2016

Derived Geometric Satake Equivalence, Springer Correspondence, and Small Representations

Jacob Paul Matherne
Louisiana State University and Agricultural and Mechanical College, jmath34@lsu.edu

DERIVED GEOMETRIC SATAKE EQUIVALENCE, SPRINGER CORRESPONDENCE, AND SMALL REPRESENTATIONS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
Jacob Paul Matherne
B.S., Northwestern State University of Louisiana, 2010
M.S., Louisiana State University, 2011
August 2016

Acknowledgments

The biggest collection of thanks is deserved by my advisor, Pramod Achar. From the very beginning, he had boundless energy to discuss anything, mathematics or otherwise. I can recall several times where he had the option of helping me to understand some concept or doing something which was honest fun—he always chose helping me. He has been a mentor and role model in mathematics as well as in various areas of life. This dissertation would have been impossible without his constant concern for my success. Not to mention, he taught me what a great thing coffee can be! I had rarely partaken in this joy before I was his student.

The second pile of thanks goes out to my wife, Maitreyee Kulkarni. No other person felt more of the pains of dissertation research than she did. She dealt with all of my frustration with ease. When times were busy for me, she did more of life's chores so that I could do less. Even though I traded many times with her for typing or thinking about math, she always showed endless support. She is the great balance in my life—she always knows when it is time for me to take a break from work, and for that I am thankful.

A third collection of thanks is for my parents, Leshia and Jimmy Matherne. Neither of them went to graduate school, but they still made sure that I had proper education. They spent most of their time working (my dad even had two jobs for many years) so that we could have a good life. They instilled in me a work ethic that I would have undoubtedly not gotten very far without.

Finally, I would like to disperse the rest of my not so small amount of thanks to my friends from school and from home. Even when I had not seen anyone for months, they treated me like no time had ever passed.

Table of Contents

Abstract

It is known that the geometric Satake equivalence is intimately related to the Springer correspondence when restricting to small representations of the Langlands dual group (see a paper by Achar and Henderson and one by Achar, Henderson, and Riche). This dissertation relates the derived geometric Satake equivalence of Bezrukavnikov and Finkelberg and the derived Springer correspondence of Rider when we restrict to small representations of the Langlands dual group under consideration. The main theorem of the before-mentioned paper of Achar, Henderson, and Riche sits inside this derived relationship as its degree zero piece.

Chapter 1
Introduction

1.1 A motivational problem

Let $\check{G}$ be a connected, reductive algebraic group over $\mathbb{C}$ with Weyl group W. A small representation of $\check{G}$ is a representation in which all weights lie in the root lattice and no weight is twice a root. It turns out that these small representations interact well with the geometry of the affine Grassmannian Gr and nilpotent cone $\mathcal{N}$ of the Langlands dual group G. In [AH13], this correspondence was made precise in characteristic zero, and in [AHR15], it was verified for a split reductive group scheme over any Noetherian commutative ring of finite global dimension (the level of generality in which the geometric Satake equivalence holds [MV07]). In both of these works, the commutativity of the following diagram is proven

$$\begin{tikzcd}
\mathrm{Perv}_{G(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}) \arrow[r, "\mathcal{S}_G^{\mathrm{sm}}", "\sim"'] \arrow[d, "\Psi_G"'] & \mathrm{Rep}(\check{G})_{\mathrm{sm}} \arrow[d, "\Phi_{\check{G}}"] \\
\mathrm{Perv}_G(\mathcal{N}) \arrow[r, "\mathbb{S}_G"'] & \mathrm{Rep}(W)
\end{tikzcd}$$

where the four functors in the diagram are the following:

- The geometric Satake equivalence $\mathcal{S}_G$ defined in [MV07] restricts to an equivalence $\mathcal{S}_G^{\mathrm{sm}}$ between $\mathrm{Perv}_{G(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}})$, where $\mathrm{Gr}^{\mathrm{sm}}$ is a certain closed subvariety of Gr, and the category $\mathrm{Rep}(\check{G})_{\mathrm{sm}}$ of small representations.

- By [AH13, Theorem 1.1], there is a finite map $\pi : \mathcal{M} \to \mathcal{N}$ where $\mathcal{M}$ is open in $\mathrm{Gr}^{\mathrm{sm}}$, giving rise to a functor $\Psi_G : \mathrm{Perv}_{G(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}) \to \mathrm{Perv}_G(\mathcal{N})$.

- W acts on the zero weight space of any representation of $\check{G}$. Tensoring this action with the sign representation ε of W, there is a functor $\Phi_{\check{G}} : \mathrm{Rep}(\check{G})_{\mathrm{sm}} \to \mathrm{Rep}(W)$.

- W also acts on the Springer sheaf $\underline{\mathrm{Sp}}$ in $\mathrm{Perv}_G(\mathcal{N})$, giving rise to a functor $\mathbb{S}_G = \mathrm{Hom}(\underline{\mathrm{Sp}}, -) : \mathrm{Perv}_G(\mathcal{N}) \to \mathrm{Rep}(W)$.

The proof in [AHR15] involves only two steps:

Step 1. For G of semisimple rank 1, the diagram commutes by direct computation.

Step 2. Every functor in the diagram commutes with 'restriction to a Levi subgroup of semisimple rank 1'.

1.2 Statement of the main problem

Let $\mathcal{D}^{\mathrm{b,mix}}_{G(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}})$ be the bounded, $G(\mathfrak{O})$-equivariant, mixed derived category of constructible sheaves on $\mathrm{Gr}^{\mathrm{sm}}$ (for more details on mixed categories see [BGS96], and for equivariant derived categories see [BL94]). The category $\mathcal{D}^{\mathrm{b,mix}}_{G}(\mathcal{N})$ is defined similarly. The main result of this paper is the following theorem.

Theorem 1.2.1. *Consider the following diagram.*

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{b,mix}}_{G(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}) & \xrightarrow[\sim]{\mathrm{der}\mathcal{S}^{\mathrm{sm}}_G} & \mathcal{D}^{\mathrm{b}}\mathrm{Coh}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*)_{\mathrm{sm}} \\ {\scriptstyle \Psi_G}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{der}\Phi_{\check{G}}} \\ \mathcal{D}^{\mathrm{b,mix}}_{G}(\mathcal{N}) & \xrightarrow[\mathrm{der}\mathbb{S}_G]{} & \mathcal{D}^{\mathrm{b}}\mathrm{Coh}^{W\times\mathbb{G}_m}(\check{\mathfrak{h}}^*) \end{array} \tag{1.1}$$

There is a natural isomorphism of functors

$$\mathrm{der}\Phi_{\check{G}} \circ \mathrm{der}\mathcal{S}^{\mathrm{sm}}_G \Longleftrightarrow \mathrm{der}\mathbb{S}_G \circ \Psi_G$$

making the diagram commute.

The functors involved are described below:

- The derived Satake equivalence $\mathrm{der}\mathcal{S}_G$ defined in [BF08] restricts to an equivalence $\mathrm{der}\mathcal{S}_G^{\mathrm{sm}}$ between $\mathcal{D}^{\mathrm{b,mix}}_{G(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}})$ and $\mathcal{D}^{\mathrm{b}}\mathrm{Coh}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*)_{\mathrm{sm}}$.

- Ψ_G is already defined on the level of derived categories in [AH13].

- $\mathrm{der}\Phi_{\check{G}}$ is the composition $(-\otimes_{\mathcal{O}_{\check{\mathfrak{g}}^*}}\mathcal{O}_{\check{\mathfrak{h}}^*})\circ\Phi_{\check{G}}$.

- There is an isomorphism of algebras $\mathrm{Hom}^\bullet(\underline{\mathrm{Spr}},\underline{\mathrm{Spr}})\cong A_W:=\mathbb{C}[W]\#\mathcal{O}_{\check{\mathfrak{h}}^*}$. Using this fact, and many others, [Rid13] gives the functor

$$\mathrm{der}\mathbb{S}_G:=\mathrm{Hom}^\bullet(\underline{\mathrm{Spr}},-)$$

  between $\mathcal{D}^{\mathrm{b,mix}}_G(\mathcal{N})$ and $\mathcal{D}^{\mathrm{b}}\mathrm{Coh}^{W\times\mathbb{G}_m}(\check{\mathfrak{h}}^*)$.[1]

Our method of proof still involves the same two steps as above. However, the first step now involves calculations of $\mathrm{Hom}^\bullet$-algebras instead of just Hom-algebras; i.e., this is no longer only a degree zero computation. For the same reason, there are many added complications in the second step as well.

One way to simplify the problem is to restrict to a diagram of additive subcategories.

$$\begin{tikzcd} \mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}) \arrow[r, "\mathrm{der}\mathcal{S}_G^{\mathrm{sm}}", "\sim"'] \arrow[d, "\Psi_G"'] & \mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*)_{\mathrm{sm}} \arrow[d, "\mathrm{der}\Phi_{\check{G}}"] \\ \mathrm{Semis}_G(\mathcal{N}) \arrow[r, "\mathrm{der}\mathbb{S}_G"'] & \mathrm{Coh}_{\mathrm{fr}}^{W\times\mathbb{G}_m}(\check{\mathfrak{h}}^*) \end{tikzcd} \tag{1.2}$$

Each of the functors above restricts to this setting. The commutativity of the first diagram will follow from the commutativity of this diagram after passing to the bounded homotopy categories $K^{\mathrm{b}}(-)$—this uses the notion of Orlov categories (see

[1] The functor $\mathrm{der}\mathbb{S}_G$ is an equivalence if one restricts to the Springer block $\mathcal{D}^{\mathrm{b,mix}}_{G,\mathrm{Spr}}(\mathcal{N})$, the triangulated subcategory of $\mathcal{D}^{\mathrm{b,mix}}_G(\mathcal{N})$ generated by the simple summands of $\underline{\mathrm{Spr}}$.

Section 4.2) to lift the natural isomorphism η_G for Diagram (1.2) to a unique one proving Theorem 1.2.1. Therefore, we concentrate on proving that Diagram (1.2) commutes using Steps 1 and 2.

Remark 1.2.2. *As we will see in Section 3 (and as was mentioned in Steps* 1 *and* 2 *above), the natural isomorphism for Diagram (1.2) will be constructed by pasting together natural isomorphisms for Levi subgroups of semisimple rank* 1 *(i.e., for each simple reflection). However, an object in* $\mathrm{Coh}_{\mathrm{fr}}^{W\times\mathbb{G}_m}(\check{\mathfrak{h}}^*)$ *is not determined by objects in* $\mathrm{Coh}_{\mathrm{fr}}^{W_L\times\mathbb{G}_m}(\check{\mathfrak{h}}^*)$ *for each* L *of semisimple rank* 1*. We need, in addition, an identification of the underlying vector spaces. Both compositions of functors,* $\mathrm{der}\Phi_{\check{G}}\circ\mathrm{der}\mathcal{S}_G^{\mathrm{sm}}$ *and* $\mathrm{der}\mathbb{S}_G\circ\Psi_G$*, give an identification, but the commutativity of* (1.2) *can only be deduced if the identifications are the same. Thus, we must work* 2*-categorically to keep track of the identifications along the way.*

Throughout the paper, we will perform many 2*-categorical computations involving pasting together various* 2*-categorical diagrams. We will also prove the commutativity of several* 2*-categorical diagrams. For background on* 2*-categorical computations of this flavor as well as various natural isomorphisms for sheaf functors, see the appendices in [AHR15].*

1.3 Organization of the paper

In Section 2, we fix notation for the rest of the paper, and we spend some time defining all of the categories and functors in the main diagram before giving a precise statement of the main theorem. In Section 3, we lay out all of the ingredients needed for the proof of the main theorem, and we give a walkthrough of the proof modulo details which will be verified in later sections. Sections 4, 5, and 8 give some general results needed later. In Section 6, we define appropriate restriction functors for each of the categories in the main diagram, and in Section 7, we prove that each of the functors in the main diagram are compatible with restriction to a Levi subgroup. In Section 10, we prove that the main diagram commutes for G a group of semisimple rank 1, and in Section 11, we prove some lemmas needed for the commutativity in Section 10.

Chapter 2
Notation and Preliminaries

In this chapter, we fix notation and discuss preliminaries needed for understanding the statement of Theorem 1.2.1.

2.1 Notation

Throughout the paper, G will denote a connected, reductive algebraic group over $\mathbb{C}$. Fix a Borel and maximal torus $B \supset T$. The weight lattice will be denoted Λ with the set of dominant integral weights Λ^+. Let $\check{G}$ denote the Langlands dual group of G, with maximal torus, Borel, weights, and dominant integral weights given by $\check{T}$, $\check{B}$, $\check{\Lambda}$, and $\check{\Lambda}^+$ respectively. Let W be the Weyl group of either G or $\check{G}$ (they have a canonical identification).

Sometimes we will instead work in a parabolic setting. That is, we will fix a parabolic subgroup P of G that contains the Borel B. There is a Levi decomposition $P = LU_P$ with $L \supset T$. Set $C = B \cap L$, a Borel subgroup of L which contains T. Since L and T are also connected reductive groups, we can perform any construction equivalently for $G \supset B \supset T$, $L \supset C \supset T$, and $T \supset T \supset T$. Denote the corresponding Weyl groups with subscripts—for example, W_G, W_L, and W_T. Write $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{l}$, and $\mathfrak{h}$ for the Lie algebras of G, B, L, and T, respectively.

Let X be a variety with a stratification, and let $\mathcal{O}_X$ denote its structure sheaf. If X carries an action of an algebraic group H, we write $\mathrm{Perv}_H(X)$ (resp. $\mathcal{D}^{\mathrm{b}}_H(X)$) for the category of H-equivariant constructible perverse sheaves on X (resp. the H-equivariant bounded derived category of constructible sheaves on X). If $\mathcal{F}$ is a complex of sheaves, we write $\mathbb{H}^\bullet(\mathcal{F})$ for the hypercohomology $\bigoplus_{i\in\mathbb{Z}} H^i(R\Gamma(\mathcal{F}))$ (and $\mathbb{H}^i(\mathcal{F})$ for $H^i(R\Gamma(\mathcal{F}))$), where H^i denotes the ith cohomology of a complex of vector spaces.

We will often consider the derived category of mixed sheaves $\mathcal{D}^{\mathrm{b}}_{\mathrm{m}}(X)$. When there is a Frobenius action, we denote by (n) the Tate twist which decreases weight by $2n$. If $\mathcal{F}, \mathcal{G} \in \mathcal{D}^{\mathrm{b}}_{\mathrm{m}}(X)$, we write

$$\mathbb{H}\mathrm{om}(\mathcal{F}, \mathcal{G}) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}_{\mathrm{m}}(X)}(\mathcal{F}, \mathcal{G}[2n](n)).$$

Moreover, $\mathbb{H}\mathrm{om}^i(\mathcal{F}, \mathcal{G}) := \mathrm{Hom}(\mathcal{F}, \mathcal{G}[2i](i))$. Note that $[n]$ increases weight by n, so $[2n](n)$ preserves weight. We will sometimes work in categories where there is no notion of weight. In this case, we write

$$\mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{G}) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}(\mathcal{F}, \mathcal{G}[n]).$$

Moreover, $\mathrm{Hom}^i(\mathcal{F}, \mathcal{G}) := \mathrm{Hom}(\mathcal{F}, \mathcal{G}[i])$.

If $K \subset H$ and K acts on X, then we define the variety

$$H \times^K X = (H \times X)/\sim$$

where $(hk, x) \sim (h, k^{-1}x)$ for all $h \in h, k \in K$, and $x \in X$.

If R is a ring, we write R-mod for the category of left R-modules. When we consider graded R-modules, we write $\langle n\rangle$ to denote grading shift; that is, if M is a graded R-module, then $(M\langle n\rangle)_i = M_{i+n}$. If M and N are two graded R-modules (or $\mathbb{G}_m$-equivariant quasi-coherent sheaves), we write

$$\underline{\mathrm{Hom}}(M, N) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{R\text{-mod}}(M, N\langle n\rangle).$$

Moreover, $\underline{\mathrm{Hom}}^i(M, N) := \mathrm{Hom}(M, N\langle i\rangle)$.

We use double arrows $\Longrightarrow$ for natural transformations of functors and $\Longleftrightarrow$ for natural isomorphisms of functors. When we want to specify the direction of a natural isomorphism, we write $\overset{\sim}{\Longrightarrow}$.

2.2 Reduction to semisimple categories

Throughout this paper, we will need to consider varieties over various fields and sheaves on these varieties with various coefficients. Let X be a variety with a stratification and with an action of a group G. When we want to distinguish among versions of this variety defined over various fields, we will write $X_{\mathbb{F}_q}$, $X_{\overline{\mathbb{F}}_q}$, and $X_{\mathbb{C}}$. Consider the following diagram

$$\begin{tikzcd} \mathcal{D}^{\mathrm{b,mix}}_G(X_{\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell) \arrow[r] & \mathcal{D}^{\mathrm{b}}_G(X_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell) \arrow[d, dash, "\sim"] \\ & \mathcal{D}^{\mathrm{b}}_G(X_{\mathbb{C}}, \mathbb{C}), \end{tikzcd} \tag{2.1}$$

where we give $X_{\mathbb{F}_q}$ and $X_{\overline{\mathbb{F}}_q}$ the étale topology and $X_{\mathbb{C}}$ the complex analytic topology.

Diagram (2.1) restricts to the following diagram

$$\begin{tikzcd} \mathrm{Pure}_G(X_{\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell) \arrow[r, "\text{fully faithful}"] & \text{(match)} \\ & \mathrm{Semis}_G(X_{\mathbb{C}}, \mathbb{C}), \arrow[u, "\text{fully faithful}"'] \end{tikzcd} \tag{2.2}$$

where $\mathrm{Pure}_G(X_{\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell)$ is the additive category consisting of objects

$$\mathcal{F} \simeq \mathrm{IC}_1[i_1]\left(\frac{i_1}{2}\right) \bigoplus \ldots \bigoplus \mathrm{IC}_n[i_n]\left(\frac{i_n}{2}\right),$$

and $\mathrm{Semis}_G(X_{\mathbb{C}}, \mathbb{C})$ is the additive category consisting of objects

$$\mathcal{G} \simeq \mathrm{IC}_1[j_1] \bigoplus \ldots \bigoplus \mathrm{IC}_m[j_m].$$

The autoequivalence $[n]\left(\frac{n}{2}\right)$ on $\mathrm{Pure}_G(X_{\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell)$ in Diagram (2.2) coincides with the autoequivalence $[n]$ on the other two categories.

Instead of lugging weights around, we will often make use of the equivalence in Diagram (2.2) in the computations to follow. Throughout the paper (since our strategy is to prove commutativity of Diagram (1.2)), we will work exclusively with $\mathrm{Semis}_G(X) := \mathrm{Semis}_G(X_{\mathbb{C}}, \mathbb{C})$, except for in Section 4.4, where we specifically need to work with weights. We also define $\mathcal{D}_G^{\mathrm{b,mix}}(X) := K^{\mathrm{b}}(\mathrm{Semis}_G(X))$.

Remark 2.2.1. *This definition of $\mathcal{D}_G^{\mathrm{b,mix}}(X)$ coincides with the definition from other sources that we could have used to make sense of Diagram (2.1) before explaining Diagram (2.2).*

2.3 The affine Grassmannian and the geometric Satake equivalence

Let $\mathfrak{O} = \mathbb{C}[[t]]$, power series in t, and $\mathfrak{K} = \mathbb{C}((t))$, Laurent series in t. Define the affine Grassmannian Gr to be the ind-variety $G(\mathfrak{K})/G(\mathfrak{O})$. It carries an action by $G(\mathfrak{O})$ which gives it a stratification

$$\mathrm{Gr} = \bigsqcup_{\check{\lambda} \in \check{\Lambda}^+} \mathrm{Gr}^{\check{\lambda}}$$

by finite-dimensional, quasi-projective varieties.

The celebrated geometric Satake equivalence (see [Lus83, Gin95, MV07]) gives an equivalence of tensor categories

$$\mathcal{S}_G : (\mathrm{Perv}_{G(\mathfrak{O})}(\mathrm{Gr}), *) \longrightarrow (\mathrm{Rep}(\check{G}), \otimes)$$

where $*$ is convolution of perverse sheaves and $\otimes$ is tensor product of representations.

The proof of the above theorem involves producing a fiber functor

$$\mathbb{H}^\bullet : \mathrm{Perv}_{G(\mathfrak{O})}(\mathrm{Gr}) \to \mathrm{Vect}_{\mathbb{C}}$$

which satisfies the properties laid out by the Tannakian formalism (see [DM82]). From this general theory, one learns that $\mathrm{Perv}_{G(\mathfrak{O})}(\mathrm{Gr})$ is equivalent to the category of representations of some algebraic group. It can be shown that this algebraic group is in fact $\check{G}$ (see [MV07]).

There is a more general result (see [ABG04, BF08]) which establishes an equivalence at the level of derived categories. We will give an outline of the proof presented in [BF08], which is not independent of the proof in the abelian category setting given in [MV07].

One has three categories that are related by functors:

$$\begin{tikzcd} \mathcal{D}^{\mathrm{b,mix}}_{G(\mathfrak{O})}(\mathrm{Gr}) \arrow[r, "\mathbb{H}^{\bullet}_{G(\mathfrak{O})}"'] & H^{\bullet}_{G(\mathfrak{O})}(\mathrm{Gr})\text{-mod} & \mathrm{Coh}^{\check{G}\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{g}}^*) \arrow[l, "\kappa"'] \arrow[ll, bend right, "\widetilde{\mathcal{S}}_G^{-1}"'] \end{tikzcd} \tag{2.3}$$

The Satake equivalence $\mathcal{S}_G^{-1} : \mathrm{Rep}(\check{G}) \to \mathrm{Perv}_{G(\mathfrak{O})}(\mathrm{Gr})$ extends to a full embedding $\widetilde{\mathcal{S}}_G^{-1} : \mathrm{Coh}^{\check{G}\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{g}}^*) \to \mathcal{D}^{\mathrm{b,mix}}_{G(\mathfrak{O})}(\mathrm{Gr})$ (see [BF08]), where $\mathrm{Coh}^{\check{G}\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{g}}^*)$ is the full subcategory of $\mathrm{Coh}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*)$ consisting of all objects of the form $V \otimes \mathcal{O}_{\check{\mathfrak{g}}^*}$ with $V \in \mathrm{Rep}(\check{G} \times \mathbb{G}_m)$. One of the main results of [BF08] is that there exists an isomorphism of functors

$$\kappa \Longleftrightarrow \mathbb{H}^{\bullet}_{G(\mathfrak{O})} \circ \widetilde{\mathcal{S}}_G^{-1}. \tag{2.4}$$

The functor κ will be defined in Section 8.

In [BF08], Bezrukavnikov and Finkelberg show that the functor $\mathbb{H}^{\bullet}_{G(\mathfrak{O})}$ restricts to an equivalence of categories

$$\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}) \xrightarrow{\sim} \mathcal{Y} := \{\text{certain } H^{\bullet}_{G(\mathfrak{O})}(\mathrm{Gr})\text{-modules}\}.$$

Remark 2.3.1. *Note that $\mathcal{Y}$ is the image of $\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr})$ in $H^{\bullet}_{G(\mathfrak{O})}(\mathrm{Gr})$-mod under the functor $\mathbb{H}^{\bullet}_{G(\mathfrak{O})}$.*

Similarly, the main results of [BF08] show that κ gives an equivalence of categories

$$\mathcal{Y} \xleftarrow{\sim} \mathrm{Coh}^{\check{G}\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{g}}^*).$$

Combining these equivalences with (2.4), it follows that

$$\widetilde{\mathcal{S}}_G^{-1} : \mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*) \xrightarrow{\sim} \mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr})$$

is an equivalence of categories. Taking bounded homotopy categories $K^b(-)$ gives the desired derived geometric Satake equivalence

$$\mathrm{der}\mathcal{S}_G^{-1} : \mathcal{D}^{\mathrm{b}}\mathrm{Coh}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b,mix}}_{G(\mathfrak{O})}(\mathrm{Gr}).$$

2.4 The Springer correspondence

Consider the Grothendieck-Springer resolution

$$\begin{array}{ccc} G\times^B \mathfrak{b} & \xrightarrow{\mu} & \mathfrak{g} \\ (g,x) & \longmapsto & g\cdot x \end{array}$$

We have inclusions $j^{\mathrm{rs}} : \mathfrak{g}^{\mathrm{rs}} \hookrightarrow \mathfrak{g}$ where $\mathfrak{g}^{\mathrm{rs}}$ is the open subset of regular semisimple elements, and $i : \mathcal{N} \hookrightarrow \mathfrak{g}$ where $\mathcal{N}$ is the set of nilpotent elements. Let μ^{rs} and $\mu_{\mathcal{N}}$ be the restrictions of μ to $\widetilde{\mathfrak{g}}^{\mathrm{rs}} = \mu^{-1}(\mathfrak{g}^{\mathrm{rs}})$ and $\widetilde{\mathcal{N}} = G\times^B \mathfrak{n}$, respectively, and consider the diagram

$$\begin{tikzcd} \widetilde{\mathcal{N}} \arrow[r, hook] \arrow[d, "\mu_{\mathcal{N}}"] & G\times^B \mathfrak{b} \arrow[d, "\mu"] & \widetilde{\mathfrak{g}}^{\mathrm{rs}} \arrow[l, hook'] \arrow[d, "\mu^{\mathrm{rs}}"] \\ \mathcal{N} \arrow[r, hook, "i"] & \mathfrak{g} & \mathfrak{g}^{\mathrm{rs}} \arrow[l, hook', "j^{\mathrm{rs}}"'] \end{tikzcd}$$

where each square is Cartesian. Let $\underline{\mathrm{Gro}} = \mu_! \underline{\mathbb{C}}_{G\times^B\mathfrak{b}}[\dim\mathfrak{g}]$, which is a perverse sheaf since μ is proper and small. There is a canonical isomorphism

$$\underline{\mathrm{Gro}} \simeq j^{\mathrm{rs}}_{!*}(\mu^{\mathrm{rs}}_! \underline{\mathbb{C}}_{\widetilde{\mathfrak{g}}^{\mathrm{rs}}}[\dim\mathfrak{g}])$$

which gives an action of W (by automorphisms in $\mathrm{Perv}(\mathfrak{g})$) on $\underline{\mathrm{Gro}}$, since μ^{rs} is a Galois covering with group W.

Now define the Springer sheaf

$$\underline{\mathrm{Sp}} = (\mu_{\mathcal{N}})_! \underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}[\dim \mathcal{N}].$$

Since the left square is Cartesian, we can base change to get an isomorphism

$$\underline{\mathrm{Sp}} \simeq i^* \underline{\mathrm{Gro}}[\dim \mathcal{N} - \dim \mathfrak{g}].$$

Thus, we get an action of W on $\underline{\mathrm{Sp}}$ which allows us to define a functor

$$\begin{array}{rcl} \mathrm{Perv}_G(\mathcal{N}) & \xrightarrow{\mathbb{S}_G} & \mathrm{Rep}(W) \\ \mathcal{F} & \longmapsto & \mathrm{Hom}_{\mathrm{Perv}_G(\mathcal{N})}(\underline{\mathrm{Sp}}, \mathcal{F}) \end{array}$$

called the Springer correspondence.

There is a more general derived Springer correspondence (see [Rid13]) that we will review here. In this setting, define the Springer sheaf to be

$$\underline{\mathrm{Spr}} = (\mu_{\mathcal{N}})_! \underline{\mathbb{C}}_{\mathcal{N}}[\dim \mathcal{N}]$$

where $(\mu_{\mathcal{N}})_! : \mathcal{D}^{\mathrm{b,mix}}_G(\widetilde{\mathcal{N}}) \to \mathcal{D}^{\mathrm{b,mix}}_G(\mathcal{N})$ is now the derived proper pushforward functor. In general, all functors will be derived from now on.

Let $\mathrm{Coh}^{W \times \mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{h}}^*)$ be the full subcategory of $\mathcal{D}^{\mathrm{b}}\mathrm{Coh}^{W \times \mathbb{G}_m}(\check{\mathfrak{h}}^*)$ consisting of objects M of the form $M = (V_1 \otimes \mathcal{O}_{\check{\mathfrak{h}}^*}\langle i_1 \rangle) \oplus \ldots \oplus (V_n \otimes \mathcal{O}_{\check{\mathfrak{h}}^*}\langle i_n \rangle)$. In [Rid13], Rider proves an equivalence of additive categories

$$\begin{array}{rcl} \mathrm{Semis}_{G,\mathrm{Spr}}(\mathcal{N}) & \xrightarrow{\sim} & \mathrm{Coh}^{W \times \mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{h}}^*) \\ \mathcal{F} & \longmapsto & \displaystyle\bigoplus_{m \in \mathbb{Z}} \mathrm{Hom}(\underline{\mathrm{Spr}}[-2m], \mathcal{F}), \end{array}$$

where the subscript Spr denotes the Springer block; that is, the subcategory generated by simple summands of $\underline{\mathrm{Spr}}$.

We will give an idea of some of the elements of the proof. In [BM81], the authors prove an isomorphism

$$\mathrm{End}(\underline{\mathrm{Spr}}) \simeq \mathbb{C}[W].$$

This leads to a decomposition of the Springer sheaf

$$\underline{\mathrm{Spr}} \simeq \bigoplus_{\chi \in \mathrm{Irr}(W)} \mathrm{IC}_\chi \otimes V_\chi.$$

Using this decomposition, and the fact that $\mathrm{Hom}^\bullet_G(\underline{\mathrm{Spr}}, \underline{\mathrm{Spr}}) \simeq C[W]\#\mathcal{O}_{\check{\mathfrak{h}}^*}$, Rider was able to prove the equivalence. If we apply $K^{\mathrm{b}}(-)$ to Rider's equivalence of additive categories above and note that $\mathcal{D}^{\mathrm{b,mix}}_{G,\mathrm{Spr}}(\mathcal{N}) := K^{\mathrm{b}}(\mathrm{Semis}_{G,\mathrm{Spr}}(\mathcal{N}))$ (see Section 2.2), we get her mixed equivalence [Rid13, Theorem 6.3]:

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{b,mix}}_{G,\mathrm{Spr}}(\mathcal{N}) & \xrightarrow{\mathrm{der}\mathbb{S}_G} & \mathcal{D}^{\mathrm{b}}\mathrm{Coh}^{W \times \mathbb{G}_m}(\check{\mathfrak{h}}^*) \\ \mathcal{F} & \longmapsto & \bigoplus_{m \in \mathbb{Z}} \mathrm{Hom}(\underline{\mathrm{Spr}}[-2m], \mathcal{F}) \end{array}$$

2.5 The functor Ψ_G

Let $\mathbf{o}$ be the base point of the affine Grassmannian; that is, the image of the identity element of $G(\mathfrak{K})$ in Gr. Let $\mathfrak{O}^- = \mathbb{C}[t^{-1}]$. Let $\mathrm{Gr}^-_{\mathbf{o}}$ be the $G(\mathfrak{O}^-)$-orbit of $\mathbf{o}$. Consider the evaluation map

$$\begin{array}{ccc} G(\mathfrak{O}^-) & \longrightarrow & G \\ t^{-1} & \longmapsto & 0 \end{array}$$

Let $\mathfrak{G}$ be its kernel. This map factors through $G(\mathbb{C}[t^{-1}]/(t^{-2}))$. The kernel of the evaluation map $G(\mathbb{C}[t^{-1}]/(t^{-2})) \to G$ can be identified with $\mathfrak{g}$. Thus, we have a homomorphism

$$\mathfrak{G} \to \mathfrak{g}$$

of the kernels.

Note that the stabilizer of $\mathbf{o} \in \mathrm{Gr}$ in $G(\mathfrak{O}^-)$ is G. Thus, the action of $G(\mathfrak{O}^-)$ on $\mathrm{Gr}^-_{\mathbf{o}}$ induces a G-equivariant morphism

$$\mathrm{Gr}^-_{\mathbf{o}} \simeq \mathfrak{G}.$$

Composing this map with the one above, we get a G-equivariant morphism

$$\pi^\dagger : \mathrm{Gr}^-_{\mathbf{o}} \to \mathfrak{g}.$$

Recall from the introduction that a small representation of $\check{G}$ is one where all weights belong to the root lattice and no weight is twice a root. Define the small part of the affine Grassmannian of G

$$\mathrm{Gr}^{\mathrm{sm}} = \bigsqcup_{\check{\lambda} \in \check{\Lambda}^+_{\mathrm{sm}}} \mathrm{Gr}^{\check{\lambda}}$$

to be the union of the $G(\mathfrak{O})$-orbits corresponding to the small representations of $\check{G}$. Define the open subvariety $\mathcal{M}$ of $\mathrm{Gr}^{\mathrm{sm}}$ to be

$$\mathcal{M} = \mathrm{Gr}^{\mathrm{sm}} \cap \mathrm{Gr}^-_{\mathbf{o}}$$

and let $j : \mathcal{M} \hookrightarrow \mathrm{Gr}^{\mathrm{sm}}$ be the inclusion map.

Now, let π be the restriction

$$\pi = \pi^\dagger|_{\mathcal{M}} : \mathcal{M} \to \mathfrak{g}.$$

By [AH13, Theorem 1.1], we have that $\pi^\dagger(\mathcal{M}) \subset \mathcal{N}$ and π is a finite G-equivariant morphism.[1] We can form the composition

$$\Psi_G = \pi_* \circ j^! : \mathcal{D}^{\mathrm{b,mix}}_{G(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}) \to \mathcal{D}^{\mathrm{b,mix}}_{G}(\mathcal{N}).$$

Remark 2.5.1. *Since π is finite and j is an open inclusion, they are both t-exact, and Ψ_G restricts to an exact functor* $\mathrm{Perv}_{G(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}) \to \mathrm{Perv}_G(\mathcal{N})$. *The functor Ψ_G also gives a functor on semisimple categories:* $\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}) \to \mathrm{Semis}_G(\mathcal{N})$, *which will be denoted in the same way.*

[1] The map π can be viewed as a generalization of Lusztig's embedding $\mathcal{N}_{\mathbf{GL}_n} \hookrightarrow \mathrm{Gr}_{\mathbf{GL}_n}$ (see [Lus81]) to other types. Many of its properties are explicitly computed in [AH13].

2.6 The functor $\mathrm{der}\Phi_{\check{G}}$

Let $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}\mathrm{Coh}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*)$. The functor $\mathrm{der}\Phi_{\check{G}}$ is defined by a composition of functors. First, define $\mathrm{Res}^{\check{G}}_{N_{\check{G}}(\check{T})} : \mathcal{D}^{\mathrm{b}}\mathrm{Coh}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*) \to \mathcal{D}^{\mathrm{b}}\mathrm{Coh}^{N_{\check{G}}(\check{T})\times\mathbb{G}_m}(\check{\mathfrak{g}}^*)$ to be the restriction of the equivariance on $\mathcal{F}$. Now, define $R^{G'}_T : \mathcal{D}^{\mathrm{b}}\mathrm{Coh}^{N_{\check{G}}(\check{T})\times\mathbb{G}_m}(\check{\mathfrak{g}}^*) \to \mathcal{D}^{\mathrm{b}}\mathrm{Coh}^{N_{\check{G}}(\check{T})\times\mathbb{G}_m}(\check{\mathfrak{h}}^*)$ to be derived coherent restriction (see §6.3 for the definition of $R^{G'}_T$ for free coherent sheaves). Lastly, define $(-)^{\check{T}} : \mathcal{D}^{\mathrm{b}}\mathrm{Coh}^{N_{\check{G}}(\check{T})\times\mathbb{G}_m}(\check{\mathfrak{h}}^*) \to \mathcal{D}^{\mathrm{b}}\mathrm{Coh}^{W\times\mathbb{G}_m}(\check{\mathfrak{h}}^*)$ to be the functor of taking $\check{T}$-invariants. The functor $\mathrm{der}\Phi_{\check{G}}$ is the composition of these three functors together with a twist by the sign representation ε of W:

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{b}}\mathrm{Coh}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*) & \xrightarrow{\mathrm{der}\Phi_{\check{G}}} & \mathcal{D}^{\mathrm{b}}\mathrm{Coh}^{W\times\mathbb{G}_m}(\check{\mathfrak{h}}^*) \\ \mathcal{F} & \longmapsto & ((-)^{\check{T}}\otimes\varepsilon)\circ R^{G'}_T \circ \mathrm{Res}^{\check{G}}_{N_{\check{G}}(\check{T})}(\mathcal{F}). \end{array} \tag{2.5}$$

Chapter 3
Walkthrough of Proof of Main Theorem

In this section, we will give the proof of Theorem 1.2.1 modulo details that will be provided in later sections. As mentioned in Section 1.2, we will concentrate on proving the commutativity of the diagram of additive categories in Diagram (1.2).

3.1 A preliminary result

Let W' be a subgroup of W. Then, we define

$$R^W_{W'} : \mathrm{Coh}^{W \times \mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{h}}^*) \to \mathrm{Coh}^{W' \times \mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{h}}^*)$$

as in Section 6.4. In this section, we will take W' to be the subgroup W_s generated by a single simple reflection s. The following lemma, whose proof we omit, allows us to piece together information about a W-equivariant coherent sheaf from W_s-equivariant coherent sheaves for all s.

Lemma 3.1.1. *Suppose that $G, H : \mathcal{A} \to \mathrm{Coh}^{W \times \mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{h}}^*)$ are two $\mathbb{C}$-linear functors and $\mathcal{A}$ is a $\mathbb{C}$-linear category. Suppose that we have an isomorphism of functors*

$$\phi : \mathrm{For}^W \circ G \xRightarrow{\sim} \mathrm{For}^W \circ H.$$

Suppose that for any simple reflection $s \in W$, we have an isomorphism of functors

$$\phi^{W_s} : R^W_{W_s} \circ G \xRightarrow{\sim} R^W_{W_s} \circ H \text{ such that } \mathrm{For}^{W_s} \circ \phi^{W_s} = \phi.$$

Then, there is a unique isomorphism of functors $\phi^W : G \xRightarrow{\sim} H$ such that $\mathrm{For}^W \circ \phi^W = \phi$.

3.2 Restriction functors

In Section 6, we will define appropriate restriction functors for each of the additive categories in Diagram (1.2). In Section 6.1, we define a restriction functor

$$\mathfrak{R}_L^G : \mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}_G) \to \mathrm{Semis}_{L(\mathfrak{O})}(\mathrm{Gr}_L).$$

We similarly define restriction functors for the other additive categories in Diagram (1.2): in Section 6.2, we define

$$\widetilde{\mathcal{R}}_L^G : \mathrm{Semis}_G(\mathcal{N}_G) \to \mathrm{Semis}_L(\mathcal{N}_L),$$

in Section 6.3, we define

$$R_L^G : \mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*) \to \mathrm{Coh}_{\mathrm{fr}}^{\check{L}\times\mathbb{G}_m}(\check{\mathfrak{l}}^*),$$

and in Section 6.4, we define

$$R_{W_L}^{W_G} : \mathrm{Coh}_{\mathrm{fr}}^{W_G\times\mathbb{G}_m}(\check{\mathfrak{h}}^*) \to \mathrm{Coh}_{\mathrm{fr}}^{W_L\times\mathbb{G}_m}(\check{\mathfrak{h}}^*).$$

We also construct transitivity of restriction natural isomorphisms for each of the restriction functors above: in (6.2) we construct

$$\mathfrak{R}_T^G \Longleftrightarrow \mathfrak{R}_T^L \circ \mathfrak{R}_L^G : \mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}_G) \to \mathrm{Semis}_{T(\mathfrak{O})}(\mathrm{Gr}_T),$$

in (6.4), we construct

$$\widetilde{\mathcal{R}}_T^G \Longleftrightarrow \widetilde{\mathcal{R}}_T^L \circ \widetilde{\mathcal{R}}_L^G : \mathrm{Semis}_G(\mathcal{N}_G) \to \mathrm{Semis}_T(\mathcal{N}_T),$$

in (6.7), we construct

$$R_T^G \Longleftrightarrow R_T^L \circ R_L^G : \mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*) \to \mathrm{Coh}_{\mathrm{fr}}^{\check{T}\times\mathbb{G}_m}(\check{\mathfrak{h}}^*),$$

and in (6.8), we construct

$$R_{W_T}^{W_G} = R_{W_T}^{W_L} \circ R_{W_L}^{W_G} : \mathrm{Coh}_{\mathrm{fr}}^{W_G\times\mathbb{G}_m}(\check{\mathfrak{h}}^*) \to \mathrm{Coh}_{\mathrm{fr}}^{W_T\times\mathbb{G}_m}(\check{\mathfrak{h}}^*).$$

3.3 Compatibility for restriction functors

In Section 7, we first define several intertwining isomorphisms between the restriction functors in Section 3.2 and the functors in Diagram (1.2). In (7.16), we define

$$R_L^G \circ \mathrm{der}\mathcal{S}_G^{\mathrm{sm}} \Longleftrightarrow \mathrm{der}\mathcal{S}_L^{\mathrm{sm}} \circ \mathfrak{R}_L^G,$$

in (7.24), we define

$$R_{W_L}^{W_G} \circ \mathrm{der}\mathbb{S}_G \Longleftrightarrow \mathrm{der}\mathbb{S}_L \circ \widetilde{\mathcal{R}}_L^G,$$

in Diagram (7.5), we define

$$\widetilde{\mathcal{R}}_L^G \circ \Psi_G \Longleftrightarrow \Psi_L \circ \mathfrak{R}_L^G,$$

and in (7.2), we define

$$\mathrm{der}\Phi_{\check{L}} \circ R_L^G \Longleftrightarrow R_{W_L}^{W_G} \circ \mathrm{der}\Phi_{\check{G}}.$$

Then, we show that the various functors in Diagram (1.2) are compatible with the transitivity of restriction natural isomorphisms mentioned in Section 3.2. That is, we will prove the following series of theorems.

Theorem 3.3.1. *The following prism commutes:*

$$\begin{tikzcd}
\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}_G^{\mathrm{sm}}) \arrow[rr, "\mathrm{der}\mathcal{S}_G^{\mathrm{sm}}"] \arrow[dr, "\mathfrak{R}_L^G"] \arrow[dd, "\mathfrak{R}_T^G"'] & & \mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*)_{\mathrm{sm}} \arrow[dr, "R_L^G"] \arrow[dd, dotted, "R_T^G"] & \\
& \mathrm{Semis}_{L(\mathfrak{O})}(\mathrm{Gr}_L^{\mathrm{sm}}) \arrow[rr, "\mathrm{der}\mathcal{S}_L^{\mathrm{sm}}"] \arrow[dl, "\mathfrak{R}_T^L"] & & \mathrm{Coh}_{\mathrm{fr}}^{\check{L}\times\mathbb{G}_m}(\check{\mathfrak{l}}^*)_{\mathrm{sm}} \arrow[dl, "R_T^L"] \\
\mathrm{Semis}_{T(\mathfrak{O})}(\mathrm{Gr}_T^{\mathrm{sm}}) \arrow[rr, "\mathrm{der}\mathcal{S}_T^{\mathrm{sm}}"'] & & \mathrm{Coh}_{\mathrm{fr}}^{\check{T}\times\mathbb{G}_m}(\check{\mathfrak{h}}^*)_{\mathrm{sm}} &
\end{tikzcd} \tag{3.1}$$

Theorem 3.3.2. *The following prism commutes:*

$$\begin{tikzcd}
\mathrm{Semis}_G(\mathcal{N}_G) \arrow[rr, "\mathrm{der}\mathbb{S}_G"] \arrow[dr, "\widetilde{\mathcal{R}}_L^G"] \arrow[dd, "\widetilde{\mathcal{R}}_T^G"'] & & \mathrm{Coh}_{\mathrm{fr}}^{W_G\times\mathbb{G}_m}(\check{\mathfrak{h}}^*) \arrow[dr, "R_{W_L}^{W_G}"] \arrow[dd, dotted, "R_{W_T}^{W_G}"] & \\
& \mathrm{Semis}_L(\mathcal{N}_L) \arrow[rr, "\mathrm{der}\mathbb{S}_L"] \arrow[dl, "\widetilde{\mathcal{R}}_T^L"] & & \mathrm{Coh}_{\mathrm{fr}}^{W_L\times\mathbb{G}_m}(\check{\mathfrak{h}}^*) \arrow[dl, "R_{W_T}^{W_L}"] \\
\mathrm{Semis}_T(\mathcal{N}_T) \arrow[rr, "\mathrm{der}\mathbb{S}_T"'] & & \mathrm{Coh}_{\mathrm{fr}}^{W_T\times\mathbb{G}_m}(\check{\mathfrak{h}}^*) &
\end{tikzcd} \tag{3.2}$$

Theorem 3.3.3. *The following prism commutes:*

$$
\begin{tikzcd}
\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}_G^{\mathrm{sm}}) \arrow[rr, "\Psi_G"] \arrow[dr, "\mathfrak{R}_L^G"] \arrow[dd, "\mathfrak{R}_T^G"'] & & \mathrm{Semis}_G(\mathcal{N}_G) \arrow[dr, "\widetilde{\mathcal{R}}_L^G"] \arrow[dd, dotted, "\widetilde{\mathcal{R}}_T^G" near start] & \\
& \mathrm{Semis}_{L(\mathfrak{O})}(\mathrm{Gr}_L^{\mathrm{sm}}) \arrow[rr, "\Psi_L" near start] \arrow[dl, "\mathfrak{R}_T^L"] & & \mathrm{Semis}_L(\mathcal{N}_L) \arrow[dl, "\widetilde{\mathcal{R}}_T^L"] \\
\mathrm{Semis}_{T(\mathfrak{O})}(\mathrm{Gr}_T^{\mathrm{sm}}) \arrow[rr, "\Psi_T"'] & & \mathrm{Semis}_T(\mathcal{N}_T) &
\end{tikzcd}
\tag{3.3}
$$

Theorem 3.3.4. *The following prism commutes:*

$$
\begin{tikzcd}
\mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*)_{\mathrm{sm}} \arrow[rr, "\mathrm{der}\Phi_{\check{G}}"] \arrow[dr, "R_L^G"] \arrow[dd, "R_T^G"'] & & \mathrm{Coh}_{\mathrm{fr}}^{W_G\times\mathbb{G}_m}(\check{\mathfrak{h}}^*) \arrow[dr, "R_{W_L}^{W_G}"] \arrow[dd, dotted, "R_{W_T}^{W_G}" near start] & \\
& \mathrm{Coh}_{\mathrm{fr}}^{\check{L}\times\mathbb{G}_m}(\check{\mathfrak{l}}^*)_{\mathrm{sm}} \arrow[rr, "\mathrm{der}\Phi_{\check{L}}" near start] \arrow[dl, "R_T^L"] & & \mathrm{Coh}_{\mathrm{fr}}^{W_L\times\mathbb{G}_m}(\check{\mathfrak{h}}^*) \arrow[dl, "R_{W_T}^{W_L}"] \\
\mathrm{Coh}_{\mathrm{fr}}^{\check{T}\times\mathbb{G}_m}(\check{\mathfrak{h}}^*)_{\mathrm{sm}} \arrow[rr, "\mathrm{der}\Phi_{\check{T}}"'] & & \mathrm{Coh}_{\mathrm{fr}}^{W_T\times\mathbb{G}_m}(\check{\mathfrak{h}}^*) &
\end{tikzcd}
\tag{3.4}
$$

3.4 Sketch of proof of Theorem 1.2.1

In this section, we will give a proof Theorem 1.2.1 assuming that we have verified all of the theorems in Section 3.3. It suffices to prove the commutativity of Diagram (1.2) (see Section 4.2). We have the following more precise statement of the commutativity of Diagram (1.2).

Theorem 3.4.1. *There exists a unique natural isomorphism*

$$\eta_G : \mathrm{der}\Phi_{\check{G}} \circ \mathrm{der}\mathcal{S}_G^{\mathrm{sm}} \Longleftrightarrow \mathrm{der}\mathbb{S}_G \circ \Psi_G$$

making the diagram below commute.

$$
\begin{tikzcd}
\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}_G^{\mathrm{sm}}) \arrow[rr, "\mathrm{der}\mathcal{S}_G^{\mathrm{sm}}"] \arrow[dr, "\Psi_G"] \arrow[dd, "\mathfrak{R}_T^G"'] & & \mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*)_{\mathrm{sm}} \arrow[dr, "\mathrm{der}\Phi_{\check{G}}"] \arrow[dd, dotted, "R_T^G" near end] & \\
& \mathrm{Semis}_G(\mathcal{N}_G) \arrow[rr, "\mathrm{der}\mathbb{S}_G" near start] \arrow[dd, "\widetilde{\mathcal{R}}_T^G" near start] & & \mathrm{Coh}_{\mathrm{fr}}^{W_G\times\mathbb{G}_m}(\check{\mathfrak{h}}^*) \arrow[dd, "R_{W_T}^{W_G}"] \\
\mathrm{Semis}_{T(\mathfrak{O})}(\mathrm{Gr}_T^{\mathrm{sm}}) \arrow[rr, dotted, "\mathrm{der}\mathcal{S}_T^{\mathrm{sm}}" near start] \arrow[dr, "\Psi_T"'] & & \mathrm{Coh}_{\mathrm{fr}}^{\check{T}\times\mathbb{G}_m}(\check{\mathfrak{h}}^*)_{\mathrm{sm}} \arrow[dr, dotted, "\mathrm{der}\Phi_{\check{T}}"] & \\
& \mathrm{Semis}_T(\mathcal{N}_T) \arrow[rr, "\mathrm{der}\mathbb{S}_T"'] & & \mathrm{Coh}_{\mathrm{fr}}^{W_T\times\mathbb{G}_m}(\check{\mathfrak{h}}^*)
\end{tikzcd}
\tag{3.5}
$$

Proof. In Lemma 4.5.1, we construct a natural isomorphism

$$\eta_T : \mathrm{der}\Phi_{\check{T}} \circ \mathrm{der}\mathcal{S}_T^{\mathrm{sm}} \Longleftrightarrow \mathrm{der}\mathbb{S}_T \circ \Psi_T$$

making the bottom face of Diagram (3.5) commute. The intertwining isomorphisms give natural isomorphisms for all of the side faces of Diagram (3.5). Using these five isomorphisms, notice that we can write down the following chain of isomorphisms.

$$\begin{tikzcd}
 & R_{W_T}^{W_G} \circ \mathrm{der}\Phi_{\check{G}} \circ \mathrm{der}\mathcal{S}_G^{\mathrm{sm}} \arrow[dr, Leftrightarrow] & \\
R_{W_T}^{W_G} \circ \mathrm{der}\mathbb{S}_G \circ \Psi_G \arrow[d, Leftrightarrow] & & \mathrm{der}\Phi_{\check{T}} \circ R_T^G \circ \mathrm{der}\mathcal{S}_G^{\mathrm{sm}} \arrow[d, Leftrightarrow] \\
\mathrm{der}\mathbb{S}_T \circ \widetilde{\mathcal{R}}_T^G \circ \Psi_G \arrow[dr, Leftrightarrow] & & \mathrm{der}\Phi_{\check{T}} \circ \mathrm{der}\mathcal{S}_T^{\mathrm{sm}} \circ \mathfrak{R}_T^G \arrow[dl, Leftrightarrow] \\
 & \mathrm{der}\mathbb{S}_T \circ \Psi_T \circ \mathfrak{R}_T^G &
\end{tikzcd}$$

The composition gives us a natural isomorphism

$$\phi_{G,T} : R_{W_T}^{W_G} \circ \mathrm{der}\Phi_{\check{G}} \circ \mathrm{der}\mathcal{S}_G^{\mathrm{sm}} \Longleftrightarrow R_{W_T}^{W_G} \circ \mathrm{der}\mathbb{S}_G \circ \Psi_G.$$

Having η_G which makes Diagram (3.5) commutative is equivalent to having η_G such that $R_{W_T}^{W_G} \circ \eta_G = \phi_{G,T}$.

By Lemma 3.1.1, we can get a unique natural isomorphism η_G satisfying $R_{W_T}^{W_G} \circ \eta_G = \phi_{G,T}$ if whenever L has semisimple rank 1, there exists a natural isomorphism

$$\phi^{W_L} : R_{W_L}^{W_G} \circ \mathrm{der}\Phi_{\check{G}} \circ \mathrm{der}\mathcal{S}_G^{\mathrm{sm}} \Longleftrightarrow R_{W_L}^{W_G} \circ \mathrm{der}\mathbb{S}_G \circ \Psi_G$$

such that $R_{W_T}^{W_L} \circ \phi^{W_L} = \phi_{G,T}$.

To this end, let L have semisimple rank 1. From Theorem 11.4.1, we have a natural isomorphism

$$\eta_L : \mathrm{der}\Phi_{\check{L}} \circ \mathrm{der}\mathcal{S}_L^{\mathrm{sm}} \Longleftrightarrow \mathrm{der}\mathbb{S}_L \circ \Psi_L$$

making Diagram (3.5) with G replaced by L commutative; that is, $R^{W_L}_{W_T} \circ \eta_L = \phi_{L,T}$. To this commutative cube, we glue the prisms (3.1), (3.2), (3.3), and (3.4) to get the following diagram. We have shortened several notations to make the diagram fit. We have written S for Semis and C for Coh, and we have suppressed writing the $\mathbb{G}_m$-equivariance and the subscript $_{\mathrm{sm}}$ in the various categories of coherent sheaves.

$$
\begin{tikzcd}
\mathrm{S}_{G(\mathfrak{D})}(\mathrm{Gr}_G^{\mathrm{sm}}) \arrow[rrrr, "\mathrm{der}\mathcal{S}_G^{\mathrm{sm}}"] \arrow[drr, "\Psi_G"] \arrow[ddr, "\mathfrak{R}^G_L"] \arrow[dddd, "\mathfrak{R}^G_T"] & & & & \mathrm{C}^{\check{G}}_{\mathrm{fr}}(\check{\mathfrak{g}}^*) \arrow[drrr, "\mathrm{der}\Phi_{\check{G}}"] \arrow[ddl, "R^G_L"] \arrow[dddd, "R^G_T"] & & & \\
& & \mathrm{S}_G(\mathcal{N}_G) \arrow[rrrr, "\mathrm{der}\mathbb{S}_G"] \arrow[ddr, "\widetilde{\mathcal{R}}^G_L"] \arrow[dddd, "\widetilde{\mathcal{R}}^G_T"] & & & & \mathrm{C}^{W_G}_{\mathrm{fr}}(\check{\mathfrak{h}}^*) \arrow[ddl, "R^{W_G}_{W_L}"] \arrow[dddd, "R^{W_G}_{W_T}"] & \mathrm{C}^{W_G}_{\mathrm{fr}}(\check{\mathfrak{h}}^*) \arrow[ddll, "R^{W_G}_{W_L}"] \arrow[ddddl, "R^{W_G}_{W_T}"] \\
& \mathrm{S}_{L(\mathfrak{D})}(\mathrm{Gr}_L^{\mathrm{sm}}) \arrow[rr, "\mathrm{der}\mathcal{S}_L^{\mathrm{sm}}"] \arrow[drr, "\Psi_L"] \arrow[ddl, "\mathfrak{R}^L_T"] & & \mathrm{C}^{\check{L}}_{\mathrm{fr}}(\check{\mathfrak{l}}^*) \arrow[drr, "\mathrm{der}\Phi_{\check{L}}"] \arrow[ddr, "R^L_T"] & & & & \\
& & & \mathrm{S}_L(\mathcal{N}_L) \arrow[rr, "\mathrm{der}\mathbb{S}_L"] \arrow[ddl, "\widetilde{\mathcal{R}}^L_T"] & & \mathrm{C}^{W_L}_{\mathrm{fr}}(\check{\mathfrak{h}}^*) \arrow[ddr, "R^{W_L}_{W_T}"] & & \\
\mathrm{S}_{T(\mathfrak{D})}(\mathrm{Gr}_T^{\mathrm{sm}}) \arrow[rrrr, "\mathrm{der}\mathcal{S}_T^{\mathrm{sm}}"] \arrow[drr, "\Psi_T"] & & & & \mathrm{C}^{\check{T}}_{\mathrm{fr}}(\check{\mathfrak{h}}^*) \arrow[drr, "\mathrm{der}\Phi_{\check{T}}"] & & & \\
& & \mathrm{S}_T(\mathcal{N}_T) \arrow[rrrr, "\mathrm{der}\mathbb{S}_T"] & & & & \mathrm{C}^{W_T}_{\mathrm{fr}}(\check{\mathfrak{h}}^*) &
\end{tikzcd}
\tag{3.6}
$$

This pasted diagram has boundary consisting of two diagrams with domain $R^{W_G} \circ \mathrm{der}\Phi_{\check{G}} \circ \mathrm{der}\mathcal{S}_G^{\mathrm{sm}}$ and codomain $R^{W_G}_{W_T} \circ \mathrm{der}\mathbb{S}_G \circ \Psi_G$. Composing the natural isomorphisms for one of these diagrams gives $R^{W_L}_{W_T} \circ \phi_{G,L}$ and the other gives $\phi_{G,T}$. Since the whole of Diagram (3.6) is commutative (see the pasting techniques given in the appendices of [AHR15]), we have that $R^{W_L}_{W_T} \circ \phi_{G,L} = \phi_{G,T}$. Hence, Lemma 3.1.1 gives a unique η_G making Diagram (3.5) commute. $\square$

Chapter 4
Some General Results

In this section, we give some general results that will be used throughout the paper.

4.1 Equivariant derived category of a point

Theorem 4.1.1 ([BL94], Theorem 12.7.2)**.** *Let G be a connected Lie group, $A_G = H(BG)$. Let $\mathcal{A}_G = (A_G, d = 0)$ be the corresponding DG-algebra.*

1. *There exists an equivalence of triangulated categories*

$$\mathcal{L}_G : \mathcal{D}^+_{\mathcal{A}_G} \xrightarrow{\sim} \mathcal{D}^+_G(\mathrm{pt})$$

which is unique up to a canonical isomorphism. It commutes with $\otimes^L$ and the cohomology functor $(\cdot) \xrightarrow{H} \mathrm{Mod}_{A_G}$.

2. *The above equivalence restricts to the functor between the full subcategories*

$$\mathcal{L}_G : \mathcal{D}^f_{\mathcal{A}_G} \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}_{G,c}(\mathrm{pt}).$$

Remark 4.1.2. *Borel gave a description of A_G as $H^\bullet_G(\mathrm{pt})$.*

4.2 Orlov categories and the restriction to Semis

In this section, we will introduce the notion of an Orlov category. They are useful in constructing natural transformations (and isomorphisms) of functors between triangulated categories [Orl97]. Orlov categories and many of their properties are discussed in [AR13].

Definition 4.2.1 ([AR13] Definition 4.1)**.** *Let $\mathcal{A}$ be an additive category together with a map* $\deg : \mathrm{Ind}(\mathcal{A}) \to \mathbb{Z}$*. Then, $\mathcal{A}$ is called an Orlov category if the following are satisfied:*

1. *All* Hom*-spaces in $\mathcal{A}$ are finite-dimensional.*

2. *For any $S \in \mathrm{Ind}(\mathcal{A})$, we have* $\mathrm{End}(S) \simeq \mathbb{C}$*.*

3. *If $S, S' \in \mathrm{Ind}(\mathcal{A})$ with $\deg(S) \leq \deg(S')$ and $S \not\simeq S'$, then* $\mathrm{Hom}(S, S') = 0$*.*

An object $A \in \mathcal{A}$ is said to be homogeneous of degree n if it is isomorphic to a direct sum of indecomposable objects of degree n. An additive functor $F : \mathcal{A} \to \mathcal{B}$ between Orlov categories $\mathcal{A}$ and $\mathcal{B}$ is called homogeneous if it sends homogeneous objects of degree n in $\mathcal{A}$ to homogeneous objects of degree n in $\mathcal{B}$.

The next theorem allows us to take a natural isomorphism for Diagram (1.2) and lift it to a unique natural isomorphism proving Theorem 1.2.1.

Theorem 4.2.2 ([AR13] Theorem 4.7, [Orl97] Proposition 2.16))**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be Orlov categories, and let $F, F' : K^{\mathrm{b}}(\mathcal{A}) \to K^{\mathrm{b}}(\mathcal{B})$ be functors of triangulated categories. Assume that $F(\mathcal{A}) \subset \mathcal{B}$ and $F'(\mathcal{A}) \subset \mathcal{B}$, and that the induced functors $F|_{\mathcal{A}}, F'|_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}$ are homogeneous. Any morphism of additive functors*

$$\theta^\circ : F|_{\mathcal{A}} \to F'|_{\mathcal{A}}$$

can be extended to a morphism $\theta : F \to F'$ of functors of triangulated categories in such a way that if θ° is an isomorphism, then θ is as well.

The following two lemmas verify that the categories playing the role of $\mathcal{A}$ and $\mathcal{B}$ (from Theorem 4.2.2) in Diagram (1.2) are Orlov.

Lemma 4.2.3. *The additive category* $\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}})$ *together with the map* $\deg(S[n]) = -n$ *is an Orlov category.*

Proof. Properties (1) and (2) of being an Orlov category are obviously satisfied. To verify Property (3), let $S[n_1]$ and $S'[n_2]$ be two non-isomorphic indecomposable objects in $\mathrm{Semis}_G(X)$; that is, they are shifted simple perverse sheaves. Notice that,

$$\mathrm{Hom}_{\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}})}(S[n_1], S'[n_2]) = \mathrm{Hom}^{n_2-n_1}_{\mathcal{D}^{\mathrm{b}}_{\mathrm{m}}(\mathrm{Gr}^{\mathrm{sm}})}(S, S').$$

If $-n_1 < -n_2$, then $n_2 - n_1 < 0$. But S, S' are simple perverse sheaves, so they live in the heart of a t-structure, which forces the Hom-space to vanish since the axioms of a t-structure disallow maps between S and S'[negative]. In the case when $n_1 = n_2$, the above Hom-space vanishes (by Schur's Lemma) since S and S' are non-isomorphic simple objects. □

Lemma 4.2.4. *The additive category* $\mathrm{Coh}^{W\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{h}}^*)_{\mathrm{sm}}$ *together with the map*

$$\deg(V \otimes \mathcal{O}_{\check{\mathfrak{h}}^*}\langle n\rangle) = -n$$

is an Orlov category.

Proof. Again Properties (1) and (2) of being an Orlov category are clear. For Property (3), let V and V' be two simple W-representations and let $n_1, n_2 \in \mathbb{Z}$. Then, it follows that $\mathrm{Hom}_{\mathrm{Coh}^{W\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{h}}^*)}(V \otimes \mathcal{O}_{\check{\mathfrak{h}}^*}\langle n_1\rangle, V' \otimes \mathcal{O}_{\check{\mathfrak{h}}^*}\langle n_2\rangle)$ is isomorphic to $\mathrm{Hom}_{\mathrm{Coh}^{W\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{h}}^*)}(V \otimes \mathcal{O}_{\check{\mathfrak{h}}^*}, V' \otimes \mathcal{O}_{\check{\mathfrak{h}}^*}\langle n_2 - n_1\rangle)$, and a map in this Hom-space is determined by where the generators are sent. But, $V \otimes \mathcal{O}_{\check{\mathfrak{h}}^*}$ is generated in degree 0 and $V' \otimes \mathcal{O}_{\check{\mathfrak{h}}^*}\langle n_2 - n_1\rangle$ is generated in degree $-(n_2 - n_1)$. Thus, if $-n_1 < -n_2$, then $-(n_2 - n_1)$ is positive and the Hom-space vanishes since $V' \otimes \mathcal{O}_{\check{\mathfrak{h}}^*}\langle n_2 - n_1\rangle$ is generated in positive degrees (and these modules are non-negatively graded). On the other hand, if $n_1 = n_2$ and $V \not\cong V'$, then the Hom-space also vanishes (again by Schur's Lemma). □

4.3 Actions on $\mathrm{Hom}^\bullet$-algebras

In this section, we will construct various actions of rings on $\mathrm{Hom}^\bullet$-algebras. In the next lemma, we do this for the constructible side.

Lemma 4.3.1. *Let X be a variety with an action of an algebraic group G, and let $\mathcal{F}, \mathcal{G} \in \mathrm{Semis}_G(X)$. Then, $\mathrm{Hom}^\bullet_{\mathrm{Semis}_G(X)}(\mathcal{F}, \mathcal{G})$ is an $H^\bullet_G(\mathrm{pt})$-module.*

Proof. Let $a : X \to \mathrm{pt}$ be the map to a point. Notice that,

$$\begin{aligned}
\mathrm{Hom}^\bullet_{\mathrm{Semis}_G(X)}(\mathcal{F}, \mathcal{G}) &= H^\bullet \mathrm{R}\Gamma \mathrm{R}\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G}) \\
&= H^\bullet(\mathrm{RHom}_{\mathcal{D}^{\mathrm{b}}_G(X)}(\underline{\mathbb{C}}, \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}^{\mathrm{b}}_G(X)}(\mathcal{F}, \mathcal{G}))) \\
&= H^\bullet(\mathrm{RHom}_{\mathcal{D}^{\mathrm{b}}_G(X)}(a^*\underline{\mathbb{C}}_{\mathrm{pt}}, \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}^{\mathrm{b}}_G(X)}(\mathcal{F}, \mathcal{G}))) \\
&= H^\bullet(\mathrm{RHom}_{\mathcal{D}^{\mathrm{b}}_G(\mathrm{pt})}(\underline{\mathbb{C}}_{\mathrm{pt}}, a_*\mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}^{\mathrm{b}}_G(X)}(\mathcal{F}, \mathcal{G}))) \\
&= \mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}_G(\mathrm{pt})}(\underline{\mathbb{C}}_{\mathrm{pt}}, a_*\mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}^{\mathrm{b}}_G(X)}(\mathcal{F}, \mathcal{G})).
\end{aligned}$$

Since $H^\bullet_G(\mathrm{pt}) \cong \mathrm{Hom}^\bullet_{\mathcal{D}^{\mathrm{b}}_G(\mathrm{pt})}(\underline{\mathbb{C}}_{\mathrm{pt}}, \underline{\mathbb{C}}_{\mathrm{pt}})$, we see that there is a natural action of $H^\bullet_G(\mathrm{pt})$ on $\mathrm{Hom}^\bullet_{\mathcal{D}^{\mathrm{b}}_G(X)}(\mathcal{F}, \mathcal{G})$ by composition. This makes $\mathrm{Hom}^\bullet_{\mathcal{D}^{\mathrm{b}}_G(X)}(\mathcal{F}, \mathcal{G})$ into an $H^\bullet_G(\mathrm{pt})$-module, as desired. □

We will construct an action of an isomorphic ring on $\mathrm{Hom}^\bullet$-algebras on the coherent side. The following lemma will help us to understand what an object in $\mathrm{Coh}^{W \times \mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{h}}^*)$ looks like. That is, if $M \in \mathrm{Coh}^{W \times \mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{h}}^*)$, then M carries compatible actions of both W and $\mathcal{O}_{\check{\mathfrak{h}}^*}$, which is the same as saying that M is a $\mathbb{C}W\#\mathcal{O}_{\check{\mathfrak{h}}^*}$-module.

Lemma 4.3.2. *Let G be a group and R be a ring carrying an action of G. Then M has compatible G and R actions if and only if it is a $\mathbb{C}G\#R$-module.*

Proof. Recall that the multiplication on $\mathbb{C}G\#R$ is given by

$$(g, r)(h, s) = (gh, (h^{-1} \cdot r)s). \tag{4.1}$$

Suppose that M has compatible G and R actions. That is,

$$g \cdot (r \cdot m) = (g \cdot r) \cdot (g \cdot m). \tag{4.2}$$

Define an action of $\mathbb{C}G\#R$ on M by

$$(g, r) \cdot m := g \cdot (r \cdot m). \tag{4.3}$$

Let us check that this action makes M into a $\mathbb{C}G\#R$-module. We will verify

$$(g, r) \cdot ((h, s) \cdot m) = ((g, r)(h, s)) \cdot m. \tag{4.4}$$

Starting from the left-hand side of (4.4), we have

$$(g,r)\cdot((h,s)\cdot m) = g\cdot(r\cdot(h\cdot(s\cdot m))) = g\cdot(r\cdot((h\cdot s)\cdot(h\cdot m))) = (g\cdot r)\cdot(g\cdot((h\cdot s)\cdot(h\cdot m))$$

by applications of (4.3) and (4.2). Continuing with another application of (4.2) as well as the G action on R, we yield

$$(g \cdot r) \cdot ((g \cdot (h \cdot s)) \cdot (g \cdot (h \cdot m))) = (g \cdot r) \cdot ((gh \cdot s) \cdot (gh \cdot m)).$$

Now, using the fact that M is an R-module and the G action on R, we get

$$((g\cdot r)(gh\cdot s))\cdot(gh\cdot m) = ((gh\cdot h^{-1}r)(gh\cdot s))\cdot(gh\cdot m) = (gh\cdot((h^{-1}\cdot r)s))\cdot(gh\cdot m).$$

Equations (4.2), (4.3), and (4.1) give

$$gh \cdot (((h^{-1} \cdot r)s) \cdot m) = (gh, (h^{-1} \cdot r)s) \cdot m = ((g, r)(h, s)) \cdot m.$$

On the other hand, suppose that M is a $\mathbb{C}G\#R$-module. Define a simultaneous action of G and R on M by

$$g \cdot (r \cdot m) = (g, r) \cdot m.$$

We will show that this action gives compatible G and R actions on M; that is (4.2) holds. The left-hand side of (4.2) is equal to

$$(g,r)\cdot m = (g, g^{-1}(g\cdot r)1)\cdot m = ((1, g\cdot r)(g,1))\cdot m = (1, g\cdot r)\cdot((g,1)\cdot m) = (g\cdot r)(g\cdot m)$$

by (4.1) and the $\mathbb{C}G\#R$-module structure on M. $\square$

Proposition 4.3.3. *Let $M, N \in \mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*)$. Then, $\underline{\mathrm{Hom}}_{\mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*)}(M,N)$ is an $\mathcal{O}_{\check{\mathfrak{g}}^*}^{\check{G}}$-module.*

Proof. Define the action of $\mathcal{O}_{\check{\mathfrak{g}}^*}^{\check{G}}$ on $\underline{\mathrm{Hom}}_{\mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*)}(M,N)$ by

$$(a\cdot f)(m) := a\cdot f(m).$$

where $a \in \mathcal{O}_{\check{\mathfrak{g}}^*}^{\check{G}}$, $f \in \underline{\mathrm{Hom}}_{\mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*)}(M,N)$, and $m \in M$. Note that a acts on N since $\mathcal{O}_{\check{\mathfrak{g}}^*}$ does and since $\mathcal{O}_{\check{\mathfrak{g}}^*}^{\check{G}}$ is a subring of $\mathcal{O}_{\check{\mathfrak{g}}^*}$. $\square$

Remark 4.3.4. *In the proof above, if we instead took $a \in \mathcal{O}_{\check{\mathfrak{g}}^*}$, then $a\cdot f$ would have failed to be a $\check{G}$-module homomorphism.*

Remark 4.3.5. *If $M, N \in \mathrm{Coh}_{\mathrm{fr}}^{W\times\mathbb{G}_m}(\check{\mathfrak{h}}^*)$, then a similar argument asserts that $\underline{\mathrm{Hom}}_{\mathrm{Coh}_{\mathrm{fr}}^{W\times\mathbb{G}_m}(\check{\mathfrak{h}}^*)}(M,N)$ is an $\mathcal{O}_{\check{\mathfrak{h}}^*}^{W}$-module. Again, the whole of $\mathcal{O}_{\check{\mathfrak{h}}^*}$ does not act, only the invariant ring does.*

4.4 Some purity results

For this section only, we will consider all of our varieties over $\mathbb{F}_q$ with the étale topology, and our sheaves will have $\overline{\mathbb{Q}}_\ell$-coefficients. Let X be a variety with a stratification $X = \bigsqcup_{w\in\mathcal{W}} X_w$ and with an action of an algebraic group G. Consider the triangulated category $\mathcal{D}_G^{\mathrm{b,mix}}(X)$.

Corollary 4.4.1. *If the stalks of $\mathrm{IC}(\overline{X}_w)$ are pure of weight 0, then the costalks are also pure of weight 0.*

Proof. Consider the inclusion $i : \mathrm{pt} \hookrightarrow \overline{X}_a$. We will show that $i^!\mathrm{IC}(\overline{X}_w)$ is pure. Notice that,

$$i^!\mathrm{IC}(\overline{X}_w) \cong \mathbb{D}\mathbb{D}i^!\mathrm{IC}(\overline{X}_w) \cong \mathbb{D}i^*\mathbb{D}\mathrm{IC}(\overline{X}_w)$$

Since ICs are fixed under $\mathbb{D}$, it follows that $\mathbb{D}\mathrm{IC}(\overline{X}_w) = \mathrm{IC}(\overline{X}_w)$. But $\mathrm{IC}(\overline{X}_w)$ is pure of weight 0. By assumption, we have that $i^*\mathbb{D}\mathrm{IC}(\overline{X}_w)$ is also pure of weight 0. Since $\mathbb{D}$ sends objects of weight $\leq v$ to objects of weight $\geq -v$ (see [BBD82]), we have that $\mathbb{D}i^*\mathbb{D}\mathrm{IC}(\overline{X}_w)$ is pure of weight 0, as desired. □

We will make use of the following facts in the proof of the next lemma. Let $\mathcal{F} \in \mathcal{D}_G^{\mathrm{b,mix}}(X)$.

- If X is simply connected, then $\mathcal{F}$ is locally constant if and only if $\mathcal{F}$ is constant. This is true because there is a bijection between locally constant sheaves and representations of $\pi_1(X)$. Since X is simply connected, $\pi_1(X) = 0$, and there are no nontrivial local systems.

- If $\mathcal{F}$ is constructible on a smooth variety X with trivial stratification, then $\mathcal{F}$ is locally constant by definition.

- If X is smooth and simply connected with trivial stratification, then a constructible sheaf (necessarily constant) is determined by its stalk at any point.

We will also make use of the following two extra assumptions:[1]

- If $\mathcal{F}$ is constructible with respect to the trivial stratification on a smooth simply connected variety X and $H^i(\mathcal{F}_x) = \mathcal{H}^i(\mathcal{F})_x$ is pure of weight i, then

$$^p\mathcal{H}^{i+\dim X}(\mathcal{F})[-\dim X] = \mathcal{H}^i(\mathcal{F})$$

[1]There is no problem in making these assumptions because they hold for Gr and $\mathcal{N}$.

is a direct sum of pure $\overline{\mathbb{Q}}_\ell\left(-\frac{i}{2}\right)$. This is an eigenspace decomposition where the number of copies of $\overline{\mathbb{Q}}_\ell\left(-\frac{i}{2}\right)$ is given by the rank.

- If $\mathcal{F}$ has pure stalks, then it is pure of weight 0 since

$$^p\mathcal{H}^i(\mathcal{F}) = \mathcal{H}^{i-\dim X}(\mathcal{F})[\dim X]$$

is a direct sum of $\overline{\mathbb{Q}}_\ell\left(\frac{\dim X - i}{2}\right)[\dim X]$, each of weight i (see [BBD82]).

Lemma 4.4.2. *Let X be a variety stratified by affine spaces $X = \bigsqcup_{w\in\mathcal{W}} X_w$ and carrying an action of an algebraic group H. Let $\mathcal{F}, \mathcal{G} \in \mathcal{D}_H^{\mathrm{b,mix}}(X)$. Suppose that $\mathcal{F}$ has pure stalks of weight* 0 *and $\mathcal{G}$ has pure costalks of weight* 0*. Then, for every i, $\underline{\mathrm{Hom}}^i(\mathcal{F}, \mathcal{G})$ is pure of weight i.*

Proof. We proceed by induction on the number of strata on which $\mathcal{F}$ and $\mathcal{G}$ are supported. For the base case, suppose that $\mathcal{F}$ and $\mathcal{G}$ are supported on a single stratum $\overline{X}_w$. Since $\mathcal{F}$ is constructible on the smooth variety $\overline{X}_w$ with trivial stratification, it follows that $\mathcal{F}$ is locally constant. But $\overline{X}_w$ is also simply connected. Thus, $\mathcal{F}$ is determined by its stalk at any point. Since $\mathcal{F}$ has pure stalks of weight 0, it follows that $H^i(\mathcal{F}_x) = \mathcal{H}^i(\mathcal{F})_x$ is pure of weight i. Now apply the bullet points above to deduce that $\mathcal{F}$ is pure of weight 0.

Since $\mathcal{G}$ has pure costalks of weight 0, we know that $\mathbb{D}\mathcal{G}$ has pure stalks of weight 0, and we can apply the method of the first paragraph to get that $\mathbb{D}\mathcal{G}$ is pure of weight 0. The purity of $\mathcal{F}$ and $\mathbb{D}\mathcal{G}$ gives decompositions

$$\mathcal{F} \simeq \bigoplus_i {}^p\mathcal{H}^i\mathcal{F}[-i] \text{ and } \mathbb{D}\mathcal{G} \simeq \bigoplus_i {}^p\mathcal{H}^i\mathbb{D}\mathcal{G}[-i].$$

The perverse cohomologies have eigenspace decompositions as described above. Putting this all together, we have $\mathcal{F} \simeq \bigoplus_i \bigoplus_{\mathrm{rank}} \overline{\mathbb{Q}}_\ell\left(\frac{\dim X - i}{2}\right)[\dim X - i]$ and $\mathbb{D}\mathcal{G} \simeq \bigoplus_i \bigoplus_{\mathrm{rank}} \overline{\mathbb{Q}}_\ell\left(\frac{\dim X - i}{2}\right)[\dim X - i]$. These ranks may be different, so we will relabel to make the argument more transparent. Let $j = \frac{\dim X - i}{2}$ in the first decomposition and $k = \frac{\dim X - i}{2}$ in the second decomposition to stress this difference. Then, we have

$$\mathcal{F} \simeq \bigoplus_j \underline{\overline{\mathbb{Q}}_\ell}(j)[2j] \text{ and } \mathbb{D}\mathcal{G} \simeq \bigoplus_k \underline{\overline{\mathbb{Q}}_\ell}(k)[2k].$$

But, we are actually interested in $\mathcal{G}$, not in $\mathbb{D}\mathcal{G}$. Thus, we apply $\mathbb{D}$ to the second sum to get

$$\mathcal{G} \simeq \bigoplus_k \underline{\overline{\mathbb{Q}}_\ell}\,(-k)\,[-2k].$$

Now, notice that

$$\begin{aligned}
\underline{\mathrm{Hom}}^i(\mathcal{F}, \mathcal{G}) &\simeq \underline{\mathrm{Hom}}^i\left(\bigoplus_j \underline{\overline{\mathbb{Q}}_\ell}\,(j)\,[2j], \bigoplus_k \underline{\overline{\mathbb{Q}}_\ell}\,(-k)\,[-2k]\right) \\
&\simeq \bigoplus_{j,k} \underline{\mathrm{Hom}}^i\left(\underline{\overline{\mathbb{Q}}_\ell}\,(j)\,[2j], \underline{\overline{\mathbb{Q}}_\ell}\,(-k)\,[-2k]\right) \\
&\simeq \bigoplus_{j,k} \underline{\mathrm{Hom}}^{i-2j-2k}\left(\underline{\overline{\mathbb{Q}}_\ell}, \underline{\overline{\mathbb{Q}}_\ell}\right)(-k-j)
\end{aligned}$$

But this is just cohomology of affine space (which is contractible); therefore,

$$\underline{\mathrm{Hom}}^i(\mathcal{F}, \mathcal{G}) \simeq \begin{cases} \underline{\overline{\mathbb{Q}}_\ell}\,(-k-j) & \text{if } i = 2j + 2k \\ 0 & \text{else} \end{cases}$$

Hence, for every i, $\underline{\mathrm{Hom}}^i(\mathcal{F}, \mathcal{G})$ is pure of weight i, and the base case is done.

Now suppose that if $\mathcal{H}_1$ and $\mathcal{H}_2$ are supported on $\overline{X}_{<w}$, $\mathcal{H}_1$ has pure stalks of weight 0, and $\mathcal{H}_2$ has pure costalks of weight 0, then we have that for every i, $\underline{\mathrm{Hom}}^i(\mathcal{H}_1, \mathcal{H}_2)$ is pure of weight i. Also, suppose that $\mathcal{F}$ and $\mathcal{G}$ are supported on $\overline{X}_{\leq w}$. Consider the inclusions

$$i_{<w} : \overline{X}_{<w} \hookrightarrow \overline{X}_w \hookleftarrow X_w : j_w$$

We have a distinguished triangle

$$i_{<w*} i^!_{<w} \mathcal{G} \to \mathcal{G} \to j_{w*} j^*_w \mathcal{G} \xrightarrow{+}$$

Applying $\underline{\mathrm{Hom}}(\mathcal{F}, —)$, we get the long exact sequence

$$\ldots \to \underline{\mathrm{Hom}}^i(\mathcal{F}, i_{<w*}i^!_{<w}\mathcal{G}) \to \underline{\mathrm{Hom}}^i(\mathcal{F}, \mathcal{G}) \to \underline{\mathrm{Hom}}^i(\mathcal{F}, j_{w*}j^*_w\mathcal{G}) \to \ldots$$

Since we have the adjoint pair $({}^*, {}_*)$, it follows that

$$\ldots \to \underline{\mathrm{Hom}}^i(i^*_{<w}\mathcal{F}, i^!_{<w}\mathcal{G}) \to \underline{\mathrm{Hom}}^i(\mathcal{F}, \mathcal{G}) \to \underline{\mathrm{Hom}}^i(j^*_w\mathcal{F}, j^*_w\mathcal{G}) \to \ldots$$

Now, $i^*_{<w}\mathcal{F}$ has pure stalks of weight 0 since $\mathcal{F}$ does, and $i^!_{<w}\mathcal{G}$ has pure costalks of weight 0 since $\mathcal{G}$ does. By the induction hypothesis, it follows that for every i, $\underline{\mathrm{Hom}}^i(i^*_{<w}\mathcal{F}, i^!_{<w}\mathcal{G})$ is pure of weight i.

Turning our attention to $\underline{\mathrm{Hom}}^i(j^*_w\mathcal{F}, j^*_w\mathcal{G})$, we see that $j^*_w\mathcal{F}$ has pure stalks of weight 0, $j^*_w\mathcal{G}$ has pure costalks of weight 0, and both are supported on a single stratum. This is an instance of the base case, and $\underline{\mathrm{Hom}}^i(j^*_w\mathcal{F}, j^*_w\mathcal{G})$ is pure of weight i for every i.

Now, let us look at more of the long exact sequence from above

$$\ldots \to \underline{\mathrm{Hom}}^{i-1}(j^*_w\mathcal{F}, j^*_w\mathcal{G}) \to \underline{\mathrm{Hom}}^i(i^*_{<w}\mathcal{F}, i^!_{<w}\mathcal{G}) \to \underline{\mathrm{Hom}}^i(\mathcal{F}, \mathcal{G})$$
$$\to \underline{\mathrm{Hom}}^i(j^*_w\mathcal{F}, j^*_w\mathcal{G}) \to \underline{\mathrm{Hom}}^{i+1}(i^*_{<w}\mathcal{F}, i^!_{<w}\mathcal{G}) \to \ldots$$

But $\underline{\mathrm{Hom}}^{i-1}(j^*_w\mathcal{F}, j^*_w\mathcal{G})$ is pure of weight $i-1$ and $\underline{\mathrm{Hom}}^i(i^*_{<w}\mathcal{F}, i^!_{<w}\mathcal{G})$ is pure of weight i. Thus, there can be no morphism which preserves the eigenvalues of Frobenius. The same is true for the right side of the sequence. Therefore, our long exact sequence breaks into a short exact sequence for every i

$$0 \to \underline{\mathrm{Hom}}^i(i^*_{<w}\mathcal{F}, i^!_{<w}\mathcal{G}) \to \underline{\mathrm{Hom}}^i(\mathcal{F}, \mathcal{G}) \to \underline{\mathrm{Hom}}^i(j^*_w\mathcal{F}, j^*_w\mathcal{G}) \to 0$$

Since for every i, we know $\underline{\mathrm{Hom}}^i(i^*_{<w}\mathcal{F}, i^!_{<w}\mathcal{G})$ and $\underline{\mathrm{Hom}}^i(j^*_w\mathcal{F}, j^*_w\mathcal{G})$ are both pure of weight i, it follows that $\underline{\mathrm{Hom}}^i(\mathcal{F}, \mathcal{G})$ is pure of weight i, as desired. $\square$

Remark 4.4.3. *An analogous proof using the technique of induction on the support is given in [BY13, Lemma 3.1.5].*

Let $\mathbb{U}$ be the functor $\mathbb{U} : \mathcal{D}^{\mathrm{b}}_{G,\mathrm{m}}(X) \to \mathcal{D}^{\mathrm{b}}_{\mathrm{m}}(X)$ that forgets the G-action. For more about the following lemma and its proof, see [AR13].

Lemma 4.4.4. *If for every i, $\underline{\mathrm{Hom}}^i(\mathbb{U}\mathcal{F}, \mathbb{U}\mathcal{G})$ is pure of weight i, then $\mathrm{Hom}^\bullet_G(\mathcal{F}, \mathcal{G})$ is a free $H^\bullet_G(\mathrm{pt})$-module.*

Proof. This is similar to the proof of Théorème 5.3.5 in [BBD82]. We have several different types of Hom. Consider the following:

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = H^0(\mathrm{R}a_*\mathrm{R}\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G})) \text{ where } \mathcal{F}, \mathcal{G} \in \mathcal{D}^{\mathrm{b}}(X)$$

$$\mathrm{Hom}_G(\mathcal{F}, \mathcal{G}) = H^0_{\mathrm{dg}}(\mathrm{R}a_*\mathrm{R}\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G})) \text{ where } \mathcal{F}, \mathcal{G} \in \mathcal{D}^{\mathrm{b}}_G(X)$$

$$\mathbb{H}\mathrm{om}(\mathcal{F}, \mathcal{G}) = \mathbb{U}\ {}^tH^0\mathrm{R}\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G}) \text{ where } \mathcal{F}, \mathcal{G} \in \mathcal{D}^{\mathrm{b}}_G(X)$$

Three facts are essential for this proof.

1. Note that $\mathrm{Hom}^\bullet(\mathbb{U}\mathcal{F}, \mathbb{U}\mathcal{G}) = \mathbb{H}\mathrm{om}^\bullet(\mathcal{F}, \mathcal{G})$ for $\mathcal{F}, \mathcal{G} \in \mathcal{D}^{\mathrm{b}}_G(X)$.

2. If $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(\mathrm{pt})$ is pure, then $\mathcal{F} \cong \bigoplus_{n \in \mathbb{Z}} {}^tH^n(\mathcal{F})[-n]$. This is an equivariant version of Théorème 5.4.5 in [BBD82].

3. Under the equivalence $\mathcal{D}^{\mathrm{b}}_G(\mathrm{pt}) \cong \mathrm{dg} - \mathrm{mod}$ over $H^\bullet_G(\mathrm{pt})$, we know that $\underline{\mathbb{C}}_{\mathrm{pt}}$ corresponds to the free module $H^\bullet_G(\mathrm{pt})$, and a constant sheaf $\underline{V}_{\mathrm{pt}}$ with underlying vector space V corresponds to $V \otimes H^\bullet_G(\mathrm{pt})$ (see Section 4.1).

4. Since $\mathbb{U}$ preserves weights, it kills nothing.

By assumption, $\mathrm{Hom}^\bullet(\mathbb{U}\mathcal{F}, \mathbb{U}\mathcal{G})$ is pure. By 1 and 4, we have that $\mathrm{Hom}^\bullet_G(\mathcal{F}, \mathcal{G})$ is pure. Applying 2, we see that

$$\mathrm{R}a_*\mathrm{R}\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G}) \cong \bigoplus_{n \in \mathbb{Z}} {}^tH^n(\mathrm{R}a_*\mathrm{R}\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G}))[-n]$$

By 3, this corresponds to a dg-module with free underlying module and differential equal to 0. Thus, $H^\bullet_{\mathrm{dg}}\mathrm{R}a_*\mathrm{R}\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G})$ is a free $H^\bullet_G(\mathrm{pt})$-module, as desired. $\square$

Since $\mathbb{H}\mathrm{om}(\mathcal{F},\mathcal{G})$ is a free $H^\bullet_G(\mathrm{pt})$-module, the equivalence in Section 2.2 asserts that $\mathrm{Hom}^\bullet(\mathcal{F},\mathcal{G})$ is also a free $H^\bullet_G(\mathrm{pt})$-module.

4.5 The diagram commutes in some simple cases

Lemma 4.5.1. *Let T be a maximal torus of G. Then, Diagram (1.2) for T*

$$\begin{tikzcd} \mathrm{Semis}_{T(\mathfrak{O})}(\mathrm{Gr}_T^{\mathrm{sm}}) \arrow[r, "\mathrm{der}\mathcal{S}_T^{\mathrm{sm}}"] \arrow[d, "\Psi_T"'] & \mathrm{Coh}_{\mathrm{fr}}^{\check{T}\times\mathbb{G}_m}(\check{\mathfrak{h}}^*)_{\mathrm{sm}} \arrow[d, "\mathrm{der}\Phi_{\check{T}}"] \\ \mathrm{Semis}_T(\mathcal{N}_T) \arrow[r, "\mathrm{der}\mathbb{S}_T"'] & \mathrm{Coh}_{\mathrm{fr}}^{W_{\check{T}}\times\mathbb{G}_m}(\check{\mathfrak{h}}^*) \end{tikzcd}$$

commutes.

Proof. Let $n = \mathrm{rank}(T)$. Notice that $\mathrm{Gr}_T \simeq \mathbb{Z}^n$, and that $\mathrm{Gr}_T^{\mathrm{sm}} \simeq \mathrm{pt}$. The nilpotent cone $\mathcal{N}_T$ is also a point, since zero is the only element of $\mathfrak{h}$ that acts nilpotently on every representation of T. Thus, $\underline{\mathrm{Spr}} \simeq \underline{\mathbb{C}}_{\mathrm{pt}}$. The Weyl group of $\check{T}$ is the trivial group. Therefore, our diagram becomes

$$\begin{tikzcd} \mathrm{Semis}_{T(\mathfrak{O})}(\mathrm{pt}) \arrow[r, "\mathbb{H}^\bullet_{T(\mathfrak{O})}"] \arrow[d, "\pi_* j^!"'] & \mathrm{Coh}_{\mathrm{fr}}^{\check{T}\times\mathbb{G}_m}(\check{\mathfrak{h}}^*)_{\mathrm{sm}} \arrow[d, "(-)^{\check{T}}"] \\ \mathrm{Semis}_T(\mathrm{pt}) \arrow[r, "\mathrm{Hom}^\bullet(\underline{\mathbb{C}}_{\mathrm{pt}}, -)"'] & \mathrm{Coh}_{\mathrm{fr}}^{\mathbb{G}_m}(\check{\mathfrak{h}}^*) \end{tikzcd}$$

Consider $\mathrm{IC}_{\mathrm{pt}} \simeq \underline{\mathbb{C}}_{\mathrm{pt}} \in \mathrm{Semis}_{T(\mathfrak{O})}(\mathrm{pt})$. Now, $\mathbb{H}^\bullet_{T(\mathfrak{O})}(\underline{\mathbb{C}}_{\mathrm{pt}}) = V(0) \otimes \mathcal{O}_{\check{\mathfrak{h}}^*}$, where $V(0)$ is the trivial representation. Thus, when we apply the functor $(-)^{\check{T}}$, we obtain $\mathbb{C} \otimes \mathcal{O}_{\check{\mathfrak{h}}^*}$. Now, let us travel the other way around the diagram. Notice that since $\mathcal{M}$ is a point, the functor $\pi_* : \mathcal{M} \to \mathcal{N}_T$ is now the identity functor. The functor $j^! : \mathcal{M} \hookrightarrow \mathrm{Gr}_T^{\mathrm{sm}}$ is also the identity. Hence, $\pi_* j^! = \mathrm{id}$. Thus, when applying the bottom arrow, we have

$$\mathrm{Hom}^\bullet(\underline{\mathbb{C}}_{\mathrm{pt}}, \underline{\mathbb{C}}_{\mathrm{pt}}) \simeq H^\bullet_T(\mathrm{pt}) \simeq \mathbb{C} \otimes \mathcal{O}_{\check{\mathfrak{h}}^*}.$$

Since both of these isomorphisms come from cohomology, there is a canonical natural isomorphism between the functors corresponding to the two paths around the diagram. Hence, the diagram commutes. $\square$

Lemma 4.5.2. *Let G_1 and G_2 be connected complex linear algebraic groups. If the diagrams*

$$\begin{tikzcd}
\mathrm{Semis}_{G_1(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}_{G_1}) \arrow[r, "\mathrm{der}\mathcal{S}^{\mathrm{sm}}_{G_1}"] \arrow[d, "\Psi_{G_1}"'] & \mathrm{Coh}^{\check{G}_1\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{g}}^*_1)_{\mathrm{sm}} \arrow[d, "\mathrm{der}\Phi_{\check{G}_1}"] \\
\mathrm{Semis}_{G_1}(\mathcal{N}_{G_1}) \arrow[r, "\mathrm{der}\mathbb{S}_{G_1}"'] & \mathrm{Coh}^{W_{G_1}\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{h}}^*_1)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Semis}_{G_2(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}_{G_2}) \arrow[r, "\mathrm{der}\mathcal{S}^{\mathrm{sm}}_{G_2}"] \arrow[d, "\Psi_{G_2}"'] & \mathrm{Coh}^{\check{G}_2\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{g}}^*_2)_{\mathrm{sm}} \arrow[d, "\mathrm{der}\Phi_{\check{G}_2}"] \\
\mathrm{Semis}_{G_2}(\mathcal{N}_{G_2}) \arrow[r, "\mathrm{der}\mathbb{S}_{G_2}"'] & \mathrm{Coh}^{W_{G_2}\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{h}}^*_2)
\end{tikzcd}$$

commute, then the diagram

$$\begin{tikzcd}
\mathrm{Semis}_{(G_1\times G_2)(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}_{G_1\times G_2}) \arrow[r, "\mathrm{der}\mathcal{S}^{\mathrm{sm}}_{G_1\times G_2}"] \arrow[d, "\Psi_{G_1\times G_2}"'] & \mathrm{Coh}^{\check{G}_1\times\check{G}_2\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{g}}^*_1\times\check{\mathfrak{g}}^*_2)_{\mathrm{sm}} \arrow[d, "\mathrm{der}\Phi_{\check{G}_1\times\check{G}_2}"] \\
\mathrm{Semis}_{G_1\times G_2}(\mathcal{N}_{G_1\times G_2}) \arrow[r, "\mathrm{der}\mathbb{S}_{G_1\times G_2}"'] & \mathrm{Coh}^{W_{G_1}\times W_{G_2}\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{h}}^*_1\times\check{\mathfrak{h}}^*_2)
\end{tikzcd}$$

also commutes.

Proof. Consider the diagram

$$\begin{tikzcd}
\mathrm{S}_{G_1(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}_{G_1})\times\mathrm{S}_{G_2(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}_{G_2}) \arrow[rr, "\mathrm{der}\mathcal{S}^{\mathrm{sm}}_{G_1}\times\mathrm{der}\mathcal{S}^{\mathrm{sm}}_{G_2}"] \arrow[dr, "\boxtimes"] \arrow[dd, "\Psi_{G_1}\times\Psi_{G_2}"] & & \mathrm{C}^{\check{G}_1}_{\mathrm{fr}}(\check{\mathfrak{g}}^*_1)\times\mathrm{C}^{\check{G}_2}_{\mathrm{fr}}(\check{\mathfrak{g}}^*_2) \arrow[dr, "\boxtimes\ (\text{or } \otimes_{\mathbb{C}})"] \arrow[dd, "\mathrm{der}\Phi_{\check{G}_1}\times\mathrm{der}\Phi_{\check{G}_2}"'] & \\
& \mathrm{S}_{(G_1\times G_2)(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}_{G_1\times G_2}) \arrow[rr, "\mathrm{der}\mathcal{S}_{G_1\times G_2}"] \arrow[dd, "\Psi_{G_1\times G_2}"] & & \mathrm{C}^{\check{G}_1\times\check{G}_2}_{\mathrm{fr}}(\check{\mathfrak{g}}^*_1\times\check{\mathfrak{g}}^*_2)_{\mathrm{sm}} \arrow[dd, "\mathrm{der}\Phi_{\check{G}_1\times\check{G}_2}"] \\
\mathrm{S}_{G_1}(\mathcal{N}_{G_1})\times\mathrm{S}_{G_2}(\mathcal{N}_{G_2}) \arrow[rr, "\mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}_1{,}-)\times\mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}_2{,}-"] \arrow[dr, "\boxtimes"] & & \mathrm{C}^{W_{G_1}}_{\mathrm{fr}}(\check{\mathfrak{h}}^*_1)\times\mathrm{C}^{W_{G_2}}_{\mathrm{fr}}(\check{\mathfrak{h}}^*_2) \arrow[dr, "\otimes_{\mathbb{C}}"] & \\
& \mathrm{S}_{G_1\times G_2}(\mathcal{N}_{G_1\times G_2}) \arrow[rr, "\mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}_1\boxtimes\underline{\mathrm{Spr}}_2{,}-)"] & & \mathrm{C}^{W_{G_1}\times W_{G_2}}_{\mathrm{fr}}(\check{\mathfrak{h}}^*_1\times\check{\mathfrak{h}}^*_2)
\end{tikzcd}$$

where we use the same shorthand as in Diagram (3.6). Proving the commutativity of the front face will give the proof of the theorem. Let us show that all the other faces commute.

The top face commutes by the Künneth formula and by Diagram (2.3). The left face commutes because sheaf functors commute with $\boxtimes$ (see [BBD82]). The right face commutes by associativity of $\otimes_{\mathbb{C}}$. The bottom face commutes because $R\mathrm{Hom}$ commutes with $\boxtimes$ (see [BBD82]). The back face commutes by assumption.

Now, we would like to deduce the commutativity of the front face from the commutativity of the other five faces. Notice that any indecomposable object in $\mathrm{Semis}_{(G_1\times G_2)(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}_{G_1\times G_2})$ is in the (essential) image of the functor $\boxtimes$. Thus, since all functors are additive, we can deduce commutativity of the front square by commutativity on indecomposable objects, and we are done. $\square$

Lemma 4.5.3. *Let G be a connected complex reductive linear algebraic group, and let Z be a finite subgroup of the center of G. Then, Diagram (1.2) for G commutes if and only if it commutes for G/Z.*

Proof. Consider the following diagram

$$\begin{tikzcd}
\mathrm{Semis}_{G/Z(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}_{G/Z}) \arrow[rr, "\mathrm{der}\mathcal{S}^{\mathrm{sm}}_{G/Z}"] \arrow[dr, hook, "\sim", "\mathrm{Rgd}"'] \arrow[dd, "\Psi_{G/Z}"] & & \mathrm{Coh}^{\check{G/Z}\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{g}}^*_{G/Z})_{\mathrm{sm}} \arrow[dr, "\sim"] \arrow[dd, "\mathrm{der}\Phi_{\check{G/Z}}"' near start] & \\
& \mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}_{G}) \arrow[rr, "\mathrm{der}\mathcal{S}^{\mathrm{sm}}_{G}" near start] \arrow[dd, "\Psi_G" near end] & & \mathrm{Coh}^{\check{G}\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{g}}^*)_{\mathrm{sm}} \arrow[dd, "\mathrm{der}\Phi_{\check{G}}"] \\
\mathrm{Semis}_{G/Z}(\mathcal{N}_{G/Z}) \arrow[rr, "\mathrm{der}\mathbb{S}_{G/Z}" near start] \arrow[dr, hook, "\mathrm{f.f.}", "\mathrm{Rgd}"'] & & \mathrm{Coh}^{W_{G/Z}\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{h}}^*_{G/Z}) \arrow[dr, "\sim"] & \\
& \mathrm{Semis}_{G}(\mathcal{N}_{G}) \arrow[rr, "\mathrm{der}\mathbb{S}_{G}"] & & \mathrm{Coh}^{W\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{h}}^*)
\end{tikzcd}$$

Notice that $\check{\mathfrak{g}}^*_{G/Z} \simeq \check{\mathfrak{g}}^*$ and $\check{\mathfrak{h}}^*_{G/Z} \simeq \check{\mathfrak{h}}^*$. The center of G acts trivially on small representations, which gives the two equivalences between the coherent sheaf categories in the diagram. Thus, the right square commutes. Since the center of G

acts trivially on small representations, $\mathrm{Gr}^{\mathrm{sm}}_{G/Z} \simeq \mathrm{Gr}^{\mathrm{sm}}_{G}$, which gives the equivalence between the semisimple categories. Thus, the top square also commutes. Regarding a G/Z-equivariant sheaf as a G equivariant sheaf commutes with Ψ, and the left square commutes. In a similar way, the bottom square commutes. Hence, the front square commutes if and only if the back square commutes, and the lemma is proved. $\square$

Chapter 5

Some Remarks on Line Bundles, Chern Classes, and the $\mathrm{Hom}^\bullet$-algebra of the Springer Sheaf

In this section, we reinterpret a construction of Lusztig concerning line bundles and maps in the derived category of sheaves on the base in terms of Euler classes.

5.1 Lusztig's construction

Let us recall Lusztig's construction [Lus95, Section 1.8]. Let X be a complex algebraic variety, and let $p : E \to X$ be a rank n vector bundle with zero section $i : X \to E$. In this setting, $p_! \simeq i^!$ and $p_* \simeq i^*$. Consider the bounded, constructible derived category $\mathcal{D}^{\mathrm{b}}(X)$. If $A \in \mathcal{D}^{\mathrm{b}}(X)$, then Lusztig constructs a degree $2n$ morphism in $\mathrm{Hom}^{2n}_{\mathcal{D}^{\mathrm{b}}(X)}(A, A)$ in the following way.

Let $A' = p^*A \in \mathcal{D}^{\mathrm{b}}(E)$. Then, there exists a canonical adjunction morphism

$$A' \to i_* i^* A' = i_! i^* A' = i_! i^* p^* A = i_! A.$$

Apply $p_!$ to get a morphism

$$p_! A' \to p_! i_! A = A. \tag{5.1}$$

Now, note that $p_! \underline{\mathbb{C}}_E = \underline{\mathbb{C}}_X[-2n]$, and that we have the following isomorphism

$$p_! A' = p_! (\underline{\mathbb{C}}_E \otimes p^* A) = (p_! \underline{\mathbb{C}}_E) \otimes A = \underline{\mathbb{C}}_X[-2n] \otimes A = A[-2n]. \tag{5.2}$$

Composing this isomorphism with (5.1) above, we get a morphism $A[-2n] \to A$, as we had hoped. We will denote this morphism by Lus_E.

Remark 5.1.1. *If we take $p : E \to X$ to be a line bundle ($n = 1$), and $A = \underline{\mathbb{C}}_X$, then we have that*

$$A' = p^* \underline{\mathbb{C}}_X = \underline{\mathbb{C}}_E.$$

In this case, (5.2) is just the isomorphism $p_!\underline{\mathbb{C}}_E = \underline{\mathbb{C}}_X[-2n]$ that we had before.

5.2 Euler classes

This section follows the book [Sch00]. We will build up the definition of the Euler class of a vector bundle in several steps. Throughout, let $p : E \to X$ be a complex rank n vector bundle, $i : X \to E$ be its zero section, $\mathrm{or}_{X/E}$ be its orientation sheaf, and $\omega_{X/E}$ be its dualizing complex. Then $\omega_{X/E} \simeq \mathrm{or}_{X/E}[-2n]$.

Definition 5.2.1. *An orientation class of the vector bundle $p : E \to X$ is a choice of element $\mu_{X/E} \in \Gamma(X; \mathrm{or}_{X/E})$. The Thom class τ_E of $p : E \to X$ is the image of the orientation class $\mu_{X/E}$ under the canonical isomorphism*

$$\Gamma(X; \mathrm{or}_{X/E}) \simeq H^n_X(E).$$

The Euler class $e_E \in H^n(X)$ of $p : E \to X$ is defined by

$$e_E := i^* \tau'_E,$$

where τ'_E is the image of τ_E in $H^n(E)$.

Remark 5.2.2. *For line bundles $p : E \to X$, the Euler class e_E coincides with the first Chern class c^1_E.*

5.3 Lusztig's morphism is an Euler class

Theorem 5.3.1. *Lusztig's construction gives the top Chern class of the vector bundle $p : E \to X$.*

Proof. Lusztig chooses an isomorphism $p_!\underline{\mathbb{C}}_E \simeq \underline{\mathbb{C}}_X[-2n]$. Let us choose an orientation class $\mu_{X/E} : \underline{\mathbb{C}}_X \xrightarrow{\sim} i^!\underline{\mathbb{C}}_E[2n]$ that coincides with Lusztig's isomorphism (recall that $p_! \simeq i^!$). We will move $\mu_{X/E}$ through to a Chern class and show that Lusztig's construction is the same.

Consider the canonical isomorphism $\Gamma(X; \mathrm{or}_{X/E}) \simeq H^{2n}_X(E)$. We can rewrite this in sheaf-theoretic language as

$$\mathrm{Hom}(\underline{\mathbb{C}}_X, \mathrm{or}_{X/E}) \simeq \mathrm{Hom}(i_!\underline{\mathbb{C}}_X, \underline{\mathbb{C}}_E[2n]).$$

Since $\omega_{X/E}[2n] \simeq \mathrm{or}_{X/E}$ and $\omega_{X/E} := i^!\underline{\mathbb{C}}_E$, the isomorphism above becomes

$$\mathrm{Hom}(\underline{\mathbb{C}}_X, i^!\underline{\mathbb{C}}_E[2n]) \simeq \mathrm{Hom}(i_!\underline{\mathbb{C}}_X, \underline{\mathbb{C}}_E[2n]).$$

This is just the isomorphism induced by adjunction $i_!i^!\underline{\mathbb{C}}_E[2n] \xrightarrow{\mathrm{adj}_1} \underline{\mathbb{C}}_E[2n]$. Pulling $\mu_{X/E}$ through this isomorphism gives the Thom class τ_E.

Now, we would like to rewrite the construction of τ'_E and e_E in sheaf-theoretic language. Recall that τ'_E is the image of τ_E under the isomorphism $H^{2n}_X(E) \simeq H^{2n}(E)$. We can rewrite this isomorphism as

$$\mathrm{Hom}(i_!\underline{\mathbb{C}}_X, \underline{\mathbb{C}}_E[2n]) \simeq \mathrm{Hom}(\underline{\mathbb{C}}_E, \underline{\mathbb{C}}_E[2n]).$$

This isomorphism is induced by the adjunction map $\underline{\mathbb{C}}_E \xrightarrow{\mathrm{adj}_2} i_*i^*\underline{\mathbb{C}}_E = i_!\underline{\mathbb{C}}_X$, so we can define

$$\tau'_E : \tau_E \circ \mathrm{adj}_2.$$

Lastly, the Euler class (and top Chern class) is given by

$$e_E := i^*\tau'_E \in \mathrm{Hom}(\underline{\mathbb{C}}_X, \underline{\mathbb{C}}_X[2n]).$$

Let us rewrite the construction of τ_E in one fell swoop. It is the composition

$$i_!\underline{\mathbb{C}}_X \xrightarrow{i_!\mu_{X/E}} i_!i^!\underline{\mathbb{C}}_E[2n] \xrightarrow{\mathrm{adj}_1} \underline{\mathbb{C}}_E[2n].$$

To construct τ'_E, we need only compose with adj_2. That is, τ'_E is given by

$$\underline{\mathbb{C}}_E \xrightarrow{\mathrm{adj}_2} i_!\underline{\mathbb{C}}_X \xrightarrow{i_!\mu_{X/E}} i_!i^!\underline{\mathbb{C}}_E[2n] \xrightarrow{\mathrm{adj}_1} \underline{\mathbb{C}}_E[2n].$$

To form e_E, we apply i^* to this composition to get

$$\underline{\mathbb{C}}_X \xrightarrow{\mathrm{id}} \underline{\mathbb{C}}_X \xrightarrow{\mu_{X/E}} i^! \underline{\mathbb{C}}_E[2n] \xrightarrow{\mathrm{adj}} \underline{\mathbb{C}}_X[2n].$$

It is clear that this is the construction of Lusztig's element in $\mathrm{Hom}(\underline{\mathbb{C}}_X, \underline{\mathbb{C}}_X[2n])$.

$\square$

Remark 5.3.2. *In the case that $p : E \to X$ is a line bundle ($n = 1$), then the image of* Lus_E *under the isomorphism* $\mathrm{Hom}(\underline{\mathbb{C}}_X, \underline{\mathbb{C}}_X[2]) \simeq H^2(X)$ *is the Chern class c_E^1.*

5.4 The $\mathrm{Hom}^\bullet$-algebra of the Springer sheaf

For this section, take $X = G/B$ and let $p : \mathcal{L}_\lambda \to G/B$ be a line bundle of weight λ on the flag variety. Consider the diagram

$$\begin{tikzcd} & \widetilde{\mathcal{N}} \arrow[dl, "\mu"'] \arrow[dr, "\pi"] & \\ \mathcal{N} & & G/B \end{tikzcd}$$

Pulling back along π gives a line bundle $\pi^*\mathcal{L}_\lambda$ on $\widetilde{\mathcal{N}}$. Applying Lusztig's construction gives a degree-two map $\mathrm{Lus}_{\pi^*\mathcal{L}_\lambda} : \underline{\mathbb{C}}_{\widetilde{\mathcal{N}}} \to \underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}[2]$. Applying $\mu_!$ gives a degree-two map

$$\mu_!(\mathrm{Lus}_{\pi^*\mathcal{L}_\lambda}) \in \mathrm{Hom}^2(\underline{\mathrm{Spr}}, \underline{\mathrm{Spr}}).$$

We would like to understand the isomorphism

$$A_W \simeq \mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}, \underline{\mathrm{Spr}}). \tag{5.3}$$

constructed in [Lus95]. Let $'H$ denote the free associative $\mathbb{C}$-algebra with unit on the set of generators $\lambda \in \mathfrak{h}^*$ and $s_i \in \Delta$. There is a unique surjective algebra homomorphism

$$\begin{aligned} {}'H &\longrightarrow A_W \\ s_i &\longmapsto s_i \\ \lambda &\longmapsto 1 \otimes \lambda \end{aligned}$$

Theorem 5.4.1 ([Lus95], Theorem 8.11)**.** *There is a unique isomorphism* $A_W \xrightarrow{\sim} \mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}, \underline{\mathrm{Spr}})$ *of graded algebras with unit whose composition with the homomorphism* ${}'H \to A_W$ *is equal to the homomorphism in [Lus95, 8.10(a)].*

Remark 5.4.2. *For our purposes, we take the negative of Lusztig's isomorphism* $A_W \xrightarrow{\sim} \mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}, \underline{\mathrm{Spr}})$ *defined in [Lus95, Theorem 8.11]. This allows us to prove commutativity of Diagram 1.2 for a group of semisimple rank* 1 *(see Theorem 11.4.1).*

Also, in [Lus95], Lusztig does not deal with the Springer sheaf and works in a slightly different levle of generality. However, we can deduce our results easily from the results in [Lus95].

Chapter 6
Restriction Functors

In this section, we define appropriate restriction functors for each category in Diagram (1.2).

6.1 Restriction for the affine Grassmannian

Let P be a parabolic subgroup of G that contains the Borel B. Let L be the unique Levi factor of P containing T. Let $P \hookrightarrow G$ denote the inclusion and $P \twoheadrightarrow L$ denote the projection. These maps induce maps between their corresponding affine Grassmannians

$$\mathrm{Gr}_L \xleftarrow{q_P} \mathrm{Gr}_P \xrightarrow{i_P} \mathrm{Gr}_G. \tag{6.1}$$

We can define a first-try restriction functor

$$\widetilde{\mathfrak{R}}_L^G := (q_P)_* \circ (i_P)^! : \mathcal{D}^b(\mathrm{Gr}_G) \to \mathcal{D}^b(\mathrm{Gr}_L).$$

However, this functor does not preserve perversity. To this end, recall that the connected components of Gr_L are parametrized by characters of $Z(\widetilde{L})$ where $\widetilde{L} \subset \check{G}$ is the Levi subgroup of $\check{G}$ containing $\check{T}$ whose roots are dual to the roots of L. Now consider the functor $\overline{\mathfrak{R}}_L^G : \mathcal{D}^b(\mathrm{Gr}_G) \to \mathcal{D}^b(\mathrm{Gr}_L)$ defined by

$$\overline{\mathfrak{R}}_L^G(\mathcal{F}) = \bigoplus_{\chi \in \mathbb{X}^*(Z(\widetilde{L}))} (\widetilde{\mathfrak{R}}_L^G(M))_\chi[\langle \chi, 2\rho_L - 2\rho_G \rangle],$$

where $(-)_\chi$ denotes the restriction to the connected component of Gr_L associated to the character $\chi \in \mathbb{X}^*(Z(\widetilde{L}))$, and ρ_L and ρ_G are the half sums of positive roots for L and G, respectively. This functor shifts by the appropriate amount on each connected component to assure preservation of perversity. Recall the following theorem of T. Braden.

Theorem 6.1.1 ([Bra03, Theorem 2])**.** *The functor* $\overline{\mathfrak{R}}_L^G$ *restricts to a functor (denoted in the same way) from* $\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}_G)$ *to* $\mathrm{Semis}_{L(\mathfrak{O})}(\mathrm{Gr}_L)$*.*

Remark 6.1.2. *In fact,* $\widetilde{\mathfrak{R}}_L^G = \overline{\mathfrak{R}}_L^G$ *on the identity connected component of* Gr_G*, which is where* $\mathrm{Gr}_G^{\mathrm{sm}}$ *lives. Thus, we need not define* $\overline{\mathfrak{R}}_L^G$ *(even more so since Braden's theorem also implies preservation of semisimplicity for* $\widetilde{\mathfrak{R}}_L^G$*). We chose to use it only because it is more commonly used in the literature.*

Let us define an analogous functor for equivariant derived categories. In this setting, define $(\mathfrak{R}_L^G)^\dagger$ as the composition of functors

$$\mathcal{D}^{\mathrm{b}}_{G(\mathfrak{O})}(\mathrm{Gr}_G) \xrightarrow{\mathrm{For}_{P(\mathfrak{O})}^{G(\mathfrak{O})}} \mathcal{D}^{\mathrm{b}}_{P(\mathfrak{O})}(\mathrm{Gr}_G) \xrightarrow{(i_P)^!} \mathcal{D}^{\mathrm{b}}_{P(\mathfrak{O})}(\mathrm{Gr}_P) \xrightarrow{(q_P)_*} \mathcal{D}^{\mathrm{b}}_{P(\mathfrak{O})}(\mathrm{Gr}_L) \xrightarrow{\mathrm{For}_{L(\mathfrak{O})}^{P(\mathfrak{O})}} \mathcal{D}^{\mathrm{b}}_{L(\mathfrak{O})}(\mathrm{Gr}_L)$$

and $\mathfrak{R}_L^G$ as the composition

$$\mathcal{D}^{\mathrm{b}}_{G(\mathfrak{O})}(\mathrm{Gr}_G) \xrightarrow{\mathrm{For}_{P(\mathfrak{O})}^{G(\mathfrak{O})}} \mathcal{D}^{\mathrm{b}}_{P(\mathfrak{O})}(\mathrm{Gr}_G) \xrightarrow{\overline{\mathfrak{R}}_L^G} \mathcal{D}^{\mathrm{b}}_{P(\mathfrak{O})}(\mathrm{Gr}_L) \xrightarrow{\mathrm{For}_{L(\mathfrak{O})}^{P(\mathfrak{O})}} \mathcal{D}^{\mathrm{b}}_{L(\mathfrak{O})}(\mathrm{Gr}_L).$$

Note that $\mathfrak{R}_L^G$ still preserves semisimplicity.

Now, we will construct a transitivity isomorphism $(\mathfrak{R}_T^G)^\dagger \iff (\mathfrak{R}_T^L)^\dagger \circ (\mathfrak{R}_L^G)^\dagger$. We have the following Cartesian square:

$$\begin{tikzcd}
\mathrm{Gr}_B \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Gr}_P \arrow[d] \\
\mathrm{Gr}_C \arrow[r] & \mathrm{Gr}_L
\end{tikzcd}$$

Define the transitivity isomorphism $(\mathfrak{R}_T^G)^\dagger \Longleftrightarrow (\mathfrak{R}_T^L)^\dagger \circ (\mathfrak{R}_L^G)^\dagger$ by using the following pasting diagram

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{b}}_{G(\mathfrak{D})}(\mathrm{Gr}_G) \arrow[r, "\mathrm{For}^{G(\mathfrak{D})}_{P(\mathfrak{D})}"] \arrow[dr, "\mathrm{For}^{G(\mathfrak{D})}_{B(\mathfrak{D})}"] & \mathcal{D}^{\mathrm{b}}_{P(\mathfrak{D})}(\mathrm{Gr}_G) \arrow[r, "(\cdot)^!"] \arrow[d, "\mathrm{For}^{P(\mathfrak{D})}_{B(\mathfrak{D})}"] & \mathcal{D}^{\mathrm{b}}_{P(\mathfrak{D})}(\mathrm{Gr}_P) \arrow[r, "(\cdot)_*"] \arrow[d, "\mathrm{For}^{P(\mathfrak{D})}_{B(\mathfrak{D})}"] & \mathcal{D}^{\mathrm{b}}_{P(\mathfrak{D})}(\mathrm{Gr}_L) \arrow[r, "\mathrm{For}^{P(\mathfrak{D})}_{L(\mathfrak{D})}"] \arrow[d, "\mathrm{For}^{P(\mathfrak{D})}_{B(\mathfrak{D})}"] \arrow[dr, "\mathrm{For}^{P(\mathfrak{D})}_{C(\mathfrak{D})}"] & \mathcal{D}^{\mathrm{b}}_{L(\mathfrak{D})}(\mathrm{Gr}_L) \arrow[d, "\mathrm{For}^{L(\mathfrak{D})}_{C(\mathfrak{D})}"] \\
& \mathcal{D}^{\mathrm{b}}_{B(\mathfrak{D})}(\mathrm{Gr}_G) \arrow[r, "(\cdot)^!"] \arrow[dr, "(\cdot)^!"] & \mathcal{D}^{\mathrm{b}}_{B(\mathfrak{D})}(\mathrm{Gr}_P) \arrow[r, "(\cdot)_*"] \arrow[d, "(\cdot)^!"] & \mathcal{D}^{\mathrm{b}}_{B(\mathfrak{D})}(\mathrm{Gr}_L) \arrow[r, "\mathrm{For}^{B(\mathfrak{D})}_{C(\mathfrak{D})}"] \arrow[d, "(\cdot)^!"] & \mathcal{D}^{\mathrm{b}}_{C(\mathfrak{D})}(\mathrm{Gr}_L) \arrow[d, "(\cdot)^!"] \\
& & \mathcal{D}^{\mathrm{b}}_{B(\mathfrak{D})}(\mathrm{Gr}_B) \arrow[r, "(\cdot)_*"] \arrow[dr, "(\cdot)_*"] & \mathcal{D}^{\mathrm{b}}_{B(\mathfrak{D})}(\mathrm{Gr}_C) \arrow[r, "\mathrm{For}^{B(\mathfrak{D})}_{C(\mathfrak{D})}"] \arrow[d, "(\cdot)_*"] & \mathcal{D}^{\mathrm{b}}_{C(\mathfrak{D})}(\mathrm{Gr}_C) \arrow[d, "(\cdot)_*"] \\
& & & \mathcal{D}^{\mathrm{b}}_{B(\mathfrak{D})}(\mathrm{Gr}_T) \arrow[r, "\mathrm{For}^{B(\mathfrak{D})}_{C(\mathfrak{D})}"] \arrow[dr, "\mathrm{For}^{B(\mathfrak{D})}_{T(\mathfrak{D})}"] & \mathcal{D}^{\mathrm{b}}_{C(\mathfrak{D})}(\mathrm{Gr}_T) \arrow[d, "\mathrm{For}^{C(\mathfrak{D})}_{T(\mathfrak{D})}"] \\
& & & & \mathcal{D}^{\mathrm{b}}_{T(\mathfrak{D})}(\mathrm{Gr}_T)
\end{tikzcd}$$

Restricting to semisimple sheaves, this isomorphism induces an isomorphism

$$\mathfrak{R}_T^G \Longleftrightarrow \mathfrak{R}_T^L \circ \mathfrak{R}_L^G : \mathrm{Semis}_{G(\mathfrak{D})}(\mathrm{Gr}_G) \to \mathrm{Semis}_{T(\mathfrak{D})}(\mathrm{Gr}_T). \tag{6.2}$$

6.2 Restriction and induction for the nilpotent cone

Let $P \hookrightarrow G$ be the inclusion of the parabolic and $P \twoheadrightarrow L$ be the projection onto the Levi. These maps induce the following

$$\mathcal{N}_L \xleftarrow{p_P} \mathcal{N}_P \xrightarrow{m_P} \mathcal{N}_G. \tag{6.3}$$

We can define a restriction functor

$$\mathcal{R}_L^G := (p_P)_* \circ (m_P)^! : \mathcal{D}^{\mathrm{b}}(\mathcal{N}_G) \to \mathcal{D}^{\mathrm{b}}(\mathcal{N}_L).$$

Let us define an analogous functor on the level of equivariant derived categories. In this setting, define $\widetilde{\mathcal{R}}_L^G$ as the composition of functors

$$\mathcal{D}^{\mathrm{b}}_G(\mathcal{N}_G) \xrightarrow{\mathrm{For}_P^G} \mathcal{D}^{\mathrm{b}}_P(\mathcal{N}_G) \xrightarrow{(m_P)^!} \mathcal{D}^{\mathrm{b}}_P(\mathcal{N}_P) \xrightarrow{(p_P)_*} \mathcal{D}^{\mathrm{b}}_P(\mathcal{N}_L) \xrightarrow{\mathrm{For}_L^P} \mathcal{D}^{\mathrm{b}}_L(\mathcal{N}_L).$$

Theorem 6.2.1 ([Bra03, Theorem 2])**.** *The functor $\widetilde{\mathcal{R}}^G_L$ restricts to a functor (denoted in the same way) from* $\mathrm{Semis}_G(\mathcal{N}_G)$ *to* $\mathrm{Semis}_L(\mathcal{N}_L)$*.*

Now let us construct a transitivity isomorphism $\widetilde{\mathcal{R}}^G_T \Longleftrightarrow \widetilde{\mathcal{R}}^L_T \circ \widetilde{\mathcal{R}}^G_L$. We have the following Cartesian square:

$$\begin{tikzcd}
\mathcal{N}_B \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{N}_P \arrow[d] \\
\mathcal{N}_C \arrow[r] & \mathcal{N}_L
\end{tikzcd}$$

Define the transitivity isomorphism

$$\widetilde{\mathcal{R}}^G_T \Longleftrightarrow \widetilde{\mathcal{R}}^L_T \circ \widetilde{\mathcal{R}}^G_L \tag{6.4}$$

by restricting the following pasting diagram to the appropriate semisimple categories

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{b}}_G(\mathcal{N}_G) \arrow[r, "\mathrm{For}^G_P"] \arrow[dr, "\mathrm{For}^G_B"] & \mathcal{D}^{\mathrm{b}}_P(\mathcal{N}_G) \arrow[r, "(\cdot)^!"] \arrow[d, "\mathrm{For}^P_B"] & \mathcal{D}^{\mathrm{b}}_P(\mathcal{N}_P) \arrow[r, "(\cdot)_*"] \arrow[d, "\mathrm{For}^P_B"] & \mathcal{D}^{\mathrm{b}}_P(\mathcal{N}_L) \arrow[r, "\mathrm{For}^P_L"] \arrow[d, "\mathrm{For}^P_B"] \arrow[dr, "\mathrm{For}^P_C"] & \mathcal{D}^{\mathrm{b}}_L(\mathcal{N}_L) \arrow[d, "\mathrm{For}^L_C"] \\
& \mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_G) \arrow[r, "(\cdot)^!"] \arrow[dr, "(\cdot)^!"] & \mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_P) \arrow[r, "(\cdot)_*"] \arrow[d, "(\cdot)^!"] & \mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_L) \arrow[r, "\mathrm{For}^B_C"] \arrow[d, "(\cdot)^!"] & \mathcal{D}^{\mathrm{b}}_C(\mathcal{N}_L) \arrow[d, "(\cdot)^!"] \\
& & \mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_B) \arrow[r, "(\cdot)_*"] \arrow[dr, "(\cdot)_*"] & \mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_C) \arrow[r, "\mathrm{For}^B_C"] \arrow[d, "(\cdot)_*"] & \mathcal{D}^{\mathrm{b}}_C(\mathcal{N}_C) \arrow[d, "(\cdot)_*"] \\
& & & \mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_T) \arrow[r, "\mathrm{For}^B_C"] \arrow[dr, "\mathrm{For}^B_T"] & \mathcal{D}^{\mathrm{b}}_C(\mathcal{N}_T) \arrow[d, "\mathrm{For}^C_T"] \\
& & & & \mathcal{D}^{\mathrm{b}}_T(\mathcal{N}_T)
\end{tikzcd}$$

In the sequel, we will need to consider the induction functor $\widetilde{\mathcal{I}}^G_L : \mathcal{D}^{\mathrm{b}}_L(\mathcal{N}_L) \to \mathcal{D}^{\mathrm{b}}_G(\mathcal{N}_G)$ which is the left adjoint of the restriction functor $\widetilde{\mathcal{R}}^G_L$. It is defined as the following composition

$$\mathcal{D}^{\mathrm{b}}_G(\mathcal{N}_G) \xleftarrow{\gamma^G_P} \mathcal{D}^{\mathrm{b}}_P(\mathcal{N}_G) \xleftarrow{(m_P)_!} \mathcal{D}^{\mathrm{b}}_P(\mathcal{N}_P) \xleftarrow{(p_P)^*} \mathcal{D}^{\mathrm{b}}_P(\mathcal{N}_L) \xleftarrow{\gamma^P_L} \mathcal{D}^{\mathrm{b}}_L(\mathcal{N}_L)$$

where γ_K^H is the left adjoint of For_K^H (see [BL94, §3.7.1]). We have a transitivity isomorphism

$$\widetilde{\mathcal{I}}_T^G \Longleftrightarrow \widetilde{\mathcal{I}}_T^L \circ \widetilde{\mathcal{I}}_L^G : \mathcal{D}_T^{\mathrm{b}}(\mathcal{N}_T) \to \mathcal{D}_G^{\mathrm{b}}(\mathcal{N}_G) \tag{6.5}$$

defined by the following pasting diagram:

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{b}}_G(\mathcal{N}_G) & \mathcal{D}^{\mathrm{b}}_P(\mathcal{N}_G) \arrow[l, "\gamma_P^G"'] & \mathcal{D}^{\mathrm{b}}_P(\mathcal{N}_P) \arrow[l, "(\cdot)_!"'] & \mathcal{D}^{\mathrm{b}}_P(\mathcal{N}_L) \arrow[l, "(\cdot)^*"'] & \mathcal{D}^{\mathrm{b}}_L(\mathcal{N}_L) \arrow[l, "\gamma_L^P"'] \\
& \mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_G) \arrow[ul, "\gamma_B^G"] \arrow[u, "\gamma_B^P"] & \mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_P) \arrow[l, "(\cdot)_!"'] \arrow[u, "\gamma_B^P"] & \mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_L) \arrow[l, "(\cdot)^*"'] \arrow[u, "\gamma_B^P"] & \mathcal{D}^{\mathrm{b}}_C(\mathcal{N}_L) \arrow[l, "\gamma_C^B"'] \arrow[u, "\gamma_C^L"] \arrow[ul, "\gamma_C^P"] \\
& & \mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_B) \arrow[ul, "(\cdot)_!"] \arrow[u, "(\cdot)_!"] & \mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_C) \arrow[l, "(\cdot)^*"'] \arrow[u, "(\cdot)_!"] & \mathcal{D}^{\mathrm{b}}_C(\mathcal{N}_C) \arrow[l, "\gamma_C^B"'] \arrow[u, "(\cdot)_!"] \\
& & & \mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_T) \arrow[ul, "(\cdot)^*"] \arrow[u, "(\cdot)^*"] & \mathcal{D}^{\mathrm{b}}_C(\mathcal{N}_T) \arrow[l, "\gamma_C^B"'] \arrow[u, "(\cdot)^*"] \\
& & & & \mathcal{D}^{\mathrm{b}}_T(\mathcal{N}_T) \arrow[ul, "\gamma_T^B"] \arrow[u, "\gamma_T^C"]
\end{tikzcd}$$

6.3 Restriction for coherent sheaves on $\check{\mathfrak{g}}^*$

We will define a restriction functor

$$R_L^G : \mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*) \to \mathrm{Coh}_{\mathrm{fr}}^{\check{L}\times\mathbb{G}_m}(\check{\mathfrak{l}}^*).$$

First, define

$$\mathrm{Res}_L^G : \mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*) \to \mathrm{Coh}_{\mathrm{fr}}^{\check{L}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*),$$

which just restricts the action of the group on each factor. Also define

$$\begin{array}{ccc}
\mathrm{Coh}_{\mathrm{fr}}^{\check{L}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*) & \xrightarrow{R_L^{G\prime}} & \mathrm{Coh}_{\mathrm{fr}}^{\check{L}\times\mathbb{G}_m}(\check{\mathfrak{l}}^*) \\
V_\lambda \otimes \mathcal{O}_{\check{\mathfrak{g}}^*} & \longmapsto & (V_\lambda \otimes \mathcal{O}_{\check{\mathfrak{g}}^*}) \otimes_{\mathcal{O}_{\check{\mathfrak{g}}^*}} \mathcal{O}_{\check{\mathfrak{l}}^*}
\end{array}$$

to be the coherent restriction functor.

Remark 6.3.1. *$\mathcal{O}_{\check{\mathfrak{l}}^*}$ is not a module over $\mathcal{O}_{\check{\mathfrak{g}}^*}$ since there is not an obvious map $\check{\mathfrak{l}}^* \hookrightarrow \check{\mathfrak{g}}^*$. However, there always exists a non-degenerate bilinear form on $\check{\mathfrak{g}}^*$ that* *restricts to one on $\check{\mathfrak{l}}^*$. By identifying $\check{\mathfrak{g}}^*$ with $\check{\mathfrak{g}}$ and $\check{\mathfrak{l}}^*$ with $\check{\mathfrak{l}}$ using this form, we can make $\mathcal{O}_{\check{\mathfrak{l}}^*}$ a module over $\mathcal{O}_{\check{\mathfrak{g}}^*}$ because there is a natural inclusion $\check{\mathfrak{l}} \hookrightarrow \check{\mathfrak{g}}$ and hence a natural map $\mathcal{O}_{\check{\mathfrak{g}}} \to \mathcal{O}_{\check{\mathfrak{l}}}$.*

Define our restriction functor as the composition of these. That is,

$$R_L^G := R_L^{G'} \circ \mathrm{Res}_L^G.$$

Now we will define a transitivity isomorphism for R_L^G. Consider the diagram

$$\begin{tikzcd}
\mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*) \arrow[r, "\mathrm{Res}_L^G"] \arrow[dr, "\mathrm{Res}_T^G"'] & \mathrm{Coh}_{\mathrm{fr}}^{\check{L}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*) \arrow[r, "R_L^{G'}"] \arrow[d, "\mathrm{Res}_T^L"] & \mathrm{Coh}_{\mathrm{fr}}^{\check{L}\times\mathbb{G}_m}(\check{\mathfrak{l}}^*) \arrow[d, "\mathrm{Res}_T^L"] \\
 & \mathrm{Coh}_{\mathrm{fr}}^{\check{T}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*) \arrow[r, "R_L^{G'}"] \arrow[dr, "R_T^{G'}"'] & \mathrm{Coh}_{\mathrm{fr}}^{\check{T}\times\mathbb{G}_m}(\check{\mathfrak{l}}^*) \arrow[d, "R_T^{L'}"] \\
 & & \mathrm{Coh}_{\mathrm{fr}}^{\check{T}\times\mathbb{G}_m}(\check{\mathfrak{h}}^*)
\end{tikzcd} \tag{6.6}$$

The top left triangle commutes, and in this case the natural isomorphism is actually an equality $\mathrm{Res}_T^G = \mathrm{Res}_T^L \circ \mathrm{Res}_L^G$. Similarly, the bottom right triangle commutes by the equation

$$(V_\lambda \otimes \mathcal{O}_{\check{\mathfrak{g}}^*}) \otimes_{\mathcal{O}_{\check{\mathfrak{g}}^*}} \mathcal{O}_{\check{\mathfrak{h}}^*} = ((V_\lambda \otimes \mathcal{O}_{\check{\mathfrak{g}}^*}) \otimes_{\mathcal{O}_{\check{\mathfrak{g}}^*}} \mathcal{O}_{\check{\mathfrak{l}}^*}) \otimes_{\mathcal{O}_{\check{\mathfrak{l}}^*}} \mathcal{O}_{\check{\mathfrak{h}}^*},$$

so we have a natural isomorphism $R_T^{G'} \Longleftrightarrow R_T^{L'} \circ R_L^{G'}$. The top right square commutes, so pasting together the natural isomorphisms in the diagram gives a transitivity isomorphism

$$R_T^G \Longleftrightarrow R_T^L \circ R_L^G. \tag{6.7}$$

6.4 Restriction for coherent sheaves on $\check{\mathfrak{h}}^*$

Define a restriction functor

$$R_{W_L}^{W_G} : \mathrm{Coh}_{\mathrm{fr}}^{W_G\times\mathbb{G}_m}(\check{\mathfrak{h}}^*) \to \mathrm{Coh}_{\mathrm{fr}}^{W_L\times\mathbb{G}_m}(\check{\mathfrak{h}}^*)$$

by restricting the action of the group on both factors from W_G to W_L. Notice that

$$R^{W_G}_{W_T} = R^{W_L}_{W_T} \circ R^{W_G}_{W_L}, \tag{6.8}$$

which we will use for our transitivity isomorphism.

Chapter 7
Functors are Compatible with Restriction

In this section, we will prove that each of the functors in our main diagram are compatible with the various transitivity isomorphisms defined in Section 6.

7.1 $\mathrm{der}\Phi_{\check{G}}$

We want to define an isomorphism for the square

$$\begin{tikzcd}
\mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*) \arrow[r, "R_L^G"] \arrow[d, "\mathrm{der}\Phi_{\check{G}}"'] & \mathrm{Coh}_{\mathrm{fr}}^{\check{L}\times\mathbb{G}_m}(\check{\mathfrak{l}}^*) \arrow[d, "\mathrm{der}\Phi_{\check{L}}"] \\
\mathrm{Coh}_{\mathrm{fr}}^{W_G\times\mathbb{G}_m}(\check{\mathfrak{h}}^*) \arrow[r, "R_{W_L}^{W_G}"'] & \mathrm{Coh}_{\mathrm{fr}}^{W_L\times\mathbb{G}_m}(\check{\mathfrak{h}}^*)
\end{tikzcd}$$

Consider the following elaboration of the diagram above:

$$\begin{tikzcd}
\mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*) \arrow[r, "\mathrm{Res}_{\check{L}}^{\check{G}}"] \arrow[d, "\mathrm{Res}_{N_{\check{G}}(\check{T})}^{\check{G}}"'] & \mathrm{Coh}_{\mathrm{fr}}^{\check{L}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*) \arrow[r, "R_L^{G\prime}"] \arrow[d, "\mathrm{Res}_{N_{\check{L}}(\check{T})}^{\check{L}}"] & \mathrm{Coh}_{\mathrm{fr}}^{\check{L}\times\mathbb{G}_m}(\check{\mathfrak{l}}^*) \arrow[d, "\mathrm{Res}_{N_{\check{L}}(\check{T})}^{\check{L}}"] \\
\mathrm{Coh}_{\mathrm{fr}}^{N_{\check{G}}(\check{T})\times\mathbb{G}_m}(\check{\mathfrak{g}}^*) \arrow[r, "\mathrm{Res}_{N_{\check{L}}(\check{T})}^{N_{\check{G}}(\check{T})}"] \arrow[d, "R_T^{G\prime}"'] & \mathrm{Coh}_{\mathrm{fr}}^{N_{\check{L}}(\check{T})\times\mathbb{G}_m}(\check{\mathfrak{g}}^*) \arrow[r, "R_L^{G\prime}"] \arrow[d, "R_T^{G\prime}"] & \mathrm{Coh}_{\mathrm{fr}}^{N_{\check{L}}(\check{T})\times\mathbb{G}_m}(\check{\mathfrak{l}}^*) \arrow[d, "R_T^{L\prime}"] \\
\mathrm{Coh}_{\mathrm{fr}}^{N_{\check{G}}(\check{T})\times\mathbb{G}_m}(\check{\mathfrak{h}}^*) \arrow[r, "\mathrm{Res}_{N_{\check{L}}(\check{T})}^{N_{\check{G}}(\check{T})}"] \arrow[d, "(-)^{\check{T}}\otimes\varepsilon"'] & \mathrm{Coh}_{\mathrm{fr}}^{N_{\check{L}}(\check{T})\times\mathbb{G}_m}(\check{\mathfrak{h}}^*) \arrow[r, equal] \arrow[d, "(-)^{\check{T}}\otimes\varepsilon"] & \mathrm{Coh}_{\mathrm{fr}}^{N_{\check{L}}(\check{T})\times\mathbb{G}_m}(\check{\mathfrak{h}}^*) \arrow[dl, "(-)^{\check{T}}\otimes\varepsilon"] \\
\mathrm{Coh}_{\mathrm{fr}}^{W_G\times\mathbb{G}_m}(\check{\mathfrak{h}}^*) \arrow[r, "R_{W_L}^{W_G}"] & \mathrm{Coh}_{\mathrm{fr}}^{W_L\times\mathbb{G}_m}(\check{\mathfrak{h}}^*) &
\end{tikzcd} \tag{7.1}$$

The top and bottom compositions are R_L^G and $R_{W_L}^{W_G}$, respectively, and the left and right compositions are $\mathrm{der}\Phi_{\check{G}}$ and $\mathrm{der}\Phi_{\check{L}}$, respectively. Each of the faces obviously commute, and the composition of the natural isomorphisms on these faces give a natural intertwining isomorphism

$$\mathrm{der}\Phi_{\check{L}} \circ R_L^G \Longleftrightarrow R_{W_L}^{W_G} \circ \mathrm{der}\Phi_{\check{G}}. \tag{7.2}$$

Now, we will prove that the intertwining isomorphism in (7.2) is compatible with transitivity of restriction defined in (6.7) and (6.8).

Theorem 7.1.1. *The following triangular prism commutes:*

$$\begin{tikzcd}
\mathrm{Coh}^{\check{G}\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{g}}^*) \arrow[rr, "\mathrm{der}\Phi_{\check{G}}"] \arrow[dr, "R^G_L"] \arrow[dd, "R^G_T"'] & & \mathrm{Coh}^{W_G\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{h}}^*) \arrow[dr, "R^{W_G}_{W_L}"] \arrow[dd, "R^{W_G}_{W_T}"] & \\
& \mathrm{Coh}^{\check{L}\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{l}}^*) \arrow[rr, "\mathrm{der}\Phi_{\check{L}}"] \arrow[dl, "R^L_T"] & & \mathrm{Coh}^{W_L\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{h}}^*) \arrow[dl, "R^{W_L}_{W_T}"] \\
\mathrm{Coh}^{\check{T}\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{h}}^*) \arrow[rr, "\mathrm{der}\Phi_{\check{T}}"] & & \mathrm{Coh}^{W_T\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{h}}^*) &
\end{tikzcd}$$

Proof. Certainly the left-hand triangle commutes by (6.7) and the right-hand triangle commutes by (6.8). The front two rectangles commute by Diagram (7.1) for G and for L. Thus, we can define a natural isomorphism for the back face using the natural isomorphisms for the other four faces, and hence the triangular prism commutes. □

7.2 Ψ_G

We follow exactly the treatment in [AHR15] and point to it for more details. We include a brief summary in this paper for the sake of completeness. Our goal is to define an intertwining natural isomorphism $R^{W_G}_{W_L} \circ \Psi_G \Longleftrightarrow \Psi_L \circ R^G_L$ for the square

$$\begin{tikzcd}
\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}_G) \arrow[r, "\Psi_G"] \arrow[d, "\mathfrak{R}^G_L"] & \mathrm{Semis}_G(\mathcal{N}_G) \arrow[d, "\widetilde{\mathcal{R}}^G_L"] \\
\mathrm{Semis}_{L(\mathfrak{O})}(\mathrm{Gr}_L) \arrow[r, "\Psi_L"] & \mathrm{Semis}_L(\mathcal{N}_L)
\end{tikzcd} \tag{7.3}$$

and to prove that this intertwining isomorphism is compatible with the transitivity of restriction defined in (6.2) and (6.4).

First, we lay down some preliminary results.

Lemma 7.2.1 ([AHR15], Lemma 5.2). *The following square is cartesian:*

$$\begin{tikzcd} \mathcal{M}_L \arrow[r, hook] \arrow[d, "\pi_L"] & \mathcal{M}_G \arrow[d, "\pi_G"] \\ \mathcal{N}_L \arrow[r, hook] & \mathcal{N}_G \end{tikzcd}$$

Recall Diagrams (6.1) and (6.3). We will produce a similar diagram relating $\mathcal{M}_L$ and $\mathcal{M}_G$.

Proposition 7.2.2 ([AHR15], Proposition 5.3). *We have $i_p(\mathcal{M}_P) \subset \mathcal{M}_G$, and there is a morphism $\pi_P : \mathcal{M}_P \to \mathcal{N}_P$ making the following square cartesian:*

$$\begin{tikzcd} \mathcal{M}_P \arrow[r, hook, "i_P"] \arrow[d, "\pi_P"] & \mathcal{M}_G \arrow[d, "\pi_G"] \\ \mathcal{N}_P \arrow[r, hook, "m_P"] & \mathcal{N}_G \end{tikzcd}$$

Let $i_P^{\mathcal{M}} : \mathcal{M}_P \to \mathcal{M}_G$ and $q_P^{\mathcal{M}} : \mathcal{M}_P \to \mathcal{M}_L$ be the restrictions of i_P and q_P, respectively. Then, we have the following diagram analogous to Diagrams (6.1) and (6.3):

$$\mathcal{M}_L \xleftarrow{q_P^{\mathcal{M}}} \mathcal{M}_P \overset{i_P^{\mathcal{M}}}{\hookrightarrow} \mathcal{M}_G.$$

Fitting everything together, we have a diagram of commutative squares:

$$\begin{tikzcd} \mathrm{Gr}_G^{\mathrm{sm}} & \mathcal{M}_G \arrow[l, hook', "j_G"'] \arrow[r, "\pi_G"] & \mathcal{N}_G \\ \mathrm{Gr}_P^{\mathrm{sm}} \arrow[u, hook, "i_P^{\mathrm{sm}}"] \arrow[d, "q_P^{\mathrm{sm}}"] & \mathcal{M}_P \arrow[l, hook', "j_P"'] \arrow[r, "\pi_P"] \arrow[u, hook, "i_P^{\mathcal{M}}"] \arrow[d, "q_P^{\mathcal{M}}"] & \mathcal{N}_P \arrow[u, hook, "m_P"] \arrow[d, "p_P"] \\ \mathrm{Gr}_L^{\mathrm{sm}} & \mathcal{M}_L \arrow[l, hook', "j_L"'] \arrow[r, "\pi_L"] & \mathcal{N}_L \end{tikzcd} \tag{7.4}$$

where the top right square is cartesian by Proposition 7.2.2 and the bottom left square is cartesian by the definition of $\mathcal{M}_P$.

Recall that the functors $\Psi_G, \Psi_L, \mathfrak{R}_L^G$, and $\widetilde{\mathcal{R}}_L^G$ are gotten by restricting functors from derived categories to semisimple categories. Thus, we can define the intertwining natural isomorphism for Diagram (7.3) by defining it for the corresponding derived categories. Using the morphisms in Diagram (7.4), we can make the following pasting diagram and use it to define our intertwining natural isomorphism:

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{b}}_{G(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}_G) \arrow[r, "\mathrm{For}^{G(\mathfrak{O})}_{G(\mathbb{C})}"] \arrow[d, "\mathrm{For}^{G(\mathfrak{O})}_{P(\mathfrak{O})}"'] & \mathcal{D}^{\mathrm{b}}_G(\mathrm{Gr}^{\mathrm{sm}}_G) \arrow[r, "j^!_G"] \arrow[d, "\mathrm{For}^G_P"] & \mathcal{D}^{\mathrm{b}}_G(\mathcal{M}_G) \arrow[r, "(\pi_G)_*"] \arrow[d, "\mathrm{For}^G_P"] & \mathcal{D}^{\mathrm{b}}_G(\mathcal{N}_G) \arrow[d, "\mathrm{For}^G_P"] \\
\mathcal{D}^{\mathrm{b}}_{P(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}_G) \arrow[r, "\mathrm{For}^{P(\mathfrak{O})}_{P(\mathbb{C})}"] \arrow[d, "(i^{\mathrm{sm}}_P)^!"'] & \mathcal{D}^{\mathrm{b}}_P(\mathrm{Gr}^{\mathrm{sm}}_G) \arrow[r, "j^!_G"] \arrow[d, "(i^{\mathrm{sm}}_P)^!"] & \mathcal{D}^{\mathrm{b}}_P(\mathcal{M}_G) \arrow[r, "(\pi_G)_*"] \arrow[d, "(i^{\mathcal{M}}_P)^!"] & \mathcal{D}^{\mathrm{b}}_P(\mathcal{N}_G) \arrow[d, "m^!_P"] \\
\mathcal{D}^{\mathrm{b}}_{P(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}_P) \arrow[r, "\mathrm{For}^{P(\mathfrak{O})}_{P(\mathbb{C})}"] \arrow[d, "\mathrm{For}^{P(\mathfrak{O})}_{L(\mathfrak{O})}"'] & \mathcal{D}^{\mathrm{b}}_P(\mathrm{Gr}^{\mathrm{sm}}_P) \arrow[r, "j^!_P"] \arrow[d, "\mathrm{For}^P_L"] & \mathcal{D}^{\mathrm{b}}_P(\mathcal{M}_P) \arrow[r, "(\pi_P)_*"] \arrow[d, "\mathrm{For}^P_L"] & \mathcal{D}^{\mathrm{b}}_P(\mathcal{N}_P) \arrow[d, "\mathrm{For}^P_L"] \\
\mathcal{D}^{\mathrm{b}}_{L(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}_P) \arrow[r, "\mathrm{For}^{L(\mathfrak{O})}_L"] \arrow[d, "(q^{\mathrm{sm}}_P)_*"'] & \mathcal{D}^{\mathrm{b}}_L(\mathrm{Gr}^{\mathrm{sm}}_P) \arrow[r, "j^!_P"] \arrow[d, "(q^{\mathrm{sm}}_P)_*"] & \mathcal{D}^{\mathrm{b}}_L(\mathcal{M}_P) \arrow[r, "(\pi_P)_*"] \arrow[d, "(q^{\mathcal{M}}_P)_*"] & \mathcal{D}^{\mathrm{b}}_L(\mathcal{N}_P) \arrow[d, "(p_P)_*"] \\
\mathcal{D}^{\mathrm{b}}_{L(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}_L) \arrow[r, "\mathrm{For}^{L(\mathfrak{O})}_L"] & \mathcal{D}^{\mathrm{b}}_L(\mathrm{Gr}^{\mathrm{sm}}_L) \arrow[r, "j^!_L"] & \mathcal{D}^{\mathrm{b}}_L(\mathcal{M}_L) \arrow[r, "(\pi_L)_*"] & \mathcal{D}^{\mathrm{b}}_L(\mathcal{N}_L)
\end{tikzcd} \tag{7.5}$$

We are ready to prove that the intertwining natural isomorphism $R^{W_G}_{W_L} \circ \Psi_G \Longleftrightarrow \Psi_L \circ R^G_L$ defined in Diagram (7.5) is compatible with the transitivity of restriction in (6.2) and (6.4).

Proposition 7.2.3 ([AHR15], Proposition 5.4)**.** *The following prism is commutative:*

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{b}}_{G(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}_G) \arrow[rrr, "\Psi_G"] \arrow[dr, "\mathfrak{R}^G_L"] \arrow[dd, "\mathfrak{R}^G_T"] & & & \mathcal{D}^{\mathrm{b}}_G(\mathcal{N}_G) \arrow[dr, "\widetilde{\mathcal{R}}^G_L"] \arrow[dd, "\widetilde{\mathcal{R}}^G_T"] & \\
& \mathcal{D}^{\mathrm{b}}_{L(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}_L) \arrow[rrr, "\Psi_L"] \arrow[dl, "\mathfrak{R}^L_T"] & & & \mathcal{D}^{\mathrm{b}}_L(\mathcal{N}_L) \arrow[dl, "\widetilde{\mathcal{R}}^L_T"] \\
\mathcal{D}^{\mathrm{b}}_{T(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}_T) \arrow[rrr, "\Psi_T"] & & & \mathcal{D}^{\mathrm{b}}_T(\mathcal{N}_T) &
\end{tikzcd} \tag{7.6}$$

Proof. We will paste together two large commutative prisms to give the commutative prism in the statement of the proposition. In these large commutative prisms, we shorten the notation for the forgetful functor For by writing F. Consider the following large prism.

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{b}}_{G(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}_G) \arrow[rrr, "j_G^! \circ F^{G(\mathfrak{O})}_{G(\mathbb{C})}"] \arrow[dr, "(\cdot)^! \circ F^{G(\mathfrak{O})}_{P(\mathfrak{O})}"] \arrow[dd, "(\cdot)^! \circ F^{G(\mathfrak{O})}_{B(\mathfrak{O})}"] & & & \mathcal{D}^{\mathrm{b}}_G(\mathcal{M}_G) \arrow[dr, "(\cdot)^! \circ F^G_P"] \arrow[dd, "(\cdot)^! \circ F^G_B"] & & \\
& \mathcal{D}^{\mathrm{b}}_{P(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}_P) \arrow[rrr, "j_P^! \circ F^{P(\mathfrak{O})}_{P(\mathbb{C})}"] \arrow[dl, "(\cdot)^! \circ F^{P(\mathfrak{O})}_{B(\mathfrak{O})}"] \arrow[ddr, "(\cdot)_* \circ F^{P(\mathfrak{O})}_{L(\mathfrak{O})}"] & & & \mathcal{D}^{\mathrm{b}}_P(\mathcal{M}_P) \arrow[dl, "(\cdot)^! \circ F^P_B"] \arrow[ddr, "(\cdot)_* \circ F^P_L"] & \\
\mathcal{D}^{\mathrm{b}}_{B(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}_B) \arrow[rrr, "j_B^! \circ F^{B(\mathfrak{O})}_{B(\mathbb{C})}"] \arrow[ddr, "(\cdot)_* \circ F^{B(\mathfrak{O})}_{C(\mathfrak{O})}"] \arrow[ddd, "(\cdot)_* \circ F^{B(\mathfrak{O})}_{T(\mathfrak{O})}"] & & & \mathcal{D}^{\mathrm{b}}_B(\mathcal{M}_B) \arrow[ddr, "(\cdot)_* \circ F^B_C"] \arrow[ddd, "(\cdot)_* \circ F^B_T"] & & \\
& & \mathcal{D}^{\mathrm{b}}_{L(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}_L) \arrow[rrr, "j_L^! \circ F^{L(\mathfrak{O})}_{L(\mathbb{C})}"] \arrow[dl, "(\cdot)^! \circ F^{L(\mathfrak{O})}_{C(\mathfrak{O})}"] & & & \mathcal{D}^{\mathrm{b}}_L(\mathcal{M}_L) \arrow[dl, "(\cdot)^! \circ F^L_C"] \\
& \mathcal{D}^{\mathrm{b}}_{C(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}_C) \arrow[rrr, "j_C^! \circ F^{C(\mathfrak{O})}_{C(\mathbb{C})}"] \arrow[dl, "(\cdot)_* \circ F^{C(\mathfrak{O})}_{T(\mathfrak{O})}"] & & & \mathcal{D}^{\mathrm{b}}_C(\mathcal{M}_C) \arrow[dl, "(\cdot)_* \circ F^C_T"] & \\
\mathcal{D}^{\mathrm{b}}_{T(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}_T) \arrow[rrr, "j_T^! \circ F^{T(\mathfrak{O})}_{T(\mathbb{C})}"] & & & \mathcal{D}^{\mathrm{b}}_T(\mathcal{M}_T) & &
\end{tikzcd} \tag{7.7}$$

All of its constituent cubes and prisms commute after noticing that we have the following two cartesian diagrams which are defined in [AHR15, (5.6)].

$$\begin{tikzcd}
\mathcal{M}_B \arrow[r] \arrow[d] & \mathcal{M}_C \arrow[d] \\
\mathrm{Gr}^{\mathrm{sm}}_B \arrow[r] & \mathrm{Gr}^{\mathrm{sm}}_C
\end{tikzcd} \qquad \begin{tikzcd}
\mathcal{M}_B \arrow[r] \arrow[d] & \mathcal{M}_P \arrow[d] \\
\mathcal{M}_C \arrow[r] & \mathcal{M}_L
\end{tikzcd} \tag{7.8}$$

Consider another large prism.

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{b}}_G(\mathcal{M}_G) \arrow[rrr, "(\pi_G)_*"] \arrow[dr, "(\cdot)^! \circ F^G_P"] \arrow[dd, "(\cdot)^! \circ F^G_B"] & & & \mathcal{D}^{\mathrm{b}}_G(\mathcal{N}_G) \arrow[dr, "(\cdot)^! \circ F^G_P"] \arrow[dd, "(\cdot)^! \circ F^G_B"] & & \\
& \mathcal{D}^{\mathrm{b}}_P(\mathcal{M}_P) \arrow[rrr, "(\pi_P)_*"] \arrow[dl, "(\cdot)^! \circ F^P_B"] \arrow[ddr, "(\cdot)_* \circ F^P_L"] & & & \mathcal{D}^{\mathrm{b}}_P(\mathcal{N}_P) \arrow[dl, "(\cdot)^! \circ F^P_B"] \arrow[ddr, "(\cdot)_*"] & \\
\mathcal{D}^{\mathrm{b}}_B(\mathcal{M}_B) \arrow[rrr, "(\pi_B)_*"] \arrow[ddr, "(\cdot)_* \circ F^B_C"] \arrow[ddd, "(\cdot)_* \circ F^B_T"] & & & \mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_B) \arrow[ddr, "(\cdot)_* \circ F^B_C"] \arrow[ddd, "(\cdot)_* \circ F^B_T"] & & \\
& & \mathcal{D}^{\mathrm{b}}_L(\mathcal{M}_L) \arrow[rrr, "(\pi_L)_*"] \arrow[dl, "(\cdot)^! \circ F^L_C"] & & & \mathcal{D}^{\mathrm{b}}_L(\mathcal{N}_L) \arrow[dl, "(\cdot)^! \circ F^L_C"] \\
& \mathcal{D}^{\mathrm{b}}_C(\mathcal{M}_C) \arrow[rrr, "(\pi_C)_*"] \arrow[dl, "(\cdot)_* \circ F^C_T"] & & & \mathcal{D}^{\mathrm{b}}_C(\mathcal{N}_C) \arrow[dl, "(\cdot)_* \circ F^C_T"] & \\
\mathcal{D}^{\mathrm{b}}_T(\mathcal{M}_T) \arrow[rrr, "(\pi_T)_*"] & & & \mathcal{D}^{\mathrm{b}}_T(\mathcal{N}_T) & &
\end{tikzcd} \tag{7.9}$$

All of its constituent cubes and prisms commute after noticing that we have the following cartesian diagram.

$$\begin{tikzcd}
\mathcal{M}_B \arrow[r] \arrow[d] & \mathcal{N}_B \arrow[d] \\
\mathcal{M}_P \arrow[r] & \mathcal{N}_P
\end{tikzcd} \tag{7.10}$$

which is defined using the cartesian diagram in Proposition 7.2.2 and its analogue with B playing the role of P.

Now, we are able to paste the two large prisms, (7.7) and (7.9), along the face labeled by $\mathcal{D}^{\mathrm{b}}_G(\mathcal{M}_G), \mathcal{D}^{\mathrm{b}}_L(\mathcal{M}_L)$, and $\mathcal{D}^{\mathrm{b}}_T(\mathcal{M}_T)$ to get the commutative prism (7.6).

□

7.3 $\mathrm{der}\mathcal{S}^{\mathrm{sm}}_G$

In this section, we will set out to define an isomorphism for the square

$$\begin{tikzcd}
\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}_G) \arrow[r, "\mathrm{der}\mathcal{S}_G"] \arrow[d, "\mathfrak{R}^G_L"] & \mathrm{Coh}^{\check{G}\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{g}}^*) \arrow[d, "R^G_L"] \\
\mathrm{Semis}_{L(\mathfrak{O})}(\mathrm{Gr}_L) \arrow[r, "\mathrm{der}\mathcal{S}_L"] & \mathrm{Coh}^{\check{L}\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{l}}^*)
\end{tikzcd} \tag{7.11}$$

that is compatible with transitivity.

We will deduce this from the nonequivariant version in [AR15a]. Consider the following diagram

$$\begin{tikzcd}
\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}_G) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathrm{Coh}^{\check{G}\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{g}}^*) \arrow[dr] \arrow[dd] & \\
& \mathrm{Semis}_{L(\mathfrak{O})}(\mathrm{Gr}_L) \arrow[rr] \arrow[dd] & & \mathrm{Coh}^{\check{L}\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{l}}^*) \arrow[dd] \\
\mathrm{Semis}_{(G(\mathfrak{O}))}(\mathrm{Gr}_G) \arrow[rr] \arrow[dr] & & \mathrm{Coh}^{\check{G}\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathcal{N}}^*_G) \arrow[dr] & \\
& \mathrm{Semis}_{(L(\mathfrak{O}))}(\mathrm{Gr}_L) \arrow[rr] & & \mathrm{Coh}^{\check{L}\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathcal{N}}^*_L)
\end{tikzcd} \tag{7.12}$$

where the vertical functors on the coherent categories are quotient (or tensoring) functors.

We will produce a natural isomorphism for each of the faces except the top one. Then we will use these natural isomorphisms to produce one for Diagram (7.11) (the top face of Diagram (7.12)).

The left square commutes because hyperbolic localization (and in fact, all sheaf functors) commute with the forgetful functor (see [BL94]). The right square commutes because all of the functors in the square involve tensoring. The bottom square commutes by [AR15a, Theorem 2.6].

We will require some work to prove an isomorphism for the front and back squares of Diagram (7.12). They are essentially the same except for swapping the roles of G and L, so we will make our argument for the back square. To this end, consider the following triangular prism whose front face is the back square of Diagram (7.12).

$$\begin{tikzcd}
 & H_{G(\mathfrak{O})}(\mathrm{Gr}_G)-\mathrm{mod} \arrow[dd, "H^\bullet(\mathrm{Gr}_G)\otimes_{H^\bullet_{G(\mathfrak{O})}(\mathrm{Gr}_G)}-"] & \\
\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}_G) \arrow[ur, "\mathbb{H}^\bullet_{G(\mathfrak{O})}"] \arrow[rr] \arrow[dd] & & \mathrm{Coh}^{\check{G}\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{g}}^*) \arrow[ul, "\kappa"'] \arrow[dd] \\
 & H(\mathrm{Gr}_G)-\mathrm{mod} & \\
\mathrm{Semis}_{(G(\mathfrak{O}))}(\mathrm{Gr}_G) \arrow[ur, "\mathbb{H}^\bullet"] \arrow[rr] & & \mathrm{Coh}^{\check{G}\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathcal{N}}^*_G) \arrow[ul, "\Xi"']
\end{tikzcd} \tag{7.13}$$

As mentioned in Section 2.3, there exists an isomorphism of functors

$$\kappa \Longleftrightarrow \mathbb{H}^\bullet_{G(\mathfrak{O})} \circ \widetilde{\mathcal{S}}_G^{-1}.$$

Thus, the top triangle commutes. The back right square also commutes by an argument in [BF08]. The bottom triangle commutes by [YZ11] and [AR15b].

Commutativity of the back left square of Diagram (7.13) remains. To verify this, we will carefully study the interaction between forgetting equivariance and taking cohomology.

Theorem 7.3.1. *Consider the following diagram of categories and functors.*

$$\begin{tikzcd}
\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}_G) \arrow[r, "\mathbb{H}^\bullet_{G(\mathfrak{O})}"] \arrow[d, "\mathrm{For}"] & H^\bullet_{G(\mathfrak{O})}(\mathrm{Gr}_G)-\mathrm{mod} \arrow[d, "H^\bullet(\mathrm{Gr}_G)\otimes_{H^\bullet_{G(\mathfrak{O})}(\mathrm{Gr}_G)}-"] \\
\mathrm{Semis}_{(G(\mathfrak{O}))}(\mathrm{Gr}_G) \arrow[r, "\mathbb{H}^\bullet"] & H^\bullet(\mathrm{Gr}_G)-\mathrm{mod}
\end{tikzcd}$$

There exists a natural isomorphism of functors making this diagram commute.

Proof. Consider the following diagram which is an extension of the one in the statement of the theorem.

$$\begin{tikzcd}
\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}_G) \arrow[r, "\mathbb{H}^\bullet_{G(\mathfrak{O})}"] \arrow[dd, "\mathrm{For}"] & H^\bullet_{G(\mathfrak{O})}(\mathrm{Gr}_G)-\mathrm{mod} \arrow[r, "\mathrm{Rgd}"] \arrow[d, "H^\bullet(\mathrm{Gr}_G)\otimes_{H^\bullet_{G(\mathfrak{O})}(\mathrm{Gr}_G)}-"] & H^\bullet_{G(\mathfrak{O})}(\mathrm{pt})-\mathrm{mod} \subset \mathcal{D}(\mathcal{A}_G) \arrow[r, "\sim"] \arrow[d, "\mathbb{C}\otimes_{H^\bullet_{G(\mathfrak{O})}(\mathrm{pt})}-"] & \mathcal{D}^{\mathrm{b}}_G(\mathrm{pt}) \arrow[d, "\mathrm{For}"] \\
 & H^\bullet(\mathrm{Gr}_G)-\mathrm{mod} \arrow[r, "\mathrm{Rgd}"] \arrow[d, dashed] & H^\bullet(\mathrm{pt})-\mathrm{mod} \subset \mathcal{D}(\mathrm{Vect}_{\mathbb{C}}) \arrow[r, "\sim"] & \mathcal{D}^{\mathrm{b}}(\mathrm{pt}) \\
\mathrm{Semis}_{(G(\mathfrak{O}))}(\mathrm{Gr}_G) \arrow[r, "\mathbb{H}^\bullet"] & H^\bullet(\mathrm{Gr}_G)-\mathrm{mod} \arrow[ur, "\mathrm{Rgd}"'] & &
\end{tikzcd}$$

The right-most square commutes by [BL94]. The square whose horizontal functors are Rgd commutes by the fact that

$$H^\bullet(\mathrm{Gr}_G) \simeq \mathbb{C} \otimes_{H^\bullet_{G(\mathfrak{O})}(\mathrm{pt})} H^\bullet_{G(\mathfrak{O})}(\mathrm{Gr}_G). \tag{7.14}$$

The composition of all the functors along the top and bottom rows are just push-forwards to a point. Since sheaf functors commute with forgetting (see [BL94]), the subdiagram formed by traveling along the perimeter of the diagram also commutes.

Let $\mathcal{F} \in \mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}_G)$. In the previous paragraph, we constructed an isomorphism

$$\mathrm{Rgd}(H^\bullet(\mathrm{Gr}_G) \otimes_{H^\bullet_{G(\mathfrak{O})}(\mathrm{Gr}_G)} \mathbb{H}^\bullet_{G(\mathfrak{O})}(\mathcal{F})) \simeq \mathrm{Rgd}(\mathbb{H}^\bullet(\mathrm{For}(\mathcal{F}))). \tag{7.15}$$

We would like to "zip up" the diagram

$$\begin{tikzcd}
\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}_G) \arrow[r, "\mathbb{H}^\bullet_{G(\mathfrak{O})}"] \arrow[dd, "\mathrm{For}"] & H^\bullet_{G(\mathfrak{O})}(\mathrm{Gr}_G)\text{−mod} \arrow[d, "H^\bullet(\mathrm{Gr}_G)\otimes_{H^\bullet_{G(\mathfrak{O})}(\mathrm{Gr}_G)} -"] & \\
 & H^\bullet(\mathrm{Gr}_G)\text{−mod} \arrow[r, "\mathrm{Rgd}"] \arrow[d, dashed] & H^\bullet(\mathrm{pt})\text{−mod} \subset \mathcal{D}(\mathrm{Vect}_{\mathbb{C}}) \\
\mathrm{Semis}_{(G(\mathfrak{O}))}(\mathrm{Gr}_G) \arrow[r, "\mathbb{H}^\bullet"] & H^\bullet(\mathrm{Gr}_G)\text{−mod} \arrow[ur, "\mathrm{Rgd}"'] &
\end{tikzcd}$$

thereby proving that our isomorphism is in fact one of $H^\bullet(\mathrm{Gr}_G)$-modules rather than only $H^\bullet(\mathrm{pt})$-modules.

To this end, consider the following diagram which encodes the module structure on the objects in our diagram.

$$\begin{tikzcd}
\mathrm{Hom}^\bullet_{G(\mathfrak{O})}(\underline{\mathbb{C}}, \underline{\mathbb{C}}) \times \mathrm{Hom}^\bullet_{G(\mathfrak{O})}(\underline{\mathbb{C}}, \mathcal{F}) \arrow[r, "\circ"] \arrow[d, "\mathbb{C}\otimes_{H^\bullet_{G(\mathfrak{O})}(\mathrm{pt})} -"] \arrow[dd, bend right=60, "\mathrm{For}"'] & \mathrm{Hom}^\bullet_{G(\mathfrak{O})}(\underline{\mathbb{C}}, \mathcal{F}) \arrow[d, "\mathbb{C}\otimes_{H^\bullet_{G(\mathfrak{O})}(\mathrm{pt})} -"'] \arrow[dd, bend left=60, "\mathrm{For}"] \\
(\mathbb{C} \otimes_{H^\bullet_{G(\mathfrak{O})}(\mathrm{pt})} \mathrm{Hom}^\bullet_{G(\mathfrak{O})}(\underline{\mathbb{C}}, \underline{\mathbb{C}})) \times (\mathbb{C} \otimes_{H^\bullet_{G(\mathfrak{O})}(\mathrm{pt})} \mathrm{Hom}^\bullet_{G(\mathfrak{O})}(\underline{\mathbb{C}}, \mathcal{F})) \arrow[r, "\circ"] \arrow[d, "\sim"', "(7.14)"] & \mathbb{C} \otimes_{H^\bullet_{G(\mathfrak{O})}(\mathrm{pt})} \mathrm{Hom}^\bullet_{G(\mathfrak{O})}(\underline{\mathbb{C}}, \mathcal{F}) \arrow[d, "(7.14)"', "\sim"] \\
\mathrm{Hom}^\bullet(\underline{\mathbb{C}}, \underline{\mathbb{C}}) \times \mathrm{Hom}^\bullet(\underline{\mathbb{C}}, \mathcal{F}) \arrow[r, "\circ"] & \mathrm{Hom}^\bullet(\underline{\mathbb{C}}, \mathcal{F})
\end{tikzcd}$$

Commutativity of this diagram is equivalent to (7.15) being an isomorphism of $H^\bullet(\mathrm{Gr}_G)$-modules instead of only $H^\bullet(\mathrm{pt})$-modules. But the diagram certainly commutes since For is a quotient functor by the positive degree elements in $H^\bullet_{G(\mathfrak{O})}(\mathrm{pt})$; therefore, it coincides with the functor $\mathbb{C} \otimes_{H^\bullet_{G(\mathfrak{O})}(\mathrm{pt})} -$. $\square$

The functors κ and Ξ in Diagram (7.13) are equivalences—thus, κ^{-1} and Ξ^{-1} exist. The new triangular prism with κ and Ξ reversed still has all the faces except the front one commuting. This implies the commutativity of the front face as well.

Now we return to Diagram (7.12). We have just constructed a natural isomorphism for the front and back faces. Our goal was to construct a natural isomorphism for the top face.

Theorem 7.3.2. *There exists a natural isomorphism of functors*

$$R_L^G \circ \mathrm{der}\mathcal{S}_G \Longleftrightarrow \mathrm{der}\mathcal{S}_L \circ \mathfrak{R}_L^G \tag{7.16}$$

making Diagram (7.11):

$$\begin{tikzcd}
\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}_G) \arrow[r, "\mathrm{der}\mathcal{S}_G"] \arrow[d, "\mathfrak{R}_L^G"] & \mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*) \arrow[d, "R_L^G"] \\
\mathrm{Semis}_{L(\mathfrak{O})}(\mathrm{Gr}_L) \arrow[r, "\mathrm{der}\mathcal{S}_L"] & \mathrm{Coh}_{\mathrm{fr}}^{\check{L}\times\mathbb{G}_m}(\check{\mathfrak{l}}^*)
\end{tikzcd}$$

commute.

Proof. We have constructed natural isomorphisms for all faces in Diagram (7.12) except for the top face. Thus, if $\mathcal{F} \in \mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}_G)$, we have an isomorphism

$$\mathrm{der}\mathcal{S}_L(\mathfrak{R}_L^G(\mathcal{F})) \otimes_{\mathcal{O}_{\check{\mathfrak{l}}^*}} \mathcal{O}_{\check{\mathcal{N}}_L^*} \simeq R_L^G(\mathrm{der}\mathcal{S}_G(\mathcal{F})) \otimes_{\mathcal{O}_{\check{\mathfrak{l}}^*}} \mathcal{O}_{\check{\mathcal{N}}_L^*}.$$

We would like to lift this isomorphism (i.e. erase $\otimes_{\mathcal{O}_{\check{\mathfrak{l}}^*}} \mathcal{O}_{\check{\mathcal{N}}_L^*}$ from both sides). Thus we have

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(V(\lambda) \otimes \mathcal{O}_{\check{\mathfrak{l}}^*}, V(\lambda) \otimes \mathcal{O}_{\check{\mathfrak{l}}^*}) \arrow[d] \\
\underline{\mathrm{Hom}}(V(\lambda) \otimes \mathcal{O}_{\check{\mathcal{N}}_L^*}, V(\lambda) \otimes \mathcal{O}_{\check{\mathcal{N}}_L^*})
\end{tikzcd}$$

where

$$\underline{\mathrm{Hom}}(V(\lambda) \otimes \mathcal{O}_{\check{\mathfrak{l}}^*}, V(\lambda) \otimes \mathcal{O}_{\check{\mathfrak{l}}^*}) \text{ and } \underline{\mathrm{Hom}}(V(\lambda) \otimes \mathcal{O}_{\check{\mathcal{N}}_L^*}, V(\lambda) \otimes \mathcal{O}_{\check{\mathcal{N}}_L^*})$$

are free $\mathcal{O}_{\check{\mathfrak{l}}^*}$- and $\mathcal{O}_{\check{\mathcal{N}}_L^*}$-modules, respectively. And the map is quotienting by the ideal of the nilpotent cone, which is generated by positive degree elements. Thus this is an isomorphism in degree zero. Hence, our isomorphism lifts to a unique isomorphism, and we are done. □

7.4 $\mathrm{der}\mathbb{S}_G$

Our goal is to define an isomorphism for the square

$$\begin{tikzcd} \mathrm{Semis}_G(\mathcal{N}_G) \arrow[r, "\mathrm{der}\mathbb{S}_G"] \arrow[d, "\widetilde{\mathcal{R}}^G_L"] & \mathrm{Coh}^{W_G\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{h}}^*) \arrow[d, "R^{W_G}_{W_L}"] \\ \mathrm{Semis}_L(\mathcal{N}_L) \arrow[r, "\mathrm{der}\mathbb{S}_L"] & \mathrm{Coh}^{W_L\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{h}}^*) \end{tikzcd}$$

that is compatible with transitivity of restriction from (6.4) and (6.8).

It suffices to define an intertwining isomorphism

$$\begin{tikzcd} \mathcal{D}^{\mathrm{b}}_G(\mathcal{N}_G) \arrow[r, "\mathrm{der}\mathbb{S}_G"] \arrow[d, "\widetilde{\mathcal{R}}^G_L"] & \mathrm{Coh}^{W_G\times\mathbb{G}_m}(\check{\mathfrak{h}}^*) \arrow[d, "R^{W_G}_{W_L}"] \\ \mathcal{D}^{\mathrm{b}}_L(\mathcal{N}_L) \arrow[r, "\mathrm{der}\mathbb{S}_L"] & \mathrm{Coh}^{W_L\times\mathbb{G}_m}(\check{\mathfrak{h}}^*) \end{tikzcd} \tag{7.17}$$

that is compatible with transitivity of restriction.

Now, define a category $\mathbb{A}_W$ whose objects are integers and whose morphisms are given by

$$\mathrm{Hom}(n,m) = \begin{cases} 0 & \text{if } m < n \text{ or if } m \not\equiv n (\mathrm{mod}\ 2) \\ W \ltimes \mathrm{Sym}^{\frac{m-n}{2}}(\mathbb{X}) & \text{if } m \geq n \text{ and } m \equiv n (\mathrm{mod}\ 2). \end{cases}$$

This category comes equipped with an autoequivalence [1] such that $n[1] = n+1$. A number of objects of interest have a categorical description in terms of $\mathbb{A}_W$. For example, we can rewrite the Springer sheaf $\underline{\mathrm{Spr}}_G$ as the functor

$$\begin{aligned} \mathbb{A}_W &\xrightarrow{\mathrm{Spr}_G} \mathcal{D}^{\mathrm{b}}_G(\mathcal{N}_G) \\ n &\longmapsto \underline{\mathrm{Spr}}_G[n] \end{aligned}$$

Lemma 7.4.1. *Let $\mathrm{Vect}_{\mathbb{C}}^{\mathbb{A}_W}$ be the category of functors $\mathbb{A}_W \to \mathrm{Vect}_{\mathbb{C}}$. There is an equivalence of categories $\mathrm{Vect}_{\mathbb{C}}^{\mathbb{A}_W} \xrightarrow{\sim} \mathrm{Coh}^{W\times\mathbb{G}_m}(\check{\mathfrak{h}}^*)$.*

Proof. Let $F : \mathbb{A}_W \to \mathrm{Vect}_{\mathbb{C}}$ be a functor. Then, $M = \bigoplus_{n\in\mathbb{Z}} F(n)$ is the underlying vector space of the module. A morphism in $\mathbb{A}_W$ determines the action of W and of $\mathcal{O}_{\check{\mathfrak{h}}^*}$. $\square$

Armed with this categorical machinery, we can factor $\mathrm{der}\mathbb{S}_G$ as

$$\mathcal{D}^{\mathrm{b}}_G(\mathcal{N}_G) \xrightarrow{\mathbb{Y}} \mathrm{Vect}_{\mathbb{C}}^{\mathcal{D}^{\mathrm{b}}_G(\mathcal{N}_G)^{\mathrm{op}}} \xrightarrow{\circ \mathrm{Spr}_G} \mathrm{Vect}_{\mathbb{C}}^{\mathbb{A}_W}$$

where $\mathbb{Y}$ is the Yoneda embedding. Doing this for both G and L and considering restriction functors, we get the following diagram:

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{b}}_G(\mathcal{N}_G) \arrow[r, "\mathbb{Y}"] \arrow[d, "\widetilde{\mathcal{R}}^G_L"] & \mathrm{Vect}_{\mathbb{C}}^{\mathcal{D}^{\mathrm{b}}_G(\mathcal{N}_G)^{\mathrm{op}}} \arrow[r, "\circ \mathrm{Spr}_G"] \arrow[d, "\circ(\widetilde{\mathcal{I}}^G_L)^{\mathrm{op}}"] & \mathrm{Vect}_{\mathbb{C}}^{\mathbb{A}_{W_G}} \arrow[d] \\
\mathcal{D}^{\mathrm{b}}_L(\mathcal{N}_L) \arrow[r, "\mathbb{Y}"] & \mathrm{Vect}_{\mathbb{C}}^{\mathcal{D}^{\mathrm{b}}_L(\mathcal{N}_L)^{\mathrm{op}}} \arrow[r, "\circ \mathrm{Spr}_L"] & \mathrm{Vect}_{\mathbb{C}}^{\mathbb{A}_{W_L}}
\end{tikzcd} \tag{7.18}$$

Defining an isomorphism for the outside rectangle in Diagram (7.18) is equivalent to defining one for Diagram (7.17). We will give a natural isomorphism for the left rectangle first, and then show that it is compatible with transitivity of restriction.

Lemma 7.4.2. *Consider the following diagram:*

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{b}}_G(\mathcal{N}_G) \arrow[r, "\mathbb{Y}"] \arrow[d, "\widetilde{\mathcal{R}}^G_L"] & \mathrm{Vect}_{\mathbb{C}}^{\mathcal{D}^{\mathrm{b}}_G(\mathcal{N}_G)^{\mathrm{op}}} \arrow[d, "\circ(\widetilde{\mathcal{I}}^G_L)^{\mathrm{op}}"] \\
\mathcal{D}^{\mathrm{b}}_L(\mathcal{N}_L) \arrow[r, "\mathbb{Y}"] & \mathrm{Vect}_{\mathbb{C}}^{\mathcal{D}^{\mathrm{b}}_L(\mathcal{N}_L)^{\mathrm{op}}}
\end{tikzcd}$$

There exists a natural isomorphism of functors making this square commute.

Proof. Consider the following diagram:

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{b}}_G(\mathcal{N}_G) \arrow[r, "\mathrm{For}^G_P"] \arrow[d, "\mathbb{Y}"] & \mathcal{D}^{\mathrm{b}}_P(\mathcal{N}_G) \arrow[r, "(m_P)^!"] \arrow[d, "\mathbb{Y}"] & \mathcal{D}^{\mathrm{b}}_P(\mathcal{N}_P) \arrow[r, "(p_P)_*"] \arrow[d, "\mathbb{Y}"] & \mathcal{D}^{\mathrm{b}}_P(\mathcal{N}_L) \arrow[r, "\mathrm{For}^P_L"] \arrow[d, "\mathbb{Y}"] & \mathcal{D}^{\mathrm{b}}_L(\mathcal{N}_L) \arrow[d, "\mathbb{Y}"] \\
\mathrm{Vect}_{\mathbb{C}}^{\mathcal{D}^{\mathrm{b}}_G(\mathcal{N}_G)^{\mathrm{op}}} \arrow[r, "\circ(\gamma^G_P)^{\mathrm{op}}"] & \mathrm{Vect}_{\mathbb{C}}^{\mathcal{D}^{\mathrm{b}}_P(\mathcal{N}_G)^{\mathrm{op}}} \arrow[r, "\circ(m_P)_!^{\mathrm{op}}"] & \mathrm{Vect}_{\mathbb{C}}^{\mathcal{D}^{\mathrm{b}}_P(\mathcal{N}_P)^{\mathrm{op}}} \arrow[r, "\circ(p_P)^{*,\mathrm{op}}"] & \mathrm{Vect}_{\mathbb{C}}^{\mathcal{D}^{\mathrm{b}}_P(\mathcal{N}_L)^{\mathrm{op}}} \arrow[r, "\circ(\gamma^P_L)^{\mathrm{op}}"] & \mathrm{Vect}_{\mathbb{C}}^{\mathcal{D}^{\mathrm{b}}_L(\mathcal{N}_L)^{\mathrm{op}}}
\end{tikzcd}$$

Each of the faces are labeled by an adjunction isomorphism, and their composition gives the desired natural isomorphism. □

Lemma 7.4.3. *The triangular prism below is commutative.*

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{b}}_{G}(\mathcal{N}_G) \arrow[rr, "\mathbb{Y}"] \arrow[dr, "\widetilde{\mathcal{R}}^G_L"] \arrow[dd, "\widetilde{\mathcal{R}}^G_T"'] & & \mathrm{Vect}_{\mathbb{C}}^{\mathcal{D}^{\mathrm{b}}_{G}(\mathcal{N}_G)^{\mathrm{op}}} \arrow[dr, "\circ(\widetilde{\mathcal{I}}^G_L)^{\mathrm{op}}"] \arrow[dd, "\circ(\widetilde{\mathcal{I}}^G_T)^{\mathrm{op}}" near start] & \\
& \mathcal{D}^{\mathrm{b}}_{L}(\mathcal{N}_L) \arrow[rr, "\mathbb{Y}" near start] \arrow[dl, "\widetilde{\mathcal{R}}^L_T"] & & \mathrm{Vect}_{\mathbb{C}}^{\mathcal{D}^{\mathrm{b}}_{L}(\mathcal{N}_L)^{\mathrm{op}}} \arrow[dl, "\circ(\widetilde{\mathcal{I}}^L_T)^{\mathrm{op}}"] \\
\mathcal{D}^{\mathrm{b}}_{T}(\mathcal{N}_T) \arrow[rr, "\mathbb{Y}"] & & \mathrm{Vect}_{\mathbb{C}}^{\mathcal{D}^{\mathrm{b}}_{T}(\mathcal{N}_T)^{\mathrm{op}}} &
\end{tikzcd}$$

Proof. For the left-hand triangle and right-hand triangle, we use the natural isomorphisms (6.4) and (6.5), respectively. The front two rectangles commute using the natural isomorphism constructed in Lemma 7.4.2 for G and for L. Thus, we can define a natural isomorphism for the back face using the natural isomorphisms for the other four faces, and hence the triangular prism commutes. $\square$

It remains to produce a natural isomorphism for the right square in Diagram (7.18) and show that this isomorphism is compatible with transitivity of restriction. This is equivalent to exhibiting commutativity of the diagram

$$\begin{tikzcd}
\mathbb{A}_{W_G} \arrow[r, "\mathrm{Spr}_G"] & \mathcal{D}^{\mathrm{b}}_{G}(\mathcal{N}_G) \\
\mathbb{A}_{W_L} \arrow[r, "\mathrm{Spr}_L"] \arrow[u] & \mathcal{D}^{\mathrm{b}}_{L}(\mathcal{N}_L) \arrow[u, "\widetilde{\mathcal{I}}^G_L"']
\end{tikzcd}$$

as well as of the following prism

$$\begin{tikzcd}
\mathbb{A}_{W_G} \arrow[rr, "\mathrm{Spr}_G"] & & \mathcal{D}^{\mathrm{b}}_{G}(\mathcal{N}_G) & \\
& \mathbb{A}_{W_L} \arrow[ul] \arrow[rr, "\mathrm{Spr}_L" near start] & & \mathcal{D}^{\mathrm{b}}_{L}(\mathcal{N}_L) \arrow[ul, "\widetilde{\mathcal{I}}^G_L"'] \\
\mathbb{A}_{W_T} \arrow[uu] \arrow[ur] \arrow[rr, "\mathrm{Spr}_T"] & & \mathcal{D}^{\mathrm{b}}_{T}(\mathcal{N}_T) \arrow[uu, "\widetilde{\mathcal{I}}^G_T" near start] \arrow[ur, "\widetilde{\mathcal{I}}^L_T"] &
\end{tikzcd}$$

Rather than doing this, it is easier to restate the problem in the following way: define a $(W_L \times \mathbb{X})$-equivariant isomorphism $\widetilde{\mathcal{I}}^G_L(\underline{\mathrm{Spr}}_L) \xrightarrow{\sim} \underline{\mathrm{Spr}}_G$ such that the following square commutes

$$\begin{tikzcd} \widetilde{\mathcal{I}}_L^G(\underline{\mathrm{Spr}}_L) \arrow[r, "\sim"] & \underline{\mathrm{Spr}}_G \\ \widetilde{\mathcal{I}}_L^G\widetilde{\mathcal{I}}_T^L(\underline{\mathrm{Spr}}_T) \arrow[u, "\widetilde{\mathcal{I}}_L^G\phi_L"', "\sim"] \arrow[r, "\sim", "\widetilde{\mathcal{I}}_T^G=\widetilde{\mathcal{I}}_L^G\widetilde{\mathcal{I}}_T^L"'] & \widetilde{\mathcal{I}}_T^G(\underline{\mathrm{Spr}}_T) \arrow[u, "\phi_G", "\sim"'] \end{tikzcd}$$

Lemma 7.4.4. *There exists a $(W_L \times \mathbb{X})$-equivariant isomorphism $\widetilde{\mathcal{I}}_L^G(\underline{\mathrm{Spr}}_L) \xrightarrow{\sim} \underline{\mathrm{Spr}}_G$ such that the following square commutes*

$$\begin{tikzcd} \widetilde{\mathcal{I}}_L^G(\underline{\mathrm{Spr}}_L) \arrow[r, "\sim", "{[AHR15]}"'] & \underline{\mathrm{Spr}}_G \\ \widetilde{\mathcal{I}}_L^G\widetilde{\mathcal{I}}_T^L(\underline{\mathrm{Spr}}_T) \arrow[u, "\widetilde{\mathcal{I}}_L^G\phi_L"', "\sim"] \arrow[r, "\sim", "\widetilde{\mathcal{I}}_T^G=\widetilde{\mathcal{I}}_L^G\widetilde{\mathcal{I}}_T^L"'] & \widetilde{\mathcal{I}}_T^G(\underline{\mathrm{Spr}}_T) \arrow[u, "\phi_G", "\sim"'] \end{tikzcd}$$

Proof. In [AHR15], a W_L-equivariant isomorphism $\widetilde{\mathcal{I}}_L^G(\underline{\mathrm{Spr}}_L) \xrightarrow{\sim} \underline{\mathrm{Spr}}_G$ is constructed which makes the square commute. It remains to show that this isomorphism is also $\mathbb{X}$-equivariant. However, we will instead show that the isomorphisms $\phi_G : \widetilde{\mathcal{I}}_T^G(\underline{\mathrm{Spr}}_T) \xrightarrow{\sim} \underline{\mathrm{Spr}}_G$ and $\phi_L : \widetilde{\mathcal{I}}_T^L\underline{\mathrm{Spr}}_T \xrightarrow{\sim} \underline{\mathrm{Spr}}_L$ are $\mathbb{X}$-equivariant and deduce from this the $\mathbb{X}$-equivariance of $\widetilde{\mathcal{I}}_L^G(\underline{\mathrm{Spr}}_L) \xrightarrow{\sim} \underline{\mathrm{Spr}}_G$ using Lemma 7.4.6.

We focus on verifying the $\mathbb{X}$-equivariance of ϕ_G (the argument for ϕ_L is similar). Consider the following diagram:

$$\begin{tikzcd} \mathbb{X} \arrow[r, hook, "p"] & \mathrm{Hom}^2_{\mathcal{D}^{\mathrm{b}}_T(\mathcal{N}_T)}(\underline{\mathrm{Spr}}_T, \underline{\mathrm{Spr}}_T) \arrow[d, "\Theta"] \arrow[dr, "\widetilde{\mathcal{I}}_T^G"] & \\ & \mathrm{Hom}^2_{\mathcal{D}^{\mathrm{b}}_G(\widetilde{\mathcal{N}}_G)}(\underline{\mathbb{C}}_{\widetilde{\mathcal{N}}_G}, \underline{\mathbb{C}}_{\widetilde{\mathcal{N}}_G}) \arrow[r, "\mu_*"] & \mathrm{Hom}^2_{\mathcal{D}^{\mathrm{b}}_G(\mathcal{N}_G)}(\underline{\mathrm{Spr}}_G, \underline{\mathrm{Spr}}_G) \\ & \mathbb{X} \arrow[u, hook, "q"] & \end{tikzcd} \tag{7.19}$$

This diagram describes the action of $\mathbb{X}$ on both $\widetilde{\mathcal{I}}_T^G(\underline{\mathrm{Spr}}_T)$ and $\underline{\mathrm{Spr}}_G$. That is, the action of $\mathbb{X}$ on $\widetilde{\mathcal{I}}_T^G(\underline{\mathrm{Spr}}_T)$ is given by the composition $\widetilde{\mathcal{I}}_T^G \circ p$, and the action of $\mathbb{X}$ on $\underline{\mathrm{Spr}}_G$ is given by the composition $\mu_* \circ q$. Thus, proving that $\phi_G : \widetilde{\mathcal{I}}_T^G(\underline{\mathrm{Spr}}_T) \xrightarrow{\sim} \underline{\mathrm{Spr}}_G$ is $\mathbb{X}$-equivariant is equivalent to verifying the following:

1. The triangle commutes (i.e., $\mu_* \circ \Theta \Longleftrightarrow \widetilde{\mathcal{I}}_T^G$), and

2. the images of $\mathbb{X}$ under $\Theta \circ p$ and under q coincide in $\mathrm{Hom}^2_{\mathcal{D}^{\mathrm{b}}_G(\widetilde{\mathcal{N}}_G)}(\underline{\mathbb{C}}_{\widetilde{\mathcal{N}}_G}, \underline{\mathbb{C}}_{\widetilde{\mathcal{N}}_G})$.

First we focus on proving (1) above. The functor Θ is given by composing the functors in the following diagram.

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{b}}_T(\mathcal{N}_T) \arrow[r, "\gamma_T^B", "\sim"'] & \mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_T) \arrow[r, "(p_B)^*"] & \mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_B) \arrow[d, "\text{i.e.}"] \\
 & & \mathcal{D}^{\mathrm{b}}_G(\widetilde{\mathcal{N}}_G)
\end{tikzcd}$$

Now recall that $\widetilde{\mathcal{I}}_T^G$ is defined by the following composition:

$$\mathcal{D}^{\mathrm{b}}_G(\mathcal{N}_G) \xleftarrow{\gamma_B^G} \mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_G) \xleftarrow{(m_B)_!} \mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_B) \xleftarrow{(p_B)^*} \mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_T) \xleftarrow{\gamma_T^B} \mathcal{D}^{\mathrm{b}}_T(\mathcal{N}_T).$$

Our goal is to prove that $\mu_* \circ \Theta \Longleftrightarrow \widetilde{\mathcal{I}}_T^G$, but since the first two functors in both definitions are equal, this reduces to showing that there exists a natural isomorphism $\gamma_B^G \circ (m_B)_! \Longleftrightarrow \mu_* \circ$ i.e. Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{N}_B \arrow[r, "i_B"] \arrow[rr, bend left, "m_B"] & \widetilde{\mathcal{N}}_G := G \times^B \mathcal{N}_B \arrow[r, "\mu"] & \mathcal{N}_G,
\end{tikzcd} \tag{7.20}$$

where i_B is the inclusion map. With this in mind, we need to prove that there exists a natural isomorphism making the following diagram commute.

$$\begin{tikzcd}
 & \mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_B) \arrow[dl, "(i_B)_!"] \arrow[dd, "\text{i.e.}"] \arrow[r, "(m_B)_!"] & \mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_G) \arrow[dd, "\gamma_B^G"] \\
\mathcal{D}^{\mathrm{b}}_B(\widetilde{\mathcal{N}}_G) \arrow[dr, "\gamma_B^G"] & & \\
 & \mathcal{D}^{\mathrm{b}}_G(\widetilde{\mathcal{N}}_G) \arrow[r, "\mu_* \simeq \mu_!"] & \mathcal{D}^{\mathrm{b}}_G(\mathcal{N}_G)
\end{tikzcd}$$

To do this, consider the diagram gotten by taking right adjoints of each functor:

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_B) & \mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_G) \arrow[l, "(m_B)^!"'] \arrow[dl, "\mu^!"] \\
\mathcal{D}^{\mathrm{b}}_B(\widetilde{\mathcal{N}}_G) \arrow[u, "(i_B)^!"] & \\
\mathcal{D}^{\mathrm{b}}_G(\widetilde{\mathcal{N}}_G) \arrow[u, "\mathrm{For}^G_B"] \arrow[uu, bend left=60, "\text{i.e.}"] & \mathcal{D}^{\mathrm{b}}_G(\mathcal{N}_G) \arrow[l, "\mu^!"] \arrow[uu, "\mathrm{For}^G_B"']
\end{tikzcd}$$

The bottom quadrilaterial commutes because sheaf functors commute with forgetting [BL94], the left triangle commutes by the definition of induction equivalence [BL94], and the top triangle commutes by Diagram (7.20). We have constructed a natural isomorphism

$$(m_B)^! \circ \mathrm{For}^G_B \Longleftrightarrow \text{i.e.} \circ \mu^!. \tag{7.21}$$

Since $\gamma^G_B \circ (m_B)_!$ and $\mu_! \circ$i.e. are the right adjoints of the left and right hand sides of (7.21) respectively, and right adjoints are unique, we deduce that there is a unique natural isomorphism induced from (7.21) that makes the diagram commute. Hence, we have constructed a natural isomorphism $\mu_* \circ \Theta \Longleftrightarrow \widetilde{\mathcal{I}}^G_T$.

Now, we will verify (2). First, we will define p and q from Diagram (7.19). Note that $\mathrm{Hom}^2_{\mathcal{D}^{\mathrm{b}}_T(\mathcal{N}_T)}(\underline{\mathrm{Spr}}_T, \underline{\mathrm{Spr}}_T) = \check{\mathfrak{h}}^*$. Thus, p is just the inclusion map. For q, recall that $\mathbb{X}$ can be viewed as the collection of line bundles on G/B. Thus, $q(\mathcal{L}_\lambda) = \mathrm{Lus}_{\pi^*\mathcal{L}_\lambda}$ (see Section 5.4). Consider the following diagram of functors whose composition also defines the functor Θ.[1]

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{b}}_T(\mathcal{N}_T) \arrow[r, "\gamma^B_T", "\sim"'] & \mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_T) \arrow[d, "\text{ind. equiv.}"', "\sim"] & \\
& \mathcal{D}^{\mathrm{b}}_G(G/B) \arrow[r, "\pi^*"] & \mathcal{D}^{\mathrm{b}}_G(\widetilde{\mathcal{N}}_G)
\end{tikzcd}$$

[1]In Lemma 7.4.5, we verify that the two given definitions of Θ coincide.

It is clear using this definition of Θ that the images of $\mathbb{X}$ under $\Theta \circ p$ and under q coincide in $\mathrm{Hom}^2_{\mathcal{D}^{\mathrm{b}}_G(\widetilde{\mathcal{N}}_G)}(\underline{\mathbb{C}}_{\widetilde{\mathcal{N}}_G}, \underline{\mathbb{C}}_{\widetilde{\mathcal{N}}_G})$. $\square$

Lemma 7.4.5. *The two definitions of the functor Θ coincide.*

Proof. We use a similar argument (using right adjoints) as in the above proof. Recall that the two definitions of Θ are given by the following two compositions

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{b}}_T(\mathcal{N}_T) \arrow[r, "\gamma_T^B", "\sim"'] & \mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_T) \arrow[d, "\text{ind. equiv.}"', "\sim"] & \\
& \mathcal{D}^{\mathrm{b}}_G(G/B) \arrow[r, "\pi^*"] & \mathcal{D}^{\mathrm{b}}_G(\widetilde{\mathcal{N}}_G)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{b}}_T(\mathcal{N}_T) \arrow[r, "\gamma_T^B", "\sim"'] & \mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_T) \arrow[r, "(p_B)^*"] & \mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_B) \arrow[d, "\sim"', "\text{i.e.}"] \\
& & \mathcal{D}^{\mathrm{b}}_G(\widetilde{\mathcal{N}}_G)
\end{tikzcd}$$

That is, we must show that the following square commutes:

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_T) \arrow[r, "(p_B)^*"] \arrow[d, "\text{ind. equiv.}"', "\sim"] & \mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_B) \arrow[d, "\sim"', "\text{i.e.}"] \\
\mathcal{D}^{\mathrm{b}}_G(G/B) \arrow[r, "\pi^*"] & \mathcal{D}^{\mathrm{b}}_G(\widetilde{\mathcal{N}}_G)
\end{tikzcd} \tag{7.22}$$

Factor ind. equiv. and i.e. as $\gamma_B^G \circ (i_T)_!$ and $\gamma_B^G \circ (i_B)_!$ respectively, where $i_T : \mathcal{N}_T \hookrightarrow G/B$ is the inclusion of a point. Take the right adjoints of each functor to get:

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_T) & \mathcal{D}^{\mathrm{b}}_B(\mathcal{N}_B) \arrow[l, "(p_B)_*"'] \\
\mathcal{D}^{\mathrm{b}}_B(G/B) \arrow[u, "(i_T)^!"] & \mathcal{D}^{\mathrm{b}}_B(\widetilde{\mathcal{N}}_G) \arrow[l, "\pi_*"] \arrow[u, "(i_B)^!"'] \\
\mathcal{D}^{\mathrm{b}}_G(G/B) \arrow[u, "\mathrm{For}_B^G"] \arrow[uu, bend left=60, "\text{ind. equiv.}"] & \mathcal{D}^{\mathrm{b}}_G(\widetilde{\mathcal{N}}_G) \arrow[l, "\pi_*"] \arrow[u, "\mathrm{For}_B^G"'] \arrow[uu, bend right=60, "\text{i.e.}"']
\end{tikzcd}$$

The bottom square commutes because sheaf functors commute with forgetting [BL94], and the top square commutes by base change since

$$\begin{tikzcd}
\mathcal{N}_B \arrow[r, hook, "i_B"] \arrow[d, "p_B"'] & \widetilde{\mathcal{N}}_G \arrow[d, "\pi"] \\
\mathcal{N}_T = \mathrm{pt} \arrow[r, hook, "i_T"] & G/B
\end{tikzcd}$$

is a Cartesian square. Thus, the diagram of right adjoints commutes, which implies that Diagram (7.22) also commutes. $\square$

Lemma 7.4.6. *If the isomorphisms $\phi_G : \widetilde{\mathcal{I}}_T^G \underline{\mathrm{Spr}}_T \xrightarrow{\sim} \underline{\mathrm{Spr}}_G$ and $\phi_L : \widetilde{\mathcal{I}}_T^L \underline{\mathrm{Spr}}_T \xrightarrow{\sim} \underline{\mathrm{Spr}}_L$ are $\mathbb{X}$-equivariant, then the isomorphism $\widetilde{\mathcal{I}}_L^G \underline{\mathrm{Spr}}_L \xrightarrow{\sim} \underline{\mathrm{Spr}}_G$ constructed in [AHR15] is $\mathbb{X}$-equivariant.*

Proof. First note that each isomorphism $f : A \to B$ in the statement of Lemma 7.4.4 induces an isomorphism $\mathrm{End}^\bullet(A) \xrightarrow{\sim} \mathrm{End}^\bullet(B)$ by precomposition with f^{-1} and postcomposition with f. The action of X on A is a map $X \to \mathrm{End}^2(A)$ that is injective for every case in Lemma 7.4.4. The isomorphism $f : A \xrightarrow{\sim} B$ being $\mathbb{X}$-equivariant is equivalent to $\mathbb{X} \subset \mathrm{End}^2(A)$ being identified with $\mathbb{X} \subset \mathrm{End}^2(B)$ under the induced isomorphism $\mathrm{End}^\bullet(A) \xrightarrow{\sim} \mathrm{End}^\bullet(B)$. In this case, we will say that the induced isomorphism is $\mathbb{X}$-matching.

Now, consider the following diagram of induced maps

$$\begin{tikzcd}
 & \mathrm{End}^\bullet(\widetilde{\mathcal{I}}_L^G \underline{\mathrm{Spr}}_L) \arrow[r, "\sim", "\text{[AHR15]}"'] & \mathrm{End}^\bullet(\underline{\mathrm{Spr}}_G) \\
\mathrm{End}^\bullet(\underline{\mathrm{Spr}}_L) \arrow[ur, "\widetilde{\mathcal{I}}_L^G"] & \mathrm{End}^\bullet(\widetilde{\mathcal{I}}_L^G \widetilde{\mathcal{I}}_T^L \underline{\mathrm{Spr}}_T) \arrow[u, "\sim", "\widetilde{\mathcal{I}}_L^G \phi_L"'] \arrow[r, "\sim", "\widetilde{\mathcal{I}}_T^G = \widetilde{\mathcal{I}}_L^G \widetilde{\mathcal{I}}_T^L"'] & \mathrm{End}^\bullet(\widetilde{\mathcal{I}}_T^G \underline{\mathrm{Spr}}_T) \arrow[u, "\phi_G", "\sim"'] \\
 & \mathrm{End}^\bullet(\widetilde{\mathcal{I}}_T^L \underline{\mathrm{Spr}}_T) \arrow[u, "\widetilde{\mathcal{I}}_L^G"'] \arrow[ul, "\sim", "\phi_L"'] & \mathrm{End}^\bullet(\underline{\mathrm{Spr}}_T) \arrow[l, "\widetilde{\mathcal{I}}_T^L"'] \arrow[u, "\widetilde{\mathcal{I}}_T^G"] \arrow[uu, bend right=60, "\mathbb{X}\text{-matching by def.}"'] \arrow[ll, bend left=40, "\mathbb{X}\text{-matching by def.}"]
\end{tikzcd} \tag{7.23}$$

The morphisms $\widetilde{\mathcal{I}}_L^G, \widetilde{\mathcal{I}}_T^L$, and $\widetilde{\mathcal{I}}_T^G$ are $\mathbb{X}$-matching because the $\mathbb{X}$-action on $\widetilde{\mathcal{I}}_L^G \underline{\mathrm{Spr}}_L$, $\widetilde{\mathcal{I}}_T^L \underline{\mathrm{Spr}}_T$, and $\widetilde{\mathcal{I}}_T^G \underline{\mathrm{Spr}}_T$ are defined using these respective morphisms. By assumption, we have that ϕ_G and ϕ_L are $\mathbb{X}$-matching. The left square (which looks like a triangle) clearly commutes, and the top square commutes by [AHR15].

We will prove that the isomorphism labeled by [AHR15] is $\mathbb{X}$-matching. Since $\widetilde{\mathcal{I}}_L^G$ and ϕ_L are $\mathbb{X}$-matching and the left square is commutative, it follows that $\widetilde{\mathcal{I}}_L^G \phi_L$ is $\mathbb{X}$-matching. Now, since $\widetilde{\mathcal{I}}_L^G \phi_L, \widetilde{\mathcal{I}}_T^G$, and ϕ_G are $\mathbb{X}$-matching and the top square is commutative, it follows that the isomorphism labeled by [AHR15] is $\mathbb{X}$-matching. Hence, the isomorphism $\widetilde{\mathcal{I}}_L^G \underline{\mathrm{Spr}}_L \xrightarrow{\sim} \underline{\mathrm{Spr}}_G$ constructed in [AHR15] is $\mathbb{X}$-equivariant. $\square$

The previous lemmas in this section, when taken together, give a construction of an intertwining natural isomorphism

$$R_{W_L}^{W_G} \circ \mathrm{der}\mathbb{S}_G \Longleftrightarrow \mathrm{der}\mathbb{S}_L \circ \widetilde{\mathcal{R}}_L^G. \tag{7.24}$$

Chapter 8
The Kostant Functor κ

In this section, we develop what is needed to define the functor $\kappa : \mathrm{Coh}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*) \to \mathrm{Coh}^{\mathbb{G}_m}(\mathbf{T}(\check{\mathfrak{h}}^*/W))$ (see [BF08]) mentioned in Section 2.3. A similar explanation was recently worked out independently by S. Riche [Ric14].

Let $M := V \otimes \mathcal{O}_{\check{\mathfrak{g}}^*}$ be a free $\check{G} \times \mathbb{G}_m$-equivariant coherent sheaf on $\check{\mathfrak{g}}^*$. We may instead view M as the space of regular maps

$$M = \{u : \check{\mathfrak{g}}^* \to V\}.$$

Note that $\mathcal{O}_{\check{\mathfrak{g}}^*}, \check{G}$, and $\check{\mathfrak{g}}$ all act on M. We will elaborate on the action of $\mathcal{O}_{\check{\mathfrak{g}}^*}$. Let $r \in \mathcal{O}_{\check{\mathfrak{g}}^*}$ and $\lambda \in \check{\mathfrak{g}}^*$. Then,

$$(r \cdot u)(\lambda) = r(\lambda)u(\lambda).$$

Define a sheaf $\mathfrak{z}$ on $\check{\mathfrak{g}}^*$ whose fiber at any point $\lambda \in \check{\mathfrak{g}}^*$ is its centralizer $(\check{\mathfrak{g}}^*)^\lambda$. Its global sections are given by

$$\Gamma(\mathfrak{z}) = \{f : \check{\mathfrak{g}}^* \to \check{\mathfrak{g}} \mid ad^*(f(\lambda))(\lambda) = 0,\ \forall \lambda \in \check{\mathfrak{g}}^*\}.$$

Equip $\Gamma(\mathfrak{z})$ with a Lie algebra structure via

$$[f, g](\lambda) := [f(\lambda), g(\lambda)]$$

for all $f, g \in \Gamma(\mathfrak{z})$, and $\lambda \in \check{\mathfrak{g}}^*$. Define an action of $\mathfrak{z}$ on M

$$f \cdot u : \check{\mathfrak{g}}^* \to V, \tag{8.1}$$

via $(f \cdot u)(\lambda) = (f(\lambda) \cdot u)(\lambda)$.

Lemma 8.0.7. *The action (8.1) makes M into a $\mathfrak{z}$-Lie algebra module. Furthermore, this action is compatible with the action of $\mathcal{O}_{\check{\mathfrak{g}}^*}$; i.e.,*

$$r \cdot (f \cdot u) = (r \cdot f) \cdot u = f \cdot (r \cdot u), \tag{8.2}$$

for all $r \in \mathcal{O}_{\check{\mathfrak{g}}^*}$.

Proof. First we will verify that the action of $\mathfrak{z}$ on M satisfies (8.2). Let $r \in \mathcal{O}_{\check{\mathfrak{g}}^*}$, $f \in \mathfrak{z}, u \in M$, and $\lambda \in \check{\mathfrak{g}}^*$. The left-hand side of (8.2) becomes

$$(r \cdot (f \cdot u))(\lambda) = r(\lambda)(f \cdot u)(\lambda) = r(\lambda)(f(\lambda) \cdot u)(\lambda).$$

The middle of (8.2) is

$$((r \cdot f) \cdot u)(\lambda) = ((r \cdot f)(\lambda) \cdot u)(\lambda) = (r(\lambda)f(\lambda) \cdot u)(\lambda) = r(\lambda)(f(\lambda) \cdot u)(\lambda).$$

For the right-hand side of (8.2), we have

$$(f \cdot (r \cdot u))(\lambda) = (f(\lambda) \cdot (r \cdot u))(\lambda) = ((f(\lambda) \cdot r) \cdot u)(\lambda) + (r \cdot (f(\lambda) \cdot u))(\lambda)$$

by the Leibniz rule. This simplifies to

$$((f(\lambda) \cdot r) \cdot u)(\lambda) + r(\lambda)(f(\lambda) \cdot u)(\lambda).$$

Thus, to verify (8.2), we must show that $((f(\lambda) \cdot r) \cdot u)(\lambda) = 0$. To this end, notice that

$$((f(\lambda) \cdot r) \cdot u)(\lambda) = (f(\lambda) \cdot r)(\lambda)u(\lambda) = (f \cdot r)(\lambda)u(\lambda).$$

So, it suffices to show that $f \cdot r = 0$. First, suppose that r is a constant function. Then $f \cdot r = 0$ since the $\check{\mathfrak{g}}$-action on constant functions is zero. Now, suppose that r is homogeneous. By the Leibniz rule, it is enough to consider the case when r is linear. To this end, suppose that $r : \check{\mathfrak{g}}^* \to \mathbb{C}$ is linear. Then for all $x \in \check{\mathfrak{g}}$,

$$(x \cdot r)(\lambda) = r(-ad^*(x)(\lambda)).$$

So, notice that

$$(f \cdot r)(\lambda) = (f(\lambda) \cdot r)(\lambda) = r(-ad^*(f(\lambda))(\lambda)) = r(0) = 0.$$

Now we will show that the action of $\mathfrak{z}$ on M in the statement of the lemma makes M into a $\mathfrak{z}$-Lie algebra module. That is, we will verify that

$$[f, g] \cdot u = f \cdot (g \cdot u) - g \cdot (f \cdot u), \tag{8.3}$$

for all $f, g \in \mathfrak{z}$ and $u \in M$. Let $\lambda \in \check{\mathfrak{g}}^*$. Then, for the left-hand side of (8.3), we have

$$([f, g] \cdot u)(\lambda) = ([f(\lambda), g(\lambda)] \cdot u)(\lambda) = (f(\lambda) \cdot (g(\lambda) \cdot u))(\lambda) - (g(\lambda) \cdot (f(\lambda) \cdot u))(\lambda).$$

The right-hand side of (8.3) gives

$$(f \cdot (g \cdot u))(\lambda) - (g \cdot (f \cdot u))(\lambda) = (f(\lambda) \cdot (g \cdot u))(\lambda) - (g(\lambda) \cdot (f \cdot u))(\lambda).$$

Thus, it remains to show that

$$(f(\lambda) \cdot (g(\lambda) \cdot u))(\lambda) = (f(\lambda) \cdot (g \cdot u))(\lambda) \tag{8.4}$$

and similarly for f and g swapped. Since the arguments are exactly the same, we will verify the first one. First, suppose that g is a constant function. That is $g(\mu) = g(\lambda)$ for all $\mu \in \check{\mathfrak{g}}^*$. Notice that

$$(g \cdot u)(\mu) = (g(\mu) \cdot u)(\mu) = (g(\lambda) \cdot u)(\mu).$$

Hence $g \cdot u = g(\lambda) \cdot u$ and we have verified (8.4) when g is constant.

Now suppose that $g = r \cdot x$ where $r \in \mathcal{O}_{\check{\mathfrak{g}}^*}$ and $x \in \check{\mathfrak{g}}$. Note that every element of M is a linear combination of functions of the form $r \cdot x$. Thus, it suffices to verify (8.4) for $g = r \cdot x$. We can view $x \in \check{\mathfrak{g}}$ as a constant function $\check{\mathfrak{g}}^* \to \check{\mathfrak{g}}$; that is, $g(\lambda) = r(\lambda)x$. In this case, we have

$$(f(\lambda) \cdot (r(\lambda)x \cdot u))(\lambda) = r(\lambda)(f(\lambda) \cdot (x(\lambda) \cdot u))(\lambda) = r(\lambda)(f(\lambda) \cdot (x \cdot u))(\lambda)$$

by the constant function case of the theorem. Continuing, we have

$$r(\lambda)(f(\lambda)\cdot(x\cdot u))(\lambda) = (r\cdot(f(\lambda)\cdot(x\cdot u)))(\lambda) = (f(\lambda)\cdot(r\cdot x\cdot u))(\lambda) = (f(\lambda)\cdot(g\cdot u))(\lambda)$$

by (8.2), and we have verified (8.4). $\square$

Remark 8.0.8. *Let $r \in \mathcal{O}_{\check{\mathfrak{g}}^*}$, $\lambda \in \check{\mathfrak{g}}^*$, and $f, g \in \mathfrak{z}$. Notice that,*

$$[r \cdot f, g](\lambda) = [(r \cdot f)(\lambda), g(\lambda)] = [r(\lambda)f(\lambda), g(\lambda)], \tag{8.5}$$

and we can now slide the constant $r(\lambda)$ throughout. That is, $[r \cdot f, g] = [f, r \cdot g] = r \cdot [f, g]$. In Lemma 8.0.7, we showed that $\Gamma(\mathfrak{z})$ acts on M, but the compatibility of $\mathcal{O}_{\check{\mathfrak{g}}^*}$ with $\mathfrak{z}$ guarantees that the action (8.1) extends to an action of the sheaf of Lie algebras $\mathfrak{z}$ on the coherent sheaf M.

We can now define the Kostant functor $\kappa : \mathrm{Coh}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*) \to \mathrm{Coh}^{\mathbb{G}_m}(\mathbf{T}(\check{\mathfrak{h}}^*/W))$. Any coherent sheaf $\mathcal{F} \in \mathrm{Coh}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*)$ when restricted to $\check{\mathfrak{g}}^*_{\mathrm{reg}}$ is a sheaf of $\mathfrak{z}|_{\check{\mathfrak{g}}^*_{\mathrm{reg}}}$-representations. We can form the universal enveloping sheaf $\mathcal{U}(\mathfrak{z}|_{\check{\mathfrak{g}}^*_{\mathrm{reg}}})$ which is a sheaf of associative algebras. Then, we get a sheaf $\mathcal{F}|_{\check{\mathfrak{g}}^*_{\mathrm{reg}}}$ of $\mathcal{U}(\mathfrak{z}|_{\check{\mathfrak{g}}^*_{\mathrm{reg}}})$-modules. We have the following result of Kostant.

Proposition 8.0.9 ([Kos63] Proposition 11)**.** *Let $x \in \mathfrak{g}$ be arbitrary. Then $\mathfrak{g}^x$ contains an l-dimensional commutative subalgebra, where $l = \mathrm{rank}\ \mathfrak{g}$.*

In our case, we are considering $x \in \check{\mathfrak{g}}^*_{\mathrm{reg}}$, so $\dim(\check{\mathfrak{g}}^*_{\mathrm{reg}})^x = \mathrm{rank}\ \check{\mathfrak{g}}^*_{\mathrm{reg}} = l$. By the proposition above, $(\check{\mathfrak{g}}^*_{\mathrm{reg}})^x$ is a commutative Lie algebra. Hence, $\mathcal{U}(\mathfrak{z}|_{\check{\mathfrak{g}}^*_{\mathrm{reg}}})$ is a sheaf of commutative associative algebras. Thus, in fact, $\mathcal{F}|_{\check{\mathfrak{g}}^*_{\mathrm{reg}}}$ is a $\mathrm{Sym}(\mathfrak{z}|_{\check{\mathfrak{g}}^*_{\mathrm{reg}}})$-module. We can apply the global spectrum functor $\underline{\mathrm{Spec}}$ to $\mathcal{F}|_{\check{\mathfrak{g}}^*_{\mathrm{reg}}}$ to get a coherent sheaf on $\underline{\mathrm{Spec}}\ \mathrm{Sym}(\mathfrak{z}|_{\check{\mathfrak{g}}^*_{\mathrm{reg}}})$.

We have started with an object in $\mathrm{Coh}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*)$ and constructed an object in $\mathrm{Coh}^{\mathbb{G}_m}(\underline{\mathrm{Spec}}\ \mathrm{Sym}(\mathfrak{z}|_{\check{\mathfrak{g}}^*_{\mathrm{reg}}}))$. However, we want an object in $\mathrm{Coh}^{\mathbb{G}_m}(\mathbf{T}(\check{\mathfrak{h}}^*/W))$. Let's continue.

Let $\mathcal{T}^*$ be the cotangent sheaf on $\check{\mathfrak{h}}^*/W$. Apply the Grothendieck-Hartshorne procedure to construct a vector bundle $\underline{\mathrm{Spec}}\ \mathrm{Sym}(\mathcal{T}^*)$ from this sheaf. This is, in fact, the tangent bundle $\mathbf{T}(\check{\mathfrak{h}}^*/W)$. But, since $\mathbf{T}(\check{\mathfrak{h}}^*/W)$ is a tangent bundle, it comes with a canonical surjection

$$\begin{tikzcd} \mathbf{T}(\check{\mathfrak{h}}^*/W) \arrow[d, "p_1"] \\ \check{\mathfrak{h}}^*/W \end{tikzcd}$$

This is an affine morphism, so we have that $\mathbf{T}(\check{\mathfrak{h}}^*/W) = \underline{\mathrm{Spec}}\ p_{1*}\mathcal{O}_{\mathbf{T}(\check{\mathfrak{h}}^*/W)}$, where $\mathcal{O}_{\mathbf{T}(\check{\mathfrak{h}}^*/W)}$ is the structure sheaf on $\mathbf{T}(\check{\mathfrak{h}}^*/W)$. But, notice that $\underline{\mathrm{Spec}}\ \mathrm{Sym}(\mathcal{T}^*) = \underline{\mathrm{Spec}}\ p_{1*}\mathcal{O}_{\mathbf{T}(\check{\mathfrak{h}}^*/W)}$, so $\mathrm{Sym}(\mathcal{T}^*) = p_{1*}\mathcal{O}_{\mathbf{T}(\check{\mathfrak{h}}^*/W)}$. In addition to the vector bundle projection, we have another map $\mathrm{pr} : \check{\mathfrak{g}}^*_{\mathrm{reg}} \to \check{\mathfrak{h}}^*/W$, which is the projection to the spectrum of invariant polynomials. These maps fit into a diagram

$$\begin{tikzcd} & \mathbf{T}(\check{\mathfrak{h}}^*/W) \arrow[d, "p_1"] \\ \check{\mathfrak{g}}^*_{\mathrm{reg}} \arrow[r, "\mathrm{pr}"] & \check{\mathfrak{h}}^*/W \end{tikzcd}$$

We can pullback the vector bundle along pr to get a new vector bundle (which is necessarily the fibered product). Thus, we have a cartesian square

$$\begin{tikzcd} \check{\mathfrak{g}}^*_{\mathrm{reg}} \times_{\check{\mathfrak{h}}^*/W} \mathbf{T}(\check{\mathfrak{h}}^*/W) \arrow[r, "\widetilde{\mathrm{pr}}"] \arrow[d, "p_2"] & \mathbf{T}(\check{\mathfrak{h}}^*/W) \arrow[d, "p_1"] \\ \check{\mathfrak{g}}^*_{\mathrm{reg}} \arrow[r, "\mathrm{pr}"] & \check{\mathfrak{h}}^*/W \end{tikzcd}$$

But, since p_1 was affine, we know that $\check{\mathfrak{g}}^*_{\mathrm{reg}} \times_{\check{\mathfrak{h}}^*/W} \mathbf{T}(\check{\mathfrak{h}}^*/W) = \underline{\mathrm{Spec}}\ \mathrm{pr}^* p_{1*}\mathcal{O}_{\mathbf{T}(\check{\mathfrak{h}}^*/W)}$. Above we saw that $\mathrm{Sym}(\mathcal{T}^*) = p_{1*}\mathcal{O}_{\mathbf{T}(\check{\mathfrak{h}}^*/W)}$, so we now have that $\check{\mathfrak{g}}^*_{\mathrm{reg}} \times_{\check{\mathfrak{h}}^*/W} \mathbf{T}(\check{\mathfrak{h}}^*/W) = \underline{\mathrm{Spec}}\ \mathrm{pr}^*\mathrm{Sym}(\mathcal{T}^*)$. In [BF08], it is shown that $\mathrm{pr}^*\mathcal{T}^* \simeq \mathfrak{z}|_{\check{\mathfrak{g}}^*_{\mathrm{reg}}}$. This implies that $\mathrm{pr}^*\mathrm{Sym}(\mathcal{T}^*) \simeq \mathrm{Sym}(\mathfrak{z}|_{\check{\mathfrak{g}}^*_{\mathrm{reg}}})$. Putting all of these isomorphisms together, we see that $\check{\mathfrak{g}}^*_{\mathrm{reg}} \times_{\check{\mathfrak{h}}^*/W} \mathbf{T}(\check{\mathfrak{h}}^*/W) \simeq \underline{\mathrm{Spec}}\ \mathrm{Sym}(\mathfrak{z}|_{\check{\mathfrak{g}}^*_{\mathrm{reg}}})$. Making this identification in our diagram, we have

$$\begin{tikzcd} \underline{\mathrm{Spec}}\ \mathrm{Sym}(\mathfrak{z}|_{\check{\mathfrak{g}}^*_{reg}}) \arrow[r, "\widetilde{\mathrm{pr}}"] \arrow[d, "p_2"] & \mathbf{T}(\check{\mathfrak{h}}^*/W) \arrow[d, "p_1"] \\ \check{\mathfrak{g}}^*_{\mathrm{reg}} \arrow[r, "\mathrm{pr}"] & \check{\mathfrak{h}}^*/W \end{tikzcd}$$

Now let $e, h, f \in \check{\mathfrak{g}}$ be a principal $\mathfrak{sl}_2$ triple with $f \in \check{\mathfrak{n}}_-$, and $e \in \check{\mathfrak{n}}_+$. Let $e+\mathfrak{z}(f)$ be the Kostant slice to the principal nilpotent orbit, and let Σ be the image of $e+\mathfrak{z}(f)$ under a $\check{G}$-invariant isomorphism $\check{\mathfrak{g}} \simeq \check{\mathfrak{g}}^*$. Thus, there is an inclusion $\iota : \Sigma \hookrightarrow \check{\mathfrak{g}}^*_{\mathrm{reg}}$. As mentioned in [BF08], we have a canonical isomorphism $\Sigma \simeq \check{\mathfrak{h}}^*/W$. Updating our diagram yields

$$\begin{tikzcd} & \underline{\mathrm{Spec}}\ \mathrm{Sym}(\mathfrak{z}|_{\check{\mathfrak{g}}^*_{\mathrm{reg}}}) \arrow[r, "\widetilde{\mathrm{pr}}"] \arrow[d, "p_2"] & \mathbf{T}(\check{\mathfrak{h}}^*/W) \arrow[d, "p_1"] \\ \Sigma \arrow[r, hook, "\iota"] \arrow[rr, bend right, "\sim"'] & \check{\mathfrak{g}}^*_{\mathrm{reg}} \arrow[r, "\mathrm{pr}"] & \check{\mathfrak{h}}^*/W \end{tikzcd}$$

Since we have an isomorphism $\Sigma \simeq \check{\mathfrak{h}}^*/W$, it follows that pulling back the vector bundle $p_2 : \underline{\mathrm{Spec}}\ \mathrm{Sym}(\mathfrak{z}|_{\check{\mathfrak{g}}^*_{\mathrm{reg}}}) \to \check{\mathfrak{g}}^*_{\mathrm{reg}}$ along ι gives a vector bundle isomorphic to $\mathbf{T}(\check{\mathfrak{h}}^*/W)$. That is, we have the diagram

$$\begin{tikzcd} \mathbf{T}(\check{\mathfrak{h}}^*/W) \arrow[r, "\tilde{\iota}"] \arrow[d] & \underline{\mathrm{Spec}}\ \mathrm{Sym}(\mathfrak{z}|_{\check{\mathfrak{g}}^*_{\mathrm{reg}}}) \arrow[r, "\widetilde{\mathrm{pr}}"] \arrow[d, "p_2"] & \mathbf{T}(\check{\mathfrak{h}}^*/W) \arrow[d, "p_1"] \\ \Sigma \arrow[r, hook, "\iota"] \arrow[rr, bend right, "\sim"'] & \check{\mathfrak{g}}^*_{\mathrm{reg}} \arrow[r, "\mathrm{pr}"] & \check{\mathfrak{h}}^*/W \end{tikzcd}$$

Continuing the definition of the Kostant functor, we take the object $\underline{\mathrm{Spec}}\ \mathcal{F}|_{\check{\mathfrak{g}}^*_{\mathrm{reg}}} \in \mathrm{Coh}^{\mathbb{G}_m}(\underline{\mathrm{Spec}}\ \mathrm{Sym}(\mathfrak{z}|_{\check{\mathfrak{g}}^*_{\mathrm{reg}}}))$ that we constructed before, and we restrict to $\mathbf{T}(\check{\mathfrak{h}}^*/W)$.

Let us give a concise definition of the Kostant functor.

Definition 8.0.10. *The construction of the Kostant functor κ is given in the paragraphs above. That is, $\kappa : \mathrm{Coh}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*) \to \mathrm{Coh}^{\mathbb{G}_m}(\boldsymbol{T}(\check{\mathfrak{h}}^*/W))$ is defined by*

$$\kappa(\mathcal{F}) = \tilde{\iota}^* \underline{\mathrm{Spec}}\ \mathcal{F}|_{\check{\mathfrak{g}}^*_{\mathrm{reg}}}$$

Remark 8.0.11. *It is shown in [BF08] that $H^\bullet_{G(\mathfrak{O})}(\mathrm{Gr}) \simeq \mathcal{O}(\boldsymbol{T}(\check{\mathfrak{h}}^*/W))$ (see Remark 9.2.5). Thus, the functor κ takes values in $H^\bullet_{G(\mathfrak{O})}(\mathrm{Gr})$-mod.*

Chapter 9

Modules, Module Homomorphisms, and Free Modules

Consider the following diagram on $\mathrm{Hom}^\bullet$-algebras induced by the functors in Diagram 1.2.

$$\begin{tikzcd}
\mathrm{Hom}^\bullet_{\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}_G^{\mathrm{sm}})}(\mathcal{F},\mathcal{G}) \arrow[r, "\mathrm{der}\mathcal{S}_G^{\mathrm{sm}}"] \arrow[dd, "\Psi_G"] & \underline{\mathrm{Hom}}_{\mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*)_{\mathrm{sm}}}(\mathrm{der}\mathcal{S}_G^{\mathrm{sm}}(\mathcal{F}),\mathrm{der}\mathcal{S}_G^{\mathrm{sm}}(\mathcal{G})) \arrow[d, "\mathrm{der}\Phi_{\check{G}}"] \\
 & \underline{\mathrm{Hom}}_{\mathrm{Coh}_{\mathrm{fr}}^{W\times\mathbb{G}_m}(\check{\mathfrak{h}}^*)}(\mathrm{der}\Phi_{\check{G}}\circ\mathrm{der}\mathcal{S}_G^{\mathrm{sm}}(\mathcal{F}),\mathrm{der}\Phi_{\check{G}}\circ\mathrm{der}\mathcal{S}_G^{\mathrm{sm}}(\mathcal{G})) \\
\mathrm{Hom}^\bullet_{\mathrm{Semis}_G(\mathcal{N})}(\Psi_G(\mathcal{F}),\Psi_G(\mathcal{G})) \arrow[r, "\mathrm{der}\mathbb{S}_G"] & \underline{\mathrm{Hom}}_{\mathrm{Coh}_{\mathrm{fr}}^{W\times\mathbb{G}_m}(\check{\mathfrak{h}}^*)}(\mathrm{der}\mathbb{S}_G\circ\Psi_G(\mathcal{F}),\mathrm{der}\mathbb{S}_G\circ\Psi_G(\mathcal{G}))
\end{tikzcd} \tag{9.1}$$

In this section, we show that each of the Hom-algebras in Diagram (9.1) is a free module over the appropriate manifestation of the ring $H_G^\bullet(\mathrm{pt})$. We also show that each arrow in the diagram is a module homomorphism.

9.1 The Hom-algebras are modules

We have shown in Lemma 4.3.1, Proposition 4.3.3, and Remark 4.3.5 that the Hom-algebras are modules over $H_G^\bullet(\mathrm{pt}) \simeq \mathcal{O}_{\check{\mathfrak{g}}^*}^{\check{G}} \simeq \mathcal{O}_{\check{\mathfrak{h}}^*}^{W}$, respectively.

9.2 The arrows are module homomorphisms

9.2.1 The top arrow

In this subsection, we show that the top arrow in Diagram (9.1) is an $\mathcal{O}_{\check{\mathfrak{g}}^*}^{\check{G}}$-equivariant map.

Lemma 9.2.1. *The functor κ induces an $\mathcal{O}_{\check{\mathfrak{g}}^*}^{\check{G}}$-equivariant map*

$$\underline{\mathrm{Hom}}_{\mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*)}(M,N) \to \underline{\mathrm{Hom}}_{H^\bullet_{G(\mathfrak{O})}(\mathrm{Gr})-\mathrm{mod}}(\kappa(M),\kappa(N)).$$

Proof. By Proposition 4.3.3, we know that $\underline{\mathrm{Hom}}_{\mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*)}(M,N)$ is a free $\mathcal{O}_{\check{\mathfrak{g}}^*}^{\check{G}}$-module. Now equip both M and N with $\mathfrak{z}$-module structures. Again, we get that the space of maps is an $\mathcal{O}_{\check{\mathfrak{g}}^*}^{\check{G}}$-module (here we use the compatibility of the actions of $\mathcal{O}_{\check{\mathfrak{g}}^*}^{\check{G}}$ and $\mathfrak{z}$ given in (8.5)). At each step in the definition of the Kostant functor (see Definition 8.0.10), the ring $\mathcal{O}_{\check{\mathfrak{g}}^*}^{\check{G}}$ still acts, and we can follow it through the definition of κ. □

Remark 9.2.2. *This implies that the functor κ^{-1} also induces an $\mathcal{O}_{\check{\mathfrak{g}}^*}^{\check{G}}$-equivariant map*

$$\underline{\mathrm{Hom}}_{H^\bullet_{G(\mathfrak{O})}(\mathrm{Gr})-\mathrm{mod}}(M,N) \to \underline{\mathrm{Hom}}_{\mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*)}(\kappa^{-1}(M),\kappa^{-1}(N)).$$

We also have the following easy lemma.

Lemma 9.2.3. *The functor $\mathbb{H}^\bullet_{G(\mathfrak{O})}$ induces an $H^\bullet_{G(\mathfrak{O})}(\mathrm{pt})$-equivariant map*

$$\mathrm{Hom}^\bullet_{\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr})}(\mathcal{F},\mathcal{G}) \to \underline{\mathrm{Hom}}_{H^\bullet_{G(\mathfrak{O})}(\mathrm{Gr})-\mathrm{mod}}(\mathbb{H}^\bullet_{G(\mathfrak{O})}(\mathcal{F}),\mathbb{H}^\bullet_{G(\mathfrak{O})}(\mathcal{G})).$$

We need a theorem from the paper [BF08]. This theorem will allow us to prove that on $\underline{\mathrm{Hom}}_{H^\bullet_{G(\mathfrak{O})}(\mathrm{Gr})-\mathrm{mod}}(M,N)$, the actions of $H^\bullet_{G(\mathcal{O})}(\mathrm{pt})$ and $\mathcal{O}_{\check{\mathfrak{g}}^*}^{\check{G}}$ coincide.

Theorem 9.2.4. *[BF08, Theorem 1 (a)] Assume G is simply connected. We have a canonical isomorphism $H^\bullet_{G(\mathfrak{O})\rtimes\mathbb{G}_m}(\mathrm{Gr}) \simeq \mathcal{O}(N_{\check{\mathfrak{t}}^*/W\times\check{\mathfrak{t}}^*/W}\Delta)$ where $\Delta\subset(\check{\mathfrak{t}}^*/W)^2$ is the diagonal. Here the projection $N_{(\check{\mathfrak{t}}^*/W)^2}\Delta\to\mathbb{A}^1$ corresponds to the homomorphism $H^\bullet_{\mathbb{G}_m}(\mathrm{pt})\to H^\bullet_{G(\mathfrak{O})\rtimes\mathbb{G}_m}(\mathrm{Gr})$; and the two projections $N_{(\check{\mathfrak{t}}^*/W)^2}\Delta\to\check{\mathfrak{t}}^*/W = \mathfrak{t}/W$ correspond to the two homomorphisms $H^\bullet_{G(\mathfrak{O})}(\mathrm{pt})\to H^\bullet_{G(\mathfrak{O})\rtimes\mathbb{G}_m}(\mathrm{Gr})$. The isomorphism is specified uniquely by these requirements.*

Remark 9.2.5. *R. Bezrukavnikov and M. Finkelberg (see [BF08, Section 2.6]) note that the fiber of $N_X Z$ over $0 \in \mathbb{A}^1$ is the normal cone to Z in X. In particular, the fiber of $N_{(\check{\mathfrak{t}}^*/W)^2}\Delta$ over $0 \in \mathbb{A}^1$ is the total space of the tangent bundle $\boldsymbol{T}(\check{\mathfrak{t}}^*/W)$. Thus, Theorem 9.2.4 implies the canonical isomorphism $H^\bullet_{G(\mathfrak{O})}(\mathrm{Gr}) \simeq \mathcal{O}(\boldsymbol{T}(\check{\mathfrak{t}}^*/W))$.*

Corollary 9.2.6. *The actions of $H^\bullet_{G(\mathfrak{O})}(\mathrm{pt})$ and $\mathcal{O}^{\check{G}}_{\check{\mathfrak{g}}^*}$ coincide on*

$$\underline{\mathrm{Hom}}_{H^\bullet_{G(\mathfrak{O})}(\mathrm{Gr})-\mathrm{mod}}(M, N).$$

Proof. Let M and N be $H^\bullet_{G(\mathfrak{O})}(\mathrm{Gr})$-modules. Then $\underline{\mathrm{Hom}}_{H^\bullet_{G(\mathfrak{O})}(\mathrm{Gr})-\mathrm{mod}}(M, N)$ carries actions of both $H^\bullet_{G(\mathfrak{O})}(\mathrm{pt})$ and $\mathcal{O}^{\check{G}}_{\check{\mathfrak{g}}^*}$. Let $p : \mathrm{Gr} \to \mathrm{pt}$ be the map to a point, and let $\pi : \mathbf{T}(\check{\mathfrak{t}}^*/W) \to \check{\mathfrak{t}}^*/W$ be the tangent bundle map. By Theorem 9.2.4, we have the commutative square

$$\begin{tikzcd}
H^\bullet_{G(\mathfrak{O})}(\mathrm{Gr}) \arrow[r, dash, "\beta", "\sim"'] & \mathcal{O}(\mathbf{T}(\check{\mathfrak{t}}^*/W)) \\
H^\bullet_{G(\mathfrak{O})}(\mathrm{pt}) \arrow[r, dash, "\sim", "\varphi"'] \arrow[u, hook, "p^\#"] & \mathcal{O}(\check{\mathfrak{t}}^*/W) \arrow[u, hook, "\pi^\#"']
\end{tikzcd}$$

Let $f \in \underline{\mathrm{Hom}}_{H^\bullet_{G(\mathfrak{O})}(\mathrm{Gr})-\mathrm{mod}}(M, N)$ and let $r \in H^\bullet_{G(\mathfrak{O})}(\mathrm{pt})$. In what follows, $\cdot$ denotes the module action of $H^\bullet_{G(\mathfrak{O})}(\mathrm{Gr})$. Then the two actions $p^\#(r) \cdot f$ and $(\beta \circ \pi^\# \circ \varphi(r)) \cdot f$, mentioned before, make sense and are equal by the commutativity of the diagram. $\square$

We will need one more theorem from [BF08].

Theorem 9.2.7. *[BF08, Theorem 4] The functor*

$$\mathrm{der}\mathcal{S}_G^{-1} : \mathrm{Coh}^{\check{G}\times\mathbb{G}_m}_{\mathrm{free}}(\check{\mathfrak{g}}^*) \to \mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr})$$

is a full imbedding, and there exists a natural isomorphism of functors

$$\kappa \Longleftrightarrow \mathbb{H}^\bullet_{G(\mathfrak{O})} \circ \mathrm{der}\mathcal{S}_G^{-1}.$$

Now we are ready to prove that the top arrow in Diagram (9.1) is an $(H^\bullet_{G(\mathfrak{O})}(\mathrm{pt}) \simeq \mathcal{O}^{\check{G}}_{\check{\mathfrak{g}}^*})$-equivariant map.

Theorem 9.2.8. *The functor* $\mathrm{der}\mathcal{S}_G$ *gives an* $(H^\bullet_{G(\mathfrak{O})}(\mathrm{pt}) \simeq \mathcal{O}^{\check{G}}_{\check{\mathfrak{g}}^*})$*-equivariant map*

$$\mathrm{Hom}^\bullet_{\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr})}(\mathcal{F},\mathcal{G}) \to \underline{\mathrm{Hom}}_{\mathrm{Coh}^{\check{G}\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{g}}^*)}(\mathrm{der}\mathcal{S}_G(\mathcal{F}), \mathrm{der}\mathcal{S}_G(\mathcal{G})).$$

Proof. Let $f \in \mathrm{Hom}^\bullet_{\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr})}(\mathcal{F},\mathcal{G})$ and let $r \in H^\bullet_{G(\mathfrak{O})}(\mathrm{pt})$. We keep the notation of Corollary 9.2.6. Notice that,

$$\begin{aligned}
\kappa^{-1} \circ \mathbb{H}^\bullet_{G(\mathfrak{O})}(r \cdot f) &= \kappa^{-1}(p^{\#}(r) \cdot \mathbb{H}^\bullet_{G(\mathfrak{O})}(f)) && \text{Lemma 9.2.3}\\
&= \kappa^{-1}((\beta \circ \pi^{\#} \circ \varphi)(r) \cdot \mathbb{H}^\bullet_{G(\mathfrak{O})}(f)) && \text{Corollary 9.2.6}\\
&= (\beta \circ \pi^{\#} \circ \varphi(r)) \cdot (\kappa^{-1} \circ \mathbb{H}^\bullet_{G(\mathfrak{O})}(f)). && \text{Lemma 9.2.1}
\end{aligned}$$

By Theorem 9.2.7, we know that $\kappa^{-1} \circ \mathbb{H}^\bullet_{G(\mathfrak{O})} \iff \mathrm{der}\mathcal{S}_G$. Using this in the equations above, we have that

$$\mathrm{der}\mathcal{S}_G(r \cdot f) = (\beta \circ \pi^{\#} \circ \varphi(r)) \cdot (\mathrm{der}\mathcal{S}_G(f)).$$

□

9.2.2 The bottom arrow

We would like to prove a version of Theorem 9.2.8 for the bottom arrow in Diagram 9.1. We will make heavy use of the ideas in [Lus95, Section 8]. First we give a lemma whose proof we omit.

Lemma 9.2.9. *Let* $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathrm{Semis}_G(\mathcal{N}_G)$ *and let* $v \in H^\bullet_G(\mathrm{pt})$*. Let*

$$f \in \mathrm{Hom}^\bullet_{\mathrm{Semis}_G(\mathcal{N}_G)}(\mathcal{F},\mathcal{G}) \;\; \textit{and} \;\; g \in \mathrm{Hom}^\bullet_{\mathrm{Semis}_G(\mathcal{N}_G)}(\mathcal{G},\mathcal{H}).$$

Consider the map $a : \mathcal{N}_G \to \mathrm{pt}$. Define $v \cdot f := f \circ a^* v$.[1] *Then*

$$v \cdot (g \circ f) = (v \cdot g) \circ f = g \circ (v \cdot f).$$

Theorem 9.2.10. *The functor* $\mathrm{der}\mathbb{S}$ *gives an* $(H_G^\bullet(\mathrm{pt}) \simeq \mathcal{O}_{\check{\mathfrak{h}}^*}^W)$*-equivariant map*

$$\mathrm{Hom}^\bullet_{\mathrm{Semis}_G(\mathcal{N}_G)}(\mathcal{F}, \mathcal{G}) \to \underline{\mathrm{Hom}}_{\mathrm{Coh}_{\mathrm{fr}}^{W \times \mathbb{G}_m}(\check{\mathfrak{h}}^*)}(\mathrm{der}\mathbb{S}(\mathcal{F}), \mathrm{der}\mathbb{S}(\mathcal{G})).$$

Proof. The map

$$\mathrm{Hom}^\bullet_{\mathrm{Semis}_G(\mathcal{N}_G)}(\mathcal{F}, \mathcal{G}) \to \underline{\mathrm{Hom}}_{\mathrm{Coh}_{\mathrm{fr}}^{W \times \mathbb{G}_m}(\check{\mathfrak{h}}^*)}(\mathrm{der}\mathbb{S}(\mathcal{F}), \mathrm{der}\mathbb{S}(\mathcal{G}))$$

induced by $\mathrm{der}\mathbb{S}$ is the composition of the map

$$\mathrm{Hom}^\bullet_{\mathrm{Semis}_G(\mathcal{N}_G)}(\mathcal{F}, \mathcal{G}) \xrightarrow{\mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}, -)} \mathrm{Hom}^\bullet_{\mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}, \underline{\mathrm{Spr}})-\mathrm{mod}}(\mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}, \mathcal{F}), \mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}, \mathcal{G})) \tag{9.2}$$

with the isomorphism $\mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}, \underline{\mathrm{Spr}}) \simeq A_W$ (see Section 5.4). Thus, it suffices to prove that (9.2) is an $H_G^\bullet(\mathrm{pt})$-module homomorphism.

Proving that (9.2) is an $H_G^\bullet(\mathrm{pt})$-module homomorphism requires careful consideration of the $H_G^\bullet(\mathrm{pt})$-actions on each of the spaces in the following diagram:

$$\mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{G}) \otimes \mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}, \mathcal{F}) \xrightarrow{\circ} \mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}, \mathcal{G}). \tag{9.3}$$

However, since both $\mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}, \mathcal{F})$ and $\mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}, \mathcal{G})$ are $\mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}, \underline{\mathrm{Spr}})$-modules, we need to be sure that the actions of $\mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}, \underline{\mathrm{Spr}})$ and $H_G^\bullet(\mathrm{pt})$ are compatible.

The actions of $H_G^\bullet(\mathrm{pt})$ and $\mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}, \underline{\mathrm{Spr}})$ on $\mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}, \mathcal{F})$ can be summarized in the following diagram:

$$\mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}, \underline{\mathrm{Spr}}) \otimes \mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}, \mathcal{F}) \xrightarrow{\circ} \mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}, \mathcal{F}), \tag{9.4}$$

[1] To make sense of this composition, one needs to use the image of v through the isomorphism

$$H_G^\bullet(\mathrm{pt}) \simeq \mathrm{Hom}^\bullet_{\mathrm{Semis}_G(\mathrm{pt})}(\underline{\mathbb{C}}_{\mathrm{pt}}, \underline{\mathbb{C}}_{\mathrm{pt}})$$

and to use the image of f through the isomorphism

$$\mathrm{Hom}^\bullet_{\mathrm{Semis}_G(\mathcal{N}_G)}(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}^\bullet_{\mathrm{Semis}_G(\mathcal{N}_G)}(\underline{\mathbb{C}}_{\mathcal{N}}, R\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G})).$$

where $H^\bullet_G(\mathrm{pt})$ acts on each of the spaces. By Lemma 9.2.9, the three actions of $H^\bullet_G(\mathrm{pt})$ in (9.4) agree.

Similarly, we have the following diagram:

$$\mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}, \underline{\mathrm{Spr}}) \otimes \mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}, \mathcal{G}) \xrightarrow{\circ} \mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}, \mathcal{G}), \tag{9.5}$$

which, by Lemma 9.2.9, describes the compatibility of the $H^\bullet_G(\mathrm{pt})$-actions and the action of $\mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}, \underline{\mathrm{Spr}})$ on $\mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}, \mathcal{G})$.

Putting together the compatibilities of actions on $\mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}, \mathcal{F})$ and $\mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}, \mathcal{G})$ arising from (9.4) and (9.5), the proof of the theorem follows by applying Lemma 9.2.9 to (9.3). $\square$

9.3 The $\mathrm{Hom}^\bullet$-algebras are free modules

We want to show that

$$\mathrm{Hom}^\bullet_{\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr})}(\mathcal{F}, \mathcal{G})$$

is free as an $\mathcal{O}^{\check{G}}_{\check{\mathfrak{g}}^*}$-module. To do this, we need a series of lemmas.

Lemma 9.3.1. *The stalks of* $\mathrm{IC}(\overline{X}_w)$ *are pure of weight 0.*

Remark 9.3.2. *Here* $\overline{X}_w$ *is the closure of a stratum in the Iwahori stratification of the affine Grassmannian.*

Proof. Consider the Bott-Samelson resolution $\pi : \widetilde{X}^{\mathrm{BS}}_w \to \overline{X}_w$. It is known that $\widetilde{X}^{\mathrm{BS}}_w$ is smooth and projective; therefore, π is a proper map (thus, $\pi_* = \pi_!$). Consider

$$\pi_*\overline{\mathbb{Q}}_\ell[\dim \overline{X}_w]\left(\frac{\dim \overline{X}_w}{2}\right)$$

which is a semisimple perverse sheaf by the Decomposition Theorem [BBD82]. We have

$$\pi_*\overline{\mathbb{Q}}_\ell[\dim \overline{X}_w]\left(\frac{\dim \overline{X}_w}{2}\right) \simeq \bigoplus_{\lambda \in \mathrm{Irr}(W)} \mathrm{IC}_\lambda[n]\left(\frac{n}{2}\right)$$

where each summand is pure of weight 0. We know that $\mathrm{IC}(\overline{X}_w)[\dim \overline{X}_w]\left(\frac{\dim \overline{X}_w}{2}\right)$ occurs in this direct sum with multiplicity one. Therefore, it is enough to show that stalks of $\pi_* \overline{\mathbb{Q}}_\ell[\dim \overline{X}_w]\left(\frac{\dim \overline{X}_w}{2}\right)$ are pure. Consider the following Cartesian square:

$$\begin{tikzcd}
\widetilde{X}_w^{\mathrm{BS}} \arrow[d, "\pi"] & \pi^{-1}(y) \arrow[l, hook'] \arrow[d, "a"] \\
\overline{X}_w & \{y\} \arrow[l, hook']
\end{tikzcd}$$

We can use the Proper Base Change Theorem (and the fact that π is proper) to get

$$\left(\pi_* \overline{\mathbb{Q}}_\ell[\dim \overline{X}_w]\left(\frac{\dim \overline{X}_w}{2}\right)\right)\Big|_y \simeq a_* \overline{\mathbb{Q}}_{\ell,\pi^{-1}(y)} = H_{\acute{e}t}(\pi^{-1}(y))$$

But $\pi^{-1}(y)$ has an affine paving by [Hai]. Thus $H_{c,\acute{e}t}(\pi^{-1}(y))$ is pure, and this implies that $H_{\acute{e}t}(\pi^{-1}(y))$ is pure as well. □

Lemma 9.3.3. *The stalks of* $\mathrm{IC}(\overline{\mathcal{O}})$ *on the nilpotent cone are pure.*

Remark 9.3.4. *Here* $\overline{\mathcal{O}}$ *is the closure of a nilpotent orbit.*

Proof. In [Spr84], the stalks of ICs appearing in the Springer correspondence are shown to be pure. For other ICs, the result follows because the nilpotent cone together with its G-action fits into the general framework of [MS98, Proposition 2.3.3]. □

Theorem 9.3.5. $\mathbb{H}\mathrm{om}^i_{\mathcal{D}^{\mathrm{b,mix}}_G(\mathcal{N})}(\mathcal{F},\mathcal{G})$ *is pure of weight* i*.*

Proof. Lemma 9.3.3 asserts that the stalks of $\mathrm{IC}(\overline{\mathcal{O}})$ are pure. The proof of all subsequent lemmas follows exactly as before. □

Corollary 9.3.6. $\mathrm{Hom}^\bullet_{\mathrm{Semis}_G(\mathcal{N})}(\mathcal{F},\mathcal{G})$ *is a free* $H^\bullet_G(\mathrm{pt})$*-module.*

Proof. Combine Theorem 9.3.5 and Lemma 4.4.4. □

Theorem 9.3.7. *All objects in the diagram are free modules over* $H^\bullet_G(\mathrm{pt}) \simeq \mathcal{O}^{\check{G}}_{\check{\mathfrak{g}}^*} \simeq \mathcal{O}^{W}_{\check{\mathfrak{h}}^*}$.

Proof. Lemma 4.4.4 and Corollary 9.3.6 give the freeness for the constructible categories in the diagram. Since $\mathrm{der}\mathbb{S}_G$ and $\mathrm{der}\mathcal{S}_G$ are equivalences of categories (so the maps they induce on $\mathrm{Hom}^\bullet$-algebras are isomorphisms), and by Theorem 9.2.8 and bot arrow Theorem, we get the freeness of $\underline{\mathrm{Hom}}$-algebras on the coherent side. □

Chapter 10

The Diagram Commutes for Groups of Semisimple Rank 1

In this section we will prove the theorem below.

Theorem 10.0.8. *Let G be a reductive group of semisimple rank 1. Then Diagram (1.2):*

$$\begin{tikzcd}
\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}) \arrow[r, "\mathrm{der}\mathcal{S}_G^{\mathrm{sm}}"] \arrow[d, "\Psi_G"'] & \mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*)_{\mathrm{sm}} \arrow[d, "\mathrm{der}\Phi_{\check{G}}"] \\
\mathrm{Semis}_G(\mathcal{N}) \arrow[r, "\mathrm{der}\mathbb{S}_G"'] & \mathrm{Coh}_{\mathrm{fr}}^{W\times\mathbb{G}_m}(\check{\mathfrak{h}}^*)
\end{tikzcd}$$

commutes.

10.1 Classification of semisimple rank 1 groups and reduction to $G = (\mathbb{C}^\times)^{n-1} \times \mathbf{PGL}(2,\mathbb{C})$

It will be useful to know all of the groups of semisimple rank 1. They are classified in the following lemma.

Lemma 10.1.1 ([GKM04] Lemma 8.1)**.** *Let K be a connected reductive complex linear algebraic group of rank n and of semisimple rank one. Then K is isomorphic to either*

1. $(\mathbb{C}^\times)^{n-1} \times \mathbf{SL}(2,\mathbb{C})$,

2. $(\mathbb{C}^\times)^{n-1} \times \mathbf{PGL}(2,\mathbb{C})$, *or*

3. $(\mathbb{C}^\times)^{n-2} \times \mathbf{GL}(2,\mathbb{C})$.

Define $\bar{K} := K/\{\pm I\}$ where I is the identity matrix. If K is isomorphic to (1) in Lemma 10.1.1, then $\bar{K}$ is isomorphic to (2). If K is isomorphic to (3), then $\bar{K}$ is isomorphic to (2). Lemma 4.5.3 shows that it suffices to prove Theorem 10.0.8 for only $G = (\mathbb{C}^\times)^{n-1} \times \mathbf{PGL}(2,\mathbb{C})$. Lemma 4.5.2 asserts that the proof of the theorem for $G = (\mathbb{C}^\times)^{n-1} \times \mathbf{PGL}(2,\mathbb{C})$ follows from the proofs for $(\mathbb{C}^\times)^{n-1}$ and $\mathbf{PGL}(2,\mathbb{C})$. But the theorem was proven for $G = (\mathbb{C}^\times)^{n-1}$ in Lemma 4.5.1, so Theorem 10.0.8 will follow from the proof for $G = \mathbf{PGL}(2,\mathbb{C})$.

10.2 Sketch of proof

Henceforth, in view of Section 10.1, we proceed with $G = \mathbf{PGL}(2,\mathbb{C})$; thus, $\check{G} = \mathbf{SL}(2,\mathbb{C})$. Notice that $\{-2,0,2\}$ is the set of small weights for $\check{G}$, so $V(0)$ and $V(2)$ are the two irreducible small representations of $\check{G}$, and $\mathrm{Gr}_G^{\mathrm{sm}} = \mathrm{Gr}^0 \sqcup \mathrm{Gr}^2$.

Define $\mathrm{Spr}_{\mathrm{Gr}} := \mathrm{IC}(\mathrm{Gr}^0) \oplus \mathrm{IC}(\mathrm{Gr}^2)$. The category $\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}})$ is generated by $\mathrm{Spr}_{\mathrm{Gr}}$. Thus, in order to prove Theorem 10.0.8, we need an isomorphism of objects

$$\eta : \mathrm{der}\Phi_{\check{G}}(\mathrm{der}\mathcal{S}_G^{\mathrm{sm}}(\mathrm{Spr}_{\mathrm{Gr}})) \xrightarrow{\sim} \mathrm{der}\mathbb{S}_G(\Psi_G(\mathrm{Spr}_{\mathrm{Gr}}))$$

that is natural; i.e., for every $f : \mathrm{Spr}_{\mathrm{Gr}} \to \mathrm{Spr}_{\mathrm{Gr}}[n]$, the diagram

$$\begin{tikzcd}
\mathrm{der}\Phi_{\check{G}}(\mathrm{der}\mathcal{S}_G^{\mathrm{sm}}(\mathrm{Spr}_{\mathrm{Gr}})) \arrow[r, "\eta"] \arrow[d, "\mathrm{der}\Phi_{\check{G}}(\mathrm{der}\mathcal{S}_G^{\mathrm{sm}}(f))"'] & \mathrm{der}\mathbb{S}_G(\Psi_G(\mathrm{Spr}_{\mathrm{Gr}})) \arrow[d, "\mathrm{der}\mathbb{S}_G(\Psi_G(f))"] \\
\mathrm{der}\Phi_{\check{G}}(\mathrm{der}\mathcal{S}_G^{\mathrm{sm}}(\mathrm{Spr}_{\mathrm{Gr}}[n])) \arrow[r, "\eta"] & \mathrm{der}\mathbb{S}_G(\Psi_G(\mathrm{Spr}_{\mathrm{Gr}}[n]))
\end{tikzcd} \tag{10.1}$$

P. Achar, A. Henderson, and S. Riche [AHR15] have constructed an isomorphism η for $n = 0$. In Section 11.3, we will show that any f can be obtained by composing linear combinations of degree zero maps and a fixed degree two map. We will compute $\mathrm{der}\Phi_{\check{G}}(\mathrm{der}\mathcal{S}_G^{\mathrm{sm}}(f))$ (resp. $\mathrm{der}\mathbb{S}_G(\Psi_G(f))$) in Section 11.1 (resp. Section 11.2). Finally, in Section 11.4, we will deduce the commutativity of Diagram (10.1).

Chapter 11
Proofs of Lemmas in §10.2

We will define a degree two map $f : \mathrm{Spr}_{\mathrm{Gr}} \to \mathrm{Spr}_{\mathrm{Gr}}[2]$ which can be combined with degree zero maps to make any map $\mathrm{Spr}_{\mathrm{Gr}} \to \mathrm{Spr}_{\mathrm{Gr}}[n]$. Consider the diagram

$$\begin{tikzcd} \mathrm{Gr}^1_G \widetilde{\times} \mathrm{Gr}^1_G \arrow[r, "\iota"] \arrow[dr, "m"'] & \mathrm{Gr}_G \widetilde{\times} \mathrm{Gr}_G \arrow[r, "\mathrm{pr}_1"] & \mathrm{Gr}_G \\ & \mathrm{Gr}^{\mathrm{sm}} & \end{tikzcd} \tag{11.1}$$

Let $\mathcal{L}_{\det}$ be the determinant line bundle on Gr_G. Then, $\iota^*\mathrm{pr}_1^*(\mathcal{L}_{\det})$ is a line bundle on $\mathrm{Gr}^1_G \widetilde{\times} \mathrm{Gr}^1_G$. The image of its first Chern class $c_1(\iota^*\mathrm{pr}_1^*(\mathcal{L}_{\det})) \in H^2(\mathrm{Gr}^1_G \widetilde{\times} \mathrm{Gr}^1_G)$ under the isomorphism

$$H^2(\mathrm{Gr}^1_G \widetilde{\times} \mathrm{Gr}^1_G) \simeq \mathrm{Hom}^2(\underline{\mathbb{C}}_{\mathrm{Gr}^1_G \widetilde{\times} \mathrm{Gr}^1_G}, \underline{\mathbb{C}}_{\mathrm{Gr}^1_G \widetilde{\times} \mathrm{Gr}^1_G}).$$

defines a degree-two map of constant sheaves. We define f as the pushforward along m of this degree-two map:

$$f := m_!(c_1(\iota^*\mathrm{pr}_1^*(\mathcal{L}_{\det}))) \in \mathrm{Hom}^2(\mathrm{Spr}_{\mathrm{Gr}}, \mathrm{Spr}_{\mathrm{Gr}}).$$

11.1 Computing $\mathrm{der}\Phi_{\check{G}}(\mathrm{der}\mathcal{S}^{\mathrm{sm}}_G(f))$

Recall that Diagram (7.13) commutes; therefore, the construction of the map $\mathrm{der}\Phi_{\check{G}}(\mathrm{der}\mathcal{S}^{\mathrm{sm}}_G(f))$ in the nonequivariant case (for the bottom triangle) will lift to a unique map in the equivariant case (for the top triangle). Hence, we content ourselves with making the argument in the nonequivariant case.

Consider the determinant line bundle $\mathcal{L}_{\det}$ on Gr. Let $\mathcal{F} \in \mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr})$ and its first Chern class $c_1(\mathcal{L}_{\det}) \in H^2(\mathrm{Gr})$. Then $\mathrm{der}\mathcal{S}_G(\mathcal{F}) = \mathbb{H}^\bullet(\mathcal{F}) \otimes \mathcal{O}_{\check{\mathfrak{g}}^*} \in \mathrm{Coh}^{\check{G}\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{g}}^*)$

is a module for $H^\bullet(\mathrm{Gr})$. Notice that

$$\mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*) \xrightarrow{-\smile c_1(\mathcal{L}_{\det})} \mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*)$$

is a degree two map. In [YZ11], Yun and Zhu assert that this map is the action of a principal nilpotent element $e \in \check{\mathfrak{g}}^*$.

On the other hand, there is a way to get from $\mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}})$ to $H^\bullet(\mathrm{Gr}) - \mathrm{mod}$.

Lemma 11.1.1. *There is a fully faithful functor* $\mathrm{ev}_e \circ \mathrm{re} : \mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*) \to \mathrm{Rep}((\check{G}\times \mathbb{G}_m)^e)$.

Proof. Identify $\check{\mathfrak{g}}$ with $\check{\mathfrak{g}}^*$ via the Killing form $K(-,-)$, and let $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathbf{SL}_2(\mathbb{C})$. Let $V \otimes \mathcal{O}_{\check{\mathfrak{g}}}\langle k\rangle \in \mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}})$. This free coherent sheaf can be regarded as the space of functions $\check{\mathfrak{g}} \to V$ with $\check{G} \times \mathbb{G}_m$ acting on both $\check{\mathfrak{g}}$ and V where $\mathbb{G}_m$ acts by k via the regard functor re. Consider the functor

$$\begin{array}{ccc} \mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}) & \xrightarrow{\mathrm{ev}_e \circ \mathrm{re}} & \mathrm{Rep}((\check{G}\times\mathbb{G}_m)^e) \\ V \otimes \mathcal{O}_{\check{\mathfrak{g}}}\langle k\rangle & \longmapsto & V \end{array}$$

where ev_e is the map $K(-,e)$. This functor is fully faithful. $\square$

Let $V = V(1)$, the natural representation of $\check{G} = \mathbf{SL}_2(\mathbb{C})$. Let $M = V \otimes V \otimes \mathcal{O}_{\check{\mathfrak{g}}^*}$. Using the fact from [YZ11] above and Lemma 11.1.1, we can find a unique map $M \to M\langle 2\rangle$ that corresponds to the action of $e : V \otimes V \to V \otimes V\langle 2\rangle$ on the first factor.

Lemma 11.1.2. *The map*

$$\begin{array}{rcl} V \otimes V & \xrightarrow{\phi} & V \otimes V \otimes \check{\mathfrak{g}}^* \\ x_1 \otimes x_2 & \longmapsto & \frac{1}{8}(x_1 \otimes x_2 \otimes h) + \frac{1}{4}(y_1 \otimes x_2 \otimes e) \\ x_1 \otimes y_2 & \longmapsto & \frac{1}{8}(x_1 \otimes y_2 \otimes h) + \frac{1}{4}(y_1 \otimes y_2 \otimes e) \\ y_1 \otimes x_2 & \longmapsto & \frac{1}{4}(x_1 \otimes x_2 \otimes f) - \frac{1}{8}(y_1 \otimes x_2 \otimes h) \\ y_1 \otimes y_2 & \longmapsto & \frac{1}{4}(x_1 \otimes y_2 \otimes f) - \frac{1}{8}(y_1 \otimes y_2 \otimes h) \end{array}$$

is the unique $\check{G}$-equivariant map that corresponds to the action of $e : V \to V\langle 2\rangle$.

Proof. Since $M = V \otimes V \otimes \check{\mathfrak{g}}^*$ is a free coherent sheaf, a map $M \to M\langle 2\rangle$ is determined by a map of $\check{G}$-representations $V \otimes V \to V \otimes V \otimes \check{\mathfrak{g}}^*$ where $\check{\mathfrak{g}}^*$ is the degree 2 part of $\mathcal{O}_{\mathcal{N}}$. We want a $\check{G}$-equivariant map $\phi : V \otimes V \to V \otimes V \otimes \check{\mathfrak{g}}^*$ that makes the following diagram commute

$$\begin{tikzcd} V \otimes V \arrow[r, "\phi"] \arrow[d, "\mathrm{id}\times\mathrm{id}"'] & V \otimes V \otimes \mathfrak{g} \arrow[d, "\mathrm{ev}_e"] \\ V \otimes V \arrow[r, "e\cdot(-)\otimes\mathrm{id}"'] & V \otimes V \end{tikzcd}$$

Choose a basis $\{x_1 \otimes x_2, x_1 \otimes y_2, y_1 \otimes x_2, y_1 \otimes y_2\}$ and $\{x_1 \otimes x_2 \otimes e, x_1 \otimes x_2 \otimes f, x_1 \otimes x_2 \otimes h, x_1 \otimes y_2 \otimes e, x_1 \otimes y_2 \otimes f, x_1 \otimes y_2 \otimes h, y_1 \otimes x_2 \otimes e, y_1 \otimes x_2 \otimes f, y_1 \otimes x_2 \otimes h, y_1 \otimes y_2 \otimes e, y_1 \otimes y_2 \otimes f, y_1 \otimes y_2 \otimes h\}$ for $V \otimes V$ and $V \otimes V \otimes \mathfrak{g}$ respectively. Since ϕ is a $\check{G}$-equivariant map, ϕ is required to preserve weight.

Let us compute the weights of each of these basis elements. Notice that $h \cdot x_i = x_i$ and $h \cdot y_i = -y_i$ for $i \in \{1, 2\}$. Also, $[h, e] = 2e$, $[h, f] = -2f$, and $[h, h] = 0$. The weights of the basis elements can be found by summing the weights of the factors.

We list the weights of elements in the following table:

wt -4	wt -2	wt 0	wt 2	wt 4
$y_1 \otimes y_2 \otimes f$	$y_1 \otimes y_2$	$x_1 \otimes y_2$	$x_1 \otimes x_2$	$x_1 \otimes x_2 \otimes e$
	$x_1 \otimes y_2 \otimes f$	$y_1 \otimes x_2$	$x_1 \otimes x_2 \otimes h$	
	$y_1 \otimes x_2 \otimes f$	$x_1 \otimes x_2 \otimes f$	$x_1 \otimes y_2 \otimes e$	
	$y_1 \otimes y_2 \otimes h$	$x_1 \otimes y_2 \otimes h$	$y_1 \otimes x_2 \otimes e$	
		$y_1 \otimes x_2 \otimes h$		
		$y_1 \otimes y_2 \otimes e$		

Since ϕ is required to preserve weight, we have the following first approximation of ϕ:

$$
\begin{array}{rcl}
V \otimes V & \xrightarrow{\phi} & V \otimes V \otimes \mathfrak{g} \\
x_1 \otimes x_2 & \longmapsto & a_{x_1x_2h}(x_1 \otimes x_2 \otimes h) + a_{x_1y_2e}(x_1 \otimes y_2 \otimes e) + a_{y_1x_2e}(y_1 \otimes x_2 \otimes e) \\
x_1 \otimes y_2 & \longmapsto & b_{x_1x_2f}(x_1 \otimes x_2 \otimes f) + b_{x_1y_2h}(x_1 \otimes y_2 \otimes h) + b_{y_1x_2h}(y_1 \otimes x_2 \otimes h) \\
 & & +\ b_{y_1y_2e}(y_1 \otimes y_2 \otimes e) \\
y_1 \otimes x_2 & \longmapsto & c_{x_1x_2f}(x_1 \otimes x_2 \otimes f) + c_{x_1y_2h}(x_1 \otimes y_2 \otimes h) + c_{y_1x_2h}(y_1 \otimes x_2 \otimes h) \\
 & & +\ c_{y_1y_2e}(y_1 \otimes y_2 \otimes e) \\
y_1 \otimes y_2 & \longmapsto & d_{x_1y_2f}(x_1 \otimes y_2 \otimes f) + d_{y_1x_2f}(y_1 \otimes x_2 \otimes f) + d_{y_1y_2h}(y_1 \otimes y_2 \otimes h)
\end{array}
$$

First, let us impose that ϕ be $\check{G}$-equivariant; that is, $\phi(g \cdot x) = g \cdot \phi(x)$ for every $x \in V \otimes V$ and $g \in \{e, f, h\}$. Checking this condition on every basis vector of $V \otimes V$, we get a system of 22 equations in the 14 unknown coefficients above.

Solving this system, we get the following

$$\begin{array}{rcl}
V \otimes V & \xrightarrow{\phi} & V \otimes V \otimes \mathfrak{g} \\
x_1 \otimes x_2 & \longmapsto & -m(x_1 \otimes x_2 \otimes h) + n(x_1 \otimes y_2 \otimes e) + (-n-2m)(y_1 \otimes x_2 \otimes e) \\
x_1 \otimes y_2 & \longmapsto & k(x_1 \otimes x_2 \otimes f) + (-\frac{1}{2}k - \frac{1}{2}n - m)(x_1 \otimes y_2 \otimes h) \\
& & + (-\frac{1}{2}k + \frac{1}{2}n)(y_1 \otimes x_2 \otimes h) + (-k-2m)(y_1 \otimes y_2 \otimes e) \\
y_1 \otimes x_2 & \longmapsto & (-k-2m)(x_1 \otimes x_2 \otimes f) + (\frac{1}{2}k - \frac{1}{2}n)(x_1 \otimes y_2 \otimes h) \\
& & + (\frac{1}{2}k + \frac{1}{2}n + m)(y_1 \otimes x_2 \otimes h) + k(y_1 \otimes y_2 \otimes e) \\
y_1 \otimes y_2 & \longmapsto & (-n-2m)(x_1 \otimes y_2 \otimes f) + n(y_1 \otimes x_2 \otimes f) + m(y_1 \otimes y_2 \otimes h)
\end{array}$$

where k, m, and n are free variables.

Imposing that ϕ make the diagram above commute, we are able to discern that $k = 0$, $m = -\frac{1}{8}$, and $n = 0$ and get a unique map

$$\begin{array}{rcl}
V \otimes V & \xrightarrow{\phi} & V \otimes V \otimes \mathfrak{g} \\
x_1 \otimes x_2 & \longmapsto & \frac{1}{8}(x_1 \otimes x_2 \otimes h) + \frac{1}{4}(y_1 \otimes x_2 \otimes e) \\
x_1 \otimes y_2 & \longmapsto & \frac{1}{8}(x_1 \otimes y_2 \otimes h) + \frac{1}{4}(y_1 \otimes y_2 \otimes e) \\
y_1 \otimes x_2 & \longmapsto & \frac{1}{4}(x_1 \otimes x_2 \otimes f) - \frac{1}{8}(y_1 \otimes x_2 \otimes h) \\
y_1 \otimes y_2 & \longmapsto & \frac{1}{4}(x_1 \otimes y_2 \otimes f) - \frac{1}{8}(y_1 \otimes y_2 \otimes h)
\end{array}$$

□

Applying the functor $(- \otimes_{\mathcal{O}_{\check{\mathfrak{g}}}} \mathcal{O}_{\check{\mathfrak{h}}})^{\check{T}}$ to ϕ, we get the map

$$\begin{array}{rcl}
(V \otimes V)^{\check{T}} & \xrightarrow{\widetilde{\phi}} & (V \otimes V)^{\check{T}} \otimes \mathfrak{h} \\
x_1 \otimes y_2 & \longmapsto & \frac{1}{8}(x_1 \otimes y_2 \otimes h) \\
y_1 \otimes x_2 & \longmapsto & -\frac{1}{8}(y_1 \otimes x_2 \otimes h)
\end{array}$$

Let us store in our memories the element $\begin{bmatrix} \frac{1}{8} & 0 \\ 0 & -\frac{1}{8} \end{bmatrix} \in \mathfrak{h}$. This element of $\mathfrak{h}$ was gotten by going along the top then right arrow in Diagram (1.2).

11.2 Computing $\mathrm{der}\mathbb{S}_G(\Psi_G(f))$

Consider the following extension of (11.1):

$$\begin{tikzcd}
\widetilde{\mathcal{N}} \arrow[r, hook, "\widetilde{j}"] \arrow[d, "\mu"'] & \mathrm{Gr}^1_G \widetilde{\times} \mathrm{Gr}^1_G \arrow[r, "\iota"] \arrow[dr, "m"] & \mathrm{Gr}_G \widetilde{\times} \mathrm{Gr}_G \arrow[r, "\mathrm{pr}_1"] & \mathrm{Gr}_G \\
\mathcal{N} \simeq (\mathrm{Gr}^{\mathrm{sm}})^\circ \arrow[rr, hook, "j"] & & \mathrm{Gr}^{\mathrm{sm}} &
\end{tikzcd} \tag{11.2}$$

where the square on the left is Cartesian. Recall that

$$f := m_!(c_1(\iota^* \mathrm{pr}_1^*(\mathcal{L}_{\det}))) \in \mathrm{Hom}^2(\mathrm{Spr}_{\mathrm{Gr}}, \mathrm{Spr}_{\mathrm{Gr}}).$$

For $G = \mathbf{PGL}(2, \mathbb{C})$, the map $\pi : \mathcal{M} \to \mathcal{N}$ is an isomorphism. Thus, $\Psi_G = \pi_* j^* = j^*$. Hence $\Psi_G(f) = j^*(f)$, and since the left square is Cartesian, we have

$$\Psi_G(f) = j^* m_!(c_1(\iota^* \mathrm{pr}_1^*(\mathcal{L}_{\det}))) \simeq \mu_! \widetilde{j}^*(c_1(\iota^* \mathrm{pr}_1^*(\mathcal{L}_{\det}))). \tag{11.3}$$

Note that $\mathrm{Gr}^1_G \widetilde{\times} \mathrm{Gr}^1_G \simeq G/B \widetilde{\times} G/B$ and that

$$\iota^* \mathrm{pr}_1^*(\mathcal{L}_{\det}) \simeq \mathcal{L}_{\lambda'} \widetilde{\boxtimes} \mathcal{L}_{\mathrm{triv}} \tag{11.4}$$

is a line bundle on this space. We will now focus on determining the weight $\lambda' \in \mathfrak{h}^*$.

Let V be the adjoint representation of G. The homomorphism $\phi : G \to \mathbf{GL}(V)$ induces a morphism of ind-schemes $\mathrm{Gr}_\phi : \mathrm{Gr}_G \to \mathrm{Gr}_{\mathbf{GL}(V)}$ over $\mathbb{C}$. Identify $\mathrm{Gr}_{\mathbf{GL}(V)}$ with the set of $\mathcal{O}$-lattices in $V \otimes F$. Consider the lattice $\Lambda_0 := V \otimes_{\mathbb{C}} \mathcal{O}$. We can define a line bundle $\mathcal{L}_{\Lambda_0}$ whose value at a point $\Lambda \in \mathrm{Gr}_{\mathbf{GL}(V)}$ is the line

$$\det(\Lambda : \Lambda_0) := \det(\Lambda / \Lambda \cap \Lambda_0) \otimes \det(\Lambda_0 / \Lambda \cap \Lambda_0)^{\otimes -1}$$

where $\det(-) = \wedge^{\mathrm{top}}(-)$. Let $\mathcal{L}^\phi_{\Lambda_0} := \mathrm{Gr}_\phi^* \mathcal{L}_{\Lambda_0}$ be the pullback line bundle on Gr_G. The next lemma computes the restriction $\mathcal{L}^\phi_{\Lambda_0}|_{\mathrm{Gr}^1}$.

Lemma 11.2.1 ([YZ11], (4.1))**.** *The element*

$$-\sum_{\chi^\vee \in \mathrm{weight}(V)} \dim V_{\chi^\vee} \cdot \langle \chi^\vee, \lambda \rangle \chi^\vee$$

is the restriction of $c_1^T(\mathcal{L}^\phi_{\Lambda_0})$ to the point t^λ, if we identify $X^\bullet(T)$ with $H^2_T(\mathrm{pt})$.

Recall that the adjoint representation V has weights $\{-2, 0, 2\}$. Lemma 11.2.1 asserts that $\mathcal{L}^\phi_{\Lambda_0}|_{\mathrm{Gr}^1}$ is the line bundle on G/B of weight

$$-(1 \cdot \langle -2, 1 \rangle (-2) + 1 \cdot \langle 0, 1 \rangle (0) + 1 \cdot \langle 2, 1 \rangle (2)) = -4,$$

since the pairing of a weight $\chi^\vee$ and a coweight χ is given by $\langle \chi^\vee, \chi \rangle = \frac{\chi^\vee \cdot \chi}{2}$.

Lemma 11.2.2 ([YZ11])**.** *We have an isomorphism of line bundles on* Gr_G*:*

$$\mathcal{L}_{\det}^{\otimes d_V} \simeq \mathcal{L}^\phi_{\Lambda_0}.$$

Here,

$$d_V = \frac{1}{2} \sum_{\chi^\vee \in \mathrm{weight}(V)} \dim V_{\chi^\vee} \cdot \langle \chi^\vee, \theta \rangle^2$$

where θ is the coroot corresponding to the highest root $\theta^\vee$ of G.

Let us compute d_V for V the adjoint representation. In this case, $\theta = 2$, and we have

$$d_V = \frac{1}{2}(1 \cdot \langle -2, 2 \rangle^2 + 1 \cdot \langle 0, 2 \rangle^2 + 1 \cdot \langle 2, 2 \rangle^2) = 4.$$

Corollary 11.2.3. *In (11.4), $\lambda' = 1$.*

Proof. Note that $\mathcal{L}_{\lambda'}$ is the restriction $\mathcal{L}_{\det}|_{\mathrm{Gr}^1}$. However, Lemma 11.2.2 asserts that $\mathcal{L}_{\det}^{\otimes 4} \simeq \mathcal{L}^\phi_{\Lambda_0}$. But $\mathcal{L}_{\det}^{\otimes 4}|_{\mathrm{Gr}^1} \simeq \mathcal{L}^\phi_{\Lambda_0}|_{\mathrm{Gr}^1} = \mathcal{L}_4$ by Lemma 11.2.1. Hence, $\mathcal{L}_{\det}|_{\mathrm{Gr}^1} = \mathcal{L}_1$ as desired. $\square$

Combining Corollary 11.2.3 with (11.3), we have that

$$\Psi_G(f) = \mu_! \widetilde{j}^*(c_1(\mathcal{L}_1 \widetilde{\boxtimes} \mathcal{L}_{\mathrm{triv}})). \tag{11.5}$$

As in Section 5.4, we see that

$$\Psi_G(f) = \mu_!(\mathrm{Lus}_{\widetilde{j}^*(\mathcal{L}_1 \widetilde{\boxtimes} \mathcal{L}_{\mathrm{triv}})}) = \mu_!(\mathrm{Lus}_{\pi^*(\mathcal{L}_1)}), \tag{11.6}$$

where $\pi : \widetilde{\mathcal{N}} \to G/B$ is the projection map.

11.3 Generators of the Hom-algebras

In Section 9.3, we established that all the Hom-algebras in Diagram (9.1) are free modules. In this section, we will determine the graded ranks of the Hom-algebras as modules over $H^\bullet_G(\mathrm{pt})$.

First, let us compute the Hom-algebras (as graded complex vector spaces) of ICs on $\mathrm{Gr}^{\mathrm{sm}}$ corresponding to small representations of $\check{G}$.

Lemma 11.3.1. *We have*

$$\mathrm{Hom}^n_{\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr})}(\mathrm{IC}(\overline{\mathrm{Gr}^0}), \mathrm{IC}(\overline{\mathrm{Gr}^0})) \simeq \begin{cases} \mathbb{C} & \textit{if } n = 0, 4, 8, \ldots \\ 0 & \textit{else} \end{cases}$$

Proof. Recall that for any $\lambda \in \check{\Lambda}^+$, we have $\mathrm{IC}(\overline{\mathrm{Gr}^\lambda}) \simeq \underline{\mathbb{C}}_{\overline{\mathrm{Gr}^\lambda}}[\dim_{\mathbb{C}} \mathrm{Gr}^\lambda]$. Notice that

$$\begin{aligned} \mathrm{Hom}^n_{\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr})}(\mathrm{IC}(\overline{\mathrm{Gr}^0}), \mathrm{IC}(\overline{\mathrm{Gr}^0})) &\simeq \mathrm{Hom}^n_{\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr})}(\underline{\mathbb{C}}_{\overline{\mathrm{Gr}^0}}, \underline{\mathbb{C}}_{\overline{\mathrm{Gr}^0}}) \\ &\simeq H^n_G(\mathrm{pt}). \end{aligned}$$

But we know that

$$H^n_G(\mathrm{pt}) \simeq \begin{cases} \mathbb{C} & \text{if } n = 0, 4, 8, \ldots \\ 0 & \text{else} \end{cases}$$

∎

Lemma 11.3.2. *We have*

$$\mathrm{Hom}^n_{\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr})}(\mathrm{IC}(\overline{\mathrm{Gr}^2}), \mathrm{IC}(\overline{\mathrm{Gr}^0})) \simeq \begin{cases} \mathbb{C} & \textit{if } n = 2, 6, 10, \ldots \\ 0 & \textit{else} \end{cases}$$

Proof. Let $a : \overline{\mathrm{Gr}^2} \to (\overline{\mathrm{Gr}^0} \simeq \mathrm{pt})$ be the map to a point. Notice that

$$\begin{aligned}
\mathrm{Hom}^n_{\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr})}(\mathrm{IC}(\overline{\mathrm{Gr}^2}), \mathrm{IC}(\overline{\mathrm{Gr}^0})) &\simeq \mathrm{Hom}^n_{\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr})}(\underline{\mathbb{C}}_{\overline{\mathrm{Gr}^2}}[2], \underline{\mathbb{C}}_{\overline{\mathrm{Gr}^0}}) \\
&\simeq \mathrm{Hom}^{n-2}_{\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr})}(a^*\underline{\mathbb{C}}_{\overline{\mathrm{Gr}^0}}, \underline{\mathbb{C}}_{\overline{\mathrm{Gr}^0}}) \\
&\simeq \mathrm{Hom}^{n-2}_{\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr})}(\underline{\mathbb{C}}_{\overline{\mathrm{Gr}^0}}, a_*\underline{\mathbb{C}}_{\overline{\mathrm{Gr}^0}}) \\
&\simeq \mathrm{Hom}^{n-2}_{\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr})}(\underline{\mathbb{C}}_{\overline{\mathrm{Gr}^0}}, \underline{\mathbb{C}}_{\overline{\mathrm{Gr}^0}}) \\
&\simeq H^{n-2}_{G(\mathfrak{O})}(\mathrm{pt}).
\end{aligned}$$

But we know that

$$H^{n-2}_{G(\mathfrak{O})}(\mathrm{pt}) \simeq \begin{cases} \mathbb{C} & \text{if } n = 2, 6, 10, \ldots \\ 0 & \text{else} \end{cases}$$

and the result follows. $\square$

Lemma 11.3.3. *We have*

$$\mathrm{Hom}^n_{\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr})}(\mathrm{IC}(\overline{\mathrm{Gr}^0}), \mathrm{IC}(\overline{\mathrm{Gr}^2})) \simeq \begin{cases} \mathbb{C} & \textit{if } n = 2, 6, 10, \ldots \\ 0 & \textit{else} \end{cases}$$

Proof. Notice that

$$\begin{aligned}
\mathrm{Hom}^n_{\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr})}(\mathrm{IC}(\overline{\mathrm{Gr}^0}), \mathrm{IC}(\overline{\mathrm{Gr}^2})) &\simeq \mathbb{D}\mathbb{D}\mathrm{Hom}^n_{\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr})}(\mathrm{IC}(\overline{\mathrm{Gr}^0}), \mathrm{IC}(\overline{\mathrm{Gr}^2})) \\
&\simeq \mathbb{D}\mathrm{Hom}^n_{\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr})}(\mathbb{D}\mathrm{IC}(\overline{\mathrm{Gr}^2}), \mathbb{D}\mathrm{IC}(\overline{\mathrm{Gr}^0})) \\
&\simeq \mathbb{D}\mathrm{Hom}^n_{\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr})}(\mathrm{IC}(\overline{\mathrm{Gr}^2}), \mathrm{IC}(\overline{\mathrm{Gr}^0})) \\
&\simeq \mathbb{D}H^{n-2}_{G(\mathfrak{O})}(\mathrm{pt})
\end{aligned}$$

from the computation in the proof of Lemma 11.3.2. But this is just the dual vector space, so as a graded vector space, our answer does not change. $\square$

Lemma 11.3.4. *We have the following*

$$\mathrm{Hom}^n_{\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr})}(\mathrm{IC}(\overline{\mathrm{Gr}^2}), \mathrm{IC}(\overline{\mathrm{Gr}^2})) \simeq \begin{cases} \mathbb{C}^2 & \textit{if } n = 4, 8, 12, \ldots \\ \mathbb{C} & \textit{if } n = 0 \textit{ and } n = 2, 6, 10, \ldots \\ 0 & \textit{else} \end{cases}$$

Proof. This lemma takes significantly more work than the previous ones. First, we will need the equivariant cohomology of $\mathbb{P}^1$. First note that $G(\mathfrak{O})$ does not act on $\mathbb{P}^1$, but $G = G(\mathbb{C})$ does. We are interested in extracting the $G(\mathfrak{O})$-equivariant cohomology of $\overline{\mathrm{Gr}^2}$ from these calculations, though. A useful observation is that the functor $\mathcal{D}^{\mathrm{b}}_{G(\mathfrak{O})}(\mathrm{Gr}) \hookrightarrow \mathcal{D}^{\mathrm{b}}_{G(\mathbb{C})}(\mathrm{Gr})$ is fully faithful. Hence, we have an isomorphism of Hom groups. Thus, we can compute $G(\mathfrak{O})$-equivariant cohomology from G-equivariant cohomology.

Let us first compute the G-equivariant cohomology of $\mathbb{P}^1$. We have a closed and complementary open inclusion

$$i : \mathrm{pt} \hookrightarrow \mathbb{P}^1 \hookleftarrow \mathbb{A}^1 : j$$

This gives a distinguished triangle

$$i_* i^! \mathrm{IC}(\mathbb{P}^1) \to \mathrm{IC}(\mathbb{P}^1) \to j_* j^* \mathrm{IC}(\mathbb{P}^1) \xrightarrow{+}$$

Now apply $\mathrm{Hom}^n_{\mathrm{Semis}_G(\mathbb{P}^1)}(\mathrm{IC}(\mathbb{P}^1), -)$ and use adjunction of $({}^*, {}_*)$ and $({}_!, {}^!)$. We get a long exact sequence

$$\ldots \to \mathrm{Hom}^n_{\mathrm{Semis}_G(\mathrm{pt})}(i^* \mathrm{IC}(\mathbb{P}^1), i^! \mathrm{IC}(\mathbb{P}^1)) \to \mathrm{Hom}^n_{\mathrm{Semis}_G(\mathbb{P}^1)}(\mathrm{IC}(\mathbb{P}^1), \mathrm{IC}(\mathbb{P}^1))$$
$$\to \mathrm{Hom}^n_{\mathrm{Semis}_G(\mathbb{A}^1)}(j^* \mathrm{IC}(\mathbb{P}^1), j^* \mathrm{IC}(\mathbb{P}^1)) \to \ldots$$

Using the fact that $\mathrm{IC}(\mathbb{P}^1) \simeq \underline{\mathbb{C}}_{\mathbb{P}^1}[1]$ and $\omega_{\mathbb{P}^1} \simeq \underline{\mathbb{C}}_{\mathbb{P}^1}[2]$, we have

$$\ldots \to \mathrm{Hom}^n_{\mathrm{Semis}_G(\mathrm{pt})}(i^* \underline{\mathbb{C}}_{\mathbb{P}^1}[1], i^! \omega_{\mathbb{P}^1}[-1]) \to \mathrm{Hom}^n_{\mathrm{Semis}_G(\mathbb{P}^1)}(\underline{\mathbb{C}}_{\mathbb{P}^1}[1], \underline{\mathbb{C}}_{\mathbb{P}^1}[1])$$
$$\to \mathrm{Hom}^n_{\mathrm{Semis}_G(\mathbb{A}^1)}(j^* \underline{\mathbb{C}}_{\mathbb{P}^1}[1], j^* \underline{\mathbb{C}}_{\mathbb{P}^1}[1]) \to \ldots$$

Now, we collect the shifts and apply the pullback functors to get

$$\ldots \to \mathrm{Hom}^{n-2}_{\mathrm{Semis}_G(\mathrm{pt})}(\underline{\mathbb{C}}_{\mathrm{pt}}, \omega_{\mathrm{pt}}) \to \mathrm{Hom}^n_{\mathrm{Semis}_G(\mathbb{P}^1)}(\underline{\mathbb{C}}_{\mathbb{P}^1}, \underline{\mathbb{C}}_{\mathbb{P}^1})$$
$$\to \mathrm{Hom}^n_{\mathrm{Semis}_G(\mathbb{A}^1)}(\underline{\mathbb{C}}_{\mathbb{A}^1}, \underline{\mathbb{C}}_{\mathbb{A}^1}) \to \ldots$$

But the dualizing complex on a point is just the constant sheaf on a point. Thus, we have

$$\ldots \to \mathrm{Hom}^{n-2}_{\mathrm{Semis}_G(\mathrm{pt})}(\underline{\mathbb{C}}_{\mathrm{pt}}, \underline{\mathbb{C}}_{\mathrm{pt}}) \to \mathrm{Hom}^{n}_{\mathrm{Semis}_G(\mathbb{P}^1)}(\underline{\mathbb{C}}_{\mathbb{P}^1}, \underline{\mathbb{C}}_{\mathbb{P}^1}) \\ \to \mathrm{Hom}^{n}_{\mathrm{Semis}_G(\mathbb{A}^1)}(\underline{\mathbb{C}}_{\mathbb{A}^1}, \underline{\mathbb{C}}_{\mathbb{A}^1}) \to \ldots$$

But, we know that

$$\mathrm{Hom}^{n-2}_{\mathrm{Semis}_G(\mathrm{pt})}(\underline{\mathbb{C}}_{\mathrm{pt}}, \underline{\mathbb{C}}_{\mathrm{pt}}) \simeq H^{n-2}_G(\mathrm{pt}) \simeq \begin{cases} \mathbb{C} & \text{if } n = 2, 6, 10, \ldots \\ 0 & \text{else} \end{cases}$$

and since $\mathbb{A}^1$ is contractible, we also have that

$$\mathrm{Hom}^{n}_{\mathrm{Semis}_G(\mathbb{A}^1)}(\underline{\mathbb{C}}_{\mathbb{A}^1}, \underline{\mathbb{C}}_{\mathbb{A}^1}) \simeq H^{n}_G(\mathrm{pt}) \simeq \begin{cases} \mathbb{C} & \text{if } n = 0, 4, 8, \ldots \\ 0 & \text{else} \end{cases}$$

Putting this together, we get the equivariant cohomology of $\mathbb{P}^1$

$$\mathrm{Hom}^{n}_{\mathrm{Semis}_G(\mathbb{P}^1)}(\underline{\mathbb{C}}_{\mathbb{P}^1}, \underline{\mathbb{C}}_{\mathbb{P}^1}) \simeq H^{n}_G(\mathbb{P}^1) \simeq \begin{cases} \mathbb{C} & \text{if } n = 0, 2, 4, 6, \ldots \\ 0 & \text{else} \end{cases}$$

Let us get back to computing $\mathrm{Hom}^{n}_{\mathrm{Semis}_G(\mathrm{Gr})}(\mathrm{IC}(\overline{\mathrm{Gr}^2}), \mathrm{IC}(\overline{\mathrm{Gr}^2}))$. We have the closed and complementary open inclusions

$$i : \mathbb{P}^1 \hookrightarrow \mathrm{Gr}^{\mathrm{sm}} \hookleftarrow \mathcal{M} : j$$

Consider the distinguished triangle

$$i_* i^! \mathrm{IC}(\overline{\mathrm{Gr}^2}) \to \mathrm{IC}(\overline{\mathrm{Gr}^2}) \to j_* j^* \mathrm{IC}(\overline{\mathrm{Gr}^2}) \xrightarrow{+}$$

Apply $\mathrm{Hom}^{n}_{\mathrm{Semis}_G(\mathrm{Gr})}(\mathrm{IC}(\overline{\mathrm{Gr}_2}), -)$ and by a similar process as before, we end up with the long exact sequence

$$\ldots \to \mathrm{Hom}^{n-2}_{\mathrm{Semis}_G(\mathbb{P}^1)}(\underline{\mathbb{C}}_{\mathbb{P}^1}, \underline{\mathbb{C}}_{\mathbb{P}^1}) \to \mathrm{Hom}^{n}_{\mathrm{Semis}_G(\mathrm{Gr}^{\mathrm{sm}})}(\mathrm{IC}(\overline{\mathrm{Gr}^2}), \mathrm{IC}(\overline{\mathrm{Gr}^2})) \\ \to \mathrm{Hom}^{n}_{\mathrm{Semis}_G(\mathcal{M})}(\underline{\mathbb{C}}_{\mathcal{M}}, \underline{\mathbb{C}}_{\mathcal{M}}) \to \ldots$$

Together with the equivariant cohomology of $\mathbb{P}^1$ above and the fact that $\mathcal{M}$ is contractible (so has the same cohomology as a point), we get the result

$$\mathrm{Hom}^n_{\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}})}(\mathrm{IC}(\overline{\mathrm{Gr}^2}), \mathrm{IC}(\overline{\mathrm{Gr}^2})) \simeq \begin{cases} \mathbb{C}^2 & \text{if } n = 4, 8, 12, \ldots \\ \mathbb{C} & \text{if } n = 0 \text{ and } n = 2, 6, 10, \ldots \\ 0 & \text{else} \end{cases}$$

□

Armed with this information and the fact that

$$H^\bullet_{G(\mathfrak{O})}(\mathrm{pt}) \simeq \begin{cases} \mathbb{C} & \text{if } n = 0, 4, 8, \ldots \\ 0 & \text{else} \end{cases}$$

we can find the generators and ranks of our groups as modules over $H^\bullet_{G(\mathfrak{O})}(\mathrm{pt})$.

Corollary 11.3.5. *We have the following:*

- $\mathrm{Hom}^\bullet_{\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}})}(\mathrm{IC}(\overline{\mathrm{Gr}^0}), \mathrm{IC}(\overline{\mathrm{Gr}^0}))$ *is a free* $H^\bullet_{G(\mathfrak{O})}(\mathrm{pt})$*-module with rank* 1 *and generator in degree* 0*.*
- $\mathrm{Hom}^\bullet_{\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}})}(\mathrm{IC}(\overline{\mathrm{Gr}^0}), \mathrm{IC}(\overline{\mathrm{Gr}^2}))$ *is a free* $H^\bullet_{G(\mathfrak{O})}(\mathrm{pt})$*-module with rank* 1 *and generator in degree* 2*.*
- $\mathrm{Hom}^\bullet_{\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}})}(\mathrm{IC}(\overline{\mathrm{Gr}^2}), \mathrm{IC}(\overline{\mathrm{Gr}^0}))$ *is a free* $H^\bullet_{G(\mathfrak{O})}(\mathrm{pt})$*-module with rank* 1 *and generator in degree* 2*.*
- $\mathrm{Hom}^\bullet_{\mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}})}(\mathrm{IC}(\overline{\mathrm{Gr}^2}), \mathrm{IC}(\overline{\mathrm{Gr}^2}))$ *is a free* $H^\bullet_{G(\mathfrak{O})}(\mathrm{pt})$*-module with rank* 3 *and generators in degrees* 0*,* 2*, and* 4*.*

Since we have shown that $\mathrm{der}\mathcal{S}^{\mathrm{sm}}_G$ preserves the action of $H^\bullet_{G(\mathfrak{O})}(\mathrm{pt}) \simeq \mathcal{O}^{\check{G}}_{\check{\mathfrak{g}}^*}$ on Hom groups in Theorem 9.2.8, we obtain the following corollary.

Corollary 11.3.6. *We have the following:*

- $\underline{\mathrm{Hom}}_{\mathrm{Coh}^{\check{G}\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{g}}^*)}(V(0) \otimes \mathcal{O}_{\check{\mathfrak{g}}^*}, V(0) \otimes \mathcal{O}_{\check{\mathfrak{g}}^*})$ *is a free* $\mathcal{O}^{\check{G}}_{\check{\mathfrak{g}}^*}$*-module with rank* 1 *and generator in degree* 0*.*
- $\underline{\mathrm{Hom}}_{\mathrm{Coh}^{\check{G}\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{g}}^*)}(V(0) \otimes \mathcal{O}_{\check{\mathfrak{g}}^*}, V(2) \otimes \mathcal{O}_{\check{\mathfrak{g}}^*})$ *is a free* $\mathcal{O}^{\check{G}}_{\check{\mathfrak{g}}^*}$*-module with rank* 1 *and generator in degree* 2*.*

- $\underline{\mathrm{Hom}}_{\mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*)}(V(2)\otimes\mathcal{O}_{\check{\mathfrak{g}}^*}, V(0)\otimes\mathcal{O}_{\check{\mathfrak{g}}^*})$ *is a free* $\mathcal{O}_{\check{\mathfrak{g}}^*}^{\check{G}}$*-module with rank* 1 *and generator in degree* 2.

- $\underline{\mathrm{Hom}}_{\mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*)}(V(2)\otimes\mathcal{O}_{\check{\mathfrak{g}}^*}, V(2)\otimes\mathcal{O}_{\check{\mathfrak{g}}^*})$ *is a free* $\mathcal{O}_{\check{\mathfrak{g}}^*}^{\check{G}}$*-module with rank* 3 *and generators in degrees* 0, 2, *and* 4.

11.4 Rank 1 commutativity

Let us return to the situation of Section 10.2 to prove that Diagram (10.1) commutes.

Theorem 11.4.1. *Let* G *be a group of semisimple rank* 1 *and* $\check{G}$ *be its Langlands dual group. Then, there exists a natural isomorphism of functors*

$$\eta_G : \mathrm{der}\Phi_{\check{G}} \circ \mathrm{der}\mathcal{S}_G^{\mathrm{sm}} \Longleftrightarrow \mathrm{der}\mathbb{S}_G \circ \Psi_G.$$

Proof. By Section 10.1, it suffices to consider the case where $G = (\mathbb{C}^\times)^{n-1} \times \mathbf{PGL}(2,\mathbb{C})$. In this case, $\check{G} = (\mathbb{C}^\times)^{n-1} \times \mathbf{SL}(2,\mathbb{C})$. We would like to prove that there exists an isomorphism of objects

$$\eta : \mathrm{der}\Phi_{\check{G}}(\mathrm{der}\mathcal{S}_G^{\mathrm{sm}}(\mathrm{Spr}_{\mathrm{Gr}})) \xrightarrow{\sim} \mathrm{der}\mathbb{S}_G(\Psi_G(\mathrm{Spr}_{\mathrm{Gr}}))$$

that is natural; i.e., for every $f : \mathrm{Spr}_{\mathrm{Gr}} \to \mathrm{Spr}_{\mathrm{Gr}}[n]$, the diagram

$$\begin{tikzcd}
\mathrm{der}\Phi_{\check{G}}(\mathrm{der}\mathcal{S}_G^{\mathrm{sm}}(\mathrm{Spr}_{\mathrm{Gr}})) \arrow[r, "\eta"] \arrow[d, "\mathrm{der}\Phi_{\check{G}}(\mathrm{der}\mathcal{S}_G^{\mathrm{sm}}(f))"'] & \mathrm{der}\mathbb{S}_G(\Psi_G(\mathrm{Spr}_{\mathrm{Gr}})) \arrow[d, "\mathrm{der}\mathbb{S}_G(\Psi_G(f))"] \\
\mathrm{der}\Phi_{\check{G}}(\mathrm{der}\mathcal{S}_G^{\mathrm{sm}}(\mathrm{Spr}_{\mathrm{Gr}}[n])) \arrow[r, "\eta"] & \mathrm{der}\mathbb{S}_G(\Psi_G(\mathrm{Spr}_{\mathrm{Gr}}[n]))
\end{tikzcd} \tag{11.7}$$

P. Achar, A. Henderson, and S. Riche have constructed a W-equivariant η_0 for $n = 0$. By Section 11.3, it suffices to prove the commutativity of Diagram (11.7) by considering only

$$f := m_!(c_1(\iota^*\mathrm{pr}_1^*(\mathcal{L}_{\det}))) \in \mathrm{Hom}^2(\mathrm{Spr}_{\mathrm{Gr}}, \mathrm{Spr}_{\mathrm{Gr}}).$$

In Section 11.1, we computed $\mathrm{der}\Phi_{\check{G}}(\mathrm{der}\mathcal{S}^{\mathrm{sm}}_{\check{G}}(f))$, and in Section 11.2, we computed $\mathrm{der}\mathbb{S}_G(\Psi_G(f))$. Rewriting Diagram (11.7) with these computations, we have

$$\begin{tikzcd}
(V\otimes V)^{\check{T}} \arrow[r, "\eta_0"] \arrow[d, "\mathrm{der}\Phi_{\check{G}}(\mathrm{der}\mathcal{S}^{\mathrm{sm}}_{\check{G}}(f))"'] & \mathbb{C}[W] \arrow[d, "\mathrm{der}\mathbb{S}_G(\Psi_G(f))"] \\
(V\otimes V)^{\check{T}}\otimes\check{\mathfrak{h}}^* \arrow[r, "\eta_0\otimes\mathrm{id}"] & \mathbb{C}[W]\otimes\check{\mathfrak{h}}^*
\end{tikzcd} \tag{11.8}$$

Let $\{u = x\otimes y + y\otimes x, w = x\otimes y - y\otimes x\}$ be a generating set for $(V\otimes V)^{\check{T}}$ and $\{\tau = 1+s, \sigma = 1-s\}$ be a generating set for $\mathbb{C}[W]$. Since η_0 is W-equivariant, we have $\eta_0(u) = \sigma$ and $\eta_0(w) = \tau$. We have the following two diagrams

$$\begin{tikzcd}
u \arrow[r, mapsto] \arrow[d, mapsto] & \sigma \\
\frac{1}{8}w\otimes h \arrow[r, mapsto] & \frac{1}{8}\tau\otimes h
\end{tikzcd}
\qquad
\begin{tikzcd}
w \arrow[r, mapsto] \arrow[d, mapsto] & \tau \\
\frac{1}{8}u\otimes h \arrow[r, mapsto] & \frac{1}{8}\sigma\otimes h
\end{tikzcd}$$

which chase u and w around Diagram (11.8). Using Lusztig's isomorphism $A_W \simeq \mathrm{Hom}^\bullet(\underline{\mathrm{Spr}}, \underline{\mathrm{Spr}})$ (see Remark 5.4.2) and an identification $\mathfrak{h}\simeq\mathfrak{h}^*$ using an invariant, symmetric, bilinear form, we compute $\mathrm{der}\mathbb{S}_G(\Psi_G(f))$ and fill in the diagrams below:

$$\begin{tikzcd}
u \arrow[r, mapsto] \arrow[d, mapsto] & \sigma \arrow[d, mapsto] \\
\frac{1}{8}w\otimes h \arrow[r, mapsto] & \frac{1}{8}\tau\otimes h
\end{tikzcd}
\qquad
\begin{tikzcd}
w \arrow[r, mapsto] \arrow[d, mapsto] & \tau \arrow[d, mapsto] \\
\frac{1}{8}u\otimes h \arrow[r, mapsto] & \frac{1}{8}\sigma\otimes h
\end{tikzcd}$$

$\square$

References

[ABG04] S. Arkhipov, R. Bezrukavnikov, and V. Ginzburg, *Quantum groups, the loop Grassmannian, and the Springer resolution*, J. Amer. Math. Soc. **17** (2004), no. 3, 595–678.

[AH13] P. Achar and A. Henderson, *Geometric Satake, Springer correspondence and small representations*, Selecta Math. (N.S.) **19** (2013), no. 4, 949–986.

[AHR15] Pramod N. Achar, Anthony Henderson, and Simon Riche, *Geometric Satake, Springer correspondence, and small representations II*, Represent. Theory **19** (2015), 94–166.

[AR13] Pramod N. Achar and Simon Riche, *Koszul duality and semisimplicity of Frobenius*, Ann. Inst. Fourier (Grenoble) **63** (2013), no. 4, 1511–1612.

[AR15a] ———, *Constructible sheaves on affine Grassmannians and geometry of the dual nilpotent cone*, Israel J. Math. **205** (2015), no. 1, 247–315.

[AR15b] Pramod N. Achar and Laura Rider, *Parity sheaves on the affine Grassmannian and the Mirković–Vilonen conjecture*, Acta Math. **215** (2015), no. 2, 183–216.

[BBD82] A. A. Beĭlinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171.

[BF08] R. Bezrukavnikov and M. Finkelberg, *Equivariant Satake category and Kostant-Whittaker reduction*, Mosc. Math. J. **8** (2008), no. 1, 39–72, 183.

[BGS96] A. Beilinson, V. Ginzburg, and W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), no. 2, 473–527.

[BL94] J. Bernstein and V. Lunts, *Equivariant sheaves and functors*, Lecture Notes in Mathematics, vol. 1578, Springer-Verlag, Berlin, 1994.

[BM81] W. Borho and R. MacPherson, *Représentations des groupes de Weyl et homologie d'intersection pour les variétés nilpotentes*, C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), no. 15, 707–710.

[Bra03] T. Braden, *Hyperbolic localization of intersection cohomology*, Transform. Groups **8** (2003), no. 3, 209–216.

[BY13] R. Bezrukavnikov and Z. Yun, *On Koszul duality for Kac-Moody groups*, Represent. Theory **17** (2013), 1–98.

[DM82] P. Deligne and J. Milne, *Tannakian categories in "Hodge cycles and motives"*, Springer, Lecture notes **900** (1982), 101–228.

[Gin95] V. Ginzburg, *Perverse sheaves on a loop group and Langlands' duality*, arxiv:alg-geom/9511007 (1995).

[GKM04] M. Goresky, R. Kottwitz, and R. MacPherson, *Homology of affine Springer fibers in the unramified case*, Duke Math. J. **121** (2004), no. 3, 509–561.

[Hai] T. Haines, *A proof of the Kazhdan–Lusztig purity theorem via the decomposition theorem of BBD*, preprint.

[Kos63] B. Kostant, *Lie group representations on polynomial rings*, Amer. J. Math. **85** (1963), 327–404.

[Lus81] G. Lusztig, *Green polynomials and singularities of unipotent classes*, Adv. in Math. **42** (1981), no. 2, 169–178.

[Lus83] ______, *Singularities, character formulas, and a q-analog of weight multiplicities*, Analysis and topology on singular spaces, II, III (Luminy, 1981), Astérisque, vol. 101, Soc. Math. France, Paris, 1983, pp. 208–229.

[Lus95] ______, *Cuspidal local systems and graded Hecke algebras. II*, Representations of groups (Banff, AB, 1994), CMS Conf. Proc., vol. 16, Amer. Math. Soc., Providence, RI, 1995, With errata for Part I [Inst. Hautes Études Sci. Publ. Math. No. 67 (1988), 145–202; MR0972345 (90e:22029)], pp. 217–275.

[MS98] J. G. M. Mars and T. A. Springer, *Hecke algebra representations related to spherical varieties*, Represent. Theory **2** (1998), 33–69 (electronic).

[MV07] I. Mirković and K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. (2) **166** (2007), no. 1, 95–143.

[Orl97] D. O. Orlov, *Equivalences of derived categories and $K3$ surfaces*, J. Math. Sci. (New York) **84** (1997), no. 5, 1361–1381, Algebraic geometry, 7.

[Ric14] S. Riche, *Kostant section, universal centralizer, and a modular derived Satake equivalence*, ArXiv e-prints (2014).

[Rid13] L. Rider, *Formality for the nilpotent cone and a derived Springer correspondence*, Adv. Math. **235** (2013), 208–236.

[Sch00] J. P. Schneiders, *Introduction to characteristic classes and index theory*, Textos de Matemàtica, vol. 13, Universidade de Lisboa, 2000.

[Spr84] T. A. Springer, *A purity result for fixed point varieties in flag manifolds*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **31** (1984), no. 2, 271–282.

[YZ11] Z. Yun and X. Zhu, *Integral homology of loop groups via Langlands dual groups*, Represent. Theory **15** (2011), 347–369.

Vita

Jacob P. Matherne was born in Opelousas, Louisiana. He completed his undergraduate studies at Northwestern State University of Louisiana, earning a Bachelor of Science degree in Mathematics in May 2010. He began graduate studies at Louisiana State University in August 2010. He earned a Master of Science degree in Mathematics from Louisiana State University in December 2011. He is currently a candidate for the Doctor of Philosophy degree in Mathematics to be awarded in August 2016.