2012

A numerical investigation of Apéry-like recursions and related Picard-Fuchs equations

Maiia J. Bakhova
Louisiana State University and Agricultural and Mechanical College, mayab@math.lsu.edu

A NUMERICAL INVESTIGATION OF APÉRY-LIKE RECURSIONS AND RELATED PICARD-FUCHS EQUATIONS.

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
Maiia J. Bakhova
M.S. in Math., Krasnoyarsk State University, 1985
M.A., University of Oregon, 2002
May 2012

Acknowledgments

This dissertation would not be possible without several contributions. It is a pleasure to express my huge gratitude to Dr. Helena Verrill for the years she spent working with me in spite of my non-adequate health.

This work was motivated by equation (1) in [10] and 4.5 in [9], brought to my attention by Dr. Helena Verrill.

It is a pleasure also to thank Dr. J.W. Hoffman for providing another idea to investigate which turned out to be helpful for my main example. A special thanks to Dr. R. Perlis for allowing me to finish my work.

This dissertation is dedicated to my daughter for her understanding and support.

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Abstract

In this work we investigate a generalization of a recursion which was used by Apéry in his proof of irrationality of the zeta function values $\zeta(2)$. It is a continuation of the work of Zagier [10], who considered generalization of the first equation and numerically investigated it. The study is made for two generalizations of the second equation, one used the mirror symmetry idea from the theory of Calabi-Yau varieties and another worked with recursion. There were discovered connections between them.

Chapter 1
Introduction

1.1 History and Motivation

This dissertation concerns a numerical investigation of power series $F(t) = \sum_{n\geq 0} u_n t^n$ with integer coefficients $u_0,\ u_1,\ u_2,\ \ldots$ which are solutions of one of the following recursions:

$$(n+1)^2 u_{n+1} - (11n(n+1)+3)u_n - n^2 u_{n-1} = 0 \quad \text{and} \tag{1.1}$$

$$(n+1)^3 u_{n+1} - (2n+1)(17n(n+1)+5)u_n + n^3 u_{n-1} = 0. \tag{1.2}$$

Where did these equations come from and why are they interesting? The story begins with the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ for real s. Every calculus student learns that the series diverges to ∞ for $s = 1$ and converges for every $s > 1$, defining a function of s, called the zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s > 1.$$

This function was first considered by Euler. In 1735 he proved $\zeta(2) = \frac{\pi^2}{6}$; in later years he showed when $s = 2k$ (k is a positive integer), then $\zeta(2k) = R_k \pi^{2k}$, where R_k is a rational number. In the intervening two and a half centuries, despite many attempts, not much has been proved about values of the zeta function at odd positive integers except for the pole at $s = 1$.

In 1978 R. Apéry announced a proof of the irrationality of $\zeta(3)$ and offered a new way to show the irrationality of $\zeta(2)$. The above-mentioned recursions are crucial for his method. However, no one, including Apéry himself has come up with a motivation or an explanation for the recurrences (1.1) and (1.2).

Since the appearance of Apéry's paper many mathematicians have investigated similar recursions in the hope of either understanding where these sequences came from or of proving more irrationality results. See [10], [11], [1].

There is a well-known connection between sequences given by a recurrence and power series solutions of linear differential equations.

Proposition 1.1. *Given a sequence* $\{u_n\}$, $n \in \mathbb{Z}$, *and a formal power series* $f = \sum_{n\geq 0} u_n t^n$. *Then for polynomials* $P_i(x)$ *with* $i = 1, \ldots, m$ *and* $\Theta = t\frac{d}{dt}$, *the following are equivalent:*

(A) For all natural $n \geq m$, *the* u_n *satisfy the recurrence equation*

$$u_n P_0(n) + u_{n-1} P_1(n) + \cdots + u_{n-m} P_m(n) = 0. \tag{1.3}$$

(B) The formal power series f *satisfies the differential equation*

$$\big[P_0(\Theta) + tP_1(\Theta + 1) + \cdots + t^m P_m(\Theta + m)\big] f = 0. \tag{1.4}$$

Proof. This follows from the facts that $\Theta t^n = nt^n$ and $t^m(\Theta + m)^k \sum_{n\geq 0} u_n t^n = \sum_{n\geq 0} (n+m)^k u_n t^{n+m}$. See also [7]. □

Applying the Proposition to the recursions (1.1) and (1.2) shows that the formal power series $F(t) = \sum_{n\geq 0} u_n t^n$ satisfies the corresponding linear differential equations of second and third order, respectfully:

$$t(1 - 11t - t^2)F''(t) - (3t^2 + 22t - 1)F'(t) + (t - 3)F(t) = 0 \quad \text{and} \tag{1.5}$$

$$(t^4 - 34t^3 + t^2)F'''(t) + (6t^3 - 153t^2 + 3t)F''(t) + (7t^2 - 112t + 1)F'(t) + (t - 5)F(t) = 0. \tag{1.6}$$

It turns out that these equations are Picard-Fuchs equations arising from a family of Calabi-Yau varieties, which basically means that there is a modular function

$t = t(z)$ such that $F(t(z)) = \sum u_n t^n(z)$ is a modular form of weight 1 for (1.5) and weight 2 for (1.6).

We note in passing that Calabi-Yau varieties are connected with string theory in physics and modular functions are used to solve Diophantine equations, in construction of Ramanujan graphs for communication network theory, in cryptography and coding theory, and many other areas.

Zagier in [10] generalized equation (1.5) as follows:

$$(tp(t)F'(t))' + (t - \lambda)F(t) = 0, \qquad (1.7)$$

which expands to

$$tp(t)F''(t) + (p(t) + tp'(t))F'(t) + (t - \lambda)F(t) = 0,$$

where $p(t) = t^2 + at + b$ and a, b, λ are rational parameters. Choosing $a = -11$, $b = -1$, $\lambda = 3$ gives us back (1.5). Zagier in [10] made a numerical study of power series $F(t) = \sum_{n\geq 0} u_n t^n$ with integer coefficients u_n that satisfy (1.7) for a, b, λ running over a specified range. Actually he made a change of variables $A = -a/b$, $B = 1/b$, $\Lambda = \lambda/b$. In term of the new parameters the recurrence relation (1.1) becomes

$$u_{n+1} = \frac{(An(n+1) + \Lambda)u_n - Bn^2 u_{n-1}}{(n+1)^2} \quad (n \geq 1), \quad u_1 = \Lambda u_0. \qquad (1.8)$$

From this we see that if $u_0 = 0$ then $u_1 = 0$ and $u_n = 0$ for all n. So we are only intersted in $u_0 \neq 0$. If $\{u_n\}$ is a solution to (1.8) with $u_0 \neq 0$ then $v_n = \frac{u_n}{u_0}$ is a solution to (1.8) with $v_0 = 1$. From now on we take $u_0 = 1$. It follows that $u_1 = \Lambda$ and $B = (2A + \Lambda)\Lambda - 4u_2$, and $A = \frac{9u_3 + 4\Lambda^3 - 17\Lambda u_2}{6u_2 - 8\Lambda^2}$. We now ask the following question: choose four integers $u_0 = 1$, u_1, u_2, u_3 and define $\Lambda = u_1$, $A = \frac{9u_3 + 4\Lambda^3 - 17\Lambda u_2}{6u_2 - 8\Lambda^2}$, and $B = (2A + \Lambda)\Lambda - 4u_2$ and then use the recursion (1.8) to define u_n for $n > 3$. If all the u_n are integers, does it follow that the parameters are integer as well? In particular in [10] the search was performed in the range $0 \le u_1 < 30$, $|u_2| \le 100$, $|u_3| \le 2000$, checking integrality of u_n up to $n = 25$. Zagier stated a conjecture:

Conjecture 1.2. *If 1.8 has a solution with $u_0 = 1$ and $u_n \in \mathbb{Z}$ for all n, then A, B and Λ are integral.*

Actually we only need to check if A is integral, since it is the only value for which its formula can yield rational values.

I did the same investigation for the range $0 \le u_1 < 300$, $|u_2| \le 5000$ and $|u_3| \le 100000$ and the conjecture still holds. I have been using Magma [5] for my calculations. Zagier claimed without any explanations that it was enough for his range to check integrality of the sequences up to 23d term, but in my range I've checked integrality up to $n = 25$ and found a case with $A = -731$, $B = -309960$ and $\lambda = 252$ which did not fall into range described in [10], but it turned out that 28th term is not an integer.

I need to introduce the definitions, concepts and notation I will use in the text.

1.2 Hypergeometric Sequences, Gauss Series and Modular Forms

Definition 1.3. *A generalized hypergeometric function has a series representation $\sum_{n\ge0} c_n$, with $\frac{c_{n+1}}{c_n}$ as a rational function of n, usually written as*

$$\frac{c_{n+1}}{c_n} = \frac{(n+a_1)(n+a_2)\dots(n+a_p)x}{(n+b_1)\dots(n+b_q)(n+1)}.$$

Introduce the shifted factorial: $(a)_0 = 1$, $(a)_n = a(a+1)\dots(a+n_1)$. Then we can write the general formula for the term

$$c_n = \frac{(a_1)(a_2)\dots(a_p)}{(b_1)\dots(b_q)}\frac{x^n}{n!}.$$

The usual notation for a generalized hypergeometric function is

$$ {}_pF_q \left[\begin{matrix} a_1, & \dots, & a_p \\ b_1, & \dots, & b_q \end{matrix} ; x \right] = \sum_{n \geq 0} \frac{(a_1)(a_2)\dots(a_p)}{(b_1)\dots(b_q)} \frac{x^n}{n!}. $$

A particular case of hypergeometric function is Gauss function

$$ F(a, b; c; x) = \quad {}_2F_1 \left[\begin{matrix} a, b \\ c \end{matrix} ; x \right]. $$

Modular functions are meromorphic functions defined on the complex upper half plane with specific invariant properties under some group action.

Definition 1.4 (Upper Half Plane)**.** *The* **complex upper half plane** *is* $\mathfrak{h} = \{z \in \mathbb{C} : Im(z) > 0\}$.

Sometimes the set is extended in the following way: $\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{Q} \cup \{\infty\}$, where $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$ is called a **set of cusps.**

Definition 1.5 (Modular Group). *The* <u>*modular group*</u> *is defined as*

$$ \mathbf{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1, \ \textit{and } a, b, c, d \in \mathbb{Z} \right\}. $$

It acts on $\mathfrak{h}$ via **linear fractional transformations**. Recall that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Z})$ the linear fractional transformation is $\gamma(z) = \dfrac{az+b}{cz+d}$ and $\mathrm{Im}\dfrac{az+b}{cz+d} = \dfrac{\mathrm{Im}(z)}{|cz+d|^2}$. Now we can see that $\mathfrak{h}$ is closed under the action and the set of cusps is an orbit.

Definition 1.6 (Congruence Subgroup). *A* <u>*congruence subgroup*</u> Γ *of* $\mathbf{SL}_2(\mathbb{Z})$ *is one that contains* $\Gamma(N) = \ker(\mathbf{SL}_2(\mathbb{Z}) \to \mathbf{SL}_2(\mathbb{Z}/N\mathbb{Z}))$ *for some* N*, which is called level.*

The most important congruence subgroups are

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N \right\}$$

and

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N \right\},$$

where $*$ means any element. Both groups have level N.

Definition 1.7 (**Weakly Modular Function for** Γ)**.** *A* <u>*weakly modular function of weight*</u> *$k \in \mathbb{Z}$ for Γ is a meromorphic function f on $\mathfrak{h}$ such that for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and all $z \in \mathfrak{h}$ we have*

$$f(z) = (cz + d)^{-k} f(\gamma(z)). \tag{1.9}$$

Definition 1.8 (**Modular form**)**.** *A* <u>*modular form*</u> *of integer weight k for a congruence group Γ is a weakly modular function $f : \mathfrak{h} \to \mathbb{C}$ that is holomorphic on $\mathfrak{h}^*$. We let* <u>$M_k(\Gamma)$</u> *denote the space of weight k modular forms of weight k for Γ.*

When we replace Γ with a congruence subgroup of certain level N we call the corresponding weakly modular function as a weakly modular function of the level N. The modular group is generated by the matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Suppose f is a weakly modular function of weight k for Γ. Then $f(z) = f(z + h)$ for some integer $h > 0$ and $f(-1/z) = z^k f(z)$. The properties can be used as definition of a weakly modular function as well. Note that T is always in any congruence group, although S may be not. Therefore the modular functions of any level are periodic and we can consider their Fourier expansion.

Definition 1.9 (Fourier Expansion). *A Fourier expansion of f, if it exists, is a representation of f as $f(z) = \sum_{n=m}^{\infty} a_n q^n$, for all $z \in \mathfrak{h}$, where $q = e^{2\pi i z}$.*

We can use differential equations to define modular forms. The linear differential equation which define modular functions are called Picard-Fuchs equations. The reversed problem is to find out what equations would yield modular solutions. A very useful property of modular forms is that there are only finite number of linearly independent functions for a given level and weight, so we can specify them by a certain number of terms of their Taylor series.

Chapter 2
Apéry-Zagier Equations

2.1 The First Recursion and the Corresponding Differential Equation

The first recurrence equation (1.1) was instrumental for Apéry's proof of irrationality of $\zeta(2)$ and has been a motivation to study such type of the equations in the form

$$(tp(t)F'(t))' + (t-\lambda)F(t) = 0, \text{ which can be rewritten as}$$

$$tp(t)F''(t) + (p(t) + tp'(t))F'(t) + (t-\lambda)F(t) = 0, \tag{2.1}$$

where $p(t) = t^2 + at + b$ and a, b, λ are rational parameters. They have been considered by Beukers [1], who improved Apéry's original work, and by Zudilin [11], who studied irrationality of zeta function values at greater odd integers. The solutions which Zagier was interested in are the series: $F(t) = \sum_{n\geq 0} u_n t^n$ for which there is an integer D such that $F(Dt) \in \mathbb{Z}[[t]]$, or $D^n u_n \in \mathbb{Z}$ for any natural n. We can find a recursive equation for the coefficients $\{u_n\}$ by equating the coefficients of the same powers of t. The corresponding recurrence equation for (2.1) is

$$b(n+1)^2 u_{n+1} + (an(n+1) - \lambda)u_n + n^2 u_{n-1} = 0 \quad (n \geq 1)$$

with initial value $u_1 = \lambda u_0$. Solving for u_{n+1} gives

$$u_{n+1} = -\frac{(an(n+1) - \lambda)u_n + n^2 u_{n-1}}{b(n+1)^2} \tag{2.2}$$

We can see now that the parameters a, b, λ and initial value of u_0 define the solution.

Note that if $F(t)$ is such a solution, then the function $H(t) = F(Dt)$ is a solution for another equation of the same type. To show this note that $H'(t) = DF'(Dt)$

and $H''(t) = D^2F''(Dt)$, and choose $\bar{p}(t) = \frac{1}{k^2}p(Dt) = t^2 + \frac{a}{D}t + \frac{b}{D^2}$. Then we consider a differential operator:

$$
\begin{aligned}
&(t\bar{p}(t)H'(t))' + \left(t - \frac{\lambda}{D}\right) H(t) \\
&\quad = t\bar{p}(t)H''(t)' + [\bar{p}(t) + t\bar{p}'(t)]\, H'(t) + \left(t - \frac{\lambda}{D}\right) H(t) \\
&\quad = t\bar{p}(t)D^2F''(Dt) + [\bar{p}(t) + t\bar{p}'(t)]\, DF'(Dt) + \left(t - \frac{\lambda}{D}\right) F(Dt) \\
&= \frac{1}{D}\left(Dt\frac{1}{D^2}p(Dt)D^2F''(Dt) + D\left[\frac{1}{D^2}p(Dt) + t\frac{1}{D}p'(Dt)\right] DF'(Dt) + (Dt-\lambda)F(Dt)\right) \\
&\quad = \frac{1}{D}\Big((Dt)p(Dt)F''(Dt) + [p(Dt) + Dtp'(Dt)]\, F'(Dt) + (Dt-\lambda)F(Dt)\Big) \\
&\quad = \frac{1}{D}\Big((zp(z)F'(z))' + (z-\lambda)F(z)\Big) = 0, \quad (2.3)
\end{aligned}
$$

by (2.1), where $z = Dt$. Therefore instead of rational coefficients u_n we can limit our consideration to integer sequences, and even more, by choosing the sign of D it's enough to consider only cases when $\lambda \leq 0$. We will say this new function H and the original function F are *equivalent up to scaling*. Looking at the recurrence condition(2.2) we see now that it is more convenient for computer code to divide everything by b and replace $1/b = B$, $A = -a/b = -Ba$, $\Lambda = \lambda/b = B\lambda$ and $P(t) = Bp(t) = Bt^2 - At + 1$. The reworked equation is

$$(tP(t)F'(t))' + (Bt - \Lambda)F(t) = 0. \tag{2.4}$$

The corresponding recurrence equation is

$$u_{n+1} = \frac{(An(n+1) + \Lambda)u_n - Bn^2u_{n-1}}{(n+1)^2} \qquad (n \geq 1), \quad u_1 = \Lambda u_0.$$

In [2] Zagier explains that he made an investigation taking $u_0 = 1$, u_1, u_2, u_3 as integers, finding the parameters A, B, Λ and checking if the following terms are integers. For the range he considered: $0 \leq u_1 \leq 30$, $|u_1| \leq 100$, $|u_3| \leq 2000$ all new parameters turned out to be integer. I considered $u_0 = 1$, $0 \leq u_1 \leq 300$, $|u_1| \leq 2000$, $|u_3| \leq 1000000$ and did not find any non integer sequences as well.

Zagier grouped the sequences he calculated into 5 classes. He calls them terminating, hypergeometric, polynomial, Legendrian and sporadic.

2.2 Terminating Cases

Some of the sequences vanish after a certain term. The case is simple and can be described completely.

Proposition 2.1. *Up to scaling the equation (2.4) has solutions which are polynomials when $A = -1$, $B = 0$ and $\Lambda = k^2 + k$, where k is a natural number.*

Proof. Assume that u_{n-1} is the last non zero term. Then $u_{n+1} = 0$, $u_n = 0$, so we have a system of equations:

$$\begin{cases} (An(n+1) + \Lambda)u_n - Bn^2 u_{n-1} = 0 \\ (A(n-1)n + \Lambda)u_{n-1} - B(n-1)^2 u_{n-2} = 0 \end{cases}$$

But because $u_n = 0$ and $u_{n-1} \neq 0$ then $B = 0$ is forced. This means n is a positive integer root of the quadratic equation $A(x-1)x + \Lambda = 0$. Since the root formula yields $r_{1,2} = \left(1 \pm \sqrt{1 - 4\frac{\Lambda}{A}}\right)/2$, then we want the discriminant $1 - 4\frac{\Lambda}{A}$ to be the square of a positive odd number, or $-\frac{\Lambda}{A} = k^2 + k$ for some natural number k.. Then the roots of the polynomial are $k+1$, $-k$ and the solution can be written as a Gauss series $F(t) = u_0 F(k, (-k-1); n; At)$, see [4]. It clearly will be a polynomial of degree k. □

2.3 Hypergeometric Cases

The case which Zagier calls "hypergeometric" corresponds to the following conditions:

i) $A = \Lambda = 0$,

ii) $B = 0$.

i) Looking at the recursive formula we see that first condition transforms the recursion into a simple ratio:

$$\text{If } A = \Lambda = 0, \text{ then } \frac{u_{n+1}}{u_{n-1}} = \frac{-Bn^2}{(n+1)^2} \quad (n \geq 1), \quad u_0 = 1, \quad u_1 = 0.$$

Note that all odd indexed terms vanish. Substituting $n + 1 = 2r$, $n - 1 = 2r - 2$ we get for odd numbered terms $u_{2r+1} = 0$ and for even numbered we can write the formula (using $(2r-1)(2r-3) \cdot \ldots \cdot 1 = \frac{(2r)!}{2^n r!}$ for the product of odd numbers) as

$$u_{2r} = \frac{-B(2r-1)^2}{(2r)^2} u_{2r-2} = \frac{(-B)^r((2r)!)^2}{2^{4r}(r!)^4} = \frac{(-B)^r}{2^{4r}} \binom{2r}{r}^2, \quad r \geq 0.$$

It is clear that B should be a multiple of 16 for the terms to be integer, and that up to the scaling property there is only one solution.

Proposition 2.2. *When $A = \Lambda = 0$ there is only one solution for (2.4) as a power series $F(t) = \sum_{n \geq 0} u_n t^n$ with $u_0 = 1$ up to the scaling.*

ii) Another instance of the case is when $B = 0$, then

$$\frac{u_{n+1}}{u_n} = \frac{An(n+1) + \Lambda}{(n+1)^2} \quad (n \geq 1), \quad u_1 = \Lambda u_0.$$

Say r_1 and r_2 are the roots of the equation $A(x-1)x + \Lambda = 0$, let $s_1 = -r_1$ and $s_2 = -r_2$, so $A(n-1)n + \Lambda = A(n+s_1)(n+s_2)$. If one of roots is a positive integer, then it's a case of terminating sequence, considered above. If not, then $u_0 = 1$, $u_1 = \Lambda = \frac{As_1s_2}{1^2}$, $u_2 = \frac{A^2s_1s_2(1+s_1)(1+s_2)}{1^2 2^2}$ and in general the formula is written using shifted factorial notation, see [4]:

$$u_n = A^n \frac{(s_1)_n (s_2)_n}{(n!)^2}, \quad n \geq 1, \quad u_0 = 1,$$

where the shifted factorial is defined as $(a)_n = a(a+1)\cdots(a+n-1)$. Then the solution is given as Gauss series $F(t) = F(s_1, s_2; 1; At)$. When the roots are rational then the formula can be written using more common binomial formulas.

For example when $A = 27$, $\Lambda = 6$ then $A(x-1)x + \Lambda = 27(x+\frac{1}{3})(x+\frac{2}{3})$ and the formula for the sequence term is

$$u_n = 27^n \frac{(\frac{1}{3})_n (\frac{2}{3})_n}{(n!)^2} = \frac{(3n)!}{(n!)^3},$$

while the power series here are $F(t) = F\left(\frac{1}{3}, \frac{2}{3}; 1; 27t\right)$. I've made a search, checking 41 first terms, for $|A| \le 5000$, $1 \le \lambda \le 500$, found 235 such cases. I do not know that in each case the resulting sequence will be integer after 41st term. One needs additional investigation. But the case is degenerate (meaning that $P(t)$ is not quadratic) and does not present much interest.

2.4 Polynomial Cases

By "polynomial" Zagier refers to the case when u_n is a polynomial in n. To spot them in the calculations of sequences we can consider differences of consecutive terms, and then the same differences for the new formed sequence: $c_n = u_{n+1} - u_n$, $d_n = c_{n+1} - c_n$, and so on. Since for any polynomial $Q(x)$ the difference $Q(x+1) - Q(x)$ is a polynomial of lesser degree, then at some point they should vanish, see below for A=2, $B = 1$, $\Lambda = 7$ (the sequences are truncated on the right):

1	7	19	37	61	91	127	169	217
6	12	18	24	30	36	42	48	
6	6	6	6	6	6	6	6	
0	0	0	0	0	0	0		

Here $u_n = 3n^2 + 3n + 1$. Note that near zero $\sum_{n=0}^{\infty} t^n = \frac{1}{1-t}$, $\sum_{n=0}^{\infty} n t^n = t\left(\frac{1}{1-t}\right)'$, $\sum_{n=0}^{\infty} n^2 t^n = t\left(t\left(\frac{1}{1-t}\right)'\right)'$, and so on. Hence the solution $F(t)$ in this case is a rational function with a denominator $(1-t)^{d+1}$:

$$\sum_{n=0} (3n^2 + 3n + 1) t^n = \frac{t^2 + 4t + 1}{(1-t)^3}.$$

where $d+1$ is the degree of the polynomial u_n. By substituting $F(t) = \frac{G(t)}{(1-t)^{d+1}}$ we'll get another differential equation which has a polynomial solution and therefore it is possible to proceed like it was done in terminating case. The result is that $\Lambda = d^2 + d + 1$ and

$$F(t) = \frac{1}{1-t} P_d \left(\frac{1+t}{1-t} \right), \text{ where } P_d(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \big([x^2 - 1] \big)$$

is Legendre polynomial.

Proposition 2.3. *The substitution* $t = \frac{x-1}{1+x}$ *turns equation (2.1)*

$$(tP(t)F'(t))' + (Bt - \Lambda)F(t) = 0, \text{ where } P(t) = (1-t)^2$$

into Legendre differential equation.

Proof. To see how the Legendre polynomial appears consider the substitution $x = \frac{1+t}{1-t}$ (derived from the Zagiers's paper), so $t = \frac{x-1}{1+x}$ and introduce $G(x)$ such that

$$F(t) = \frac{1}{1-t} P_d \left(\frac{1+t}{1-t} \right) = \frac{x+1}{2} G(x).$$

We'll use $t - 1 = \frac{-2}{x+1}$, $\frac{1}{1-t} = \frac{x+1}{2}$, $\frac{dx}{dt} = \frac{2}{(1-t)^2} = \frac{(x+1)^2}{2}$ and

$$\begin{aligned} F'(t) = \left(\frac{x+1}{2} G(x) \right)'_t &= \left(\frac{1}{2} G(x) + \frac{x+1}{2} G'(x) \right) \frac{(x+1)^2}{2} \\ &= \frac{1}{4} \Big(G(x) + (x+1)G'(x) \Big) (x+1)^2. \end{aligned}$$

Putting all together with $P(t) = (t-1)^2 = \frac{4}{(x+1)^2}$ in the equation (2.4) yields

$$\left[\frac{x-1}{1+x} \left(\frac{-2}{x+1} \right)^2 \cdot \frac{1}{4} \left(G(x) + (x+1)G'(x) \right) (x+1)^2 \right]'_t \\ + \left(\frac{x-1}{1+x} - \Lambda \right) \frac{x+1}{2} G(x) = 0$$

$$\left[\frac{x-1}{1+x} G(x) + (x-1)G'(x) \right]'_t + \frac{1}{2} \Big((x-1) - (1+x)\Lambda \Big) G(x) = 0$$

$$G(x) + \frac{x-1}{1+x} G'(x) \frac{(x+1)^2}{2} + \frac{(x+1)^2}{2} G'(x) + (x-1)G''(x) \frac{(x+1)^2}{2} \\ + \frac{1}{2} \Big((x-1) - (1+x)\Lambda \Big) G(x) = 0$$

$$(x-1)\frac{(x+1)^2}{2}G''(x)\frac{(x+1)^2}{2} + \big((x-1)\frac{x+1}{2} + \frac{(x+1)^2}{2}\big)G'(x)$$
$$+ \frac{1}{2}\big((x+1) - (1+x)\Lambda\big)G(x) = 0$$

Since we're working around $t = 0$ which corresponds to $x = 1$ then we can cancel $\frac{x+1}{2}(= \frac{1}{1-t})$, and the result is a Legendre equation up to negation:

$$(x^2-1)G''(x) + 2xG'(x) + (1-\Lambda)G(x) = 0$$

□

Therefore $-(1-\Lambda) = d(d+1)$, and $\Lambda = d^2 + d + 1$ as it was claimed above. See [7] for more information on this type of equation.

2.5 Legendrian Cases

There are cases for some other values of A and $B \neq 1$ such that $P(t)$ is a complete square: $A^2 = 4B$. Then by scaling $t = \frac{2}{A}z$ we can change our function to $H(t) = F(z) = F(\frac{A}{2}t)$, so by the calculations (2.3) we have a case similar to the previous one, with the last term on the left side in the corresponding equation (2.4) as $(z - \frac{2\Lambda}{A})H(t)$. This time we need $\frac{2\Lambda}{A} = d^2 + d + 1$ for some d as well, but d could be rational and the solution will be given using $P_d(x)$, although this time it will be a Legendre function and not a polynomial. There are actually two kind of Legendre functions, first kind and second kind. But second kind is defined using the logarithm, which with our substitution will be considered at 0 and therefore cannot be a power series of considered type. Hence it should be a Legendre function of first kind. I remind that such for the equation

$$(1-z^2)f''(z) - 2xf'(z) + \nu(\nu+1)f(z) = 0$$

can be written as Gauss series $f(z) = F(-\nu, \nu+1, 1; \frac{1-z}{2})$.

2.6 Sporadic Cases

These are other cases which do not fall into the previous classes. Note that although only some of the previous cases were called hypergeometrical, but actually almost all of them had the property of ratio $\frac{u_{n+1}}{u_n}$ to be a rational function, except for the case $A = \Lambda = 0$, with zero odd terms. Zagier called them "sporadic" and he added it at the end of his list.

TABLE 2.1. Sporadic cases with formulae.

A	B	Λ	$u_0, u_1, u_2, u_3, \dots$	u_n
7	-8	2	1, 2, 10, 56, ...	$\sum_{k=0}^{n} \binom{n}{k}^3$
9	27	3	1, 3, 9, 21, ...	$\sum_{k=0}^{[n/3]} (-1)^k 3^{n-3k} \binom{n}{3k} \binom{3k}{k} \binom{2k}{k}$
10	9	3	1, 3, 15, 93, ...	$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}$
11	-1	3	1, 3, 19, 147, ...	$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}$
12	32	4	1, 4, 20, 112, ...	$\sum_{k=0}^{n} \binom{n}{k} \binom{2n-2k}{n-k} \binom{2k}{k}$
17	72	6	1, 6, 42, 312, ...	$\sum_{0 \le k \le i \le n} (-1)^i 8^{n-i} \binom{n}{i} \binom{n}{k}^3$
0	-16	0	1, 0, 4, 0, ...	$\binom{2r}{r}^2$ if $n = 2r$ and 0 otherwise.

2.7 Numerical Investigation

All sequences with a property $u_n \mid D^n$ for some integer D have been excluded, because if $F(t)$ satisfies a differential equation of this type, then $F(kt)$ will satisfy one, too. In addition I've excluded cases with $B = 0$ or when $A = \Lambda = 0$ as degenerate although they are present in Zagier's table. The first index is the number of solution in my list, the second is from Zagier's table. Time spent is a bit under 34 hours. In the following table each entry in the second column is the row number in Zagier's table with that sequence. All sequences can be computed using Magma. The program is appended.

TABLE 2.2. All non-degenerate cases.

	Z.	A	B	Λ	u_0	u_1	u_2	u_3
1	3	2	1	1	1	1	1	1
2	5	7	-8	2	1	2	10	56
3	6	2	1	3	1	3	5	7
4	7	9	27	3	1	3	9	21
5	8	10	9	3	1	3	15	93
6	9	11	-1	3	1	3	19	147
7	10	12	32	4	1	4	20	112
8	13	17	72	6	1	6	42	312
9	15	2	1	7	1	7	19	37
10	19	32	256	12	1	12	164	2352
11	21	2	1	13	1	13	55	147
12		354	31329	21	1	21	-4005	-1246929
13	24	2	1	21	1	21	131	471
14	25	54	729	21	1	21	495	12171
15	26	32	256	28	1	28	580	10992
16	29	2	1	31	1	31	271	1281
17		54	729	39	1	39	1251	37821
18	31	2	1	43	1	43	505	3067
19		128	4096	52	1	52	2980	176848
20	33	2	1	57	1	57	869	6637
21		54	729	57	1	57	2169	73353
22		538	72361	59	1	59	-1349	-2390151
23		2	1	73	1	73	1405	13237
24		32	256	76	1	76	2596	68656
25		842	177241	91	1	91	-3929	-9413619
26		794	157609	91	1	91	-1205	-7024439
27		2	1	91	1	91	2161	24691
28		2	1	111	1	111	3191	43561
29		2	1	133	1	133	4555	73327
30		-146	5329	287	1	287	-1691	-569077

Chapter 3
Symmetric Square of Original Equation

3.1 The Equation and Recursion

There are different way to generalize the Zagier's equation by going to 3d order differential equation. Dr. Hoffman suggested considering symmetric square of the original equation, because it will yield a Picard-Fuchs equation, as described in [6]. The formula for the symmetric square of a differential equation can be derived explicitly. By construction all our previously found power series, squared, should be solutions of the new equation, too. So the interesting question arises: are there any solutions which are not the squares of the functions already investigated? Regretfully I did not find any. But the computations turned out to to be useful for another case. For an equation $u'' + h(t)u' + g(t)u = 0$, the symmetric square is

$$u''' + 3h(t)u'' + (4g(t) + 2h^2(t) + h'(t))u' + (4h(t)g(t) + 2g'(t))u = 0.$$

Clearly in the Beuker's equation we have $h(t) = \frac{P(t)+tP'(t)}{tP(t)}$ and $g(t) = \frac{(Bt-\lambda)}{tP(t)}$, therefore the symmetric square for the Beukers equation is

$$\begin{aligned} &t^2P^2(t)u''' + 3tP(t)\left[P(t) + tP'(t)\right]u'' \\ &+ \Big[4(Bt-\lambda)tP(t) + 2(P(t) + tP'(t))^2 + t^2P(t)P''(t) - t^2(P'(t))^2 - P^2(t)\Big]u' \\ &+ \Big[4(P(t) + tP'(t))(Bt-\lambda) + 2(tBP(t) - (Bt-\lambda)(P(t) + tP'(t))\Big]u = 0 \quad (3.1) \end{aligned}$$

If we substitute $P(t) = Bt^2 + At + 1$ then the equation is

$$\begin{aligned}&\left[B^2t^4 + 2ABt^3 + (2B + A^2)t^2 + 2At + 1\right] \sum_{n\geq 3} u_n n(n-1)(n-2)t^{n-1} \\ &+ \left[9B^2t^4 + 15ABt^3 + (12B + 6A^2)t^2 + 9At + 3\right] \sum_{n\geq 2} u_n n(n-1)t^{n-1} \\ &+ \left[19B^2t^4 + (24A - 4\lambda)Bt^3 + (16B + 6A^2 - 4\lambda A)t^2 \right.\\ &\left. + (6A - 4\lambda)t + 1\right] \sum_{n\geq 1} u_n n t^{n-1} \\ &+ \left[8B^2t^3 + (6AB - 6B\lambda)t^2 + (4B - 4A\lambda)t - 2\lambda\right] \sum_{n\geq 0} u_n t^n = 0\end{aligned}$$

The work with coefficients for power series $F(t) = \sum_{n\geq 0} u_n t^n$ is kind of involved.

$$\begin{aligned}&B^2 \sum_{n\geq 2} u_{n+1}(n+1)n(n-1)t^{n+4} + 2AB \sum_{n\geq 2} u_{n+1}(n+1)n(n-1)t^{n+3} \\ &+ (2B + A^2) \sum_{n\geq 2} u_{n+1}(n+1)n(n-1)t^{n+2} + 2A \sum_{n\geq 2} u_{n+1}(n+1)n(n-1)t^{n+1} \\ &+ \sum_{n\geq 2} u_{n+1}(n+1)n(n-1)t^n + 9B^2 \sum_{n\geq 1} u_{n+1}(n+1)nt^{n+4} \\ &+ 15AB \sum_{n\geq 1} u_{n+1}(n+1)nt^{n+3} + (12B + 6A^2) \sum_{n\geq 1} u_{n+1}(n+1)nt^{n+2} \\ &+ 9A \sum_{n\geq 1} u_{n+1}(n+1)nt^{n+1} + 3 \sum_{n\geq 1} u_{n+1}(n+1)nt^n \\ &+ 19B^2 \sum_{n\geq 1} u_{n+1}(n+1)t^{n+4} + (24A - 4\lambda)B \sum_{n\geq 1} u_{n+1}(n+1)t^{n+3} \\ &+ (16B + 6A^2 - 4\lambda A) \sum_{n\geq 1} u_{n+1}(n+1)t^{n+2} \\ &+ (6A - 4\lambda) \sum_{n\geq 1} u_{n+1}(n+1)t^{n+1} + \sum_{n\geq 1} u_{n+1}(n+1)t^n \\ &+ 8B^2 \sum_{n\geq 0} u_n t^{n+3} + (6AB - 6B\lambda) \sum_{n\geq 0} u_n t^{n+2} \\ &+ (4B - 4A\lambda) \sum_{n\geq 0} u_n t^{n+1} - 2\lambda \sum_{n\geq 0} u_n t^n = 0\end{aligned}$$

$$B^2\sum_{n\geq 6} u_{n-3}(n-3)(n-4)(n-5)t^n + 2AB\sum_{n\geq 5} u_{n-2}(n-2)(n-3)(n-4)t^n$$
$$+ (2B + A^2)\sum_{n\geq 4} u_{n-1}(n-1)(n-2)(n-3)t^n + 2A\sum_{n\geq 3} u_n n(n-1)(n-2)t^n$$
$$+ \sum_{n\geq 2} u_{n+1}(n+1)n(n-1)t^n + 9B^2\sum_{n\geq 5} u_{n-3}(n-3)(n-4)t^n$$
$$+ 15AB\sum_{n\geq 4} u_{n-2}(n-2)(n-3)t^n + (12B + 6A^2)\sum_{n\geq 3} u_{n-1}(n-1)(n-2)t^n$$
$$+ 9A\sum_{n\geq 2} u_n n(n-1)t^n + 3\sum_{n\geq 1} u_{n+1}(n+1)nt^n$$
$$+ 19B^2\sum_{n\geq 5} u_{n-3}(n-3)t^n + (24A - 4\lambda)B\sum_{n\geq 4} u_{n-2}(n-2)t^n$$
$$+ (16B + 6A^2 - 4\lambda A)\sum_{n\geq 3} u_{n-1}(n-1)t^n$$
$$+ (6A - 4\lambda)\sum_{n\geq 2} u_n nt^n + \sum_{n\geq 1} u_{n+1}(n+1)t^n$$
$$+ 8B^2\sum_{n\geq 3} u_{n-3}t^n + (6AB - 6B\lambda)\sum_{n\geq 2} u_{n-2}t^n$$
$$+ (4B - 4A\lambda)\sum_{n\geq 1} u_{n-1}t^n - 2\lambda\sum_{n\geq 0} u_n t^n = 0$$

The corresponding recursive condition for the coefficients in general case is

$$B^2(n-1)^3 u_{n-3} + \left[An(2n^2 - 3n + 1) - \lambda(4n-2)\right] Bu_{n-2}$$
$$+ \left[A^2(n^3 - n) + 2B(n^3 + n) - 4\lambda A\right] u_{n-1}$$
$$+ \left[An(2n^2 + 3n + 1) - (4n+2)\lambda\right] u_n + (n+1)^3 u_{n+1} = 0$$

As we can see to use it we need to know first 5 terms. I used Pari to compute coefficients of powers of t up to t^4 of the differential operator on the left hand of the equation (3.1), and then checked it with Magma. But one can do everything by hand, of course. The condition for them to vanish yields the following equations for coefficients of $F(t) = \sum_{n\geq 0} u_n t^n$:

$$-2\lambda u_0 + u_1 = 0$$

$$(4B - 4A\lambda)u_0 + (6A - 6\lambda)u_1 + 8u_2 = 0$$

$$(6A - 6\lambda)Bu_0 + (20B + 6A^2 - 8\lambda A)u_1 + (30A - 10\lambda)u_2 + 27u_3 = 0$$

$$8B^2u_0 + (30A - 10\lambda)Bu_1 + (60B + 24A^2 - 12\lambda A)u_2 + (84A - 14\lambda)u_3 + 64u_4 = 0$$

$$27B^2u_1 + (84A - 14\lambda)Bu_2 + (136B + 60A^2 - 16\lambda A)u_3 + (180A - 18\lambda)u_4 + 125u_5 = 0$$

By construction among all solutions should be squared functions of Zagier's case, with the same paremeters A, B, $\lambda = \Lambda$. So the interesting question if there are any solutions which are not the squares of functions already investigated. Regretfully I did not find any. But the computations turned out to be useful for another case.

3.2 Terminating Cases

Some of the sequences vanish after a certain term as before. This is the only case when the computer calculation yields true solutions. All sequences I've found turned out to be squares of already found solutions for Zagier's case and others of the same type with $A = 1$, $B = 0$ and $\Lambda = k^2 + k$.

TABLE 3.1. Terminating cases.

	A	B	Λ	u_0	u_1	u_2	u_3	u_4	u_5
1.	1	0	2	1	4	4	0	0	0
2.	1	0	6	1	12	48	72	36	0
3.	1	0	12	1	24	204	760	1380	1200
4.	1	0	20	1	40	580	3880	13840	28000
5.	1	0	30	1	60	1320	13720	78960	273504
6.	1	0	42	1	84	2604	38640	323820	1681344
7.	1	0	56	1	112	4648	93072	1065036	7677264

Table 3.1, continued

8.	1	0	72	1	144	7704	199920	2987460	28418544
9.	1	0	90	1	180	12060	393360	7426980	89901504
10.	1	0	110	1	220	18040	722040	16791720	251620512
11.	1	0	132	1	264	26004	1252680	35160840	638269632
12.	1	0	156	1	312	36348	2074072	69105036	1493283792
13.	1	0	182	1	364	49504	3301480	128777740	3264893632
14.	1	0	210	1	420	65940	5081440	229330920	6738880512
15.	1	0	240	1	480	86160	7596960	392714280	13236429024
16.	1	0	272	1	544	110704	11073120	649921560	24900447264
17.	1	0	306	1	612	140148	15783072	1043752536	45099523584

3.3 Hypergeometric Cases

I do not get here simple equations to solve. So I just compare values of the parameters of Zagier's cases, and see that they are the same. It means that the the corresponding power series are the squares of already known Zagier's cases.

i) $A = \Lambda = 0$,

ii) $B = 0$.

i) The is only one case here, up to scaling, as before, with $B = -16$.

ii) I will provide only numbers here.

3.4 Polynomial Cases

By "polynomial" Zagier calls a case when a formula for u_n is a polynomial, and as before, there is a way to determine them by computing differences between

TABLE 3.2. Hypergeometric cases.

	A	B	Λ	$u_0, u_1, u_2, u_3, u_4, u_5, \ldots$
1.	-16	0	4	1, 8, 88, 1088, 14296, 195008, ...
2.	-27	0	6	1, 12, 216, 4440, 97560, 2231712, ...
3.	16	0	12	1, 24, 24, -320, 4440, -63168, ...
4.	27	0	12	1, 24, -108, 1176, -15624, 217728, ...
5.	64	0	20	1, 40, -680, 22080, -876840, 38490816, ...
6.	27	0	30	1, 60, 540, -5520, 93780, -1881792, ...
7.	16	0	60	1, 120, 4440, 47040, 2520, -36288, ...
8.	27	0	84	1, 168, 8316, 94920, -356580, 3891888, ...
9.	64	0	84	1, 168, 5208, -93632, 3394776, -152142144, ...

consecutive sequence terms, although all of them have the same parameters as in the Zagier's calculations: $A = -2$, $B = 1$.

TABLE 3.3. Polynomial cases.

	A	B	Λ	$u_0, u_1, u_2, u_3, u_4, u_5, \ldots$
1.	-2	1	1	1, 2, 3, 4, 5, ...
2.	-2	1	3	1, 6, 19, 44, 85, 146, ...
3.	-2	1	7	1, 14, 87, 340, 1001, 2442, ...
4.	-2	1	13	1, 26, 279, 1724, 7465, 25326, ...
5.	-2	1	21	1, 42, 703, 6444, 39445, 181446, ...
6.	-2	1	31	1, 62, 1503, 19364, 161365, 980370, ...
7.	-2	1	43	1, 86, 2859, 49564, 543905, 4257210, ...
8.	-2	1	57	1, 114, 4987, 112340, 1578001, 15558206, ...
9.	-2	1	73	1, 146, 8139, 231604, 4065545, 49498350, ...
10.	-2	1	91	1, 182, 12603, 442684, 9516185, 140607690, ...

3.5 "Legendrian" Cases

This time the solution won't be a Legendre function, but a square of it, so they can be called "Legendrian" only conditionally. They can be recognized by their parameters.

TABLE 3.4. Legendrian cases.

	A	B	Λ	u_0, u_1, u_2, u_3, u_4, u_5, ...
1.	-32	256	12	1, 24, 472, 8640, 152536, 2635584, ...
2.	-54	729	21	1, 42, 1431, 45132, 1368045, 40486230, ...
3.	-32	256	28	1, 56, 1944, 54464, 1351000, 30984768, ...
4.	-54	729	39	1, 78, 4023, 173220, 6736941, 245443122, ...
5.	-54	729	57	1, 114, 7587, 393972, 17752185, 729916110, ...
6.	-32	256	76	1, 152, 10968, 531904, 20375000, 668820288, ...

3.6 Sporadic Cases

The sporadic cases, as before, have the same parameters as for Zagier's equation.

The applied formulas for u_n can be then reworked as double summation formulas.

TABLE 3.5. Sporadic cases.

#	A	B	Λ	u_0, u_1, u_2, u_3, u_4, u_5...
1.	-7	-8	2	1, 4, 24, 152, 1016, 7008...
2.	-11	-1	3	1, 6, 47, 408, 3745, 35598...
3.	-10	9	3	1, 6, 39, 276, 2061, 15930...
4.	-9	27	3	1, 6, 27, 96, 225, -162...
5.	-12	32	4	1, 8, 56, 384, 2648, 18496...
6.	-17	72	6	1, 12, 120, 1128, 10296, 92448...

Chapter 4
Another Apéry-like Equation

4.1 The Recursion and Equation

There is another recursion equation used by Apéry to prove irrationality of $\zeta(3)$. It looks like

$$(n+1)^3 u_{n+1} - (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1} = 0.$$

It is clear that there is a number of ways to generalize and investigate it, but if there would be too many parameters then it will take too much time to compute. So let's start with an equation in a way similar to one in Chapter 2, offered by H.Verrill:

$$(n+1)^3 u_{n+1} - (2n+1)(An^2+An+\lambda)u_n - Bn^3 u_{n-1} = 0. \tag{4.1}$$

We can get a corresponding differential equation by applying the Proposition 1.1. The resulting equation is

$$(t^2 - 2At^3 - Bt^4)F'''(t) + (3t - 9At^2 - 6Bt^3)F''(t) \\ + (1 - 6At - 2\lambda t - 7Bt^2)F'(t) + (-\lambda - Bt)F(t) = 0. \tag{4.2}$$

Taking $A = 17$, $B = -1$, $\lambda = 5$ we get again as in Chapter 1, Introduction:

$$(t^4 - 34t^3 + t^2)F'''(t) + (6t^3 - 153t^2 + 3t)F''(t) + (7t^2 - 112t + 1)F'(t) + (t-5)F(t) = 0.$$

$$(1 - 2At - Bt^2)\sum_{n\geq 2} n(n-1)(n-2)u_n t^{n-1} + (3 - 9At - 6Bt^2)\sum_{n\geq 1} n(n-1)u_n t^{n-1} \\ + (1 - 6At - 2\lambda t - 7Bt^2)\sum_{n\geq 1} n u_n t^{n-1} + (-Bt - \lambda)\sum_{n\geq 0} u_n t^n = 0$$

Pari calculations:

```
f=u0+u1*t+u2*t^2+u3*t^3+u4*t^4+u5*t^5+u6*t^6+u7*t^7+u8*t^8+u9*t^9;
 do=(t^2-2*A*t^3-B*t^4)*deriv(deriv(deriv(f,t),t),t)
+(3*t-9*A*t^2-6*B*t^3)*deriv(deriv(f,t),t)
+(1-6*A*t-2*la*t-7*B*t^2)*deriv(f,t)-(la+B*t)*f;
?  polcoeff(do,0,t)
%20 = -u0*la + u1
?  polcoeff(do,1,t)
%21 = -B*u0 + ((-6*A - 3*la)*u1 + 8*u2)
?  polcoeff(do,2,t)
%22 = -8*B*u1 + ((-30*A - 5*la)*u2 + 27*u3)
```

The initial condition here is $u_1 = \lambda u_1$. Therefore again we have a solution uniquely defined by the parameters and value of u_0. We can show again like for the Apéry-Zagier equations that if $F(t)$ the desired solution, then there is an equation of the same type for which $H(t) = F(kt)$ is a solution for a rational k, too.

4.2 Terminating Cases

Let's see for what parameter values the corresponding sequences vanish after a certain term. Assume that u_{n-1} is the last non zero term, so for $u_{n+1} = 0$, $u_n = 0$ we have a system of equations:

$$\begin{cases} (2n+1)(An^2 + An + \lambda)u_n + Bn^3 u_{n-1} = 0, \\ (2n-1)(An^2 - An + \lambda)u_{n-1} + B(n-1)^3 u_{n-2} = 0 \end{cases}$$

Because $u_n = 0$ and $u_{n-1} \neq$ then it forces $B = 0$, so we are left with the condition for the quadratic equation $A(x-1)x + \lambda = 0$ to have a positive integer root. Since the root formula yields $r_{1,2} = \left(1 \pm \sqrt{1 - 4\frac{\lambda}{A}}\right)/2$, then we need for the discriminant to be a square of an odd number, or $-\frac{\lambda}{A} = k^2 + k$, where k is an arbitrary integer. Then the roots of the polynomial are $k+1$, $-k$ and the solution can be written as

Gauss series $F(t) = u_0 F(k, (-k-1); n; At)$, see [4]. It clearly will be a polynomial of degree k.

We can consider here what happens when parameters vanish as with first equation.

i) $A = \lambda = 0$, the corresponding differential equation is

$$(t^2 - Bt^4)F'''(t) + (3t - 6Bt^3)F''(t) + (1 - 7Bt^2)F'(t) - BtF(t) = 0.$$

ii) $B = 0$,

$$(t^2 - 2At^3)F'''(t) + (3t - 9At^2)F''(t) + (1 - 6At - 2\lambda t)F'(t) - F(t) = 0.$$

The second case is degenerate since the polynomial $P(t)$ does not have second degree. I've decided to rename them since actually only second one satisfies the definition of "hypergeometric" and to consider these cases separately.

4.3 One Parameter Cases

When $A = \lambda = 0$ we have only one non-zero parameter left. Looking at the recursive formula we see that the condition transforms it into hypergeometric sequence:

$$\text{If } A = \lambda = 0, \text{ then } u_{n+1} = \frac{-Bn^3}{(n+1)^3} u_{n-1} \quad (n \geq 1), \quad u_0 = 1, \quad u_1 = 0.$$

Note that all odd indexed terms vanish. Substituting $n + 1 = 2r$, $n - 1 = 2r - 2$ we get get for odd numbered terms $u_{2r+1} = 0$ and for even numbered we can write the formula (using $(2r-1)(2r-3) \cdot \ldots \cdot 1 = \frac{(2r)!}{2^n r!}$ for the product of odd numbers) as

$$u_{2r} = \frac{-B(2r-1)^3}{(2r)^3} u_{2r-2} = \frac{(-B)^r((2r)!)^3}{2^{6r}(r!)^6} = \frac{(-B)^r}{2^{6r}} \binom{2r}{r}^3, \quad r \geq 0.$$

It is clear that B should be a multiple of 8 for the terms to be integer.

4.4 Two Parameter Cases (Degenerate)

Another instance of the case is when $B = 0$, then

$$\frac{u_{n+1}}{u_n} = (2n+1)\frac{An(n+1)+\lambda}{(n+1)^3} \qquad (n \geq 1), \quad u_1 = \lambda u_0.$$

Say r_1 and r_2 are the roots of the equation $A(x-1)x+\lambda = 0$, let $s_1 = -r_1$ and $s_2 = -r_2$, so $A(n-1)n+\lambda = A(n+s_1)(n+s_2)$. If one of roots is a positive integer, then it's a case of terminating sequence, considered above. If not, then $u_0 = 1$, $u_1 = \lambda = \frac{As_1s_2}{1^2}$, $u_2 = 3\frac{A^2s_1s_2(1+s_1)(1+s_2)}{1^3 2^3}$ and in general the formula is written using shifted factorial notation, see [4]:

$$u_n = 2^n A^n \frac{(1/2)_n (s_1)_n (s_2)_n}{(n!)^3}, \ n \geq 1, \ u_0 = 1.$$

$n \geq 1$, $u_0 = 1$. Then the solution is given as sum of hypergeometric series $F(t) = F(1/2, s_1, s_2; 1, 1; 2At)$. For example when $A = 32$ and $\lambda = 8$ we get $u_n = \binom{2n}{n}^3$, $F(t) = F(1/2, 1/2, 1/2; 1, 1; 64t)$.

4.5 Polynomial Cases

There are cases when a formula for u_n is a polynomial as well, the value of the parameters are $A = 1$, $B = -1$ and $\lambda = 2k(k+1)+1$ for some integer k. The leading polynomial coefficient in the differential equation (4.2) is a complete square: $t^2(t-1)^2$. Some of the first sequences are below. The formulas for the terms are found in “The On-Line Encyclopedia of Integer Sequences™ (OEIS™), http://oeis.org/, where the sequences are called ”Crystal ball sequences for the $A_k \times A_k$ lattice“. Since $A = 1$ and $B = -1$, then I removed them from the table of formulas. The interesting thing is that second sequence yields values of λ for all cases.

TABLE 4.1. Polynomial cases.

	A	B	λ	u_0	u_1	u_2	u_3	u_4
1	1	-1	1	1	1	1	1	1
2	1	-1	5	1	5	13	25	41
3	1	-1	13	1	13	73	253	661
4	1	-1	25	1	25	253	1445	5741
5	1	-1	41	1	41	661	5741	33001
6	1	-1	61	1	61	1441	17861	142001
7	1	-1	85	1	85	2773	46705	494341
...	1	-1	...	1	...	...	...	...

λ	u_n
1	1
5	$2n(n+1)+1$
13	$(3n^4+6n^3+9n^2+6n+2)/2$
25	$(10n^6+30n^5+85n^4+120n^3+121n^2+66n+18)/18$
41	$(35n^8+140n^7+630n^6+1400n^5+2595n^4+3020n^3+2500n^2$
	$+1200n+288)/288$
...	...

And so on. Then $F(t)=\frac{1}{1-t}\left[P_k\left(\frac{1+t}{1-t}\right)\right]^2$, where P_k is a Legendre polynomial. Now if we remember polynomial cases for the symmetric square case then the formulas there were $\frac{1}{(1-t)^2}\left[P_k\left(\frac{1+t}{1-t}\right)\right]^2=\left[\frac{1}{1-t}P_k\left(\frac{1+t}{1-t}\right)\right]^2$, which leads to a helpful assumption that when $P(t)$ is a complete square, although not $(1-t)^2$, then we may deal with a modified Legendre function. It could be checked numerically, by multiplying the power series using Pari by $(1-t)$, which I did, and in more general way (by checking if modified funciton is a solution for another equation), for which I did not have enough time.

4.6 Legendrian Cases

It is now clear that because there are Legendre polynomials as before then there will be Legendre functions as well when the leading polynomial coefficient in the differential equation (4.2) is a complete square. Looking at the data we see that

TABLE 4.2. Legendrian cases.

	A	B	λ	u_0	u_1	u_2	u_3	u_4
1	16	-256	8	1	8	88	1088	14296
2	27	-729	15	1	15	297	6495	149481
3	16	-256	40	1	40	1048	23360	479576
4	27	-729	51	1	51	1917	64599	2060001
5	27	-729	87	1	87	4509	189123	7114941
6	16	-256	136	1	136	8536	356416	11864536
23	27	-729	195	1	195	18117	1155615	60027381
...	...	...	...	1	...	...	...	...

there are 2 kinds of leading polynomial coefficients in the differential equation (4.2): $t^2(1-16t)^2$ and $t^2(1-27t)^2$. To get the sequences to look like the sequences from the symmetric square case one need to multiply the corresponding power series by $1-4t$ or $1-8t$ or $1-16t$; a similar factors are needed in the case of $t^2(1-27t)^2$.

4.7 Sporadic Cases

Finally there are sequences which do not fall into any of the previous categories. The property of the terms to be integer is checked up to $n = 50$.

TABLE 4.3. Sporadic cases

	A	B	λ	u_0	u_1	u_2	u_3	u_4	u_5
1	7	-81	3	1	3	9	3	-279	-2997
2	9	27	3	1	3	27	309	4059	57753
3	10	-64	4	1	4	28	256	2716	31504
4	12	-16	4	1	4	40	544	8536	145504
5	11	-125	5	1	5	35	275	2275	19255
6	17	-1	5	1	5	73	1445	33001	819005

Some of these sequences can be found in the "The On-Line Encyclopedia of Integer Sequences$^{\text{TM}}$". We refer to it as OEIS. At first I will describe the OEIS sequences and will provide proofs that they satisfy their recursions later.

OEIS says that the displayed terms of the sequence $\{u_n\}$ in the first row are also given by the Almkvist-Zudilin numbers, with the formula

$$a_n = \sum_{k=0}^{n} (-1)^k \frac{3^{n-3k}(3k)!}{(k!)^3} \binom{n}{3k} \binom{n+k}{k}, \tag{4.3}$$

The sequence got its name in [3], where the authors indicate the technique of proof that it is the same as $\{u_n\}$ using the famous algorithm of Wilf-Zeilberger. The sequence $\{a_n\}$ has integer terms as we will explain. When the exponent $n-3k$ of 3 is negative then the left binomial symbol $\binom{n}{3k}$ is zero. So we consider k in the range $0 \leq k \leq \frac{n}{3}$. For these k the expression 3^{n-3k} is an integer. Moreover, $\frac{(3k)!}{(k!)^3} = \binom{3k}{2k}\binom{2k}{k}$ is a product of two binomial coefficients and is therefore an integer.

When the third sequence $\{u_n\}$ is entered into the searchable OEIS $\{u_n\}$ is identified as $\{v_n\}$, with formula

$$v_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2n-2k}{n-k} \binom{2k}{k}.$$

All the terms are products of binomial coefficients, and hence $\{v_n\}$ is a sequence of integers. OEIS states that $\{v_n\}$ satisfies the recursion (4.1) with $A = 10$, $B = -64$, $\lambda = 4$ and attributes this to Vladeta Jovovic. We will not prove it in this dissertation, but the proof is very similar to the two given below.

The fifth row of the table contain a sequence which was described in [9] together with the corresponding recursion and differential equation, and even has the known modular interpretation, but there is no general formula for its terms yet.

When the sixth row of the table is entered into OEIS it is identified as the sequence of Apéry numbers $\{w_n\}$ defined by

$$w_n = \sum_{k=0}^{n} \left(\binom{n}{k} \binom{n+k}{k} \right)^2 .$$

One way to prove that the sequence formulas satisfy the corresponding recursions is using Wilf-Zeilberger algorithm, which is implemented in particular on Maple and Mathematica. The Wilf-Zeilberger algorithm constructs identities of certain types involving sums of products and quotients of binomial coefficients which are instrumental for proofs of more complicated identities. The crucial step is to find the identity.

For the first sequence (Almkvist-Zudilin numbers) using package EKHAD with Maple and denoting the summand in (4.3) as $b_{n,k} = (-1)^k \frac{3^{n-3k}(3k)!}{(k!)^3} \binom{n}{3k} \binom{n+k}{k}$ we obtain the identity

$$\begin{aligned} &-(n+2)^3 b_{n+2,k} + (2n+3)(7n^2+21n+17) b_{n+1,k} - 81(n+1)^3 b_{n,k} \\ &\qquad = G(n,k+1) - G_1(n,k), \end{aligned} \tag{4.4}$$

where

$$G(n,k) = 324 \frac{(4n+5)k^4}{(n-3k+2)(n-3k+1)} b_{n,k}. \tag{4.5}$$

The corresponding recursion (4.1) for the sequence with $A = 7$, $B = -81$, $\lambda = 3$ looks like

$$(n+1)^3 u_{n+1} - (2n+1)(7n^2+7n+3) u_n + 81 n^3 u_{n-1} = 0.$$

Therefore we need to do some index shifting: $m = n - 1$ yields

$$\begin{aligned} &-(m+1)^3 b_{m+1,k} + (2m+1)(7m^2+7m+3) b_{m,k} - 81 m^3 b_{m-1,k} \\ &\qquad = G(m-1,k+1) - G(m-1,k), \end{aligned}$$

where

$$G(m-1,k) = 324\frac{(4m+1)k^4}{(m-3k+1)(m-3k)}b_{m-1,k}.$$

Hence the right hand side is looking now close to the desired recursion. But it's better to check everything done with computer since there could be bugs in the code or just wrong buttons pushed. Luckily the checking is easier then the algorithm itself involving going over huge range of possible solutions.

We begin with an elementary Lemma.

Lemma 4.1. *For* $0 \leq s \leq r$ *the product* $b_{r,s} = (-1)^s \frac{3^{r-3s}(3s)!}{(s!)^3}\binom{r}{3s}\binom{r+s}{s}$ $= \frac{3^r(r+s)!}{(-27)^s(s!)^4(r-3s)!}$.

Proof.

$$(-1)^s\frac{3^{r-3s}(3s)!}{(s!)^3}\binom{r}{3s}\binom{r+s}{s} = 3^r(-1)^s 3^{-3s}\frac{(3s)!r!(r+s)!}{(s!)^3(3s)!(r-3s)!s!r!}$$
$$= \frac{3^r(r+s)!}{(-27)^s(s!)^4(r-3s)!}.$$

□

Corollary 4.2. *Under the usual convention* $\binom{i}{j} = 0$ *if* $i < j$ *or* $j < 0$

1. $b_{n+1,k} = 3\frac{n+k+1}{n-3k+1}b_{n,k}$.
2. $b_{n-1,k} = \frac{n-3k}{3(n+k)}b_{n,k}$.
3. $b_{n,k+1} = \frac{-(n+k+1)(n-3k)(n-3k-1)(n-3k-2)}{27(k+1)^4}b_{n,k}$.

Proof. These are direct applications of the Lemma using the following substitutions: for 1 put $r = n+1$, $s = k$ on the left and $r = n$, $s = k$ on the right; for 2 put $r = n-1$, $s = k$ on the left and $r = n$, $s = k$ on the right; and for 3 put $r = n$, $s = k+1$ on the left and $r = n$, $s = k$ on the right. □

Now we are ready to prove

Proposition 4.3. *The sequence* $\{a_n\}$ *defined by*

$$a_n = \sum_{k=0}^{n} (-1)^k \frac{3^{n-3k}(3k)!}{(k!)^3} \binom{n}{3k} \binom{n+k}{k}$$

satisfies the recursion

$$(n+1)^3 a_{n+1} - (2n+1)(7n^2+7n+3)a_n + 81n^3 a_{n-1} = 0, \quad n \geq 1.$$

Proof. We will use the expression from EKHAD, modified using Lemma 4.1:

$$\begin{aligned} H(n,k) = G(n-1,k) &= 324 \frac{(4n+1)k^4}{(n-3k+1)(n-3k)} b_{n-1,k} \\ &= 324 \frac{(4n+1)k^4}{(n-3k+1)(n-3k)} \cdot \frac{n-3k}{3(n+k)} b_{n,k} = 108 \frac{(4n+1)k^4}{(n-3k+1)(n+k)} b_{n,k}. \end{aligned}$$

We assert that the difference $H(n,k+1) - H(n,k)$ is given by the formula

$$c(n,k) = -(n+1)^3 b_{n+1,k} + (2n+1)(7n^2+7n+3)b_{n,k} - 81n^3 b_{n-1,k}.$$

Equivalently we assert that $H(n,k+1) - H(n,k) - c(n,k) = 0$. Let $R(n,k)$ be this difference:

$$\begin{aligned} R(n,k) = &\; H(n,k+1) - H(n,k) \\ &+ (n+1)^3 b_{n+1,k} - (2n+1)(7n^2+7n+3)b_{n,k} + 81n^3 b_{n-1,k}. \end{aligned}$$

We will need to calculate $H(n,k+1)$ using the Lemma 4.1:

$$\begin{aligned} H(n,k+1) &= 108 \frac{(4n+1)(k+1)^4}{(n-3(k+1)+1)(n+k+1)} b_{n,k+1} \\ &= 108 \frac{(4n+1)(k+1)^4}{(n-3k-2)(n+k+1)} \\ &\quad \cdot \frac{-(n+k+1)(n-3k)(n-3k-1)(n-3k-2)}{27(k+1)^4} b_{n,k} \\ &= -4(4n+1)(n-3k)(n-3k-1)b_{n,k}. \end{aligned}$$

We will show that $R(n,k) = 0$. Temporarily assume that $0 \leq 3k \leq n$, and therefore $b_{n,k} = (-1)^k \frac{3^{n-3k}(3k)!}{(k!)^3} \binom{n}{3k} \binom{n+k}{k} \neq 0$. We will remove the assumption later. Divide

$R(n,k)$ by $b_{n,k}$ and apply the Lemma 4.1. Denoting $R(n,k) \div b_{n,k} = Q(n,k)$ we see that we need to show $Q(n,k) = 0$, where

$$Q(n,k) = (4n+1)\left(-4(n-3k)(n-3k-1) \right. \\ \left. -108\frac{k^4}{(n-3k+1)(n+k)}\right) + (n+1)^3 3\frac{n+k+1}{n-3k+1} \\ -(2n+1)(7n^2+7n+3) + 81n^3\frac{n-3k}{3(n+k)}.$$

$Q(n,k)$ is a rational function in two variables n, k whose denominator is degree 2 and the numerator is of degree at most 5. In fact $Q(n,k)$ is identically 0. This is kind of tedious to check out, but can be verified instantaneously using any symbolic algebra system, like Mathematica or Pari. The computation is not numerical, but is exact. So $Q(n,k) = 0$. Then $R(n,k) = Q(n,k)b_{n,k}$ is equal to 0, too. So far we've been working under the temporary assumption that the product $(-1)^k \frac{3^{n-3k}(3k)!}{(k!)^3}\binom{n}{3k}\binom{n+k}{k}$ is not 0. When the product is 0, then $R(n,k) = Q(n,k)\left(\binom{n}{k}\binom{n+k}{k}\right)^2$ is also 0. This proves our assertion. Hence $H(n,k) - H(n,k-1) = c(n,k)$.

For each fixed n we will sum this difference as k runs from $-\infty$ to ∞, remembering that $c(n,k) = 0$ whenever $k < 0$ or $k \geq n+1$, so this sum is a finite sum. Then

$$\sum_{k=-\infty}^{\infty}\left[H(n,k) - H(n,k-1)\right] = \sum_{k=0}^{n+1}(H(n,k) - H(n,k-1)) \\ = H(n,0) - H(n,-1) + H(n,1) - H(n,0) + \ldots + H(n,n+1) - H(n,n) \\ = H(n,n+1) - H(n,-1),$$

since the series telescopes. But the explicit formula for $H(n,k)$ shows that $H(n,-1) = 0$ and $H(n,n+1) = 0$. Thus $\sum_{k=-\infty}^{\infty}(H(n,k) - H(n,k-1)) = 0$ and hence

$\sum_{k=-\infty}^{\infty} c(n,k) = 0$. Explicitly

$$0 = \sum_{k=-\infty}^{\infty} \left[(n+1)^3 b_{n+1,k} - (2n+1)(7n(n+1)+3)b_{n,k} + n^3 b_{n-1,k}\right]$$

$$= (n+1)^3 a_{n+1} - (2n+1)(17n(n+1)+5)a_n + n^3 a_{n-1}$$

This hold for all $n \geq 0$ and shows that a_n satisfies the recursion claimed by the proposition. □

We can do the same proof for the sixth sequence, which is Apéry numbers

$$w_n = \sum_{k=0}^{n} \left(\binom{n}{k}\binom{n+k}{k}\right)^2.$$

Lemma 4.4. *For $0 \leq s \leq r$ the product $\binom{r}{s}\binom{r+s}{s} = \frac{(r+s)!}{(s!)^2(r-s)!}$.*

Proof.

$$\binom{r}{s}\binom{r+s}{s} = \frac{r!}{s!(r-s)!}\frac{(r+s)!}{s!r!} = \frac{(r+s)!}{(s!)^2(r-s)!}.$$

□

Corollary 4.5. *Under the usual convention $\binom{i}{j} = 0$ if $i < j$ or $j < 0$*

1.

$$\binom{n+1}{k}\binom{n+k+1}{k} = \frac{n+k+1}{n-k+1}\binom{n}{k}\binom{n+k}{k}$$

2.

$$\binom{n-1}{k}\binom{n+k-1}{k} = \frac{n-k}{n+k}\binom{n}{k}\binom{n+k}{k}$$

3.

$$\binom{n}{k-1}\binom{n+k-1}{k-1} = \frac{k^2}{(n+k)(n-k+1)}\binom{n}{k}\binom{n+k}{k}.$$

Proof. These are direct applications of the Lemma 4.4 using the following substitutions: for 1 put $r = n+1$, $s = k$ on the left and $r = n$, $s = k$ on the right; for 2 put $r = n-1$, $s = k$ on the left and $r = n$, $s = k$ on the right; and for 3 put $r = n$, $s = k-1$ on the left and $r = n$, $s = k$ on the right. □

Now we are ready to prove

Proposition 4.6. *The sequence $\{w_n\}$ defined by*

$$w_n = \sum_{k=0}^{n} \left(\binom{n}{k} \binom{n+k}{k} \right)^2$$

satisfies the recursion

$$(n+1)^3 w_{n+1} - (2n+1)(17n(n+1)+5)w_n + n^3 w_{n-1} = 0, \quad n \geq 0.$$

Proof. Introduce the expression (this was Zagier's insight):

$$B_{n,k} = 4(2n+1)\big[k(2k+1) - (2n+1)^2\big] \left(\binom{n}{k} \binom{n+k}{k} \right)^2.$$

We assert that the difference $B_{n,k} - B_{n,k-1}$ is given by the formula

$$c(n,k) = (n+1)^3 \left(\binom{n+1}{k} \binom{n+k+1}{k} \right)^2$$
$$- (2n+1)(17n(n+1)+5) \left(\binom{n}{k} \binom{n+k}{k} \right)^2 + n^3 \left(\binom{n-1}{k} \binom{n+k-1}{k} \right)^2.$$

Equivalently we assert that $B_{n,k} - B_{n,k-1} - c(n,k) = 0$. Let $R(n,k)$ be this difference:

$$R(n,k) = B_{n,k} - B_{n,k-1} - (n+1)^3 \left(\binom{n+1}{k} \binom{n+k+1}{k} \right)^2$$
$$+ (2n+1)(17n(n+1)+5) \left(\binom{n}{k} \binom{n+k}{k} \right)^2 - n^3 \left(\binom{n-1}{k} \binom{n+k-1}{k} \right)^2.$$

We will show that $R(n,k) = 0$. Temporarily assume that $0 \leq k \leq n$, and therefore $\binom{n}{k}\binom{n+k}{k} \neq 0$. We will remove the assumption later. Divide $R(n,k)$ by $\left(\binom{n}{k}\binom{n+k}{k}\right)^2$. Denoting $R(n,k) \div \left(\binom{n}{k}\binom{n+k}{k}\right) = Q(n,k)$ we see that we need to show $Q(n,k) = 0$, where

$$Q(n,k) = 4(2n+1) \left(k(2k+1) - (2n+1)^2 + \frac{k^4\big[(2n+1)^2 - (k-1)(2k-1)\big]}{(n+k)^2(n-k+1)^2} \right)$$
$$- (n+1)^3 \left(\frac{n+k+1}{n-k+1} \right)^2 + (2n+1)(17n^2+17n+5) - n^3 \left(\frac{n-k}{n+k} \right)^2.$$

$Q(n,k)$ is a rational function in two variables n, k whose denominator is degree 4 and the numerator is of degree at most 7. In fact $Q(n,k)$ is identically 0. This is complicated to verify by hand, but can be verified instantaneously using any symbolic algebra system, like Mathematica or Pari. The computation is not numerical, but is exact. So $Q(n,k) = 0$. Then $R(n,k) = Q(n,k)\left(\binom{n}{k}\binom{n+k}{k}\right)^2$ is equal to 0, too. So far we've been working under the temporary assumption that the product $\binom{n}{k}\binom{n+k}{k}$ is not 0. When the product is 0, then $R(n,k) = Q(n,k)\left(\binom{n}{k}\binom{n+k}{k}\right)^2$ is also 0. This proves our assertion. Hence $B_{n,k} - B_{n,k-1} = c(n,k)$.

For each fixed n we will sum this difference as k runs from $-\infty$ to ∞, remembering that $c(n,k) = 0$ whenever $k < 0$ or $k \geq n+1$, so this sum is a finite sum. Then

$$\sum_{k=-\infty}^{\infty}(B_{n,k} - B_{n,k-1}) = \sum_{k=0}^{n+1}(B_{n,k} - B_{n,k-1})$$

$$= B_{n,0} - B_{n,-1} + B_{n,1} - B_{n,0} + \ldots + B_{n,n+1} - B_{n,n}$$

$$= B_{n,n+1} - B_{n,-1},$$

since the series telescopes. But the explicit formula for $B(n,k)$ shows that $B_{n,-1} = 0$ and $B_{n,n+1} = 0$. Thus $\sum_{k=-\infty}^{\infty}(B_{n,k} - B_{n,k-1}) = 0$ and hence $\sum_{k=-\infty}^{\infty} c(n,k) = 0$. Explicitly

$$0 = \sum_{k=-\infty}^{\infty}\left[(n+1)^3\left(\binom{n+1}{k}\binom{n+k+1}{k}\right)^2 -(2n+1)(17n(n+1)+5)\left(\binom{n}{k}\binom{n+k}{k}\right)^2 + n^3\left(\binom{n-1}{k}\binom{n+k-1}{k}\right)^2\right]$$

$$= (n+1)^3 w_{n+1} - (2n+1)(17n(n+1)+5)w_n + n^3 w_{n-1}$$

This hold for all $n \geq 0$ and shows that w_n satisfies the recursion claimed by the proposition. □

Corollary 4.7. *For the sixth sequence we have $u_n = w_n$ for all $n \geq 0$.*

Proof. $\{u_n\}$ and $\{w_n\}$ satisfy the same recursion and the first two terms agree: $u_0 = w_0 = 1$ and $u_1 = 5 = w_1 = \left(\binom{1}{0}\binom{1}{0}\right)^2 + \left(\binom{1}{1}\binom{1+1}{1}\right)^2$. $\square$

There are similar proofs that can be adapted in case of third sequence. We use the computer package EKHAD to find the pair of expressions as in (4.4) and (4.5). Here are the details. For the third sequence denote $c_{n,k} = \binom{n}{k}^2\binom{2n-2k}{n-k}\binom{2k}{k}$, so a term of the sequence is given by $\sum_{k=0}^{n} c_{n,k} = v_n$. Then the computer program EKHAD suggests the identity:

$$\begin{aligned}&-64(n+1)^3 c_{n,k} + (2(2n+3))(5n^2+15n+12)c_{n,k+1} - (n+2)^3 c_{n,k+2}\\ &\qquad = G_2(n,k+1) - G_2(n,k),\end{aligned}$$

where in this case

$$\begin{aligned}G_2(n,k) &= 4(-2n+2k-1))k^3(n^2+2n+1)c_{n,k}\cdot\\ &\quad \frac{(-12n^3-62n^2-104n-56+26n^2k+89nk+74k-18nk^2-30k^2+4k^3)}{(n+1-k)^3(n+2-k)^3}\end{aligned}$$

which can be verified as for the Almkvist-Zudilin sequence. Summing $G_2(n,k+1) - G_2(n,k)$ from $k = -\infty$ to ∞ then shows that $\{v_n\}$ satisfy the desired recurrence.

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Appendix A: Magma Code for Zagier Case

The program is made for non-terminating cases and non-zero parameters, but can be easily modified.

```
t := Cputime();
u:=[1..25];
K:=#u;
print "There are "; K;"coefficients computed.";
count:=0;
for la:=0 to 100 do
   u[1]:=la;
       for a:=-1000 to 1000 do
         for B:=-5000 to 5000 do
          remndr1:=((la+2*a)*u[1]-B) mod 4;
             if   (remndr1 eq 0)  then
             u[2]:=((la+2*a)*u[1]-B) div 4;
             remndr2:=((la+6*a)*u[2]-4*B*u[1]) mod 9;
                 if (remndr2 eq 0) then
               n:=2;
                while ( (remndr2 eq 0) and (n le K) ) do
u[n+1] := ((a*n*(n+1)+ la)*u[n] - B*n^2*u[n-1] ) div ((n+1)^2);
                             n:=n+1;
remndr2:= ((a*n*(n+1)+la)*u[n]-B*n^2*u[n-1]) mod ((n+1)*(n+1));
                end while; // if remndr1 eq 0 and n<=K
                    if  ( (u[K] ne 0) or ( u[K-1] ne 0))
                    and ( (a ne 0) or ( la ne 0))   then
// Removing terminating sequences and a=la=0
                        remndr3:=1;
                      D:=2;
                          if u[1] ne 0 then UB:=u[1];
                               else UB:=u[2];
                          end if;
                      while ((D le UB) and (remndr3 ne 0)) do
// Finding such D so u_n | D^n
                        remndr3:=u[1] mod D;
                         n:=2;
                         while ((remndr3 eq 0)  and (n le K)) do
// Checking if u_n | D^n
                              remndr3:=u[n] mod D^n;
                              n:=n+1;
                          end while; // if remndr3 eq 0
                           if n le K then
                               D:=NextPrime(D);
```

```
                            end if;
                         end while; // Finding such D so u_n | D^n
                         if (remndr2 eq 0)  and (remndr3 ne 0) then
// Printing to a file.
                               count:=count+1;
 ans:=[count, a, B, la, 1, u[1], u[2], u[3], u[4], u[5], u[6],
u[7], u[8], u[9], u[10]];
                              Write("ZagierCase2.txt", ans);
                        end if;   // Printing to a file.
                    end if; // Removing terminating sequences
             end if; // if remndr2 eq 0
          end if; // if remndr1 eq 0
       end for; // B
   end for; // a
end for; // la
print "the number of cases is "; count;
print Cputime(t);
```

Appendix B: Magma Code for Symmetric Square Case

```
t := Cputime();
u:=[1..20];
K:=#u;
print "There are "; K;"coefficients computed";
u0:=1;
count:=0;
for la:=0 to 100 do
   u[1]:=2*la*u0;
   for a:=-100 to 100 do
   for B:=-1000 to 1000 do
      remndr1:=((2*la*a-2*B)*u0-(3*a-3*la)*u[1]) mod 4;
          if   (remndr1 eq 0) and (B ne 0) then
          u[2]:=((2*la*a-2*B)*u0-(3*a - 3*la)*u[1]) / 4;
          remndr2:=((6*a - 6*la)*B*u0+(20*B+6*a*a-8*la*a)*u[1]
+ (30*a - 10*la)*u[2]) mod 27;
           if (remndr2 eq 0) then
              u[3]:=-((6*a-6*la)*B*u0+(20*B+6*a*a-8*la*a)*u[1]
+ (30*a - 10*la)*u[2]) / 27;
              remndr3:=(8*B*B*u0+(30*a-10*la)*B*u[1]
+(60*B+24*a*a-12*la*a)*u[2]+(84*a-14*la)*u[3]) mod  64;
                if (remndr3 eq 0)  then
                    u[4]:=-(8*B*B*u0+(30*a-10*la)*B*u[1]
+ (60*B + 24*a*a - 12*la*a)*u[2] + (84*a - 14*la)*u[3]) / 64;
                   remndr4:=(27*B*B*u[1] + (84*a - 14*la)*B*u[2]
+(136*B+60*a*a-16*la*a)*u[3]+(180*a-18*la)*u[4]) mod 125;
                         if  (remndr4 eq 0)  then
                            n:=4;
                            while (remndr4 eq 0) and (n lt K) do
u[n+1]:=-(B*B*(n-1)*(n-1)*(n-1)*u[n-3]+(a*n*(2*n*n-3*n+1)
-la*(4*n-2))*B*u[n-2]+(a*a*(n^3-n)+2*B*(n^3+n)-4*la*a*n )*u[n-1]
+ (a*n*(2*n*n+3*n+1)-  (4*n+ 2)*la )*u[n] ) div ((n+1)^3);
                            n:=n+1;
remndr4:=(B*B*(n-1)*(n-1)*(n-1)*u[n-3]+(a*n*(2*n*n-3*n+1)
-la*(4*n-2))*B*u[n-2]+(a*a*(n^3-n)
+2*B*(n^3+n)-4*la*a*n )*u[n-1] + (a*n*(2*n*n+3*n+1)
- (4*n+ 2)*la )*u[n] ) mod ((n+1)^3);
                            end while;
// if remndr4 eq 0  and inside of a loop
                            if (remndr4 eq 0) and (u[K] ne 0)
and (u[K-1] ne 0)  then
                              count:=count+1;
```

```
                              ans:=[count, a, B, la, 1,
u[1], u[2], u[3], u[4], u[5], u[6], u[7], u[8], u[9], u[10] ];
                              Write("SymSqr0_1.txt", ans);
                             end if;
                   end if; // if remndr4 eq 0
                end if; // if remndr3 eq 0
             end if; // if remndr2 eq 0
          end if; // if remndr1 eq 0
       end for; // B
   end for; // a
end for; // la
print "the number of cases is "; count;
print "Time spent ", Cputime(t);
```

Appendix C: Magma Code for Another Apéry-like Case

This is a general case, for all paramers non vanishing. Regretfully for it to fit the standards of publishing of our Graduate Scholl requirements I should have violate the structure.

```
t := Cputime(); count:=0;
u:=[1..40]; K:=#u; // There should be at least 10 terms.
print "There are "; K;"coefficients computed.";
count:=0; u0:=1;
for la:= 1 to 1000 do
    u[1]:=la;
    for A:=-1000 to 1000 do
       for B:=-10000 to 10000 do
           remndr1:=(3*(A*2+la)*u[1]+B) mod 8;
                 if (remndr1 eq 0) and (A ne 0)
and (la ne 0) and (B ne 0) then
// excluding cases which can be done algebraically
                      u[2]:= (3*(A*2+la)*u[1]+B) div 8;
                      remndr2:=(5*(A*6+la)*u[2]+8*B*u[1]) mod 27;
                      n:=2;
                       while (remndr2 eq 0) and (n le K) do
u[n+1]:=((2*n+1)*(A*n*(n+1)+la)*u[n]+B*n^3 *u[n-1]) div (n+1)^3;
                             n:=n+1;
remndr2:=((2*n+1)*(A*n*(n+1)+la)*u[n]+B*n^3*u[n-1]) mod (n+1)^3;
                       end while;
// if remndr2 eq 0  and inside of a loop
                     if   (u[K] ne 0) or ( u[K-1] ne 0)  then
// Removing terminating sequences
                            remndr3:=1;
                           Dv:=2;
                           if u[1] ne 0
                               then UB:=AbsoluteValue(u[1]);
                               else UB:=AbsoluteValue(u[2]);
                           end if;
                           while (Dv le UB )
                           and  (remndr3 ne 0)  do
                                 remndr3:=u[1] mod Dv;
                                  n:=2;
                                while (remndr3 eq 0)
                                and (n le K) do
                                   remndr3:=u[n] mod Dv^n;
                                   n:=n+1;
                               end while; // if remndr3 = 0
```

```
                               if n le K then
                                 Dv:=NextPrime(Dv);
                               end if;
                          end while;
                           if (remndr2 eq 0) and
                           (remndr3 ne 0) then
                               count:=count+1;
                               ans:=[count, A, B,la, 1,
u[1], u[2], u[3], u[4], u[5], u[6], u[7], u[8], u[9], u[10] ];
                              Write("Verrill_PsNo0.txt", ans);
                           end if;
// Printing to a file for the correct result
                       end if; // Removing terminating sequences
                   end if; // if remndr1 eq 0
       end for; // B
   end for; //  la
   La:=la;
end for; // A
ans:=[A,B, La];
print "the file is Verrill_PsNo0L.txt, the parameter bounds:";
            A; B; La;
print "the number of cases is "; count;
print Cputime(t);
```

Vita

Maiia Jurevna Bakhova was born in 1962, in Zheleznogorsk, Russia. She had been studying physics and applied mathematics in Moscow Institute of Physics and Technology from September, 1979 to January, 1981. She then transferred to Krasnoyarsk State University where she got her master's degree in mathematics in June 1985. She earned a master of science degree in mathematics from University of Oregon in May 2001. In August 2004 she came to Louisiana State University to pursue graduate studies in mathematics. She is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in May 2012. Her first publication was in year 1983, it was an abstract of her talk on Novosibirsk Student Conference. Her master's thesis was accepted for publication in "Mathematicheskij Vestink", but due to bad advice she did not follow it. She has experienced discrimination against her desire to pursue mathematics as a result of being female, and it is because of this and raising a family that she has large gaps in her academic career.