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Computation on metric spaces via domain theory

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Abstract

The purpose of this paper is to survey recent approaches to realizing (or embedding) a Polish space as the set of maximal points of a continuous domain. Such realizations provide a convenient framework in which to model certain computational algorithms on the space and a useful alternate approach via the probabilistic power domain to measure theory and integration on the space. © 1998 Elsevier Science B.V.

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1. Introduction

In an early paper, Scott [27] suggested that partially ordered structures such as the set of all closed subintervals of a closed real interval (viewed as a partially ordered set ordered by reverse inclusion) should be useful for the study of continuous and computable functions on the closed interval. Consider, for example, a standard basic method for numerically approximating a zero of a continuous function, the bisection method. If a continuous function \( f \) defined on \( I := [0, 1] \) satisfies \( f(0) < 0 \) and \( f(1) > 0 \), then we repeatedly divide into two subintervals of equal length, at each stage choosing one for which \( f \) is positive on the right-hand endpoint and negative on the left-hand endpoint (this is always possible unless we reach a zero of \( f \) as an endpoint at some stage). By this procedure we obtain a sequence of nested intervals having a one point intersection, and it follows from continuity that this point must be a zero of \( f \). A purpose of this paper is to identify essential mathematical features of such computational examples, abstract and axiomatize these features, and indicate how domain theory can provide general
constructions that provide an appropriate computational framework for the space \( I \) and a much wider class of spaces besides. The construction of these computational frameworks will typically be "hyperspace-like" constructions.

**Example 1.1.** We model the bisection algorithm and other computational algorithms on \( I = [0, 1] \) in the "approximate unit interval"

\[
P_I := \{ [a, b]: 0 < a < b < 1 \}.
\]

Points of \( I \) are identified with the degenerate closed intervals. Since successful algorithms for computing some real number compute smaller and smaller intervals ("approximate reals") containing that number, we order \( P_I \) with the "information ordering" (i.e., smaller intervals give more information about the point in question):

\[
[a, b] \subseteq [c, d] \iff [a, b] \supseteq [c, d].
\]

The computation of the bisection algorithm may be represented in the model \( P_I \) by an increasing sequence \([a_n, b_n]\) \((m < n \implies [a_m, b_m] \subseteq [a_n, b_n])\) with intersection point \( \bar{x} = \bigcup [a_n, b_n] \). (The supremum with arrow on the right-hand side of the equality indicates the sequence is increasing and has least upper bound \( \bar{x} \).)

Since typical computational algorithms involve infinitely many iterations of increasing accuracy, in actual computations one must settle for approximate solutions. In the approximate unit interval \( P_I \), an \( \varepsilon \)-approximation of \( a = [a, a] \) is given by the interval \([a - \varepsilon, a + \varepsilon]\). More generally, \([a, \bar{a}]\) approximates \([b, \bar{b}]\) if

\[
[b, \bar{b}] \subseteq \text{int}([a, \bar{a}]), \quad \text{that is, } \ a < b < \bar{b} < \bar{a}.
\]

In this case we write \([a, \bar{a}] \ll [b, \bar{b}]\).

Ideas of computability on metric spaces related to those of Scott's had surfaced much earlier, for example, in [17] (see also [23]). Scott's suggestion was developed along various lines in [28] and [16]. Two recent developments have given a new impetus to such considerations. One of these is the introduction and ripening theory of the probabilistic power domain of a continuous poset. If a continuous poset can be appropriately associated with a topological space, then one has at hand via the probabilistic power domain what can be a very effective and powerful tool for the study of a variety of problems associated with integration, measure and probability. In addition, Edalat has effectively used such structures in a wide variety of applications involving measures, dynamical systems, and iterated function systems and fractals [5,6]. The central purpose of this survey is to consider in some detail how metric spaces arise as the set of maximal points of a continuous poset with a countable base and characterize such spaces as the class of Polish spaces, where the maximal points are endowed with the relative Scott topology.

The earlier sections introduce the basic concepts and machinery that is needed in the paper. For other introductory references and points of connection with theoretical computer science we refer the reader to [3, Chapter 3], [20] and [1].
2. Continuous domains

The example of the approximate unit interval in the preceding section and the related comments are suggestive of a more general approach we take to constructing computational models for a variety of topological spaces. Given a topological space $X$, we seek to embed it in a larger "computational environment", namely a partially ordered set in which the points of the space sit as maximal "ideal" elements with "sufficiently many" approximating elements below. The partial order represents intuitively the information order. Sequential computations will be viewed in the models as increasing sequences of elements. Hence the appropriate type of completeness in this context is the existence of suprema for increasing sequences, and more generally for directed sets.

**Definition 2.1.** A partially ordered set $P$ is said to be directed complete and is called a dcpo (directed complete partially ordered set) if every directed set $D (a, b \in D$ implies there exists $c \in D$ with $a \leq c, b \leq c)$ has a supremum. We assume always that directed sets are nonempty. If the empty set is also required to have a supremum, then $D$ must have a least or bottom element, denoted $\bot$. A pointed dcpo is one with a bottom element.

The theory of "continuous partial orders" that has emerged in the last twenty years provides a suitable notion of approximation. An element $x$ is viewed as approximating an element $y$ if any computation of $y$ (an increasing sequence with supremum greater than or equal to $y$) yields $x$ at some finite stage.

**Definition 2.2.** Let $P$ be a partially ordered set. For $x \leq y \in P$, we say that $x$ approximates $y$, written $x \ll y$, if

$$D \text{ directed, } w = \bigsqcup D, \ y \sqsubseteq w \Rightarrow x \sqsubseteq d, \text{ for some } d \in D.$$  

A continuous poset is a partially ordered set $P$ in which each element is the directed supremum of all elements which approximate it, i.e.,

$$\forall x \in P, \ x = \bigsqcup \{ y \in P: y \ll x \}.$$  

A continuous poset which is also a dcpo is called a continuous domain.

The following proposition lists basic properties of the approximation relation.

**Proposition 2.3.** Let $P$ be a continuous domain. The following properties hold:

1. $x \ll y \Rightarrow x \sqsubseteq y$;
2. $x \sqsubseteq y \ll u \sqsubseteq v \Rightarrow x \ll v$;
3. (interpolation property) $x \ll z \Rightarrow \exists y \in P, \ x \ll y \ll z$.

In what follows we concentrate on continuous domains. The algebraic domains form an important special class of domains.

**Definition 2.4.** Let $P$ be a partially ordered set. If $x \in P$ approximates itself, $x \ll x$, then $x$ is called a compact element. A dcpo $P$ is called an algebraic domain if every element is the directed supremum of all the compact elements less than or equal to it.
Just as topological spaces can alternately be defined in terms of a basis of open sets, a continuous domain can be defined in terms of an appropriate notion of basis.

**Definition 2.5.** Let $P$ be a dcpo. A subset $B$ of $P$ is a *basis* for $P$ if for each $x \in P$, there exists a directed set $B_x \subseteq B$ such that each element of $B_x$ approximates $x$ and $\bigcup B_x = x$. An $\omega$-continuous domain is a (continuous) dcpo which possesses a countable basis.

If $P$ is a continuous domain, then $P$ is a basis for itself (with $B_x$ the subset of all elements approximating $x$). Conversely, it is fairly straightforward to verify that a dcpo $P$ which possesses a basis is a continuous domain (see, for example, [11]). This is the reason that the word “continuous” appears in parenthesis in the definition of an $\omega$-continuous domain.

Besides $\omega$-continuous domains there are other important subclasses of continuous domains.

**Definition 2.6.** Let $P$ be a pointed continuous domain. Then $P$ is a bounded complete continuous domain if each pair $x, y \in P$ which is bounded above has a least upper bound. This turns out to be equivalent to requiring that $P$ be a complete continuous semilattice, i.e., a continuous domain in which every nonempty set has an infimum. A complete continuous semilattice which contains a largest element $T$ is actually a complete lattice and is called a continuous lattice. An pointed algebraic domain which is $\omega$-continuous and bounded complete is called a Scott domain.

We illustrate these ideas with some basic examples.

**Example 2.7.** The approximate unit interval $P_I$ ordered by reverse inclusion is a bounded complete $\omega$-continuous domain with the approximation relation described earlier.

**Example 2.8.** Let $X$ be a Hausdorff space. Let $O(X)$ denote the lattice of open sets ordered by inclusion. Then $O(X)$ is a complete lattice. It is a continuous lattice if and only if $X$ is locally compact. In this case the approximation relation is given by $U \ll V$ if and only if there exists a compact set $K$ such that $U \subseteq K \subseteq V$. If $X$ is locally compact and totally disconnected, then $O(X)$ is an algebraic lattice, and the compact elements are the open sets which are also compact. In the locally compact case, $O(X)$ is $\omega$-continuous if and only if $X$ is second countable, i.e., has a countable base for the topology (this is equivalent to $X$ being separable metrizable). Most of these results extend in the non-Hausdorff setting to sober spaces, spaces in which every irreducible closed set is the closure of a unique point in the space (see [13,10]).

**Example 2.9.** Let $X$ be a Hausdorff space, and let $K(X)$ be the space of nonempty compact subsets ordered by reverse inclusion. Then $K(X)$ is a dcpo, with directed suprema being intersections of decreasing families. The dcpo $K(X)$ is a continuous domain (actually a semilattice with respect to union, but not complete) if $X$ is locally
compact. In this case \( K_1 \ll K_2 \) if and only if \( K_2 \) is contained in the interior of \( K_1 \). Again in the locally compact case, \( K(X) \) is \( \omega \)-continuous if and only if \( X \) is second countable. If \( X \) is compact, then \( K(X) \) is a complete continuous semilattice.

3. The Scott topology

We think of the elements of an ascending sequence providing increasingly better approximations to the supremum of the sequence and, in the information ordering, to elements below the supremum. These considerations yield a notion of convergence that can be precisely captured topologically.

**Definition 3.1.** Let \( P \) be a dcpo. A subset \( U \) is **Scott open** if
- \( U = \uparrow U := \{ z \in P : \exists x \in U, \; x \subseteq z \} \), and
- \( D \) directed, \( \bigcup D = d \in U \Rightarrow D \) is eventually in \( U \), i.e., there exists \( b \in D \) such that \( d \in U \) for \( b \subseteq d \).

The Scott open sets form a topology called the **Scott topology**. Dually a subset \( A \) is **Scott closed** if
- \( A = \downarrow A := \{ y \in P : \exists x \in A, \; y \subseteq x \} \),
- \( D \) directed, \( D \subseteq A \Rightarrow \bigcup D \in A \).

Given a topology on a dcpo \( P \), a directed set \( D \) is said to **converge to** \( x \in P \) if given any open set \( U \) containing \( x \), there exists \( b \in D \) such that \( d \in U \) if \( b \subseteq d \). In the Scott topology a directed set converges to the elements it "computes".

**Proposition 3.2.** Let \( P \) be a dcpo. A directed set \( D \) converges to \( x \in P \) in the Scott topology if and only if \( x \subseteq \bigcup D \).

The Scott topology is very natural and useful in the study of continuous domains. Via the Scott topology the fundamental concepts of domain theory have alternate topological descriptions. However, it departs radically from classical topology since it is a non-Hausdorff topology. But it is precisely such topologies that lend themselves to the study of partially ordered sets.

**Definition 3.3.** Let \( X \) be a topological space. The **order of specialization** of \( X \) is defined by

\[ x \subseteq y \iff x \in \overline{\{y\}}. \]

If \( P \) is a partially ordered set, then a topology on \( P \) is called **compatible** if its order of specialization agrees with the original partial order.

Note that in general the order of specialization is only a quasiorder (reflexive and transitive), that it is a partial order precisely when \( X \) is a \( T_0 \) space, and that it is the
diagonal relation precisely when \( X \) is \( T_1 \). Thus the order of specialization becomes mathematically interesting precisely in the context of \( T_0 \) spaces.

In a continuous domain, there are close connections between the Scott open sets and the approximation relation.

**Proposition 3.4.** Let \( P \) be a continuous domain equipped with the Scott topology.

(i) A subset \( U \) of \( P \) is open if and only if

\[
U = \uparrow U, \quad \text{and} \quad y \in U \Rightarrow \exists x \in U, \ x \ll y.
\]

(ii) \( x \ll y \iff \uparrow x \) is a neighborhood of \( y \).

(iii) The sets \( \uparrow x, \ x \in P \) form an open basis for the topology, where

\[
\uparrow x := \{ y \in P : x \ll y \}.
\]

(iv) The Scott topology has a countable base if and only if \( P \) is \( \omega \)-continuous. In this case, if \( B \) is a countable basis for \( P \), then \( \uparrow b, \ b \in B \) is a countable base for the Scott topology.

The directed complete partially ordered sets form the objects of a category DCPO. The appropriate morphisms are the continuous functions, the order preserving functions which also preserve suprema of directed sets. Such functions may be viewed as the "computation preserving" functions. They have a natural topological characterization, one which provides another motivation for the Scott topology.

**Proposition 3.5.** Let \( P, Q \) be directed complete partially ordered sets equipped with the Scott topology and let \( f : P \to Q \) be a function. The following are equivalent:

(i) The function \( f \) is order preserving and preserves directed suprema.

(ii) The function \( f \) is (Scott) continuous.

An irreducible closed set in a topological space is a nonempty closed set which cannot be written as the union of two strictly smaller closed sets. A topological space is said to be sober if every irreducible closed set is the closure of an unique singleton subset. Sober spaces are automatically \( T_0 \)-spaces. In Hausdorff spaces irreducible closed sets are precisely the singleton subsets, and hence all Hausdorff spaces are sober. A space is locally compact if each point has a basis of compact neighborhoods (where the neighborhoods are not required to be open sets). Since some properties of locally compact Hausdorff spaces carry over to locally compact sober spaces, the next theorem (see [18] or [10]) is often useful.

**Theorem 3.6.** A continuous domain endowed with the Scott topology is a locally compact sober space.
4. Ordered spaces

As we have seen in the previous section, the theory of $T_0$-spaces provides a convenient mathematical framework for relating topological and order theoretic notions. But there is an earlier approach to relating ordered and topological structures that dates back to the work of Nachbin [24] in the middle of this century.

Definition 4.1. An ordered topological space is a topological space $X$ equipped with a partial order with closed graph, i.e., the set $\{(x, y): x \leq y\}$ is a closed subset of $X \times X$ (equipped with the product topology).

Since in an ordered topological space the diagonal of $X \times X$ is given by $\leq \cap \leq$, and is thus closed, it follows that an ordered topological spaces is always Hausdorff.

Continuous domains admit a natural topology which refines the Scott topology and provides the structure of an ordered topological space.

Definition 4.2. Let $P$ be a partially ordered set. The $d$-weak topology on $P$ is defined by taking all sets $P \setminus \downarrow z$ (complements of principal ideals), $z \in P$, as a subbasis of open sets. The Lawson-topology on $P$ is defined as the join of the Scott and the $d$-weak topologies.

Proposition 4.3. If $P$ is a continuous domain, then $P$ equipped with the $L$-topology is an ordered topological space with a regular topology. If $P$ is $\omega$-continuous, then the $L$-topology is separable metrizable.

Proof. We show the complement of the graph of the order is open. Suppose $x \not\leq y$. Then there exists $z \ll x$ such that $z \not\leq y$. Then the set $(x, y)$ is a product of $L$-open sets which contains $(x, y)$ and misses the graph of the order. Thus the order is closed.

To show that a topology is regular, it suffices to show that each subbasic open set containing a point contains a closed neighborhood of the point. Let $x \in P$ and let $\uparrow z$ be a basic Scott open set containing $x$, where $z \ll x$. By the interpolation property (Proposition 2.3), there exists $y \in P$ such that $z \ll y \ll x$. Then $\uparrow y$ is a closed neighborhood of $x$ (Proposition 3.4) which is contained in $\uparrow z$. The case that the subbasic open set containing $x$ is of the form $P \setminus \uparrow y$ proceeds in a manner similar to the argument of the preceding paragraph.

If $P$ has a countable basis $B$, then the Scott topology has a countable basis by Proposition 3.4 and the sets $P \setminus \uparrow b$, $b \in B$, form a countable basis for the $d$-weak topology. Thus $P$ equipped with the $L$-topology is a regular Hausdorff space equipped with a countable basis, and is hence separable and metrizable. That it is also a Polish space is tied to the following discussion. $\square$

The preceding construction of a partially ordered topological space is a special example of a more general topological construction. Let $X$ be a locally compact sober space. A subset $A$ is called a saturated set if it is the intersection of open sets (this is equivalent
to its being an upper set, \( A = \uparrow A \), in the specialization order). The patch topology on \( X \) is the topology generated by the join of the original open sets together with the complements of all compact saturated sets. Then \( X \) equipped with the patch topology is a partially ordered topological space. If \( X \) is second countable, then \( X \) equipped with the patch topology is separable metrizable and is further a \( G_δ \) set in some compact metric space. Thus \( X \) is in this case a Polish space, a separable metrizable space that admits a complete metric. These results are all obtained via the spectral theory of continuous lattices (see [13] or [10], particularly Chapter V). For the following theorem, see [21] or [10].

**Theorem 4.4.** Let \( P \) be an \( \omega \)-continuous domain. Then the patch topology of the Scott topology is the \( L \)-topology, and hence \( P \) equipped with the \( L \)-topology is a Polish space.

### 5. Spaces of maximal points

In the preceding sections we have developed the prerequisite notions needed for defining “computational environments” for metric spaces.

We consider \( \omega \)-continuous domains \( P \) satisfying the condition

\[ p \in P \Rightarrow \exists A \text{ Scott closed in } P, \uparrow p \cap \text{Max}(P), \]

where \( \text{Max}(P) \) is the set of elements in \( P \) which are maximal in the partial order. Alternately

\[ \text{Scott topology } | \text{Max}(P) = \text{L-topology } | \text{Max}(P), \]

i.e., the Scott and \( L \)-topologies restricted to the set of maximal elements agree. Indeed the subbasic closed sets in the \( L \)-topology on \( P \) are either Scott closed or of the form \( \uparrow p \) for some \( p \in P \), and from this it follows easily that (1) and (2) are equivalent. In this case, \( \text{Max}(P) \) is a separable metric space, since the \( L \)-topology is separable metric for \( \omega \)-continuous domains by Proposition 4.3.

**Definition 5.1.** A separable metric space \( X \) is called a maximal point space if there exists an \( \omega \)-continuous domain \( P \) satisfying condition (1) (or equivalently (2)) such that \( X \) is homeomorphic to \( \text{Max}(P) \) equipped with the relative Scott topology. In this case the embedding

\[ X \hookrightarrow \text{Max}(P) \hookrightarrow P \]

is called a domain hull for \( X \).

Maximal point spaces were studied by Kamimura and Tang [16] for the case that the domain hulls were Scott continuous retracts of Scott domains. They called such spaces “total spaces”.

**Example 5.2.** The approximate unit interval \( P \) is a domain hull for \( I \), where the inclusion is the obvious one, \( x \mapsto \{x\} : I \to P \).
Example 5.3 (The upper space). For a locally compact metric space $X$, there is a standard domain hull for $X$ called the upper space $UX$, which consists of the set of all nonempty compact subsets of $X$ ordered by reverse inclusion. As in Example 2.9, $UX$ is an $\omega$-continuous domain and the homeomorphic injection $x \mapsto \{x\} : X \to UX$ is a domain hull for $X$. The Scott topology on $UX$ is the upper topology with basis

$$\Lambda(U) := \{K \in UX: \ K \subseteq U\}, \ \forall U \ \text{open in } X.$$ 

The $L$-topology is the usual Vietoris topology, or equivalently the topology on $UX$ induced by the Hausdorff metric.

The next example is a typical one arising in computer science settings.

Example 5.4 (The Cantor tree). Consider the set $P$ consisting of all finite and infinite strings of $\{0, 1\}$ (including the empty string $\bot$). The strings are ordered by the prefix order, i.e., one string is less than or equal to a second string if and only if it is a prefix of the second. The set $P$ endowed with this order is an $\omega$-continuous domain (actually a Scott domain) and the set of maximal elements $\text{Max}(P)$ consists of all infinite chains. The restriction of the Scott (or $L$-) topology to $\text{Max}(P)$ gives a space homeomorphic to the usual Cantor set. Hence the Cantor tree is a domain hull for the Cantor set.

The next theorem, which characterizes maximal point spaces, is the central result of [20].

Theorem 5.5. A metric space $X$ is a maximal point space if and only if it is a Polish space.

To show that a Polish space is a maximal point space, one needs a generalization of the upper space construction.

Proposition 5.6. Let $X$ be a dense $G_δ$-subset of a compact metric space $Y$, and let $UY$ be the upper space of nonempty compact subsets of $Y$. Then there exists a domain hull $x \mapsto \{x\} : X \to P \subseteq UY$ for $X$ satisfying

1. $P$ is a subset of $UY$ and is closed under directed sups in the inherited order;
2. $P$ is a continuous domain, and the relation $\ll_P$ of $P$ is the restriction of $\ll_{UY}$ to $P$;
3. for any countable base of $UY$, the members of the base belonging to $P$ form a countable base for $P$;
4. $P$ contains all nonempty compact subsets of $X$.

It is a standard topological result that a Polish space $X$ is homeomorphic to a $G_δ$-subset of $I^\omega$, a countable product of intervals (see, e.g., [2]). If $X$ is thus identified with this image in $I^\omega$, then one sees directly that $X$ is a dense $G_δ$ in its closure $Y$, a compact metric space, and then the preceding proposition can be applied to obtain that $X$ is a maximal point space.
To obtain the reverse implication one needs some further facts about \( \omega \)-continuous domains satisfying (1). One first proves the following useful lemma.

**Lemma 5.7.** Let \( P \) be an \( \omega \)-continuous domain satisfying (1). Then there exists a descending sequence \( \{U_n\} \) of Scott open sets such that

\[
\text{Max}(P) = \bigcap_n U_n.
\]

That a maximal point space \( X \) is Polish now follows from the facts that

(i) its hull \( P \) is the prime spectrum of the continuous lattice \( \sigma(P) \) of Scott open sets, which is compact metrizable under the \( L \)-topology,

(ii) the spectrum \( P \) is a \( G_\delta \) and its \( L \)-topology is the same as the relative \( L \)-topology,

(iii) \( X \) sits as a \( G_\delta \) in \( P \) by the preceding lemma, and

(iv) a \( G_\delta \) subset of a Polish space is again a Polish space.

6. The domain of closed formal balls

Edalat and Heckmann [9] later gave a much more direct and natural construction for a domain hull for a complete separable metric space than the one mentioned in the previous section. Their approach has the additional advantage of being functorial for the category of Lipschitz maps.

**Definition 6.1.** Let \( (X, d) \) be a metric space. The set of **closed formal balls** is given by

\[
B_X := X \times \mathbb{R}^+, \quad \text{where } \mathbb{R}^+ = [0, \infty).
\]

(Intuitively the pair \((x, r)\) represents the closed formal ball of radius \( r \) around \( x \).) A partial order \( \sqsubseteq \) of formal reverse inclusion is defined on \( B_X \) by

\[
(x, r) \sqsubseteq (y, s) \text{ if } d(x, y) \leq r - s.
\]

**Example 6.2.** Let \( X \) be a normed linear space. Then the ordered set of closed formal balls is order isomorphic to the set of closed balls ordered by reverse inclusion,

\[
(B_X, \sqsubseteq) \approx \left( \overline{B(x, \varepsilon)} : x \in X, \varepsilon \geq 0 \right),
\]

where the order isomorphism is the obvious one taking \((x, r)\) to the closed ball of radius \( r \) around \( x \). However, for more general metric spaces, this function need no longer be an order isomorphism.

**Proposition 6.3.** The partially ordered set \( B_X \) is a continuous poset with

\[
(x, r) \ll (y, s) \iff d(x, y) < r - s.
\]

It has maximal elements \( \text{Max}(B_X) \) consisting of all \((x, 0)\), \( x \in X \) and satisfies condition (1) of Section 5.
Proof. We verify only the validity of condition (1). The rest is straightforward and may also be found in [9]. Suppose that \((y, 0) \notin \uparrow(x, r)\). Then \(\varepsilon := d(y, x) - r > 0\). Then \(\uparrow(y, \varepsilon)\) is a Scott open set containing \((y, 0)\) and missing \(\uparrow(x, r)\), and thus we conclude that the complement of \(\uparrow(x, r)\) intersected with \(\text{Max}(B_X)\) is relatively Scott open.

Via the continuous poset of closed ordered balls, one has available an order theoretic approach to the theory of metric spaces. Standard properties of metric spaces often have very natural order theoretic counterparts in the corresponding ordered set of closed formal balls.

Proposition 6.4. Let \((X, d)\) be a metric space, and let \(B_X\) be the ordered set of closed formal balls.

1. \(X\) is complete \(\iff B_X\) is directed complete (a dcpo).
2. \(X\) is separable \(\iff B_X\) has a countable base, i.e., is \(\omega\)-continuous.

In light of the previous proposition, \(B_X\) is a continuous domain in the case that the metric space is complete. In this case we shall speak of \(B_X\) as the domain of closed formal balls.

Corollary 6.5. For a complete separable metric space \(X\), the domain of closed formal balls is a domain hull for \(X\) via

\[ x \mapsto (x, 0): X \to B_X. \]

Observe that the preceding construction carries through for the more general case of complete metric spaces, except that the domain hull of closed formal balls is no longer \(\omega\)-continuous if the metric space in not separable.

Let \(\mathcal{L}\) denote the category with objects metric spaces and morphisms Lipschitz maps \((f, c)\) where \(f: X \to Y\) satisfies \(d(fx, fx') \leq c d(x, x')\) for all \(x, x' \in X\). Let \(\mathcal{C}\) denote the category with objects continuous posets and morphisms Scott continuous functions. Then there is a functor \(B: \mathcal{L} \to \mathcal{C}\) which sends a space \(X\) to \(B_X\), the continuous poset of closed formal balls, and sends \((f, c): X \to Y\) to \(B(f, c): B_X \to B_Y\) defined by \(B(f, c)(x, r) = (fx, cr)\). Thus the construction of the poset of closed formal balls is functorial on \(\mathcal{L}\), and hence on any subcategory. In particular, it carries the complete separable metric spaces to domain hulls for those spaces.

7. Fixed point theory

The following elementary fixed point theorem is the basis for recursive constructions in theoretical computer science as they are modeled in the category of dcpo's.

Proposition 7.1. Let \(P\) be a dcpo with \(\bot\), and let \(f: P \to P\) be a Scott continuous function. The \(f\) has a least fixed point.
The least fixed point is easily seen to be the supremum of the sequence
\[ \bot \subseteq f(\bot) \subseteq f^2(\bot) \cdots \subseteq f^n(\bot) \cdots. \]
We observe that if one can show that the least fixed point is a maximal element, then the fixed point must be unique.

The construction of the domain of closed formal balls provides an alternate order theoretic approach to the theory of metric spaces in contrast to the usual topological methods. We illustrate this principle by giving an alternate order theoretic proof of the Banach Fixed Point Theorem.

**Theorem 7.2.** Suppose that \( X \) is a complete metric space and that \( f : X \to X \) is a Lipschitz map with Lipschitz constant \( c < 1 \). Then \( f \) has a unique fixed point.

**Proof.** Set \( g := Bf : BX \to BX \). For \( x \in X \), set \( R_x = d(x, fx)/(1 - c) \). Then for \( r \geq R_x \),
\[ d(x, fx) \leq (1 - c)r = r - cr, \text{ so } (x, r) \subseteq (fx, cr) = g(x, r). \]
Applying Proposition 7.1 to \( \uparrow(x, r) \), which is invariant under \( g \) by the preceding step, we obtain a fixed point. It is easy to see that a fixed point for \( g \) must have second coordinate equal to 0, hence must be a member of \( X \), and thus must be a fixed point for \( f \). By the remark after Proposition 7.1 we conclude that the fixed point in \( \uparrow(x, r) \) is unique. Since \( BX \) is the union of all \( \uparrow(x, r) \) for \( r \geq R_x \), we conclude the fixed point is unique. \( \square \)

Similar order theoretic methods can be used to establish the existence and uniqueness of limits of hyperbolic iterated function systems, see [5] and [9]. We recall that such systems are typically used for the generation of fractals. While the topological approaches are typically more direct and straightforward in the more elementary theory, the order theoretic approach sometimes gives slightly stronger results and provides significant new tools and insights for certain more complex problems, such as the study of the invariant measure on the limit of a weighted hyperbolic iterated function system.

### 8. The probabilistic power domain

The machinery of domain theory has in recent times been fruitfully applied to the theory of measure and integration and to applications of measure theory to dynamical systems and fractals (see [5,7,8]). These applications have come via the machinery of the (normalized) probabilistic power domain. Our main purpose in this section is to introduce this structure and connect it with the earlier material of this paper by deriving the new result that if \( P \) is a domain hull for the maximal point space \( X \), then the normalized probabilistic power domain is a domain hull for the space of Borel probability measures on \( X \) equipped with the topology of weak convergence.

**Definition 8.1.** A valuation on a lattice \((L, \vee, \wedge)\) is a function \( \nu : L \to [0, \infty) \) satisfying
\( (i) \ \nu(a) + \nu(b) = \nu(a \vee b) + \nu(a \wedge b), \)
(ii) $\nu(\bot) = 0$,
(iii) $a \leq b \Rightarrow \nu(a) \leq \nu(b)$.

Our case of interest will be the lattice $(\Omega(Y), \cup, \cap)$ of all open sets of a topological space $Y$. In this case, we say that a valuation $\nu$ is continuous if

$$\nu\left( \bigcup_{O \in \mathcal{D}} O \right) = \sup\{\nu(O) : O \in \mathcal{D}\},$$

for $\mathcal{D}$ a directed set (with respect to $\subseteq$) of open sets in $Y$.

For any $b \in Y$, the valuation $\eta_b : \Omega(Y) \to [0, \infty)$ is given by

$$\eta_b(O) = \begin{cases} 1 & \text{if } b \subseteq O; \\ 0 & \text{otherwise} \end{cases}$$

(3)

is called the point valuation at $b$. Any finite linear combination of point valuations

$$\sum_{i=1}^{n} t_i \eta_{b_i}, \text{ where } t_i \in [0, \infty)$$

is a continuous valuation on $Y$, called a simple valuation.

The restriction of a bounded regular Borel measure on a space $X$ to the lattice of open sets yields a continuous valuation, and for many spaces this assignment is bijective. The following theorem from [19] or [25] is one of the more general results.

**Theorem 8.2.** A continuous valuation on a second countable locally compact sober space extends uniquely to a regular Borel measure on the space equipped with the patch topology. The same Borel algebra is generated by both.

In particular, by results of [19] an $\omega$-continuous poset $P$ equipped with the Scott topology is a second countable locally compact sober space (but $T_0$, not $T_2$). Thus there is a 1-1 correspondence between the continuous valuations $\mu$ on $P$ and their extensions $\mu^*$ to bounded regular Borel measures on $P$.

Suppose that $Y$ is an maximal point space, which we identify with the set of maximal points of an $\omega$-continuous domain $P$. Then a bounded regular Borel measure on $Y$ can be identified in an obvious way with a bounded regular Borel measures on $P$ with support contained in $Y$ whose restriction to $Y$ gives the original measure.

We introduce the important notion of the probabilistic power domain of a continuous domain, a notion which had its origins in the work of Saheb-Djahromi [26] and was further developed by Graham [11], Jones [14], and Jones and Plotkin [15].

**Definition 8.3.** Let $P$ be an $\omega$-continuous domain equipped with the Scott topology. The probabilistic power domain $M_{\mu}(P)$ is the set consisting of all continuous valuations $\nu : \Omega(P) \to [0, 1]$ on $P$ equipped with the pointwise order. The normalized probabilistic power domain $M_{\mu}^*(P)$ is that subset of $M_{\mu}(P)$ consisting of all valuations such that $\nu(P) = 1$. 
The following important theorem was established in [14] and appears again in [15].

**Theorem 8.4.** The (normalized) probabilistic power domain of an \( \omega \)-continuous domain is again an \( \omega \)-continuous domain which has a basis of simple valuations.

Thus for an maximal point space \( X \), identified with the maximal points of \( P \), the bounded regular Borel measures on \( X \), which are maximal elements in the poset \( M_{\mu}(P) \) (see below), can be approximated from below by simple valuations on \( P \). This provides a constructive framework for doing measure theory and an alternate approach to Riemann integration on more general spaces, where the classical Riemann sums are replaced by integrating a function with respect to a simple valuation approximating the given measure (see [5]). In the case of an interval the two approaches are closely reminiscent of each other.

**Example 8.5.** Let \( P \) be the set of approximate and real numbers on \( [0, 1] \) with set of maximal elements \( [0, 1] \). Let \( \lambda \) be Lebesgue measure on \( [0, 1] \) (restricted to the Borel sets). For each \( i \in \mathbb{N} \), let the simple valuation (measure) \( \mu_i \) be defined by

\[
\mu_i = \sum_{j=1}^{2^i} \frac{1}{2^i} \eta_{b_j}, \quad b_j := \left[ \frac{j-1}{2^i}, \frac{j}{2^i} \right].
\]

Then \( \{\mu_i\} \) is a directed sequence converging up to \( \lambda \).

**Theorem 8.6.** Let \( P \) be an \( \omega \)-continuous domain. A Borel measure \( \nu \) in the probabilistic power domain \( M_{\mu}(P) \) is maximal in \( M_{\mu}(P) \) only if \( \nu \) is supported in the set \( \text{Max}(P) \). The converse holds in the normalized probabilistic power domain.

**Proof.** Suppose that \( \nu \) is maximal in \( M_{\mu}(P) \), and that \( \nu(Q) > 0 \), where \( Q := P \setminus \text{Max}(P) \). Since the measure \( \nu \) is regular, there exists an \( L \)-compact set \( K \subseteq Q \) such that \( \nu(K) > 0 \). The set \( K \times \text{Max}(P) \) is a Polish space, since \( \text{Max}(P) \) is, and the subset

\[
H := \{(x, y) \in K \times \text{Max}(P) : x \leq y\}
\]

is closed in \( K \times \text{Max}(P) \), hence Borel, since \( P \) with the \( L \)-topology is a partially ordered topological space. By the von Neumann Selection Theorem (see, for example, [4, Theorem 7, p. 215]) there exists a measurable function \( f : K \to \text{Max}(P) \) such that \( f(x) \geq x \) for each \( x \in K \); here the measurability is with respect to the \( \sigma \)-algebra of analytic sets (continuous images of Polish spaces) of \( K \) and the \( \sigma \)-algebra of Borel sets of \( \text{Max}(P) \). But it is standard that the analytic sets are measurable with respect to any finite regular Borel measure, and thus \( f \) is \( \nu \)-measurable. We now define a new Borel measure \( \nu^* \) by

\[
\nu^*(A) := \nu((P \setminus K) \cap A) + \nu(f^{-1}(A \cap \text{Max}(P))).
\]

This construction transports the part of the measure \( \nu \) supported by \( K \) to \( f(K) \subseteq \text{Max}(P) \), and one sees directly that \( \nu \leq \nu^* \), but \( \nu \neq \nu^* \). This contradicts the maximality of \( \nu \).
A proof of the converse may be found in [5]. □

It follows from the preceding theorem that the space of probability measures on Max(P) is the space of maximal points for the normalized probabilistic power domain.

**Theorem 8.7.** Let X be a maximal point space and let X ↔ Max(P) ↔ P be a domain hull for X. Then Prob(X) ↔ MaxM^1_μ(P) ↔ M^1_μ(P) is a domain hull for the space of probability measures Prob(X) on X endowed with the weak topology. In particular, Prob(X) is a maximal point space.

**Proof.** The identification Prob(X) ↔ MaxM^1_μ(P) of the probability measures on X with the maximal members of the normalized probabilistic power domain comes from the previous theorem. It is a result of Edalat [7] that the usual weak topology considered by probabilists on Prob(X) (convergence of integrals on every bounded continuous function) agrees with the relative Scott topology inherited from M^1_μ(P). To complete the proof, we need to see that condition (1) of Section 5 is satisfied. Let ν be a valuation (= probability measure) on P. If σ is maximal and not above ν, then there exists a Scott open set U in P such that σ(U) < ν(U). For each finite set F contained in U set

\[ \uparrow F := \{ x \in P : \exists y \in F \text{ such that } y \ll x \} = \bigcup_{x \in F} \uparrow x. \]

It follows from the fact that P is continuous that each \( \uparrow F \) is Scott open and that U is the directed union of these sets. Since the union has a countable cofinal subset, there exists some finite subset \( F \subseteq U \) such that σ(U) < ν(\( \uparrow F \)). By condition (1) of Section 5 there exists a Scott open set W such that \( \sigma(W) = 1 - \sigma(\uparrow F) \geq 1 - \sigma(U) > 1 - \nu(\uparrow F) \).

Note that \( W \cap \uparrow F = \emptyset \) since if W met \( \uparrow F \), then there would be a maximal point above a point in the intersection (directed completeness and Zorn’s Lemma), and this maximal point would be in both \( W = \uparrow W \) and \( \uparrow F \), a contradiction. Note also that

\[ \sigma(W) - 1 - \sigma(\uparrow F) \geq 1 - \sigma(U) > 1 - \nu(\uparrow F). \]

since σ is maximal, hence supported in the maximal points, and the intersections of \( \uparrow F \) and W in Max(P) are complementary. Since \( M^1_μ(P) \) is continuous, we can pick \( \tau \ll \sigma \) such that \( \tau(W) > 1 - \nu(\uparrow F) \). Then for any probability measure ρ such that \( \tau \ll \rho \),

\[ \rho(W) \geq \tau(W) > 1 - \nu(\uparrow F), \]

and so since W is contained in the complement of \( \uparrow F \),

\[ \rho(\uparrow F) \leq 1 - \rho(W) < \nu(\uparrow F). \]

We conclude that the Scott open set \( \uparrow \tau \) containing \( \sigma \) misses \( \uparrow \nu \). It follows that the complement of \( \uparrow \nu \) in the maximal measures is relatively Scott open, i.e., condition (1) of Section 5 is satisfied. □
In closing we note that the (normalized) probabilistic power domain construction is functorial on the category of $\omega$-continuous domains and Scott continuous mappings:

$$M_\mu f: M_\mu(P) \rightarrow M_\mu(Q)$$

is defined by $M_\mu f(\nu)(U) = \nu(f^{-1}U)$

for a Scott continuous $f: P \rightarrow Q$, $\nu \in M_\mu(P)$, and $U$ Scott open in $Q$.

References


