Two-Nucleon Scattering Using Velocity-Dependent Interactions.

Onofre Rojo-asenjo

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in

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by

Onofre Reyes-Asenjo
M.S., Louisiana State University, 1955
June, 1961
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ABSTRACT

The problem of two-nucleon scattering is solved assuming a well-behaved interaction which consists of a static attractive part and a velocity-dependent part depending on the square of momentum. The form of the potential used is $v = V(r) + \frac{1}{M} \left( p^2 \omega(r) + \omega(r) p^2 \right)$. With this potential Schrödinger equation is solved numerically for different sets of values of $\omega(r)$ and $V(r)$. The values of the $^1S$ and $^1D$ phase shifts for energies from 20 to 340 Mev are compared with Breit's values. The results are in good agreement for the $^1S$ phase shifts, but for $^1D$ the fit varies from poor to fair for the potentials used.

Born approximation using an "effective potential" is used to calculate $^1S$ and $^1D$ phase shifts at 310 Mev. Results are in good agreement with the ones obtained numerically.

To compare a velocity-dependent potential and a static potential with a core, we use a velocity-dependent potential of the form $v = -V_0 J_1(r) - \frac{\lambda}{M} p \cdot J_2(r) p$. With $J_1(r)$ and $J_2(r)$ square wells, we find the analytical solution and calculate the integrated cross section for deuteron photoeffect. We compare with a static square well with an infinite repulsive core. We find that both potentials give very similar integrated cross sections.
Phenomena involving two nucleons have received great attention due to the comparative mathematical simplicity of the problem, and to the belief that a complete knowledge of the two-nucleon interaction would allow computation of all nuclear properties. The knowledge that we have of the interaction is very limited. The properties of the deuteron and the nucleon scattering experiments contain information about the interaction between pairs. We can determine the phase shifts as functions of energy from scattering experiments but it is not possible so far to derive the phase shifts from a well-defined and unique potential. As Weisskopf (1957) points out, any definite conclusion from the two-body scattering can be drawn only with a knowledge of the scattering potential and not from the mere phase shifts. On the other hand Bég (1960) tries to derive all the properties of the interaction from the knowledge of the phase shifts for the $^1S_0$ state. Due to the impossibility at present of deriving such a potential with existing meson theories we use a phenomenological approach aiming to interpret the observed properties in terms of some model.
The phenomenological approach (Phillips, 1959) is based on the solution of the non-relativistic Schrödinger equation. The models used are either potentials, or boundary conditions on the wave function. Limitations on any phenomenological nucleon potential were proposed by Elsenbud and Wigner (1941). These are:

a) The potential depends only on the spins, the separation r, and the relative momentum P.

b) The potential is invariant against spatial rotations or inversion of the coordinate system.

Both restrictions insure the separability of the center of mass motion, the conservation of total angular momentum, and of parity.

Data known to date do not solve important issues such as the shape and exchange character of the potential or its possible velocity dependence.

Scattering processes can be analyzed in terms of the phase shifts and these in turn, from the specification of the S-matrix. At low energy where only S-scattering is important the phase shifts are uniquely determined from the angular distribution data, but at high energy where states of different angular momentum are mixed the angular distribution data alone cannot specify the S-matrix. The specification of the scattering amplitude requires more data and these are supplied by the experiments in double and triple scattering with polarized protons at 310 Mev by Chamberlain et al. (1957) and the phase shift analysis of Stapp et al. (1957). Noyes and MacGregor (1958) claim that an unique phase shift analysis is impossible even
with polarization data. More refined treatment of their data eliminates some of their multiple solutions (Gamme1 and Thaler, 1960). The experiment that would make this analysis unique is the measurement of the spin correlation parameter as a function of the angle $\theta$.

Breit et al. (1960) made a thorough analysis of all the available data to date for all energies from 20 to 340 Mev. They adjust the data by the least square method and, using the gradient method with an IBM 704 computer, search for the best phase-shift fit of the scattering cross section. Our $\delta_0$ and $\delta_2$ are compared with their $K_0$ and $K_2$ of their best fit (YLAM).

Experiments at low energy define only two properties of the potential well: the "effective range" and the "depth." (Blatt and Jackson, 1949). Another conclusion can be also drawn from the low energy experiments, namely that the nuclear force between protons is the same as that between a neutron and a proton in the same state (charge independence).

At high energy, scattering data give some information concerning the shape and the exchange character of the potential. The fact that the $S$ phase shift becomes negative around 220 Mev is a strong argument against a static potential that is attractive at all distances. This suggests the introduction of a very strong repulsive potential of very short range. Jastrow (1951) and Case and Pais (1950) introduced interactions which preserved the charge independence. The former contains an intense short range repulsion (repulsive core) while Case and Pais depend particularly on interactions of spin-orbit
type. Gammel and Thaler (1957 and 1960) were very successful in fitting two-body data with a static potential of the Yukawa type with a repulsive core.

However, ordinary perturbation theory cannot be used in nuclear matter for an infinite two-body repulsive core. A "modified perturbation theory" that replaces the Gammel-Thaler core by a pseudopotential gives very large second order terms of the order 20 MeV/particle (Levinger et al., 1960). Levinger (1960) has argued that any static potential that produces saturation near the empirical density must have second-order terms of magnitude 10 MeV/particle or more.

The failure of a static potential to give a rapidly converging series suggested the use of a different type of potential. Razavy and Levinger (1960) have used a well-behaved velocity-dependent potential to fit the two-body S-phase shifts up to 310 Mev. This velocity-dependent potential does not require more parameters than a potential with a repulsive core and should give small second-order terms in ordinary perturbation theory.

Using a non-static potential we can account for the interaction of two particles in an infinity of ways. But using a criterion of simplicity we can start, (Peierls, 1960) from a non-local potential \( v = F(r) - G(r) p^2 + \ldots \) which has one static part plus another part which is proportional to the square of the relative momentum. Such a velocity-dependent potential should be of an opposite sign to the static force, and also must be of shorter range since the S-wave phase shift changes sign at an energy where the D-interaction is still attractive.
Breit's analysis shows that D-phase shifts are monotonically increasing up to at least 340 Mev.

The idea of a velocity-dependent potential is not completely new in physics. Classical electromagnetism (Panofsky and Phillips, 1955) used a potential of the form

\[ \frac{1}{4\pi\epsilon_0} \left( V - \frac{u^2}{c^2} V \right) \]

to calculate the force exerted by an electron on another electron moving with a velocity \( u \) parallel to that of the original electron. Here \( V \) is the Coulomb potential.

Wheeler (1936) showed that Majorana exchange interaction may be described as an ordinary potential without exchange but with dependence on both the separation and the relative velocity of the nucleons, i.e. a non-local potential. Expanding his non-local potential, and using \( \partial^2/\partial x^2 \) instead of \((ih)^{-1} p\) he obtains a potential that depends on velocity. With an interaction of this form he accounts for the saturation properties of nuclear binding in the same way as do Heisenberg or Majorana potentials.

Yamaguchi (1954) with a non-local separable potential of the form \( \langle p' V p \rangle = (\hbar/M) g(p) g(p') \) was able to find exact solutions for bound states and continuum states of the two nucleon system. The photodisintegration of the deuteron is also examined with this potential and is found that the D-wave part of the deuteron plays an important role at high energies leading to a larger cross section. This is similar to the work done by Wheeler but tensor forces and a separable potential are used.
Invariance arguments were used by Okubo and Marshak (1958) to derive the most general velocity-dependent charge-independent two-nucleon interaction in a non-relativistic approximation. They attempt to identify the meson-theoretic origin of the velocity-dependent terms in the potential that they derive, and find that the new term is a \((\mu/M)^2\) correction to the static potential in second order. \(\mu/M\) is the ratio of pion mass to nucleon mass.

Moshinsky (1957, 1958) explores velocity-dependent central forces from the level ordering in nuclear shell theory considering the simplest velocity-dependent forces that depend on the second power of the momentum. The fact that the spin-orbit force (e.g. that of Gammel-Thaler) has a very short range permits him to show that the effects of the spin-orbit force are similar in nuclear shell theory to those of a short range velocity-dependent central force. He obtains restrictions on the strength and range of the velocity-dependent force that may be added to the static one, by comparing the theoretical ordering of the levels with the experimental one.

In Chapter II we solve numerically several types of velocity-dependent interactions to find the S-phase shift and D-phase shift. We use a potential of the form

\[ V(r) + \left(\frac{1}{M}\right) (p \cdot \omega(r) + \omega(r)p) \]

This potential is of second order in \(p\). It is invariant against spatial rotations or inversion of the coordinate system and is Hermitian.
Work has also been done with a potential of the form:

\[ B) \quad -V_0 I_1(r) - (\lambda/M)p \cdot J_2(r) p \]

that has the same properties (Razavy, Rojo, Levinger, 1960, and Razavy, 1961). Published data on the latter form (RRL 1960) contain some errors.

For reasons of convenience we use form B to study the photodisintegration of the deuteron, and form A for numerical work and effective potential. We emphasize the accuracy of the S-phase shift to test Bég's argument that any potential that fits the S-phase for the two-body interaction would give the same binding energy/particle in the many-body problem.

Green (1960) in Birmingham applies with success a modified Born approximation to a potential of type A. He transforms the potential operator in Schrödinger's equation into a function of k and r and uses this function denoted as an "effective" potential to find the S, and D, phase shifts.

Plots of the effective potential are given in Chapter II.

It seems to be only a matter of taste (Gammel, 1960) whether we prefer a phenomenological velocity-dependent interaction rather than a phenomenological repulsive core. The good fit which we can achieve to experimental two-body data makes it possible to believe in the reality of a velocity-dependent potential. On the other hand, experiments on scattering of electrons by nuclei are an argument for a repulsive core. (Fowler and Watson, 1959) A proving ground to...
decide which type of interaction is the true one could be the photo-
disintegration of the deuteron (Levinger, 1960). The photo-disintegration
of the deuteron is the inverse process of the capture of neutrons by
protons. The study of the absorption or emission of gamma-rays by the
neutron-proton system should give information concerning the structure
of the deuteron. In Chapter III we compare a velocity-dependent (well-
behaved) potential and a static potential with a repulsive core, for
sum-rule calculations of the integrated cross section of the deuteron.

Levinger and Bethe (1950) used this procedure for dipole
transitions in the study of the photo-electric cross section of the
deuteron and compared their results with direct calculations. They
found that the summed oscillator strength was increased by an exchange
interaction. This was in agreement with Feenberg (1936) and Siegert
(1937) who have shown that attractive exchange potentials would
increase the summed oscillator strength.

The first application of a velocity-dependent potential to the
calculation of the integrated cross section of the deuteron was done
by K. Way (1937), who used a separable velocity-dependent potential.
She compares with the integrated cross section for a static potential
of Gaussian shape, with Majorana exchange. The cross sections for
these two potentials agree; but they do not agree with the integrated
cross section of a static square well, with Majorana exchange, that is
used for comparison.

We find in Chapter III that our velocity-dependent interaction
increases the value of $\sigma_{int}$ above the value for repulsive core; but
the increase is not enough to draw definite conclusions.
CHAPTER II

S, D PHASE SHIFTS AND EFFECTIVE RANGE FOR VELOCITY-DEPENDENT INTERACTIONS OF DIFFERENT SHAPES

We have solved the Schrödinger equation numerically for different shapes of interactions using an IBM 650 computer.

For some assumed shape of the potential we calculate the values of the wave function describing the scattering process, and compare with the wave function of a free particle. We need first to reduce the two-body problem to the problem of scattering of a single particle by a center of force.

The Schrödinger equation in terms of the separation \( r \) of the two nucleons can be written as

\[
\nabla^2 \psi - \frac{M}{\hbar^2} \nabla \psi + \frac{M E}{\hbar^2} \psi = 0
\]  

(2.1)

We assume the type of interaction to be form A of Chapter I

\[
\nabla = V(r) + \frac{1}{M} \left( \hat{P}^2 \omega(r) + \omega(r) \hat{P}^2 \right)
\]  

(2.2)

where \( V(r) \) is the static part and \( \omega(r) \) is a dimensionless quantity.

With these assumptions above equation becomes,

\[
\nabla^2 \psi + \frac{2}{1 + 2 \omega(r)} \frac{d \omega(r)}{dr} \frac{d \psi}{dr} + \frac{1}{1 + 2 \omega(r)} \left[ k^2 - \frac{M}{\hbar^2} V(r) \right. \\
\left. + \nabla^2 \omega(r) \right] \psi = 0
\]  

(3.3)
for a central potential, the radial part of the wave function $R(r)$ obeys

$$\frac{d^2R}{dr^2} + 2 \left[ \frac{1}{r} + \frac{1}{1+2\omega(r)} \frac{d\omega(r)}{dr} \right] \frac{dR}{dr} + \frac{1}{1+2\omega(r)} \left[ k^2 - \frac{M}{k^2} V(r) + \Omega^2(\omega(r)) \right] R = \frac{\ell(\ell+1)}{r^2} R = 0 \tag{2.4}$$

Eliminating the first derivative through the transformation

$$U(r) = r \left[ 1 + 2\omega(r) \right] \frac{1}{2} R(r) \tag{2.5}$$

we obtain

$$U'' + \frac{1}{1+2\omega(r)} \left[ k^2 - \frac{2V(r)}{83} + \left[ \frac{\omega(r)}{2} \right]^2 \right] U - \frac{\ell(\ell+1)}{r^2} U = 0 \tag{2.6}$$

The transformation (2.5) was made independently by Green at Birmingham and by our group at Louisiana State University.

For $\ell = 0$, the solution $u_0(r)$ for large $r$ is shifted with respect to $\sin kr$ for a free particle:

$$u_0(r) \approx \Delta \ln (kr + \delta_0) \tag{2.7}$$

$\delta_0$ being the phase shift. Note that the Coulomb interaction, if any, has been neglected in solution of both (2.6) and (2.7).

For $\ell > 0$ one must compare $u_\ell(r)$ for large $r$ with spherical Bessel functions.

The asymptotic solution $v_\ell$ is

$$v_\ell(kr) = r \left[ J_\ell(kr) - \tan \delta_\ell \eta_\ell(kr) \right] \tag{2.8}$$
where \( j_\ell (kr) \) is the regular Bessel function and \( n^\ell (kr) \) is the non-regular Bessel function.

If a zero is found in \( u_0 (r) \) at \( r_N \) (for large enough \( r \) that \( u_o = v_o \)) we find the phase shift through the formula

\[
k r_N + \delta_o = n \pi
\]

subjecting \( \delta_o \) to the condition that

\[
-\frac{\pi}{2} \leq \delta_o \leq \frac{\pi}{2}
\]

Using (2.7) for \( \ell = 2 \) one obtains at the zero of the wave function

\[
u_2 (kr_N) = 0 = \frac{\partial}{\partial r} j_2 (kr_N) - \tan \delta_2 \eta_2 (kr_N)
\]

\[
\tan \delta_2 = \frac{\partial j_2 (kr_N)}{\eta_2 (kr_N)}
\]

From this formula \( \delta_2 \) can be easily calculated.

These calculated phase shifts \( \delta_0 (k) \) and \( \delta_2 (k) \) are then compared with Breit's phase shifts \( K_0 \) and \( K_2 \) for \( ^1S \) and \( ^1D \) waves respectively.

The Milne method used to solve the equation uses as predictor

\[
U_{n+1} = U_n + U_{n-2} - U_{n-3} + \frac{h^2}{4} \left( 5 U_n + 2 U_{n-1} \right.
\]

\[
\left. + 5 U_{n-2} \right) + \frac{17}{240} h^6 U^{(4)} (\xi)
\]

(2.12)

and as corrector:

\[
U_{n+1} = 2 U_n - U_{n-1} + \frac{h^2}{12} \left[ U_n + 10 U_n + U_{n-1} \right]
\]

\[- \frac{1}{240} h^6 U^{(6)} (\xi)
\]

(2.13)
where \( h \) is the length of the interval (Kunz, 1957).

To start the step-by-step solution (2.6) we required the knowledge of four values. The Runge-Kutta method supplies these four starting values. The reason for using the Milne method instead of the Runge-Kutta for all the solutions is that the Milne method is faster and gives the same degree of accuracy.

For S-waves the boundary conditions used were

\[
U(0) = 0 \\
U'(0) = 1
\]

Change of \( u'(u) \) merely multiplies \( u(r) \) by factor, not changing the phase shift.

For D-waves, since the equation (2.6) becomes singular at the origin due to the \( 6/r^2 \) centrifugal term, values very close to zero were substituted in the asymptotic solution; to obtain the initial boundary conditions of the differential equation.

\[
U \sim r^3 \\
U' \sim 3r^2
\]

(This form (2.15) holds for any potential for which \( \lim_{r \to 0} (r^2V(r)) = 0 \) and \( \lim_{r \to 0} (r\omega(r)) = 0 \).

The choice of the interval \( h \) of integration was made empirically. We selected length intervals and we used them to solve a differential equation whose solution was known and compared the obtained results. We concluded that \( h = 0.1 \) f was the optimum interval. We estimate that truncation errors and accumulated errors are less than \( h^4 \) in the
final solution (likely less than 0.001 radians in the phase shift).

It took about 20 min. of computer time to run phase shifts for energies from 20 to 340 Mev in steps of 40 Mev.

Green (1960) at Birmingham transforms the Schrödinger equation with potential (2.2) into the form:

\[ U'' + \left( k^2 U - \frac{l(l+1)}{r^2} \right) U - W U = 0 \]  
(2.16)

where

\[ W = \frac{1}{1 + 2 \omega(r)} \left\{ 2 \omega(r) k^2 - \frac{2M}{\hbar^2} V(r) - \frac{[\omega'(r)]^2}{1 + 2 \omega(r)} \right\} \]  
(2.17)

\( W(k,r) \), called the effective potential, is not an operator. It depends on wave-number \( k \) as a parameter, becoming repulsive for large \( k \), and thus giving negative \( L \) phase shifts at high energy. (Note that Eq. (2.16) is written so that if \( W = 0 \), the phase shifts are zero).

Green uses \( W(k,r) \) in a Born approximation calculation of \( \delta_0(k) \) and \( \delta_2(k) \). His Born approximation proves to be quite good as is seen from our results given in Table IV. Green uses this Born approximation for the case of a potential whose static part is an exponential and the velocity-dependent part is also exponential. We have used Born approximation in two cases: one with exponential-exponential like Green and another with Yukawa-exponential.
We have used phenomenological potentials of form (2.2) with different choices of shapes for static $V_i(r)$ and the velocity-dependent term $\omega(r)$: $V(r)$ is either Yukawa or exponential while $\omega(r)$ is exponential. (A Yukawa shape for $\omega(r)$ gives a highly singular term at the origin in Eq. (2.6)). For a given choice of shapes (e.g., Yukawa-exponential meaning $V(r)$ has a Yukawa shape and $\omega(r)$ an exponential shape) we want to determine optimum values for the two parameters in $V(r)$ and the two parameters in $\omega(r)$. As a first trial, we chose $V(r)$ to agree with effective range parameters (scattering length $a$ and effective range $r_0$ (Blatt and Weisskopf, 1952)). We then adjusted $\omega(r)$ to fit Breit's $^1S$ phase shifts above 100 Mev. Then $V(r)$ was modified to fit Breit's phase shifts from 20 to 100 Mev. After this fit to Breit's $^1S$ phase shifts (involving five or more trials) the potential was used to calculate $^3D$ phase shifts, which were subsequently compared with Breit's values.

In the following paragraphs we give the five potentials used in (2.6). These are designated by $\hat{\mathcal{V}}_1$, $\hat{\mathcal{V}}_2$, $\hat{\mathcal{V}}_3$, $\hat{\mathcal{V}}_4$, and $\hat{\mathcal{V}}_5$. It must be understood all through this chapter that lengths are expressed in fermis and energies in Mev.

The general form of the potential is

$$\hat{\mathcal{V}}_i = \hat{V}_i(r) + \frac{1}{M} \left( \hat{p}^2 \hat{\omega}_i(r) + \hat{\omega}_i \hat{p}^2 \right)$$

$$\hat{V}_i (\text{Yuk-exp}) \begin{cases} V_i(r) = -53 \exp\left(-0.815 r\right)/0.815 r \\ \omega_i(r) = 2.75 \exp\left(-2.8 r\right) \end{cases}$$

This is a Yukawa-exponential potential whose static part has an intrinsic range of $b = 2.12/0.815 = 2.60$ f and the velocity-dependent
part, (assuming that effective range theory is valid for this type of potential) has an intrinsic range of $b = 3.54/2.8 = 1.26$ f.

\[
\hat{U}_2(r) (\text{Yuk-exp}) \left\{ \begin{array}{l}
V_2(r) = -25 \exp(-1.1 r) / 0.81 \text{ MeV} \\
\omega_2(r) = 11 \exp(-4.0 r)
\end{array} \right.
\]

This is another Yukawa-exponential. The static part has an intrinsic range of 2.62 f and the velocity-dependent part has a range of .89 f.

\[
\hat{U}_3(r) (\text{exp-exp}) \left\{ \begin{array}{l}
V_3(r) = -55 \exp(-1.0 r) \text{ MeV} \\
\omega_3(r) = 5.0 \exp(-3.6 r)
\end{array} \right.
\]

In this exponential-exponential potential, the static part has a range 3.54 f. The velocity-dependent part has a range of .98 f.

\[
\hat{U}_4(r) (\text{exp-exp}) \left\{ \begin{array}{l}
V_4(r) = -100 \exp(-1.25 r) \text{ MeV} \\
\omega_4(r) = 2.0 \exp(-2.8 r)
\end{array} \right.
\]

This is another exponential-exponential potential with ranges 2.83 f for the static part and 1.26 f for the velocity-dependent term.

\[
\hat{U}_5(r) (\text{exp-exp}) \left\{ \begin{array}{l}
V_5(r) = -124.5 \exp(-1.43 r) \text{ MeV} \\
\omega_5(r) = 2.5 \exp(-2.86 r)
\end{array} \right.
\]

This potential $\hat{U}_5$ is taken from Green. Its range is 2.48 f for the static part and 1.24 f for the velocity-dependent part.
Table I and Fig. 1 show $^1S$ phase shifts for different shapes of velocity-dependent potentials. $\hat{V}_1$ and $\hat{V}_2$ (Yukawa-exponential) both give as compared to Breit's reasonable values, $\hat{V}_1$ being better than $\hat{V}_2$. The exponential-exponential shape $\hat{V}_4$ is not so good, but $\hat{V}_3$ and $\hat{V}_5$ (Green's) give very nearly the same good values.

Table II shows $^1D$ phase shifts. The Yukawa-exponential $\hat{V}_1$ is better than $\hat{V}_2$ but is not really good. The exponential-exponential $\hat{V}_4$ is the best one of all three but as shown in Table I it does not give very good $^1S$ phase shifts. $\hat{V}_3$ is better than $\hat{V}_5$ and thus $\hat{V}_3$ seems to be the best overall fit to $^1S$ and $^1D$.

Table III gives the values of effective range and scattering length for different potentials. Values are in approximate agreement with accepted ones, e.g. from the Gammel-Thaler n-n potential. Note scattering length $a$ is very sensitive to strength of potential since the strength parameter $b \propto 1$.

Table IV compares Born approximation calculations with numerical calculations. $\hat{V}_5$ (Born) done by Green; $\hat{V}_2$ (Born) with similar calculation at LSU. The Born approximation good to $\approx .01$ radian for $E > 150$ Mev. Difference between numerical and Born values are in the expected direction in each case (Born < exact).
Table 1

$^1S$ Phase Shifts

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<th>E(Mev)</th>
<th>$\delta_0$ Calculated for Potential</th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
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<th>$v_5$</th>
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<tr>
<td>220</td>
<td>0.0485</td>
<td>0.0467</td>
<td>0.0485</td>
<td>0.0730</td>
<td>0.0459</td>
<td>0.048</td>
<td></td>
</tr>
<tr>
<td>260</td>
<td>-0.0411</td>
<td>-0.0511</td>
<td>-0.0365</td>
<td>-0.0135</td>
<td>-0.0372</td>
<td>-0.033</td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>-0.1146</td>
<td>-0.1405</td>
<td>-0.1139</td>
<td>-0.0745</td>
<td>-0.1116</td>
<td>-0.120</td>
<td></td>
</tr>
<tr>
<td>340</td>
<td>-0.1853</td>
<td>-0.2233</td>
<td>-0.1849</td>
<td>-0.1385</td>
<td>-0.1797</td>
<td>-0.195</td>
<td></td>
</tr>
</tbody>
</table>

The $^1S$ phase shift $\delta_0$ (in radians) was calculated from numerical solutions of Eq. 2.6: the column labeled $v_1$ used the potential Eq. 2.2. These 5 calculations of $\delta_0$ versus lab energy $E$ are compared with Breit's YLAM curve for $K_o$ (Breit et al. 1960).
### Table II

**1D Phase Shifts**

<table>
<thead>
<tr>
<th>E(Mev)</th>
<th>$\hat{v}_1$</th>
<th>$\hat{v}_2$</th>
<th>$\hat{v}_3$</th>
<th>$\hat{v}_4$</th>
<th>$\hat{v}_5$</th>
<th>Breit's $K_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.0193</td>
<td>0.0092</td>
<td>0.0311</td>
<td>0.0180</td>
<td>0.0105</td>
<td>0.015</td>
</tr>
<tr>
<td>60</td>
<td>0.0667</td>
<td>0.0320</td>
<td>0.1016</td>
<td>0.0796</td>
<td>0.0551</td>
<td>0.045</td>
</tr>
<tr>
<td>100</td>
<td>0.0904</td>
<td>0.0439</td>
<td>0.1338</td>
<td>0.1168</td>
<td>0.0873</td>
<td>0.072</td>
</tr>
<tr>
<td>140</td>
<td>0.0968</td>
<td>0.0473</td>
<td>0.1418</td>
<td>0.1363</td>
<td>0.1044</td>
<td>0.097</td>
</tr>
<tr>
<td>180</td>
<td>0.0921</td>
<td>0.0440</td>
<td>0.1396</td>
<td>0.1339</td>
<td>0.1047</td>
<td>0.120</td>
</tr>
<tr>
<td>220</td>
<td>0.0800</td>
<td>0.0353</td>
<td>0.1304</td>
<td>0.1267</td>
<td>0.0980</td>
<td>0.140</td>
</tr>
<tr>
<td>260</td>
<td>0.0626</td>
<td>0.0224</td>
<td>0.1164</td>
<td>0.1138</td>
<td>0.0853</td>
<td>0.160</td>
</tr>
<tr>
<td>300</td>
<td>0.0418</td>
<td>0.0058</td>
<td>0.0994</td>
<td>0.0972</td>
<td>0.0685</td>
<td>0.175</td>
</tr>
<tr>
<td>340</td>
<td>0.0187</td>
<td>-0.0135</td>
<td>0.0798</td>
<td>0.0780</td>
<td>0.0488</td>
<td>0.184</td>
</tr>
</tbody>
</table>

$\hat{v}_2$ was calculated from Eq. (2.6) using potentials $\hat{v}_1$, $\hat{v}_2$, ... $\hat{v}_5$ (Eqs. 2.2). Breit's $K_2$ is from his YLAM fit (Breit et al., 1960).
Table III
Values of Effective Range and Scattering Length

<table>
<thead>
<tr>
<th></th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
<th>$v_4$</th>
<th>$v_5$</th>
<th>Potential</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_o$</td>
<td>3.00</td>
<td>2.65</td>
<td>3.30</td>
<td>3.1</td>
<td>3.0</td>
<td>2.65</td>
</tr>
<tr>
<td>$a$</td>
<td>-55.5</td>
<td>-10.53</td>
<td>&gt;100</td>
<td>-30.3</td>
<td>-33.0</td>
<td>-23.6</td>
</tr>
</tbody>
</table>

Effective range $r_o$ and scattering length $a$ are expressed in fermis. The values for the Gammel-Thaler potential are taken from Brueckner et al. (1960).

Table IV
Born Approximation Values for Phase Shifts

<table>
<thead>
<tr>
<th></th>
<th>$v_2$</th>
<th>$v_5$</th>
<th>150 Mev $\delta_0$</th>
<th>310 Mev $\delta_2$ (310 Mev)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Born Approximation</td>
<td>-</td>
<td>.2</td>
<td>-.170</td>
<td>-.003</td>
</tr>
<tr>
<td>Exact</td>
<td>-</td>
<td>.223</td>
<td>-.162</td>
<td>-.001</td>
</tr>
<tr>
<td>Born Approximation</td>
<td>-</td>
<td>.2</td>
<td>-.14</td>
<td>.05</td>
</tr>
<tr>
<td>Exact</td>
<td>-</td>
<td>.223</td>
<td>-.123</td>
<td>.004</td>
</tr>
</tbody>
</table>

The values are expressed in radians. Born approximation values for $v_2$ are taken from Green (1960).
Fig. 1 is a plot of $\tilde{B}_0$ versus $E_{lab}$. We compare calculations for our $\hat{V}_3$ (marked x) with Breit's $K_0$ (the solid curve).

Fig. 2 and Fig. 3 show the effective potentials $W$ calculated with Eq. (2.17) for $E_{lab} = 310$ Mev. Fig. 2 shows $W_2$ and $W_3$ for $\hat{V}_2$ and $\hat{V}_3$; observe the $W_2$ goes to infinity at the origin due to the negative value of the Yukawa static potential. Fig. 3 is a graph of $W_4$ and $W_5$. Note that $W > 0$ for values of $r \leq 1$ f and $W < 0$ for large $r$ ($W_5$ is taken from Green, 1960).
For our $v_3$ curve: Breit's $K_0$
Fig. 2
Fig. 3. Effective potentials for two different potentials $v_4$ and $v_5$. 

$W_4$ 

$W_5$ 

$(\text{fermi})^{-2}$
CHAPTER III

INTEGRATED CROSS-SECTION FOR STATIC AND VELOCITY-DEPENDENT POTENTIALS

The cross section for photon absorption by a proton in a deuteron is defined as \( \frac{\text{Transitions sec}}{\text{photon flux}} \). The cross section integrated over an absorption line gives

\[
\int \omega \, d\omega = \pi^2 c^2 (E_n - E_0) |Z_{on}|^2
\]

(3.1)

where \( |z_{on}| \) is the matrix element of the component of the proton-neutron separation along the direction of polarization, and \( E_0 \) and \( E_n \) are the energies of the ground and excited states (J. S. Levinger, 1960). We are using the electric-dipole approximation.

The integrated cross section can be expressed in terms of a dimensionless quantity \( f_{on} \) called oscillator strength defined as

\[
f_{on} = \frac{2M}{k^2} (E_n - E_0) |Z_{on}|^2
\]

(3.2)

The cross section for the line integrated over energy \((dW = \hbar \, d\omega)\) written in terms of the oscillator strength is as follows:

\[
\sum_n f_{on} = \frac{2M}{k^2} \sum_n (E_n - F_o) Z_{on} Z_{on} = -M \sum_n \left\{ [H, Z]_{on} Z_{on} - Z_{on} [H, Z]_{on} \right\}
\]

and using closure:

\[
\sum_n f_{on} = -\frac{M}{k^2} \left\{ \left[ [H, Z]_{on}, Z \right]_{on} \right\}
\]

(3.3)
The cross section summed over all discrete states is

\[ \sigma_{\text{int}} = \sum_n \int \sigma \, d\omega = \frac{\pi^2 e^2}{2 Me} \sum f_{\text{on}} \]

(3.4)

For continuous states (as in the case of the deuteron) the sum over \( n \) becomes an integral over the energy of the excited states.

We define the integrated cross section as

\[ \sigma_{\text{int}} = \int_0^\infty \sigma(\omega) \, d\omega = \frac{\pi^2 e^2}{2 Me} \sum f_{\text{on}} \]

(3.5)

Using the summed oscillator strength in the evaluation of the cross section of the deuteron photoeffect has the advantage that only knowledge of the wave functions for the ground state and of the Hamiltonian \( H \) are required.

To evaluate the double-commutator we write \( H = T + V \) = kinetic energy + potential energy, and

\[- \frac{M}{k^2} \left[ [H, Z], Z \right]_{00} = - \frac{M}{k^2} \left\{ \left[ [T, Z], Z \right]_{00} + \left[ [V, Z], Z \right]_{00} \right\} \]

(3.6)

The first double-commutator is model-independent

\[- \frac{M}{k^2} \left[ [T, Z], Z \right]_{00} = 2 \]

(3.7)

(for the deuteron \( T = P^2/M \) and we use the fundamental relation \( [P_z, Z] = -i\hbar \).

For simplicity we assume pure central forces: either velocity-dependent or static. We are interested in comparing these two oversimplified cases with each other: neither calculation should be compared with experimental values of \( \sigma_{\text{int}} \) for a real deuteron.
To calculate $\left[ [v, z], z \right]_\infty$ we assume a velocity-dependent potential of the form

$$\hat{U} = -V_0 j_1(r) - \frac{\lambda}{M} \vec{P} \cdot j_2(r) \vec{P} \quad \text{(Razavy 1961)} \quad (3.8)$$

i) $V_0 j_1(r)$ is a mixture containing $(1-x)$ Wigner interaction and $x$ exchange Majorana interaction $P^M$ thus

$$V_0 j_1(r) = \left[ (1-x) j_1(r) + x j_1(r) P^M \right] V_0$$

We shall use $x = 0.5$ for a Serber mixture.

iii) The velocity-dependent part $(\lambda/M) \vec{P} \cdot j_2(r) \vec{P}$ is a pure ordinary (or Wigner) force.

Since the ordinary part of the static potential will commute with $z$ we only need to evaluate the double commutator of $-xV_0 j_1(r)P^M$. By definition of the exchange operator,

$$P^M Z^2 = Z^2 P^M$$

and calling $Z_i - Z_j = Z = \text{separation of neutron and proton}$

$$P^M z = (Z_j - Z_i) P^M = -z P^M$$

$$\left[ \left[ P^M, z \right], z \right] = 4z^2 P^M$$

Then

$$-xV_0 \left[ \left[ j_1(r) P^M Z, Z \right], Z \right]_\infty =$$

$$-xV_0 \left[ 4z^2 j_1(r) P^M \right]_\infty - \frac{4xV_0}{3} \langle r^2 j_1(r) P^M \rangle_\infty \quad (3.9)$$

The velocity-dependent part gives

$$- \left[ \left[ \frac{1}{M} \vec{P} \cdot j_2(r) \vec{P}, Z \right], Z \right] = \frac{2\lambda \hbar'}{M} \langle j_1(r) \rangle_\infty \quad (3.10)$$

Substituting in (3.5)

$$\sum_n f_{on} = \left( 2 + \frac{4xMV_0}{3\hbar^2} \right) \int_0^\infty \psi_n^* \hbar^2 j_1(r) P^M \psi_o \, dr + \lambda \int_0^\infty \psi_n^* j_1(r) \psi_o \, dr \quad (3.11)$$
From (3.5) the integrated cross section is
\[ \int d\omega = \frac{\pi^2 e^2}{Mc} \left( 1 + \frac{2xMV_c}{\lambda h^2} \right) \int_0^\infty \psi_0^{*} \nabla \cdot J_i(r) P \psi_0 \, dr - \lambda \int_0^\infty \psi_0^{*} J_i(r) \psi_0 \, dr. \]  

(3.12)

It is seen at once that the integrated cross-section is increased if the value of \( \lambda \) is negative and decreased if it is positive. We now apply above relation to the case of a velocity-dependent potential; and also to a static square well with an infinite repulsive core both fitting the same two-body data and compare the results. (The latter case is treated using (3.12) with \( \lambda = 0 \) and assuming that the core is a pure ordinary force.)

With both potentials we fit the following data about the bound two-body system:

a) The binding energy \( c \) of the deuteron is equal to 2.226 Mev.

b) Both potentials give \( ^3S \) phase shift going through zero at the same energy. (This energy is chosen between 200 and 300 Mev to give qualitative agreement with the \( ^1S \) phase shifts discussed in the previous chapter.)

c) The mean square radius of the deuteron is 155 mb (equivalent to \( \rho(-\ell,-\ell) = 1.79 \) f).

To find the bound-state energy levels of the deuteron when a velocity-dependent potential of the form (3.8) is used, the Schrödinger equation is solved (Razavy, 1961) making use of the boundary conditions for the wave function and its derivatives. We find the bound-state that corresponds to particular choices of \( \hat{v}(r) \) and the angular momentum \( \ell \)
by solving the radial wave equations for a negative eigenvalue of the Hamiltonian, namely \(-e\). For the particular choice of \(l = 0\) we choose square shapes: 

\[ J_1(r) = J_2(r) = 1 - U(r-b) \]

(Here \(b\) is the range of the well and \(U\) is the step function)

Schrödinger's equation can be written

\[
\left[ -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + M V \right] u + \lambda \left( u - \frac{U}{\hbar^2} \right) \delta(r-b) = 0 
\]

Applying the equation in two regions one inside the well and another outside the well,

\[
\begin{align*}
\frac{d^2u}{dr^2} + k'^2 u &= 0 & \text{for} & \quad k < b \\
\frac{d^2u}{dr^2} - \gamma^2 u &= 0 & \text{for} & \quad k > b 
\end{align*}
\]

where

\[
k'^2 = \frac{M}{\hbar^2} \frac{(V - e)}{1 - \lambda} \quad \text{and} \quad \gamma^2 = \frac{M e}{\hbar^2}
\]

Now we look for the solutions of this system subject to the boundary conditions

\[
\begin{align*}
u_1(r) &= 0 & \text{as} & \quad r \to \infty \\
u_\Pi(r) &= 0 & \text{as} & \quad r \to \infty 
\end{align*}
\]

This requires that

\[
\begin{align*}
u_1 &= A e^{k' r} & \text{for} & \quad r < b \\
u_\Pi &= C e^{-\gamma r} & \text{for} & \quad r > b 
\end{align*}
\]

(3.14)
At \( r = b \) the boundary conditions obtained by Razavy (1961) (Eq. 3.20) are

\[
\begin{align*}
\omega_1(b) &= \omega_2(b) = \omega(b) \\
\frac{\partial \omega_1}{\partial r}(b) - (1-\lambda) \frac{\partial \omega_2}{\partial r}(b) &= \frac{\lambda}{b} \omega(b)
\end{align*}
\]

(The first derivative is discontinuous at \( r = b \) due to the delta function in the Schrödinger equation)

Solving this system of equations one obtains as the relation between parameters, \( V_0 \), \( \lambda \) and \( b \) for the potential to give a single bound-state of energy \(-\epsilon_i\),

\[
(1-\lambda)k C_0^2 k^3 b + \frac{\lambda}{b} = -\gamma \quad \text{(vel-dependent)}
\]

A similar procedure is followed to find the s-bound state energy levels using an infinite repulsive core plus a square well potential. This static potential is defined as

\[
\begin{align*}
U &= \infty & & k \leq C \\
U &= -V_0 & & C \leq r \leq b+C \\
U &= 0 & & r > b+C
\end{align*}
\]

We use a boundary condition

\[
U(c) = 0
\]

and \( U \) and \( U' \) continuous at \( r = c \).
The condition between parameters is
\[ k' c_0 + k' b = - \gamma \]  
(static + core) \quad (3.19)

with
\[ k'^2 = \frac{14}{h^2} (\sqrt{\gamma - \epsilon}) \]

The \( ^3S \) phase shifts of both potentials are obtained through the formulae

\[ \tan (\delta_o + k'b) = \frac{k' \sin k'b}{(1-\lambda) k' c_0 + k'b + \frac{i}{2} \Delta \sin k'b} \]  
(vel-dependent) \quad (3.20)

for the velocity-dependent. Here \( k' = \sqrt{\frac{E_{\text{lab}} + 2V_0}{8\gamma(1-\lambda)}} \) and

\[ k = \sqrt{\frac{E_{\text{lab}}}{8\gamma}} \]  
is the wave number outside the well (Razavy 1961, Eq. 3.21) also

\[ \tan \left[ k (b+c) + \delta_o \right] = \frac{k}{k'} \tan k'b \]  
(static + core) \quad (3.21)

for the static square plus core. Here \( k' = \left[ \frac{(E + 2V_0)}{8\gamma} \right]^{\frac{1}{2}} \)

and \( k \) is the same as above.

In fitting the mean-square radius of the deuteron \( <r^2>_{\infty} \) the value considered was 155, mb (Levinger and Rustgi, 1957) which corresponds to a bound-state effective range \( p(-\epsilon,-\epsilon) \) of 1.79 f. For our velocity-dependent potential,

\[
<r^2> = \left\{ B^2 \frac{1}{k'^3} \left[ \frac{(k'b)^3}{6} - \left( \frac{(k'b)^2}{4} - \frac{1}{8} \right) \Delta \sin 2k'b - \frac{k'b c_0 2k'b}{4} \right] \right.

+ e^{-2\gamma b} \left( \frac{b^2}{2Y} + \frac{2b}{4Y^2} + \frac{2}{8Y^3} \right) \right\} / B^2 \left( \frac{b}{2} - \frac{\Delta \sin k'b}{4k'} \right) + \frac{e^{-2\gamma b}}{2Y} \quad (3.22)
\]
Here $B^2 = A^2/C^2 = .400$.

For our static square-well plus core,

$$
\langle n^2 \rangle = \begin{cases} 
B^2 \left[ k^{-3} \left( \frac{(k' b)^3}{6} - \frac{(k' b)^2}{4} - \frac{1}{8} \right) \Delta m z k' b - \frac{k' b C_0 z k' b}{4} \right] \\
+ \frac{2C}{k'^2} \left[ \frac{(k' b)^2}{4} - \frac{k' b \Delta m z k' b}{4} - \frac{C_0 z k' b}{8} + \frac{1}{8} \right] + C^2 \left( \frac{b}{2} - \frac{\Delta m z k' b}{4 k'} \right) \\
+ e^{-2Yb} \left( \frac{b^2}{4Y} + \frac{2b}{4Y} + \frac{2}{8Y^3} \right) \right] / B^2 \left( \frac{b}{2} - \frac{\Delta m z k' b}{4 k'} \right) + e^{-2Yb} 
\end{cases}
$$

Here $B^2 = A^2/C^2 = .400$.

The numerical values for the three parameters obtained were in each case:

<table>
<thead>
<tr>
<th>Potential</th>
<th>b (in fm)</th>
<th>$V_0$ (in Mev)</th>
<th>$k'$ (in fm$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square-square (velocity-dependent)</td>
<td>0.0</td>
<td>-0.25</td>
<td>38.5</td>
</tr>
<tr>
<td>Square plus core (static)</td>
<td>1.84</td>
<td>0.24 fm</td>
<td>42.0</td>
</tr>
</tbody>
</table>

The static and velocity-dependent potentials give exactly the same $c$. Both potentials give zero $3S$ phase shift (within 0.02 radians) at 260 Mev and both give $\langle n^2 \rangle_{\infty} = 155$ mb (within 3 mb). Integral (3.11) evaluated with the above potentials gives 2.61 for the case of a square-square velocity-dependent well and 2.56 for the case of a static square well plus core.

$$
\sum_{n} f_{0n} = (2 + 0.44 + 0.17) = 2.61 
$$

(vel-dependent)

The 2 comes from the kinetic energy, the next term comes from $P^M$ for the attractive static potential and the last term comes from the velocity-dependence.
\[
\sum f_{on} = (2 + 0.56) = 2.56 \quad \text{(static + core)}
\]

The second term comes from \( P^M \) for the attractive square well. With these values of the summed oscillator strength the integrated cross section gives 39.2 MeV-nb for the case of velocity-dependent and 38.4 for the case of a static square well plus core.

The two values are so close (about 3% difference) that we could not tell which one corresponds to the true potential. At this stage it seems that velocity-dependent square wells and static square well plus core each adjusted to the two body data (binding energy, mean square radius and high energy \( ^3S \) phase shift) give almost identical integrated cross section for the electron-photoneffect. Further work done with more realistic potentials (velocity-dependent and static) may give more information as to whether we can differentiate one from another.

From these results we observe:

1) Model independent term \( \Sigma f_{on} \) dominates, decreasing the sensitivity of \( \Sigma f_{on} \) to choice of potential (model dependent) (0.610 and 0.56 differ by over 10%).

2) Bound state wave functions \( \Psi \) are rather similar for velocity-dependent and static potential used.
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EXAMINATION AND THESIS REPORT

Candidate: Onofre Rojo-Asenjo

Major Field: Physics


Approved:

[Signatures]

Major Professor and Chairman

Dean of the Graduate School

EXAMINING COMMITTEE:

[Signatures]

Date of Examination:

January 30, 1961