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A C^0 Interior Penalty Method for the von Kármán Equations

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A C^0 INTERIOR PENALTY METHOD FOR THE VON KÁRMÁN EQUATIONS

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To my wife Shelley
and our precious son Oliver.

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Abstract

In this dissertation we develop a C^0 interior penalty method for the von Kármán equations for nonlinear elastic plates. We begin with a brief survey on frequently used finite element methods for the von Kármán equations. After addressing some topics from functional analysis in the preliminaries, we present existence, uniqueness and regularity results for the solutions of the von Kármán equations in Chapter 3. In the next chapter we review the C^0 interior penalty method for the biharmonic problem. Motivated by these results, we propose a C^0 interior penalty method for the linearized von Kármán equations in Chapter 5 and show the well-posedness and stability of this method. We then introduce the new C^0 interior penalty method for von Kármán equations, and establish the corresponding a priori error estimate by a fixed point argument. Numerical examples are presented that confirm the theoretical results.

Chapter 1
Introduction

1.1 Von Kármán Equations

The von Kármán equations are two coupled partial differential equations of fourth order that describe the behavior of an elastic plate of thickness δ subject to transversal loads and lateral forces. In this model, the geometry of the plate $\mathcal{P}$ is prescribed by a middle surface. The middle surface of $\mathcal{P}$ is a bounded subset $\Omega \subset \mathbb{R}^2$ that lies in the (x_1, x_2) plane and cuts the plate into two equal halves of thickness $\frac{\delta}{2}$; i.e.,

$$\mathcal{P} = \Omega \times [-\frac{\delta}{2}, \frac{\delta}{2}].$$

The plate is assumed to be of St. Venant-Kirchhoff material with Lamé constants λ and μ, Young's modulus E, and Poisson ratio ν. The transversal load that acts on the plate is given by a function $f^\delta \colon \Omega \to \mathbb{R}$. The resulting transverse displacement is denoted by the function $\xi_1^\delta \colon \Omega \to \mathbb{R}$. The stresses on the plate are given in terms of the Airy stress function $\xi_2^\delta \colon \Omega \to \mathbb{R}$ that relates to the Cauchy stress tensor by

$$\sigma_{11} = \partial_{11}\xi_2^\delta, \quad \sigma_{22} = \partial_{22}\xi_2^\delta, \quad \sigma_{21} = -\partial_{21}\xi_2^\delta.$$

If the deflection ξ_1^δ is small relative to the thickness δ, the linear plate theory due to Kirchhoff and Love [36] provides a reasonable approximation for the behavior of the plate. In this model, ξ_1^δ is the solution of the biharmonic equation

$$D\Delta^2\xi_1^\delta = f^\delta, \tag{1.1}$$

where

$$D = \frac{E\delta^3}{12(1-\nu^2)}$$

is the flexural rigidity and Δ^2 is the biharmonic operator, given by

$$\Delta^2 u = \partial_{1111}u + 2\partial_{1122}u + \partial_{2222}u.$$

In 1910, Theodore von Kármán [39] observed that the existing theories for plates were limited to the case of small deflections. To capture the behavior of the plate for large deflections more accurately, von Kármán [39] included in his model terms of higher order for the strain tensor. In this way, he obtained the following system of partial differential equations

$$D\Delta^2 \xi_1^\delta = \delta[\xi_1^\delta, \xi_2^\delta] + f^\delta \quad \text{in } \Omega, \tag{1.2a}$$

$$\Delta^2 \xi_2^\delta = -\frac{E}{2}[\xi_1^\delta, \xi_1^\delta] \quad \text{in } \Omega, \tag{1.2b}$$

where $[\cdot,\cdot]$ is the Monge-Ampère form defined by

$$[u, v] = \partial_{11}u\partial_{22}v + \partial_{22}u\partial_{11}v - 2\partial_{12}u\partial_{12}v.$$

Originally, von Kármán did not specify boundary conditions for the equations, and the physical reasoning behind these equations remained controversial until 1980, when Ciarlet [13] showed that the two dimensional von Kármán system is "*the leading term of a formal asymptotic expansion (in terms of the thickness of the plate as the "small" parameter) of the exact three-dimensional equations of nonlinear elasticity associated with a specific class of boundary conditions.*" For a clamped plate, the boundary conditions are

$$\xi_1^\delta = \partial_n \xi_1^\delta = 0 \text{ on } \partial\Omega, \tag{1.2c}$$

$$\xi_2^\delta = \psi_0^\delta, \;\; \partial_n \xi_2^\delta = \psi_1^\delta \text{ on } \partial\Omega, \tag{1.2d}$$

where ∂_n denotes the normal derivative, and ψ_0^δ and ψ_1^δ are real-valued functions on $\partial\Omega$ that are related to the surface forces acting on the lateral face $\partial\Omega \times [\frac{\delta}{2}, \frac{\delta}{2}]$ of the plate.

By introducing the functions ξ_1, ξ_2, f, ψ_0, and ψ_1 with

$$\xi_1^\delta = \sqrt{\frac{2D}{\delta E}}\,\xi_1, \quad f^\delta = \sqrt{\frac{2}{\delta E}}\,D^{3/2}\,f,$$
$$\xi_2^\delta = \frac{D}{\delta}\xi_2, \quad \psi_0^\delta = \frac{D}{\delta}\psi_0, \quad \psi_1^\delta = \frac{D}{\delta}\psi_1,$$

the von Kármán equations can be written in a form free of any physical constants. These equations are referred to as the canonical von Kármán equations and are stated as follows. Find (ξ_1, ξ_2) such that

$$\Delta^2 \xi_1 = [\xi_1, \xi_2] + f \quad \text{in } \Omega, \tag{1.3a}$$

$$\Delta^2 \xi_2 = -[\xi_1, \xi_1] \quad \text{in } \Omega, \tag{1.3b}$$

$$\xi_1 = \partial_n \xi_1 = 0 \text{ on } \partial\Omega, \tag{1.3c}$$

$$\xi_2 = \psi_0, \; \partial_n \xi_2 = \psi_1 \text{ on } \partial\Omega. \tag{1.3d}$$

Due to the combination of nonlinear, second-order terms involving the Monge-Ampère form and linear terms of fourth order from the biharmonic operator, the von Kármán system is called a quasi-linear, fourth order problem. Von Kármán did not have solutions for his problem at hand [39], and although widely investigated, exact solutions were only obtained in rare cases for special geometries and boundary conditions [36, 26]. In 1934 Way [41] was celebrated for finding an exact solution for a circular, uniformly loaded plate (see also [26]). Eight years later, Levy [23] used double Fourier series to solve the von Kármán equations exactly on a simply supported, rectangular plate. Although finding an exact solution remained difficult in general, it was shown that solutions of the von Kármán equations exist for arbitrarily shaped plates [21, 24, 16, 14].

Another feature of the von Kármán equations due to their nonlinear nature is that the solutions are not unique in general. If only small forces are applied, Knightly [21] proved that the solutions are unique - the plate behaves in a predictable way. For larger loads, the plate starts to buckle: one deformation is realized among several possible distinct solutions of the von Kármán problem. The critical loading that causes the occurrence of bifurcating solution branches is of particular interest [4, 16].

1.2 Overview on Finite Element Methods for the von Kármán Equations

Various finite element methods have been developed for the von Kármán equations. The finite element method with the easiest formulation is the conforming method that is based on the weak form of the von Kármán problem.

Find $(\xi_1, \xi_2) \in V$ such that for all $(v_1, v_2) \in V$, there holds

$$\int_\Omega (\Delta\xi_1 \Delta v_1 + \Delta\xi_2 \Delta v_2 - [\xi_1, \xi_2]v_1 + [\xi_1, \xi_1]v_2)dx = \int_\Omega f v_1 dx, \tag{1.4}$$

where V denotes here the Sobolev space $(H_0^2(\Omega))^2$. The von Kármán equation is then discretized by restricting the variational formulation (1.4) to a finite dimensional subspace V_h of V. A function $(\xi_{h,1}, \xi_{h,2}) \in V_h$ is called a discrete solution of the conforming method if for all $(v_1, v_2) \in V_h$, there holds

$$\int_\Omega (\Delta\xi_{h,1} \Delta v_1 + \Delta\xi_{h,2} \Delta v_2 - [\xi_{h,1}, \xi_{h,2}]v_1 + [\xi_{h,1}, \xi_{h,1}]v_2)dx = \int_\Omega f v_1 dx. \tag{1.5}$$

In the case that the exact solution of the von Kármán equations is isolated, Brezzi [10] proved the existence and uniqueness of the discrete solution, and derived optimal error bounds for this method. More recently, the convergence of the conforming method was also established for a von Kármán plate with more general boundary conditions where the plate is partly clamped and partly free [15].

Conforming finite element methods for the space $(H_0^2(\Omega))^2$ require the use of C^1 elements, which are rather complicated and have a large number of degrees of freedom [43]. The simplest C^1 element is the Argyris element with 21 degrees of freedom and polynomial shape functions of degree 5. This means that 42 variables have to be known to describe the discrete solution locally on one finite element. Consequently, the size of the global discrete problem is very large for the conforming method, even if the triangulation consists of a relatively small number of elements.

One class of finite methods that can be used to avoid C^1 elements are mixed finite element methods. If the solution (ξ_1, ξ_2) of the von Kármán equations has the regularity $(\xi_1, \xi_2) \in (H^3(\Omega))^2$, the von Kármán equations can be written in an alternative weak form on the space $V = H_0^1(\Omega) \times (H^1(\Omega))^3$, treating the Hessians $\nabla^2\xi_1, \nabla^2\xi_2 \in (H^1(\Omega))^4$ as additional unknowns. This weak formulation is given as follows. Find $(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \in V \times V$ such that

$$\begin{aligned} a(\boldsymbol{\xi}_1, \boldsymbol{v}) + \int_\Omega [\boldsymbol{\xi}_1, \boldsymbol{\xi}_2] v \, dx &= -\int_\Omega f v \, dx \quad \forall \boldsymbol{v} \in V, \\ a(\boldsymbol{\xi}_2, \boldsymbol{v}) - \int_\Omega [\boldsymbol{\xi}_1, \boldsymbol{\xi}_1] v \, dx &= 0 \qquad\qquad \forall \boldsymbol{v} \in V, \end{aligned}$$

where $a(\cdot, \cdot)$ and $[\cdot, \cdot]$ are defined here for $\boldsymbol{\xi} = (\xi, u_{11}, u_{12}, u_{22}) \in V$, and $\boldsymbol{v} = (v, v_{11}, v_{12}, v_{22}) \in V$ with $u_{21} = u_{12}$ and $v_{21} = v_{12}$ by

$$\begin{aligned} [\boldsymbol{\xi}, \boldsymbol{v}] &= u_{11}v_{22} + u_{22}v_{11} - 2u_{12}v_{12}, \\ a(\boldsymbol{\xi}, \boldsymbol{v}) &= \sum_{i \le j} \int_\Omega \left(\partial_j \xi \partial_i v_{ij} + u_{ij} v_{ij}\right) dx + \sum_{i,j} \int_\Omega \partial_i u_{ij} \partial_j v \, dx. \end{aligned}$$

Based on this formulation, Miyoshi [28] defines a mixed method for the von Kármán equations. Since functions in V only need to be weakly differentiable once, C^0 elements can be used to discretize V.

The discrete problem is then to find $(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \in V_h \times V_h$ such that

$$\begin{aligned} a(\boldsymbol{\xi}_1, \boldsymbol{v}) + \int_\Omega [\boldsymbol{\xi}_1, \boldsymbol{\xi}_2] v \, dx &= -\int_\Omega f v \, dx \quad \forall \boldsymbol{v} \in V_h, \\ a(\boldsymbol{\xi}_2, \boldsymbol{v}) - \int_\Omega [\boldsymbol{\xi}_1, \boldsymbol{\xi}_1] v \, dx &= 0 \qquad\quad \forall \boldsymbol{v} \in V_h, \end{aligned}$$

where $V_h \subset V$ is the space of piecewise linear polynomials $(v, v_{11}, v_{12}, v_{22})$ with $v \equiv 0$ on $\partial\Omega$.

Since a linear polynomial on a triangle is determined by the three values on the vertices, it follows that the discrete solution of the mixed method depends on 24 unknowns per element. Compared to the conforming method with 42 unknowns per element, the size of the discrete problem is significantly smaller for the mixed method. Miyoshi [28] showed existence and convergence of the approximate solution, provided the exact solution of the von Kármán equation is isolated. Mixed methods for the von Kármán equations were further analyzed in [20, 11, 33] for the approximation of bifurcating solution branches.

Another approach to the approximation of the von Kármán equations is the hybrid finite element method by Quarteroni [31]. For this method, the variational formulation (1.4) is restricted to a subspace $V_h \subset H_0^2(\Omega) \times H_0^2(\Omega)$ defined as follows. A function (v_1, v_2) belongs to V_h if, for any triangle T in a regular triangulation of Ω, and $i = 1, 2$

$$\begin{aligned} \Delta^2 v_i =& 0 \text{ on } T, \\ v_i\big|_{\partial T} & \text{ is a cubic polynomial on } \partial T, \\ \partial_n v_i\big|_{\partial T} & \text{ is a linear polynomial on } \partial T. \end{aligned}$$

Since $(v_1, v_2) \in V_h$ are not polynomials inside T, the integrals in the formulation (1.5) cannot be evaluated easily in general. This issue can be overcome by discretizing the space V_h further with nonconforming Zienkiewicz elements and a space of piecewise polynomials for the approximation of $\nabla^2 v_1$ and $\nabla^2 v_2$. Quarteroni showed that this hybrid method is convergent if the exact solution of the von Kármán equation is isolated [31].

All of the methods presented above lead to a discrete nonlinear problem that is typically solved by Newton's method. The complexity of the computations in Newton's method depends heavily on the size of the Jacobian of the discrete problem. For this reason, discretizations with fewer degrees of freedom are preferable for this problem.

In this dissertation, we introduce a finite element method for the von Kármán equation with only 12 degrees of freedom per element. This is achieved by using a quadratic Lagrange finite element space $\boldsymbol{V}_h$ for the discretization. Our main result, proved in Section 5.2, is that the discrete solution of this method is well defined and converges to the exact solution of the von Kármán equations, provided the exact solution is unique.

The price we pay for using this simple finite element space is that the method is nonconforming. That is, $\boldsymbol{V}_h$ is not a subspace of the space $\boldsymbol{V} = (H_0^2(\Omega))^2$ where the exact solution lives. Hence, the finite element method cannot be obtained by just restricting the weak formulation (1.4) to the finite element space. Instead, one has to modify (1.4) such that is also defined for functions in the nonconforming finite element space $\boldsymbol{V}_h$. This modification can be obtained by considering the integration by parts formulas, that lead to the weak formulation (1.4), on a local scale, element by element. To mitigate the deficiency of $\boldsymbol{V}_h$ not being contained in $(C^1(\Omega))^2$, we penalize the jumps of the normal derivatives along inter-element boundaries. We call the resulting method the C^0 interior penalty method for the von Kármán equations (see Section 5.2 and Section 4.2 for details).

Before discretizing the von Kármán equations in Section 5.2, we study the theory of the von Kármán problem first. Following the outline of the analysis for the von Kármán equations in [14, 16], we establish in Section 3.2 conditions under which the von Kármán problem for a uniform loaded plate has a unique solution $\boldsymbol{\xi} \in (H_0^2(\Omega))^2$. Under these conditions, we show that the linearization of the von Kármán problem at $\boldsymbol{\xi}$ and the corresponding dual problem are invertible.

After reviewing the C^0 interior penalty method for the biharmonic equation [9] in Section 4.2, we formulate in Section 5.1 a C^0 interior penalty method for the linearized von Kármán equations. With an adaption of an argument from Schatz [34] and using the invertibility of the linearized von Kármán problem, we then prove that this method has a unique solution and derive an a priori error estimate for the approximation. Furthermore, we show with a new argument that this method is stable, which is a key component in the proof of our main result on the convergence of the C^0 interior penalty method for the von Kármán equations in Section 5.2.

We conclude this dissertation with numerical experiments of the new interior penalty method for the von Kármán equations. The theoretically expected rates of convergence are confirmed by our computational results.

Chapter 2
Preliminaries

2.1 Spectral Theory of Compact Operators on Hilbert Spaces

In this section, we review some facts from spectral theory of compact operators on Hilbert spaces. We begin with the definition of a compact operator and the definition of the spectrum.

Definition 2.1. *Let X, Y be Banach spaces. A bounded linear operator $T : X \to Y$ is compact if, for every bounded sequence $\{x_n\}$ in X, the sequence $\{Tx_n\}$ in Y has a convergent subsequence.*

Definition 2.2. *Let $T : X \to X$ be a bounded linear operator. The spectrum $\sigma(T)$ of T is defined by*

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid \lambda I - T \text{ is not bijective}\}.$$

A complex number $\lambda \in \mathbb{C}$ is said to be an eigenvalue of T, if $\lambda I - T$ is not injective, or equivalently if $Tx = \lambda x$ for some $0 \neq x \in H$.

Remark 2.3. *If $\lambda \notin \sigma(T)$, then the Inverse Mapping Theorem [32] implies that the resolvent $(\lambda I - T)^{-1}$ is a bounded linear operator.*

Clearly, every eigenvalue of a bounded linear operator T is contained in the spectrum of T. More can be said about the spectrum of a compact operator.

Theorem 2.4. *(Riesz-Schauder Theorem, [32])*
Let H be a Hilbert space. The spectrum of a compact operator $T : H \to H$ is discrete with no limit points other than possibly 0. Moreover, any nonzero $\lambda \in \sigma(T)$ is an eigenvalue of T with finite multiplicity.

Definition 2.5. *Let H be a Hilbert space with inner product $(\cdot,\cdot)$. A linear operator $T\colon H \to H$ is called symmetric if*

$$(Tx, y) = (x, Ty) \quad \forall\, x, y \in H.$$

A standard result in spectral theory is that the eigenvalues of a symmetric operator are real-valued [25]. A more precise statement on the distribution of the eigenvalues holds for a compact, symmetric operator (see [25], for instance).

Theorem 2.6. *Let H be a Hilbert space with scalar product $(\cdot,\cdot)$ and T be a symmetric, compact, linear operator on H. Then*

$$M = \sup_{\substack{x \in H \\ x \neq 0}} \frac{(Tx, x)}{(x, x)}, \quad \textit{and} \quad m = \inf_{\substack{x \in H \\ x \neq 0}} \frac{(Tx, x)}{(x, x)} \tag{2.1}$$

belong to the spectrum $\sigma(T)$, and $\sigma(T) \subseteq [m, M]$.

Definition 2.7. *Let H be a Hilbert space with inner product $(\cdot,\cdot)$. A linear operator $T : H \to H$ is called positive definite if*

$$(Tx, x) > 0 \quad \forall\, 0 \neq x \in H.$$

From Theorem 2.4 and Theorem 2.6, we conclude the following for the eigenvalues of a positive definite operator.

Corollary 2.8. *Let $T\colon H \to H$ be a compact, symmetric, positive definite operator on a Hilbert space H. Then M defined in (2.1) is the largest eigenvalue of T. Moreover, the eigenvalues of T form a non-increasing sequence $\{\lambda_k\} \subseteq (0, M]$ that is either finite, or converges to 0.*

2.2 Representation Theorems on Hilbert Spaces

This section covers the solvability of the abstract problem below.

Given $f \in H'$, find $u \in H$ such that

$$a(u, v) = f(v) \;\; \forall v \in H,$$

where $a(\cdot,\cdot): H \times H \to \mathbb{R}$ is a bounded bilinear form. That is, there exists $C > 0$ such that

$$a(u,v) \leq C||u||_H||v||_H \;\; \forall u,v \in H. \tag{2.2}$$

For the special case that $a(\cdot,\cdot)$ is the inner product associated with H, the Riesz-Representation Theorem [32] states the following.

Theorem 2.9. *(Riesz-Representation Theorem)*
Let H be a Hilbert space with inner product $(\cdot,\cdot)$. Then, for any $f \in H'$, there exists a unique $u \in H$ with $||u||_H = ||f||_{H'}$ such that

$$(u,v) = f(v) \;\; \forall v \in H.$$

Suppose now that the bounded bilinear form $a(\cdot,\cdot)$ is also coercive; i.e., there exists a constant $\alpha > 0$ such that

$$a(v,v) \geq \alpha||v||_H^2 \quad \forall v \in H. \tag{2.3}$$

If additionally $a(\cdot,\cdot)$ is symmetric, it follows that $a(\cdot,\cdot)$ is itself an inner product on H. By (2.2) and (2.3), we have that

$$\alpha||v||_H^2 \leq a(v,v) \leq C||v||_H^2 \;\; \forall v \in H. \tag{2.4}$$

Consequently, the inner product $a(\cdot,\cdot)$ induces a norm equivalent to the standard norm on H, and we can apply Theorem 2.9.

Corollary 2.10. *Let H be a Hilbert space and $a(\cdot,\cdot)$ be a symmetric bilinear form on $H \times H$. If $a(\cdot,\cdot)$ is bounded and coercive, then $a(\cdot,\cdot)$ forms an inner product on H. Its induced norm is equivalent to the standard norm in H, and for every $f \in H'$, there exists a unique $u \in H$ such that*

$$a(u,v) = f(v) \;\; \forall v \in H. \tag{2.5}$$

The next theorem shows that the symmetry of $a(\cdot,\cdot)$ is not necessary for the representation (2.5) to hold. A proof can be found in [18, 8, 12].

Theorem 2.11. *(Lax-Milgram Lemma)*
Let H be a Hilbert space and $a(\cdot,\cdot)$ be a bounded bilinear form on $H \times H$. If $a(\cdot,\cdot)$ is coercive with coercivity constant $\alpha > 0$, then for any $f \in H'$, there exists a unique $u \in H$ such that

$$a(u, v) = f(v) \quad \forall v \in H. \tag{2.6}$$

Remark 2.12. *If $u \in H$ is the unique solution of (2.6), then*

$$\alpha||u||_H^2 \leq a(u, u) = f(u) \leq ||f||_{H'}||u||_H.$$

This implies the stability estimate $||u||_H \leq \frac{1}{\alpha}||f||_{H'}$.

2.3 Weak Convergence

In this section, we introduce the notion of weak convergence in a Hilbert space H. It is a meaningful tool in many compactness arguments.

Definition 2.13. *Let $\{x_n\}$ be a sequence in a Hilbert space H, and $x \in H$. We say $x_n \rightharpoonup x$ converges weakly if*

$$(x_n, y) \to (x, y) \;\; \forall y \in H.$$

Remark 2.14. *By the Cauchy-Schwarz inequality, it follows that strong convergence implies weak convergence.*

Lemma 2.15. *Let H be a Hilbert space. If $x_k \to x$ converges strongly in H and $y_k \rightharpoonup y$ converges weakly in H, then $(x_k, y_k) \to (x, y)$.*

Proof. Since (y_k, z) is convergent for any $z \in H$, the sequence $\{(y_k, z)\}$ is bounded for every $z \in H$. By the principle of uniform boundedness [32], there exists $M > 0$

such that $||y_k||_H < M$ for all $k \in \mathbb{N}$. Hence, as $k \to \infty$, we have that

$$\begin{aligned}|(x_k, y_k) - (x, y)| &= |(x, y_k - y) + (x_k - x, y_k)| \\ &\leq |(x, y_k - y)| + M||x_k - x||_H \to 0.\end{aligned}$$

□

Lemma 2.16. *(Sub-Subsequence Lemma)*

Let $\{x_k\}$ be a sequence in a Hilbert space H, and $x \in H$.

Then the following is equivalent.

(i) The sequence $\{x_k\}$ converges strongly to x.

(ii) Every subsequence of $\{x_k\}$ has a subsequence that converges to x.

Proof. The implication $(i) \Rightarrow (ii)$ is obvious. For the other direction, suppose $\{x_k\}$ does not converge to x. Then we can find $\epsilon > 0$ such that for any $k \in \mathbb{N}$ there exists $k_0 > k$ with $||x_{k_0} - x||_H \geq \epsilon$. In the same manner, we find $k_1 > k_0$ with $||x_{k_1} - x||_H \geq \epsilon$. Repeating this argument, we obtain a subsequence $\{x_{k_n}\}$ that satisfies $||x_{k_n} - x||_H \geq \epsilon$ for all $n \in \mathbb{N}$. Clearly, the sequence $\{x_{k_n}\}$ does not have a subsequence converging to x. □

Lemma 2.17. *Let H be a Hilbert space that is compactly embedded into a Hilbert space X. If $x_k \rightharpoonup x$ weakly in H, then $x_k \to x$ strongly in X.*

Proof. As seen in the proof of Lemma 2.15, it follows from the principle of uniform boundedness that the sequence $\{x_k\}$ is bounded in H. Now let $\{x_l\}$ be an arbitrary subsequence of $\{x_k\}$. Since H is compactly embedded in X, the sequence $\{x_l\}$ has a subsequence $\{x_m\}$ that converges strongly in X, say $x_m \to y$. It follows that $x_m \rightharpoonup y$ in X. But it also holds that $x_m \rightharpoonup x$ in H, because $\{x_m\}$ is a subsequence of $\{x_k\}$. Since the weak limit is unique in H, we conclude that $y = x$. Hence, $x_m \to x$ strongly in X. The Sub-Subsequence Lemma completes the proof. □

The next lemma can be seen as a compactness result in the topology of weak convergence. Its proof involves a diagonal argument (cf. [42, Theorem III.3.7]).

Lemma 2.18. *Every bounded sequence in a Hilbert space H has a weakly convergent subsequence.*

Definition 2.19. *Let H be a Hilbert space. A functional $f : H \to \mathbb{R}$ is called weakly lower semicontinuous, if*

$$f(x) \leq \liminf f(x_k) \quad \text{whenever } x_k \rightharpoonup x. \tag{2.7}$$

Example 2.20. *Let H be a Hilbert space. Then the functional $f(x) = ||x||_H$ is weakly lower semicontinuous. This follows from the observation that*

$$||x||_H^2 = (x,x) = \lim_k (x, x_k) = \liminf_k (x, x_k) \leq \liminf_k ||x||_H \, ||x_k||_H \quad \text{as } x_k \rightharpoonup x.$$

Definition 2.21. *Let H be a Hilbert space. We call $f \in H'$ coercive, if*

$$f(x) \to \infty \text{ as } ||x||_H \to \infty.$$

The final theorem of this section is a classical result from the direct methods in the calculus of variations [35].

Theorem 2.22. *Let H be a Hilbert space, and $f \in H'$ be a weakly lower semicontinuous, coercive functional. Then f is bounded below, and there exists $x \in H$ such that*

$$f(x) = \inf_{y \in H} f(y). \tag{2.8}$$

Proof. If f is not bounded from below, there is a sequence $\{x_k\}$ in H such that $f(x_k) \to -\infty$. From the coercivity of f, it follows that $\{x_k\}$ is bounded. By Lemma 2.18, $\{x_k\}$ has a convergent subsequence, say $x_{k_l} \rightharpoonup x$ for some $x \in H$. Then the weakly lower semicontinuity of f implies

$$f(x) \leq \liminf_l f(x_{k_l}) = -\infty.$$

Therefore, f is bounded below. Let $\{x_k\}$ be a minimizing sequence in H, i.e.,

$$f(x_k) \to \inf_{y \in H} f(y). \tag{2.9}$$

As $\{x_k\}$ is bounded by coercivity, we can find by Lemma 2.18 a subsequence $\{x_{k_l}\}$ that converges weakly to some $x \in H$. From (2.9) and (2.7), it follows that

$$f(x) \leq \liminf_{l} f(x_{k_l}) = \inf_{y \in H} f(y).$$

This concludes the theorem. □

2.4 Sobolev Spaces

Let Ω be a bounded open subset in $\mathbb{R}^n$. The space of infinitely differentiable functions on Ω with compact support is denoted by $C_c^\infty(\Omega)$, and the space of locally integrable functions is defined by

$$L^1_{\text{loc}}(\Omega) = \{f : \Omega \to \mathbb{R} \mid f \in L^1(D) \text{ for any compact subset } D \subset \Omega\}.$$

A function $f \in L^1_{\text{loc}}(\Omega)$ is weakly differentiable of order α if there exists a function $g \in L^1_{\text{loc}}(\Omega)$ such that

$$\int_\Omega f \partial^\alpha v \, dx = (-1)^{|\alpha|} \int_\Omega g v \, dx \quad \forall v \in C_c^\infty(\Omega), \tag{2.10}$$

where α is a vector $(\alpha_1, \alpha_2, ..., \alpha_n)$ of nonnegative integers with $|\alpha| = \sum_{i=1}^n \alpha_i$, and $\partial^\alpha f$ denotes the partial derivatives in the multiindex notation $(\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n} f$. We call the function g in (2.10) the α^{th} weak derivative of f, and write $\partial^\alpha f$ for g. For the weak derivatives $\partial^{(2,0)} f$, $\partial^{(0,2)} f$, and $\partial^{(1,1)} f$, we often use the notation $\partial_{11} f, \partial_{22} f$, and $\partial_{12} f$.

Definition 2.23. *(Sobolev Spaces, [1, 18, 37])*

For an integer $k \geq 0$ and $1 \leq p \leq \infty$, the Sobolev space $W_p^k(\Omega)$ is defined by

$$W_p^k(\Omega) = \{f \in L^1_{loc}(\Omega) \mid \partial^\alpha f \in L^p(\Omega) \quad \forall \, |\alpha| \leq k\}.$$

The Sobolev space $W_p^k(\Omega)$ is equipped with the norm

$$||f||_{W_p^k(\Omega)} = \begin{cases} \left(\sum_{|\alpha|\leq k} ||\partial^\alpha f||^p_{L^p(\Omega)}\right)^{1/p} & \text{for } 1 \leq p < \infty \\ \max_{|\alpha|\leq k} ||\partial^\alpha f||_{L^\infty(\Omega)} & \text{for } p = \infty. \end{cases}$$

and the seminorm

$$|f|_{W_p^k(\Omega)} = \begin{cases} \left(\sum_{|\alpha|= k} ||\partial^\alpha f||^p_{L^p(\Omega)}\right)^{1/p} & \text{for } 1 \leq p < \infty \\ \max_{|\alpha|= k} ||\partial^\alpha f||_{L^\infty(\Omega)} & \text{for } p = \infty. \end{cases}.$$

For $s > 0$ with $s = k + \sigma$ for some nonnegative integer k and some $0 < \sigma < 1$, the fractional Sobolev space $W_p^s(\Omega)$ is defined for $1 \leq p < \infty$ by

$$W_p^s(\Omega) = \{f \in W_p^k(\Omega) \mid ||f||_{W_p^s(\Omega)} < \infty\},$$

where

$$||f||^p_{W_p^s(\Omega)} = ||f||^p_{W_p^k(\Omega)} + \sum_{|\alpha|=m} \left(\int_\Omega \int_\Omega \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|^p}{|x-y|^{n+\sigma p}}\, dxdy \right). \tag{2.11}$$

Remark 2.24. *One can also define the fractional Sobolev space $W_p^s(\Omega)$ by means of interpolation between $W_p^k(\Omega)$ and $W_p^{k+1}(\Omega)$ (see [1, 37, 38]).*

Theorem 2.25. *The space $W_p^s(\Omega)$ is a Banach space.*

For $p = 2$, the Sobolev space $W_p^s(\Omega)$ is a Hilbert space, and we refer to it by $H^s(\Omega)$. For $k \in \mathbb{N}$, the inner product on $H^k(\Omega)$ is given by

$$(f, g)_{H^k(\Omega)} = \sum_{|\alpha|\leq k} \int_\Omega \partial^\alpha f \partial^\alpha g\, dx.$$

The next theorem shows that functions in a Sobolev space can be approximated by smooth functions. A proof of this theorem can be found in [18, 37, 1].

Theorem 2.26. *Let $1 \leq p < \infty$, and k be a nonnegative integer. Then $C^\infty(\Omega)$ is dense in $W_p^k(\Omega)$.*

Definition 2.27. *Let $s > 0$. The closure of $C_c^\infty(\Omega)$ in $H^s(\Omega)$ is denoted by $H_0^s(\Omega)$.*

Definition 2.28. *The dual space of $H_0^s(\Omega)$ is called $H^{-s}(\Omega)$, and its norm is defined for $F \in H^{-s}(\Omega)$ by*

$$||F||_{H^{-s}(\Omega)} = \sup_{\substack{v \in H_0^s(\Omega) \\ v \neq 0}} \frac{F(v)}{||v||_{H^s(\Omega)}}. \tag{2.12}$$

Remark 2.29. *Since the dual space of $H^{-s}(\Omega)$ is isometric to $H_0^s(\Omega)$, the norm of $u \in H_0^2(\Omega)$ can be calculated with the formula*

$$||u||_{H^s(\Omega)} = \sup_{\substack{F \in H^{-s}(\Omega) \\ F \neq 0}} \frac{F(u)}{||F||_{H^{-s}(\Omega)}}. \tag{2.13}$$

Theorem 2.30. *(Sobolev Embedding Theorem, [38])*

Let Ω be an arbitrary domain in $\mathbb{R}^n$, let $m \geq 0$ and $t \geq 0$, and $1 < p < \infty$. Then the following continuous embeddings exist.

(i) $W_p^{m+t}(\Omega) \hookrightarrow C^t(\Omega)$, provided $m > \frac{n}{p}$,

(ii) $W_p^{m+t}(\Omega) \hookrightarrow W_q^t(\Omega)$, provided $m \geq \frac{n}{p} - \frac{n}{q}$ and $p \leq q < \infty$.

Theorem 2.31. *(Rellich-Kondrachov, [1])*

Let Ω be a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary, let $t \geq 0$ and $k \geq 1$ be integers, and $1 \leq p < \infty$. Then the following embeddings are compact.

(i) $W_p^{k+t}(\Omega) \hookrightarrow W_q^t(\Omega)$ with $1 \leq q < \frac{np}{n-kp}$, provided $k < \frac{n}{p}$.

(ii) $W_p^{k+t}(\Omega) \hookrightarrow W_q^t(\Omega)$ with $1 \leq q < \infty$ provided $k \geq \frac{n}{p}$.

(iii) $W_p^{k+t}(\Omega) \hookrightarrow C^t(\Omega)$, provided $k > \frac{n}{p}$.

Corollary 2.32. *Let Ω be a bounded domain with piecewise smooth boundary. Then $L^1(\Omega)$ is continuously embedded into $H^{-2}(\Omega)$.*

Proof. By the Sobolev embedding theorem (Theorem 2.30), there exists a continuous embedding $H^2(\Omega) \hookrightarrow C^0(\Omega)$, that is, there exists $C > 0$ such that

$$||\phi||_\infty \leq C||\phi||_{H^2(\Omega)} \quad \forall \phi \in C_c^\infty(\Omega).$$

For $g \in L^1(\Omega)$, we have

$$||g||_{H^{-2}(\Omega)} = \sup_{\substack{\phi \in C_c^\infty(\Omega) \\ \phi \neq 0}} \frac{|\langle g, \phi \rangle|}{||\phi||_{H^2(\Omega)}} \leq \frac{||g||_{L^1(\Omega)} ||\phi||_\infty}{||\phi||_{H^2(\Omega)}} \leq C||g||_{L^1(\Omega)},$$

where $\langle \cdot, \cdot \rangle$ denotes here the duality between $H_0^2(\Omega)$ and $H^{-2}(\Omega)$. $\square$

Theorem 2.33. *(Trace Theorem, [38])*

Let Ω be a polygon with piecewise smooth boundary, $s > \frac{1}{2}$, and k be the largest integer smaller than $s - \frac{1}{2}$. Then there exists a bounded linear bounded operator TR *from $H^s(\Omega)$ onto $\prod_{l=0}^k H^{s-\frac{1}{2}-l}(\partial\Omega)$ such that*

$$\mathrm{TR}\, v = \left(v\Big|_{\partial\Omega}, \frac{\partial}{\partial n} v\Big|_{\partial\Omega}, \cdots, \frac{\partial^k}{\partial n^k} v\Big|_{\partial\Omega} \right) \quad \forall v \in C_c^\infty(\Omega),$$

where n is the outer normal with respect to $\partial\Omega$.

With the bounded linear operator TR, the space $H_0^s(\Omega)$ can be characterized as follows (see [38]).

Lemma 2.34. *Under the assumptions of Theorem 2.33, the following holds*

$$H_0^s(\Omega) = \{v \in H^s(\Omega) \mid \mathrm{TR}\, v = 0\}.$$

Theorem 2.35. *(Poincaré's Inequality, [1])*

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then there exists a constant $K > 0$ such that

$$||v||_{L^2(\Omega)} \leq K|v|_{H^1(\Omega)} \quad \forall v \in C_c^\infty(\Omega). \tag{2.14}$$

Since $H_0^1(\Omega)$ is the closure of $C_c^\infty(\Omega)$ with respect to $||\cdot||_{H^1(\Omega)}$, inequality (2.14) also holds for all $v \in H_0^1(\Omega)$. Then it is easy to see that on $H_0^1(\Omega)$ the seminorm $|\cdot|_{H^1(\Omega)}$ turns out to be a norm equivalent to $||\cdot||_{H^1(\Omega)}$. By applying Poincaré's inequality successively on derivatives of higher order, we obtain the following.

Corollary 2.36. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and k be a positive integer. Then $|\cdot|_{H^k(\Omega)}$ is a norm on $H_0^k(\Omega)$ equivalent to the norm $||\cdot||_{H^k(\Omega)}$.*

Remark 2.37. *To avoid the accumulation of constants, we will sometimes use the notation $A \lesssim B$ indicating an inequality of the type $A \leq C\,B$, where C is a positive generic constant, independent from any discretization parameters. If this relation holds in both directions, we write $A \approx B$.*

Thus, Corollary 2.36 becomes

$$|\cdot|_{H^k(\Omega)} \approx ||\cdot||_{H^k(\Omega)}. \tag{2.15}$$

Chapter 3

Theoretical Results on the von Kármán Equations

As the biharmonic operator is an integral part of the von Kármán problem, we begin this chapter with a section on the biharmonic problem.

3.1 The Biharmonic Problem

On a bounded domain $\Omega \subset \mathbb{R}^2$, we consider the biharmonic problem (1.1) with homogeneous boundary conditions, that is,

$$\Delta^2 u = f \quad \text{in } \Omega, \tag{3.1a}$$

$$u = \partial_n u = 0 \quad \text{on } \partial\Omega. \tag{3.1b}$$

If $u \in H^4(\Omega) \cap H_0^2(\Omega)$, integration by parts (see [12]) yields

$$\int_\Omega f v \, dx = \int_\Omega \Delta^2 u \, v \, dx = \int_\Omega \Delta u \Delta v \, dx \quad \forall v \in H_0^2(\Omega). \tag{3.2}$$

This motivates the weak formulation of the biharmonic problem.

For $F \in H^{-2}(\Omega)$, find $u \in H_0^2(\Omega)$ such that

$$(u, v)_\Delta = F(v) \quad \forall v \in H_0^2(\Omega), \tag{3.3}$$

where $(u, v)_\Delta$ is the symmetric bilinear form

$$(u, v)_\Delta = \int_\Omega \Delta u \Delta v \, dx. \tag{3.4}$$

For $v \in C_c^\infty(\Omega)$, integration by parts [12] yields

$$\begin{aligned} (v, v)_\Delta = ||\Delta v||^2_{L^2(\Omega)} &= \int_\Omega \Big(\sum_i (\partial_{ii} v)^2 + \sum_{i \neq j} \partial_{ii} v \partial_{jj} v \Big) \, dx \\ &= \int_\Omega \Big(\sum_i (\partial_{ii} v)^2 + \sum_{i \neq j} (\partial_{ij} v)^2 \Big) \, dx = |v|^2_{H^2(\Omega)}. \end{aligned} \tag{3.5}$$

Thus, Corollary 2.36 implies that $(\cdot,\cdot)_\Delta$ is an inner product on $H_0^2(\Omega)$. Moreover, $|v|_\Delta = \sqrt{(v,v)_\Delta}$ defines a norm on $H_0^2(\Omega)$ that coincides with $|\cdot|_{H^2(\Omega)}$. Hence, the norm $|\cdot|_\Delta$ is equivalent to the Sobolev norm $||\cdot||_{H^2(\Omega)}$ on $H_0^2(\Omega)$; i.e.,

$$|v|_\Delta \approx ||v||_{H^2(\Omega)} \ \ \forall v \in H_0^2(\Omega). \tag{3.6}$$

From the above, we conclude that $(\cdot,\cdot)_\Delta$ is a coercive, bounded bilinear form on $H_0^2(\Omega) \times H_0^2(\Omega)$. Furthermore, Theorem 2.9 implies the following.

Theorem 3.1. *For any $F \in H^{-2}(\Omega)$, the biharmonic problem* (3.3) *has a unique solution $u \in H_0^2(\Omega)$. Moreover, there exists $C > 0$ such that*

$$||u||_{H^2(\Omega)} \leq C||f||_{H^{-2}(\Omega)}. \tag{3.7}$$

An alternative way of carrying out the integration by parts in (3.2) is

$$\begin{aligned}\int_\Omega f v\,dx &= \int_\Omega \Delta^2 u\, v\,dx = -\int_\Omega \nabla(\Delta u)\cdot \nabla v\,dx = -\sum_{i=1}^{2}\int_\Omega (\Delta\partial_i u)\,\partial_i v\,dx \\ &= \sum_{i=1}^{2}\int_\Omega \nabla\partial_i u\cdot\nabla\partial_i v\,dx = \sum_{i,j=1}^{2}\int_\Omega \partial_{ij}u\,\partial_{ij}v\,dx.\end{aligned}$$

This leads to another weak formulation of the biharmonic problem.
Given $F \in H^{-2}(\Omega)$, find $u \in H_0^2(\Omega)$ such that

$$a(u,v) = F(v), \tag{3.8}$$

where $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is the symmetric bilinear form defined by

$$a(w,v) = \sum_{i,j=1}^{2}\int_\Omega \partial_{ij}w\,\partial_{ij}v\,dx. \tag{3.9}$$

Remark 3.2. *By Lemma 2.11, there exists a unique solution of* (3.8)*. Moreover, one can show that the unique solutions of the weak problems* (3.3) *and* (3.8) *coincide.*

3.2 The von Kármán Equations

Having introduced the biharmonic problem and its underlying Sobolev spaces, we proceed with theoretical considerations on the von Kármán equations. The analysis here essentially follows Ciarlet [14].

First, let us endow the von Kármán equations (1.3) with the appropriate function spaces from Section 2.4. Interpreting the equations (1.3) in the weak sense, the von Kármán problem is to find $(\xi, \psi) \in H_0^2(\Omega) \times H^2(\Omega)$ such that

$$\Delta^2 \xi = [\xi, \psi] + f \quad \text{in } \Omega, \tag{3.10a}$$

$$\Delta^2 \psi = -[\xi, \xi] \quad \text{in } \Omega, \tag{3.10b}$$

$$\psi = \psi_0, \;\; \partial_n \psi = \psi_1 \text{ on } \partial\Omega. \tag{3.10c}$$

where $f \in H^{-2}(\Omega), \psi_0 \in H^{3/2}(\partial\Omega)$, and $\psi_1 \in H^{1/2}(\Omega)$. Note that the boundary condition (1.3c) is enforced by the space $H_0^2(\Omega)$. Moreover, the spaces $H^{3/2}(\Omega)$ and $H^{1/2}(\Omega)$ are by the trace theorem (Theorem 2.33) the appropriate function spaces for the boundary conditions (3.10c).

To formulate the von Kármán problem on $H_0^2(\Omega) \times H_0^2(\Omega)$, we decompose ψ into

$$\psi = \xi_2 + \theta,$$

where $\xi_2 \in H_0^2(\Omega)$, and $\theta \in H^2(\Omega)$ is the unique solution of

$$\Delta^2 \theta = 0 \;\; \text{in } \Omega, \tag{3.11a}$$

$$\theta = \psi_0, \;\; \partial_n \theta = \psi_1 \text{ on } \partial\Omega. \tag{3.11b}$$

Writing ξ_1 for ξ and replacing ψ by $\theta + \xi_2$, we obtain the following version of the von Kármán equations.

Given $f \in H^{-2}(\Omega)$, find $\boldsymbol{\xi} = (\xi_1, \xi_2) \in H_0^2(\Omega) \times H_0^2(\Omega)$ such that

$$\Delta^2 \xi_1 = [\xi_1, \xi_2] + [\xi_1, \theta] + f \quad \text{in } \Omega, \tag{3.12a}$$

$$\Delta^2 \xi_2 = -[\xi_1, \xi_1] \quad \text{in } \Omega. \tag{3.12b}$$

where θ is defined by (3.11). For the remainder of this thesis, we use this formulation of the von Kármán equations, keeping in mind that the Airy stress function of the plate is actually $\psi = \theta + \xi_2$.

Remark 3.3. *To simplify the notation in the product spaces that arise in the von Kármán problem, we frequently use the boldface style for elements in a product space: We denote elements in the product space $X \times X = X^2$ of a Banach space X with norm $||\cdot||_X$ by $\boldsymbol{x} = (x_1, x_2)$. The product norm of $\boldsymbol{x}$ is then given by $||\boldsymbol{x}||_{X^2}^2 = ||x_1||_X^2 + ||x_2||_X^2$.*

By Theorem 3.1 the two biharmonic equations (3.12a) and (3.12b) of the von Kármán system are well defined, if the corresponding right hand sides, in particular the terms with the Monge-Ampère form, are in $H^{-2}(\Omega)$. As we will see now, this is indeed the case.

For $\eta, \chi \in H^2(\Omega)$, the Cauchy-Schwarz inequality yields

$$\begin{aligned} ||\,[\eta,\chi]\,||_{L^1(\Omega)} &\leq \int_\Omega \big(|\partial_{11}\eta\partial_{22}\chi| + |\partial_{22}\eta\partial_{11}\chi| + 2|\partial_{12}\eta\partial_{12}\chi|\big)dx \\ &\leq |\chi|_{H^2(\Omega)}|\eta|_{H^2(\Omega)}. \end{aligned} \tag{3.13}$$

The continuous embedding $L^1(\Omega) \hookrightarrow H^{-2}(\Omega)$ of Corollary 2.32 then implies that $[\eta, \chi] \in H^{-2}(\Omega)$. In other words, the Monge-Ampère form maps functions from $H^2(\Omega) \times H^2(\Omega)$ to $H^{-2}(\Omega)$. Thus, the following operator is defined by Theorem 3.1.

Definition 3.4. *Define the operator $B\colon H^2(\Omega) \times H^2(\Omega) \to H_0^2(\Omega)$ that maps (η, χ) to the unique solution $B(\eta, \chi) \in H_0^2(\Omega)$ of*

$$\Delta^2 B(\eta, \chi) = [\eta, \chi].$$

Remark 3.5. *The operator $B(\cdot,\cdot)$ is symmetric and bilinear. This follows directly from the linearity of the biharmonic operator and the fact that the Monge-Ampère form is symmetric and bilinear.*

The operator $B(\cdot,\cdot)$ reduces the von Kármán equations to a single equation in $H_0^2(\Omega)$, the reduced von Kármán equation (see [14]).

Theorem 3.6. *For $f \in H^{-2}(\Omega)$, let $\bar{f} \in H_0^2(\Omega)$ be the unique solution of the biharmonic problem $\Delta^2 \bar{f} = f$, let θ be defined by* (3.11)*, and $\mathcal{C}$ be the operator on $H_0^2(\Omega)$ defined by*

$$\mathcal{C}u = B(B(u,u),u). \tag{3.14}$$

Then $(\xi_1, \xi_2) \in (H_0^2(\Omega))^2$ is a solution of the von Kármán equations (3.12) *if and only if $\xi_1 \in H_0^2(\Omega)$ satisifies the reduced von Kármán equation*

$$\xi_1 + \mathcal{C}\xi_1 - B(\theta, \xi_1) + \bar{f} = 0, \tag{3.15}$$

and ξ_2 is determined by ξ_1 via

$$\xi_2 = -B(\xi_1, \xi_1). \tag{3.16}$$

Proof. The equivalence of (3.12b) and (3.16) follows immediately from the definition of $B(\cdot,\cdot)$. Thus, ξ_1 is a solution of the von Kármán equations (3.12) if and only if ξ_1 satisfies

$$\Delta^2 \xi_1 = -[\xi_1, B(\xi_1, \xi_1)] + [\theta, \xi_1] + f.$$

By Definition 3.4 and the definition of $\bar{f}$, this is equivalent to

$$\xi_1 = -B(\xi_1, B(\xi_1, \xi_1)) + B(\theta, \xi_1) + \bar{f}.$$

This proves the theorem. □

In the next lemma, we summarize the main properties of the Monge-Ampère form for the analysis of the von Kármán equations [14].

Lemma 3.7. *The mappings*

$$(u,v,w) \mapsto \int_\Omega [u,v]w\,dx, \tag{3.17}$$

$$b\colon (u,v,w) \mapsto \int_\Omega \big(\partial_{12}u(\partial_1 v\partial_2 w + \partial_2 v\partial_1 w) - (\partial_{11}u\partial_2 v\partial_2 w + \partial_{22}u\partial_1 v\partial_1 w)\big)dx \tag{3.18}$$

are bounded trilinear forms on $H^2(\Omega)\times H^2(\Omega)\times H^2(\Omega)$ with the following properties.

(i) If one of the arguments u, v, w belongs to $H_0^2(\Omega)$, then

$$\int_\Omega [u,v]w\,dx = b(u,v,w) = \int_\Omega [u,w]v\,dx. \tag{3.19}$$

(ii) There exists $C>0$ such that

$$\Big|\int_\Omega [u,v]w\,dx\Big| \le C|u|_{H^2(\Omega)}|v|_{W_4^1(\Omega)}|w|_{W_4^1(\Omega)}. \tag{3.20}$$

(iii) If $u\in W_\infty^2(\Omega)$, then there exists $C>0$ such that

$$|b(u,v,w)| \le ||u||_{W_\infty^2(\Omega)}||v||_{H^1(\Omega)}||w||_{H^1(\Omega)}. \tag{3.21}$$

Proof. Due to the Sobolev embedding $H^2(\Omega)\hookrightarrow C^0(\Omega)$ (see Theorem 2.30), there exists $C>0$ such that $||w||_{L^\infty(\Omega)} \le C||w||_{H^2(\Omega)}$. Thus, we can estimate using (3.13)

$$\Big|\int_\Omega [u,v]w\,dx\Big| \le ||w||_{L^\infty(\Omega)}||[u,v]||_{L^1(\Omega)} \le C||u||_{H^2(\Omega)}||v||_{H^2(\Omega)}||w||_{H^2(\Omega)}.$$

We conclude that the mapping (3.17) is a continuous trilinear form.

In a similar way, one can show that the trilinear form $b(\cdot,\cdot,\cdot)$ is continuous. The Sobolev embedding $H^2(\Omega)\hookrightarrow W_4^1(\Omega)$ (see Theorem 2.30) and the Cauchy-Schwarz inequality yield

$$\begin{aligned} |b(u,v,w)| &\lesssim |u|_{H^2(\Omega)} \sum_{1\le i,j\le 2}\Big(\int_\Omega |\partial_i v\partial_j w|^2\,dx\Big)^{\frac12} \\ &\lesssim |u|_{H^2(\Omega)}|v|_{W_4^1(\Omega)}|w|_{W_4^1(\Omega)} \\ &\lesssim ||u||_{H^2(\Omega)}||v||_{H^2(\Omega)}||w||_{H^2(\Omega)}. \end{aligned} \tag{3.22}$$

Since both trilinear forms are continuous, it suffices to show that equality (3.19) holds for functions in a dense subspace of $H^2(\Omega)$, with one of the arguments belonging to a dense subspace in $H_0^2(\Omega)$.

Let $u, v, w \in C^\infty(\Omega)$ and suppose one of the three functions has compact support. By integration by parts, we verify that

$$\begin{aligned}\int_\Omega [u,v]w\,dx &= \int_\Omega (\partial_{11}v\partial_{22}u)w + (\partial_{22}v\partial_{11}u)w - 2(\partial_{12}u\partial_{12}v)w\,dx \\ &= -\int_\Omega \partial_1 v\partial_1(w\partial_{22}u)\,dx - \int_\Omega \partial_2 v\partial_2(w\partial_{11}u)\,dx \\ &\quad + \int_\Omega \partial_2 v\partial_1(w\partial_{12}u)\,dx + \int_\Omega \partial_1 v\partial_2(w\partial_{12}u)\,dx \\ &= \int_\Omega \partial_{12}u(\partial_1 v\partial_2 w + \partial_2 v\partial_1 w) - (\partial_{11}u\partial_2 v\partial_2 w + \partial_{22}u\partial_1 v\partial_1 w)\,dx \\ &= b(u,v,w).\end{aligned}$$

In consideration of Theorem 2.26 and Definition 2.27, this implies the first equality in (3.19). The second equality in (3.19) is then a consequence of the fact that $b(u,v,w)$ is symmetric with respect v and w. Moreover, inequality (3.20) follows from (3.19) and estimate (3.22). Finally, if $u \in W^2_\infty(\Omega)$ we can estimate

$$|b(u,v,w)| \le \max_{i,j}\{||\partial_{ij}u||_{L^\infty(\Omega)}\}\sum_{i,j=1}^{2}\int_\Omega |\partial_i v\partial_j w|\,dx \le ||u||_{W^2_\infty}||v||_{H^1(\Omega)}||w||_{H^1(\Omega)}.$$

□

Corollary 3.8. *Let $u \in H^2(\Omega)$ and $v, w \in H_0^2(\Omega)$. Then*

$$(B(u,v),w)_\Delta = (B(u,w),v)_\Delta.$$

Proof. By (3.19) and Definition 3.4, we have

$$\begin{aligned}(B(u,v),w)_\Delta &= \int_\Omega \Delta B(u,v)\Delta w\,dx = \int_\Omega [u,v]w\,dx \\ &= \int_\Omega [u,w]v\,dx = \int_\Omega \Delta B(u,w)\Delta v\,dx = (B(u,w),v)_\Delta.\end{aligned}$$

□

With the previous lemma and the associated corollary one can derive the main properties of the operators in the reduced von Kármán equation (cf. [14]).

Lemma 3.9. *Let $\theta \in H^2(\Omega)$ and $B(\cdot,\cdot)$ be the operator from Definition 3.4. Then the following holds.*

(i) The operator $B\colon H_0^2(\Omega)\times H_0^2(\Omega) \to H_0^2(\Omega)$ is bounded; i.e., there exists $C > 0$ such that

$$|B(u,v)|_\Delta \leq C|u|_\Delta |v|_\Delta \quad \forall u,v \in H_0^2(\Omega). \tag{3.23}$$

Moreover, B is sequentially compact; i.e.,

if $(u_k, v_k) \rightharpoonup (u,v)$ converges weakly in $H^2(\Omega)\times H^2(\Omega)$,

then $B(u_k,v_k) \to B(u,v)$ converges strongly in $H_0^2(\Omega)$.

(ii) The operator $\mathcal{C} : H_0^2(\Omega) \to H_0^2(\Omega)$ defined in (3.14) is sequentially compact and positive definite in the sense that

$$(\mathcal{C}u, u)_\Delta > 0 \quad \forall\, 0 \neq u \in H_0^2(\Omega).$$

(iii) The mapping $u \mapsto B(\theta, u)$ defines a compact operator on $H_0^2(\Omega)$ that is symmetric with respect to $(\cdot,\cdot)_\Delta$.

Proof. By (3.20) and (3.6), we have for $u, v \in H^2(\Omega)$

$$\begin{aligned} |B(u,v)|_\Delta &= \sup_{\substack{w\, \in\, H_0^2(\Omega)\\ w\neq 0}} \frac{(B(u,v),w)_\Delta}{|w|_\Delta} \\ &= \sup_{\substack{w\, \in\, H_0^2(\Omega)\\ w\neq 0}} \frac{\int_\Omega [u,v] w\, dx}{|w|_\Delta} \lesssim |u|_{W_4^1(\Omega)} |v|_{W_4^1(\Omega)}. \end{aligned} \tag{3.24}$$

Thus, the Sobolev embedding $H^2(\Omega) \hookrightarrow W_1^4(\Omega)$ together with (3.6) imply (3.23). Furthermore, we obtain from (3.24) that

$$|B(u_k,v_k) - B(u,v)|_\Delta \leq |B(u_k - u, v)|_\Delta + |B(u, v_k - v)|_\Delta + |B(u_k - u, v_k - v)|_\Delta$$
$$\lesssim (|u_k - u|_{W_4^1(\Omega)} + |v_k - v|_{W_4^1(\Omega)} + |u_k - u|_{W_4^1(\Omega)}|v_k - v|_{W_4^1(\Omega)}). \quad (3.25)$$

Since $(u_k, v_k) \rightharpoonup (u, v)$ in $H^2(\Omega) \times H^2(\Omega)$, and since $H^2(\Omega)$ is compactly embedded in $W_4^1(\Omega)$ (see Theorem 2.31), Lemma 2.17 implies that $(u_k, v_k) \to (u, v)$ converges strongly in $W_4^1(\Omega) \times W_4^1(\Omega)$. This together with estimate (3.25) proves (i).

Let $u \in H_0^2(\Omega)$. Since $B(u, u) \in H_0^2(\Omega)$, we obtain from Corollary 3.8 that

$$(\mathcal{C}u, u)_\Delta = (B(B(u, u), u), u)_\Delta = (B(u, u), B(u, u))_\Delta \geq 0. \quad (3.26)$$

Moreover, it follows that $(\mathcal{C}u, u) = 0$ if and only if $B(u, u) = 0$. By Definition 3.4, this is equivalent to $[u, u] = 0$. Thus, for the positive definiteness of $\mathcal{C}$, we only need to show that $[u, u] = 0$ implies $u = 0$. Consider the function $v \in H^2(\Omega)$ defined by

$$v(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2).$$

Then

$$[u, v] = \partial_{11}u\partial_{22}v + \partial_{22}u\partial_{11}v = \Delta u.$$

If $[u, u] = 0$, integration by parts yields together with identity (3.19) that

$$0 = -\int_\Omega [u, u]v\,dx = -\int_\Omega [u, v]u\,dx = -\int_\Omega (\Delta u)u\,dx = \int_\Omega \nabla u \nabla u\,dx = |u|_{H^1(\Omega)}^2.$$

Therefore, u is constant on Ω. From the fact that $u = 0$ on $\partial\Omega$ (cf. Lemma 2.34), we conclude that $u = 0$. Since the sequential compactness of $\mathcal{C}$ is a direct consequence of (i), this establishes (ii).

The symmetry of the operator $u \mapsto B(\theta, u)$ follows directly from Corollary 3.8. To show compactness, let $\{u_k\}$ be a bounded sequence in $H_0^2(\Omega)$. By Lemma 2.18,

$\{u_k\}$ has a weakly convergent subsequence, say $\{u_{k_l}\}$. Because of the sequential compactness of B, the sequence $\{B(\theta, u_{k_l})\}$ is a convergent subsequence of $\{B(\theta, u_k)\}$. Thus, by Definition 2.1, the map $u \mapsto B(\theta, u)$ is a compact operator. This completes the proof. $\square$

In the preceding lemma, we have seen that the operator $B(\cdot,\cdot)$ is bounded in the sense of (3.23). Therefore, we can define

$$|B|_\Delta = \sup_{\substack{u, v \in H_0^2(\Omega) \\ u \neq 0, v \neq 0}} \frac{|B(u,v)|_\Delta}{|u|_\Delta |v|_\Delta}. \tag{3.27}$$

The boundedness of $B(\cdot,\cdot)$ allows us to investigate the operator $u \mapsto \mathcal{C}u$ with respect to perturbations in u. The next lemma results from an exercise in [14].

Lemma 3.10. *Let $u, v \in H_0^2(\Omega)$. Then*

$$|\mathcal{C}u - \mathcal{C}v|_\Delta \leq 3|B|_\Delta^2 |u-v|_\Delta \max\{|u|_\Delta^2, |v|_\Delta^2\}. \tag{3.28}$$

Proof. Since $B(\cdot,\cdot)$ is bilinear and symmetric, we have for any $u, v \in H_0^2(\Omega)$

$$\begin{aligned}
\mathcal{C}u - \mathcal{C}v &= B(u, B(u,u)) - B(v, B(v,v)) \\
&= B(u, B(u, u-v)) + B(v, B(v, u-v)) + B(u-v, B(u,v)).
\end{aligned}$$

Hence, (3.27) and the geometric mean inequality imply

$$\begin{aligned}
|\mathcal{C}u - \mathcal{C}v|_\Delta &\leq |B|_\Delta^2 |u-v|_\Delta (|u|_\Delta^2 + |v|_\Delta^2 + |u|_\Delta |v|_\Delta) \\
&\leq |B|_\Delta^2 |u-v|_\Delta (|u|_\Delta^2 + |v|_\Delta^2 + \frac{1}{2}|u|_\Delta^2 + \frac{1}{2}|v|_\Delta^2) \\
&= \frac{3}{2}|B|_\Delta^2 |u-v|_\Delta (|u|_\Delta^2 + |v|_\Delta^2) \\
&\leq 3|B|_\Delta^2 |u-v|_\Delta \max\{|u|_\Delta^2, |v|_\Delta^2\}.
\end{aligned}$$

$\square$

3.2.1 Existence of Solutions

Having derived the essential properties of the operators in the reduced von Kármán equation, we address now the existence of solutions of the von Kármán equations. In the first lemma, which can be found in [14], we see that the solutions of the von Kármán equations correspond to the stationary points of a certain functional.

Lemma 3.11. *Suppose the operator $\mathcal{C}$, and the functions $\theta \in H^2(\Omega)$ and $\bar{f} \in H_0^2(\Omega)$ are given as in Theorem 3.6. Define the functional $j\colon H_0^2(\Omega) \to \mathbb{R}$ by*

$$j(u) = \frac{1}{4}(\mathcal{C}u, u)_\Delta + \frac{1}{2}(u, u)_\Delta - \frac{1}{2}(B(\theta, u), u)_\Delta - (\bar{f}, u)_\Delta. \tag{3.29}$$

Then j is differentiable on $H_0^2(\Omega)$, and $\xi_1 \in H_0^2(\Omega)$ is a solution of (3.15)*, if and only if $j'(\xi_1) = 0$.*

Proof. Let $u \in H_0^2(\Omega)$ be arbitrary. As $B(\cdot, \cdot)$ is bilinear, bounded and symmetric (see Lemma 3.9), we have

$$B(u + v, u + v) - B(u, u) = 2B(u, v) + B(v, v),$$

$$\text{and} \quad \frac{|B(v, v)|_\Delta}{|v|_\Delta} \to 0 \quad \text{as} \quad |v|_\Delta \to 0.$$

Thus, $B(\cdot, \cdot)$ is differentiable and its derivative at u is $2B(u, \cdot)$.

Similarly, since $(\cdot, \cdot)_\Delta$ and $(B(\theta, \cdot), \cdot)_\Delta$ are bilinear, bounded and symmetric, it follows that their derivatives at u are $2(u, \cdot)_\Delta$ and $2(B(\theta, u), \cdot)_\Delta$, respectively. Lastly, it follows from the chain rule that the derivative of $(B(\cdot, \cdot), B(\cdot, \cdot))_\Delta$ at u is $2(B(u, u), 2B(u, \cdot))_\Delta$. Thus, by (3.26), the derivative of $(\mathcal{C}\cdot, \cdot)_\Delta$ at u is $4(\mathcal{C}u, \cdot)_\Delta$. Combing the calculations above, we obtain for the derivative of j at u the functional

$$j'(u) = u + \mathcal{C}u - B(\theta, u) - \bar{f}.$$

Theorem 3.6 completes the proof. □

In consideration of Lemma 3.11, the existence of solutions for the von Kármán equations can be proved by finding stationary points of the functional j. This is done in the next theorem (see [14]).

Theorem 3.12. *Under the conditions in* (3.10)*, there exists a solution of the von Kármán equations* (3.12) *that globally minimizes the functional* j *in* (3.29)*.*

Proof. By Lemma 3.11, it suffices to show that there exists a minimizer $\xi_1 \in H_0^2(\Omega)$ of the functional j, i.e.,

$$j(\xi_1) = \inf_{u \in H_0^2(\Omega)} j(u).$$

By Theorem 2.22, we only need to check that j is weakly lower semicontinuous and coercive.

First we prove that j is weakly lower semicontinuous. Suppose $u_k \rightharpoonup u$ converges weakly in $H_0^2(\Omega)$. From Example 2.20 we know that

$$|u|_\Delta \leq \liminf_k |u_k|_\Delta.$$

Taking the square of this inequality, we obtain that $(\cdot,\cdot)_\Delta$ is weakly lower semicontinuous. Moreover, since $\mathcal{C}$ and $B(\theta,\cdot)_\Delta$ are sequentially compact (see Lemma 3.9), $\mathcal{C}u_k \to \mathcal{C}u$ and $B(\theta, u_k) \to B(\theta, u)$ converge strongly in $H_0^2(\Omega)$. Thus, Lemma 2.15 implies

$$(\mathcal{C}u_k, u_k)_\Delta \to (\mathcal{C}u, u)_\Delta, \tag{3.30}$$

$$(B(\theta, u_k), u_k)_\Delta \to (B(\theta, u), u)_\Delta. \tag{3.31}$$

Note that the convergence $(\bar{f}, u_k)_\Delta \to (\bar{f}, u)_\Delta$ is a direct consequence of the weak convergence $u_k \rightharpoonup u$ in $H_0^2(\Omega)$. Therefore, j is weakly lower semicontinuous.

Next, suppose j is not coercive. Then by Definition 2.21, there exists a constant $M > 0$ and an unbounded sequence $\{u_k\}$ in $H_0^2(\Omega)$ such that

$$j(u_k) \leq M \quad \forall k \in \mathbb{N}. \tag{3.32}$$

Without loss of generality, we can assume that $u_k \neq 0$ for all $k \in \mathbb{N}$. Then

$$\frac{j(u_k) + (\bar{f}, u_k)_\Delta}{|u_k|_\Delta^2} \leq \frac{M + (\bar{f}, u_k)_\Delta}{|u_k|_\Delta^2}. \tag{3.33}$$

Since the sequence $\{v_k\}$, defined by

$$v_k = \frac{u_k}{|u_k|_\Delta} \quad \text{with} \quad |v_k|_\Delta = 1 \quad \forall k \in \mathbb{N}$$

is bounded, there exists by Theorem 2.18 a weakly convergent subsequence of $\{v_k\}$, say $v_{k_l} \rightharpoonup v$. Passing inequality (3.33) to the subsequence $\{u_{k_l}\}$ yields

$$\frac{1}{2} - \frac{1}{2}(B(\theta, v_{k_l}), v_{k_l})_\Delta + \frac{1}{4}|u_{k_l}|_\Delta^2 (\mathcal{C}v_{k_l}, v_{k_l})_\Delta \leq \frac{M}{|u_{k_l}|_\Delta^2} + \frac{(\bar{f}, v_{k_l})_\Delta}{|u_{k_l}|_\Delta} \quad \forall l \in \mathbb{N}. \tag{3.34}$$

As in (3.30), the weak convergence $v_{k_l} \rightharpoonup v$ implies $(\mathcal{C}v_{k_l}, v_{k_l})_\Delta \to (\mathcal{C}v, v)_\Delta$. According to Lemma 3.9(ii) there are the following two cases:

If $(\mathcal{C}v, v)_\Delta > 0$, then the left hand side in (3.34) is unbounded. This contradicts the fact that the right hand side tends to 0 as $l \to \infty$.

Otherwise, we have that $v = 0$. Then it follows from (3.31) that $(B(\theta, v_{k_l}), v_{k_l})_\Delta$ approaches 0 as $l \to \infty$. Consequently, the left hand side in (3.34) exceeds $\frac{1}{2}$ as $l \to \infty$, whereas the right hand side approaches 0. We conclude in both cases that j is coercive. □

3.2.2 Uniqueness and Non-Uniqueness of Solutions

Although a solution of the von Kármán equations always exists (see Theorem 3.12), the solution is not necessarily unique. In this section, we determine conditions for the uniqueness of solutions of the von Kármán plate (3.10) subject to uniform loading along the lateral faces. The analysis here differs only slightly from [14].

The boundary conditions that correspond to the uniform lateral loading are

$$\psi_0 = -\frac{p}{2}(x_1^2 + x_2^2) \text{ and } \psi_1 = -\frac{p}{2}\frac{\partial}{\partial n}(x_1^2 + x_2^2), \tag{3.35}$$

where $p \in \mathbb{R}$ is a parameter proportional to the magnitude of the lateral force. For this special choice of boundary conditions,

$$\theta(x_1, x_2) = -\frac{p}{2}(x_1^2 + x_2^2) \in H^2(\Omega) \tag{3.36}$$

is the unique solution of (3.11). The Monge-Ampère form $[\theta, u]$ for $u \in H_0^2(\Omega)$ reduces then to

$$[\theta, u] = -p\Delta u. \tag{3.37}$$

Hence, the von Kármán equations (3.12) can be stated as follows.

Given $f \in H^{-2}(\Omega)$, find $(\xi_1, \xi_2) \in H_0^2(\Omega) \times H_0^2(\Omega)$ such that

$$\Delta^2 \xi_1 = [\xi_1, \xi_2] - p\Delta\xi_1 + f \quad \text{in } \Omega, \tag{3.38a}$$

$$\Delta^2 \xi_2 = -[\xi_1, \xi_1] \quad \text{in } \Omega. \tag{3.38b}$$

Likewise, the reduced von Kármán equation (3.15) reduces to the following problem. Find $\xi_1 \in H_0^2(\Omega)$ such that

$$\mathcal{C}\xi_1 + \xi_1 - p\Lambda\xi_1 - \bar{f} = 0, \tag{3.39}$$

where $\Lambda \colon H_0^2(\Omega) \to H_0^2(\Omega)$ is the operator defined by

$$\Lambda u := \frac{1}{p} B(\theta, u). \tag{3.40}$$

As we will see in this section, the uniqueness of the von Kármán equations is closely linked to the spectral properties of Λ.

Lemma 3.13. *The operator $\Lambda : H_0^2(\Omega) \to H_0^2(\Omega)$ is a compact and symmetric positive definite.*

Proof. For $u \in H_0^2(\Omega)$, integration by parts together with (3.37) implies

$$\begin{aligned}(\Lambda u, u)_\Delta &= \frac{1}{p}(B(\theta, u), u)_\Delta = \frac{1}{p}\int_\Omega \Delta B(\theta, u)\Delta u \, dx \\ &= \frac{1}{p}\int_\Omega [\theta, u] u \, dx = -\int_\Omega \Delta u \, u \, dx = \int_\Omega (\nabla u)^2 \, dx = |u|_{H^1(\Omega)}^2 \geq 0.\end{aligned}$$

Since $|\cdot|_{H^1(\Omega)}$ is a norm on $H_0^1(\Omega)$ and since $H_0^2(\Omega) \subset H_0^1(\Omega)$, this implies that $(\Lambda u, u)_\Delta = 0$ if and only if $u = 0$. Thus, Λ is positive definite. The remaining properties of Λ follow directly from the properties of the operator $B(\theta, \cdot)$ in Lemma 3.9(iii). □

As Λ is symmetric, positive definite, and compact, we obtain from Corollary 2.8 that the largest eigenvalue λ_1 of Λ is given by

$$\lambda_1 = \sup_{\substack{u \in H_0^2(\Omega) \\ u \neq 0}} \frac{(\Lambda u, u)_\Delta}{(u, u)_\Delta}.$$

Thus, we can define

$$p_1 = \frac{1}{\lambda_1} = \inf_{\substack{u \in H_0^2(\Omega) \\ u \neq 0}} \frac{(u, u)_\Delta}{(\Lambda u, u)_\Delta}. \tag{3.41}$$

This constant determines if the homogeneous von Kármán problem (3.38) has unique or non-unique solutions, as we see in the next theorem from [14].

Theorem 3.14. *Let $f = 0$ and $p_1 > 0$ be the constant defined in* (3.41)*. Then the following holds.*

(i) If $p \leq p_1$, then $\xi_1 = 0$ is the unique solution of problem (3.39)*.*

(ii) If $p > p_1$, then problem (3.39) *has at least three solutions: Besides $\xi_1 = 0$, there also exist two nontrivial solutions $\widetilde{\xi}_1 \in H_0^2(\Omega)$ and $-\widetilde{\xi}_1 \in H_0^2(\Omega)$.*

Proof. For the proof of (i), we refer to Remark 3.16.

To show (ii), consider the normalized eigenvector $u \in H_0^2(\Omega)$ with $|u|_\Delta^2 = 1$ that corresponds to the largest eigenvalue $\frac{1}{p_1}$ of Λ. By (3.29) we have

$$j(\alpha u) = \frac{\alpha^4}{4}(\mathcal{C}u, u)_\Delta + \frac{\alpha^2}{2}(1 - \frac{p}{p_1}).$$

Since $1 - \frac{p}{p_1}$ is negative for $p > p_1$, it follows that $j(\alpha u) < 0$ for some α sufficiently small. Consequently, the solution $\xi_1 = 0$ is not the global minimizer of the functional j. Hence, by Theorem 3.12, there exists another solution $\widetilde{\xi}_1$ that minimizes the functional j globally. In view of the fact that $\mathcal{C}$ is cubic and Λ is linear, it is clear that $-\widetilde{\xi}_1$ is another solution of the homogenous reduced von Kármán equation (3.39). □

In the next lemma, we show a stability estimate for the reduced von Kármán equation. The case $p \geq 0$ of this lemma is stated as an exercise in [14].

Lemma 3.15. *Let $p < p_1$. Then the following holds.*

(i) Any $u \in H_0^2(\Omega)$ satisfies

$$\min\{1, 1 - \frac{p}{p_1}\}|u|_\Delta^2 \leq |u|_\Delta^2 - p(\Lambda u, u)_\Delta. \tag{3.42}$$

(ii) If $\xi_1 \in H_0^2(\Omega)$ is a solution of the reduced von Kármán equation (3.39)*, then*

$$|\xi_1|_\Delta \leq \max\{1, \frac{p_1}{p_1 - p}\}|\bar{f}|_\Delta. \tag{3.43}$$

Proof. If $p \geq 0$, then it follows from (3.41) that

$$p(\Lambda\xi_1, \xi_1)_\Delta \leq \frac{p}{p_1}|\xi_1|_\Delta^2.$$

Consequently, we have

$$(1 - \frac{p}{p_1})|u|_\Delta^2 \leq |u|_\Delta^2 - p(\Lambda u, u)_\Delta.$$

Otherwise, if $p < 0$, then the fact that Λ is a positive definite operator (see Lemma 3.13) implies

$$|u|_\Delta^2 \leq |u|_\Delta^2 - p(\Lambda u, u)_\Delta.$$

This proves statement (i).

Suppose $\xi_1 \in H_0^2(\Omega)$ is a solution of the reduced von Kármán equation (3.39). Then

$$|\xi_1|_\Delta^2 + (\mathcal{C}\xi_1, \xi_1)_\Delta - p(\Lambda\xi_1, \xi_1)_\Delta - (\bar{f}, \xi_1)_\Delta = 0. \tag{3.44}$$

Using the result of statement (i), we obtain from this the estimate

$$\min\{1, 1 - \frac{p}{p_1}\}|u|^2_\Delta \le |u|^2_\Delta - p(\Lambda u, u)_\Delta = (\bar{f}, \xi_1)_\Delta - (\mathcal{C}\xi_1, \xi_1)_\Delta. \tag{3.45}$$

By Lemma 3.9(ii), $(\mathcal{C}\xi_1, \xi_1)_\Delta$ is nonnegative. Thus, (3.45) and the Cauchy-Schwarz inequality yield

$$\min\{1, 1 - \frac{p}{p_1}\}|u|^2_\Delta \le (\bar{f}, \xi_1)_\Delta \le |\bar{f}|_\Delta |\xi_1|_\Delta.$$

Dividing both sides by $\min\{1, 1 - \frac{p}{p_1}\}$ leads to

$$|\xi_1|^2_\Delta \le \max\{1, \frac{p_1}{p_1 - p}\}|\bar{f}|_\Delta |\xi_1|_\Delta.$$

The fact that, for $\bar{f} \neq 0$, the solution ξ_1 of the reduced von Kármán equation (3.39) is nontrivial completes the proof. □

Remark 3.16. *The statement (ii) of this lemma proves Theorem 3.14 for $p \le p_1$.*

In the next theorem, we show that the von Kármán equations (3.38) has a unique solution if certain conditions on p and f hold. Similar versions of this theorem can be found in [14, 16].

Theorem 3.17. *Suppose $\bar{f} \in H^2_0(\Omega)$ is sufficiently small such that*

$$3|B|^2_\Delta |\bar{f}|^2_\Delta < 1. \tag{3.46}$$

Moreover, assume that

$$p < p_1(1 - \sqrt[3]{3}|B|^{\frac{2}{3}}_\Delta |\bar{f}|^{\frac{2}{3}}_\Delta). \tag{3.47}$$

Then the solution of the reduced von Kármán equation (3.39) *is unique.*

Proof. First, let us bring the two conditions (3.47) and (3.46) into an easier form that is advantageous for our estimates. The assumption (3.47) can be written as

$$\frac{p_1 - p}{p_1} > \sqrt[3]{3}\, |B|^{2/3}_\Delta |\bar{f}|^{2/3}_\Delta.$$

Clearly, the two conditions (3.47) and (3.46) imply $p < p_1$. Thus, we have

$$\sqrt[3]{3}\,|B|_\Delta^{2/3}|\bar{f}|_\Delta^{2/3}\,\frac{p_1}{p_1 - p} < 1.$$

Since cubing preserves inequalities, we obtain

$$3\,|B|_\Delta^2|\bar{f}|_\Delta^2\left(\frac{p_1}{p_1 - p}\right)^3 < 1.$$

Therefore, the two conditions (3.46) and (3.47) yield

$$3\,|B|_\Delta^2|\bar{f}|_\Delta^2\max\{1, \left(\frac{p_1}{p_1 - p}\right)^3\} < 1. \tag{3.48}$$

Next, suppose ξ_1 and $\widetilde{\xi_1}$ are two distinct solutions of the reduced von Kármán equation (3.39). Then we have by (3.39) that

$$\mathcal{C}\xi_1 - \mathcal{C}\widetilde{\xi_1} = \widetilde{\xi_1} - \xi_1 - p\Lambda(\widetilde{\xi_1} - \xi_1).$$

This together with the results in Lemma 3.15 and Lemma 3.10 implies

$$\begin{aligned}
\min\{1, 1 - \frac{p}{p_1}\}|\widetilde{\xi_1} - \xi_1|_\Delta^2 &\le |\widetilde{\xi_1} - \xi_1|_\Delta^2 - p(\Lambda(\widetilde{\xi_1} - \xi_1), \widetilde{\xi_1} - \xi_1)_\Delta \\
&= (\widetilde{\xi_1} - \xi_1 - p\Lambda(\widetilde{\xi_1} - \xi_1), \widetilde{\xi_1} - \xi_1)_\Delta \\
&= (\mathcal{C}\xi_1 - \mathcal{C}\widetilde{\xi_1}, \widetilde{\xi_1} - \xi_1)_\Delta \\
&\le 3\,|B|_\Delta^2|\xi_1 - \widetilde{\xi_1}|_\Delta^2\max\{|\xi_1|_\Delta^2, |\widetilde{\xi_1}|_\Delta^2\} \\
&\le 3\,|B|_\Delta^2|\xi_1 - \widetilde{\xi_1}|_\Delta^2|\bar{f}|_\Delta^2\max\{1, \left(\frac{p_1}{p_1 - p}\right)^2\}.
\end{aligned}$$

Therefore, we obtain the inequality

$$1 \le 3\,|B|_\Delta^2|\bar{f}|_\Delta^2\max\{1, \left(\frac{p_1}{p_1 - p}\right)^3\}, \tag{3.49}$$

which contradicts (3.48). This proves the uniqueness of ξ_1. □

Remark 3.18. *In the proof of Theorem 3.17, we showed that the conditions* (3.47) *and* (3.46) *imply* $p < p_1$ *and* (3.48)*. One can easily check that the converse implication also holds.*

3.3 The Linearized von Kármán Equations

Let $\boldsymbol{\xi} = (\xi_1, \xi_2) \in (H_0^2(\Omega))^2$ be the unique solution of the von Kármán equations (3.38) under the uniqueness conditions (3.46) and (3.47) from Theorem 3.17. By linearizing the nonlinear terms $[\xi_1, \xi_1]$ and $[\xi_1, \xi_2]$ in the von Kármán equations at $\boldsymbol{\xi}$, we obtain the linearized von Kármán equations.

Given $\boldsymbol{F} = (F_1, F_2) \in (H^{-2}(\Omega))^2$, find $\boldsymbol{u} = (u_1, u_2) \in (H_0^2(\Omega))^2$ such that

$$\Delta^2 u_1 + p\Delta u_1 - [\xi_2, u_1] - [u_2, \xi_1] = F_1, \tag{3.50a}$$

$$\Delta^2 u_2 + 2[\xi_1, u_1] = F_2. \tag{3.50b}$$

In this section we prove that this problem as well as the corresponding dual problem is well-posed. To begin, we show that solving the linearized von Kármán equations is equivalent of inverting a certain linear operator.

Lemma 3.19. *Given $\boldsymbol{F} = (F_1, F_2) \in (H^{-2}(\Omega))^2$, the linearized von Kármán equations* (3.50) *have a unique solution if and only if the operator $I - S$ is invertible, where $S\colon H_0^2(\Omega) \to H_0^2(\Omega)$ is the linear operator defined by*

$$Su = -2B(B(\xi_1, u), \xi_1) - B(B(\xi_1, \xi_1), u) + p\Lambda u. \tag{3.51}$$

Proof. Let $\bar{F}_1, \bar{F}_2 \in H_0^2(\Omega)$ be the unique solutions of

$$\Delta^2 \bar{F}_1 = F_1,$$

$$\Delta^2 \bar{F}_2 = F_2.$$

By Definition 3.4 we can write equation (3.50b) equivalently as

$$u_2 = \bar{F}_2 - 2B(\xi_1, u_1).$$

In consideration of this and the fact from Theorem 3.6 that $\xi_2 = -B(\xi_1, \xi_1)$, equation (3.50a) reduces to

$$\Delta^2 u_1 + [B(\xi_1, \xi_1), u_1] + 2[B(\xi_1, u_1), \xi_1] + p\Delta u_1 = F_1 + [\bar{F}_2, \xi_1].$$

By (3.37), (3.40) and Definition 3.4, this is equivalent to

$$(I-S)u_1 = \bar{F}_1 + B(\bar{F}_2, \xi_1), \tag{3.52}$$

where S is the operator defined in (3.51). Thus, the linearized von Kármán equations (3.50) are uniquely solvable, if the operator $I-S$ is invertible. □

Remark 3.20. *The operator $I-S$ in Lemma 3.19 coincides with the linearization of the reduced von Kármán equation* (3.39) *at $\boldsymbol{\xi}$.*

Remark 3.21. *From the proof of Lemma 3.19, we infer that the unique solution of the linearized von Kármán equations* (3.50) *is given by*

$$u_1 = (I-S)^{-1}(\bar{F}_1 + B(\bar{F}_2, \xi_1)), \tag{3.53a}$$

$$u_2 = \bar{F}_2 - 2B(\xi_1, u_1), \tag{3.53b}$$

provided the operator $I-S$ is invertible.

Lemma 3.22. *Let $\boldsymbol{F} \in (H^{-2}(\Omega))^2$. Then under the uniqueness conditions* (3.46) *and* (3.47)*, the linearized von Kármán equations* (3.50) *admit a unique solution $\boldsymbol{u} \in (H_0^2(\Omega))^2$. Moreover, there exists $C > 0$ such that*

$$||\boldsymbol{u}||_{(H^2(\Omega))^2} \leq C||\boldsymbol{F}||_{(H^{-2}(\Omega))^2}. \tag{3.54}$$

Proof. In consideration of Lemma 3.19 and Definition 2.2, to prove the existence and uniqueness of $\boldsymbol{u}$, we only need to show that $\lambda = 1$ does not belong to the spectrum of S. From the sequential compactness of $B(\cdot,\cdot)$, one can deduce similarly as in the proof of Lemma 3.9(iii) that S is a compact operator. Therefore, by the Riesz-Schauder Theorem (Theorem 2.4), $I-S$ is invertible if $\lambda = 1$ is not an eigenvalue of S. By Corollary 3.8, we have for all $u, v \in H_0^2(\Omega)$ that

$$(B(B(\xi_1, u), \xi_1), v)_\Delta = (B(v, \xi_1), B(u, \xi_1))_\Delta,$$

$$(B(B(\xi_1, \xi_1), u), v)_\Delta = (B(u, v), B(\xi_1, \xi_1))_\Delta.$$

From this and Lemma 3.13, it follows that S is a symmetric operator. Thus, by Corollary 2.8, we can estimate the largest eigenvalue of S with the Rayleigh quotient. For any nonzero $u \in H_0^2(\Omega)$, we have

$$\begin{aligned}\frac{(Su,u)_\Delta}{|u|_\Delta^2} &= \frac{(-2B(B(\xi_1,u),\xi_1) - B(B(\xi_1,\xi_1),u),u)_\Delta}{|u|_\Delta^2} + p\frac{(\Lambda u,u)_\Delta}{|u|_\Delta^2} \\ &\le 3|B|_\Delta^2|\xi_1|_\Delta^2 + p\frac{(\Lambda u,u)_\Delta}{|u|_\Delta^2} \\ &\le 3|B|_\Delta^2|\bar{f}|_\Delta^2 \max\{1, (\frac{p_1}{p_1-p})^2\} + p\frac{(\Lambda u,u)_\Delta}{|u|_\Delta^2}. \end{aligned} \tag{3.55}$$

If $p < 0$, this estimate yields

$$\sup_{\substack{u \in H_0^2(\Omega) \\ u \neq 0}} \frac{(Su,u)_\Delta}{|u|_\Delta^2} \le 3|B|_\Delta^2|\bar{f}|_\Delta^2 < 1.$$

Thus, 1 is not an eigenvalue of S, and $I - S$ is invertible.

In the other case, if $p \ge 0$, we obtain from (3.55) that

$$\sup_{\substack{u \in H_0^2(\Omega) \\ u \neq 0}} \frac{(Su,u)_\Delta}{|u|_\Delta^2} \le 3|B|_\Delta^2|\bar{f}|_\Delta^2\Big(\frac{p_1}{p_1-p}\Big)^2 + \frac{p}{p_1}. \tag{3.56}$$

By Remark 3.18, we have that $p < p_1$, and

$$3|B|_\Delta^2|\bar{f}|_\Delta^2\Big(\frac{p_1}{p_1-p}\Big)^3 < 1.$$

This is equivalent to

$$3|B|_\Delta^2|\bar{f}|_\Delta^2\Big(\frac{p_1}{p_1-p}\Big)^2 + \frac{p}{p_1} < 1.$$

Combining this with estimate (3.56) implies that $\lambda = 1$ is not an eigenvalue of S, and the solution of the linearized von Kármán equation is uniquely determined by (3.53). Moreover, by Remark 2.3, the resolvent $(I - S)^{-1}$ is a bounded operator on $H_0^2(\Omega)$.

Next we prove inequality (3.54). By Theorem 3.1, there exists $\bar{C} > 0$ such that

$$||\bar{F}_1||_{H^2(\Omega)} \le \bar{C}||F_1||_{H^{-2}(\Omega)},$$

$$||\bar{F}_2||_{H^2(\Omega)} \le \bar{C}||F_2||_{H^{-2}(\Omega)}.$$

Hence, (3.53a) and the inequality $a + b \le 2\sqrt{a^2 + b^2}$ imply

$$\begin{aligned}
|u_1|_\Delta &= |(I - S)^{-1}(\bar{F}_1 + B(\bar{F}_2, \xi_1))|_\Delta \\
&\le |(I - S)^{-1}|_\Delta \, (|\bar{F}_1|_\Delta + |B|_\Delta |\bar{F}_2|_\Delta |\xi_1|_\Delta) \\
&\le \bar{C}\, |(I - S)^{-1}|_\Delta \, (||\bar{F}_1||_{H^2(\Omega)} + |B|_\Delta |\xi_1|_\Delta ||\bar{F}_2||_{H^2(\Omega)}) \\
&\le C_1 ||\boldsymbol{F}||_{(H^{-2}(\Omega))^2}, \qquad (3.57)
\end{aligned}$$

where $C_1 = 2\bar{C}\, |(I - S)^{-1}|_\Delta \max\{1, |B|_\Delta |\xi_1|_\Delta\}$.

Now, it follows from (3.53b) and (3.57) that

$$|u_2|_\Delta = |\bar{F}_2 - 2B(\xi_1, u_1)|_\Delta \le |\bar{F}_2|_\Delta + 2|B|_\Delta |\xi_1|_\Delta |u_1|_\Delta \le C_2 ||\boldsymbol{F}||_{(H^{-2}(\Omega))^2}, \quad (3.58)$$

where $C_2 = 2\max\{\bar{C}, 2C_1 |B|_\Delta |\xi_1|_\Delta\}$. Finally, inequality (3.54) is a consequence of (3.6), (3.57), and (3.58). This completes the proof. □

After establishing the well-posedness of the linearized von Kármán problem, we formulate the linearized von Kármán problem in a weak form which is the starting point for the finite element discretization in Section 5.1. From the considerations on the biharmonic problem in Section 3.1 and on the Monge-Ampère form in Lemma 3.7, we obtain the weak formulation of the linearized von Kármán problem. Given $\boldsymbol{F} \in (H^{-2}(\Omega))^2$, find $\boldsymbol{u} \in (H_0^2(\Omega))^2$ such that

$$L(\boldsymbol{u}, \mathbf{v}) = \boldsymbol{F}(\boldsymbol{v}) \;\; \forall \boldsymbol{v} \in (H_0^2(\Omega))^2, \qquad (3.59)$$

where

$$\begin{aligned}
L(\boldsymbol{u}, \boldsymbol{v}) = &\int_\Omega (\Delta u_1 \Delta v_1 + \Delta u_2 \Delta v_2)\, dx - p \int_\Omega \nabla u_1 \nabla v_1 \, dx \qquad (3.60) \\
&- b(\xi_1, u_2, v_1) - b(\xi_2, u_1, v_1) + 2b(\xi_1, u_1, v_2),
\end{aligned}$$

and $b(\cdot, \cdot, \cdot)$ is the trilinear form defined in (3.18) .

At this point, let us also formulate the corresponding dual problem. Given $\boldsymbol{G} \in (H^{-2}(\Omega))^2$, find $\boldsymbol{\phi} \in (H_0^2(\Omega))^2$ such that

$$L(\boldsymbol{v}, \boldsymbol{\phi}) = \boldsymbol{G}(\boldsymbol{v}) \ \ \forall \boldsymbol{v} \in (H_0^2(\Omega))^2. \tag{3.61}$$

The corresponding strong form of this formulation is accordingly

$$\begin{aligned} \Delta^2\phi_1 + p\Delta\phi_1 - [\xi_2, \phi_1] + 2[\xi_1, \phi_2] &= G_1, \\ \Delta^2\phi_2 - [\xi_1, \phi_1] &= G_2. \end{aligned}$$

Following the analysis in the proof of Lemma 3.19 and applying it to the dual problem leads to the equivalent problem

$$(I - S)\phi_1 = \bar{G}_1 - 2B(\bar{G}_2, \xi_1), \tag{3.62a}$$

$$\phi_2 = \bar{G}_2 + B(\xi_1, \phi_1), \tag{3.62b}$$

where $\bar{G}_1, \bar{G}_2 \in H_0^2(\Omega)$ are the unique solutions of

$$\begin{aligned} \Delta^2\bar{G}_1 &= G_1, \\ \Delta^2\bar{G}_2 &= G_2. \end{aligned}$$

From (3.62) and the arguments in Lemma 3.22, we conclude the well-posedness of the dual problem (3.61).

Corollary 3.23. *Let* $\boldsymbol{G} \in (H^{-2}(\Omega))^2$. *Then, under the conditions* (3.46) *and* (3.47)*, the dual problem* (3.61) *of the linearized von Kármán equations admit a unique solution* $\boldsymbol{\phi} \in (H_0^2(\Omega))^2$. *Moreover, there exists* $C > 0$ *such that*

$$||\boldsymbol{\phi}||_{(H^2(\Omega))^2} \leq C||\boldsymbol{G}||_{(H^{-2}(\Omega))^2}. \tag{3.63}$$

3.4 Regularity Results

3.4.1 For the Biharmonic Equation

In this section, we gather regularity results on the biharmonic problem (3.3). The classical result [29] states that, for $f \in L^2(\Omega)$, the weak solution $u \in H_0^2(\Omega)$ of the biharmonic problem (3.3) is a strong solution that belongs to the space $H^4(\Omega) \cap H_0^2(\Omega)$, provided Ω is a bounded domain with sufficiently smooth boundary. For a polygonal domain Ω with only piecewise smooth boundary, the regularity results on the biharmonic solution are obtained by Kondratiev's method [22]. We follow the exposition of the theory in [5].

With the help of a partition of unity with smooth cut-off functions, the singular behavior of u can be studied locally on sectorial neighborhoods

$$\Omega_c = \{(r, \theta) \mid 0 < r < \delta,\ 0 \leq \theta \leq \omega\}$$

of the corners c of Ω with interior angles $\omega \in (0, 2\pi]$. Here the ordered pair (r, θ) denotes the polar coordinates at c, and δ is chosen small enough so that the sectorial neighborhoods Ω_c are disjoint. Inserting the ansatz

$$u_S(r, \theta) = r^z \phi(\theta), \quad z \in \mathbb{C},$$

for the singular part of u into the equation $\Delta^2 u_S = 0$ leads to the differential equation

$$\partial_{\theta\theta\theta\theta}\phi + (2z^2 - 4z + 4)\partial_{\theta\theta}\phi + (z^4 - 4z^3 + 4z^2)\phi = 0 \quad \text{for} \ \ \theta \in (0, \omega), \tag{3.64a}$$

$$\phi = \partial_\theta \phi = 0 \quad \text{at} \ \ \theta \in \{0, \omega\}. \tag{3.64b}$$

For every zero $z \in \mathbb{C} \backslash \{0, 1, 2\}$ of the function

$$D(z) = \sin^2(z-1)\omega - (z-1)^2 \sin^2 \omega, \tag{3.65}$$

equation (3.64) has m_z nontrivial, linearly independent solutions $\phi_{z,1}, \phi_{z,2}, ..., \phi_{z,m_z}$ (see [5] for details). These nontrivial solutions describe the singular behavior of u on Ω_c as follows.

Theorem 3.24. *Let $f \in H^{-2}(\Omega) \cap H^{-k}(\Omega)$ for some $k \in \{0, 1\}$. Suppose that for all $z \in \mathbb{C}$ with $\operatorname{Re} z = 3 - k$, the trivial solution is the only solution of (3.64). Let Z be the set of all complex numbers z in the strip*

$$\{z \in \mathbb{C} \mid 1 < \operatorname{Re} z < 3\}$$

for which equation (3.64) admits m_z nontrivial, linearly independent solutions $\phi_{z,1}, \phi_{z,2}, ..., \phi_{z,m_z}$. Then the solution $u \in H_0^2(\Omega)$ of (3.3) can be represented by

$$u = u_S + u_R \quad \text{on } \Omega_c,$$

where $u_R \in H^{4-k}(\Omega_c)$ and

$$u_S = \sum_{z \in Z} \sum_{\mu=1}^{m_z} a_{z\mu} r^z (\ln^{\mu-1} r) \phi_{z,\mu} \tag{3.66}$$

for some constants $a_{z\mu} \in \mathbb{C}$. Moreover, there exists $C > 0$ such that

$$||u_R||_{H^{k-4}(\Omega)} + \max |a_{z\mu}| \leq C||f||_{H^{-k}(\Omega)}.$$

Since ω is the only parameter of the function $D(z)$, it is clear that the distribution of the zeros of $D(z)$ solely depends on the interior angle ω. In [27], the relation between ω and the zeros of $D(z)$ is studied in detail. One of the results is that for sufficiently small ω, the function $D(z)$ does not have any zeros in the strip $1 < \operatorname{Re} z < 3$. In this case, the set Z from Theorem 3.24 is empty, and u coincides with u_R on Ω_c. Hence, u has full regularity near the corner c. The precise regularity result for the biharmonic equation is stated as follows.

Theorem 3.25. *Let Ω be a bounded, polygonal domain and $k \in \{0, 1\}$. Assume that the largest interior angle ω of Ω is less than $\frac{7\pi}{10}$ for $k = 0$, and less than π for $k = 1$. Then, given $f \in H^{-k}(\Omega)$, the weak solution of* (3.3) *has the regularity*

$$u \in H^{4-k}(\Omega) \cap H_0^2(\Omega).$$

Furthermore, there exists $C > 0$ such that

$$||u||_{H^{4-k}(\Omega)} \leq C||f||_{H^{-k}(\Omega)}. \tag{3.67}$$

Remark 3.26. *The regularity result in Theorem 3.25 also holds for $k \in [2-\alpha, 2]$, where $\alpha = \min\{\operatorname{Re} z - 1 \mid z \in Z\} \in (\frac{1}{2}, 2)$. This is shown in [3] using interpolation techniques. The dependency of the regularity parameter α on the largest interior angle ω is graphically illustrated in [3, 27].*

3.4.2 For the von Kármán Equations

Rannacher and Blum [5] extended the regularity results of Theorem 3.25 to non-linear perturbations of the biharmonic problem of the kind

$$\Delta^2 u + F(\nabla^2 u) = f, \tag{3.68}$$

where F is a function that satisfies the growth condition $|F(u)| \leq C|u|^2$ for $u \in \mathbb{R}^{2\times 2}$. Since the von Kármán equations (3.12) are two equations of this type, we obtain the following.

Theorem 3.27. *Let Ω be a bounded, polygonal domain and ω be the largest interior angle of Ω. Then there exists $\alpha \in (\frac{1}{2}, 2)$ that depends only on ω such that for any $s \in (2-\alpha, 2]$ and $f \in H^{-s}(\Omega)$, the solution of* (3.12) *has the regularity*

$$(\xi_1, \xi_2) \in (H^{4-s}(\Omega) \cap H_0^2(\Omega))^2.$$

3.4.3 For the Linearized von Kármán Equations

Assume that the uniqueness conditions (3.46) and (3.47) hold. Then, as shown in Section 3.3, the linearized von Kármán equations (3.59), as well as the dual problem (3.61), are uniquely solvable. Our goal is to carry the regularity results for the biharmonic equation over to the linearized von Kármán equation (3.59).

To do so, let us assume that the solution of the von Kármán equations has the regularity

$$\boldsymbol{\xi} \in (H^{3+s}(\Omega))^2 \text{ for some } s \in (0,1]. \tag{3.69}$$

According to Theorem 3.27, this holds if $f \in H^{-1+s}(\Omega)$ and the largest interior angle of the polygonal domain Ω is sufficiently small. From the regularity (3.69), and the Sobolev embedding $H^{3+s}(\Omega) \hookrightarrow C^2(\overline{\Omega})$, we infer that $\boldsymbol{\xi} \in (W^2_\infty(\Omega))^2$. Therefore, we can bound the trilinear form $b(\cdot,\cdot,\cdot)$ from Lemma 3.7 by

$$|b(\xi_1, v, w)| \leq C_b ||v||_{H^1(\Omega)} ||w||_{H^1(\Omega)} \quad \forall\, v, w \in H^1(\Omega), \tag{3.70a}$$

$$|b(\xi_2, v, w)| \leq C_b ||v||_{H^1(\Omega)} ||w||_{H^1(\Omega)} \quad \forall\, v, w \in H^1(\Omega), \tag{3.70b}$$

where $C_b = |\boldsymbol{\xi}|_{(W^2_\infty(\Omega))^2}$. Now we are ready to prove the following.

Lemma 3.28. *(Regularity of the Linearized von Kármán Equations)*
Let Ω be a bounded polygonal domain with largest interior angle $\omega \in (0,\pi)$. Assume that the uniqueness conditions (3.46) *and* (3.47)*, and the regularity condition* (3.69) *hold. Then, given $\boldsymbol{F}, \boldsymbol{G} \in (H^{-1}(\Omega))^2$, the unique solutions $\boldsymbol{u}$ and $\boldsymbol{\phi}$ of* (3.59) *and* (3.61) *belong to $(H^3(\Omega))^2$. Moreover, there exists $C_R > 0$ such that*

$$||\boldsymbol{u}||_{(H^3(\Omega))^2} \leq C_R ||\boldsymbol{F}||_{(H^{-1}(\Omega))^2}, \tag{3.71}$$

$$||\boldsymbol{\phi}||_{(H^3(\Omega))^2} \leq C_R ||\boldsymbol{G}||_{(H^{-1}(\Omega))^2}. \tag{3.72}$$

Proof. By Lemma 3.22, the solution $\boldsymbol{u}$ of the linearized von Kármán equations is uniquely determined. Thus, we can consider the linearized von Kármán problem as two (uncoupled) biharmonic problems

$$\Delta^2 u_1 = \widetilde{F}_1, \tag{3.73}$$

$$\Delta^2 u_2 = \widetilde{F}_2, \tag{3.74}$$

where the right hand sides $\widetilde{\boldsymbol{F}} = (\widetilde{F}_1, \widetilde{F}_2) \in (H^{-2}(\Omega))^2$ are defined by

$$\widetilde{F}_1(v) = F_1(v) + p\int_\Omega \nabla u_1 \nabla v + b(\xi_1, u_2, v) + b(\xi_2, u_1, v), \tag{3.75}$$

$$\widetilde{F}_2(v) = F_2(v) - 2b(\xi_1, u_1, v). \tag{3.76}$$

By (3.70) and (3.54), we obtain for all $v \in H^1(\Omega)$

$$\begin{aligned}
|\widetilde{F}_1(v)| &\le \big(||F_1||_{H^{-1}(\Omega)} + (|p| + C_b)||u_1||_{H^1(\Omega)} + C_b||u_2||_{H^1(\Omega)}\big)||v||_{H^1(\Omega)} \\
&\le \big(||\boldsymbol{F}||_{(H^{-1}(\Omega))^2} + (|p|C + 2CC_b)||\boldsymbol{F}||_{(H^{-2}(\Omega))^2}\big)||v||_{H^1(\Omega)} \\
&\le (1 + |p|C + 2CC_b)||\boldsymbol{F}||_{(H^{-1}(\Omega))^2}||v||_{H^1(\Omega)}.
\end{aligned} \tag{3.77}$$

In a similar way, we can estimate $\widetilde{F}_2(v)$ for $v \in H^1(\Omega)$. We have

$$\begin{aligned}
|\widetilde{F}_2(v)| &\le ||F_2||_{H^{-1}(\Omega)}||v||_{H^1(\Omega)} + 2C_b||u_1||_{H^1(\Omega)}||v||_{H^1(\Omega)} \\
&\le (1 + 2CC_b)||\boldsymbol{F}||_{(H^{-1}(\Omega))^2}||v||_{H^1(\Omega)}.
\end{aligned} \tag{3.78}$$

From (3.77) and (3.78), we conclude that

$$||\widetilde{F}_1||_{H^{-1}(\Omega)} + ||\widetilde{F}_2||_{H^{-1}(\Omega)} \le 2(1 + |p|C + 2CC_b)||\boldsymbol{F}||_{(H^{-1}(\Omega))^2}.$$

Now, the regularity estimate (3.71) follows from (3.73), (3.74), and the regularity results in Theorem 3.25. The regularity estimate (3.72) of the dual problem can be proved with analogous arguments.

□

Chapter 4

Finite Element Methods for the Biharmonic Problem

4.1 Conforming Finite Element Methods

On a bounded, polygonal domain Ω, we consider the biharmonic problem from Section 3.1 in the form

$$(u, v)_\Delta = F(v) \;\; \forall v \in V, \tag{4.1}$$

where $(\cdot,\cdot)_\Delta$ is the inner product defined in (3.4) and $F \in H^{-2}(\Omega)$. Here, and throughout this chapter, we write V for the space $H_0^2(\Omega)$.

A conforming method approximates the exact solution $u \in V$ by functions in a finite dimensional subspace $V_h \subset V$. Recall from Section 3.1 that $(\cdot,\cdot)_\Delta$ is a coercive, bounded bilinear form on $V \times V$. Thus, also its restriction to $V_h \times V_h$ is coercive and bounded. By Corollary 2.10, there exists a unique $u_h \in V_h$ such that

$$(u_h, v)_\Delta = F(v) \;\; \forall v \in V_h. \tag{4.2}$$

We call $u_h \in V_h$ the discrete solution of problem (4.2).

By subtracting (4.2) from (4.1), we obtain the Galerkin orthogonality relation

$$(u - u_h, v)_\Delta = 0 \;\; \forall v \in V_h. \tag{4.3}$$

From the three properties, boundedness, coercivity, and Galerkin orthogonality, one deduces an abstract error estimate, also known as Céa's Lemma.

Lemma 4.1. *(Céa's Lemma, [8])*

Let $u \in V$ be the solution of (4.1) and $u_h \in V_h$ be the discrete solution of (4.2). Then there exists a constant $C > 0$ such that

$$||u - u_h||_{H^2(\Omega)} \leq C \sup_{v \in V_h} ||u - v||_{H^2(\Omega)}.$$

To obtain a concrete error bound, we need to specify the finite dimensional subspace V_h. Let $\mathcal{T}_h$ be a regular family of triangulations of Ω. That is, every triangle $T \in \mathcal{T}_h$ satisfies

$$\begin{aligned} \operatorname{diam} T &\leq h \quad \forall T \in \mathcal{T}_h, \\ \frac{\operatorname{diam} T}{\operatorname{diam} B_T} &\leq \rho \quad \forall T \in \bigcup_h \mathcal{T}_h, \end{aligned} \tag{4.4}$$

where $\rho > 0$ is a constant and B_T is the largest ball that can be inscribed into T. As one can show by plain trigonometry, condition (4.4) is equivalent to Zlámal's minimum angle condition [12, 43], that there exists an angle $\theta_{\min} > 0$ such that

$$\theta_T \geq \theta_{\min} \;\; \forall T \in \mathcal{T}_h, \tag{4.5}$$

where θ_T denotes the smallest angle of T.

To each triangle $T \in \mathcal{T}_h$, we associate a space of polynomials P_T with domain T, and a set of nodal variables $\mathcal{N} \subset P_T'$ with $|\mathcal{N}| = \dim P_T$.

Definition 4.2. *Under the notations above, a triple $(T, P_T, \mathcal{N})$ is called a finite element if every $p \in P_T$ is uniquely determined by the nodal values*

$$\{N_i(p) \mid N_i \in \mathcal{N}\}.$$

Remark 4.3. *The original definition from Ciarlet [12] is more general in the sense that it also allows closed subset with non-empty interior and Lipschitz-continuous boundary for T, and that the space P_T does not need to be a polynomial space. However, for the purposes of this dissertation, the definition above is sufficient.*

The finite element space V_h associated to the triangulation $\mathcal{T}_h$ with finite elements $(T, P_T, \mathcal{N})$ consists of all functions $v_h \colon \Omega \to \mathbb{R}$ with the following two properties.

(i) The restriction of v_h to T is a polynomial in P_T for each $T \in \mathcal{T}_h$.

(ii) The function v_h is uniquely defined by the nodal values of each $T \in \mathcal{T}_h$.

The condition $V_h \subset V$ of the conforming method poses a significant restriction on the selection of finite elements that are compatible with this method. The next theorem [12] shows that a conforming finite element space for the von Kármán equations is contained in $C^1(\Omega)$. We call such a space a C^1 finite element space.

Theorem 4.4. *Let V_h be a finite element space associated to a triangulation $\mathcal{T}_h$ with finite elements $(T, P_T, \mathcal{N})$. Then the following holds.*

(i) If $P_T \subset C^1(T)$ for all $T \in \mathcal{T}_h$ and $V_h \subset C^1(\bar{\Omega})$, then $V_h \subset H^2(\Omega)$.

(ii) If $P_T \subset H^2(T)$ for all $T \in \mathcal{T}_h$ and $V_h \subset H^2(\Omega)$, then $V_h \subset C^1(\bar{\Omega})$.

The crux is that C^1 finite element spaces are rather complicated. In fact, Ženíšek [40] showed that C^1 finite element spaces based on triangles with piecewise polynomials have at least 18 nodal variables per element.

One of the easier C^1 elements is the Argyris finite element [2]. It is a triangular finite element that relies on a polynomial space P_T of degree 5 with $\dim P_T = 21$. The nodal values of the Argyris triangle with vertices a_1, a_2, and a_3 of a function $v \in C^2(\Omega)$ are given by

$$\frac{\partial^\alpha}{\partial x^\alpha} v(a_i), \text{ and } \partial_n v(m_i), \quad (i = 1, 2, 3)$$

where $|\alpha| \leq 2$, and m_1, m_2, and m_3 are the midpoints of the three edges of T and n is the outer unit normal (see Figure 4.1 for a illustration of the Argyris triangle).

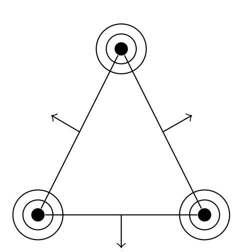

FIGURE 4.1. Argyris finite element

The operator $\Pi_h : C^2(\Omega) \to V_h$ that maps $v \in C^2(\Omega)$ to the unique function $\Pi_h v \in V_h$ whose nodal values coincide with the nodal values of v is called the global interpolant. The following is an interpolation result of the Argyris finite element space. A proof can be found in [12].

Lemma 4.5. *Let $\mathcal{T}_h$ be a regular triangulation of Ω with Argyris triangles. Then there exists a constant $C > 0$ such that*

$$||v - \Pi_h v||_{H^2(\Omega)} \leq Ch^4 |v|_{H^6(\Omega)} \quad \forall\, v \in H^6(\Omega). \tag{4.6}$$

Combining Céa's Lemma (Lemma 4.1) and the interpolation estimate (4.6) yields the following approximation result for the biharmonic problem.

Theorem 4.6. *Let $u \in V$ be the solution of* (4.1)*. Suppose $\mathcal{T}_h$ is a family of regular triangulations of Ω, and $\widetilde{V}_h$ its associated Argyris finite element space.*
Then $V_h = \widetilde{V}_h \cap H_0^2(\Omega)$ is a conforming finite element space for the biharmonic problem. Moreover, if $u \in V$ has the regularity $u \in H^6(\Omega)$, then there exists $C > 0$ such that

$$||u - u_h||_{H^2(\Omega)} \leq Ch^4 |u|_{H^6(\Omega)}, \tag{4.7}$$

where $u_h \in V_h$ is the discrete solution of (4.2)*.*

Remark 4.7. *The proof of Theorem 4.6 relied on the interpolation estimate* (4.6) *of the global interpolant Π_h. By an interpolation argument [8, Theorem 14.3.3], this regularity requirement can be lowered to $u \in H^s(\Omega)$ for $s \in (2, 6)$. The corresponding error estimate is then*

$$||u - u_h||_{H^2(\Omega)} \leq Ch^{s-2} ||u||_{H^s(\Omega)}. \tag{4.8}$$

From this and the regularity results in the previous chapter, we obtain the final error estimate for the conforming finite element method with Argyris elements.

Theorem 4.8. *Let Ω be a bounded, polygonal domain and $\alpha \in (\frac{1}{2}, 2)$ be the regularity parameter in Remark 3.26. Then, given $f \in H^{-k}(\Omega)$ with $k \in (2-\alpha, 2)$, there exists $C > 0$ such that*

$$||u - u_h||_{H^2(\Omega)} \leq Ch^{2-k}||f||_{H^{-k}(\Omega)}.$$

4.2 A C^0 Interior Penalty Method for the Biharmonic Problem

In the previous section, we used a conforming finite element method to approximate the biharmonic problem on a bounded, polygonal domain Ω. We have seen that a finite element space compatible to this kind of method requires C^1 elements, which are rather complicated. A finite element method that relies only on C^0 elements but still yields essentially the same approximation error, is the C^0 interior penalty method, developed by Brenner and Sung in [9]. The analysis in this section follows closely their paper with the difference that we use a slightly different h-dependent norm $|| \cdot ||_h$ that fits better into the framework of the new methods in the next chapter (cf. Lemma 5.5).

4.2.1 Notation and Underlying Finite Element Space

First, let us become familiar with the necessary notation and Sobolev spaces that are typical for Discontinuous Galerkin (DG) Methods.

Let $\mathcal{T}_h$ be a regular triangulation of Ω, and $H^k(\Omega, \mathcal{T}_h)$ for a nonnegative integer k the piecewise Sobolev space defined by

$$H^k(\Omega, \mathcal{T}_h) = \{w \in L^2(\Omega) \mid w \in H^k(T)\ \forall T \in \mathcal{T}_h\}.$$

We denote the set of interior edges in $\mathcal{T}_h$ by $\mathcal{E}_h^i$, and the set of edges on the boundary $\partial\Omega$ by $\mathcal{E}_h^b$ so that $\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^b$ contains all edges in $\mathcal{T}_h$. For two triangles adjacent to an interior edge $e \in \mathcal{E}_h^i$, we use the notation $T_\pm$. A triangle adjacent to an edge on the boundary $\partial\Omega$ is denoted by T without ambiguity.

For $v \in H^2(\Omega, \mathcal{T}_h)$, we define the jump

$$[\![v]\!] = \begin{cases} v_{T_+} - v_{T_-} & \text{if } e \in \mathcal{E}_h^i \\ -v_T & \text{if } e \in \mathcal{E}_h^b \end{cases} \tag{4.9}$$

and the average

$$\{\!\{v\}\!\} = \begin{cases} \frac{v_{T_+} + v_{T_-}}{2} & \text{if } e \in \mathcal{E}_h^i \\ v_T & \text{if } e \in \mathcal{E}_h^b, \end{cases} \tag{4.10}$$

where $v_T = v|_T$ is the restriction of v onto T.

Throughout this dissertation, we use the notation $n_e = (n_{e,1}, n_{e,2})$ for the unit normal along an interior edge e pointing from T_- to T_+ as illustrated in Figure 4.2. We denote the tangential vector along e with counterclockwise orientation with respect to the triangle T_- by t_e. In the unambiguous case $e \in \mathcal{E}_h^b$, the vector n_e is the outer unit normal along e with respect to Ω, and t_e is the tangential unit vector along $\partial\Omega$ in counterclockwise orientation.

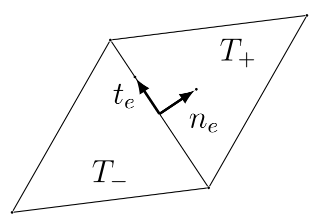

FIGURE 4.2. Definition of n_e and t_e of an interior edge $e \in \mathcal{E}_h^i$

Remark 4.9. *Since the normal n_e of an interior edge e always points from T_- to T_+, the average $\{\!\{\partial_{n_e n_e} v\}\!\}$, the jump $[\![\partial_{n_e} v]\!]$ as well as the products $[\![v]\!] n_{e,1}$ and $[\![v]\!] n_{e,2}$ do not depend on the way the adjacent triangles are labeled with T_- and T_+.*

On $H^2(\Omega, \mathcal{T}_h)$, we define the seminorm

$$|w|^2_{H^2(\Omega,\mathcal{T}_h)} = \sum_{T \in \mathcal{T}_h} |w|^2_{H^2(T)} + \sum_{e \in \mathcal{E}_h} |e|^{-1} || [\![\partial_{n_e} w]\!] ||^2_{L^2(e)}. \tag{4.11}$$

The Sobolev spaces $H^k(\Omega, \mathcal{T}_h)$ and $H^k(\Omega)$ often have properties that are alike in a certain sense. One such property is the piecewise version of the Poincaré inequality (2.14), which was discovered by S. C. Brenner [6] (see also [8, Theorem 10.6.12]). It will be a useful tool for the analysis of the C^0 penalty methods for the von Kármán equations in the next chapter.

Theorem 4.10. *(Piecewise Poincaré Inequality)*
There exists a constant $C > 0$ that depends only on the minimum angle (4.5) *such that, for all $v \in H^1(\Omega, \mathcal{T}_h)$,*

$$||v||_{L^2(\Omega)} \leq C\Big(\big|\int_{\partial\Omega} v\,ds\big| + \big(\sum_{T\in\mathcal{T}_h} |v|^2_{H^1(T)}\big)^{\frac{1}{2}} + \big(\sum_{e\in\mathcal{E}_h} |e|^{\frac{1}{2}} ||[\![v]\!] n_e||^2_{L^2(e)}\big)^{\frac{1}{2}}\Big).$$

Applying this inequality to ∇v for $v \in H^2(\Omega, \mathcal{T}_h) \cap H_0^1(\Omega)$ yields the following.

Corollary 4.11. *There exists a constant $C > 0$ that depends only on the minimum angle* (4.5) *such that*

$$|v|_{H^1(\Omega)} \leq C|v|_{H^2(\Omega,\mathcal{T}_h)} \ \ \forall v \in H^2(\Omega, \mathcal{T}_h) \cap H_0^1(\Omega).$$

Next, let us define the underlying finite element space V_h of the C^0 interior penalty method. To each triangle $T \in \mathcal{T}_h$, we assign a space P_T of quadratic polynomials on T and a set of nodal variables $\mathcal{N} = \{N_1, ..., N_6\}$ defined for $v \in C^0(\Omega)$ by

$$N_i v = v(a_i), \text{ and } N_{i+3} v = v(m_i), \ \ (i = 1, 2, 3),$$

where a_1, a_2, a_3 are the vertices of T and m_1, m_2, m_3 are the midpoints of the three edges $e \in \partial T$. The triple $(T, P_T, \mathcal{N})$ is called the $\mathcal{P}_2$ Lagrange finite element [8], shown in Figure 4.3. Let $\widetilde{V}_h$ be the $\mathcal{P}_2$ Lagrange finite element space associated to $\mathcal{T}_h$. Due to $V_h \subset H^1(\Omega)$, we can incorporate the boundary condition $u = 0$ of (3.1b) into our finite element space by setting

$$V_h = \widetilde{V}_h \cap H_0^1(\Omega). \tag{4.12}$$

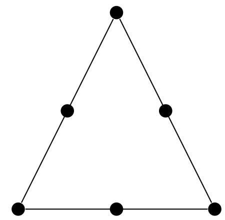

FIGURE 4.3. Quadratic Lagrange finite element

4.2.2 Derivation of the Method and Error Analysis

Since V_h is a nonconforming finite element space, i.e., $V_h \not\subset V$, we cannot just restrict the weak formulation (4.1) to functions in V_h to obtain the discrete problem. Instead, an alternative weak formulation of the biharmonic problem has to be found that is also defined on V_h. This can be achieved by carrying out integration by parts locally - triangle by triangle.

On each triangle $T \in \mathcal{T}_h$, we have for $v \in V_h$ and $w \in C^\infty(\Omega)$

$$\begin{aligned}
\int_T (\Delta^2 w) v \, dx &= \int_{\partial T} (\partial_n \Delta w) v \, ds - \int_T \nabla(\Delta w) \cdot \nabla v \, dx \\
&= \int_{\partial T} (\partial_n \Delta w) v \, ds - \sum_{i=1}^{2} \int_T (\Delta \partial_i w) \partial_i v \, dx \\
&= \int_{\partial T} (\partial_n \Delta w) v \, ds - \sum_{i=1}^{2} \left(\int_{\partial T} (\partial_n \partial_i w) \partial_i v \, ds - \int_T \nabla \partial_i w \cdot \nabla \partial_i v \, dx \right) \\
&= \int_{\partial T} (\partial_n \Delta w) v \, ds - \int_{\partial T} \nabla \partial_n w \cdot \nabla v \, ds + \sum_{i=1}^{2} \int_T \nabla \partial_i w \cdot \nabla \partial_i v \, dx, \qquad (4.13)
\end{aligned}$$

where n denotes the outer unit normal of T. Recall that for two adjacent triangles $T_\pm$ with interior edge e, the outer unit normal along e of T_- is denoted by n_e. Consequently, $-n_e$ is the outer unit normal along e with respect to T_+. Under consideration of this, summing up equation (4.13) over all triangles yields in the notation of (4.9) that

$$\int_\Omega (\Delta^2 w) v \, dx = \sum_{T \in \mathcal{T}_h} \Big(\sum_{i,j=1}^{2} \int_T \partial_{ij} w \, \partial_{ij} v \, dx \Big) + \sum_{e \in \mathcal{E}_h} \int_e [\![\nabla \partial_{n_e} w \cdot \nabla v]\!] \, ds. \qquad (4.14)$$

Note that the terms in involving $(\partial_{n_e}\Delta w)v$ in (4.13) have disappeared in (4.14), since $[\![\partial_{n_e}\Delta w v]\!]$ vanishes on every $e \in \mathcal{E}_h$ for $v \in C^0(\Omega) \cap H_0^1(\Omega)$ and $w \in C^\infty(\Omega)$.

A well known fact from linear algebra is that an orthogonal matrix $U \in \mathbb{R}^{2\times 2}$ satisfies

$$(Ux)\cdot(Uy) = x \cdot y \text{ for } x, y \in \mathbb{R}^2. \tag{4.15}$$

This means that the dot product is invariant under a coordinate transformation between two orthonormal bases. Hence, by taking (n_e, t_e) as new orthonormal basis, we obtain on $e \in \mathcal{E}_h$,

$$\begin{aligned}
[\![\nabla\partial_{n_e} w \cdot \nabla v]\!] &= [\![\partial_{n_e n_e} w\, \partial_{n_e} v + \partial_{n_e t_e} w\, \partial_{t_e} v]\!] \\
&= [\![\partial_{n_e n_e} w\, \partial_{n_e} v]\!] + [\![\partial_{n_e t_e} w\, \partial_{t_e} v]\!] \\
&= \partial_{n_e n_e} w [\![\partial_{n_e} v]\!].
\end{aligned} \tag{4.16}$$

Here, we used the fact that $w \in C^\infty(\Omega)$ and $v|_e$ is a quadratic polynomial. From (4.14) and (4.16), it follows that

$$\int_\Omega (\Delta^2 w) v\, dx = \sum_{T\in\mathcal{T}_h} \Big(\sum_{i,j=1}^{2} \int_T \partial_{ij} w\, \partial_{ij} v\, dx \Big) + \sum_{e\in\mathcal{E}_h} \int_e \partial_{n_e n_e} w [\![\partial_{n_e} v]\!]\, ds. \tag{4.17}$$

Suppose that $F \in L^2(\Omega)$. Then by Theorem 3.24, the solution of the biharmonic problem has a representation $u = u_R + u_S$, where $u_R \in H^4(\Omega)$ and u_S is of the form (3.66) near each corner of Ω. Since $C^\infty(\Omega)$ is dense in $H^4(\Omega)$ (cf. Theorem 2.26), equation (4.17) also holds for $w = u_R$. The justification for the fact that (4.17) even holds for the singular part $w = u_S$, involves a more subtle argument (see [9]). Nevertheless, we obtain

$$F(v) = \sum_{T\in\mathcal{T}_h} \Big(\sum_{i,j=1}^{2} \int_T \partial_{ij} u\, \partial_{ij} v\, dx \Big) + \sum_{e\in\mathcal{E}_h} \int_e \partial_{n_e n_e} u [\![\partial_{n_e} v]\!]\, ds \;\; \forall v \in V_h. \tag{4.18}$$

By Remark 3.26, the function u belongs to $H^{2+\alpha}(\Omega)$ for some $\alpha \in (\frac{1}{2}, 1]$ that depends on the largest interior angle of Ω. Hence, the trace theorem (Theorem 2.33) implies that the function $\partial_{n_e n_e} u$ is in $L^2(e)$ and $\partial_{n_e n_e} u = \{\!\{\partial_{n_e n_e} u\}\!\}$. Because of this and due to $[\![\partial_{n_e} u]\!] = 0$, the right hand side of (4.18) can be written in the symmetric form

$$\begin{aligned} F(v) = &\sum_{T\in\mathcal{T}_h} \Big(\sum_{i,j=1}^{2} \int_T \partial_{ij} u \, \partial_{ij} v \, dx \Big) \\ &+ \sum_{e\in\mathcal{E}_h} \int_e \Big(\{\!\{\partial_{n_e n_e} u\}\!\} [\![\partial_{n_e} v]\!] + \{\!\{\partial_{n_e n_e} v\}\!\} [\![\partial_{n_e} u]\!] \Big) ds \\ &+ \sum_{e\in\mathcal{E}_h} \frac{\eta}{|e|} \int_e [\![\partial_{n_e} u]\!] [\![\partial_{n_e} v]\!] ds, \end{aligned} \tag{4.19}$$

where the last term in equation (4.19) is a stabilization term that includes a penalty parameter $\eta > 0$ we specify later.

To simplify the notation in equation (4.19), we define for $s > \frac{5}{2}$ the bilinear form $a_h \colon (H^s(\Omega) + V_h) \times (H^s(\Omega) + V_h) \to \mathbb{R}$ by

$$\begin{aligned} a_h(w, v) = &\sum_{T\in\mathcal{T}_h} \Big(\sum_{i,j=1}^{2} \int_T \partial_{ij} w \, \partial_{ij} v \, dx \Big) \\ &+ \sum_{e\in\mathcal{E}_h} \int_e \Big(\{\!\{\partial_{n_e n_e} v\}\!\} [\![\partial_{n_e} w]\!] + \{\!\{\partial_{n_e n_e} w\}\!\} [\![\partial_{n_e} v]\!] \Big) ds \\ &+ \sum_{e\in\mathcal{E}_h} \frac{\eta}{|e|} \int_e [\![\partial_{n_e} w]\!] [\![\partial_{n_e} v]\!] ds. \end{aligned} \tag{4.20}$$

Since $L^2(\Omega)$ is dense in $H^{-2+\alpha}(\Omega)$, we obtain the following from equation (4.19).

Lemma 4.12. *Let $\alpha \in (\frac{1}{2}, 1]$ be the regularity parameter of Ω such that the solution u of the biharmonic problem is in $H^{2+\alpha}(\Omega)$ whenever $F \in H^{-2+\alpha}(\Omega)$. Then*

$$a(u, v) = F(v) \;\; \forall\, v \in V_h. \tag{4.21}$$

Having found a formulation of the biharmonic problem, that is also defined on V_h, we can define the discrete solution of the C^0 interior penalty method as follows. Given $F \in H^{-2+\alpha}(\Omega)$, find $u_h \in V_h$ such that

$$a_h(u_h, v) = F(v) \;\; \forall v \in V_h. \tag{4.22}$$

From equation (4.22) and (4.21), we obtain the consistency relation

$$a_h(u-u_h,v)=0 \;\; \forall v\in V_h. \tag{4.23}$$

To begin with the error analysis of the C^0 interior penalty method (4.22), we define on $H^{2+\alpha}(\Omega)+V_h$ the seminorm

$$||w||_h^2=\sum_{T\in\mathcal{T}_h}|w|^2_{H^2(T)}+\sum_{e\in\mathcal{E}_h}\Big(|e|\sum_{i,j=1}^{2}||\{\!\!\{\partial_{ij}w\}\!\!\}||^2_{L^2(e)}+\frac{1}{|e|}||[\![\partial_{n_e}w]\!]||^2_{L^2(e)}\Big). \tag{4.24}$$

The reason for this choice of h-dependent seminorm is that the bilinear form $a_h(\cdot,\cdot)$ is bounded with respect to $||\cdot||_h$.

Lemma 4.13. *There exists $C_a>0$ that depends only on η such that*

$$a_h(v,w)\le C_a||v||_h||w||_h \;\; \forall v,w\in H^{2+\alpha}(\Omega)+V_h. \tag{4.25}$$

Proof. By the triangle inequality, we have

$$\begin{aligned}||\{\!\!\{\partial_{n_en_e}v\}\!\!\}||_{L^2(e)}&=||\sum_{i,j=1}^{2}n_{e,i}n_{e,j}\{\!\!\{\partial_{ij}v\}\!\!\}||_{L^2(e)}\\&\le\sum_{i,j=1}^{2}|n_{e,i}|\,|n_{e,j}|\,||\{\!\!\{\partial_{ij}v\}\!\!\}||_{L^2(e)}\le\sum_{i,j=1}^{2}||\{\!\!\{\partial_{ij}v\}\!\!\}||_{L^2(e)}.\end{aligned} \tag{4.26}$$

Thus, an application of the inequality $(a+b)^2\le 2(a^2+b^2)$ yields

$$||\{\!\!\{\partial_{n_en_e}v\}\!\!\}||^2_{L^2(e)}\le 4\sum_{i,j=1}^{2}||\{\!\!\{\partial_{ij}v\}\!\!\}||^2_{L^2(e)}. \tag{4.27}$$

Now the Cauchy-Schwarz inequality together with (4.27) implies

$$\begin{aligned}\sum_{e\in\mathcal{E}_h}\int_e\{\!\!\{\partial_{n_en_e}v\}\!\!\}\,[\![\partial_{n_e}w]\!]\,ds&\le\sum_{e\in\mathcal{E}_h}|e|^{\frac12}||\{\!\!\{\partial_{n_en_e}v\}\!\!\}||_{L^2(e)}\frac{1}{|e|^{\frac12}}||[\![\partial_{n_e}w]\!]||_{L^2(e)}\\&\le\Big(\sum_{e\in\mathcal{E}_h}|e|\,||\{\!\!\{\partial_{n_en_e}v\}\!\!\}||^2_{L^2(e)}\Big)^{\frac12}\Big(\sum_{e\in\mathcal{E}_h}\frac{1}{|e|}||[\![\partial_{n_e}w]\!]||^2_{L^2(e)}\Big)^{\frac12} &(4.28)\\&\le 2\Big(\sum_{e\in\mathcal{E}_h}|e|\sum_{i,j=1}^{2}||\{\!\!\{\partial_{ij}v\}\!\!\}||^2_{L^2(e)}\Big)^{\frac12}\Big(\sum_{e\in\mathcal{E}_h}\frac{1}{|e|}||[\![\partial_{n_e}w]\!]||^2_{L^2(e)}\Big)^{\frac12}. &(4.29)\end{aligned}$$

In a similar way, we can bound

$$\sum_{e\in\mathcal{E}_h}\frac{1}{|e|}\int_e [\![\partial_{n_e}w]\!]\,[\![\partial_{n_e}v]\!]\,ds \le \sum_{e\in\mathcal{E}_h}\frac{1}{|e|^{\frac{1}{2}}}||\,[\![\partial_{n_e}w]\!]\,||_{L^2(e)}\frac{1}{|e|^{\frac{1}{2}}}||\,[\![\partial_{n_e}v]\!]\,||_{L^2(e)}$$

$$\le \Big(\sum_{e\in\mathcal{E}_h}\frac{1}{|e|}||\,[\![\partial_{n_e}v]\!]\,||^2_{L^2(e)}\Big)^{\frac{1}{2}}\Big(\sum_{e\in\mathcal{E}_h}\frac{1}{|e|}||\,[\![\partial_{n_e}w]\!]\,||^2_{L^2(e)}\Big)^{\frac{1}{2}}, \quad (4.30)$$

and

$$\sum_{T\in\mathcal{T}_h}\Big(\sum_{i,j}^{2}\int_T \partial_{ij}w\,\partial_{ij}v\,dx\Big) \le \Big(\sum_{T\in\mathcal{T}_h}|v|^2_{H^2(T)}\Big)^{\frac{1}{2}}\Big(\sum_{T\in\mathcal{T}_h}|w|^2_{H^2(T)}\Big)^{\frac{1}{2}}. \quad (4.31)$$

Now, the proof follows from (4.29), (4.30), and (4.31). □

For the Lax-Milgram Lemma (Theorem 2.11) to be applicable, the bilinear form $a_h(\cdot,\cdot)$ also needs to be coercive. In contrast to the conforming method, coercivity holds - as typical for nonconforming methods - only for the restriction of $a_h(\cdot,\cdot)$ to the finite element space $V_h \times V_h$.

Lemma 4.14. *If the penalty parameter $\eta > 0$ is sufficiently large, there exist $\kappa > 0$ that depends only on the minimum angle (4.5) such that*

$$a_h(v,v) \ge \kappa||v||_h^2 \quad \forall v \in V_h. \quad (4.32)$$

Remark 4.15. *Since the existence and uniqueness of the solution of the biharmonic problem has already been established, the lack of coercivity on V should not worry us at this point. Rather, we apply the Lax-Milgram Lemma (Theorem 2.11) to the restriction of $a_h(\cdot,\cdot)$ to $V_h \times V_h$ and conclude that the C^0 interior penalty method (4.22) is well-posed.*

The following is the key of the proof of Lemma 4.14.

Lemma 4.16. *There exists $C > 0$ that depends only on the minimum angle $\theta_{\min}$ of (4.5) such that*

$$|v|_{H^2(\Omega,\mathcal{T}_h)} \le ||v||_h \le C|v|_{H^2(\Omega,\mathcal{T}_h)} \quad \forall v \in V_h.$$

Proof. The inequality $|v|_{(H^2(\Omega,\mathcal{T}_h))^2} \leq ||v||_h$ is obvious from (4.11) and (4.24).

For the other direction, we show first that there exists $c > 0$ that depends only on $\theta_{\min}$ such that for all $e \in \mathcal{E}_h$, we have

$$|e|^2 \leq c\,|T| \quad \forall T \in \mathcal{T}_e, \tag{4.33}$$

where $\mathcal{T}_e$ denotes the set of all triangles $T \in \mathcal{T}_h$ that are adjacent to e. The law of sines implies for a triangle $T \in \mathcal{T}_e$ with edges e, e_2, e_3 and opposite angles $\theta, \theta_2, \theta_3$ that

$$\frac{|e|}{\sin\theta} = \frac{|e_2|}{\sin\theta_2} = \frac{|e_3|}{\sin\theta_3}.$$

Thus, we have, for $k = 2, 3$,

$$|e| = |e_k|\frac{\sin\theta}{\sin\theta_k} \leq |e_k|\frac{1}{\sin\theta_{\min}}.$$

Therefore,

$$\frac{1}{2}|e|^2\sin^3\theta_{\min} \leq \frac{1}{2}|e_1|\,|e_2|\sin\theta_{\min} \leq \frac{1}{2}|e_1|\,|e_2|\sin\theta = |T|,$$

which implies (4.33) with $c = 2\sin^{-3}\theta_{\min}$.

Using the fact that the second derivatives of $v \in V_h$ are constant on each $T \in T_h$, we estimate for each $e \in \mathcal{E}_h$

$$\begin{aligned} |e|\sum_{i,j=1}^{2} ||\{\!\{\partial_{ij}v\}\!\}||^2_{L^2(e)} &= |e|^2\sum_{i,j=1}^{2}\Big|\frac{1}{|\mathcal{T}_e|}\sum_{T\in\mathcal{T}_e}\partial_{ij}v_T\Big| \\ &\leq \sum_{T\in\mathcal{T}_e}|e|^2\sum_{i,j=1}^{2}|\partial_{ij}v_T| \leq c\sum_{T\in\mathcal{T}_e}|T|\sum_{i,j=1}^{2}|\partial_{ij}v_T| \\ &= c\sum_{T\in\mathcal{T}_e}|v|^2_{H^2(T)}. \end{aligned}$$

Since each triangle has three edges, this yields

$$\sum_{e\in\mathcal{E}_h}|e|\sum_{i,j=1}^{2}||\{\!\{\partial_{ij}v\}\!\}||^2_{L^2(e)} \leq c\sum_{e\in\mathcal{E}_h}\sum_{T\in\mathcal{T}_e}|v|^2_{H^2(T)} \leq 3c\sum_{T\in\mathcal{T}_h}|v|^2_{H^2(T)}. \tag{4.34}$$

Therefore, we obtain the inequality

$$||v||_h \leq (1+3c)|v|_{H^2(\Omega,\mathcal{T}_h)}.$$

This completes the proof. □

Now we are ready to prove Lemma 4.14.

Proof of Lemma 4.14. Let $c = 2\sin^{-3}\theta_{\min}$ and $C = 1+3c$ be the constants from the proof of Lemma 4.16. Then, the arithmetic-geometric mean inequality $2ab \leq \delta a^2 + \delta^{-1}b^2$ with $\delta = \frac{|e|}{6c}$, and the inequalities (4.34) and (4.26) imply

$$\begin{aligned}
2\Big|\sum_{e\in\mathcal{E}_h}\int_e \{\!\{\partial_{n_e n_e} v\}\!\}\,[\![\partial_{n_e} v]\!]\,ds\Big| &\leq 2\sum_{e\in\mathcal{E}_h} ||\{\!\{\partial_{n_e n_e} v\}\!\}||_{L^2(e)}||\,[\![\partial_{n_e} v]\!]\,||_{L^2(e)} \\
&\leq 2\sum_{e\in\mathcal{E}_h}\sum_{i,j=1}^{2} ||\{\!\{\partial_{ij} v\}\!\}||_{L^2(e)}||\,[\![\partial_{n_e} v]\!]\,||_{L^2(e)} \\
&\leq \sum_{e\in\mathcal{E}_h}\sum_{i,j=1}^{2}\Big(\frac{|e|}{6c}||\{\!\{\partial_{ij} v\}\!\}||^2_{L^2(e)} + \frac{6c}{|e|}||\,[\![\partial_{n_e} v]\!]\,||^2_{L^2(e)}\Big) \\
&\leq \frac{1}{2}\sum_{T\in\mathcal{T}_h}|v|^2_{H^2(T)} + 24c\sum_{e\in\mathcal{E}_h}\frac{1}{|e|}||\,[\![\partial_{n_e} v]\!]\,||^2_{L^2(e)}.
\end{aligned}$$

From this and Lemma 4.16, it follows for all $v \in V_h$

$$\begin{aligned}
a_h(v,v) &= \sum_{T\in\mathcal{T}_h}|v|^2_{H^2(T)} + 2\sum_{e\in\mathcal{E}_h}\int_e \{\!\{\partial_{n_e n_e} v\}\!\}\,[\![\partial_{n_e} v]\!]\,ds + \sum_{e\in\mathcal{E}_h}\frac{\eta}{|e|}||\,[\![\partial_{n_e} v]\!]\,||^2_{L^2(e)} \\
&\geq \sum_{T\in\mathcal{T}_h}|v|^2_{H^2(T)} - 2\Big|\sum_{e\in\mathcal{E}_h}\int_e \{\!\{\partial_{n_e n_e} v\}\!\}\,[\![\partial_{n_e} v]\!]\,ds\Big| + \sum_{e\in\mathcal{E}_h}\frac{\eta}{|e|}||\,[\![\partial_{n_e} v]\!]\,||^2_{L^2(e)} \\
&\geq \frac{1}{2}\sum_{T\in\mathcal{T}_h}|v|^2_{H^2(T)} + (\eta - 24c)\sum_{e\in\mathcal{E}_h}\frac{1}{|e|}||\,[\![\partial_{n_e} v]\!]\,||^2_{L^2(e)} \\
&\geq \frac{1}{2}|v|^2_{H^2(\Omega,\mathcal{T}_h)} \geq \frac{1}{2C}||v||^2_h,
\end{aligned}$$

where η was assumed to be large enough such that $\eta \geq 24c + \frac{1}{2}$ holds. □

From the consistency (4.23), boundedness (4.25), and coercivity (4.32) of the bilinear form $a_h(\cdot,\cdot)$, an abstract error estimate can be derived as follows.

Lemma 4.17. *Let $u \in H^{2+\alpha}(\Omega)$ be the solution of the biharmonic problem, and $u_h \in V_h$ be the solution of the discrete problem* (4.22). *Then*

$$||u - u_h||_h \leq \big(1 + \frac{C_a}{\kappa}\big)||u - v||_h. \quad \forall v \in V_h \tag{4.35}$$

Proof. For all $v \in V_h$, we estimate

$$\begin{aligned} ||u - u_h||_h &\leq ||u - v||_h + ||v - u_h||_h \\ &\leq ||u - v||_h + \frac{1}{\kappa}\frac{a_h(v - u_h, v - u_h)}{||v - u_h||_h} \\ &= ||u - v||_h + \frac{1}{\kappa}\frac{a_h(v - u, v - u_h)}{||v - u_h||_h} \\ &\leq \big(1 + \frac{C_a}{\kappa}\big)||u - v||_h. \end{aligned}$$

□

To turn the abstract error estimate (4.35) into a concrete error estimate, we rely, as in the conforming case, on an interpolation result of V_h (see [9]).

Lemma 4.18. *Let $\alpha \in (\frac{1}{2}, 1]$ be the parameter of elliptic regularity from Remark 3.26, and $s \geq 2$. Let Π_h be the global interpolant that projects $w \in C^0(\Omega)$ to $\Pi_h w \in V_h$. Then there exists $C_I > 0$ such that*

$$||w - \Pi_h w||_h \leq C_I h^{\alpha} ||w||_{H^{2+\alpha}(\Omega)} \quad \forall\, w \in H^{2+\alpha}(\Omega), \tag{4.36}$$

$$||w - \Pi_h w||_{H^{2-\alpha}(\Omega)} \leq C_I h^{s-2+\alpha} ||w||_{H^s(\Omega)} \quad \forall\, w \in H^s(\Omega). \tag{4.37}$$

Combining the abstract error estimate (4.35) with the interpolation estimate (4.36), and the elliptic regularity estimate (3.67) for $k = 2+\alpha$, we obtain an a priori error estimate for the C^0 interior penalty method of the biharmonic problem.

Theorem 4.19. *Let $\alpha \in (\frac{1}{2}, 1]$ be the regularity parameter from Remark 3.26 such that the solution u of the biharmonic problem is in $H^{2+\alpha}(\Omega)$ whenever $F \in$*

$H^{-2+\alpha}(\Omega)$. *Let* $u_h \in V_h$ *be the solution of the discrete problem* (4.22). *Then there exists a constant* $C > 0$ *independent from* h *such that*

$$||u - u_h||_h \leq Ch^\alpha ||F||_{H^{-2+\alpha}(\Omega)}.$$

The previous theorem concluded the error analysis of the C^0 interior penalty method. Still, we present at the end of this chapter a new estimate that is useful for analyzing DG methods for fourth order problems, in particularly for the C^0 interior penalty method for the von Kármán equations. The new estimate is a version of the Sobolev embedding $H^2(\Omega) \hookrightarrow C^0(\Omega)$ for functions in V_h. The argument is similar to the proof of the DG version of the Sobolev embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ in [19, Lemma 6.2].

Lemma 4.20. *Let* $h_0 > 0$. *Then, for* $h < h_0$, *there exists* $C > 0$ *such that*

$$||v||_{L^\infty(\Omega)} \leq C||v||_h \quad \forall\, v \in V_h.$$

Proof. Given $v \in V_h$, we consider the biharmonic problem in the weak form (3.8). Find $u \in H_0^2(\Omega)$ such that

$$a(u, w) = F(w) \quad \forall\, w \in H_0^2(\Omega), \tag{4.38}$$

where $a(\cdot, \cdot)$ is the bilinear form in (3.9) and $F \in H^{-2}(\Omega)$ is defined by

$$F(w) = \sum_{T \in \mathcal{T}_h} \Big(\sum_{i,j=1}^{2} \int_T \partial_{ij} v \, \partial_{ij} w \, dx \Big) \;\; \forall w \in H_0^2(\Omega).$$

Similar to (4.31), we can bound the functional F by

$$F(w) \leq ||v||_h ||w||_{H^2(\Omega)} \;\; \forall\, w \in H_0^2(\Omega), \tag{4.39}$$

which implies in the notation of Remark 2.37 that $||F||_{H^{-2}(\Omega)} \lesssim ||v||_h$. Hence, by Theorem 3.1, the solution $u \in H_0^2(\Omega)$ of (4.38) satisfies

$$|u|_{H^2(\Omega)} \lesssim ||F||_{H^{-2}(\Omega)} \lesssim ||v||_h.$$

Combining this with the Sobolev embedding $H^2(\Omega) \hookrightarrow C^0(\Omega)$ (Theorem 2.30), and Corollary 2.36, we obtain

$$||u||_{L^\infty(\Omega)} \lesssim ||u||_{H^2(\Omega)} \lesssim |u|_{H^2(\Omega)} \lesssim ||v||_h.$$

From the Sobolev embedding $H^{1+\varepsilon}(\Omega) \hookrightarrow C^0(\Omega)$ (Theorem 2.30) and the fact that $V_h \subset H^{1+\varepsilon}(\Omega)$ for $0 < \varepsilon < \frac{1}{2}$ (see [9]), we infer that

$$||v-u||_{L^\infty(\Omega)} \lesssim ||v-u||_{H^{1+\varepsilon}(\Omega)}.$$

Therefore, to prove the lemma we only need to show that

$$||v-u||_{H^{1+\varepsilon}(\Omega)} \lesssim ||v||_h. \tag{4.40}$$

This will be done by a duality argument. By Remark 3.26, there exists $\varepsilon \in (0, \frac{1}{2})$ such that for $G \in H^{-1-\varepsilon}(\Omega)$, the solution $\phi \in H_0^2(\Omega)$ of the biharmonic problem

$$a(w, \phi) = G(w) \quad \forall w \in H_0^2(\Omega) \tag{4.41}$$

has the regularity $\phi \in H^{3-\varepsilon}(\Omega)$, and

$$||\phi||_{H^{3-\varepsilon}(\Omega)} \lesssim ||G||_{H^{-1-\varepsilon}(\Omega)}. \tag{4.42}$$

By the consistency property in Lemma 4.12, we have that

$$a_h(\phi, w) = G(w) \quad \forall w \in V_h. \tag{4.43}$$

Then, under consideration of the facts that $[\![\partial_{n_e}\phi]\!] = 0$ and $\{\!\{\partial_{n_e n_e}\phi\}\!\} = \partial_{n_e n_e}\phi$ for all $e \in \mathcal{E}_h$, it follows from (4.20), (4.38), (4.41), (4.28) and (4.43) that

$$\begin{aligned}
G(v-u) = a_h(\phi, v) - a(u, \phi) &= \sum_{e \in \mathcal{E}_h} \int_e [\![\partial_{n_e} v]\!] \{\!\{\partial_{n_e n_e}\phi\}\!\}\, ds \\
&\leq \Big(\sum_{e\in\mathcal{E}_h} \frac{1}{|e|} ||[\![\partial_{n_e} v]\!]||^2_{L^2(e)}\Big)^{\frac{1}{2}} \Big(\sum_{e\in\mathcal{E}_h} |e|\, ||\{\!\{\partial_{n_e n_e}\phi\}\!\}||^2_{L^2(e)}\Big)^{\frac{1}{2}} \\
&\leq ||v||_h \Big(\sum_{e\in\mathcal{E}_h} |e|\, ||\partial_{n_e n_e}\phi||^2_{L^2(e)}\Big)^{\frac{1}{2}}.
\end{aligned}$$

The trace theorem with scaling (Theorem 2.33) yields

$$\sum_{e\in\mathcal{E}_h} |e| \, ||\partial_{n_e n_e}\phi||^2_{L^2(e)} \lesssim h^{2-2\epsilon}||\phi||^2_{H^{3-\varepsilon}(\Omega)}.$$

Thus, by the regularity (4.42), we have

$$G(v-u) \lesssim h^{1-\varepsilon}||v||_h||\phi||_{H^{3-\varepsilon}(\Omega)} \lesssim h_0^{1-\varepsilon}||v||_h||G||_{H^{-1-\varepsilon}(\Omega)}.$$

Consequently, the duality (2.13) implies (4.40) and completes the proof. □

Chapter 5

A C^0 Interior Penalty Method for the von Kármán Equations

This chapter is the centerpiece of this dissertation. We present a new C^0 interior penalty method for the von Kármán equations and derive a priori error estimates for it. The theoretical results will subsequently be confirmed by numerical experiments. Since the numerical analysis of nonlinear problems is often based on results of the linearized problem [30, 7], we begin with analyzing the C^0 interior penalty method for the linearized von Kármán equation.

5.1 A C^0 Interior Penalty Method for the Linearized von Kármán Equations

Consider the linearized von Kármán problem from Section 3.3.

Given $\boldsymbol{F} \in (H^{-2}(\Omega))^2$, find $\boldsymbol{u} \in (H_0^2(\Omega))^2$ such that

$$L(\boldsymbol{u}, \mathbf{v}) = \boldsymbol{F}(\boldsymbol{v}) \;\; \forall \boldsymbol{v} \in (H_0^2(\Omega))^2, \tag{5.1}$$

where

$$\begin{aligned} L(\boldsymbol{u}, \boldsymbol{v}) = (u_1, v_1)_\Delta + (u_2, v_2)_\Delta - p \int_\Omega \nabla u_1 \nabla v_1 \, dx \\ - b(\xi_1, u_2, v_1) - b(\xi_2, u_1, v_1) + 2b(\xi_1, u_1, v_2), \end{aligned} \tag{5.2}$$

and $b(\cdot,\cdot,\cdot)$ is the trilinear form defined in (3.18). Here, we implicitly assumed that the uniqueness conditions (3.46) and (3.47) hold so that the solution $\boldsymbol{\xi}$ of the von Kármán equation (3.38) is unique, and the linearization (5.1) at $\boldsymbol{\xi}$ is well-defined. Throughout this chapter, we assume that $\boldsymbol{\xi}$ has the regularity

$$\boldsymbol{\xi} \in (H^{3+s}(\Omega))^2 \text{ for some } s \in (0, 1], \tag{5.3}$$

and that Ω is a convex, bounded polygonal domain. Then we know from the regularity theory in Section 3.4.3 that the solution of the linearized problem has the regularity $\boldsymbol{u} \in (H^3(\Omega) \cap H_0^2(\Omega))^2$, provided $\boldsymbol{F} \in (H^{-1}(\Omega))^2$. For that reason, we set for $s \geq 0$

$$\boldsymbol{V}^s = (H^{3+s}(\Omega) \cap H_0^2(\Omega))^2,$$

and simply write $\boldsymbol{V}$ for $\boldsymbol{V}^0$.

Motivated by the results on the C^0 interior penalty method for the biharmonic problem in the previous section, our goal is to approximate the linearized von Kármán problem (5.1) on the finite element space

$$\boldsymbol{V}_h = V_h \times V_h.$$

On $\boldsymbol{V}_h$ we define the h-dependent seminorm $\|\!| \cdot |\!\|_h$ by

$$|\!|\!| \boldsymbol{v} |\!|\!|_h = \sqrt{||v_1||_h^2 + ||v_2||_h^2} \quad \forall \boldsymbol{v} \in \boldsymbol{V}^s + \boldsymbol{V}_h,$$

where V_h and $||\cdot||_h$ are given in (4.12) and (4.24). By setting

$$\Pi_h \boldsymbol{v} = (\Pi_h v_1, \Pi_h v_2) \quad \forall \boldsymbol{v} \in (C^0(\Omega))^2,$$

the interpolation results from Lemma 4.18 carry over to the finite element space $\boldsymbol{V}_h$ without further work.

In Section 3.4.3, we have seen that under the regularity assumption (5.3), the mappings $b(\xi_1, \cdot, \cdot)$ and $b(\xi_2, \cdot, \cdot)$ become bounded bilinear forms on $H^1(\Omega) \times H^1(\Omega)$. Since V_h is contained in $H_0^1(\Omega)$, the only terms in (5.1) that are undefined for functions in $\boldsymbol{V}_h$, are the biharmonic terms $(\cdot, \cdot)_\Delta$. But these can be replaced by the bilinear form $a_h(\cdot, \cdot)$ from Section 4.2.2. Thus, we obtain the following formulation of the linearized von Kármán problem on $\boldsymbol{V}_h$.

Given $\boldsymbol{F} \in (H^{-1}(\Omega))^2$, find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that

$$L_h(\boldsymbol{u}_h, \boldsymbol{v}) = \boldsymbol{F}(\boldsymbol{v}) \ \ \forall \boldsymbol{v} \in \boldsymbol{V}_h, \tag{5.4}$$

where $L_h(\cdot,\cdot)$ is defined for $\boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{V}^s + \boldsymbol{V}_h$ by

$$\begin{aligned} L_h(\boldsymbol{w}, \boldsymbol{v}) = a_h(w_1, v_1) + a_h(w_2, v_2) - p \int_\Omega \nabla w_1 \nabla v_1 \, dx \\ -b(\xi_1, w_2, v_1) - b(\xi_2, w_1, v_1) + 2b(\xi_1, w_1, v_2). \end{aligned} \tag{5.5}$$

Throughout this chapter, we use $\eta > 0$ as the penalty parameter for both of the bilinear forms $a_h(\cdot,\cdot)$ appearing in (5.5). For completeness, let us also define the corresponding discrete dual problem.

Given $\boldsymbol{G} \in (H^{-1}(\Omega))^2$, find $\boldsymbol{\phi}_h \in \boldsymbol{V}_h$ such that

$$L_h(\boldsymbol{v}, \boldsymbol{\phi}_h) = \boldsymbol{G}(\boldsymbol{v}) \ \ \forall \boldsymbol{v} \in \boldsymbol{V}_h. \tag{5.6}$$

Before showing that the discrete problem (5.4) is well-posed, we summarize properties of $L_h(\cdot,\cdot)$ that are important for the analysis later.

Lemma 5.1. *Given $\boldsymbol{F}, \boldsymbol{G} \in (H^{-1}(\Omega))^2$, let $\boldsymbol{u} \in \boldsymbol{V}$ be the solution of* (5.1) *and $\boldsymbol{\phi} \in \boldsymbol{V}$ the solution of the dual problem* (3.61)*. Then the following holds.*

(i) The bilinear form $L_h(\cdot,\cdot)$ is consistent and adjoint consistent; i.e.,

$$L_h(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{F}(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h, \tag{5.7}$$

$$L_h(\boldsymbol{v}, \boldsymbol{\phi}) = \boldsymbol{G}(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h. \tag{5.8}$$

(ii) The bilinear form $L_h(\cdot,\cdot)$ is bounded. That is, there exists $C_B > 0$ such that

$$L_h(\boldsymbol{w}, \boldsymbol{v}) \leq C_B \|\boldsymbol{w}\|_h \|\boldsymbol{v}\|_h \quad \forall \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}^s + \boldsymbol{V}_h, \tag{5.9}$$

where C_B depends only on p, $\boldsymbol{\xi}$, η, and the shape of Ω.

(iii) If η is sufficiently large, then $L_h(\cdot,\cdot)$ satisfies a Gårding type inequality

$$L_h(\boldsymbol{v}, \boldsymbol{v}) \ \geq C_1 \|\boldsymbol{v}\|_h^2 - C_2 ||\boldsymbol{v}||^2_{(H^1(\Omega))^2} \ \forall \boldsymbol{v} \in \boldsymbol{V}_h, \tag{5.10}$$

where $C_1 > 0$ and $C_2 \in \mathbb{R}$ are constants independent from h.

Proof. As in the proof of Lemma 3.28, we can consider the linearized von Kármán problem as two uncoupled biharmonic problems

$$(u_1, v_1)_\Delta = \widetilde{F}_1(v_1) \;\; \forall v_1 \in H_0^2(\Omega),$$
$$(u_2, v_2)_\Delta = \widetilde{F}_2(v_2) \;\; \forall v_2 \in H_0^2(\Omega),$$

where $\widetilde{F}_1$ and $\widetilde{F}_2$ are the functionals defined in (3.75) and (3.76). Thus, by the consistency (4.21) of the biharmonic problem, we have

$$a_h(u_1, v_1) = \widetilde{F}_1(v_1) \;\; \forall v_1 \in V_h,$$
$$a_h(u_2, v_2) = \widetilde{F}_2(v_2) \;\; \forall v_2 \in V_h.$$

This implies (5.7). The adjoint consistency (5.8) follows from an analogous argument for the dual problem using the fact that $a_h(\cdot,\cdot)$ is adjoint consistent.

By (4.25) and (3.70), we have for $\boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{V}^s + \boldsymbol{V}_h$

$$\begin{aligned} L_h(\boldsymbol{w}, \boldsymbol{v}) &= a_h(w_1, v_1) + a_h(w_2, v_2) - p \int_\Omega \nabla w_1 \nabla v_1 \, dx \\ &\quad - b(\xi_1, w_2, v_1) - b(\xi_2, w_1, v_1) + 2b(\xi_1, w_1, v_2) \\ &\le C_a(||w_1||_h ||v_1|| + ||w_2||_h ||v_2||_h) + |p| \, |w_1|_{H^1(\Omega)} |v|_{H^1(\Omega)} \\ &\quad + 4C_b ||\boldsymbol{w}||_{(H^1(\Omega))^2} \, ||\boldsymbol{v}||_{(H^1(\Omega))^2} \\ &\le 2C_a |||\boldsymbol{w}|||_h |||\boldsymbol{v}|||_h + (|p| + 4C_b) ||\boldsymbol{w}||_{(H^1(\Omega))^2} \, ||\boldsymbol{v}||_{(H^1(\Omega))^2}. \end{aligned} \tag{5.11}$$

Since $\boldsymbol{V}^s + \boldsymbol{V}_h \subset (H_0^1(\Omega) \cap H^2(\Omega, \mathcal{T}_h))^2$, Corollary 4.11 and Corollary 2.36 imply that there exists $C_H > 0$ such that

$$||\boldsymbol{v}||_{(H^1(\Omega))^2} \le C_H |||\boldsymbol{v}|||_h. \tag{5.12}$$

From (5.11) and (5.12), we conclude (ii).

Suppose $\eta > 0$ is sufficiently large. Then (3.70), (3.18), and (4.32) yield for $\boldsymbol{v} \in \boldsymbol{V}_h$

$$\begin{aligned} L_h(\boldsymbol{v}, \boldsymbol{v}) &= a_h(v_1, v_1) + a_h(v_2, v_2) - p|v_1|^2_{H^1(\Omega)} + b(\xi_1, v_2, v_1) - b(\xi_2, v_1, v_1) \\ &\geq \kappa(||v_1||_h^2 + ||v_2||_h^2) - |p| \, ||v_1||^2_{H^1(\Omega)} - |b(\xi_1, v_2, v_1)| - |b(\xi_2, v_1, v_1)| \\ &\geq \kappa |\!|\!| \boldsymbol{v} |\!|\!|_h - (|p| + C_b)||v_1||^2_{H^1(\Omega)} - C_b ||v_2||_{H^1(\Omega)} ||v_1||_{H^1(\Omega)} \\ &\geq \kappa |\!|\!| \boldsymbol{v} |\!|\!|_h - (|p| + 2C_b)||\boldsymbol{v}||^2_{(H^1(\Omega))^2}. \end{aligned}$$

This proves (iii). □

In the previous lemma, we have seen that $L_h(\cdot, \cdot)$ is not coercive. Thus, the Lax-Milgram Lemma (Theorem 2.11) is not applicable to prove the well-posedness of (5.4). Instead, we adapt a classical argument from Schatz [34], that employs the Gårding inequality. Since (5.4) is a nonconforming method, some additional steps will be necessary to carry out the argument.

Theorem 5.2. *Let $\boldsymbol{F} \in (H^{-1}(\Omega))^2$, and $\boldsymbol{u} \in \boldsymbol{V}$ be the solution of* (5.1). *Suppose $\eta > 0$ is sufficiently large, so that the Gårding inequality* (5.10) *holds.*
Then there exists a constant $h_0 > 0$, such that for $h \in (0, h_0)$, the discrete problem (5.4) *has a unique solution $\boldsymbol{u}_h \in \boldsymbol{V}_h$. Moreover, there exists $C > 0$ independent from h such that*

$$|\!|\!| \boldsymbol{u} - \boldsymbol{u}_h |\!|\!|_h \leq Ch||\boldsymbol{F}||_{(H^{-1}(\Omega))^2}. \tag{5.13}$$

Proof. The key of this proof is to show the following.
Under the assumption that $\boldsymbol{u}_h \in \boldsymbol{V}_h$ is a solution of the discretized problem (5.4), there exists $h_0 > 0$ such that for $h \in (0, h_0)$, there holds

$$|\!|\!| \Pi_h \boldsymbol{u} - \boldsymbol{u}_h |\!|\!|_h^2 \leq C_0 h^2 ||\boldsymbol{u}||^2_{(H^3(\Omega))^2}, \tag{5.14}$$

where $C_0 > 0$ is a constant independent of h.

To see the importance of (5.14), note that by Lemma 3.22, the homogenous problem (5.1) has exactly one solution, namely $\boldsymbol{u} = 0$. Thus, if (5.14) holds, then $\boldsymbol{u}_h = 0$ is the only solution of (5.4); i.e.

$$L_h(\boldsymbol{u}_h, \boldsymbol{v}) = 0 \; \forall \boldsymbol{v} \in \boldsymbol{V}_h \text{ if and only if } \boldsymbol{u}_h = 0.$$

In other words, the mapping $\mathcal{L}_h : \boldsymbol{u} \mapsto L_h(\boldsymbol{u}, \cdot)$ from $\boldsymbol{V}_h$ to $\boldsymbol{V}_h'$ is injective. Since $\mathcal{L}_h$ is a linear map between finite dimensional vector spaces spaces, it follows that $\mathcal{L}_h$ is invertible. Therefore, the solution of (5.4) also exists in the non-homogenous case $\boldsymbol{F} \neq 0$, and it is uniquely determined by $\boldsymbol{u}_h = \mathcal{L}_h^{-1}\boldsymbol{F}$. Finally, from (5.14) and (4.36), we conclude that

$$|||\boldsymbol{u} - \boldsymbol{u}_h|||_h \leq |||\boldsymbol{u} - \Pi_h \boldsymbol{u}|||_h + |||\Pi_h \boldsymbol{u} - \boldsymbol{u}_h|||_h \leq (C_I + \sqrt{C_0})||\boldsymbol{u}||_{(H^3(\Omega))^2}. \quad (5.15)$$

Hence, the error estimate (5.13) follows from the regularity estimate (3.71).

To complete the proof, we still need to establish (5.14). Let $\boldsymbol{u}_h \in \boldsymbol{V}_h$ be a solution of the discrete problem (5.4). Then the consistency (5.7) directly implies the Galerkin orthogonality

$$L_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}) = 0 \;\; \forall \boldsymbol{v} \in \boldsymbol{V}_h. \quad (5.16)$$

Applying the Gårding inequality (5.10) to $\Pi_h \boldsymbol{u} - \boldsymbol{u}_h$ yields

$$C_1 |||\Pi_h \boldsymbol{u} - \boldsymbol{u}_h|||_h^2 \leq L_h(\Pi_h \boldsymbol{u} - \boldsymbol{u}_h, \Pi_h \boldsymbol{u} - \boldsymbol{u}_h) + C_2 ||\Pi_h \boldsymbol{u} - \boldsymbol{u}_h||_{(H^1(\Omega))^2}^2. \quad (5.17)$$

Using (5.16), (5.9), (4.36) together with the inequality $ab \leq \frac{1}{2C_1}a^2 + \frac{C_1}{2}b^2$, we can bound the first term of (5.17) as follows

$$\begin{aligned} L_h(\Pi_h \boldsymbol{u} - \boldsymbol{u}_h, \Pi_h \boldsymbol{u} - \boldsymbol{u}_h) &= L_h(\Pi_h \boldsymbol{u} - \boldsymbol{u}, \Pi_h \boldsymbol{u} - \boldsymbol{u}_h) \\ &\leq C_B |||\Pi_h \boldsymbol{u} - \boldsymbol{u}|||_h \, |||\Pi_h \boldsymbol{u} - \boldsymbol{u}_h|||_h \\ &\leq C_B C_I h ||\boldsymbol{u}||_{(H^3(\Omega))^2} |||\Pi_h \boldsymbol{u} - \boldsymbol{u}_h|||_h \\ &\leq \frac{1}{2C_1} C_B^2 C_I^2 h^2 ||\boldsymbol{u}||_{(H^3(\Omega))^2}^2 + \frac{C_1}{2} |||\Pi_h \boldsymbol{u} - \boldsymbol{u}_h|||_h^2. \quad (5.18) \end{aligned}$$

By applying (5.18) to (5.17), and subtracting the term $\frac{C_1}{2}|||\Pi_h \boldsymbol{u} - \boldsymbol{u}_h|||_h$ from both sides, we obtain

$$\frac{C_1}{2}|||\Pi_h \boldsymbol{u} - \boldsymbol{u}_h|||_h \leq \frac{1}{2C_1} C_B^2 C_I^2 h^2 ||\boldsymbol{u}||^2_{(H^3(\Omega))^2} + C_2 ||\Pi_h \boldsymbol{u} - \boldsymbol{u}_h||^2_{(H^1(\Omega))^2}. \quad (5.19)$$

If $C_2 \leq 0$, inequality (5.14) follows without further work.

To bound the second term in inequality (5.19) in the case $C_2 > 0$, we use a duality argument to find $C_D > 0$ such that

$$||\boldsymbol{u} - \boldsymbol{u}_h||_{(H^1(\Omega))^2} \leq C_D h |||\boldsymbol{u} - \boldsymbol{u}_h|||_h. \quad (5.20)$$

For $\boldsymbol{G} \in (H^{-1}(\Omega))^2$, let $\boldsymbol{\phi} \in \boldsymbol{V}$ be the solution of the dual problem (3.61). Then we obtain from (5.8), (5.16), (4.36), and (3.72) that

$$\begin{aligned}
\boldsymbol{G}(\boldsymbol{u} - \boldsymbol{u}_h) &= L_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\phi}) \\
&= L_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\phi} - \Pi_h \boldsymbol{\phi}) \\
&\leq C_B |||\boldsymbol{u} - \boldsymbol{u}_h|||_h \, |||\boldsymbol{\phi} - \Pi_h \boldsymbol{\phi}|||_h \\
&\leq C_B C_I C_R h |||\boldsymbol{u} - \boldsymbol{u}_h|||_h \, ||\boldsymbol{G}||_{(H^{-1}(\Omega))^2}.
\end{aligned}$$

The duality (2.13) then proves (5.20), where $C_D = C_B C_I C_R > 0$.

Using the just established inequality (5.20) together with the interpolation estimates (4.36) and (4.37) we obtain an estimate for the second term in (5.19)

$$\begin{aligned}
||\Pi_h \boldsymbol{u} - \boldsymbol{u}_h||^2_{(H^1(\Omega))^2} &\leq 2||\Pi_h \boldsymbol{u} - \boldsymbol{u}||^2_{(H^1(\Omega))^2} + 2||\boldsymbol{u} - \boldsymbol{u}_h||^2_{(H^1(\Omega))^2} \\
&\leq 2C_I^2 h^4 ||\boldsymbol{u}||^2_{H^3(\Omega))^2} + 2C_D^2 h^2 |||\boldsymbol{u} - \boldsymbol{u}_h|||^2_h \\
&\leq 2C_I^2 h^4 ||\boldsymbol{u}||^2_{H^3(\Omega))^2} + 4C_D^2 h^2 |||\boldsymbol{u} - \Pi_h \boldsymbol{u}|||^2_h + 4C_D^2 h^2 |||\Pi_h \boldsymbol{u} - \boldsymbol{u}_h|||^2_h \\
&\leq 2C_I^2 h^4 ||\boldsymbol{u}||^2_{H^3(\Omega))^2} + 4C_D^2 C_I^2 h^4 ||\boldsymbol{u}||^2_{(H^3(\Omega)^2} + 4C_D^2 h^2 |||\Pi_h \boldsymbol{u} - \boldsymbol{u}_h|||^2_h. \quad (5.21)
\end{aligned}$$

Applying (5.21) to (5.19) and subtracting the term $4C_2 C_D^2 h^2 |||\Pi_h \boldsymbol{u} - \boldsymbol{u}_h|||^2_h$ from both sides yields

$$(\frac{C_1}{2} - 4C_2 C_D^2 h^2)|||\Pi_h \boldsymbol{u} - \boldsymbol{u}_h|||^2_h \leq (\frac{C_B^2}{2C_1} + 2C_2 h^2 + 4C_2 C_D^2 h^2) C_I^2 h^2 ||\boldsymbol{u}||^2_{(H^3(\Omega))^2}. \quad (5.22)$$

Now let $\varepsilon \in (0, \frac{C_1}{2})$. Then there exists $h_0 > 0$ such that $\frac{C_1}{2} - 4C_2C_D^2h^2 \geq \varepsilon$ for all $h \in (0, h_0)$. Therefore, (5.22) becomes

$$\varepsilon \|\Pi_h \boldsymbol{u} - \boldsymbol{u}_h\|_h^2 \leq (\frac{C_B^2}{2C_1} + 2C_2h_0^2 + 4C_2C_D^2h_0^2)C_I^2h^2||\boldsymbol{u}||^2_{(H^3(\Omega))^2}.$$

This implies (5.14), and completes the proof. □

Since the proof of Theorem 5.2 relies on properties of the linearized von Kármán problem (5.1) that are also valid for the dual problem (3.61) (see Corollary 3.23, Lemma 3.28, and Lemma 5.1), the argument in Theorem 5.2 can be applied analogously to the discrete dual problem (5.6).

Corollary 5.3. *Let $\boldsymbol{G} \in (H^{-1}(\Omega))^2$, and $\boldsymbol{\phi} \in \boldsymbol{V}$ be the solution of* (3.61)*. Suppose $\eta > 0$ is sufficiently large, so that the Gårding inequality* (5.10) *holds.*
Then there exists a constant $h_0 > 0$, such that for $h \in (0, h_0)$, the discrete problem (5.6) *has a unique solution $\boldsymbol{\phi}_h \in \boldsymbol{V}_h$. Moreover, there exists $C > 0$ such that*

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_h \leq Ch||\boldsymbol{G}||_{(H^{-1}(\Omega))^2}. \tag{5.23}$$

The next theorem shows that the C^0 interior penalty method for the linearized von Kármán equation is stable. It is noteworthy that the proof utilizes the results of the discrete dual problem in Corollary 5.3.

Theorem 5.4. *Let $h_0 > 0$ be small enough such that both Theorem 5.2 and Corollary 5.3 hold for $h \in (0, h_0)$. Given $\boldsymbol{F} \in \boldsymbol{V}_h'$, let $\boldsymbol{u}_h \in \boldsymbol{V}_h$ be the unique solution of* (5.4)*. Then there exists $C_s > 0$ such that*

$$\|\boldsymbol{u}_h\|_h \leq C_s \sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{|\boldsymbol{F}(\boldsymbol{v})|}{\|\boldsymbol{v}\|_h}. \tag{5.24}$$

Proof. By the Gårding inequality (5.10), and by (5.4) and (5.12), we have

$$\begin{aligned} C_1 \|\boldsymbol{u}_h\|_h^2 &\le L_h(\boldsymbol{u}_h, \boldsymbol{u}_h) + C_2 ||\boldsymbol{u}_h||^2_{(H^1(\Omega))^2} \\ &= \boldsymbol{F}(\boldsymbol{u}_h) + C_2 ||\boldsymbol{u}_h||^2_{(H^1(\Omega))^2} \\ &\le \sup_{\substack{\boldsymbol{v} \in \boldsymbol{V}_h \\ \boldsymbol{v} \neq 0}} \frac{|\boldsymbol{F}(\boldsymbol{v})|}{\|\boldsymbol{v}\|_h} \|\boldsymbol{u}_h\|_h + C_2 C_H \|\boldsymbol{u}_h\|_h ||\boldsymbol{u}_h||_{(H^1(\Omega))^2}. \end{aligned} \tag{5.25}$$

To bound the term $||\boldsymbol{u}||_{(H^1(\Omega))^2}$ in (5.25), we employ a duality argument.

Given $\boldsymbol{G} \in (H^{-1}(\Omega))^2$, let $\boldsymbol{\phi} \in \boldsymbol{V}$ be the solution of the dual problem (3.61) and $\boldsymbol{\phi}_h \in \boldsymbol{V}_h$ be the solution of the corresponding discrete problem (5.6).

By the trace theorem with scaling (Theorem 2.33), there exists $\bar{C} > 0$ such that

$$\begin{aligned} \|\boldsymbol{\phi}\|_h^2 &\le ||\boldsymbol{\phi}||^2_{(H^2(\Omega))^2} + \sum_{e \in \mathcal{E}_h} |e| \big(\sum_{i,j}^{2} ||\partial_{ij}\phi_1||^2_{L^2(e)} + ||\partial_{ij}\phi_2||^2_{L^2(e)} \big) \\ &\le (1 + \bar{C}h^2) ||\boldsymbol{\phi}||^2_{(H^3(\Omega))^2} \\ &\le C_R^2 (1 + \bar{C}h_0^2) ||\boldsymbol{G}||^2_{(H^{-1}(\Omega))^2}. \end{aligned}$$

Thus, we obtain

$$\|\boldsymbol{\phi}\|_h \le C_R \sqrt{1 + \bar{C}h_0^2} ||\boldsymbol{G}||_{(H^{-1}(\Omega))^2}.$$

This together with (5.6), (5.4), and (5.23) implies

$$\begin{aligned} \boldsymbol{G}(\boldsymbol{u}_h) = L_h(\boldsymbol{u}_h, \boldsymbol{\phi}_h) = \boldsymbol{F}(\boldsymbol{\phi}_h) &\le \sup_{\substack{\boldsymbol{v} \in \boldsymbol{V}_h \\ \boldsymbol{v} \neq 0}} \frac{|\boldsymbol{F}(\boldsymbol{v})|}{\|\boldsymbol{v}\|_h} \|\boldsymbol{\phi}_h\|_h \\ &\le \sup_{\substack{\boldsymbol{v} \in \boldsymbol{V}_h \\ \boldsymbol{v} \neq 0}} \frac{|\boldsymbol{F}(\boldsymbol{v})|}{\|\boldsymbol{v}\|_h} (\|\boldsymbol{\phi}_h - \boldsymbol{\phi}\|_h + \|\boldsymbol{\phi}\|_h) \\ &\le \sup_{\substack{\boldsymbol{v} \in \boldsymbol{V}_h \\ \boldsymbol{v} \neq 0}} \frac{|\boldsymbol{F}(\boldsymbol{v})|}{\|\boldsymbol{v}\|_h} (Ch ||\boldsymbol{G}||_{(H^{-1}(\Omega))^2} + C_R \sqrt{1 + \bar{C}h_0^2} \, ||\boldsymbol{G}||_{(H^{-1}(\Omega))^2}). \end{aligned}$$

Therefore, it follows from the duality (2.13) that

$$||\boldsymbol{u}_h||_{(H^1(\Omega))^2} \le \big(Ch_0 + C_R \sqrt{1 + \bar{C}h_0^2} \big) \sup_{\substack{\boldsymbol{v} \in \boldsymbol{V}_h \\ \boldsymbol{v} \neq 0}} \frac{|\boldsymbol{F}(\boldsymbol{v})|}{\|\boldsymbol{v}\|_h}. \tag{5.26}$$

Combining (5.25) and (5.26) completes the proof. □

As preparation for the next section, we write the results here in a different notation, compatible to the framework in [7]. In this way we will be able to adapt the arguments from [7] to the von Kármán problem easily.

Let $\mathcal{L} : \boldsymbol{V}^s + \boldsymbol{V}_h \to \boldsymbol{V}_h'$ be the bounded linear operator that maps $\boldsymbol{w}$ to the functional $L_h(\boldsymbol{w}, \cdot)$. We define $\mathcal{L}_h : \boldsymbol{V}_h \to \boldsymbol{V}_h'$ as the restriction of $\mathcal{L}$ to the finite element space $\boldsymbol{V}_h$. Under the conditions of Theorem 5.2, $\mathcal{L}_h$ has an inverse operator $\mathcal{L}_h^{-1} : \boldsymbol{V}_h' \to \boldsymbol{V}_h$ that maps $\boldsymbol{F} \in \boldsymbol{V}_h'$ to the unique solution of the problem

$$\mathcal{L}_h \boldsymbol{u}_h = \boldsymbol{F}.$$

By Theorem 5.4, $\mathcal{L}_h^{-1}$ is a bounded operator. In particular, we have that

$$\|\mathcal{L}_h^{-1}\boldsymbol{F}\|_h = \|\boldsymbol{u}_h\|_h \leq C_s \|\boldsymbol{F}\|_{-h}, \tag{5.27}$$

where $\|\cdot\|_{-h}$ denotes the dual norm in $\boldsymbol{V}_h'$ given by

$$\|\boldsymbol{F}\|_{-h} = \sup_{\substack{\boldsymbol{v}_h \in \boldsymbol{V}_h \\ \boldsymbol{v}_h \neq 0}} \frac{|\langle \boldsymbol{F}, \boldsymbol{v} \rangle|}{\|\boldsymbol{v}\|_h}.$$

Here and in the following, $\langle \cdot, \cdot \rangle$ denotes the duality between $\boldsymbol{V}_h$ and $\boldsymbol{V}_h'$.

5.2 C^0 Interior Penalty Methods for the von Kármán Equations

In this section, we propose a finite element method for the von Kármán equations (3.38) based on the C^0 finite element space $\boldsymbol{V}_h$. As for the previous C^0 interior penalty methods, we need to find a formulation that is defined on the finite element space $\boldsymbol{V}_h$ first. The obvious way to do this is to use the C^0 interior penalty method from Section 4.2 for the biharmonic part of (3.38). To discretize the Monge-Ampère part of (3.38), we need a discrete version of the identity (3.19)

$$\int_\Omega [w, u] v \, dx = b(w, u, v)$$

that relates the Monge-Ampère terms in (5.41) to the trilinear form $b(\cdot,\cdot,\cdot)$ for functions in V_h.

For $w \in C^\infty(\Omega)$, $u \in V_h + H^3(\Omega)$, and $v \in H^{1+\varepsilon}(\Omega) \cap H^1_0(\Omega)$ for $\varepsilon > 0$, integration by parts yields on any $T \in \mathcal{T}_h$

$$\begin{aligned}
\int_T [w,u]v\,dx &= \int_T (\partial_{11}w\partial_{22}u)v + (\partial_{22}w\partial_{11}u)v - 2(\partial_{12}w\partial_{12}u)v\ dx \\
&= -\int_T \partial_1 u\partial_1(v\partial_{22}w)\,dx + \int_{\partial T} (\partial_1 u\partial_{22}w)n_1 v\,ds \\
&\quad - \int_T \partial_2 u\partial_2(v\partial_{11}w)\,dx + \int_{\partial T} (\partial_2 u\partial_{11}w)n_2 v\,ds \\
&\quad + \int_T \partial_2 u\partial_1(v\partial_{12}w)\,dx - \int_{\partial T} (\partial_2 u\partial_{12}w)n_1 v\,ds \\
&\quad + \int_T \partial_1 u\partial_2(v\partial_{12}w)\,dx - \int_{\partial T} (\partial_1 u\partial_{12}w)n_2 v\,ds \\
&= \int_T \Big(\partial_{12}w(\partial_1 u\partial_2 v + \partial_2 u\partial_1 v) - (\partial_{11}w\partial_2 u\partial_2 v + \partial_{22}w\partial_1 u\partial_1 v)\Big)dx \\
&\quad + \int_{\partial T} \Big(\big(\partial_{22}w\partial_1 u - \partial_{12}w\partial_2 u\big)n_1 v + \big(\partial_{11}w\partial_2 u - \partial_{12}w\partial_1 u\big)n_2 v\Big)ds, \qquad (5.28)
\end{aligned}$$

where (n_1, n_2) denotes the outer unit normal of T.

Note that the second order partial derivatives of $w \in C^\infty(\Omega)$ agree along common edges of neighboring triangles. Thus, summing up (5.28) over all $T \in \mathcal{T}_h$ yields in the notation of Section 4.2

$$\begin{aligned}
\sum_{T\in\mathcal{T}_h} \int_T [w,u]v\,dx &= b(w,u,v) + \sum_{e\in\mathcal{E}_h^i} \int_e \big(\partial_{22}w\,[\![\partial_1 u]\!] - \partial_{12}w\,[\![\partial_2 u]\!]\big)n_{e,1}\,v\,ds \\
&\quad + \sum_{e\in\mathcal{E}_h^i} \int_e \big(\partial_{11}w\,[\![\partial_2 u]\!] - \partial_{12}w\,[\![\partial_1 u]\!]\big)n_{e,2}\,v\,ds. \qquad (5.29)
\end{aligned}$$

To extend the right hand side of (5.29) to functions $w \in V_h$, we replace the second derivatives of w by the corresponding averages, without changing the essence of relation (5.29). By setting

$$b_{\mathcal{E}}(w,u,v) = \sum_{e\in\mathcal{E}_h^i} \int_e \big(\{\{\partial_{22}w\}\}\,[[\partial_1 u]] - \{\{\partial_{12}w\}\}\,[[\partial_2 u]]\big) n_{e,1}\, v\, ds$$
$$+ \sum_{e\in\mathcal{E}_h^i} \int_e \big(\{\{\partial_{11}w\}\}\,[[\partial_2 u]] - \{\{\partial_{12}w\}\}\,[[\partial_1 u]]\big) n_{e,2}\, v\, ds, \tag{5.30}$$

we can write (5.29) in the compact form

$$\sum_{T\in\mathcal{T}_h} \int_T [w,u] v\, dx = b(w,u,v) - b_{\mathcal{E}}(w,u,v). \tag{5.31}$$

Next we show that the trilinear form $b_{\mathcal{E}}(\cdot,\cdot,\cdot)$ is bounded with respect to $||\cdot||_h$. The proof relies on the discrete Sobolev embedding from Lemma 4.20.

Lemma 5.5. *There exists $C > 0$ such that for $w, u \in V + V_h$, and $v \in V_h$, there holds*

$$b_{\mathcal{E}}(w,u,v) \le C||w||_h||u||_h||v||_h. \tag{5.32}$$

Proof. By Remark 5.8, we can assume without loss of generality that $u \in V_h$. For $e \in \mathcal{E}_h^i$, let (n_e, t_e) be the orthonormal basis depicted in Figure 4.2. Then the vector x_1 of the standard basis (x_1, x_2) of $\mathbb{R}^2$ can be written as

$$x_1 = a\, n_e + b\, t_e, \quad a, b \in [-1, 1].$$

From the fact that the dot product is invariant with respect to a change of orthonormal bases (cf. (4.15)), it follows that

$$|\,[[\partial_1 u]]\,| = |\,[[\nabla u \cdot x_1]]\,| = |\,[[a\,\partial_{n_e} u + b\,\partial_{t_e} u]]\,| \le |a|\,|\,[[\partial_{n_e} u]]\,| + |b|\,|\,[[\partial_{t_e} u]]\,|.$$

Since V_h is a C^0 finite element space, the jump $[[\partial_{t_e} u]]$ vanishes. This implies together with the analogous argument for $[[\partial_2 u]]$ that

$$|\,[[\partial_1 u]]\,| \le |\,[[\partial_{n_e} u]]\,|, \tag{5.33a}$$

$$|\,[[\partial_2 u]]\,| \le |\,[[\partial_{n_e} u]]\,|. \tag{5.33b}$$

Now the Cauchy-Schwarz inequality together with (5.33) yields

$$
\begin{aligned}
b_{\mathcal{E}}(w,u,v) \le ||v||_{L^\infty(\Omega)} &\sum_{e\in\mathcal{E}_h^i} \int_e \big(\big|\{\!\{\partial_{22}w\}\!\}\,[\![\partial_1 u]\!]\big| + \big|\{\!\{\partial_{12}w\}\!\}\,[\![\partial_2 u]\!]\big| \big)\,ds \\
&+ ||v||_{L^\infty(\Omega)} \sum_{e\in\mathcal{E}_h^i} \int_e \big(\big|\{\!\{\partial_{11}w\}\!\}\,[\![\partial_2 u]\!]\big| + \big|\{\!\{\partial_{12}w\}\!\}\,[\![\partial_1 u]\!]\big| \big)\,ds \\
\le ||v||_{L^\infty(\Omega)} &\sum_{e\in\mathcal{E}_h} \sum_{i,j=1}^{2} \int_e \big|\{\!\{\partial_{ij}w\}\!\}\,[\![\partial_{n_e} u]\!]\big|\,ds \\
\le ||v||_{L^\infty(\Omega)} &\sum_{e\in\mathcal{E}_h} \sum_{i,j=1}^{2} |e|^{\frac{1}{2}} ||\{\!\{\partial_{ij}w\}\!\}||_{L^2(e)}\, |e|^{-\frac{1}{2}}\, ||[\![\partial_{n_e} u]\!]||_{L^2(e)} \\
\le ||v||_{L^\infty(\Omega)} &\big(\sum_{e\in\mathcal{E}_h} \sum_{i,j=1}^{2} |e|\, ||\{\!\{\partial_{ij}w\}\!\}||^2_{L^2(e)} \big)^{\frac{1}{2}} \big(\sum_{e\in\mathcal{E}_h} |e|^{-1}\, ||[\![\partial_{n_e} u]\!]||^2_{L^2(e)} \big)^{\frac{1}{2}}. \qquad (5.34)
\end{aligned}
$$

Lemma 4.20 completes the proof. □

Lemma 5.6. *Identity* (5.31) *also holds for* $w \in H^3(\Omega)$.

Proof. Since $C^\infty(\Omega)$ is dense in $H^3(\Omega)$ (cf. Theorem 2.26), we only have to show

$$\sum_{T\in\mathcal{T}_h} \int_T [w,u]v\,dx \lesssim ||w||_{H^3(\Omega)} ||u||_h ||v||_{H^{1+\varepsilon}(\Omega)}, \qquad (5.35)$$

$$b(w,u,v) - b_{\mathcal{E}}(w,u,v) \lesssim ||w||_{H^3(\Omega)} ||u||_h ||v||_{H^{1+\varepsilon}(\Omega)}. \qquad (5.36)$$

Note that estimate (5.36) is a consequence of (3.21) and (5.34) and the Sobolev embedding $H^{1+\varepsilon}(\Omega) \hookrightarrow C^0(\Omega)$. Moreover the Sobolev embedding $H^{1+\varepsilon}(\Omega) \hookrightarrow C^0(\Omega)$ implies

$$
\begin{aligned}
\sum_{T\in T_h} \int_T [w,u]v\,dx &\le ||v||_{L^\infty(\Omega)} \sum_{T\in\mathcal{T}_h} \int_T [w,u]\,dx && (5.37)\\
&\lesssim ||v||_{H^{1+\varepsilon}(\Omega)} \sum_{T\in\mathcal{T}_h} |w|_{H^2(T)} |u|_{H^2(T)} && (5.38)\\
&\lesssim ||v||_{H^{1+\varepsilon}(\Omega)} ||w||_{H^3(\Omega)} ||u||_h. && (5.39)
\end{aligned}
$$

Thus, (5.35) holds, and the proof is complete. □

Remark 5.7. *Since* $V_h \subset H^{1+\varepsilon}(\Omega) \cap H_0^1(\Omega)$ *for* $\varepsilon \in [0, \frac{1}{2})$*, identity* (5.31) *also holds for* $v \in V_h$*.*

Remark 5.8. *By the Sobolev embedding* $H^3(\Omega) \hookrightarrow C^1(\Omega)$ *(Theorem 2.30), the jumps in* (5.30) *vanish for functions* $u \in H^3(\Omega)$*. Thus,*

$$b_{\mathcal{E}}(w, u, v) = 0 \ \ \forall\, u \in H^3(\Omega), \tag{5.40}$$

and identity (5.31) *becomes* (3.19)*.*

We proceed with the discretization of the von Kármán problem.

For $f \in L^2(\Omega)$, we define the operator $G_h \colon \boldsymbol{V}^s + \boldsymbol{V}_h \to \boldsymbol{V}_h'$ by

$$\begin{aligned} \langle G_h \boldsymbol{w}, \boldsymbol{v} \rangle = a_h(w_1, v_1) + a_h(w_2, v_2) - \int_\Omega p \nabla w_1 \nabla v_1 \, dx - f(v_1) \\ + \sum_{T \in \mathcal{T}_h} \int_T ([w_1, w_1] v_2 - [w_1, w_2] v_1) \, dx, \end{aligned} \tag{5.41}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality in $\boldsymbol{V}_h$, and $a_h(\cdot, \cdot)$ is the bilinear form defined in (4.20). The solution $\boldsymbol{\xi} \in \boldsymbol{V}^s$ of the von Kármán equations satisfies the following.

Lemma 5.9. *Suppose the uniqueness conditions* (3.47) *and* (3.46) *hold. Given* $f \in L^2(\Omega)$*, let* $\boldsymbol{\xi} \in \boldsymbol{V}^s$ *be the unique solution of the von Kármán equations* (3.38)*. Then*

$$G_h \boldsymbol{\xi} = 0. \tag{5.42}$$

Proof. Since $\boldsymbol{\xi} \in \boldsymbol{V}^s$ is uniquely determined as the solution of the von Kármán equations (3.38), the von Kármán equations can be considered as two uncoupled biharmonic equations

$$\Delta^2 \xi_1 = -p \Delta \xi_1 + [\xi_1, \xi_2] + f, \tag{5.43a}$$

$$\Delta^2 \xi_2 = -[\xi_1, \xi_1]. \tag{5.43b}$$

If the right hand sides of (5.43) belong to $L^2(\Omega)$, equation (5.42) follows from the consistency of the C^0 interior penalty method for the biharmonic problem (see (4.23)). For $v_1 \in L^2(\Omega)$, we have

$$\int_\Omega [\xi_1, \xi_2] v_1 \, dx \leq ||\xi_1||_{W^2_\infty(\Omega)} ||\xi_1||_{H^2(\Omega)} ||v_1||_{L^2(\Omega)}, \tag{5.44}$$

and

$$-p \int_\Omega \Delta \xi_1 v_1 \, dx = \leq |p| \, ||\xi_1||_{H^2(\Omega)} ||v_1||_{L^2(\Omega)}.$$

Thus, the right hand side of (5.43a) is in $L^2(\Omega)$. A similar argument proves that the right hand side of (5.43b) belongs to $L^2(\Omega)$. This completes the proof. □

The discrete problem is the following. Given $f \in L^2(\Omega)$, find $\boldsymbol{\xi}_h \in \boldsymbol{V}_h$ such that

$$G_h \boldsymbol{\xi}_h = 0. \tag{5.45}$$

For the analysis of this problem, it would be ideal if the linearization of G_h at $\boldsymbol{\xi}$ coincided with the operator $\boldsymbol{\mathcal{L}}$ we have investigated in the previous section. Although this is not the case here, we can achieve this by modifying G_h slightly. Let $\widetilde{G}_h \colon \boldsymbol{V} + \boldsymbol{V}_h \to \boldsymbol{V}_h'$ be the operator defined by

$$\langle \widetilde{G}_h \boldsymbol{w}, \boldsymbol{v} \rangle = \langle G_h \boldsymbol{w}, \boldsymbol{v} \rangle - b_{\mathcal{E}}(w_1, w_2, v_1) - b_{\mathcal{E}}(w_2, w_1, v_1) + 2 b_{\mathcal{E}}(w_1, w_1, v_2). \tag{5.46}$$

Note that by (5.40), the additional terms of $\widetilde{G}$ in (5.46) vanish if $\boldsymbol{w} \in \boldsymbol{V}^s$. Thus, by Lemma 5.9, the solution $\boldsymbol{\xi} \in \boldsymbol{V}^s$ of the von Kármán equations (3.38) satisfies

$$\widetilde{G}_h \boldsymbol{\xi} = 0. \tag{5.47}$$

The discrete method is then to find $\boldsymbol{\xi}_h \in \boldsymbol{V}_h$ such that

$$\widetilde{G}_h \boldsymbol{\xi}_h = 0. \tag{5.48}$$

The gain of the modification of G in (5.46) is that $\widetilde{G}$ is related to operator $\mathcal{L}$ from the previous section. Using (5.40), (5.47), and (5.31), we obtain, for $\boldsymbol{u} \in \boldsymbol{V}^s + \boldsymbol{V}_h$,

$$\begin{aligned}
\langle \widetilde{G}_h(\boldsymbol{\xi} + \boldsymbol{u}), \boldsymbol{v}\rangle &= \langle \widetilde{G}_h(\boldsymbol{\xi} + \boldsymbol{u}), \boldsymbol{v}\rangle - \langle \widetilde{G}_h \boldsymbol{\xi}, \boldsymbol{v}\rangle \\
&= a_h(u_1, v_1) + a_h(u_2, v_2) - p \int_\Omega \nabla u_1 \nabla v_1 \, dx \\
&\quad + \sum_{T \in \mathcal{T}_h} \int_T \big((2[\xi_1, u_1] + [u_1, u_1]) v_2 - ([u_1, \xi_2] + [\xi_1, u_2] + [u_1, u_2]) v_1\big) dx \\
&\quad - b_{\mathcal{E}}(\xi_1 + u_1, u_2, v_1) - b_{\mathcal{E}}(\xi_2 + u_2, u_1, v_1) + 2 b_{\mathcal{E}}(\xi_1 + u_1, u_1, v_2) \\
&= a_h(u_1, v_1) + a_h(u_2, v_2) - p \int_\Omega \nabla u_1 \nabla v_1 \, dx \\
&\quad + 2b(\xi_1, u_1, v_2) - b(\xi_2, u_1, v_1) - b(\xi_1, u_2, v_1) \\
&\quad + \sum_{T \in \mathcal{T}_h} \int_T \big([u_1, u_1] v_2 - [u_1, u_2] v_1\big) \, dx \\
&\quad - b_{\mathcal{E}}(u_1, u_2, v_1) - b_{\mathcal{E}}(u_2, u_1, v_1) + 2 b_{\mathcal{E}}(u_1, u_1, v_2) \\
&= \langle \mathcal{L}\boldsymbol{u}, \boldsymbol{v}\rangle + \langle R\boldsymbol{u}, \boldsymbol{v}\rangle,
\end{aligned} \tag{5.49}$$

where $R \colon \boldsymbol{V}^s + \boldsymbol{V}_h \to \boldsymbol{V}_h'$ is the functional given by

$$\begin{aligned}
\langle R\boldsymbol{u}, \boldsymbol{v}\rangle = \sum_{T \in \mathcal{T}_h} \int_T \big([u_1, u_1] v_2 - [u_1, u_2] v_1\big) \, dx \\
- b_{\mathcal{E}}(u_1, u_2, v_1) - b_{\mathcal{E}}(u_2, u_1, v_1) + 2 b_{\mathcal{E}}(u_1, u_1, v_2).
\end{aligned} \tag{5.50}$$

The relation between $\widetilde{G}_h$ and $\mathcal{L}$ derived in equation (5.49) can be expressed in the compact form

$$\widetilde{G}_h(\boldsymbol{\xi} + \boldsymbol{u}) = \mathcal{L}\boldsymbol{u} + R\boldsymbol{u} \quad \forall \boldsymbol{u} \in \boldsymbol{V}^s + \boldsymbol{V}_h.$$

Equivalently, we have

$$\widetilde{G}_h \boldsymbol{u} = \mathcal{L}(\boldsymbol{u} - \boldsymbol{\xi}) + R(\boldsymbol{u} - \boldsymbol{\xi}) \quad \forall \boldsymbol{u} \in \boldsymbol{V}^s + \boldsymbol{V}_h. \tag{5.51}$$

On the foundation of this identity, we will show that the discrete problem (5.48) has a solution that approximates the exact solution $\boldsymbol{\xi}$. Thereby we follow the fixed point argument in [7] that was employed for the C^0 penalty method for the fully nonlinear Monge-Ampère equation.

To begin, we define the operator $M : \boldsymbol{V}_h + \boldsymbol{V}^s \to \boldsymbol{V}_h$ by

$$M = \mathcal{L}_h^{-1}(\mathcal{L} - \widetilde{G}_h). \tag{5.52}$$

By (5.51), we obtain for $\boldsymbol{u} \in \boldsymbol{V}_h + \boldsymbol{V}^s$

$$\begin{aligned} M\boldsymbol{u} &= \mathcal{L}_h^{-1}(\mathcal{L}\boldsymbol{u} - \widetilde{G}_h\boldsymbol{u}) \\ &= \mathcal{L}_h^{-1}\big(\mathcal{L}\boldsymbol{u} - \mathcal{L}(\boldsymbol{u} - \boldsymbol{\xi}) - R(\boldsymbol{u} - \boldsymbol{\xi})\big) \\ &= \mathcal{L}_h^{-1}\mathcal{L}\boldsymbol{\xi} - \mathcal{L}_h^{-1}R(\boldsymbol{u} - \boldsymbol{\xi}) \\ &= \boldsymbol{\xi}_h^c - \mathcal{L}_h^{-1}R(\boldsymbol{u} - \boldsymbol{\xi}), \end{aligned} \tag{5.53}$$

where $\boldsymbol{\xi}_h^c \in \boldsymbol{V}_h$ is given by

$$\boldsymbol{\xi}_h^c = \mathcal{L}_h^{-1}\mathcal{L}\boldsymbol{\xi}. \tag{5.54}$$

The significant role of M in the analysis of (5.48) becomes clear when we restrict M to $\boldsymbol{V}_h$. Let $M_h : \boldsymbol{V}_h \to \boldsymbol{V}_h$ be the restriction of M to $\boldsymbol{V}_h$. By (5.52) and (5.53), M_h satisfies

$$M_h\boldsymbol{u} = \boldsymbol{u} - \mathcal{L}_h^{-1}\widetilde{G}_h\boldsymbol{u} \tag{5.55}$$

$$= \boldsymbol{\xi}_h^c - \mathcal{L}_h^{-1}R(\boldsymbol{u} - \boldsymbol{\xi}). \tag{5.56}$$

In view of equation (5.55), it is clear that $\widetilde{G}_h\boldsymbol{\xi}_h = 0$ if and only if $\boldsymbol{\xi}_h \in \boldsymbol{V}_h$ is a fixed point of M_h. Thus, a solution of (5.48) exists, if the fixed point iteration $\boldsymbol{\xi}_{n+1} = M_h\boldsymbol{\xi}_n$ is convergent. By the contraction mapping theorem (see [32]), this holds if M_h is a contraction.

Due to the representation (5.56) of M_h, we have for $\boldsymbol{u}, \bar{\boldsymbol{u}} \in \boldsymbol{V}_h$

$$M_h\boldsymbol{u} - M_h\bar{\boldsymbol{u}} = \mathcal{L}_h^{-1}\big(R(\bar{\boldsymbol{u}} - \boldsymbol{\xi}) - R(\boldsymbol{u} - \boldsymbol{\xi})\big). \tag{5.57}$$

Thus, a contraction estimate for M_h requires a contractive property of R. This will be established in the next lemma.

Lemma 5.10. *There exists $C_3 > 0$ such that for any $\boldsymbol{w}, \bar{\boldsymbol{w}} \in \boldsymbol{V}^s + \boldsymbol{V}_h$, there holds*

$$\|\!|R\boldsymbol{w} - R\bar{\boldsymbol{w}}\|\!|_{-h} \leq C_3(\|\!|\boldsymbol{w}\|\!|_h + \|\!|\bar{\boldsymbol{w}}\|\!|_h)\|\!|\boldsymbol{w} - \bar{\boldsymbol{w}}\|\!|_h. \tag{5.58}$$

Proof. For any $\boldsymbol{u}, \boldsymbol{w} \in \boldsymbol{V} + \boldsymbol{V}_h$ and $\boldsymbol{v} \in \boldsymbol{V}_h$, we obtain by Lemma 4.20

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}\int_T \big([w_1, u_1]v_2 - [w_1, u_2]v_1\big)dx &\leq ||\boldsymbol{v}||_{(L^\infty(\Omega))^2}\sum_{T\in\mathcal{T}_h}\int_T \big(|[w_1, u_1]| + |[w_1, u_2]|\big)dx \\ &\leq C\|\!|\boldsymbol{v}\|\!|_h \sum_{T\in\mathcal{T}_h}\big(||w_1||_{H^2(T)}||u_1||_{H^2(T)} + ||w_1||_{H^2(T)}||u_2||_{H^2(T)}\big) \\ &\leq 2C\|\!|\boldsymbol{w}\|\!|_h\|\!|\boldsymbol{u}\|\!|_h\|\!|\boldsymbol{v}\|\!|_h. \end{aligned}\tag{5.59}$$

To simplify the notation in this proof, we introduce the trilinear form $r(\cdot,\cdot,\cdot)$ on $(\boldsymbol{V}^s + \boldsymbol{V}_h) \times (\boldsymbol{V}^s + \boldsymbol{V}_h) \times \boldsymbol{V}_h$ by

$$\begin{aligned} r(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) = &\sum_{T\in\mathcal{T}_h}\int_T \big([w_1, u_1]v_2 - [w_1, u_2]v_1\big)dx \\ &- b_{\mathcal{E}}(w_1, u_2, v_1) - b_{\mathcal{E}}(w_2, u_1, v_1) + 2b_{\mathcal{E}}(w_1, u_1, v_2). \end{aligned}\tag{5.60}$$

As consequence of (5.59) and (5.34), there exists $C_4 > 0$ such that

$$r(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) \leq C_3\|\!|\boldsymbol{w}\|\!|_h\|\!|\boldsymbol{u}\|\!|_h\|\!|\boldsymbol{v}\|\!|_h. \tag{5.61}$$

By (5.50) and (5.60), it follows that

$$\langle R\boldsymbol{w}, \boldsymbol{v}\rangle = r(\boldsymbol{w}, \boldsymbol{w}, \boldsymbol{v}) \;\; \forall\, \boldsymbol{w} \in \boldsymbol{V}^s + \boldsymbol{V}_h, \text{ and } \forall\, \boldsymbol{v} \in \boldsymbol{V}_h.$$

Thus, we can use (5.61) to estimate

$$\begin{aligned}\langle R\boldsymbol{w} - R\bar{\boldsymbol{w}}, \boldsymbol{v}\rangle &= r(\boldsymbol{w}, \boldsymbol{w}, \boldsymbol{v}) - r(\bar{\boldsymbol{w}}, \bar{\boldsymbol{w}}, \boldsymbol{v}) \\ &= r(\boldsymbol{w} - \bar{\boldsymbol{w}}, \boldsymbol{w}, \boldsymbol{v}) + r(\bar{\boldsymbol{w}}, \boldsymbol{w} - \bar{\boldsymbol{w}}, \boldsymbol{v}) \\ &\leq C_3(\|\!|\bar{\boldsymbol{w}}\|\!|_h + \|\!|\boldsymbol{w}\|\!|_h)\|\!|\boldsymbol{w} - \bar{\boldsymbol{w}}\|\!|_h\|\!|\boldsymbol{v}\|\!|_h. \end{aligned}$$

□

Remark 5.11. *Taking $\bar{\boldsymbol{w}} = 0$ in Lemma 5.10 implies*

$$\frac{\|\!|R\boldsymbol{w}|\!\|_{-h}}{\|\!|\boldsymbol{w}|\!\|_h} \to 0 \text{ as } \|\!|\boldsymbol{w}|\!\|_h \to 0.$$

Therefore, by (5.2)*, $\mathcal{L}_h$ is indeed the linearization of $\widetilde{G}_h$ at $\boldsymbol{\xi}$.*

From inequality (5.58), we conclude that R is a contraction if $\|\!|\boldsymbol{w}|\!\|_h$ and $\|\!|\bar{\boldsymbol{w}}|\!\|_h$ remain sufficiently small. That is, for R to be a contraction mapping, we have to restrict R to a sufficiently small ball around zero. In the light of (5.56), this means that we have to restrict M_h to a subset of $\boldsymbol{V}_h$ whose functions are sufficiently close to $\boldsymbol{\xi}$. In the next lemma, we see that $\boldsymbol{\xi}_h^c \in \boldsymbol{V}_h$ belongs to that subset, provided h is chosen sufficiently small.

Lemma 5.12. *Let $\boldsymbol{\xi} \in (H^3(\Omega))^2$ and $\boldsymbol{\xi}_h^c \in \boldsymbol{V}_h$ be defined by* (5.54)*. Then there exists $C_4 > 0$ such that*

$$\|\!|\boldsymbol{\xi} - \boldsymbol{\xi}_h^c|\!\|_h \leq C_4 h ||\boldsymbol{\xi}||_{(H^3(\Omega))^2}. \tag{5.62}$$

Proof. Using (5.54), (5.27), (5.9), and (4.36), we estimate

$$\begin{aligned}
\|\!|\boldsymbol{\xi} - \boldsymbol{\xi}_h^c|\!\|_h &\leq \|\!|\boldsymbol{\xi} - \Pi_h\boldsymbol{\xi}|\!\|_h + \|\!|\Pi_h\boldsymbol{\xi} - \boldsymbol{\xi}_h^c|\!\|_h \\
&\leq \|\!|\boldsymbol{\xi} - \Pi_h\boldsymbol{\xi}|\!\|_h + \|\!|\mathcal{L}_h^{-1}\mathcal{L}_h(\Pi_h\boldsymbol{\xi} - \boldsymbol{\xi}_h^c)|\!\|_h \\
&\leq \|\!|\boldsymbol{\xi} - \Pi_h\boldsymbol{\xi}|\!\|_h + C_s\|\!|\mathcal{L}(\Pi_h\boldsymbol{\xi} - \boldsymbol{\xi})|\!\|_{-h} \\
&\leq \|\!|\boldsymbol{\xi} - \Pi_h\boldsymbol{\xi}|\!\|_h + C_sC_B\|\!|(\Pi_h\boldsymbol{\xi} - \boldsymbol{\xi})|\!\|_h \\
&\leq (1 + C_sC_B)C_I h||\boldsymbol{\xi}||_{(H^3(\Omega))^2}.
\end{aligned}$$

□

We are now ready to establish the main result of this dissertation. The proof is based on a fixed point argument applied to the restriction of M_h to a ball

$$B_\rho(\boldsymbol{\xi}_h^c) = \{\boldsymbol{v} \in \boldsymbol{V}_h \mid \|\!|\boldsymbol{\xi}_h^c - \boldsymbol{v}|\!\|_h \leq \rho\}$$

centered at $\boldsymbol{\xi}_h^c \in \boldsymbol{V}_h$ with sufficiently small radius $\rho > 0$.

Theorem 5.13. *Under the uniqueness conditions* (3.47) *and* (3.46), *given* $f \in L^2(\Omega)$, *let* $\boldsymbol{\xi} \in \boldsymbol{V}^s$ *be the unique solution of the von Kármán equations* (3.38), *and let*

$$\rho \in \left(0, \frac{1}{4C_sC_3}\right). \tag{5.63}$$

Then there exists $h_1 > 0$ *such that for any* $h \in (0, h_1)$, *the discrete von Kármán problem* (5.48) *has a unique solution* $\boldsymbol{\xi}_h \in B_\rho(\boldsymbol{\xi}_h^c)$. *Moreover, there exists* $C > 0$, *independent from* h, *such that*

$$|||\boldsymbol{\xi} - \boldsymbol{\xi}_h|||_h \leq Ch||\boldsymbol{\xi}||_{(H^3(\Omega))^2}. \tag{5.64}$$

Proof. Let

$$h_1 = \min\{h_0, \frac{\rho}{C_4||\boldsymbol{\xi}||_{(H^3(\Omega))^2}}\}, \tag{5.65}$$

and $h \in (0, h_1)$, where $h_0 > 0$ is the constant in Lemma 5.4.

Then, (5.27), (5.58), (5.62), (5.65), and (5.63), imply for any $\boldsymbol{u} \in B_\rho(\boldsymbol{\xi}_h^c)$

$$\begin{aligned}
|||\boldsymbol{\xi}_h^c - M_h\boldsymbol{u}|||_h &\leq |||\mathcal{L}_h^{-1}R(\boldsymbol{u} - \boldsymbol{\xi})|||_h \\
&\leq C_s|||R(\boldsymbol{u} - \boldsymbol{\xi})|||_{-h} \\
&\leq C_sC_3|||\boldsymbol{u} - \boldsymbol{\xi}|||_h^2 \\
&\leq 2C_sC_3(|||\boldsymbol{u} - \boldsymbol{\xi}_h^c|||_h^2 + |||\boldsymbol{\xi}_h^c - \boldsymbol{\xi}|||_h^2) \\
&\leq 2C_sC_3(\rho^2 + C_4^2h^2||\boldsymbol{\xi}||^2_{(H^3(\Omega))^2}) \\
&\leq 4C_sC_3\rho^2 < \rho.
\end{aligned} \tag{5.66}$$

Furthermore, (5.27), (5.58), and (5.62) yield for $\boldsymbol{u}, \bar{\boldsymbol{u}} \in B_\rho(\boldsymbol{\xi}_h^c)$

$$
\begin{aligned}
\|M_h \boldsymbol{u} - M_h \bar{\boldsymbol{u}}\|_h &= \|\mathcal{L}_h^{-1}\big(R(\bar{\boldsymbol{u}} - \boldsymbol{\xi}) - R(\boldsymbol{u} - \boldsymbol{\xi})\big)\|_h \\
&\le C_s C_3 (\|\bar{\boldsymbol{u}} - \boldsymbol{\xi}\|_h + \|\boldsymbol{u} - \boldsymbol{\xi}\|_h)\|\bar{\boldsymbol{u}} - \boldsymbol{u}\|_h \\
&\le C_s C_3 (\|\bar{\boldsymbol{u}} - \boldsymbol{\xi}_h^c\|_h + \|\boldsymbol{u} - \boldsymbol{\xi}_h^c\|_h + 2\|\boldsymbol{\xi} - \boldsymbol{\xi}_h^c\|_h)\|\boldsymbol{u} - \bar{\boldsymbol{u}}\|_h \\
&\le 2C_s C_3 (\rho + C_4 h ||\boldsymbol{\xi}||_{(H^3(\Omega))^2})\|\boldsymbol{u} - \bar{\boldsymbol{u}}\|_h \\
&\le 4 C_s C_3 \rho \|\boldsymbol{u} - \bar{\boldsymbol{u}}\|_h. \qquad (5.67)
\end{aligned}
$$

As a consequence of (5.66) and (5.67), and (5.63), the restriction of M_h to $B_\rho(\boldsymbol{\xi}_h^c)$ is a contraction mapping from $B_\rho(\boldsymbol{\xi}_h^c)$ into $B_\rho(\boldsymbol{\xi}_h^c)$. Thus, the contraction mapping theorem (see [32]) implies that the iteration $\boldsymbol{\xi}_{n+1} = M_h \boldsymbol{\xi}_n$ with starting point $\boldsymbol{\xi}_0 \in B_\rho(\boldsymbol{\xi}_h^c)$ converges to a unique fixed point $\boldsymbol{\xi}_h \in B_\rho(\boldsymbol{\xi}_h^c)$. As a fixed point of M_h, the function $\boldsymbol{\xi}_h$ is a solution of the discrete von Kármán equation.

To obtain the error estimate (5.64), we restrict M_h further to the ball $B_{\widetilde{\rho}}(\boldsymbol{\xi}_h^c)$ of radius $\widetilde{\rho} = C_4 h ||\boldsymbol{\xi}||_{(H^3(\Omega))^2}$. From (5.65), and(5.63), it follows that

$$
\widetilde{\rho} = C_4 h ||\boldsymbol{\xi}||_{(H^3(\Omega))^2} < \rho < \frac{1}{4 C_s C_3}.
$$

This implies that the estimates (5.67) and (5.66) hold for $\widetilde{\rho}$ as well. Consequently, M_h restricted to the ball $B_{\widetilde{\rho}}(\boldsymbol{\xi}_h^c)$ is also a contraction mapping, and there exists a unique fixed point $\widetilde{\boldsymbol{\xi}}_h$ of M_h in $B_{\widetilde{\rho}}(\boldsymbol{\xi}_h^c)$. Since $B_{\widetilde{\rho}}(\boldsymbol{\xi}_h^c) \subset B_\rho(\boldsymbol{\xi}_h^c)$ and since the fixed point of M_h in $B_\rho(\boldsymbol{\xi}_h^c)$ is unique, we conclude that $\widetilde{\boldsymbol{\xi}}_h = \boldsymbol{\xi}_h$. In particular, this means that $\boldsymbol{\xi}_h \in B_{\widetilde{\rho}}(\boldsymbol{\xi}_h^c)$. Therefore,

$$
\|\boldsymbol{\xi} - \boldsymbol{\xi}_h\|_h \le \|\boldsymbol{\xi} - \boldsymbol{\xi}_h^c\|_h + \|\boldsymbol{\xi}_h^c - \boldsymbol{\xi}_h\|_h \le C_4 h ||\boldsymbol{\xi}||_{(H^3(\Omega))^2} + \widetilde{\rho} = 2C_4 h ||\boldsymbol{\xi}||_{(H^3(\Omega))^2}.
$$

□

Chapter 6
Numerical Examples

In this chapter, we present numerical examples for the C^0 interior penalty methods (5.45) and (5.48) for the von Kármán equations. All our computational results are obtained by the software package COMSOL Multiphysics 3.5a. The arising nonlinear system is solved by COMSOL's standard nonlinear solver, which uses a damped Newton's method [17]. If not stated otherwise, we use zero as initial guess for the Newton's method. The domain Ω is discretized with quadratic Lagrange elements; isoparametric quadratic elements are used along any curved boundaries of Ω.

To investigate if our method converges in the predicted rate, we determine the error of the method in a concrete case. Provided the exact solution $\boldsymbol{\xi} \in \boldsymbol{V}^s$ is known, we can measure the error with a post-processing step in COMSOL in the broken Sobolev seminorm

$$|\boldsymbol{\xi} - \boldsymbol{\xi}_h|_{H^2(\mathcal{T}_h)} = \Big(\sum_{T \in \mathcal{T}_h} |\xi_1 - \xi_{h,1}|^2_{H^2(T)} + |\xi_2 - \xi_{h,2}|^2_{H^2(T)} \Big)^{\frac{1}{2}}.$$

First, we need to find a nontrivial von Kármán problem whose exact solution is known.

6.1 On the Unit Disk

Let $\Omega \subset \mathbb{R}^2$ be the unit disk. Our goal is to construct a von Kármán problem (3.38), whose solution $\boldsymbol{\xi} \in \boldsymbol{V}^s$ is of the form

$$\xi_1 = (1 - x^2 - y^2)^2, \tag{6.1a}$$

$$\xi_2 = a_1(1 - x^2 - y^2)^2 + a_2(1 - x^2 - y^2)^3 + a_3(1 - x^2 - y^2)^4, \tag{6.1b}$$

where a_1, a_2, and a_3 are constants. With this ansatz, we calculate

$$-[\xi_1, \xi_1] = -96(x^2+y^2)^2 + 128(x^2+y^2) - 32,$$
$$\Delta^2 \xi_2 = 2304a_1(x^2+y^2)^2 - (2304a_1 - 576a_2)(x^2+y^2) + 384a_1 + 192a_2 + 64a_3.$$

Hence, $\boldsymbol{\xi} \in \boldsymbol{V}^s$ satisfies $\Delta^2\xi_2 = -[\xi_1, \xi_1]$, if we choose

$$a_1 = -\frac{1}{12}, \quad a_2 = -\frac{1}{18}, \quad a_3 = -\frac{1}{24}. \tag{6.2}$$

By setting $f = \Delta^2\xi_1 - [\xi_1, \xi_2] + p\Delta\xi_1$, we obtain a von Kármán problem whose exact solution is $\boldsymbol{\xi} \in \boldsymbol{V}^s$. After some calculations and algebraic simplifications, we obtain f explicitly

$$f(x,y) = \frac{40}{3}(1-x^2-y^2)^4 + \frac{16}{3}(11+x^2+y^2) + 8p(2x^2+2y^2-1). \tag{6.3}$$

Thus, for the von Kármán problem (3.38) with f in (6.3), we can compare the computational results from C^0 interior penalty methods (5.48) and (5.45) to the exact solution $\boldsymbol{\xi} \in \boldsymbol{V}^s$. Table 6.1 shows the error of the two methods for the parameters with $p = 10$ and $\eta = 10$ with respect to different mesh sizes h (COMSOL code provided in Appendix).

(a) Solution of $\widetilde{G}_h\boldsymbol{\xi}_h = 0$

h	$\lvert\boldsymbol{\xi} - \boldsymbol{\xi}_h\rvert_{H^2(\mathcal{T}_h)}$	rate
1/4	2.037825	
1/8	1.014723	1.0059
1/16	0.491562	1.0456
1/32	0.235502	1.0616
1/64	0.116818	1.0115

(b) Solution of $G_h\boldsymbol{\xi}_h = 0$

h	$\lvert\boldsymbol{\xi} - \boldsymbol{\xi}_h\rvert_{H^2(\mathcal{T}_h)}$	rate
1/4	2.031396	
1/8	1.013469	1.0032
1/16	0.491388	1.0444
1/32	0.235479	1.0613
1/64	0.116815	1.0114

TABLE 6.1. Computed errors and rates of convergence of the C^0 interior penalty methods (5.45) and (5.48) with $\eta = 10$ on the unit circle.

6.2 On a Square and a L-Shaped Domain

We consider the von Kármán problem on the square $\Omega_S = [-0.5, 0.5] \times [-0.5, 0.5]$ and the L-shaped domain Ω_L that is obtained by removing $(0, 0.5] \times (0, 0.5]$ from Ω_S.

Let f be the function defined in (6.3), and $\bar{\boldsymbol{\xi}}$ be the exact solution from the previous section, given in (6.1) and (6.2). Then the restriction of $\bar{\boldsymbol{\xi}}$ to Ω is a solution of the following von Kármán problem.

Find $\boldsymbol{\xi} \in H^2(\Omega) \times H^2(\Omega)$ such that

$$
\begin{aligned}
\Delta^2 \xi_1 &= [\xi_1, \xi_2] + f \text{ in } \Omega,\\
\Delta^2 \xi_2 &= -[\xi_1, \xi_1] \text{ in } \Omega,\\
\xi_1 &= \bar{\xi}_1,\ \xi_2 = \bar{\xi}_2 \text{ on } \partial\Omega,\\
\partial_n \xi_1 &= \partial_n \bar{\xi}_1,\ \partial_n \xi_2 = \partial_n \bar{\xi}_2 \text{ on } \partial\Omega.
\end{aligned}
$$

As in the previous section, we can compare the computed solution with the exact solution $\bar{\boldsymbol{\xi}}$. To capture the nonzero boundary conditions of this problem, we have to modify the C^0 interior penalty methods accordingly. As underlying finite element space, we define

$$
\boldsymbol{V}_h^b = \{\boldsymbol{v} \in \boldsymbol{W}_h \mid \boldsymbol{v} \text{ and } \bar{\boldsymbol{\xi}} \text{ agree on all nodal points } \eta \in \partial\Omega\},
$$

where $\boldsymbol{W}_h$ denotes the $\mathcal{P}_2$ finite element space of $H^1(\Omega) \times H^1(\Omega)$.

Moreover,we have to modify the functionals G and $\widetilde{G}$ by subtracting the term

$$
\sum_{e \in \mathcal{E}_h^b} \int_e \big(\partial_n \bar{\xi}_1 (\frac{\eta}{|e|} \partial_n v_1 - \partial_{nn} v_1) + \partial_n \bar{\xi}_2 (\frac{\eta}{|e|} \partial_n v_2 - \partial_{nn} v_2)\big)\, ds
$$

from $\langle G_h \boldsymbol{w}, \boldsymbol{v} \rangle$ and $\langle \widetilde{G}_h \boldsymbol{w}, \boldsymbol{v} \rangle$ in (5.41) and (5.46). The reason for that is that these terms were added in the derivation (4.19) under the assumption $\bar{\boldsymbol{\xi}} \in (H_0^2(\Omega))^2$.

With these modifications, we obtain the results shown in Table 6.2 for the square Ω_S, and the results shown in Table 6.3 for the L shaped domain Ω_L. All the computations were carried out for $\eta = 10$ and $p = 10$. For reference, the corresponding COMSOL codes are attached in the Appendix.

(a) Solution of $\widetilde{G}_h \boldsymbol{\xi}_h = 0$

h	$\lvert\boldsymbol{\xi} - \boldsymbol{\xi}_h\rvert_{H^2(\mathcal{T}_h)}$	rate
1/8	0.335032	
1/16	0.167126	1.0034
1/32	0.083278	1.0049
1/64	0.041582	1.0020

(b) Solution of $G_h \boldsymbol{\xi}_h = 0$

h	$\lvert\boldsymbol{\xi} - \boldsymbol{\xi}_h\rvert_{H^2(\mathcal{T}_h)}$	rate
1/8	0.335065	
1/16	0.167130	1.0035
1/32	0.083279	1.0050
1/64	0.041582	1.0020

TABLE 6.2. Computed errors and rates of convergence of the C^0 interior penalty methods (5.45) and (5.48) with $\eta = 10$ on the square Ω_S.

(a) Solution of $\widetilde{G}_h \boldsymbol{\xi}_h = 0$

h	$\lvert\boldsymbol{\xi} - \boldsymbol{\xi}_h\rvert_{H^2(\mathcal{T}_h)}$	rate
1/8	0.285172	
1/16	0.144541	0.9804
1/32	0.072036	1.0047
1/64	0.036029	0.9996

(b) Solution of $G_h \boldsymbol{\xi}_h = 0$

h	$\lvert\boldsymbol{\xi} - \boldsymbol{\xi}_h\rvert_{H^2(\mathcal{T}_h)}$	rate
1/8	0.285177	
1/16	0.144542	0.9804
1/32	0.072036	1.0047
1/64	0.036029	0.9996

TABLE 6.3. Computed errors and rates of convergence of the C^0 interior penalty methods (5.45) and (5.48) with $\eta = 10$ on the L-shaped domain Ω_L.

As predicted in the analysis in the previous chapter, we see from the results in Table 6.1, Table 6.2, and Table 6.3 that the method (5.48) converges in the order $O(h)$ on convex domains, and performs still well on a L-shaped domain with re-entrant corner. It is noteworthy that the results of the unmodified method (5.45) do not differ very much - the analysis of this method still remains open for further research.

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Appendix: COMSOL CODES

This is the COMSOL Code that generates the results from Table 6.1(a). To obtain the results for Table 6.1(b), one simply has to set $buu = buv = bvu = 0$ below.

```
% COMSOL Multiphysics Model M-file
clear; flclear fem
% Geometry
g1=circ2('1','base','center','pos',{'0','0'},'rot','0');
s.objs={g1};
fem.draw=struct('s',s);
fem.geom=geomcsg(fem);

% Initialize mesh
for j=2:6
   hvec(j-1)=2^j;
   fem.mesh=meshinit(fem,'hmax',1/2^j);

% Application mode 1
  clear appl;clear equ;clear bnd;
  appl.mode.class = 'FlPDEW';
  appl.assignsuffix = '_w';
  bnd.constrf = 'test(u)';
  bnd.constr = 'u';
  bnd.weak = '(-dnnu+eta/h*dnu)*dnutest-dnu*dnnut';
  bnd.ind = [1,1,1,1];
  equ.dweak = 0;
  equ.bndweak = 'avgu*jputt+avgutt*jpu+eta/h*jpu*jputt-buv-bvu';
  equ.weak = 'd2ud2utest-mongeuv*test(u)-f*test(u)-p*dudutest';
  equ.ind = [1];appl.bnd = bnd;appl.equ = equ;
  fem.appl{1} = appl;

  % Application mode 2
  clear appl;clear bnd;clear equ;
  appl.mode.class = 'FlPDEW';
  appl.dim = {'v','v_t'};
  appl.assignsuffix = '_w2';
  bnd.constrf = 'test(v)';
  bnd.constr = 'v';
  bnd.weak = '(-dnnv+eta/h*dnv)*dnvtest-dnv*dnnvt';
  bnd.ind = [1,1,1,1];
  equ.dweak = 0;
```

```
equ.bndweak = 'avgv*jpvtt+avgvtt*jpv+eta/h*jpv*jpvtt+2*buu';
equ.weak = 'd2vd2vtest+mongeuu*test(v)';
equ.ind = [1];
appl.equ = equ;appl.bnd = bnd; fem.appl{2} = appl;
fem.frame = {'ref'};fem.border = 1;

% Global expressions
fem.globalexpr = {'mongeuv','uxx*vyy+uyy*vxx-2*uxy*vxy', ...
'mongeuu','2*(uxx*uyy-uxy*uxy)', ...
'd2ud2utest','uxx*test(uxx)+2*uyx*test(uyx)+uyy*test(uyy)',...
'd2vd2vtest','vxx*test(vxx)+2*vyx*test(vyx)+vyy*test(vyy)',...
'dudutest','(ux*test(ux)+uy*test(uy))',...
'jpu','jux*dnx+juy*dny', ...
'jpv','(up(vx)-down(vx))*dnx+(up(vy)-down(vy))*dny', ...
'jputt','juxt*dnx+juyt*dny', ...
'jpvtt','jvxt*dnx+jvyt*dny', ...
'avgu','auxx*unx^2+2*auxy*unx*uny+auyy*uny^2', ...
'avgv','avxx*unx^2+2*avxy*unx*uny+avyy*uny^2', ...
'avgutt','auxxt*unx^2+2*auxyt*unx*uny+auyyt*uny^2', ...
'avgvtt','avxxt*unx^2+2*avxyt*unx*uny+avyyt*uny^2', ...
'dnnut','test(uxx)*nx^2+2*test(uxy)*nx*ny+test(uyy)*ny^2',...
'dnnvt','test(vxx)*nx^2+2*test(vxy)*nx*ny+test(vyy)*ny^2',...
'dnu','ux*nx+uy*ny',...
'dnv','vx*nx+vy*ny',...
'dnnu','uxx*nx*nx+2*uxy*nx*ny+uyy*ny*ny', ...
'dnnv','vxx*nx*nx+2*vxy*nx*ny+vyy*ny*ny', ...
'dnutest','test(ux)*nx+test(uy)*ny',...
'dnvtest','test(vx)*nx+test(vy)*ny',...
'buv','((jvx*auyy-jvy*auxy)*dnx+(jvy*auxx-jvx*auxy)*dny)*tu',...
'bvu','((jux*avyy-juy*avxy)*dnx+(juy*avxx-jux*avxy)*dny)*tu',...
'buu','((jux*auyy-juy*auxy)*dnx+(juy*auxx-jux*auxy)*dny)*tv',...
'auxx','0.5*(up(uxx)+down(uxx))',...
'auyy','0.5*(up(uyy)+down(uyy))',...
'auxy','0.5*(up(uxy)+down(uxy))',...
'jux','up(ux)-down(ux)',...
'juy','up(uy)-down(uy)',...
'avxx','0.5*(up(vxx)+down(vxx))',...
'avyy','0.5*(up(vyy)+down(vyy))',...
'avxy','0.5*(up(vxy)+down(vxy))',...
'jvx','up(vx)-down(vx)',...
'jvy','up(vy)-down(vy)',...
'auxxt','0.5*(test(up(uxx))+test(down(uxx)))',...
'auxyt','0.5*(test(up(uxy))+test(down(uxy)))',...
'auyyt','0.5*(test(up(uyy))+test(down(uyy)))',...
```

```
  'avxxt','0.5*(test(up(vxx))+test(down(vxx)))',...
  'avxyt','0.5*(test(up(vxy))+test(down(vxy)))',...
  'avyyt','0.5*(test(up(vyy))+test(down(vyy)))',...
  'juxt','test(up(ux))-test(down(ux))',...
  'juyt','test(up(uy))-test(down(uy))',...
  'jvxt','test(up(vx))-test(down(vx))',...
  'jvyt','test(up(vy))-test(down(vy))',...
  'tu','test(u)',...
  'tv','test(v)',...
  'f','(40*r*r*r*r+16*(11+x^2+y^2)+24*p*(2*x^2+2*y^2-1))/3',...
  'p','10', ...
  'eta','10', ...
  'r','1-x^2-y^2', ...
  'xi1','(1-x^2-y^2)^2', ...
  'xi1xx','4*y^2+12*x^2-4',...
  'xi1yy','12*y^2+4*x^2-4',...
  'h2errorxi1','(xi1xx-uxx)^2+2*(8*x*y-uxy)^2+(xi1yy-uyy)^2',...
  'xi2','(-6*xi1-4*(1-x^2-y^2)^3-3*(1-x^2-y^2)^4)/72',...
  'xi2xx','(-2*x^2+(1-4*x^2)*r+(1-6*x^2)*xi1+r*r*r)/3',...
  'xi2yy','(-2*y^2+(1-4*y^2)*r+(1-6*y^2)*xi1+r*r*r)/3',...
  'xi2xy','(-2*x*y-4*x*y*r-6*x*y*xi1)/3',...
  'h2errorxi2','(xi2xx-vxx)^2+2*(xi2xy-vxy)^2+(xi2yy-vyy)^2', ...
  'h2semierror','(h2errorxi1+h2errorxi2)'};

  % Multiphysics
  fem=multiphysics(fem);
  fem.xmesh=meshextend(fem);

  % Solve problem
  fem.sol=femstatic(fem,'solcomp',{'v','u'},...
                        'outcomp',{'v','u'},'blocksize','auto');

  % Integrate error
   error(j-1)=sqrt(postint(fem,'h2semierror','unit','',...
   'recover','off','dl',1));
   if j>2
      rate(j-1)=log(error(j-2)/error(j-1))/log(2);
   end
end
%Output
errormatrix=[hvec;error;rate];
errormatrix'
```

Next we append the COMSOL code for the method with nonzero boundary conditions that produces the results in Table 6.2 for the square.

```
% COMSOL Multiphysics Model M-file
clear; flclear fem
% Geometry
g1=square2('1','base','center','pos',{'0','0'},'rot','0');
s.objs={g1};
fem.draw=struct('s',s);
fem.geom=geomcsg(fem);

% Initialize mesh
for j=2:6
   hvec(j-1)=2^j;
   fem.mesh=meshinit(fem,'hmax',1/2^j);

% Application mode 1
  clear appl;clear equ;clear bnd;
  appl.mode.class = 'FlPDEW';
  appl.assignsuffix = '_w';
  bnd.constrf = 'test(u-xi1)';
  bnd.constr = 'u-xi1';
  bnd.weak ='(-dnnu+eta/h*dnu)*dnutest-dnu*dnnut-xi1n*bndut';
  bnd.ind = [4,2,3,1];
  equ.dweak = 0;
  equ.bndweak = 'avgu*jputt+avgutt*jpu+eta/h*jpu*jputt-buv-bvu';
  equ.weak = 'd2ud2utest-mongeuv*test(u)-f*test(u)-p*dudutest';
  equ.ind = [1];appl.bnd = bnd;appl.equ = equ;
  fem.appl{1} = appl;

  % Application mode 2
  clear appl;clear bnd;clear equ;
  appl.mode.class = 'FlPDEW';
  appl.dim = {'v','v_t'};
  appl.assignsuffix = '_w2';
  bnd.constrf = 'test(v-xi2)';
  bnd.constr = 'v-xi2';
  bnd.weak = '(-dnnv+eta/h*dnv)*dnvtest-dnv*dnnvt-xi2n*bndvt';
  bnd.ind = [4,2,3,1];
  equ.dweak = 0;
  equ.bndweak = 'avgv*jpvtt+avgvtt*jpv+eta/h*jpv*jpvtt+2*buu';
  equ.weak = 'd2vd2vtest+mongeuu*test(v)';
  equ.ind = [1];
  appl.equ = equ;appl.bnd = bnd; fem.appl{2} = appl;
```

```
fem.frame = {'ref'};fem.border = 1;

% Global expressions
fem.globalexpr = {'mongeuv','uxx*vyy+uyy*vxx-2*uxy*vxy', ...
'mongeuu','2*(uxx*uyy-uxy*uxy)', ...
'd2ud2utest','uxx*test(uxx)+2*uyx*test(uyx)+uyy*test(uyy)',...
'd2vd2vtest','vxx*test(vxx)+2*vyx*test(vyx)+vyy*test(vyy)',...
'dudutest','(ux*test(ux)+uy*test(uy))',...
'bndut','eta/h*dnutest-dnnut',...
'bndvt','eta/h*dnvtest-dnnvt',...
'jpu','jux*dnx+juy*dny', ...
'jpv','(up(vx)-down(vx))*dnx+(up(vy)-down(vy))*dny', ...
'jputt','juxt*dnx+juyt*dny', ...
'jpvtt','jvxt*dnx+jvyt*dny', ...
'avgu','auxx*unx^2+2*auxy*unx*uny+auyy*uny^2', ...
'avgv','avxx*unx^2+2*avxy*unx*uny+avyy*uny^2', ...
'avgutt','auxxt*unx^2+2*auxyt*unx*uny+auyyt*uny^2', ...
'avgvtt','avxxt*unx^2+2*avxyt*unx*uny+avyyt*uny^2', ...
'dnnut','test(uxx)*nx^2+2*test(uxy)*nx*ny+test(uyy)*ny^2',...
'dnnvt','test(vxx)*nx^2+2*test(vxy)*nx*ny+test(vyy)*ny^2',...
'dnu','ux*nx+uy*ny',...
'dnv','vx*nx+vy*ny',...
'dnnu','uxx*nx*nx+2*uxy*nx*ny+uyy*ny*ny', ...
'dnnv','vxx*nx*nx+2*vxy*nx*ny+vyy*ny*ny', ...
'dnutest','test(ux)*nx+test(uy)*ny',...
'dnvtest','test(vx)*nx+test(vy)*ny',...
'buv','((jvx*auyy-jvy*auxy)*dnx+(jvy*auxx-jvx*auxy)*dny)*tu',...
'bvu','((jux*avyy-juy*avxy)*dnx+(juy*avxx-jux*avxy)*dny)*tu',...
'buu','((jux*auyy-juy*auxy)*dnx+(juy*auxx-jux*auxy)*dny)*tv',...
'auxx','0.5*(up(uxx)+down(uxx))',...
'auyy','0.5*(up(uyy)+down(uyy))',...
'auxy','0.5*(up(uxy)+down(uxy))',...
'jux','up(ux)-down(ux)',...
'juy','up(uy)-down(uy)',...
'avxx','0.5*(up(vxx)+down(vxx))',...
'avyy','0.5*(up(vyy)+down(vyy))',...
'avxy','0.5*(up(vxy)+down(vxy))',...
'jvx','up(vx)-down(vx)',...
'jvy','up(vy)-down(vy)',...
'auxxt','0.5*(test(up(uxx))+test(down(uxx)))',...
'auxyt','0.5*(test(up(uxy))+test(down(uxy)))',...
'auyyt','0.5*(test(up(uyy))+test(down(uyy)))',...
'avxxt','0.5*(test(up(vxx))+test(down(vxx)))',...
'avxyt','0.5*(test(up(vxy))+test(down(vxy)))',...
```

```
  'avyyt','0.5*(test(up(vyy))+test(down(vyy)))',...
  'juxt','test(up(ux))-test(down(ux))',...
  'juyt','test(up(uy))-test(down(uy))',...
  'jvxt','test(up(vx))-test(down(vx))',...
  'jvyt','test(up(vy))-test(down(vy))',...
  'tu','test(u)',...
  'tv','test(v)',...
  'f','(40*r*r*r*r+16*(11+x^2+y^2)+24*p*(2*x^2+2*y^2-1))/3',...
  'p','10', ...
  'eta','10', ...
  'r','1-x^2-y^2', ...
  'xi1','(1-x^2-y^2)^2', ...
  'xi1x','-4*x*(1-x^2-y^2)', ...
  'xi1y','-4*y*(1-x^2-y^2)', ...
  'xi1n','xi1x*nx+xi1y*ny',...
  'xi1xx','4*y^2+12*x^2-4',...
  'xi1yy','12*y^2+4*x^2-4',...
  'h2errorxi1','(xi1xx-uxx)^2+2*(8*x*y-uxy)^2+(xi1yy-uyy)^2',...
  'xi2','(-6*xi1-4*(1-x^2-y^2)^3-3*(1-x^2-y^2)^4)/72',...
  'xi2x','1/3*x*r*(2-x^2-y^2+(1-x^2-y^2)^2)', ...
  'xi2y','1/3*y*r*(2-x^2-y^2+(1-x^2-y^2)^2)',...
  'xi2n','xi2x*nx+xi2y*ny',...
  'xi2xx','(-2*x^2+(1-4*x^2)*r+(1-6*x^2)*xi1+r*r*r)/3',...
  'xi2yy','(-2*y^2+(1-4*y^2)*r+(1-6*y^2)*xi1+r*r*r)/3',...
  'xi2xy','(-2*x*y-4*x*y*r-6*x*y*xi1)/3',...
  'h2errorxi2','(xi2xx-vxx)^2+2*(xi2xy-vxy)^2+(xi2yy-vyy)^2', ...
  'h2semierror','(h2errorxi1+h2errorxi2)'};

  % Multiphysics
  fem=multiphysics(fem);
  fem.xmesh=meshextend(fem);

  % Solve problem
  fem.sol=femstatic(fem,'solcomp',{'v','u'},...
                     'outcomp',{'v','u'},'blocksize','auto');

  % Integrate error
   error(j-1)=sqrt(postint(fem,'h2semierror','unit','',...
   'recover','off','dl',1));
   if j>2
      rate(j-1)=log(error(j-2)/error(j-1))/log(2);
   end
end
```

```
%Output
errormatrix=[hvec;error;rate];
errormatrix'
```

The results for the L-shaped domain in Table 6.3 can be obtained with the previous code by changing the geometry only. That is, one has to replace the line

```
g1=square2('1','base','center','pos',{'0','0'},'rot','0');
```

by the lines

```
g2=rect2('0.5','1','base','corner','pos',{'-0.5','-0.5'},'rot','0');
g3=square2('0.5','base','corner','pos',{'0','-0.5'},'rot','0');
g1=geomcomp({g2,g3},'ns',{'SQ1','R1'},'sf','SQ1+R1','edge','all');
```

and

```
bnd.ind=[4,2,3,1];
```

by

```
bnd.ind =  [2,3,4,3,1,4,1];
```

Vita

Armin Reiser was born in 1981 in Tübingen, Germany. He began his studies in mathematics and physics at the University of Tübingen in 2002. After an exchange year at Louisiana State University in 2007, he joined its graduate program there, deepening his studies in mathematics. In December 2007, he earned a Master of Science degree in mathematics at Louisiana State University; and in 2008, he completed the Diplom degree in mathematics and the Erstes Staatsexamen degree in mathematics and physics at the University of Tübingen. Currently, he is a candidate for the degree of Doctor of Philosophy in mathematics at Louisiana State University, which he expects to be awarded in December 2011.