On Measure and Integration.

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by

Richard Brian Darst
B.S., Illinois Institute of Technology, 1957
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ABSTRACT

The purpose of this dissertation is to present a Lebesgue-Radon-Nikodym type decomposition for bounded and finitely additive set functions (measures) on a set algebra \((X, S)\) (i.e. \(S\) is a collection of subsets of a set \(X\), containing \(X\) and closed under finite union and complementation) and to distinguish between the basic nature of \(S\)-type (Stieltjes) and \(L\)-type (Lebesgue) integration with respect to a finitely additive set function. A bounded and finitely additive set function \(f\) on \(S\) is a real valued function on \(S\) such that 1) \(f(E + F) = f(E) + f(F)\) when each of \(E\) and \(F\) is in \(S\) and \(E \cdot F = \emptyset\) and 2) \(\sup_{E \in S} |f(E)| < \infty\). We denote the class of all such functions by \(H(X, S)\) and, for each such function and each set \(E\) in \(S\), we set \(V^+(f, E) = \sup_{F \subseteq E} |f(F)|\), \(V^-(f, E) = \inf_{F \subseteq E} |f(F)|\), and \(V(f, E) = V^+(f, E) - V^-(f, E)\). The number \(V(f, E)\) is called the variation of \(f\) on \(E\) and for \(E = X\) defines a norm for \(H(X, S)\) under which \(H(X, S)\) is a NLC or Banach space.

The principal results of this paper are Theorem 2.2 where it is proved that if each of \(f\) and \(g\) is in \(H(X, S)\), then there exist uniquely \(h\) and \(s\) in \(H(X, S)\) such that 1)
\[ f = h + s, \text{ 2) } h \text{ is absolutely continuous with respect to } g \ (i.e. \text{ if } \epsilon > 0, \text{ there exists a } \delta > 0 \text{ such that if } E \subseteq S \text{ and } V(g, E) < \delta \text{ then } V(h, E) < \epsilon), \text{ and 3) } s \text{ is singular with respect to } g \ (i.e. \text{ if } \epsilon > 0, \text{ there exists } E \subseteq S \text{ such that } V(g, E) < \epsilon \text{ and } V(s, X - E) < \epsilon), \text{ and Theorem 4.2 which asserts that the natural setting of the L-type integral is a sigma algebra. It is apparent that, for each } g \in H(X, S), \text{ each of } H_c^0(g) = \{ f \in H(X, S); f \text{ is absolutely continuous with respect to } g \text{ on } S \} \text{ and } H_s^0(g) = \{ f \in H(X, S); f \text{ is singular with respect to } g \text{ on } S \} \text{ is a subspace of } H(X, S) \text{ and that } H_c^0(g) \cap H_s^0(g) = 0. \text{ Hence, from the point of view of linear space theory, the problem in Theorem 2.2 is to show that for each } g, H(X, S) \text{ is the direct sum of } H_c^0(g) \text{ and } H_s^0(g). \text{ Also, it will follow from our proof (cf. Corollary 2.2.1) that if } (X, S) \text{ is a sigma algebra (i.e. } S \text{ is closed under countable union) and } g \text{ is completely additive on } S \ (i.e. \text{ if } \{E_i\} \text{ is a sequence of pairwise disjoint elements of } S, \text{ then } \Sigma g(E_i) = g(\Sigma E_i)), \text{ then there exists a } g\text{-summable function } y \text{ on } X \text{ such that } h(E) = \int_E y \text{ for each } E \subseteq S. \text{ Thus, an implicit result of this paper is that, in general, the Radon-Nikodym theorem can always be regarded as a special case of a Lebesgue decomposition.}

The proof presented of Theorem 2.2 is self contained.
In the third chapter of this paper, we outline an alternate proof using the theory of sequential weak convergence and compactness in the space $H(X,S)$. We feel that this is of interest because our application of this theory points out some important implications of the theory.
CHAPTER I
INTRODUCTION

The purpose of this dissertation is to present a Lebesgue-Radon-Nikodym type decomposition for bounded and finitely additive set functions (measures) on a set algebra \((X, S)\) (i.e. \(S\) is a collection of subsets of a set \(X\), containing \(X\) and closed under finite union and complementation) and to distinguish between the basic nature of \(S\)-type (Stieltjes) and \(L\)-type (Lebesgue) integration with respect to a finitely additive set function. A bounded and finitely additive set function \(f\) on \(S\) is a real valued function on \(S\) such that 1) \(f(E + F) = f(E) + f(F)\) when each of \(E\) and \(F\) is in \(S\) and \(E \cap F = \emptyset\) and 2) \(\operatorname{lub}_{E \in S} |f(E)| < \infty\). We denote the class of all such functions by \(H(X, S)\) and, for each such function and each set \(E\) in \(S\), we set \(V^+(f, E) = \operatorname{lub}_{F \in S, F \subseteq E} [f(F)]\), \(V^-(f, E) = \operatorname{glb}_{F \in S, F \subseteq E} [f(F)]\), and \(V(f, E) = V^+(f, E) - V^-(f, E)\). The number \(V(f, E)\) is called the variation of \(f\) on \(E\) and for \(E = X\) defines a norm for \(H(X, S)\) under which \(H(X, S)\) is a NLC or Banach space.

The principal results of this paper are Theorem 2.2 where it is proved that if each of \(f\) and \(g\) is in \(H(X, S)\),
then there exist uniquely $h$ and $s$ in $H(X,S)$ such that 1) $f = h + s$, 2) $h$ is absolutely continuous with respect to $g$ (i.e. if $e > 0$, there exists a $d > 0$ such that if $E \in S$ and $V(g,E) < d$ then $V(h,E) < e$), and 3) $s$ is singular with respect to $g$ (i.e. if $e > 0$, there exists $E \in S$ such that $V(g,E) < e$ and $V(s, X - E) < e$), and Theorem 4.2 which asserts that the natural setting of the $L$-type integral is a sigma algebra. It is apparent that, for each $g$ in $H(X,S)$, each of $H_0(g) = \{ f \in H(X,S); f$ is absolutely continuous with respect to $g$ on $S \}$ and $H_0(g) = \{ f \in H(X,S); f$ is singular with respect to $g$ on $S \}$ is a subspace of $H(X,S)$ and that $H_0(g) \cdot H_0(g) = 0$. Hence, from the point of view of linear space theory, the problem in Theorem 2.2 is to show that for each $g$, $H(X,S)$ is the direct sum of $H_0(g)$ and $H_0(g)$. Also, it will follow from our proof (cf. Corollary 2.2.1) that if $(X,S)$ is a sigma algebra (i.e. $S$ is closed under countable union) and $g$ is completely additive on $S$ (i.e. if $\{E_i\}$ is a sequence of pairwise disjoint elements of $S$, then $\sum g(E_i) = g(\sum E_i)$), then there exists a $g$-summable function $y$ on $X$ such that $h(E) = \int_E y dg$ for each $E \in S$. Thus, an implicit result of this paper is that, in general, the Radon-Nikodym theorem can always be regarded as a special case of a Lebesgue decomposition.

The proof presented of Theorem 2.2 is self contained.
In the third chapter of this paper, we outline an alternate proof using the theory of sequential weak convergence and compactness in the space $H(X,S)$. We feel that this is of interest because our application of this theory points out some important implications of the theory.

In view of the recent publications (cf. [3] and [5]), with extensive bibliographies in this area of mathematics, and the basic nature of our results, we find it desirable to make our presentation as self-contained as possible. In one instance in our general discussions (e.g. top of page 20), we refer to ideas in anticipation of a publication by W. G. Franzen and P. Porcelli, where they will appear with more detail.
CHAPTER II
DECOMPOSITION THEOREM

In this chapter we establish our principal result (Theorem 2.2). We first present an extension of the Hahn Decomposition Theorem and establish a theorem that shows when weak and norm convergence are equivalent for elementary sequences of functions in $H(X, S)$. Since we are only interested in integrating simple functions, the integrals may be taken either in the S- or L-sense (cf.[1]).

Definition 2.1. If $(X, S)$ is a set algebra, $f$ is in $H(X, S)$, and $\varepsilon > 0$, then the statement that: 1) $E \in S$ is an $\varepsilon$-positive set with respect to $f$ in $(X, S)$ means $\int_{E^c} f d\mu \geq -\varepsilon$ and $\int_{E^c} f d\mu \leq \varepsilon$ and 2) the pair $(E^+, E^-)$ is an $\varepsilon$-decomposition with respect to $f$ in $(X, S)$ means $E^+$ is an $\varepsilon$-positive set with respect to $f$ and $E^- = X - E^+$.

Lemma 2.1. If $f$ is in $H(X, S)$ and $\varepsilon > 0$ then there exists an $\varepsilon$-decomposition with respect to $f$ in $(X, S)$.

Proof. There exists $E \in S$ such that $\int_{E^c} f d\mu - f(E) < \varepsilon$; $(E, X - E)$ is an $\varepsilon$-decomposition.

It is readily seen that there is, in general, no Hahn decomposition with respect to $f$ (i.e. the preceding Lemma is, in general, not true if $\varepsilon = 0$), for instance, let $X$ be the set of positive integers, $E \in S$ if and only if one of
E and X - E is finite, and \( f \in H(X, S) \) such that \( f(E) = \sum_{n \in \mathbb{E}} (-2)^{-n} \).

**Definition 2.2.** If \((X, S)\) is a set algebra and \( y \) is a real valued function on \( X \), then the statement that \( y \) is an \((X, S)\)-simple function means 1) the range of \( y \) is a finite set and 2) if \( r \) is a real number then \([x \in X; y(x) = r] \in S\).

If \( g \in H(X, S) \), the sequence \( \{y_n\} \) of \((X, S)\)-simple functions converges in \( g \)-measure if for \( \epsilon > 0 \) and \( d > 0 \) there exists a positive integer \( N \) such that \( m > N \) and \( n > N \) imply \( V(g, [x; |y_m(x) - y_n(x)| > \epsilon]) < d \). Also, if \( g \in H(X, S) \), \( y \) is a \((X, S)\)-simple function with range \( R \), and \( E \in S \), then \( \int_E y \, d g \) denotes the number \( \sum_{r \in R} g(E; [x; y(x) = r]) \). We note that if \( y \) is a \((X, S)\)-simple function, \( g \in H(X, S) \), and \( h(E) = \int_E y \, d g \) for \( E \in S \), then \( h \in H(X, S) \) and \( h \) is absolutely continuous with respect to \( g \).

**Lemma 2.2.** Let \((X, S)\) be a set algebra, \( g \in H(X, S) \), \( y \) be a \((X, S)\)-simple function, and, for each \( E \in S \), let \( h(E) = \int_E y \, d g \) and \( \overline{g}(E) = V(g, E) \). Then \( V(h, E) = \int_E |y| \, d \overline{g} \) for each \( E \in S \).

**Proof.** Let \([r_1; i \leq n]\) be an enumeration of the range of \( y \), \( E_i = [x; y(x) = r_i] \) for \( i \leq n \), and \( y_1(x) = 1 \) if \( x \in E_i \) and \( y_1(x) = 0 \) otherwise. Then \( y = \sum_{i \leq n} r_i y_1 \) and \( V(h, E) = \sum_{i \leq n} V(h, E \cdot E_i) = \sum_{i \leq n} |r_i| V(g, E \cdot E_i) = \int_E |y| \, d \overline{g} \).

The following theorem is due to P. Porcelli and the author. While we use only the necessity part of the theorem.
in the proof of Theorem 2.2, we present the theorem and a proof in its entirety, since it is of interest in itself.

Theorem 2.1. If \( \{y_n\} \) is a sequence of \((X,S)\) - simple functions, \(g \in H(X,S)\), and \(h_m(E) = \int_E y_m \, dg\) for \(E \in S\), then

\[ \lim_{m,n} V(h_m - h_n, x) = 0 \text{ if, and only if, } \]

1) the sequence \(\{h_m\}\) converges weakly in \(H(X,S)\), and 2) the sequence \(\{y_n\}\) converges in \(g\)-measure.

Proof. Necessity. The \(g\)-measure convergence of the sequence \(\{y_m\}\) is an immediate consequence of Lemma 2.2 (i.e. \(\varepsilon > 0\) implies \(V(h_m - h_n, x) = \int_X |y_m - y_n| \, dg \geq \varepsilon \cdot V(g, [x; |y_m(x) - y_n(x)| > \varepsilon])\) for each pair \((m,n)\) of positive integers).

Sufficiency. Assume conditions one and two are satisfied and \(\lim_{m,n} V(h_m - h_n, x) \neq 0\). Then there exists \(\varepsilon > 0\) such that for each positive integer \(n\) there exists a positive integer \(m\) greater than \(n\) such that \(V(h_m - h_n, x) > 10\varepsilon\) which implies that there exists \(E \in S\) such that \(|h_m(E) - h_n(E)| > 10\varepsilon\). Let \(E_{m,n} = [x; |y_m(x) - y_n(x)| > \varepsilon/ (V(g, X) + 1)]\). For \(d > 0\) there exists, by 2), a positive integer \(N\) such that \(m > n > N\) implies \(V(g, E_{m,n}) < d\). If each of \(m\) and \(n\) is a positive integer and \(E \in S\) then \(|h_m(E) - h_n(E)| \leq \left| \int_E y_m \, dg - \int_E y_n \, dg \right| + \left| \int (y_m - y_n) \, dg \right| < |h_m(E \cdot E_{m,n}) - h_n(E \cdot E_{m,n})| + \varepsilon\). Therefore, \(|(h_m - h_n)(E \cdot E_{m,n})| > |(h_m - h_n)(E)| - \varepsilon\). Let \(n_1 = 1\). There exists \(m_1 > n_1\) and \(E_{m_1,n_1} \in S\) such that \(|(h_{m_1} - h_{n_1})(E_{m_1,n_1})| > 10\varepsilon\) which implies \(|(h_{m_1} - h_{n_1})(E_{m_1,n_1} \cdot F_{m_1,n_1})| > 9\varepsilon\). Let \(k_1 = h_{m_1} - h_{n_1}\) and
There exists $n_2 > m_1$ such that $m > n \geq n_2$ implies $|(h_m - h_n)(H_2)| < \varepsilon$ and $V(g, E_n, n) \cdot \|y_{m_1}\| + \|y_{n_1}\| < \varepsilon \cdot 2^{-1}$ where $\|y_n\| = \text{ub}|y_n(x)|$. There exists $m_2 > n_2$ and $E_{m_2, n_2} \in S$ such that $|(h_{m_2} - h_{n_2})(E_{m_2, n_2} \cdot F_{m_2, n_2})| > 9\varepsilon$. Let $k_2 = h_{m_2} - h_{n_2}$ and $G_2 = E_{m_2, n_2} \cdot F_{m_2, n_2}; G_2 = (H_2 - H_2) + G_2 \cdot H_2$ and $H_2 = H_2 \cdot G_2 + (H_2 - G_2)$. If $|k_2(G_2 - H_2)| > 4\varepsilon$ then $|k_2(G_2 + H_2)| \geq |k_2(G_2 - H_2)| - |k_2(H_2)| > 3\varepsilon$; otherwise, $|k_2(G_2 \cdot H_2)| > 4\varepsilon$ and $|k_2(H_2 - G_2)| \geq |k_2(H_2 \cdot G_2)| - |k_2(H_2)| > 3\varepsilon$. In the former case, $|k_1(H_2 + G_2)| \geq |k_1(H_2)| - |k_1(G_2 - H_2)| > 9\varepsilon - \varepsilon \cdot 2^{-1}$; in the latter case, $|k_1(H_2 - G_2)| \geq |k_1(H_2)| - |k_1(H_2 \cdot G_2)| > 9\varepsilon - \varepsilon \cdot 2^{-1}$.

If $|k_2(G_2 - H_2)| > 4\varepsilon$, let $H_3 = H_2 + G_2$; otherwise, let $H_3 = H_2 - G_2$. Repeating this process inductively, we find sequences $\{m_p\}$ and $\{n_p\}$ of positive integers such that $n_p < m_p < n_{p+1}$, $m > n \geq n_p$ implies $|(h_m - h_n)(H_p)| < \varepsilon$ and $V(g, E_n, n) \cdot \max\{\|y_{m_1}\| + \|y_{n_1}\|; 1 < p\} < \varepsilon \cdot 2^{-(p-1)}$, and $1 \leq p$ implies $|k_1(H_{p+1})| > 3\varepsilon - \sum_{j=p-2}^{p-1} \varepsilon \cdot 2^{-j} > 2\varepsilon$.

This is a contradiction to weak convergence (cf. [2]).

Definition 2.3. If $(X, S)$ is a set algebra and $E \in S$, then $(E, E \cdot S)$ denotes the set algebra $(E, T)$ where $T = [E \cdot F; F \in S]$.

Theorem 2.2. If $(X, S)$ is a set algebra and each of $f$ and $g$ is in $H(X, S)$, then there exist uniquely $h$ and $s$ in $H(X, S)$ such that

1) $h$ is absolutely continuous with respect to $g,$
2) \( s \) is singular with respect to \( g \),
3) \( f = h + s \), and
4) \( E \) in \( S \) implies \( V(f, E) = V(h, E) + V(s, E) \).

Moreover, there exists a sequence \( \{y_n\} \) of \((X, S)\)-simple functions which converges in \( g \)-measure and such that if \( h_n(E) = \int_E y_n \, dg \) for each \( E \in S \) then \( \lim_{n \to \infty} V(h - h_n, X) = 0 \).

Proof. Let \( n \) be a positive integer and \( \varepsilon_n = 2^{-n(n+1)} \).

There exists an \( \varepsilon_n \)-partition \((A_n^+, A_n^-)\) with respect to \( g \) in \((X, S)\) and an \( \varepsilon_n \)-partition \((B_n^+, B_n^-)\) with respect to \( f \) in \((X, S)\). Let \( E_n^{++} = A_n^+ \cdot B_n^+ \), \( E_n^{+-} = A_n^+ \cdot B_n^- \), \( E_n^{-+} = A_n^- \cdot B_n^+ \), and \( E_n^{--} = A_n^- \cdot B_n^- \).

Let \( g \) and \( f \) be restricted to \((E_n^{++}, E_n^{++} \cdot S)\) and let \( U_n^{++} \) be an \( \varepsilon_n \)-positive set for \( f - 2^n g \) in \((E_n^{++}, E_n^{++} \cdot S)\), \( E_n^{4n} \) be an \( \varepsilon_n \)-positive set for \( f - [(4^n - 1) \cdot 2^{-n}] g \) in \((E_n^{++} - U_n^{++}, (E_n^{++} - U_n^{++}) \cdot S)\), \( E_1^{++} \) be an \( \varepsilon_n \)-positive set for \( f - [(1 - 1) \cdot 2^{-n}] g \) in \((H_1^{++}, H_1^{++} \cdot S)\), where \( H_1^{++} = E_n^{++} - \sum_{j=1}^{4^n} E_n^{++} \) for \( 1 \leq j \leq 4^n \), and \( E_1^{++} = E_n^{++} - \sum_{j=1}^{4^n} E_n^{++} \).

If \( E \subseteq E_n^{++} \) then \( -\varepsilon_n < g(E) < 2^{-n} \cdot g(E) + \varepsilon_n \) and if \( E \subseteq E_n^{++} \) for \( 1 \leq i \leq 4^n \) then \( f(E) - [(i - 1) \cdot 2^{-i}] g(E) > -\varepsilon_n \) and \( f(E) - [1 \cdot 2^{-n}] g(E) < \varepsilon_n \).

Therefore, \( E \subseteq E_n^{++} \) for \( 1 \leq i \leq 4^n \) implies \( f(E) - [(1 - 1) \cdot 2^{-n}] g(E) | < 2^{-n} \cdot |g(E)| + \varepsilon_n \). Similarly \( E \subseteq U_n^{++} \) implies \( f(E) - 2^n g(E) > -\varepsilon_n \) and \( -\varepsilon_n < g(E) \) which, in turn, imply \( |g(E)| < \max \{\varepsilon_n, (\varepsilon_n + f(E)) \cdot 2^{-n} \} \). Let \( L_n^{++} = E_n^{++} - U_n^{++} \). We repeat the preceding for \( g \) and \( -\varepsilon_n \) and \( f \) restricted to \((E_n^{--}, E_n^{--} \cdot S)\), \( -\varepsilon_n \) and \( f \) restricted to \((E_n^{-+}, E_n^{-+} \cdot S)\), and
\(-g\) and \(-f\) restricted to \((E^+,E^+:S)\). Let \(U_n = U_n^{++} + U_n^{++} + U_n^{--}\) and \(L_n = X - U_n\). Let \(y_n = \sum_{1 \leq i \leq n}(1 - 1) \cdot 2^{-n} [C(E_{n,1}^{++}) - C(E_{n,1}^{+-}) - C(E_{n,1}^{-+}) + C(E_{n,1}^{--})]\), where \(C(E)(x) = 1\) if \(x \in E\), and \(C(E)(x) = 0\) otherwise.

For \(E \in S\), let \(h_n(E) = \int_E y_n d\sigma\), \(s_n(E) = f(E \cdot U_n)\), and \(f_n = h_n + s_n\), so that \(|f(E) - f_n(E)| = |(f(E \cdot L_n) + f(E \cdot U_n)) - (h_n(E) + s_n(E))| = |f(E \cdot L_n) - \int_E y_n d\sigma| \leq \sum_{1 \leq i \leq n} |f(E \cdot E_{n,i}^{++}) - \int_E E_{n,i}^{++} y_n d\sigma| + |f(E \cdot E_{n,i}^{-+}) - \int_E E_{n,i}^{-+} y_n d\sigma| + |f(E \cdot E_{n,i}^{+-}) - \int_E E_{n,i}^{+-} y_n d\sigma| + |f(E \cdot E_{n,i}^{--}) - \int_E E_{n,i}^{--} y_n d\sigma| \leq 2^{-n} V(g,X) + 4n+1 \cdot \varepsilon_n = 2^{-n}(V(g,X) + 2^{-1})\). Therefore, \(\lim V(f - f_n,X) = 0\). Moreover, \(\lim V(g \cdot U_n) = 0\) since \(E \subseteq U_n\) implies \(|g(E)| < 4\varepsilon_n + (4\varepsilon_n + V(f,X)) \cdot 2^{-n}\) and \(V(f_n,E) = V(h_n,E) + V(s_n,E)\) for \(E \in S\).

The remainder of the proof consists in showing that \(\lim V(s_m - s_n,X) = 0\). Let us assume this and show how the proof is completed. Under this assumption there exists uniquely an element \(s\) in \(H(X,S)\) such that \(\lim V(s - s_n,X) = 0\) and, moreover, \(s\) is singular with respect to \(g\) on \((X,S)\). \((\lim V(g,U_n) = 0 \text{ and } \lim V(s,X - U_n) = 0)\). Consequently \(\lim V(h_m - h_n,g) = 0\), which implies there exists uniquely an element \(h\) in \(H(X,S)\) such that \(\lim V(h - h_n,X) = 0\) and, moreover, \(h\) is absolutely continuous with respect to \(g\) since for each \(n\), \(h_n\) is absolutely continuous with respect to \(g\).

Turning now to the proof of our assumption, we note
that \( V(s_m - s_n, X) = V(f, U_m - U_n) + V(f, U_n - U_m) = \)
\( V(f, U_m \cdot L_n) + V(f, U_n \cdot L_m). \) Because of symmetry, it is
sufficient to suppose \( m > n. \) Under this supposition we
shall first show \( \lim_{m,n} V(f, U_m \cdot L_n) = 0 \) and then use it to
show \( \lim_{m,n} (f, U_n \cdot L_m) = 0. \)

Let us first note that \( V(f, E_m^{++} + E_n^{--}) \leq \epsilon_m + \epsilon_n \)
for each pair \((m, n)\) of positive integers.

Let \( E \subseteq U_m^{++} \cdot L_n^{++}. \) This implies \(-\epsilon_m < f(E) \), (a)
\( f(E) - 2^m g(E) > -\epsilon_m, \) and \( b) f(E) - 2^n g(E) < \epsilon_n \)
which imply \( (a') -\epsilon_m + 2^m g(E) < f(E) \) and \( (b') f(E) < 2^n g(E) + \epsilon_n \)
which imply \( (c) (2^m - 2^n) g(E) < \epsilon_n + \epsilon_m; \) \( (b') \)
and \( (c) \)
imply \( f(E) < 2^n (\epsilon_n + \epsilon_m) \cdot (2^m - 2^n)^{-1} + \epsilon_n \leq 2 \epsilon_n + \epsilon_m. \)
Therefore, \( V(f, U_m^{++} \cdot L_n^{++}) \leq 2(\epsilon_n + \epsilon_m). \) Likewise, each
of \( V(f, U_m^{--} \cdot L_n^{--}), V(f, U_m^{++} \cdot L_n^{--}), \) and \( V(f, U_m^{--} \cdot L_n^{--}) \) is
less than or equal to \( 2(\epsilon_n + \epsilon_m). \)

Let \( E \subseteq E_m^{++} \cdot L_n^{--}. \) This implies \(-\epsilon_m < f(E) \), \(-\epsilon_m \)
< \( g(E) < \epsilon_n, \) and \( f(E) + 2^n g(E) < \epsilon_n \) which imply \(-\epsilon_m < f(E) \)
< \( \epsilon_n - 2^n g(E) < \epsilon_n + 2^n \epsilon_m. \) Therefore, \( V(f, E_m^{++} \cdot L_n^{--}) \leq\)
\( \epsilon_n + (2^n + 1) \epsilon_m. \) Likewise, each of \( V(f, E_m^{--} \cdot L_n^{++}), \)
\( V(f, E_m^{--} \cdot L_n^{--}), \) and \( V(f, E_m^{--} \cdot L_n^{++}) \) is less than or equal
to \( \epsilon_n + (2^n + 1) \epsilon_m. \)

Now, \( V(f, U_m \cdot L_n) = V(f, (U_m^{++} + U_m^{--} + U_m^{+-} + U_m^--) \cdot \)
\( (L_n^{++} + L_n^{--} + L_n^{+-} + L_n^{--})) = V(f, U_m^{++} \cdot L_n^{++}) + V(f, U_m^{++} \cdot L_n^{+-}), \)
\( + V(f, U_m^{+-} \cdot L_n^{++}) + V(f, U_m^{+-} \cdot L_n^{+-}) + V(f, (U_m^{--} + \)
\( U_m^{--}) \cdot (L_n^{++} + L_n^{--})) + V(f, (U_m^{--} + U_m^{--}) \cdot (L_n^{++} + L_n^{--})) +\)
\( V(f, U_m^{+-} \cdot L_n^{+-}) + V(f, U_m^{+-} \cdot L_n^{--}) + V(f, U_m^{--} \cdot L_n^{+-}) +\)
\[ V(f, U_m \cdot L_n) \leq 10(\varepsilon_n + \varepsilon_m) + 4(\varepsilon_n + (2^n + 1)\varepsilon_m) = 14\varepsilon_n + (2^{n+2} + 14)\varepsilon_m < 4^{-(n-1)}. \] Therefore, \( \lim_{m,n} V(f, U_m \cdot L_n) = 0 \) for \( m > n \).

We shall now show that if \( \varepsilon > 0 \) then there exists a positive integer \( N \) such that \( m > n > N \) implies \( V(f, U_n \cdot L_n) < \varepsilon \). Assume this is false. Then there exists \( \varepsilon > 0 \) and a sequence \( \{(m_i, n_i)\}_{i \geq 1} \) of pairs of positive integers such that 1) \( n_i < m_i < n_i+1 \), 2) \( 4^{-n_i} < \varepsilon \cdot 2^{-i} \) (this implies \( 4^{-(m_i-1)} \leq 4^{-n_i} < \varepsilon \cdot 2^{-(i+1)} \) for each positive integer \( i \)), and 3) \( V(f, U_{n_i+1} \cdot L_{m_i}) \geq \varepsilon \). Let \( K_1 = U_{n_1} \cdot L_{m_1} \), \( F_1 = K_1 \), and \( F_{i+1} = K_{i+1} - \sum K_j \). \( V(f, F_1) \geq \varepsilon \) and \( V(f, F_{i+1}) = V(f, K_{i+1}) - \sum V(f, K_{i+1} \cdot F_j) \geq \varepsilon - \sum \varepsilon \cdot 2^{-(j+1)} > \varepsilon \cdot 2^{-(i+1)} \) which contradicts that \( f \) is bounded.

The measure convergence of the sequence \( \{y_n\} \) follows from the norm convergence of the sequence \( \{t_n\} \). This completes a proof of Theorem 2.2.

We have shown that the sequence \( \{y_n\} \) converges in \( g \)-measure. It is possible that in our construction no subsequence of \( \{y_n\} \) will converge almost everywhere. In fact, the following example, shows that even less is true. Let \( X \) be the set of non-negative integers, \( E \in S \) if and only if one of \( E \) and \( X - E \) is a finite subset of the positive integers, \( f(E) = \sum_{i \in E \cap [0]} 2^{-1}, g(E) = \sum_{i \in E \cap [0]} 4^{-1}, \) and \( y_n(1) = 2^1 \) if \( 1 \leq i \leq n \) and \( y_n(1) = 2^n \) otherwise; \( \lim y_n(0) = \infty \).

Hence, in this example, if \( \{z_n\} \) is a subsequence of \( \{y_n\} \)
and \( \{E_i\} \) is a sequence of elements of \( S \) such that \( V(g, E_i) = 0 \) for each positive integer \( i \), then there exists an element \( x \) in \( X - \sum E_i \) such that \( \lim_n z_n \) does not exist (i.e. there does not exist a real number \( r \) such that \( \lim_n z_n(x) = r \)). Of course, this situation (or pathology) disappears if \( (X, S) \) is a sigma algebra and \( g \) is completely additive. We note that, in this example, it is possible to redefine the sequence \( \{y_n\} \) in order to get pointwise convergence (i.e. let \( y_n(1) = 2^1 \) if \( 1 \leq i \leq 8^{n+2} \) and \( y_n(1) = 0 \) otherwise).

Corollary 2.2.1. (Radon-Nikodym). If in addition to the hypothesis of Theorem 2.2. \( (X, S) \) is a sigma algebra and \( g \) is completely additive on \( S \) then there exists a sub-sequence \( \{z_n\} \) of \( \{y_n\} \) and a \( g \)-summable function \( z \) such that 1) \( \lim_n z_n(x) = z(x) \) a.e. and 2) \( h(E) = \int_E z \, dg \) for for \( E \) in \( S \), where, in accordance with Theorem 2.2, \( h \) is the absolutely continuous part of \( f \) with respect to \( g \).

The proof of the corollary follows readily since \( \lim_{m,n} V(h_m - h_n, X) = 0 \) implies \( \lim_{m,n} \int_X |y_m - y_n| \, dg = 0 \), where \( V(g, E) = V(g, E) \) for each \( E \) in \( S \).
One may also show the norm convergence of the sequences \( \{ h_n \} \) and \( \{ s_n \} \) in a less direct but instructive way. We shall outline this procedure. As background for what follows, we refer the reader to P. Porcelli's paper on weak convergence in \( H(X,S) \) (cf. [2]).

**Theorem 3.1.** Let \( \{ f_n \} \) be a sequence of elements of \( H(X,S) \) such that if \( \{ g_n \} \) is a subsequence of \( \{ f_n \} \) and \( \varepsilon > 0 \), then there exists a \( h \) in \( H(X,S) \) and a subsequence \( \{ k_n \} \) of \( \{ g_n \} \) such that \( V(k_n,E) < V(h,E) + \varepsilon \) for each positive integer \( n \) and each \( E \) in \( S \). Then \( \{ f_n \} \) contains a weakly convergent subsequence.

**Proof.** Using a diagonalization process, we obtain a subsequence \( \{ g_n \} \) of \( \{ f_n \} \) and a sequence \( \{ h_n \} \) of elements of \( H(X,S) \) such that if \( E \) is in \( S \) and \( m \geq n \), then \( V(g_m,E) < V(h_n,E) + n^{-1} \). Let \( (X,F) \) be the set algebra generated by the finite subsets of the parameter set induced by the subspace of \( H(X,S) \) generated by the sequence \( \{ g_n \} \). ¹ It is implicit in the proof of Theorem 3.1 of Porcelli's paper that \( (X,F) \) is \( (f) \) as defined in section 3 of Porcelli's paper, by a parameter set we mean a collection of elements of \( S \) obtained as in section 2 of the same paper.

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¹(\( X,F \)) is \( (f) \) as defined in section 3 of Porcelli's paper, by a parameter set we mean a collection of elements of \( S \) obtained as in section 2 of the same paper.
that weak convergence for a subsequence \( \{k_n\} \) of \( \{g_n\} \) is equivalent to the existence of \( \lim \sum_{i \geq 1} k_n(E_i) \) for each sequence \( \{E_i\} \) of pairwise disjoint elements of \( F \) (i.e. for weak convergence we need not, in general, consider all of \( S \)); since \( F \) is countable, we diagonalize \( \{g_n\} \) to obtain a subsequence \( \{k_n\} \) of \( \{g_n\} \) such that \( \lim k_n(E) \) exists for each \( E \) in \( F \). We shall now show that the condition for weak convergence is satisfied by \( \{k_n\} \). Let \( \{E_i\} \) be a sequence of pairwise disjoint elements of \( F \). 

\( \epsilon > 0 \) implies there exist positive integers \( t, u, \) and \( V \) such that 1) \( t < V \), 2) \( 8 \cdot t^{-1} < \epsilon \), 3) \( \sum_{i > u} V(h_t, E_i) < \epsilon \cdot 8^{-1} \), and 4) \( m \) and \( n > V \) imply \( |k_m(E_i) - k_n(E_i)| < \epsilon \cdot 2^{-(i+1)} \) for \( 1 \leq u \). If each of \( m \) and \( n \) is greater than \( V \) then

\[
| \sum_{i \geq 1} k_m(E_i) - \sum_{i \geq 1} k_n(E_i) | \leq \sum_{i \leq u} |k_m(E_i) - k_n(E_i)| + \sum_{i > u} V(k_m, E_i) + \sum_{i > u} V(k_n, E_i) \leq \sum_{i \leq u} \epsilon \cdot 2^{-(i+1)} + 2(t^{-1} + \sum_{i > u} V(h_t, E_i)) < \epsilon.
\]

Therefore, the sequence \( k_n \) is weakly convergent.

The proof given of Theorem 3.1 points out the significance of the fact that for separable subspaces of \( H(X, S) \), weak convergence is determined on a countable subalgebra of \( (X, S) \).

Corollary 3.3.1. Let \( \{h_n\} \) be a sequence of elements of \( H(X, S) \) such that there exists a \( f \) in \( H(X, S) \) such that if \( \epsilon > 0 \) then there exists a positive integer \( N \) such that \( n > N \) and \( E \) in \( S \) imply \( V(h_n, E) < V(f, E) + \epsilon \). Then \( \{h_n\} \) contains a weakly convergent subsequence.
Corollary 3.1.2. The sequence \( \{h_n\} \) defined in Theorem 2.2 contains a weakly convergent subsequence.

Corollary 3.1.3. If \((X,S)\) is a set algebra, \( \{E_n\} \) is a sequence of elements of \( S \), \( g \in \mathcal{H}(X,S) \), and \( f_n(E) = g(E^* E_n) \) for each \( E \) in \( S \) then \( \{f_n\} \) contains a weakly convergent subsequence \( \{f'_n\} \).

By Theorem 2.1, the sequence \( \{f'_n\} \) is norm convergent if and only if the sequence \( \{C(E_n)\} \) of characteristic functions converges in \( g \)-measure on \((X,S)\). It is easy to give examples where no subsequence of \( \{C(E_n)\} \) converges in \( g \)-measure.

One can prove directly that the sequence \( \{y_n\} \) defined in Theorem 2.2 converges in \( g \)-measure. This combined with Corollary 3.1.2. and Theorem 2.1 will complete an alternate proof of the norm convergence of the sequences \( \{h_n\} \) and \( \{s_n\} \) (the convergence of the full sequences follows from the uniqueness of the decomposition).
A NOTE ON ABSTRACT INTEGRATION

The principal purpose of this chapter is to distinguish between the basic nature of S-type (Stieltjes) and L-type (Lebesgue) integration with respect to a finitely additive set function (measure). Interest in these types of integration was given impetus by the fundamental representation theorem of Hildebrandt-Fichtenholz-and Kantorovitch (cf.[1]), and, since that time, a considerable amount of work has been done with these integrals along the lines of developing a formal theory and as a powerful analytic tool in the study of linear spaces.

In the case of the formal theory, the usual practice has been to start with a set algebra (ring) instead of a sigma algebra (ring) and define both of the integrals in this basic setting. (This is natural since the "integrator" need not be completely additive and, at any rate, on a sigma algebra the two integrals are, for essentially bounded function, identical). This procedure has worked well for the S-integral; however, for the L-integral a great deal of difficulty arises. For example, in this primitive setting, the L-integral (in general) is neither a
absolutely continuous, nor a linear, nor a homogeneous operation. These difficulties, and the fact that, for a formal theory, no one of the three usual ways that the class of measurable functions can be defined is any more desirable than the others [in general, each yields a class of measurable functions distinct from the others], reflect the artificiality inherent in the usual ways of defining the class of measurable functions.

We use a fourth class of function (called continuous) and, by considering the relationships between these four classes, establish a theorem (Theorem 4.2) that, in effect, says the natural setting of the L-type integral is a sigma algebra. Also, we show that the class of continuous functions (which includes the various types of bounded measurable functions) can be characterized entirely in terms of the S-integral (Theorem 4.1), and this, together with the definition of continuity, implies that the natural setting of the S-type integral is a set algebra. We conclude the chapter by deriving a necessary and sufficient condition, in the case of a set algebra, in order that each continuous function belong to at least one of the classes of measurable functions and show, by example, that a set algebra need not be a sigma algebra in order to satisfy this condition.

There are many ways in which a maximal proper ideal,
in $S$, can be characterized and, for the purpose of this paper, we will use two characterizations: 1) a proper ideal $J$ in $S$ is maximal if and only if $E \subseteq S$ implies one, and only one, of $E$ and $E' = X - E \cap J$, and 2) a subset $J$ of $S$ is a maximal proper ideal if and only if there exists (uniquely) $g \in H(X,S)$ such that $g(E) = 0$ if $E \subseteq J$ and $g(E) = 1$ if $E \nsubseteq J$. If $g$ has the properties of 2) we say that $g$ is a two valued jump function. For the definitions of the $S$- and $L$-type integrals see [1] or [4]. All functions considered in this paper are assumed to be real valued.

**Definition 4.1.** If $f$ is a function on $X$, then $f$ is said to be an $(X,S)$-continuous function if, for each $e > 0$ there exists a partition $\{E_i\}_{i=1}^n$ of $X$ such that $E_i \subseteq S$ and $0(f, E_i) < e$ for $i \leq n$ where $0(f, E_i) = \text{lub}_{x,y \in E_i} |f(x) - f(y)|$ (i.e. $0(f, E)$ denotes the oscillation of $f$ on $E$). The collection of $(X,S)$-continuous functions is denoted by $C(X,S)$.

**Theorem 4.1.** Let $(X,S)$ be a set algebra and let $f$ be a function on $X$. Then $f$ is an $(X,S)$-continuous function if and only if $\int_X f dg$ exists for each $g \in H(X,S)$.

**Proof.** We will show that if $f$ is not $(X,S)$-continuous then there exists a two valued jump function $g$ on $S$ such that $\int_X f dg$ does not exist. If, for $e > 0$, we let $D(f, S, e) = \{E \subseteq S; \text{if } \{E_i\}_{i=1}^n \text{ is a partition of } E \in S, \text{i.e. } E_i \in S \text{ for } i \leq n, \text{ then } \text{lub}_{i \leq n} 0(f, E_i) \geq e\}$, then...
If and only if \( D(f,S,e) = \emptyset \) for \( e > 0 \). Suppose there exists \( e > 0 \) such that \( D(f,S,e) \neq \emptyset \). Let \( I \) denote the set of ideals in \( S \) such that \( J \in I \) and \( E \in J \) imply 
\[ E' = X - E \in D(f,S,e). \]
There exists \( J \in I \) which is maximal with respect to inclusion. Suppose there exists \( E \in S \) such that each of \( E \) and \( E' \) implies \( E' \). Then each of \( K \) and \( K_1 \), the ideals generated by \( J \) and \( E \) and \( J \) and \( E' \) respectively (i.e. \( F \in K \) if and only if there exist \( G \in J \) and \( H \in S \) such that \( F = G + H \cdot E \)), is a proper ideal in \( S \) which contains \( J \) as a proper subset. Therefore, each of \( K \) and \( K_1 \) and there exist \( F \in K \) and \( F_1 \in K_1 \) such that each of \( F' \) and \( F_1' \) implies each of \( E' \cdot F' \) and \( E' \cdot F_1' \), which imply each of \( E' \cdot F \) and \( E' \cdot F_1 \); however, each of \( E' \cdot F \) and \( E' \cdot F_1 \) does not imply each of \( E' \cdot F' \) and \( E' \cdot F_1' \). But, \( X = E + E' = E \cdot F_1 + E' \cdot F + E' \cdot F' = (E \cdot F_1 + E' \cdot F) + (E' \cdot F_1' + E' \cdot F') \), \((E \cdot F_1 + E' \cdot F) \in J \) and \((E \cdot F_1' + E' \cdot F') \notin D(f,S,e) \). This contradiction shows our supposition that \( J \) is not a maximal proper ideal in \( S \) is false. There exists \( g \in H(X,S) \) such that \( g(E) = 0 \) if \( E \in J \) and \( g(E) = 1 \) if \( E \notin J \); \( \int_X f d g \) does not exist.

Remark. Perhaps it is of interest to note that, in a sense, Theorem 4.1 is an extension to the general case of the classical theorem which states that a function \( f \) on the interval \([a,b]\) is continuous if and only if the Stieltjes integral \( \int_a^b f d g \) exists for every function \( g \) of bounded variation on \([a,b]\). Also, we note that, regarding \( C(X,S) \)
as a linear-normed-complete space, \( C(X,S) \) is isomorphically
isometric to the space of topologically continuous
functions on \( \mathcal{S}(S) \), where \( \mathcal{S}(S) \) is the space of ultrafilters
associated with \( S \) (i.e. the Stone-Cech type compactification
of \( (X,S) \)). Finally, we see that a function \( f \) on \( X \) is in
\( C(X,S) \) if and only if there exists a sequence \( \{f_i\} \) of
\( (X,S) \)-simple functions such that \( \lim \| f - f_i \| = \lim \lub_{x \in X} |f(x) - f_i(x)| = 0 \).

Definition 4.2. If \( (X,S) \) is a set algebra, then
1) \( M(X,S) = \{ f \text{ on } X; -\infty < a < b < \infty \Rightarrow f^{-1}(a,b) \in S \} \),
   \( f^{-1}(a,b) = \{ x \in X; a < f(x) < b \} \),
2) \( LM(X,S) = \{ f \text{ on } X; -\infty < a < b < \infty \Rightarrow f^{-1}[a,b) \in S \} \),
   \( f^{-1}[a,b) = \{ x \in X; a < f(x) < b \} \),
3) \( RM(X,S) = \{ f \text{ on } X; -\infty < a < b < \infty \Rightarrow f^{-1}(a,b] \in S \} \),
   \( f^{-1}(a,b] = \{ x \in X; a < f(x) \leq b \} \),
4) \( G(X) = \{ f \text{ on } X; \| f \| = \lub_{x \in X} |f(x)| < \infty \} \), and
5) \( m(X,S) = g(X) \cdot M(X,S), \quad LM(X,S) = G(X) \cdot LM(X,S), \) and
   \( Rm(X,S) = G(X) \cdot RM(X,S) \).

The following two lemmas follow readily from Definition
4.2.

Lemma 4.1. Let \( (X,S) \) be a set algebra. Then
1) if \( \{ E_i \}_{i=1}^n \) is a finite collection of pairwise disjoint
   subsets of \( X \) and \( \sum_{i \leq n} E_i \in S \) then either \( E_i \in S \) for \( i \leq n \) or
   there exist at least two indices \( i \) and \( j \) such that each of
   \( E_i \) and \( E_j \) \( \notin S \),
2) each of $m(X,S)$, $Lm(X,S)$, and $Rm(X,S)$ is a subset of $C(X,S)$,
3) if $f \in M(X,S)$ and $P$ is a real number then $f^{-1}(P) \in S$,
4) $M(X,S) = LM(X,S) \cdot RM(X,S)$, and
5) if $f$ is a function on $X$, then $f \in M(X,S)$, $LM(X,S)$, or
$RM(X,S)$ if and only if $f^n \in m(X,S)$, $Lm(S,S)$, or $Rm(X,S)$
respectively for each positive integer $n$ where $f^n(x) = f(x)$ if $|f(x)| \leq n$ and $f^n(x) = n \cdot f(x) \cdot |f(x)|^{-1}$ if $|f(x)| > n$.

Lemma 4.2. If $(X,S)$ is a sigma algebra, then
1) $M(X,S) = LM(X,S) = Rm(X,S)$, and
2) $m(X,S) = C(X,S)$.

Proof of 2). If $f \in C(X,S)$, then there exists a sequence
\{f_n\} of $(X,S)$-simple functions such that $\|f - f_n\| < n^{-1}$; $a < b \implies f^{-1}(a,b) = \bigcup_{j \geq 1} f^{-1}_j(a + j^{-1}, b - j^{-1})$.

Theorem 4.2. Let $(X,S)$ be a set algebra, $m = m(X,S)$,
$Lm = Lm(X,S)$, $Rm = Rm(X,S)$, and $C = C(X,S)$. Then $C = m = Lm = Rm$ if and only if $(X,S)$ is a sigma algebra. Moreover,
if $(X,S)$ is not a sigma algebra $Lm \not\subseteq Rm$, $m \not\subseteq Lm$, $m \not\subseteq Rm$, and
each of $Lm$ and $Rm$ is a proper subset of $C$.

Proof. If $(X,S)$ is not a sigma algebra then there
exists a sequence $\{E^*_1\}$ of pairwise disjoint elements of $S$
such that $\bigcup E^*_1 \not\subseteq S$. Let $f_L(x) = 2^{-1}$ if $x \in E^*_1$ and $f_L(x) = 0$
if $x \in X - \bigcup E^*_1$ and let $f_R = -f_L$; $f_L \in Lm - Rm$ and $f_R \in Rm - Lm$ ($f \in LM(X,S)$ if and only if $-f \in RM(X,S)$).
Definition 4.3. If \((X, S)\) is a set algebra, then by the statement that \(S\) has property \(Q\) we mean that if \(\{E_i\}\) is non-decreasing in \(S\) (i.e. \(E_i \subseteq E_{i+1}\) for \(i \geq 1\)), \(\{F_i\}\) is non-decreasing in \(S\), and \(\sum E_i \cdot \sum F_i = \emptyset\), then at least one of \(E_i \cdot \sum F_i\) is in \(S\).

Lemma 4.3. Let \((X, S)\) be a set algebra and let \(S\) have property \(Q\). Then

1) if \(\{E_i\}\) is non-increasing in \(S\), \(\{F_i\}\) is non-decreasing in \(S\), and there exists a positive integer \(j\) such that \(E_j \cdot \sum F_i = \emptyset\) then at least one of \(\sum E_i\) and \(\sum F_i\) is in \(S\).

2) if \(\{E_i\}\) is non-increasing in \(S\), \(\{F_i\}\) is non-increasing in \(S\), and there exists a positive integer \(j\) such that \(E_j \cdot \sum F_i = \emptyset\) then at least one of \(\sum E_i\) and \(\sum F_i\) is in \(S\).

Proof. 1) Let \(G_1 = E_j - E_1\), \(\{G_i\}\) is non-decreasing in \(S\), \(\sum G_i \cdot \sum F_i = \emptyset\), and \(\prod E_1 = E_j - \sum G_1\) (\(\prod E_1 \in S\) if and only if \(\sum G_1 \in S\)).

2) \(\prod F_i = (\prod F_i) \cdot E_j = \prod (F_i \cdot E_j)\). Let \(G_1 = F_i \cdot E_j\) and \(H_1 = E_j - E_1\), \(\{G_i\}\) is non-increasing in \(S\), \(\{H_i\}\) is non-decreasing in \(S\), \(\prod G_i = \prod F_i\), \(\sum H_1 = E_j - \prod E_1\), and \(G_1 \cdot \sum H_1 = \emptyset\).

Lemma 4.4. Let \((X, S)\) be a set algebra, \(S\) have property \(Q\), and \(f \in C(X, S)\). Then

1) if each of \((a, b)\) and \((c, d)\) is a segment and \((a, b) \cdot (c, d) = \emptyset\), then at least one of \(f^{-1}(a, b)\) and \(f^{-1}(c, d) \in S\),
2) If \( c \notin [a, b] \), then at least one of \( f^{-1}(c) \) and \( f^{-1}(a, b) \) is in \( S \).

3) If \( c \notin \mathcal{P} \), then at least one of \( f^{-1}(c) \) and \( f^{-1}(\mathcal{P}) \) is in \( S \).

4) If \( c \notin \mathcal{P} \) and \( e > 0 \) implies there exists \((a, b)\) such that \( |P - a| < e \), \( |P - b| < e \), and \( f^{-1}(a, b) \) is in \( S \), then \( f^{-1}(c) \) is in \( S \), and

5) If \( \mathcal{P} \notin [c, d] \) and \( e > 0 \) implies there exists \((a, b)\) such that \( |P - a| < e \), \( |P - b| < e \) and \( f^{-1}(a, b) \) is in \( S \), then \( f^{-1}(c, d) \) is in \( S \).

Proof. Since \( f \in C(X, S) \), there exists a sequence \( f_i \) of \((X, S)\)-simple functions such that \( \|f - f_i\| < (2(i + 1))^{-2} \) which implies \( f^{-1}(a + 2^{-1}[(i - 1)^{-1} + i^{-1}], b - 2^{-1}[(i - 1)^{-1} + i^{-1}]) f^{-1}(a + 2^{-1}[i^{-1} + (i + 1)^{-1}], b - 2^{-1}[i^{-1} + (i + 1)^{-1}]) \); this implies \( \{f_i^{-1}(a + i^{-1}, b - i^{-1})\} \) is non-decreasing in \( S \) and \( \mathcal{E} f_i^{-1}(a + i^{-1}, b - i^{-1}) = f^{-1}(a, b) \). The preceding, together with property \( \psi \), shows 1). To get 2) we note that

\[ f_i^{-1}(c) = \prod_{i \in \mathbb{N}} f_i^{-1}(c - i^{-1}, c + i^{-1}) = \prod_{i \in \mathbb{N}} \mathcal{E}_i \text{ where } \mathcal{E}_i = \prod_{i \in \mathbb{N}} f_i^{-1}(c - i^{-1}, c + i^{-1}) \text{ and then apply Lemma 4.3 - 1 to } \{\mathcal{E}_i\} \text{ is non-increasing in } S \text{ and } \{f_i^{-1}(a + i^{-1}, b - i^{-1})\} \text{ is non-decreasing in } S. \]

We get 3) in a similar fashion from 4.3 - 2 and 4) and 5) follow from 2) and 1).

Lemma 4.5. Let \((X, S)\) be a set algebra, \( S \) have property \( \psi \), and \( f \in C(X, S) \subset L_m(X, S) \). Then there exists uniquely a point \( P \) such that 1) \( f^{-1}(P) \) is in \( S \) and 2) \(-\infty < a < P \) implies \( f^{-1}(a, P) \) is in \( S \).
Proof. There exists \([c, d)\) such that \(f^{-1}(c, d) \notin S\).

We want to find a point \(P\) which has properties considered in Lemma 4.4. If \(f^{-1}(c) \notin S\), let \(P = c\); suppose \(f^{-1}(c) \in S\); then \(f^{-1}(c, d) \notin S\), if \(f^{-1}(2^{-1}[c + d]) \notin S\), let \(2^{-1}[c + d] = P\); otherwise, exactly one of \(f^{-1}(c, 2^{-1}[c + d])\) and \(f^{-1}(2^{-1}[c + d], d) \notin S\) (Lemma 4.4 - 1), denote that one by \((c_1, d_1)\) and repeat the preceding inductively. If there exists a positive integer \(1\) such that \(f^{-1}(2^{-1}[c_1 + d_1]) \notin S\), fine; otherwise, let \(P = \bigcap[c_1, d_1]\). It is impossible that \(c < P < d\) since \(c < P < d\) would imply \(f^{-1}[c, d) = X - [f^{-1}(-\infty, c) + f^{-1}(d) + f^{-1}(d, \infty)] \in S\) (Lemma 4.4). If \(P = d\), then \(f^{-1}(c) \in S\) (Lemma 4.4) which implies \(f^{-1}(c, P) \notin S\) which, in turn, implies \(f^{-1}(P, \infty) \notin S\) (Lemma 4.4) and thus \(f^{-1}(P) = (X - [f^{-1}(-\infty, c) + f^{-1}(c) + f^{-1}(P, \infty)]) - f^{-1}(c, P) \notin S\) (Lemma 4.1 - 1). If \(P = c\), let \(e\) be a number less than \(P\). Then \(f^{-1}(e, d) = X - [f^{-1}(-\infty, e) + f^{-1}(e) + f^{-1}(d) + f^{-1}(d, \infty)] \in S\) which implies \(f^{-1}(e, P) = f^{-1}(e, d) - f^{-1}(P, d) \notin S\) (\(P = 0\), Lemma 4.1 - 1) which, in turn, implies \(f^{-1}(P, d) \notin S\) (Lemma 4.4 - 1) and thus \(f^{-1}(P) = f^{-1}(P, d) - f^{-1}(P, d) \notin S\).

The lemma now follows from Lemma 4.4.

Remark. Since \(f \in \text{Rm}(X, S)\) if and only if \(-f \in \text{Lm}(X, S)\) we have a dual result for \(f \in \text{C}(X, S) = \text{Rm}(X, S)\) (i.e. there exists uniquely a point \(P\) such that \(f^{-1}(P) \notin S\) and \(b > P\) implies \(f^{-1}(P, b) \notin S\)).

Theorem 4.3. Let \((X, S)\) be a set algebra. Then \(C(X, S) =\)
$Lm(X,S) + Rm(X,S)$ if and only if $S$ has property $Q$.

Proof. If $S$ has property $Q$ and $f \in C(X,S) - Lm(X,S)$ then Lemma 4.5 and the remark which follows Lemma 4.5 imply $f \in Rm(X,S)$. If $S$ does not have property $Q$ then there exist sequences $\{E_i\}$ non-decreasing in $S$ and $\{F_i\}$ non-decreasing in $S$ such that $\sum E_i \cap \sum F_i = \emptyset$ and each of $\sum E_i$ and $\sum F_i \in S$. Let $f(x) = 1$ if $x \in E_1$, $f(x) = 2^{-1}$ if $1 > 1$ and $x \in E_1 \cap \sum E_j$ if $j < 1$ and $x \in F_1$ if $j > 1$ and $x \in F_1 \cap \sum F_j$, and $f(x) = 0$ if $x \in X = \sum (E_1 + F_1)$; $f \in C(X,S) - [Lm(X,S) + Rm(X,S)]$.

We conclude with an example to show that the property of being a sigma algebra (while sufficient) is not necessary in order that $S$ have property $Q$. Let $I$ be the set of positive integers, let $E \in T$ if and only if $E \subseteq I$, let $J$ be a maximal proper ideal in $T$, let $X = I + [0]$, and let $E \in S$ if and only if one of $E$ and $X - E \in J$ (i.e. we add 0 to the elements of $T - J$). $(X,S)$ is a set algebra which is not a sigma algebra and such that $S$ does have property $Q$. 
SELECTED BIBLIOGRAPHY


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