 180°]
10.49 Teacher: These are not number 6 and number 2 right? They are angles.
Even though he didn’t put that he meant angles. It’s not 6 and
2. I just wanted to make sure.
10.50 Ryan: And ∠3 & ∠7...
[He writes ∠3 + ∠7 = 180°]
10.51 Teacher: Is 180°?
10.52 Ryan: 180° and ∠4, ∠1 and ∠5 is also 180°.
10.53 Teacher: ∠4 and ∠1 and what is the other number?
10.54 Ryan: ∠5.
10.55 Teacher: ∠5. [Addressing other students] Okay. Somebody...when you
see somebody thinking about that, you can get some more
ideas too right? So think about what he is trying to do.
10.56 Ryan: Then I saw that ∠4 and ∠3 are interior...
10.57 Teacher: ∠4 and ∠3?
10.58 Laila: I can’t see. ∠4 and ∠3. Will they be corresponding?
10.57 Teacher: Okay, you were trying to say interior angles? [Addressing
Ryan]
10.58 Ryan: Yeah.
10.59 Ryan: ∠4 and ∠3 are interior angles. ∠5 and ∠2 are interior angles
and the I saw that ∠4 and ∠3 are interior angles, I can replace
the ∠4 and the ∠3; ∠5 and ∠2 are interior, so I replace the ∠5
with the ∠2 and it would be ∠3 + ∠1 + ∠2 = 180°.

Ryan presented the linear pairs ∠6+ ∠2 = 180° (line 10.44), ∠3 + ∠7 = 180° (line
10.50) and then stated that he would take ∠4 + ∠1 + ∠5 = 180° (line 10.52) formed by
the parallel line. He then expressed that he would replace \( \angle 4 \) with \( \angle 3 \) and \( \angle 5 \) with \( \angle 2 \) (line 10.59). At that point the teacher wanted to raise a question about the replacement to which Laila responded and pointed him the mistake.

10.62 Teacher: I have one question Ryan. Does anybody have a question or can I go? He says \( \angle 4 \) and \( \angle 3 \) are interior; I have a question about that.

10.63 Laila: Oh…I know something. \( \angle 4 \) is not interior… \( \angle 4 \) and \( \angle 2 \).

10.64 Teacher: It’s \( \angle 4 \) and \( \angle 2 \). What was yours? [Addressing Dalton]

10.65 Dalton: \( \angle 4 \) and \( \angle 3 \) would be exterior?

10.66 Teacher: Anybody has still a question about it? Okay, look again. He says \( \angle 4 \) and \( \angle 3 \).

She made him to see that one interior angle pairs was \( \angle 4 \) and \( \angle 2 \) (line 10.63).

Ryan rectified his mistake and proved the theorem. Ryan was operating on Level 3. Laila then presented her proof. She drew the figure below, but did not know how to arrive at the proof from that figure:

![Figure 4.135: Laila’s Triangle Sum](image)

Julia came to help her but did not use Laila’s figure. Instead, she used Ryan’s figure and they both discussed the problem:

10.81 Julia: Well… \( \angle 3 \) and \( \angle 7 \) equal 180\(^\circ\).

10.82 Teacher: Okay. He also wrote that, right?

10.83 Julia: Yes, but \( \angle 4 \) plus \( \angle 1 \) plus \( \angle 5 \) will be equal to 180 degrees too. [She writes \( \angle 1 + \angle 4 + \angle 5 = 180^\circ \)]

10.84 Teacher: Okay. Yeah, write it \( \angle 4 \) plus \( \angle 1 \) plus \( \angle 5 \).

[Julia writes \( \angle 1 + \angle 4 + \angle 5 = 180^\circ \)]

[Laila and Julia talk in low tones]

10.85 Teacher: You got it.

10.86 Laila: \( \angle 6 \) and \( \angle 2 \) would be 180\(^\circ\) too.
Julia and Laila understood what Ryan had done and proceeded on the same grounds. A few students were absent that day. So the teacher in the next session asked Jeremy, Ricky and Joey to present the proof of the theorem to the absentees. Even though they were not able to produce the proof independently on their first attempt, they were able to understand the proof presented by Ryan. They formulated the proof in the way Ryan approached it and proved the theorem. The group was operating on Level 3, as the students understood how to apply previous theorems and was able to check the conditions required in getting the desired result. Also the students were able to produce a deductive text.

One can see the role that the Vygotskian perspective, the zone of proximal development (ZPD) plays in a classroom setting. The ZPD is defined as, “the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under the adult guidance or in collaboration with more capable peers.” (Vygotsky, 1978, p. 86).

**Task 11**

In task 11, the students were required to prove the exterior angle theorem. The students worked independently and then presented their proofs on the board. Here the episode is analyzed through three aspects that were inferred from the proofs presented by the students. The first one is related to the outcome that arose due to the multiplicity of angles caused by additional constructions applied to the original figure. The second one is related to the replication of arguments related to previous theorems. The third one is
related to the change in the thinking of students when their peers questioned about inconsistencies observed in the initial arguments. First, the instances where the students lost track of their proof production when they modified the given figure, thus resulting in multiple angles either at a single vertex or at different vertices are given below:

Dalton altered the initial figure by extending one more side of the triangle to prove that \( \angle 4 = \angle 1 + \angle 2 \).

\[ \angle 3 + \angle 4 = 180^\circ \]
\[ \angle 5 + \angle 2 = 180^\circ \]
\[ \angle 1 + \angle 2 + \angle 3 = 180^\circ \]
So \( \angle 4 = \angle 1 + \angle 2 \)

When he was asked to explain his proof, he came up with this explanation:

11.52 Dalton: I said, that \( \angle 3 + \angle 4 = 180^\circ \).
11.53 Teacher: Aha
11.54 Dalton: \( \angle 5 + \angle 2 = 180^\circ \).
11.55 Teacher: Aha.
11.56 Dalton: And \( \angle 1 + \angle 2 + \angle 3 = 180^\circ \).
11.57 Teacher: And where did you get the \( \angle 1 + \angle 2 + \angle 3 \)? From your sum of the angles, you said that, right? Okay.
11.58 Dalton: And then I classified that \( \angle 4 = \angle 1 + \angle 2 \)
11.59 Teacher: \( \angle 1 + \angle 2 \). So, he… he was similar to what… uh…. Joey was saying.

Though he adopted a valid deductive approach related to his proof, the additional exterior angle caused by the construction diverted his attention to the data; \( \angle 5 + \angle 2 = \)

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180° which was irrelevant to the task. However he arrived at $\angle 4 = \angle 1 + \angle 2$ from; $\angle 3 + \angle 4 = 180°$ and $\angle 1 + \angle 2 + \angle 3 = 180°$. Though it was obvious that he arrived at the conclusion from those two steps, the fact that he was picking up additional data that would not lead him to the proof showed his immaturity in dealing with multiple angles in a figure. In Dalton’s case we cannot infer that he lost track of his approach to the proof as he organized his proof into a deductive text. He is considered to be operating on Level 3. Tommy referred to the figure below as he approached the proof:

![Figure 4.138: Tommy’s Exterior Angle](image)

Tommy seemed confused as a result of all the different angles in the modified figure. However he returned back after working on his paper for some time and finished his proof:

11.158 Tommy: I figured it out.
11.159 Teacher: He figured it out. Guys… guys listen.
11.160 Tommy: I said that $\angle 1 + \angle 6 = 4$. So, if that can happen… then…
11.161 Teacher: Let me write that for you. Okay. You want to write? Okay. Guys you should be listening.
Tommy: $\angle 4 = \angle 1 + \angle 6$. So if $\angle 1 + \angle 6 =$ that then $\angle 2$ and $\angle 6$ are interior. So...

Teacher: You mean… alternate interior, right?

Tommy: Yeah.

Teacher: Okay.

Tommy: So… then we switch $\angle 6$ with $\angle 2$.

Teacher: Oh… Hey, hey guys. He gave a very good reasoning. Listen to what he said.

Tommy: And then… after that $[\angle 1]$… that $[\angle 2]$ equals that $[\angle 4]$.

As he put some effort to understand the angles, the relationships of the angles untangled and Tommy was able to proceed efficiently form there. Tommy was operating on Level 3 by the end.

Darren presented the proof below and explained it in the following manner:

Darren: Instead of using a regular triangle, I used a right triangle instead.

\[\angle 4 + \angle 1 + \angle 5 = 180^\circ\]
\[\angle 1 + \angle 2 + \angle 3 = 180^\circ\]

Darren: And like Joey said, he has the $\angle 4$ over here and I have $\angle 4 + \angle 1 + \angle 5$.

Teacher: Okay.

Darren: And that equals $180^\circ$ [referring to $\angle 4 + \angle 1 + \angle 5$].

Teacher: How did you say that was $180^\circ$?

Darren: Because to me, it was…

Teacher: Aha…

Darren: I said this could be $90^\circ$.

Teacher: Are you guys listening to what he is saying?

Student: Yeah.
11.71 Darren: That could be a $90^\circ$ and that could be a $180^\circ$. $\angle 2 = 90^\circ$, $\angle 1 = 90^\circ$ and that is $30^\circ$. That would be $20^\circ$.

Darren’s limited understanding of the exterior angle disabled him to realize that $\angle 4$ formed by the construction could not be the exterior angle to the triangle any more. As he was not able to proceed from the data that he started with, he attempted to formulate an argument based on some measures. After he felt that the argument was not convincing enough, he returned back to his seat. He understood the need to apply prior theorems and tried to search for a convincing coherent link. He was not able to recognize the newly formed angles due to the construction added to the figure. It may be because of his limited exposure to the multiple angles at a vertex. He was operating on Level 2.

Delbert explained his proof with reference to the figure that he drew:

Delbert’s proof on the board:

\[
\angle 4 = \angle 5 + \angle 6 \\
\angle 4 = \angle 1 + \angle 2 \quad \text{[Replace $\angle 5$ with $\angle 1$ and $\angle 6$ with $\angle 2$]}
\]

He provided the following explanation for what he wrote on the board:

11.123 Delbert: Okay. $\angle 4$ and $\angle 1$ are like interior. So I draw a line up here to make an angle 5. And like I wrote this like $\angle 6$ as a substitute.

Delbert did not understand that once he drew the parallel line, $\angle 4$ will be the sum of $90$ degrees and $\angle 1$ not just $\angle 1$. He continued:

11.124 Teacher: Where is the $\angle 6$, baby? [He shows $\angle 6$ in the figure.]
11.125 Teacher: Oh… the $\angle 6$ is there. Okay. $\angle 4 = \angle 5 + \angle 6$. [The teacher reads what he wrote on the board.]

11.126 Delbert: I replaced them with $\angle 1$ and $\angle 2$.

11.127 Teacher: Oh…

11.128 Tommy: I don’t see what he did.

11.129 Julia: Even though you are my cousin, I cannot understand you.

The students did not understand what he explained. So the teacher questioned him further:

11.130 Teacher: You guys have to ask when you don’t understand, because I also didn’t understand that… or can you guys like… I have a question, Delbert. How did you replace the $\angle 5$ with the $\angle 6$ here?

11.131 Julia: Yeah.

11.132 Teacher: Julia, did you have the same question? Julia? Or ask your question.

11.133 Julia: I was kind of confused because I saw they were two right angles.

11.134 Teacher: Guys, he replaced the $\angle 5$ with the $\angle 1$ and $\angle 6$ with the $\angle 2$. Ryan has a question.

11.135 Ryan: How could you replace the $\angle 1$ and $\angle 5$ with $\angle 6$ and $\angle 2$, $\angle 6$ and $\angle 2$ equals 180° and $\angle 5$ and $\angle 1$, doesn’t it?

11.136 Teacher: So think about what they are asking, okay? We are not complete yet. Okay. We still have to go through everybody. It was good. Ryan asked how he could replace the $\angle 5$ with the $\angle 1$ and the $\angle 6$ with the $\angle 2$. There should be a logical explanation for it. So think about it. [Delbert goes back to his seat to think about it.]

In the case of Delbert also, the complexity of the angles which resulted due to the modification of the original figure seemed to be a hindrance to his approach to the proof. Even though he understood the obligation to be logically persuasive, the appropriate structure for coordinating the elements of argument into a logical form is not evident. He was operating on Level 2.

Second, the instances where students lost focus on the target to attain while trying to replicate an argument in earlier theorems are presented. When Ricky’s turn came, he
showed the figure below and provided an explanation for his proof in the following manner:

![Figure 4.141: Ricky’s Exterior Angle](image)

11.91 Ricky: \( \angle 6 \) and \( \angle 2 \) are interior.
11.92 Teacher: Ssh… Everybody should be listening. What is it? \( \angle 6 \) and \( \angle 2 \)? Guys he is saying \( \angle 6 \) and \( \angle 2 \).
11.93 Darren: Exterior… No…
11.94 Teacher: Ryan… Ryan is saying something. \( \angle 6 \) and \( \angle 3 \) are interior? Okay. What else Ricky?
11.95 Ricky: \( \angle 5 \) and \( \angle 1 \) are linear.
11.96 Teacher: Only \( \angle 5 \) and \( \angle 1 \) are linear?
11.97 Ricky: \( \angle 5 \), \( \angle 1 \) and \( \angle 6 \).
11.98 Teacher: \( \angle 5 \), \( \angle 1 \) and \( \angle 6 \). Okay. Let me write for you, because sometimes not all kids can explain, right? You said \( \angle 5 \) and \( \angle 1 \) and \( \angle 6 \) they become…
11.100 Teacher: 180°. So where are you going from there?
11.101 Ricky: So… so… I would switch the \( \angle 1 \) with the \( \angle 2 \). I will switch \( \angle 5 \) with the \( \angle 1 \) and the \( \angle 6 \) with the \( \angle 2 \).
11.102 Teacher: \( \angle 5 \) with the \( \angle 1 \)? Show me… show me what you want to switch?
11.103 Ricky: \( \angle 5 \) and \( \angle 1 \); \( \angle 6 \) and \( \angle 2 \).

In this task, Ricky used the idea of using a parallel line to the base of the triangle and tried to replicate the idea of replacing the alternate interior angles as done in the triangle sum theorem (Task 10). Even though by now he knew how to arrive at a proof and understood the structure of a proof, his dependence on arguments presented earlier made it impossible for him to prove this theorem. He was operating on Level 2.
Third, is an instance where some students who were observing and listening to the explanations provided by one of their friends, raised an objection to the inconsistencies in the data to which the student responded, modified her data and came up with a valid proof. Julia first wrote a whole set of data that she inferred from her figure:

\[ \angle 1 + \angle 2 + \angle 3 = 180^\circ \]
\[ \angle 1 \cong \angle 2 \cong \angle 3 \]
\[ \angle 2 \cong \angle 1 \cong \angle 3 \]
\[ \angle 3 \cong \angle 1 \cong \angle 2 \]
\[ \angle 4 \cong \angle 5 \cong \angle 6 \quad \angle 4 + \angle 3 = 180^\circ \]
\[ \angle 5 \cong \angle 4 \cong \angle 6 \quad \angle 2 + \angle 5 = 180^\circ \]
\[ \angle 6 \cong \angle 4 \cong \angle 5 \quad \angle 6 + \angle 1 = 180^\circ \]

Figure 4.142: Julia’s Exterior Angle

Dalton and Ryan had an objection to the data that she started with initially. They say:

11.140 Dalton: I disagree.
11.141 Teacher: You disagree. Why do you disagree?
11.142 Dalton: Because…
11.143 Teacher: Okay. Alright go… He has a doubt about the congruence.
11.144 Dalton: I say that all these congruences aren’t right.
11.145 Julia: Don’t they look similar?
11.146 Ryan: I think what Dalton is trying to say is that you know how you said \( \angle 1 \) and \( \angle 2 \) equal each other.
11.147 Dalton: How she has, \( \angle 3 \) is congruent to \( \angle 4 \). \( \angle 2 \)… \( \angle 2 \)… I say \( \angle 3 \) can’t be congruent to \( \angle 4 \). \( \angle 2 \) can’t be congruent to \( \angle 3 \) and \( \angle 3 \) can’t be congruent to \( \angle 4 \).
11.148 Ryan: Because they are off that side.
11.149 Julia: I know that, but \( \angle 3 + \angle 4 = 180^\circ \) and \( \angle 6 + \angle 1 = 180^\circ \) too, but, \( 180^\circ \) except that they are all… except that they are all equal.
11.150 Ryan: The first step…
[Teacher addressing a student]
11.151 Teacher: You are confused about that too? So let’s just go and sit down. But…
11.152 Julia: Well…
11.153 Teacher: But the thing that they asking, Julia is. They are all… guys… guys… you should listen to me, as I am trying to ask the question that you are asking me. Okay. They are saying. All of
them are interior inside the triangle. They add up to 180°. But they cannot be individually congruent, right?

11.154 Julia: Oh.
11.155 Teacher: So, go and think about it.

Julia started to prove the theorem with wrong data. The objection from Dalton and Ryan brought back her focus to the task. In this episode, one can see that the proof process was becoming more a model of mature mathematics, individual work subjected to review by the community. The feedback process when the proof was not readily accepted by her peers, led Julia to rethink.

Julia returned back after she worked on her paper for some time and proceeded with her explanation:

11.176 Julia: I see why you guys are confused. Let’s see 180°, right here.  
11.177 Teacher: Okay.
11.178 Julia: Am I supposed to put an equal sign here?
11.179 Teacher: You already… yeah… you can put, it doesn’t matter.
11.180 Julia: So I am going to take ∠1 and ∠2. 
11.181 [She writes ∠6 + ∠1 = 180° = ∠1 + ∠2 + ∠3]
11.182 Teacher: Okay. They are both ∠1s.
11.183 Julia: So you mean, they are both ∠1?
11.184 Teacher: Aha.
11.185 Julia: Look… these are… I got rid of this [referring to ∠1], I got rid of this [referring to the other ∠1] they would be equal.
11.186 Teacher: Guys. Oh… oh… guys, guys… [Students clap]

Her proof as she wrote it is presented below:

∠6 + ∠1 = 180° = ∠1 + ∠2 + ∠3  
[If we get rid of ∠1]
∠6 = ∠2 + ∠3

There were two instances where the students presented valid proofs for the theorem. Joey presented the proof below:

∠3 + ∠4 = 180°
∠1 + ∠2 + ∠3 = 180°
So $\angle 4 = \angle 1 + \angle 2$

He explained that first pair was a linear pair and in the second step he applied the triangle sum theorem and concluded the result from those two steps. Joey was operating on Level 3. Carl presented the following proof on the board:

\[
\angle 1 + \angle 2 + \angle 3 = 180^\circ \\
\angle 3 + \angle 4 = 180^\circ \\
180^\circ - \angle 3 = \angle 4 \\
\angle 4 = 180^\circ - \angle 3 \\
\angle 1 + \angle 2 = 180^\circ - \angle 3 \\
\angle 4 = \angle 1 + \angle 2
\]

Carl approached the proof algebraically. Carl was operating on Level 3. Overall the group was operating on Level 3 as the students clearly understood how to provide a logical deductive text for the proof of the theorem. Yackel’s (2000) explanation, “In order to achieve intellectual autonomy, children need a basis for making judgments about what is acceptable mathematically, for example, with respect to mathematical difference, mathematical sophistication, mathematical efficiency, mathematical elegance, and mathematical explanation and justification. However, these are precisely the types of judgments that the teacher and students negotiate when constituting socio mathematical norms that are characteristic of an inquiry tradition. In the process, students construct specifically mathematical beliefs and values that help them form their judgments”. (p. 5) aptly fits in this context of proof production.

By Task 12, as the students were no longer co–participating in the construction of proofs and rather worked independently on their own, the group analyses were not done for the remaining three tasks.
CHAPTER 5
FINDINGS AND CONCLUSIONS

Findings

Cobb (1994) identifies constructivism and socioculturalism as theoretical influences on reform in education. Construction involves thinking of the individual student as the unit of analysis; the teacher’s main goal is to have individual students arrive at cognitive conflicts with respect to the conceptual content to be mastered. The group process is a vehicle for that to happen. Enculturation as described in this study operates differently. The mixing together of construction and enculturation has tended to privilege construction, with enculturation being viewed as a flavoring to the reform pedagogy. Kirshner (2004) expressed the same sentiment by hinting that enculturation was the neglected metaphor in mathematics education. In this light, as reform pedagogy is based on an integrative discourse and does not recognize these two pedagogical objectives as distinct, reform teaching tends to be somewhat unclear and imprecise. In particular, the enculturational goal has not previously been examined as to its structure and process. The current study attempts to remedy this situation.

The findings of this study show that when students are given an opportunity to communicate mathematical ideas through discussion in exploring proofs, the results are rewarding. In particular, communication of thoughts is invariably a guiding tool in refining and shaping thought processes. The current findings show that the group of students involved in this enculturationist pedagogy benefited from the discourse in the classroom. The current study focused on the development of reasoning processes through classroom discourse and on helping the students think like mathematicians. An increase
in the students’ sophistication of reasoning in ways characteristic of mathematical proof was observed in each individual student. The analysis of each student’s work and the group analyses presented in the previous chapter give a detailed account of the students’ progress individually and in relation to the group. However, a few things observed by the teacher—researcher need to be discussed in this chapter.

It was observed that in the process of proof production, sometimes there was a regression to less sophisticated proof practices under the stress of a difficult problem. These proving practices of students illustrated their immaturity as provers. In the course of their professional work, mathematicians are routinely overwhelmed by the complexity of the concepts they are exploring, yet they do not regress in terms of their proof practices. These students’ regression can be attributed to the instability and insecurity in the new argumentation processes that they tended to adopt in these sessions. The students were gradually becoming enculturated to the new forms of reasoning that constitute mature mathematical argumentation (i.e., proof) and adopted this form of argumentation because they were surrounded with it in the classroom microculture. When they ran into trouble, they attempted to seize control of the argumentation process and use past methods that, while more secure, were also less sophisticated.

It was also observed that when students modified the given figures, multiple angles were formed at the respective vertices, which led them to misread these angle labels. The reason for this might be that they have not yet been introduced to more sophisticated practices related to labeling of angles. This gives an idea for incorporating some additional tasks to this curriculum dealing with tasks involving multiple angles at a single vertex.
Students in one of the groups adopted a system of representation of using arrows between congruent angle pairs while approaching a proof. This practice seemed to become taken– as– shared within the classroom microculture. Though it has limitations in relation to the progress of the proof, this kind of representation influenced the prover as it was a part of the group practice. The complicating factor is that mathematicians themselves might function with less than full formal rigor in order to actually accomplish a proof. So, informal ways of expressing oneself are part of what a mathematician does as well and hence can be used to justify the students’ use of these informal representations.

**Conclusions**

The aim of the study was to develop reasoning capabilities and proof production competencies in students. As research shows that there are no effective ways of developing proof production as a cultural practice, the current study employed an enculturationist approach to involve students in the process. The goal to “shape the microculture so that it comes to more closely resemble the reference culture [of disciplinary mathematics] with respect to the target dispositions” (Kirshner, 2004, p. 7) has been achieved by the study. Through *explanation, justification, and argumentation*, there was a gradual increase not only in the students’ sophistication of reasoning, but in their engagement in proof practices as well. From a Vygotskian perspective, these practices came to exist in the community of the classroom microculture through the persistent effort of the enculturationist teacher. The influence of the social norms of the microculture, namely “explaining and justifying solutions, trying to make sense of explanations presented by others, indicating agreement and disagreement, and questioning alternatives in situations in which a conflict between interpretations or
solutions” (Cobb, 1995, p. 22), seems to be spiraling up the group level and in turn the individual participants to the level at which the group is operating. At the same time, the students were also slowly becoming autonomous provers as they were trying to make arguments of their own and give proofs of their own.

The spiraling up of individual growth in relation to group effort is clearly reflected in the discussions leading to the proofs. The analyses of individual and group work provide a glimpse of how students developed a *modus scienti*, or a way of knowing during the tasks. Students’ progress from a level where they did not understand the obligation for an explanation to be logically persuasive to a level where they were able to coordinate the elements of the argument in a way that is consistent with logically sound deductive reasoning is clearly seen in the analysis. A comprehensive color coded graphical representation showing the increase in sophistication of reasoning of individuals, the group, and individuals in relation to the group in all the tasks is presented in Fig 5.1.

The graph provides direct observational evidence that students’ individual practices tend to lag behind the class as a whole and illustrates how their thinking was influenced by its taken– as– shared practices. The interpretation of the graph is an attempt to provide answers for the third research question, “Given the reflexive relationship between the group and individuals who comprise it, how do the reasoning processes of individual students evolve in relation to the group?”
Figure 5.1: Sophistication of Reasoning
The graph reveals the individual reasoning level in relation to the group. An attempt has been made to qualitatively describe the struggles and triumphs of individuals to keep up with the dynamics of the group and portray how the group tends to draw the individual students to emulate the group norms in the following discussion.

**Discussion on Evolution of Individual Students in Relation to Group**

The increasing sophistication of the group which in turn promotes more sophisticated individual participation is evident. For example, in the first tasks, Jeremy and Ricky were usually spectators to the dynamic discussions of the rest of the class. In some instances they understood that an explanation was required to be presented though they themselves could not present arguments which were logically persuasive in nature. The same was true for Carl and Joey, who were still trying to makes associations to the contents of the theorem, and engage in the practices of proof production while a few others in their respective groups were already making progress in structuring the argumentation. These students slowly began to understand the process through observation and eventually attempted to present their own arguments.

The workings of the group dynamic inevitably influenced these individuals’ reasoning capabilities. Though the change could not be perceived in the same task, the influence of the group dynamics can be clearly observed in their arguments in the subsequent tasks. Darren’s problem lay in his inability to grasp the workings of proofs until Task 8. In contrast to the other students mentioned above, he did attempt to involve himself in the discussions but could not keep with the practices that the group tried to adopt to prove the theorems. For instance, when he tried to revert to using specific measurements in proving the tasks, the others reminded him that angle measures cannot
be used. In other instances, after witnessing the presentation of a valid proof by the other students in the group, some students arrived at alternate approaches to the same proof in the lines of those arguments. For instance, in Task 1 Laila presented the proof of the vertical angle theorem in an alternative manner after observing a logical proof presented by Dalton and the group. She had been previously operating on a very basic level of reasoning but progressed to a higher level of reasoning with the influence of the way the group was operating. In Task 5 related to the alternate exterior angle theorem, Tommy and Chris proved the theorem in an alternate way using a different approach to what Delbert initially presented as a valid proof. It was an observation on the part of the teacher–researcher that Tommy and Chris collaborated more and each one understood what the other said or presented. Their train of thought seemed to be synchronous while approaching the proofs of specific tasks, though Tommy seemed to exhibit signs of understanding the nature and shape of the proof as a whole more quickly than Carl and the others in general.

The Vygotskian idea of the zone of proximal development also plays a major role in the process of learning. Enculturation of the students into the cultural practices of proving occurs through their interactions with more knowledgeable counterparts. We can see those instances in which students developed their own arguments by listening to the arguments of more knowledgeable others. In the first task, Dalton pointed out the mistakes made by his peers Laila and Delbert as they presented their arguments for the proof and offered them solutions for the errors. This benefitted the others as they were able to arrive at a valid conclusion based upon Dalton’s advice. From the beginning, Dalton exhibited greater sophistication in argumentation. This made him critically
examine the arguments presented by others. He was open in asking questions when he
was not convinced and was open to criticism as well. The other students would
reconsider what they had presented if he raised a question. Ryan was also explicit in his
judgments about the arguments of others and made sure to express his opinion if he
thought that an argument was not logically sound. In Task 11 related to the exterior angle
theorem, an objection raised by these two students to the data presented by Julia made her
re-focus on the task at hand and present a valid proof with a different set of data. These
are all examples of how the contributions of more sophisticated students could influence
the thinking of their peers.

All the students showed a steady growth while occasionally reverting back to
lower levels of reasoning, but Dalton and Delbert exhibited continuous progress in their
reasoning and proof competencies. Delbert also adopted a way of representing congruent
angles by drawing an arrow between the angles. This representation influenced the group
and the others began to use his representation in their proofs as well, resulting in a shared
meaning of representing congruent angles. Overall, the analyses of the individuals,
analyses of group and the graphical representation together give the reader an idea about
the influence of group over the progress of individuals.

Bar (2009) points out that:

Students exploring proofs, trying out their own mathematical ideas and discussing
them with others are considered to be teaching strategies that can foster learning.
Although many researchers believe that “surely students who experience such
instruction will develop different proof schemes” (Harel & Sowder, 2007, p. 40),
findings are not clear cut, and, in some cases, students even come up with ideas
and mathematical rules that do not count as proper in mathematics. (p. 92)
The findings of the current study provide clarity on these issues by demonstrating that enculturational teaching strategies can be successfully employed to promote proof competencies. The current study also bridges the gap between theory and practice.
REFERENCES


APPENDIX : IRB FORMS

Application for Exemption from Institutional Oversight

Unless qualified as meeting the specific criteria for exemption from Institutional Review Board (IRB) oversight, ALL LSU research projects using living humans as subjects, or samples, or data obtained from humans, directly or indirectly, with or without their consent, must be approved or exempted in advance by the LSU IRB. This Form helps the PI determine if a project may be exempted, and is used to request an exemption.

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Applicant, please fill out the application in its entirety and include the completed application as well as parts A-E, listed below, when submitting to the IRB. Once the application is completed, please submit two copies of the completed application to the IRB Office or to a member of the Human Subjects Screening Committee. Members of this committee can be found at http://www.lsu.edu/screeningmembers.shtml

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A Complete Application includes All of the Following:

(A) Two copies of this completed form and two copies of Part 8 thru E

(B) A brief project description (adequate to evaluate risks to subjects and to explain your responses to Parts 1&2)

(C) Copies of all instruments to be used.

*If this proposal is part of a grant proposal, include a copy of the proposal and all recruitment material.

(D) The consent form that you will use in the study (see part 3 for more information.)

(E) Certificate of Completion of Human Subjects Protection Training for all personnel involved in the project, including students who are involved with testing or handling data, unless already on file with the IRB. Training link: (http://phpb.ohsafety.com/users/LOGIN.php)

(F) IRB Security of Data Agreement: (http://www.lsu.edu/irb/IRB%20Security%20of%20Data.pdf)

1) Principal Investigator: Chillara V. Indira

Dept: Curriculum Instruction

Ph: 803-565-1025

2) Co Investigator(s): please include department, rank, phone and e-mail for each

3) Project Title: Enculturational practices in the teaching of program Mathematics

4) Proposal? (yes or no) No

If Yes, LSU Proposal Number

Also, if YES, either

- This application completely matches the scope of work in the grant

- More IRB Applications will be filed later

5) Subject pool (e.g. Psychology students) Middle school 8th grade students

*Circle any "vulnerable populations" to be used: (children, elderly, the mentally impaired, pregnant women, the ages, other). Projects with incarcerated persons cannot be exempted.

6) PI Signature C. V. Indira Date 7/6/2011

- ** I certify my responses are accurate and complete. If the project scope or design is later changed, I will resubmit for review. I will obtain written approval from the Authorized Representative of all non-LSU Institutions in which the study is conducted. I also understand that it is my responsibility to maintain copies of all consent forms at LSU for three years after completion of the study. If I leave LSU before that time the consent forms should be preserved in the Departmental Office.

Screening Committee Action: Exempted

Category/Paragraph 1

Reviewer Mathews

Signature Mathews Date 7/28/11

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Parental Consent form

Study Title: Enculturational Practices in the Teaching of Proof in Mathematics

Performance Site: Middle school

Contacts: The following investigator is available for questions,
         M-F, 8:00 a.m.-4:30 p.m.
         Indira V. Chilimara
         College of Education, LSU
         803-565-1025

1. Purpose of the Study: Students should learn mathematics through the cultural practices of exploring, arguing and thinking. This study intends to explore the effects of an enculturationist approach in students’ learning to think about and write proofs. The study intends to look into the ways in which inductive and deductive reasoning processes evolve during the instruction. The secondary focus is on the ways in which group cohesion, mutual responsiveness and sense of excitement/energy are nurtured during the sessions.

2. Subjects: 8th grade students

3. Number of Subjects: 12

4. Study Procedures: The researcher will involve the students in making sense of mathematical ideas through argumentation and justification. The students will be exposed to the area of proofs using an enculturationist instruction. The whole-class discussions will be video-taped and the data is analyzed for increase in sophistication of argumentation structures over the sessions. Two questionnaires in which the students rate their own participation and performance as a class will be administered to the students.
5. Benefits: The study benefits the students as the type of instruction used is to improve their argumentation skills and at the same time develop conceptual understanding of the various mathematical principles.

Risks/Discomforts: The study does not pose any kind of risk to the students as the data will be used solely for research and educational purposes only.

6. Commitment: The participation in the study is voluntary and the subjects can withdraw from the study at any time. Participation during the sessions, punctuality, commitment to the study and regular attendance will be very much appreciated.

7. Privacy: Confidentiality is maintained regarding the data obtained in the study. to which confidentiality of records identifying the subject will be maintained. Data will be kept confidential and will be strictly used for research and educational purposes unless release is legally compelled.

8. Financial Information: No compensation will be paid for participating in the study as it is voluntary, but occasional treats will be given.

Signatures:

The study has been discussed with me and all my questions have been answered. I may direct additional questions regarding study specifics to the investigator. If I have questions about subjects' rights or other concerns, I can contact Robert C. Mathews, Chairman, Institutional Review Board, (225) 578-8692, irb@lsu.edu, www.lsu.edu/irb. I will allow my child to participate in the study described above and acknowledge the investigator's obligation to provide me with a signed copy of this consent form.

Parent's Signature: _________________________________ Date: ________________________________
Child Assent Form

I, _____________________________________________________________________, agree to be in the study to find ways to help me understand math better. I have to involve in whole-class discussions as the study requires my participation to my fullest ability. I have to follow all the directions given by the teacher. I have to participate in all of the sessions as it is a requirement of the study. I know that I will be video-recorded and the data will be strictly used for research and educational purposes only.

Child's Signature: ___________________________________ Age: _____ Date: __________________

Witness* ___________________________________________ Date: __________________

Institutional Review Board
Dr. Robert Mathews, Chair
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Study Exempted By:
Dr. Robert C. Mathews, Chairman
Institutional Review Board
Louisiana State University
203 B-1 David Boyd Hall
225-578-8692 | www.lsu.edu/irb
Exemption Expires: 7-2-14

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VITA

After completing my B.S from Nagarjuna University, India with three majors; Mathematics, Physics and Chemistry, I had a choice to do a Master’s in Applied Mathematics and Nuclear Physics. I opted for the first one as I was always fascinated by Math. I completed my M.S in Applied Mathematics from Sri Padmavathi Mahila Viswa Vidyalayam, a prestigious women’s university in India. During those days I felt that I was good at teaching mathematics and wanted to be a math teacher someday. So, I worked on my Bachelor’s in Education and I was the university topper for the course. I subsequently got admission into the M. Ed program into one of India’s top Colleges of Education and passed it with distinction. After that I was married and I started to work in a school as a math teacher. After working as a math teacher for three years, I was offered a job in a College of Education as a lecturer of Mathematics. It was at this time, I became interested in Psychology. So I pursued a Master’s in Psychology and completed it. I worked in the college of Education for three years and I then got the opportunity to work in the South Carolina Public Schools System as an exchange visitor from India. The Indian government sponsored the program and a few math teachers from all over India were selected to participate in it. I was one of the teachers selected. I worked in a high school in South Carolina for three years and it was then, I met Dr. Kirshner at LSU and wanted to pursue the Ph.D program. After being a full time student and a research assistant for three years, I applied for an itinerary gifted teacher position and was offered a job by the Plaquemines Parish School Board. I am currently working for that parish. This job made it possible for me to get into touch with public schools again and implement my study.