2015

Topological Dynamics On Compact Phase Spaces

Lieth Abdalateef Majed
Louisiana State University and Agricultural and Mechanical College, liethen84@yahoo.com

TOPOLOGICAL DYNAMICS ON COMPACT PHASE SPACES

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
Lieth Abdalateef Majed
B.S. in Math., Al-mustansiriya university, 1998
M.S. in Math., Al-mustansiriya university, 2001
M.S.,Western Michigan University, 2010
M.S., Louisiana State University, 2013
December 2015

Acknowledgments

I would like to express my gratitude to my thesis advisor profesor. Amha Lisan for his ideas and insight as well as his patience in numerous and long discussions. I am also thankful to my topology professors at Louisiana State University, specifically Professors Richard A. Litherland and Pallavi Dani. I would like to thank the Mathematics Department at Louisiana State University for providing me an excellent scholarly environment and to my father and mother who inspire me to work hard. A very special thank to my friends for the support they have lent me over all these years. Finally, I would like to thank my wife Ghada. She was always there cheering me up and stood by me through the good times and bad. No words can adequately express how grateful I am to have and have had each of you in my life.

Table of Contents

Abstract

Our main focus will be to investigate the various facets of what are commonly called dynamical systems or flows, which are triples (S, X, π), where X is a compact Hausdorff space and $\pi : S \times X \longrightarrow X$ is a separately continuous action of a semigroup S on X.

Historically, as was introduced by R.Ellis 1960, the enveloping semigroup, which is a closure of the set of continuous functions on a compact space X, was discovered to be an important tool to study dynamical systems. Soon, a realization of the existence of a universal compactification of a phase semigroup with an extended homomorphism onto the enveloping semigroup lead to an alternate approach to study these systems via this compactification. The importance of this alternative approach in this respect derives from the fact that the dynamical and many topological properties of S can be translated into properties of its compactification.

In Chapter one we will present a brief summary of notations and basic results from topological algebra as well as some basic information on the Stone-Čech Compactification, βS.

In Chapter two we will expand on some of the work in chapter one and recall the necessary background from topological dynamics. We will define the enveloping semigroup and review some of the well known results concerning its structure. Fundamental and well known theorems which lead to the assertion as to the existence of a universal system will be presented. Utilizing this universal property we will further explain how all other dynamical systems arise as quotients of this universal system via suitable closed left congruences.

In Chapter three, we concentrate on the special case where the phase semigroup is the set of natural numbers N under addition and treat its compactification as

the set of ultrafilters on N and the extended action as that of βN. In so doing we will present results which relate notions of proximality and almost periodicity in a dynamical system to combinatorially rich central subsets of N. In lieu of an appendix we have also included in this chapter a rather deeper exposition of an example in symbolic dynamics arising from an action of N on the product space $X = \prod_{i=1}^{\infty}\{0,1\}$, which is isomorphic to a variety of other dynamic mappings, like the quadratic map on the cantor set which has significant applications in data storage and transmission, linear algebra and many other areas [5].

Chapter 1
Introduction

In this chapter, we give background on semigroups which will be used in the thesis. We refer the reader to [2] and [12].

Definition 1.1. *A semigroup is a pair $(S, *)$ where S is non-empty set and $*$ is a binary associative operation on S.*

Formally a binary operation on S is a function $* : S \times S \longrightarrow S$ and the operation is associative if $(x * y) * z = x * (y * z)$ for all x, y and z in S. We say S is closed under $*$ if $x * y \in S$ whenever $x, y \in S$.

Example 1.2. *(a) The set of natural numbers $\mathbb{N}$ under addition or multiplication is a semigroup.*

(b) $(S, *)$ where S is a non empty set and $x * y = y$ for all $x, y \in S$.

(c) $(S, \vee)$ where $x \vee y = max\{x, y\}$.

Definition 1.3. *Let S be a semigroup,*

(a) S is commutative if xy=yx for all $x, y \in S$.

(b) The center of S is $\{x \in S$:for all $y \in S, xy = yx\}$.

(c) An element $x \in S$ is right cancellable (respectively left) if whenever $y, z \in S$ and $yx = zx$ (respectively, $xy = yz$) one has $y = z$.

(d) S is right (respectively left) cancellative if and only if every $x \in S$ is (respectively left) cancellable.

(e) S is cancellative if S is both left cancellative and right cancellative.

In a semigroup S, an element z is called a zero element if $z * s = s * z = z$, for all $s \in S$.

Example 1.4. *(a) The set of positive integers under addition is a cancellative semigroup.*

(b) A left zero semigroup is right cancellative but not left cancellative.

(c) A right zero semigroup is left cancellative but not right cancellative.

Definition 1.5. *Let S be a semigroup,*

(a) An element $x \in S$ is an idempotent if and only if $xx = x$.

(b) $E(S) = \{x \in S : x \text{ is an idempotent}\}$.

(c) Let $e \in E(S)$. Then $H(e) = \cup\{G : G \text{ is a subgroup of } S \text{ and } e \in G\}$.

Example 1.6. *(a) The only idempotent in $(N, \cdot)$ is 1.*

(b) The set of idempotents of the left zero semigroup is $E(S) = S$.

Theorem 1.7. *Let S be a semigroup and let $e \in E(S)$. Then $H(e)$ is the largest subgroup of S with e as identity.*

Proof. We want to show $H(e)$ is a group. Note we have e is an identity for $H(e)$ and $H(e)$ contains every group with e as an identity. Also every $x \in H(e)$, there exist $G \in H(e)$ such that $x \in G$ which implies x has an inverse. The only remaining thing to show is that $H(e)$ is closed. Let $x, y \in H(e)$ and let G_1 and G_2 be a subgroups of S with $e \in G_1 \cap G_2$, $x \in G_1$ and $y \in G_2$. Let $G = \{\prod_{i=1}^{n} x_i : n \in N$ and $\{x_1, x_2, ...x_n\} \subseteq G_1 \cup G_2\}$. Thus $xy \in G$ and $e \in G$. To show that G is a group, the only requirement is to show the existence of inverses. So let $\prod_{i=1}^{n} x_i \in G$. For $i \in \{1, 2, ...n\}$, pick y_i such that $x_{n+1-i} y_i = e$ [Note that $\prod_{i=1}^{n} x_i = \prod_{i=n}^{1} x_{n+1-i}$]. Then from definition of G, $\prod_{i=1}^{n} y_i \in G$ and $(\prod_{i=1}^{n} x_i)(\prod_{i=1}^{n} y_i) = e$. The group $H(e)$ referred to as maximal groups. Given any group $G \subseteq S$, G has an identity e and $G \subseteq H(e)$. □

Definition 1.8. *Let S be a semigroup and let L, R and I be a nonempty subset of S.Then* *(a) L is a left ideal of S if and only if $SL \subseteq L$.*

(b) R is a right ideal of S if and only if $RS \subseteq R$.

(c) I is an ideal of S if and only if I is both a left and right ideal of S.

Definition 1.9. *Let S be a semigroup, L is left ideal of S, and R is right ideal of S. Then*

(a) L is a minimal left ideal (respectively R minimal right) of S if and only if L(respectively, R) is a left (respectively, right) ideal of S and whenever J is a left (respectively, right) ideal of S and $J \subseteq L$(respectively $J \subseteq R$) one has $J = L$ (respectively $J = R$).

(b) S is left simple (respectively, right simple) if and only if S is a minimal left (respectively minimal right) ideal of S .

(c) S is simple if and only if the only ideal of S is S.

Remark 1.10. *If S is simple, then the only ideal of S is S itself.*

Example 1.11. *(a) Semigroups with a zero has only one minimal left (right-two sided) ideal of S namely the trivial one $\{0\}$.*

(b) $(Z, +)$ has no minimal ideal.

Lemma 1.12. *Let S be a semigroup.*

(a) Let L_1 and L_2 be left ideals of S. Then $L_1 \bigcap L_2$ is a left ideal of S if and only if $L_1 \bigcap L_2 \neq \emptyset$.

(b) Let L be a left ideal of S and let R be a right ideal of S. Then $L \bigcap R$ is non empty set.

Proof. Statement (a) by definition of left ideal is immediate. To see (b), let $x \in L$ and $y \in R$. Because $x \in L$ then $yx \in L$, and also because $y \in R$ then $yx \in R$. Thus $L \cap R \neq \emptyset$. □

Lemma 1.13. *Let S be a semigroup.*

(a) *Let $x \in S$. Then xS is a right ideal, Sx is a left ideal and SxS is an ideal.*

(b) *Let $e \in E(S)$. Then e is a left identity for eS, a right identity for Se, and an identity for eSe.*

Proof. Statement (a) is immediate. For (b), let $e \in E(S)$. To see that e is a left identity for eS, let $x \in eS$ and pick $t \in S$ such that $x = et$. Then $ex = eet = et = x$. Likewise e is a right identity for Se. $\square$

Theorem 1.14. *Let S be a semigroup.*

(a) *If S is left simple and $e \in E(S)$, then e is a right identity for S.*

(b) *If L is a left ideal of S and $t \in L$, then $St \subseteq L$*

(c) *Let $\emptyset \neq L \subseteq S$. Then L is a minimal left ideal of S if and only if for each $t \in L$, $St = L$.*

Proof. (a) By Lemma 1.13(a), Se is a left ideal of S. Since S is a left simple then $Se = S$. Since $e \in E(S)$, by Lemma 1.13(b), then e is a right identity for S.

(b) This follows immediately from the definition of left ideal.

(c) Suppose that L is a minimal left ideal of S and $t \in L$. By Lemma 1.13(a), St is a left ideal and by (b) $St \subseteq L$. Since L is minimal, hence $St = L$. Conversely, since $St = L$ for some $t \in L$ and St is a left ideal, it follows that L is a left ideal. Let J be a left ideal of S with $J \subseteq L$ and pick $r \in J$. Then by (b), $Sr \subseteq J$. Thus $J \subseteq L = Sr \subseteq J$. Hence L is minimal. $\square$

Definition 1.15. *Let S be a semigroup. Then e is a minimal idempotent if and only if $e \in E(S)$ and $e = ef = fe$ for all $f \in E(S)$.*

There is a corresponding "right" version of the following well known theorem. We will omit the routine proof and refer the reader to [12].

Theorem 1.16. *Let S be a semigroup and let $e \in E(S)$.*

(a) If e is a member of some minimal left ideal (equivalently if Se is a minimal left ideal), then e is a minimal idempotent.

(b) If S is simple and e is minimal, then Se is a minimal left ideal.

(c) If every left ideal of S contains an idempotent and e is minimal, then Se is a minimal left ideal.

(d) If S is simple or every left ideal of S has an idempotent then the following statements are equivalent:

(i) e is minimal.

(ii) e is a member of some minimal left ideal of S.

(iii) Se is a minimal left ideal of S.

Proof. See [12], Theorem 1.38. □

Lemma 1.17. *Let S be a semigroup, let L be a left ideal of S, and let T be a left ideal of L.*

(a) For all $t \in T$, Lt is a left ideal of S and $Lt \subseteq T$

(b) If L is a minimal left ideal of S, then $T = L$.(So minimal left ideals are left simple.)

(c) If T is a minimal left ideal of L, then T is a minimal left ideal of S.

Of course, the right-left switch of this statement also holds.

Proof. (a) Note that $S(Lt) = (SL)t \subseteq Lt$ and since T is a left ideal of L then $Lt \subseteq Lt \subseteq T$.

(b) Pick any $t \in T$. By (a), Lt is a left ideal of S and $Lt \subseteq T \subseteq L$, so $Lt = L$ so $T = L$.

(c) To see that T is a left ideal of S, pick any $t \in T$. By (a), Lt is a left ideal of S and Lt is a left ideal of L. Since $Lt \subseteq T$ and T is a minimal left ideal of L then

$Lt = T$. Therefore, $ST = S(Lt) = (SL)t \subseteq Lt = T$. Hence T is a left ideal of S. To see that T is minimal in S, let J be a left ideal of S with $J \subseteq T$. Then J is a left ideal of L, so $J = T$. □

We now see that all minimal left ideals of a semigroup are intimately connected with each other.

Lemma 1.18. *Let S be a semigroup , let I be an ideal of S, and let L be a minimal left ideal of S. Then $L \subseteq I$.*

Proof. Since $L, I \neq \emptyset$ then $S(L \cap I) \neq \emptyset$. Also $S(L \cap I) \subseteq L \cap I$. Hence $L \cap I$ is a left ideal. But L is a minimal left ideal, implies $L \cap I = L$. Therefore $L \subseteq I$. □

Theorem 1.19. *Let S be a semigroup, let L be a minimal left ideal of S, and let $T \subseteq S$. Then T is a minimal left ideal of S if and only if there is some $a \in S$ such that $T = La$.*

Proof. Suppose T is a minimal left ideal of S. Since $SLa \subseteq La$ then La is left ideal. Let $a \in T$ then $La \subseteq T$. But T is minimal and hence $La = T$. Conversely, let $a \in S$ then $La \subseteq L$ which is a left ideal of S. Since L is a minimal left ideal then $La = L$. Thus La is a minimal left ideal and thus T is a minimal left ideal. □

Corollary 1.20. *Let S be a semigroup. If S has a minimal left ideal, then every left ideal of S contains a minimal left ideal.*

Proof. Let L be a minimal left ideal of S and let J be a left ideal of S. Pick $a \in J$. Then by above Theorem 1.19, La is a minimal left ideal which is contained in J. □

Lemma 1.21. *Let S be a semigroup and let K be an ideal of S. If K is minimal in $\{J{:}J$ is an ideal of $S\}$ and I is an ideal of S, then $K \subseteq I$.*

Proof. To show that $K \subseteq I$. Note that $I \cap K \neq \emptyset$ which is an ideal of S, also $I \cap K \subseteq I$ and $I \cap K \subseteq K$. Since K is a minimal then $I \cap K = K$ and thus $K \subseteq I$. □

Definition 1.22. *Let S be a semigroup. If S has a smallest ideal, then $K(S)$ is the smallest ideal.*

Theorem 1.23. *Let S be a semigroup. If S has a minimal left ideal, then $K(S)$ exists and $K(S) = \cup\{L : L$ is a minimal left ideal of $S\}$.*

Proof. Let $I = \cup\{L :$ is a minimal left ideal of $S\}$. First we will show I is a minimal ideal. Let $L \in I$ be a minimal left ideal. Let J is any ideal of S, then by Lemma 1.12, $J \cap L \neq \emptyset$. Let $x \in J \cap L$ and $s \in S$, then $sx \in L$ and J. So $J \cap L$ is a left ideal and a subset of minimal left ideal L. Therefore $J \cap L = L$. Hence $I \subseteq J$, which implies that I is the smallest. So it suffices to show that I is an ideal of S. We have that $I \neq \emptyset$ by assumption. Let $x \in I$. Pick a minimal left ideal L such that $x \in L$. Then $sx \in L \subseteq I$, for all $s \in S$. So I is left ideal. Also by Theorem 1.19, Ls is a minimal left ideal of S so $Ls \subseteq I$ while $xs \in Ls$. □

Lemma 1.24. *Let S be a semigroup.*

(a) *Let L be a left ideal of S. Then L is minimal if and only if $Lx = L$ for every $x \in L$.*

(b) *Let I be an ideal of S.Then I is the smallest ideal if and only if $IxI = I$ for every $x \in I$.*

Proof. (a) If L is a minimal and $x \in L$, then Lx is a left ideal of S and $Lx \subseteq L$ so $Lx = L$. Now assume $Lx = L$ for every $x \in L$ and let J be a left ideal of S with $J \subseteq L$. Pick $x \in J$. Then $L = Lx \subseteq LJ \subseteq J \subseteq L$.

(b) If I is smallest ideal, then $I \subseteq IxI$. Since $x \in I$, then $IxI \subseteq I$. Therefore $I = IxI$. $\square$

The proof of the following result is well-known and may be found in [12].

Theorem 1.25. *Let S be a semigroup, then*

(a) If L is a minimal left ideal of S and R is a minimal right ideal of S, then $K(S) = LR$.

(b) If there is a minimal left ideal of S which has an idempotent. Then every minimal left ideal has an idempotent.

(c) If there is a minimal left ideal of S which has an idempotent. Then there is a minimal right ideal of S which has an idempotent.

Theorem 1.26. *Let S be a semigroup and assume that there is a minimal left ideal of S which has an idempotent, and let $e \in E(S)$. Then the folowing are equivalent:*

(a) Se is a minimal left ideal.

(b) Se is left simple.

(c) eSe is a group.

(d) $eSe = H(e)$.

(e) $e \in K(S)$.

(f) $K(S) = SeS$.

Proof. (a) implies (b). Fallows directly from Lemma 1.17 part (b).

(b) implies (c). Trivially eSe is closed. By Lemma 1.13 e is a two sided identity for eSe. Let $x = ese \in eSe$. One has $x \in Se$, so Sx is a left ideal of Se. Since Se is simple, then $Sx = Se$. Thus $e \in Sx$. let $e = yx$ for some $y \in S$. Then $eye \in eSe$ and $eyex = eyx = ee = e$. Thus x has a left inverse in eSe. For the right inverse, start with $xeye = x(eyex)eye = (xeye)(xeye)$. Since eSe has closed binary operation, and $xeye \in ese$ has a left inverse say r; apply it to both side. Then $e = xeye$.

(c) implies (d). Since eSe is a group and $e \in eSe$, then $eSe \subseteq H(e)$. On the other hand by Theorem 1.7, e is the identity of $H(e)$. So given $x \in H(e)$, then $x = exe \in eSe$. Hence $H(e) \subseteq eSe$.

(d) implies (a). Let L be a left ideal of S with $L \subseteq Se$ and let $t \in L$. Then $t \in Se$, so $et \in eSe$. pick $x \in eSe$ such that $x(et) = e$. Then $xt = (xe)t = x(et) = e$. Thus $e \in L$, so $Se \subseteq SL \subseteq L$.

(a) implies (e). Since $K(S) = \cup\{L : L \text{ is a minimal left ideal of } S\}$. This complete the proof.

(e) implies (f). Since SeS is an ideal and and $K(S)$ is the smallest ideal then $K(S) \subseteq eSe$. But $e \in K(S)$. Thus $SeS \subseteq K(S)$.

(e) implies (a). This follows from the next Theorem. □

Theorem 1.27. *Let S be a semigroup and assume that there is a minimal left ideal of S which has an idempotent. Let $T \subseteq S$.*

(a) T is a minimal left ideal of S if and only if there is some $e \in E(K(S))$ such that $T = Se$.

(b) T is a minimal right ideal of S if and only if there is some $e \in E(K(S))$ such that $T = eS$.

Proof. (a) Let L be a minimal left ideal of S and an idempotent $f \in L$. Suppose that T is a minimal left ideal. Since Sf is a left ideal contained in L, $Sf = L$. By Theorem 1.26, fSf is a group. Let $a \in T$, then $faf \in fSf$. Pick $x \in fSf$ such that $x(faf) = f$. Then $xaxa = (xf)a(fx)a = (xfaf)xa = fxa = xa$. Hence xa is an idempotent. Also $xa \in T$ while by Theorem 1.23, $T \subseteq K(S)$. So $xa \in E(K(S))$. Finally, since Sxa is a left ideal contained in T, so $T = Sxa$. Conversely, suppose that $e \in K(S)$, by Theorem 1.23, pick a minimal left ideal I of S with $e \in I$. Then by Theorem 1.14 part (c), $Se = I$.

(b) Since S has a minimal left ideal which has an idempotent. By Theorem 1.25, S has a minimal right ideal which has an idempotent. Thus the proof follows by a left-right switch. $\square$

Theorem 1.28. *Let S be a semigroup and assume that there is a minimal left ideal of S which has an idempotent. Given any minimal left ideal L of S and any minimal right ideal R of S, there is an idempotent $e \in R \cap L$ such that $R \cap L = RL = eSe$ and eSe is a group.*

Proof. Let R and L be given. Pick by Theorem 1.27 an idempotent $f \in K(S)$ such that $L = Sf$. By Theorem 1.26, we get fSf is a group. Let $a \in R$ and let x be the inverse of faf in $fSf \subseteq Sf$. Then $x \in Sf = L$, since $fS \subseteq S$ then $fSf \subseteq Sf$ so $ax \in R \cap L$. By Theorem 1.23, $ax \in K(S)$. Also

$$\begin{aligned} axax =& a(xf)a(fx) \\ =& a(xfaf)x \\ =& afx \\ =& ax. \end{aligned} \tag{1.1}$$

Let $e = ax$. Then $eSe \subseteq Sx \subseteq L$ and $eSe \subseteq aS \subseteq R$ so $eSe \subseteq R \cap L$. To see that $R \cap L \subseteq eSe$. Let $b \in R \cap L$. By Theorem 1.14, $L = Se$ and $R = eS$ so by Lemma 1.2, $b = eb = be$. Thus $b = eb = ebe \in eSe$. Now $RL = eSSe \subseteq eSe \subseteq RL$, so $RL = eSe$. As we see $e \in K(S)$, so by Theorem 1.26, eSe is a group. $\square$

We are intrested in the relationship of a minimal left (right) ideal with other minimal left (right) ideals. The following result can be found in [12].

Theorem 1.29. *Let S be a semigroup and assume that there is a minimal left ideal of S which has an idempotent.Then all minimal left ideals of S are isomorphic.*

Proof. See Theorem 1.64 [12] □

The next theorem gives us the structure of the smallest ideal of any semigroup which has a minimal left ideal with an idempotent. The following result can be found in [12].

Theorem 1.30. *Let S be a semigroup and assume that there is a minimal left ideal of S which has an idempotent. Let R be a minimal right ideal of S, let L be a minimal left ideal of S, let $X = E(L)$, let $Y = E(R)$, and let $G = RL$. Define an operation . on $X \times G \times Y$ by $(x, g, y).(x_1, g_1, y_1) = (x, gyx_1g_1, y_1)$. Then*

(a) The minimal right ideals of S partition $K(S)$ and the minimal left ideals of S partion $K(S)$.

(b) The Maximal group in $K(S)$ partion $K(S)$.

(c) All minimal right ideals of S are isomorphic and all minimal left ideals of S are isomorphic.

(d) All maximal groups in $K(S)$ are isomorphic.

Remark 1.31. *All topological spaces we will be considering here after involve Hausdorff spaces.*

Definition 1.32. *Let S be a semigroup. For $s, t \in S$ with product st we define*

(a) Given $x \in S$, the function $\lambda_x : S \longrightarrow S$ is defined by $\lambda_x(y) = xy$

(b) Given $x \in S$, the function $\rho_x : S \longrightarrow S$ is defined by $\rho_x(y) = yx$

and we write $\rho_x(y) = yx = \lambda_x(y)$; the functions ρ_x and λ_x on S are called right and left translations respectively.

Definition 1.33. *(a) A right topological semigroup is a triple $(S, +, T)$ where $(S, +)$ is a semigroup, (S, T) is a topological space, and for all $x \in S$, $\rho_x : S \longrightarrow S$ is continuous.*

(b) A left topological semigroup is a triple $(S, +, T)$ where $(S, +)$ is a semigroup, (S, T) is a topological space, and for all $x \in S$, $\lambda_x : S \longrightarrow S$ is continuous.

(c) A semitopological semigroup is a right topological semigroup which is also a left topological semigroup.

(d) A topological semigroup is a triple $(S, +, T)$ where $(S, +)$ is a semigroup, (S, T) is a topological space, and $+ : S \times S \longrightarrow S$ is continuous.

(e) A topological group is a triple $(S, +, T)$ such that $(S, +)$ is a group, (S, T) is a topological space, $+ : S \times S \longrightarrow S$ is continuous, and $In : S \longrightarrow S$ is continuous (where In(x) is the inverse of $x \in S$).

Definition 1.34. *Let S be a right topological semigroup. The topological center of S is the set $\wedge(S) = \{x \in S : \lambda_x$ is continuous $\}$.*

Theorem 1.35. *Let S be a compact right topological semigroup. Then $E(S) \neq \emptyset$.*

Proof. Let $\mathscr{R} = \{T \subseteq S : T \neq \emptyset, T$ is compact and $T.T \subseteq T\}$ is the set of compact subsemigroups of S. By using Zorn's Lemma we will show $\mathscr{R}$ has a minimal element. Since $S \in \mathscr{R}$, $\mathscr{R} \neq \emptyset$. Let $\mathscr{C}$ be a chain in $\mathscr{R}$. Since S is Hausdorff space then $\mathscr{C}$ is a collection of closed subsets from the compact space S. Hence it has finite intersection property, $\cap\mathscr{C} \neq \emptyset$ which is a compact set since the intersection of closed set is closed set subset of a Hausdorff space. Thus $\cap\mathscr{C} \in \mathscr{R}$. So by Zorn's Lemma we may pick a minimal member A of $\mathscr{R}$.

Let $x \in A$. We will show that $x.x = x$. We start by showing that $Ax = A$. Let $B = Ax$. Then $B \neq \emptyset$ and since $B = \rho_x[A]$, then B is the continuous image of a compact space, hence is compact. Also $BB = AxAx \subseteq AAAx \subseteq Ax = B$, thus

$B \in \mathscr{R}$. Since $B = Ax \subseteq AA \subseteq A$ and A is minimal, then $B = A$. Let $C = \{y \in A : yx = x\}$. Since $x \in A = Ax$, we have $C \neq \emptyset$. Also, since $A = A \cap \rho_x^{-1}[\{x\}]$, so C is closed and hence compact. Now given $y, z \in C$ one has $yz \in AA \subseteq A$ and $yzx = yx = x$ so $yz \in C$. Thus $C \in \mathscr{R}$. Since $C \subseteq A$ and A is minimal, we have $C = A$ so $x \in C$ and so $xx = x$ as required. □

Corollary 1.36. *Let S be a compact right topological semigroup. Then every left ideal of S contains a minimal left ideal. Minimal left ideals are closed, and each minimal left ideal has an idempotent.*

Proof. If L is any left ideal of S and $x \in L$. since we have Hausdorff space then $Sx = \rho_x(S)$ is a compact left ideal contained in L. It follows any minimal left ideal is closed and by Theorem 1.35, any minimal left ideal contains an idempotent. We need to show that any left ideal of S contains a minimal left ideal. So let L be a left ideal of S and let $\mathscr{A} = \{T : T \text{ is a closed left ideal of } S \text{ and } T \subseteq L\}$ which is partially ordered set by inclusion. $\mathscr{A} \neq \emptyset$. Applying Zorn's Lemma, $\mathscr{A}$ has a minimal left ideals M. Since M is a minimal among left closed ideals contained in L. But since every left ideal contains a closed left ideal, M is a minimal left ideal. □

Theorem 1.37. *Let S be a compact right topological semigroup.*

(a) *All maximal subgroups of $K(S)$ are (algebraically) isomorphic.*

(b) *Maximal subgroups of $K(S)$ which lie in the same minimal right ideal are topologically and algebraically isomorphic. In fact if R is a minimal right ideal of S and $e, f \in E(R)$, then the restriction of ρ_f to eSe is an isomorphism and a homeomorphism onto fSf.*

(c) *All minimal left ideals of S are homeomophic. In fact, if L and J are minimal left ideals of S and $z \in J$, then $\rho_{z|L}$ is a homeomorphism from L onto J.*

Definition 1.38. *Let X be a discrete topological space. A Stone-Čech Compactification of X is a pair (φ,Z) such that:*

(a) Z is a compact space,

(b) φ is an embedding of X into Z,

(c) $\varphi[X]$ is dense in Z, and

(d) given any compact space Y and any continuous function $f : X \longrightarrow Y$ there exists a continuous function $g : Z \longrightarrow Y$ such that $g \circ \varphi = f$.

$$\begin{tikzcd}
Z \arrow[dr, "g"] & \\
X \arrow[u, "\varphi"] \arrow[r, "f"] & Y
\end{tikzcd}$$

Remark 1.39. *(a) Any two Stone-Čech Compactification of the same topological space X are homeomorphic.*

(b) The topology induced on X as a subset of Z is the original topology of X.

Definition 1.40. *Let D be any set. A filter on D is a nonempty set μ of subsets of D with the following properties:*

(a) If $A, B \in \mu$,then $A \cap B \in \mu$.

(b) If $A \in \mu$ and $A \subseteq B \subseteq D$, then $B \in \mu$.

(c) $\emptyset \notin \mu$

The following are some examples of filters.

Example 1.41. *The set of neighborhoods of a point in a topological space.*

Example 1.42. *In topological space, the open neighborhoods of a point form a filter base for the filter of neighborhoods of that point.*

Definition 1.43. *An ultrafilter on S is a filter on S which is not properly contained in any other filter on S.*

Definition 1.44. *Let D be a set and let U be a filter on D. A family $\mathscr{A}$ is a filter base for U if and only if $\mathscr{A} \subseteq U$ and for each $B \in U$ there is some $C \in \mathscr{A}$ such that $C \subseteq B$.*

Definition 1.45. *Let S be a topological space and let βS=$\{P$:P is an ultrafilter on S $\}$ the set of ultrafilters on S.*

Remark 1.46. *If S is a discrete topological space βS will be the Stone-Čech Compactification of S.*

Definition 1.47. *Let S be a topological space we define the set.*

(a) Given $A \subseteq S$, $\widehat{A}$=$\{$ $p \in \beta S : A \in P$ $\}$

(b) Let $a \in S$. then $e(a)$=$\{$ $A \subseteq S$:$a \in A$ $\}$

Remark 1.48. *(a) In the topological space βD we shall be thinking of ultrafilters as points.*

(b) For each $a \in D$, $e(a)$ is the Principal ultrafilter corresponding to a.

Remark 1.49. *We define the topology of βS to be the topology generated by the sets $\widehat{A}$ as a basis. βS with this topology is the the Stone-Čech Compactification of S.*

The following Theorem will tell us what $g(p)$ is in terms of the underlining set X.

Theorem 1.50. *Let βD be the Stone-Čech Compactification of the discrete set D. Then given any compact Hausdorff space Y and any continuous function $f : X \longrightarrow Y$ there exists a continuous function $g : \beta D \longrightarrow Y$ such that $g \circ \varphi = f$.*

$$\begin{tikzcd}
\beta D \arrow[dr, "g"] & \\
X \arrow[u, "\varphi"] \arrow[r, "f"'] & Y
\end{tikzcd}$$

Proof. Let Y be a compact space and let $f : X \longrightarrow Y$. For each $p \in \beta D$ let $\mathscr{A}_p = \{cl_Y f[a] : A \in p\}$. We claim $\mathscr{A}_p$ has finite intersection property. Let $A_1, A_2, ..., A_n \in p$ then $\cap_{i=1}^{n} \neq \emptyset$. Hence $\emptyset \neq f(\cap_{i=1}^{n} A_i) \subseteq \cap_{i=1}^{n} f(A_i) \subseteq \overline{\cap_{i=1}^{n} f(A_i)}$. Thereore $\mathscr{A}_p$ has a non empty intersection. Choose $g(p) \in \cap \mathscr{A}_p$. We need to show that the digram commutes and that g is continuous. Let $x \in D$ Then $\{x\} \in \varphi(x) = \{A \subseteq D : x \in A\}$ and so $g(\varphi(x)) \in cl_Y f[\{x\}] = cl_Y[\{f(x)\}] = \{f(x)\}$. Hence $g \circ \varphi = f$. To see that g is continuous. Let $p \in \beta D$ and let U be a neighborhood of $g(p)$ in Y. Since Y is compact Hausdorff space then y is normal implies Y is regular. Pick a neighborhood V of $g(p)$ with $cl_Y V \subseteq U$ and let $A = f^{-1}[V]$. We claim that $A \in p$. Suppose that $D \backslash A \in p$. Then $g(p) \in cl_Y f[D \backslash A]$ and V is a neighborhood of $g(p)$ so $V \cap f[D \backslash A] \neq \emptyset$ which is a contradicting the fact that $A = f^{-1}[V]$. Thus $A \in p$ and so $p \in \hat{A}$. We claim that $g[\hat{A}] \subseteq U$. Let $q \in \hat{A}$ and suppose that $g(p) \notin U$. Then $Y \backslash cl_Y V$ is a neighborhood of $g(q)$ and $g(q) \in cl_Y f[A]$. Thus $(Y \backslash cl_Y V) \cap f[A] \neq \emptyset$ which is a contradicting the fact that $A = f^{-1}[V]$. $\square$

Remark 1.51. *Given a discrete semigroup $(S, \cdot)$, one can extend the operation $\cdot$ to βS, the Stone-Čech Compactification of S. So that $(\beta S, \cdot)$ is a right topological semigroup (i.e for each $p \in \beta S$,the function $\rho_p : \beta S \longrightarrow \beta S$, defined by $\rho_p(q) = q \cdot p$ is continuous) with S contained in the topological center (i.e for each $x \in S$, the function $\lambda_p : \beta S \longrightarrow \beta S$ defined by $\lambda_x(p) = x \cdot p$ is continuous). (It is also if S is not discrete, such an extension may not be possible).*

Definition 1.52. *Let $(S,+)$ be a semigroup, for any $A \subseteq S$ and $x \in S$ we define $-x + A = \{s \in S : x + s \in A\}$. Given any two ultrafilters $p, q \in \beta S$ we define their sum by $p + q = \{A \subset S | \{x \in S | -x + A \in q\} \in p\}$.*

The following next two Lemmas are well known results.

Lemma 1.53. *Every left ideal in βS contains a minimal left ideal.*

Proof. First we know from the structure of βS that we have a left ideal since $\beta S+p$ is a left ideal $\forall p \in \beta S$. Let $r \in \beta S$, then $\beta S + r \subseteq \beta S + p$ which are both closed and compact left ideal since $\rho_r(\beta S) = \beta S + p$ is compact and since every compact subset of T_2- space is closed. We will try to show $\beta S + r$ is a minimal left ideal. Consider $\Gamma = \{\beta S + r_i$ is a closed left ideals of βS and $\beta S + \grave{r} \subseteq \beta S + p\}$. Then $\Gamma \neq \emptyset$ since we have $\beta S + r$. Also Γ is partially ordered by inclusion $\{T_1 \subseteq T_2$ then $T_1 \leq T_2\}$. Let $C = \{\beta S + r_1 \supseteq \beta S + r_2 \supseteq ...\}$ be a chain. Note that by finitely intersection property $\bigcap_i \beta S + r_i$ is a closed left ideal and $\neq \emptyset$. So C has lower bounded $M = \cap \beta S + r_i$. By Zorn's lemma Γ has a minimal left ideal $\beta S + M$ among left closed ideals contained in $\beta S + p$.

We need to show $\beta S+M$ is minimal left ideal for all space. So if $\grave{L} \subseteq \beta S+M$ and $\grave{L}$ is a left ideal. By corollary 1.36 we know every left ideal contains a closed left ideal, thus there exist $L_1 \subseteq \grave{L}$ such that L_1 is a closed left ideal. Now $L_1 \subseteq \grave{L} \subseteq M$, and hence $L = M$. □

Lemma 1.54. *Let S be a semigroup then $K(\beta S)$ is a smallest ideal*

Proof. From previous lemma we show that βS has a minimal left ideal, so $K(\beta S) \neq \emptyset$. Let $p \in K(\beta S)$. Then there exist a minimal left ideal $L \in K(\beta S)$ such that $p \in L$. Then $\beta S + p \subseteq L \subseteq K(\beta S)$ and thus $K(\beta S)$ is a left ideal. Similarly $K(\beta S)$ is a right ideal. Therefore $K(\beta S)$ is an ideal. Now, to show $K(\beta S)$ is smallest ideal we will show first $K(\beta S)$ is a minimal ideal. Let M be an ideal such that $M \subseteq K(\beta S)$. To show $M = K(\beta S)$. Let L be any minimal left ideal $\subseteq K(\beta S)$, then $M \cap L \neq \emptyset$. We claim $M \cap L$ is a left ideal in βS, let $x \in M \cap L$ and since $\beta S + x \subseteq M$ and $\beta S + x \subseteq L$ then $\beta S + x \subseteq M \cap L$. So $M \cap L$ is a left ideal. Since So $M \cap L \subseteq L$ and by minimality of L then $M \cap L = L$. Therefore $L \subseteq M$ and hence $M = K(\beta S)$is a minimal ideal.

Now, for showing $K(\beta S)$ is the smallest one. Let I be an ideal in $K(\beta S)$. We know $K(\beta S) \cap I \neq \emptyset$, then $K(\beta S) \cap I \subseteq K(\beta S)$. Let $x \in K(\beta S) \cap I$ then $\beta S + x \in K(\beta S) \cap I$ and hence $K(\beta S) \cap I$ is an ideal. Therefore $K(\beta S) \cap I = K(\beta S)$. So $K(\beta S) \subseteq I$. □

Remark 1.55. *Note that in* $(\beta N, +)$ *there are infinitely many minimal left ideals and minimal right ideals.*

Theorem 1.56. *Let* D *be any set.*

(a) βD *is compact Hausdorff space.*

(b) $D \subseteq \beta D$

(c) D *is dense in* βD

Proof. (a) Suppose p and q be two distinct ultrafilter of βD. Let $A \in p \backslash q$ then $A^c = D \backslash A \in q$. So $\widehat{A}$ and $\widehat{D \backslash A}$ are disjoint open subsets of βD containing p and q respectively. Hence βD is T_2 space.

To show βD is compact, we will show that a family μ ={The sets of the form $\widehat{A}$ with the finite intersection property } has non-empty intersection. Let γ= $\{A \subseteq D : \widehat{A} \in \mu\}$. If $F \in \rho_f(\gamma)$=$\{F : \emptyset \neq F \subseteq \gamma$ and F is finite$\}$. From definition for μ there is some $p \in \cap_{A \in F} \widehat{A}$ and by definition of $\widehat{A}$ we get $\cap F \in p$. Thus $\cap F \neq \emptyset$ and hence γ has finite intersection property. Hence there exist an ultrafilter $q \in \beta D$ such that $\gamma \subseteq q$ {see theorem 3.8 [6] }, and so $q \in \cap \mu$. Thus βD is compact.

(b) This is clear since we identify the points of D with the principle ultrafilters generated by those points.

(C) We will try to show if p is ultrafilter in βD is a limit point of D if every neighborhood of p contains at least one point of D different from p itself. Let U be a neighborhood of p, so there is a basic open subset of $\widehat{A}$ of βD for which $p \in \widehat{A} \subseteq U$. Then by definition of $\widehat{A}$ we have $\emptyset \neq A$ and any $a \in A$ satisfy $e(a) = a \in D \cap \widehat{A}$. So $D \cap \widehat{A} \neq \emptyset$ which mean D has its limit point. □

Remark 1.57. *(a) βD has a smallest ideal.*

(b) The sets of the form $\widehat{A}$ are the clopen of βD.

(c) For every $A \subseteq D$, $\widehat{A}= c\ell_{\beta D}\ e[A]$.

Remark 1.58. *(a) βD is not commutative.*

(b) βD is a right topological semigroup.

(c) The center of βD in the algebraic sense is $e(D)$.

(d) βD is contains idempotents.

We are particularly interested in the semigroup $(\beta N, +)$, as an extension of $(N, +)$. This is intrinsically interesting as being a natural extension of $(N, +)$, and the sense of Definition 1.38, the largest possible extension. It also has important applications to combinatorial number theory and topological dynamics.

Chapter 2

Dynamical Systems and Decomposing Left Congruences

2.1 The Enveloping Semigroup

Most of the result of this section can be found in [1], [12] and [14] and we will include them here because they will lead us to the important results in Theorem 2.9 and Theorem 2.10.

Definition 2.1. *Let S be a semitopological semigroup and X be a compact Hausdorff space. An S-flow or alternately S-system is a triple (S, X, π) such that $\pi : S \times X \longrightarrow X$ is an action of S on X which is separately continuous. We denote $\pi(s, p)$ by sp; to say that π is an action means that $(st)p = s(tp)$ for all $s, t \in S$. Also if S has an identity e then the identity acts as the identity mapping on X (i.e $e.p = p$ for all $p \in X$).*

Definition 2.2. *A homomorphism from an S-system X to an S-system Y is a continuous function $f : X \longrightarrow Y$ satisfying $f(sx) = sf(x)$ for all $s \in S$, $x \in X$. If additionally f is a homeomorphism from X onto Y, then it is called an isomorphism of S-systems.*

Definition 2.3. *Let X denote an S-system. For $x \in X$, we define the orbit of x to be the set $Sx = \{sx : s \in S\}$ and the orbit closure $\Gamma_{(x)}$ to be the topological closure of Sx.*

Definition 2.4. *The S-system X is is called point transitive if there exists an $x \in X$ such that $\{x\} \cup \Gamma_{(x)}$ is all of X. If we choose some distinguished point $p \in X$, then we call the pair (X, p) a pointed S-system; the point p is referred to as the base point. In that case the homomorphism also preserves the base points.*

Definition 2.5. *A right topological monoidal compactification of S (where S has an identity, otherwise one can attach an identity to S) is a pair (T, f) such that .*

(a) T is compact Hausdorff right topological semigroup with identity.

(b) f is a continuous homomorphism from S into T.

(c) $f(S) \cup \{e\}$ is dense in T, where e is the identity of T.

(d) f carries the identity of S to the identity of T.

(e) $f(S) \subseteq \wedge(T)$, where $\wedge(T) = \{x \in T : \lambda_x$ is continuous $\}$.

One motivation to introduce and to study monoidal compactifications on S is because of a close relationship that exists between these compactifications and S-flows. These connections allow one to study many aspects of the theory of S-flows from an algebraic point of view.

By standard compactification arguments (see Theorem 1.50) there exists a universal right topological compactification $(\beta S, j)$ of S, which is characterized by the universal property that if $f : S \longrightarrow T$ is a right topological compactification of S, then there exists a unique continuous homomorphism F from βS to T such that $f = Foj$. That is the following diagram is commutative.

$$\begin{tikzcd}
\beta(S) \arrow[dr, dotted, "\exists ! F"] & \\
S \arrow[u, "j"] \arrow[r, "f"] & T
\end{tikzcd}$$

Theorem 2.6. *Let (X, p) be a transitive pointed S-system and let (Y, q) be a pointed S-system. Then there exists at most one homomorphism from (X, p) to (Y, q). If $f : (X, p) \longrightarrow (Y, q)$ and $g : (Y, q) \longrightarrow (X, p)$ are homomorphisms of transitive pointed S-systems, then f and g are inverse homeomorphisms and hence S-system isomorphisms.*

Proof. Suppose $f, h : (X, p) \longrightarrow (Y, q)$ be homomorphisms of S-systems. Then for each $s \in S$,

$$\begin{aligned} f(sp) =& sf(p) \\ =& sq \\ =& sh(p) \\ =& h(sp) \end{aligned} \tag{2.1}$$

so we have f and h, two continuous functions which agree on the set $\{p\} \cup \{Sp\}$. Thus f and g agree on the dense set $\{p\} \cup \{Sp\}$ of X. Therefore f and h agree on X.

Now, suppose $f : (X, p) \longrightarrow (Y, q)$ and $g : (Y, q) \longrightarrow (X, p)$ are homomorphisms, if we can show that the composition fog and the identity map I are a homomorphisms then by the first part above, we get $fog = I$. Note that for $r \in X$

$$\begin{aligned} fog(sr) =& f(g(sr)) \\ =& sf(g(r)) \\ =& s(fog)(r) \end{aligned} \tag{2.2}$$

So fog is a homomorphism and since $I(sr) = sr = sI(r)$, I is also homomorphism. Thus by the first part of Theorem there exists at most one homomorphism and hence $fog = I_x$, similarly $gof = I_y$. This means f and g are inverse each other, and hence f is an isomorphism. Thus f is an S-system isomorphisim. □

Definition 2.7. *Let X be a compact T_2-space, and let $T = \{f : X \longrightarrow X|\ f$ is continuous $\}$ which is a semigroup under composition. We form the Ellis semigroup $\Sigma(T) = c\ell(T)$, by taking the closure of T in the Cartesian product X^X endowed with the topology of pointwise convergence. The enveloping semigroup is the Ellis semigroup with the identity mapping on X adjoined to it .*

The following theorem will help us to prove out result in Theorem 2.8.

Theorem 2.8. *Let (X,T) be any topological space and let V be the product topology on X^X. (a) $(X^X, \circ, V)$ is a right topological semigroup, where $\circ$ is composition of functions.*

(b) For each $f \in X^X$, λ_f is continuous iff f is continuous, where $\lambda_f(g) = f \circ g = \rho_g(f)$

Proof. (a) Let $f \in X^X$ and suppose that the net $\langle g_i \rangle_{i \in I}$ convergence to g in the product topology of X^X. Since a net $\langle f_i \rangle_{i \in I}$ convergence to f in X^X if and only if $\langle f_i(y) \rangle_{i \in I}$ convergence to $f(y)$ for every $y \in X$, we get $\langle g_i(f(x)) \rangle_{i \in I}$ convergence to $g(f(x))$ in X. Therefore $\langle g_i \circ f \rangle_{i \in I}$ convergence to $g \circ f$ in X^X. Hence $\langle g_i \circ f \rangle$ is continuous , which means ρ_f is continuous. Therefore X^X is a right topological semigroup.

(b) Suppose that $f \in X^X$ is continuous. Then for a given net $\langle g_i \rangle_{i \in I}$ which is convergent to g in (X^X, V) we get $\langle f(g_i(x)) \rangle_{i \in I}$ converging to $f(g(x))$. This implies that $\langle f \circ g_i(x) \rangle_{i \in I}$ convergence to $(f \circ g)x$. Therefore $f \circ g_i$ is continuous which means that λ_f is continuous. Conversely, suppose that for each $f \in X^X$, λ_f is continuous. Let $\langle x_i \rangle_{i \in I}$ be a net converging to x in X. Then $\lambda_f(x_i) \longrightarrow \lambda_f(x) = f(x)$. Similarly, $\lambda_f(x_i) = f(x_i) \longrightarrow \lambda_f(x)$. This prove that f is continuous. □

By using Theorem 2.8, we will prove the following theorem [which is posed in [12]. This theorem will allow us to prove that the enveloping semigroup is a semigroup.

Theorem 2.9. *Let S be a right topological semigroup, and let T be a subset of the topological center of S. Then $c\ell(T)$ is a semigroup if T is a semigroup.*

Proof. Let $x, y \in c\ell(T)$. We want to show that $xy \in c\ell(T)$. Let U be any neighborhood of xy. We need to show that $U \cap T \neq \phi$. By Theorem 2.8, since ρ_y is continuous, there exists a neighborhood V of x such that $\rho_y(V) = Vy \subseteq U$. Since $x \in c\ell(T)$ then $V \cap T \neq \phi$. Let $x_1 \in V \cap T$ then $\lambda_{x_1}(y) = x_1 y = \rho_y(x_1) \in U$. Since

$x_1 \in T$ which is a topological center of S, λ_{x_1} is continuous. Therefore there is a neighborhood W of y with $\lambda_{x_1}(W) \subseteq U$. Since $y \in c\ell(T)$ then $W \cap T \neq \phi$. Let $y_1 \in W \cap T$. Then we have $\lambda_{x_1}(y_1) \in \lambda_{x_1}(W) \subseteq U$ then $x_1 y_1 = \lambda_{x_1}(y_1) \in U$ and since S is a semigroup then $x_1 y_1 \in T$. Therefore $U \cap T \neq \phi$ and so $xy \in c\ell(T)$. $\square$

Theorem 2.10. *The enveloping semigroup $\Sigma(T)$ is a compact right topological semigroup.*

Proof. By Theorem 2.8, X^X is a right topological semigroup and the set of all continuous functions T is a topological center of X^X. Then by Theorem 2.9, $\Sigma(T)$ is a semigroup. Since X is compact, by Tychonoff's theorem $X \times X \times X...$ is compact. Since $T \subseteq X \times X \times X...$ and $\Sigma(T)$ is closed subset of compact Hausdorff space then $\Sigma(T) = c\ell(T) \subseteq X \times X \times X...$ is compact Hausdorff space. We need to show that ρ_f is continuous on $\Sigma(T)$. Let $f \in \Sigma(T)$, we need to show ρ_f is continuous on $\Sigma(T)$. To do that we will first show ρ_f is continuous on T. Let $\langle g_i \rangle_{i \in I}$ be a net on T convergence to g. Since g is a pointwise convergence, then for any $x \in X$, $g_n(x) \longrightarrow g(x)$. Hence $g_n(f(x)) \longrightarrow g(f(x))$. Therefore $\rho_f(g_n)$ convergence pointwise to $\rho_f(g)$. Therefore ρ_f is continuous on the dense set T and hence on its closure $\Sigma(T)$. $\square$

Theorem 2.8 and Theorem 2.10 show that the mapping $S \to \Sigma(T)$ defined by for $s \in S$, π^s the continuous mapping $x \longmapsto sx : X \longrightarrow X$ is a monoidal compactification of S, which is sometimes called the Ellis compactification of S for the S-system X.

2.2 Relation between the Enveloping Semigroup and Dynamical Systems

In this section we will be looking at some results from [15] showing the relationship between the enveloping semigroup and dynamical systems. In all of our work we will attach an identity to $(S, +)$.

The results that follow will allow us to associate a left congruence on βS to any pointed flow (X, p). Furthermore they will allow us to set up a one-to-one correspondence between isomorphism classes of transitive pointed S-flows and closed left congruences on βS.

Definition 2.11. *Let (T_1, j_1) and (T_2, j_2) are monidal compactifications of S. A homomorphism of compactifications from T_1 to T_2 is a continuous-identity preserving homomorphism $\beta : T_1 \longrightarrow T_2$ such that $\beta \circ j_1 = j_2$.*

Remark 2.12. *If $f : (X, p) \longrightarrow (Y, q)$ and $g : (Y, q) \longrightarrow (X, p)$ are homomorphism of transitive pointed S-systems, then f and g are inverse homeomorphisms of each other and hence S-system isomorphisim.*

The next few results are from [7] showing the relationship between the enveloping semigroup and the dynamical system.

Remark 2.13. *Let S be a topological semigroup, and let (T, j) be a monoidal compactification. We may view T as an S-system, where the action is defined by sending (s, t) to $j(s)t$. We will call this flow $(s, t) \longmapsto j(s)t$ on T the associated flow.*

Lemma 2.14. *Let (T_1, j_1) and (T_2, j_2) be monidal compactifications. Then $\beta : (T_1, j_1) \longrightarrow (T_2, j_2)$ is a homomorphism of compactifications if and only if β is a homomorphism of pointed S-systems.*

Proof. Suppose that $\beta : (T_1, j_1) \longrightarrow (T_2, j_2)$ is a homomorphism of compactifications. Thus by Definition 2.11, $\beta \circ j_1 = j_2$. Hence $\beta(sx) = \beta(j_1(s)x) = \beta(j_1(s))\beta(x) = j_2(s)\beta(x) = s\beta(x)$. Therefore β is a homomorphism of pointed S-systems.

$$\begin{tikzcd} T_1 \arrow[dr, "j_2"] & \\ S \arrow[u, "j_1"] \arrow[r, "\beta"] & T_2 \end{tikzcd}$$

Conversely, if β is homomorphsim of pointed S-systems, then for $s \in S$ and $x \in T_1$ we get $\beta(j_1(s)x) = \beta(sx) = s\beta(x) = j_2(s)\beta(x)$ ....$(*)$.

Letting $x = 1$, we found $\beta \circ j_1 = j_2$. thus $\beta \circ j_1(s) = j_2(s)$ for all $s \in S$. Let $x, y \in T_1$. We want to show $\beta(yx) = \beta(y)\beta(x)$. Since $y \in T_1$, then there exist a net $\{j_1(s_\alpha)\}$ converging to y. Note that $j_1(s_\alpha)x$ converge to yx by right continuity of T_1. Thus $\beta(j_1(s_\alpha)x)$ converging to $\beta(yx)$. On the other hand

$$\begin{aligned} j_1(s_\alpha)x &= j_2(s_\alpha)\beta(x) \quad \text{ by } (*) \text{ above} \\ &= \beta(j_2(s_\alpha))\beta(x) \end{aligned} \tag{2.3}$$

Since β is continuous then $\beta(j_1(s_\alpha))$ convergence to $\beta(y)$. By right continuity of T_2, $\beta(j_1(s_\alpha))\beta(x)$ converging to $\beta(y)\beta(x)$. Therefore $\beta(yx) = \beta(y)\beta(x)$. $\square$

Remark 2.15. *Let (S, X, π) be an S-system. For each $s \in S$, let π^s denote the continuous mapping $x \longmapsto sx : X \longrightarrow X$. We form the Ellis semigroup by taking the closure of $S(\pi) = \{\pi^s : s \in S\}$ in the cartesian product X^X endowed with the product topology. Alternately this is just the closure of $S(\pi)$ in the space of all functions from X to X endowed with the topology of pointwise convergence.*

Lemma 2.16. *Let (S, X, π) be an S-system. By Remark 2.13, we can consider the Ellis compactification $\Sigma(X)$ as an S-system.*

(a) For $p \in X$, the mapping $f \longmapsto f(p) : \Sigma(X) \longrightarrow X$ is a homomorphsim of S-systems, and a homomorphsim of pointed S-systems from $(\Sigma(X), I_X)$ to (X, p) where I_X is the identity function.

(b) *If* (T, j) *is a monoidal compactification of* S*, then the homomorphism of part (a) from* $(\Sigma(T), 1_T)$ *to the associated flow* $(T, 1)$ *is an isomorphism, where* 1 *is the identity of* T*.*

Proof. For (a). Let $\phi : \Sigma(X) \longrightarrow X$ which is defined by $\phi(f) = f(p)$ for given $p \in X$, then $\phi(sf) = (sf)(p) = sf(p) = s\phi(f)$ for all $s \in S$ and $f \in \Sigma(X)$. Also $\phi(Id_x) = I_X(p) = p$. Therefore ϕ is a homomorphism pointed S-systems.

For part (b). By definition of the action of S on the set of functions T^T, we have $\pi^s = \lambda_{j(s)}$ for $s \in S$. Hence $\Sigma(T)$ is the closure of $S(\pi) = \{\lambda_{j(s)}\}$. We claim that $\{\lambda_t : t \in T\}$ is compact set in T^T. It is suffices to show $\{\lambda_t : t \in T\}$ is closed set. Suppose that a net λ_{t_α} converge to f. Since $t_\alpha \in T$ and T is compact space then t_α converge to t for some $t \in T$. Note that $\lambda_{t_\alpha}(x) = t_\alpha x = \rho_x(t_\alpha)$. Since ρ_x is continues we have $\rho_x(t_\alpha) \longrightarrow \rho_x(t) = tx = \lambda_t(x)$. Hence $\lambda_{t_\alpha} \longrightarrow \lambda_t = f$, which means the set $\{\lambda_t : t \in T\}$ contains its limits. Hence its a closed set. Since $\{\lambda_s : s \in S\} \subseteq \{\lambda_t : t \in T\}$, $\Sigma(T) = \overline{\{\lambda_{j(s)} : s \in S\}} \subseteq \ \{\lambda_t : t \in T\}$. On the other hand let $t \in T$. Since $j(S)$ is dense in T, there exist a net $j(s_\alpha) \longrightarrow t$. Thus $\lambda j_{(s_\alpha)}(x) = j(s_\alpha)(x) = \rho_x j(s_\alpha)$. Since ρ_x is continuous, so $\rho_x(j(s_\alpha) \longrightarrow \rho_x(t) = tx = \lambda_t(x)$. Hence $\{\lambda_t : t \in T\} \subseteq \ \Sigma(T) = \overline{\{\lambda_j(s) : s \in S\}}$. The fact that T is a monoid and part (a) of the theorem also ensure that the identity of the enveloping semigroup go to the identity of T. Therefore $\Sigma(T) = \{\lambda_t : t \in T\}$. Define $\psi : \Sigma(T) \longrightarrow T$ by $\psi(\lambda_t) = t$. It suffices to show ψ is an isomorphisim. Since

$$
\begin{aligned}
\psi(s\lambda_t) &= \psi(\lambda_s\lambda_t) \\
&= psi(\lambda_{st}) \\
&= st \\
&= s\psi(\lambda_t).
\end{aligned}
\tag{2.4}
$$

Hence ψ is a homomorphisim. It clear ψ is onto, and since $\ker\{\psi\} = I_T$, then ψ is 1-1. Thus ψ is an isomorphisim. By Theorem 2.6 since the homomorphisim between two pointed systems are unique then $\psi = \phi$. Therefore ϕ is an isomorphisim. $\square$

Theorem 2.17. *Let (S, X) and (S, Y) be flows, and let $\theta : X \longrightarrow Y$ be a surjective homomorphisim. Then there exists a unique homomorphisim of Ellis compactifications $\psi : \Sigma(X) \longrightarrow \Sigma(Y)$, which is given by $\psi(\alpha) = \theta(\alpha(x))$, where $x \in X$ satisfies $\theta(x) = y$. Thus Σ is a functor from the category of transitive pointed S systems to the category of S-monoidal compactifications.*

Proof. See [2] Proposition 1.6.7. $\square$

Theorem 2.18. *Let (X, p) be a pointed S-flow, and let $\sigma : \Sigma(X) \longrightarrow X$ be the flow homomorphism $t \longmapsto tp$ of Lemma 2.16. Then σ is an isomorphism of flows if and only if (X, p) is isomorphic to the flow arising from some monoidal compactification (T, j).*

Proof. Suppose that σ is an isomorphisim of flows. Then (X, p) is isomorphic to the Ellis compactification. Conversely, suppose that (X, p) is isomorphic to some monidal compactification (T, j). By Theorem 2.17, there exist an isomorphisim $\psi : \Sigma(X) \longrightarrow \Sigma(Y)$ defined as in Theorem 2.17. Hence (X, p) is isomorphic to its Ellis compactification $(\Sigma(T), I_X)$. $\square$

Note for a flow X, we will to consider the possibility of extending the action of the set S on X to its monoidal compactification of βS.

Definition 2.19. *Suppose that X is an S-system and (T, j) is a monoidal compactification of S. We say that the action of S on X extends to T if there exists an action $(t, x) \longmapsto tx$ from $T \times X \longrightarrow X$ satisfying that* *(a) the action is right continuous ($t \longmapsto tx : T \longrightarrow X$ is continuous for all $x \in X$),*

(b) the identity of T acts as the identity mapping on X,

(c) the action of T extends the action of S in the sense that $sx = j(s)x$ for all $s \in S, x \in X$, where the left-hand side of the equality is the action of S on X and the right-hand side is the action of T on X.

Theorem 2.20. *Let (S, X, π) be an S-system where π is separately continuous.*

(a) The continuous π action of (S, π) on X extends uniquely to an action $\widehat{\pi}$ of βS on X such that the mapping $t \mapsto tx : \beta S \to X$ is continuous for each $x \in X$,i.e, the action $\widehat{\pi}$ is right continuous.

(b) For each $t \in \beta S$, let $\tilde{t} : X \to X$ be defined by $\tilde{t}(x) = tx$. Then the function $\phi : \beta S \longrightarrow X^X$ defined by $t \mapsto \tilde{t}$ from in to is a continuous homomorphism onto the enveloping semigroup of Σ of $\{\tilde{n} : n \in S\}$.

Proof. (a) We will show first the action $\pi : S \times X \longrightarrow X$ can be extended to the action $\widehat{\pi} : \beta S \times X \longrightarrow X$.

$$\begin{tikzcd}
\beta S \times X \arrow[dr, "\widehat{\pi}"] & \\
S \times X \arrow[u, "j \times id"] \arrow[r, "\pi"] & X
\end{tikzcd}$$

Let $K = \{\tilde{n} : n \in S\}$ where $\tilde{n} : X \longrightarrow X$ defined by $\tilde{n}(x) = nx$. Note that $\tilde{n}$ is a continuous since it is defined like the separately continuous action π. Let $T = \Sigma K$. By Theorem 2.10, $T = \Sigma K$ is a compact right topological semigroup.

Let ψ be a continuous mapping, $\psi : S \to T$ defined by $n \mapsto \tilde{n}$. Since T is compact and ψ is continuous, thus by the universal property of βS (See Definition 1.38) there exists a continuous homomorphism $\phi : \beta S \to T$ such that $\psi(n) = \tilde{n} = \phi(j(n))$

$$
\begin{tikzcd}
\beta S \arrow[dr, "\phi"] & \\
S \arrow[u, "j"] \arrow[r, "\psi"] & T
\end{tikzcd}
$$

for all $n \in S$ (That says the diagram above commutes). Moreover, βS is compact and T is Hausdorff, $\overline{\phi(j(S))} = \phi(\overline{j(S)}) = \phi(\beta S)$. On the other hand from the commutative digram above and the density of K in T, $\overline{\phi(j(S)} = \overline{\psi(S)} = \overline{K} = T$. Therefore $\phi(\beta S) = T$ which means ϕ is onto. Define $\widehat{\pi}(t, x) = tx = \phi(t)(x)$ where $t \in \beta S$. Note that $\widehat{\pi}$ is right continuous. It is also straightforward to verify that this gives an action $\widehat{\pi}(nt, x) = \phi(nt)(x)$ which implies

$$
\begin{aligned}
&= \phi(n)\phi(t)(x) \\
&= \phi(n)(tx) \\
&= (n)(tx) \\
&= \widehat{\pi}(n, tx).
\end{aligned}
\tag{2.5}
$$

The uniqueness it will come since $j(S)$ is dense in $\beta S \backslash \{e\}$.

(b) From part (a), we see that ϕ is continuous onto T. Also $\widehat{\pi}$ is a homomorphism since $\widehat{\pi}(nt, x) = \phi(nt)(x) = n\phi(t)(x) = n\widehat{\pi}(t, x)$. Hence $\widehat{\pi}$ is a homomorphism. □

Corollary 2.21. *Let X be an S-system, $x \in X$. Then $\{x\} \cup \Gamma_{(x)} = (\beta S)x$, the orbit of x under the extended action $\widehat{\pi}$.*

Proof. Note that $j(S) \cup \{\hat{0}\}$ is dense in βS. By Theorem 2.20, since the extended action is right continuous and onto. It follow that

$$
\begin{aligned}
\widehat{\pi}(j(S) \cup \{\hat{0}\}, x) =& (j(S) \cup \{\hat{0}\})x \\
=& j(S)x \cup \{x\} \\
=& Sx \cup \{x\}
\end{aligned}
\tag{2.6}
$$

is dense in the closed set $(\beta S)x$ because a continuous image of a compact set into Hausdorff space is closed. Hence $\overline{\{x\} \cup Sx} = \{x\} \cup \overline{Sx} = \{x\} \cup \Gamma_{(x)} = \{x\} \cup \beta(S)x = \beta(S)x$. $\square$

Definition 2.22. *A transitive pointed S-system (X, p) is said to be universal if there exists a homomorphism from it to every other pointed S-system.*

The problem we will now consider is the existence of a universal transitive pointed S-system (Note by the Theorem 2.6, it is unique up to isomorphism).

Theorem 2.23. *Let $(\beta S, j)$ be the universal right topological monoidal compactification of S. We define $\pi : S \times \beta S \longrightarrow \beta S$ by $\pi(n, p) = j(n)p$ where the right-hand side is addition in βS.Then $(S, \beta S, \pi)$ is an S-system, and the pointed S-system $(\beta S, \hat{0})$ is a universal transitive pointed S-system.*

$$\begin{tikzcd}
\beta S \arrow[dr] & \\
S \arrow[u, "j"] \arrow[r] & \beta S
\end{tikzcd}$$

Proof. By Remark 1.58 S is the center of βS (i.e $j(S) = S \subseteq \wedge(\beta S)$), so $\pi(n, p) = j(n) + p = \{A \subset S | \{x \in S | -x + A \in p\} \in j(n)\} = \lambda_{j(n)}(p)$. Thus that π is separately continuous. Since j is a homomorphism, $\pi(n_1 + n_2, p) = j(n_1 + n_2)p = j(n_1)j(n_2)x = j(n_1)n_2p = \pi(n_1, n_2p)$. Therefore π defines an action of S on βS. Since $j(0) = \hat{0}$ then $\pi(0, x) = j(0)x = \hat{0}x = 0x = x$ act as an identity on βS. Also $\pi(S \times \{\hat{0}\}) \cup \{\hat{0}\} = j(S)\hat{0} \cup \{\hat{0}\} = j(S) \cup \{\hat{0}\}$ is dense in βS, so the action is point transitive.

To show $(\beta S, \hat{0})$ is universal. Let (Y, q) be a pointed S-system. By Theorem 2.20, the extended action $\hat{K} : \beta S \times Y \longrightarrow Y$ is a right continuous action of βS on Y.

Define $g : \beta S \longrightarrow Y$ by $g(t) = tq$, where the right-hand side is the extended action. Since the action is an extension then $g(\hat{0}) = \hat{0}q = q$; hence g preserves distinguished points. Finally for $n \in S$,

$$\begin{aligned} g(nx) =& g(j(n)x) \\ =& g(j(n)j(r)) \quad \text{where } j(r) = x \\ =& g(j(n+r)) \\ =& j(n+r)q \end{aligned} \tag{2.7}$$

But j is a homomorphisim, thus

$$\begin{aligned} =& j(n)j(r)q \\ =& j(n)g(r) \\ =& ng(r) \\ =& ng(j(r)) \\ =& ng(x). \end{aligned} \tag{2.8}$$

Thus g is a homomorphism of S-systems. □

We see from Theorem 2.23, how the universal right topological compactification of S yields the universal transitive pointed S-flow $(\beta S, \hat{0})$. We will now consider how all other such systems may be obtained (up to isomorphism) from it.

Definition 2.24. *An equivalence relation on a topological space X is closed if it is closed as a subset of $X \times X$, and that an equivalence relation $\sim$ on a semigroup S is a left congruence if $a \sim b$ implies $sa \sim sb$ for all $a, b, s \in S$.*

Remark 2.25. *If $\sim$ is a closed left congruence on βS then the relation $\approx$ on βS defined by $s \approx t$ if $sp \sim tp$ for all $p \in \beta S$ is a closed congruence relation on βS and is the largest closed congruence contained in that $\sim$.*

Theorem 2.26. *Let (X,x) and (Y,y) be transitive pointed S-flows.*

(a) $p \approx_{(X,x)} q$ if and only if $px = qx$ is a closed left congruence, where px is the extended action to βS in Theorem 2.20.

(b) There exists a homomorphsim from (X,x) onto (Y,y) if and only if $\approx_{(X,x)} \subseteq \approx_{(Y,y)}$.

Proof. (a) To show $\approx_{(x,x)}$ is a left congruence. Let $p,q \in \beta S$ and $p \approx_{(x,x)} q$. Then $px = qx$ implies $tpe = tqe$, $t \in \beta S$ and hence $tp \approx_{(x,x)} tq$. Hence $\approx_{(x,x)}$ a left congruence relation. To show that $\approx_{(x,x)}$ is closed. Let $n_\alpha \longrightarrow n$ and $t_\alpha \longrightarrow t$ such that $n_\alpha \approx_{(x,x)} t_\alpha$. Then $n_\alpha x = t_\alpha x$. Since ρ_x is right continuous on X, $n_\alpha x \longrightarrow nx$. Similarly $t_\alpha x \longrightarrow tx$. Therefore $nx = tx$ and hence $n \approx_{(x,x)} t$, impling that $\approx_{(x,x)}$ is a closed relation.

(b) Suppose that $\phi : (X,x) \longrightarrow (Y,y)$ is a homomorphsim. Let $p,q \in \beta S$ such that $p \approx_{(x,x)} q$ then $px = qx$ which implies $\phi(px) = \phi(qx)$

$$
\begin{aligned}
&\Rightarrow p\phi(x) = q\phi(x) \\
&\Rightarrow py = qy \qquad (2.9) \\
&\Rightarrow p \approx_{(Y,y)} q \quad \text{in } Y.
\end{aligned}
$$

Hence $\approx_{(X,x)} \subseteq \approx_{(Y,y)}$. Suppose that $\approx_{(X,x)} \subseteq \approx_{(Y,y)}$. By Theorem 2.29, $(X,x) \cong (\beta S/ \approx_{(X,x)}, [\hat{0}])$ and $(Y,y) \cong (\beta S/ \approx_{(Y,y)}, [\hat{0}])$. So there exist a homomorpsim $\phi : (\beta S/ \approx_{(X,x)}, [\hat{0}]) \longrightarrow (\beta S/ \approx_{(Y,y)}, [\hat{0}])$. Hence there exist a homomorphism $g : (X,x) \longrightarrow (Y,y)$. $\square$

Lemma 2.27. *Let $\sim$ be a closed left congruence relation on βS. We obtain a pointed transitive S-flow as follows: Let $X = \{[x] : x \in \beta S\}$ be the set of equivalence classes endowed with the quotient topology, and define $\pi : S \times X \longrightarrow X$ by $\pi(n,[x]) = [n+x]$ (the extended action of βS on X is also given by the same formula). Then $(X,[\hat{0}])$ is a transitive pointed S-flow, and the map from $\beta S \longrightarrow X$ defined by $x \mapsto [x]$ is a homomorphism of S-flows.*

Proof. Clearly the quotient space obtained as the equivalence classes of a closed relation on a compact Hausdorff space is again a compact Hausdorff space. Thus X is a compact Hausdorff space. We will show that $\pi : S \times X \longrightarrow X$ yields an action on βS. To see that, since $\pi(n+t, x) = [n+t+x] = [n+(t+x)] = \pi(n, (t+x))$ and $\pi(\hat{0}, x) = [\hat{0}+x] = [x]$. By standard quotient argument we are assure that π is continuous, and thus we have an S-system . Since S is dense in βS, then by definition $\pi(S, \hat{0}) \cup [\hat{0}] = [S] \cup [\hat{0}]$ is dense in X. Therefore $(X, [\hat{0}])$ is a transitive pointed S-system and the map $\alpha : \beta S \longrightarrow X = \beta S/\sim$ defined by $x \mapsto [x]$ is a homomorphism since $\alpha(n+x) = [n+x] = \alpha(n+[x]) = n+[x]$. □

Theorem 2.28. *Let $\sim$ be a closed left congruence on βS. Let $\alpha : \beta S \longrightarrow T$ (where T is the enveloping semigroup of the a action of βS on $\beta S/\sim$) be the onto homomorphism of Theorem 2.20, which is defined by $\alpha(n) = \tilde{n}$ where $\tilde{n}([x]) = [nx]_\sim$. Then the relation $\approx$ which is defined by $s \approx t$ if $sp \sim tp$ for all $p \in \beta S$ is the kernel relation of α. There is an induced action of $\beta S/\approx$ on βS on $\beta S/\sim$ and hence a homeomorphism from $\beta S/\approx$ to the enveloping semigroup T.*

Proof. To show $\approx$ is the kernal relation. Let $n_1 \approx n_2$, then by definition

$$\begin{aligned} &\Rightarrow n_1 x \sim n_2 x \\ &\Rightarrow [n_1 x]_\sim = [n_2 x]_\sim \\ &\Rightarrow n_1[x]_\sim = n_2[x]_\sim \quad \forall x \in \beta S \end{aligned} \tag{2.10}$$

$$\begin{aligned} &\Rightarrow \tilde{n_1}([x]_\sim) = \tilde{n_2}([x]_\sim) \\ &\Rightarrow \tilde{n_1} = \tilde{n_2} \\ &\Rightarrow \alpha(n_1) = \alpha(n_2) \end{aligned} \tag{2.11}$$

Hence $\approx$ is the kernel relation of α. To show $\approx$ is a closed relation. Suppose that $n_\alpha \longrightarrow n$ and $t_\alpha \longrightarrow t$ such that $n_\alpha \approx t_\alpha$. By definition of $\approx$ relation, $n_\alpha x \sim t_\alpha x$.

Since ρ_x is continuous on βS then $\rho_x(n_\alpha) \longrightarrow nx$. Similarly $\rho_x(t_\alpha) \longrightarrow tx$. Hence $n \sim t$ because $\sim$ is closed relation. So $\approx$ is a closed relation and there exist an induced action $\beta S/\approx$ on $\beta S/\sim$. By the 1^{st} isomorphism theorem there exist a continuous isomorphism $\widehat{\alpha} : \beta S/\approx \longrightarrow T$ defined by $\widehat{\alpha}([n]) = \alpha(n)$ where $n \in \beta S/\approx$.

$$\begin{tikzcd}
\beta S/\approx \arrow[dr, "\widehat{\alpha}"] & \\
S \arrow[u] \arrow[r, "\alpha"] & T
\end{tikzcd}$$

Since the domain βS is compact then $\beta S/\approx$ is compact . Hence the inverse of function $\widehat{\pi}$ is continuous . Therefore $\widehat{\pi}$ is a homeomorphism. $\square$

The next Theorem shows that if (X, p) is a transitive pointed systems, then the associated left congruence $\approx_{(X,p)}$ on βS determines the pointed flow (X, x) up to isomorphisim.

Theorem 2.29. *Given a transitive pointed S-system (X, p), there exists a unique closed left congruence $\sim$ of βS such that (X, p) is isomorphic to the S-system $(\beta S/\sim, [\hat{0}])$. Hence the closed left congruence on βS gives rise to a transitive pointed S-system via the construction of Lemma 2.27, and these S-systems section all the isomorphism classes of transitive pointed S-systems.*

Proof. Let (X, p) be a transitive pointed S-system. By Theorem 2.23, $(\beta S, \hat{0})$ is universal pointed S-system. So there exists a homomorphism $\gamma : \beta S \longrightarrow X$ defined by $\gamma(t) = tp$. Note that since (X, p) is transitive, and by Corollary 2.18 $\{p\} \cup \beta S = \beta S.p$, γ is a surjective. Define a relation $\sim$ on βS by $u \sim v$ if $\gamma(u) = \gamma(v)$, the kernel relation of γ. Note that $\sim$ is closed since if u_α and v_α be two nets such that $u_\alpha \longrightarrow u$ and $v_\alpha \longrightarrow v$, by the continuity of γ we get $\gamma(u_\alpha) \longrightarrow \gamma(u)$ and $\gamma(v_\alpha) \longrightarrow \gamma(v)$.

So if $u_\alpha \sim v_\alpha$ then $\gamma(u_\alpha) = \gamma(v_\alpha)$ which implies that $\gamma(u) = \gamma(v)$ and so $u \sim v$ thus $\sim$ is a closed congruence. To shown that $\sim$ is left congruence. Suppose that $u \sim v$, thus $\gamma(u) = \gamma(v)$. Hence $t\gamma(u) = t\gamma(v)$. Since γ is a homomorphisim then $(tu) = \gamma(tv)$. Hence $tu \sim tv$ for $t \in \beta S$.

Let $Y = \beta S/\sim$ with the quotient topology and let $\phi : \beta S \longrightarrow Y$ be the quotient map. By Lemma 2.27, $(Y, [\hat{o}])$ is a transitive pointed S-system. Since $\sim$ is the kernel relation for γ, there exists a unique continuous injection $\theta : Y \longrightarrow X$ such that $\theta \circ \phi = \gamma$. Since γ is surjective , so is θ. Since Y is compact and θ is 1-1 and onto,

$$\begin{tikzcd}
\beta S/\sim \arrow[dr, "\theta"] & \\
\beta S \arrow[u, "\phi"] \arrow[r, "\gamma"] & X
\end{tikzcd}$$

we actually get θ is a homeomorphism.

For uniqueness, suppose that $\equiv$ is another closed left congruence on βS such that $\phi : (\beta S/\equiv, [\hat{0}]) \longrightarrow (Y, [q])$ is an isomorphisim. Note that $(Y, q) \cong (X, p)$ from above. Let $\tau : \beta S \longrightarrow \beta S/\equiv$ be the quotient mapping. Then $\phi \circ \tau$ is an S- homomorphism from $(\beta S, [\hat{0}])$ to (Y, q). By Theorem 2.6, must be equal γ (i.e $\phi \circ \tau = \gamma$). We claim $\equiv$ be the kernel relation for γ. Let $u \equiv v$ where $u, v \in \beta S$

$$\begin{aligned}
\Rightarrow \gamma(u) =& (\phi o \tau)(u) \\
=& \phi(\tau(u)) \\
=& \phi([u]) \\
=& \phi([v]) \\
=& \phi(\tau(v)) \\
=& \gamma(v).
\end{aligned} \tag{2.12}$$

Hence by definition of $\sim$, $u \sim v$ they agree with $\sim$. Thus we have uniqueness. $\square$

The associated left congruence on βS to the pointed flow (X, p) we showed so far is the first step in the algebraic approach to the study of flows, and set up a one-to-one correspondence between isomorphism class of transitive pointed S-flows and closed left congruences on βS. These results allow us to pay attention to the quotients under closed left congruences of the universal right topological monidal compactification of S. Theorem 2.28, also give us information about the enveloping semigroup in this setting.

2.3 Principal Left Congruences and Minimal S-Systems

We have seen in the previous section that transitive pointed S-systems are classified by closed left congruences on the universal right topological compactification βS. In this section we introduce and investigate an important special class of left congruences on βS which can readily be identified, and the correspondence of this relations with minimal systems.

Definition 2.30. *Two elements s,t in a semigroup S with identity are related under Green $\mathscr{H}$ relation if they generate the same principal right ideals, i.e.; if $sS = tS$*

Lemma 2.31. *Let T be a right topological semigroup with identity 1, and let $e = e^2$ be an idempotent in T.*

(a) The relation $\equiv_e$ defined by $x \equiv_e y$ if $xe = ye$ is a closed left congruence on T and is the smallest one identifying 1 and e.

(b) Every equivalence class contains precisely one element from Te.

(c) Two relations satisfy $\equiv_e \subseteq \equiv_f$, if and only if, $ef = f$. Hence $\equiv_e$ is equal to $\equiv_f$, if and only if, $ef = f$ and $fe = e$. That is if and only if, e and f are related under Green $\mathscr{H}$ relation.

Proof. (a) To show $\equiv_e$ is a left congruence. Let $x, y \in T$ and $x \equiv_e y$. Then $xe = ye$ implies $txe = tye$ and thus $tx \equiv_e ty$. Hence $\equiv_e$ a left congruence relation. To show that $\equiv_e$ is closed. Let $n_\alpha \longrightarrow n$ and $t_\alpha \longrightarrow t$ such that $n_\alpha \equiv_e t_\alpha$. Then $n_\alpha e = t_\alpha e$. Since ρ_e is right continuous on T, $n_\alpha e \longrightarrow ne$. Similarly $t_\alpha e \longrightarrow te$. Therefore $ne = te$ and hence $n \equiv_e t$, impling that $\equiv_e$ is a closed relation. Since $1.e = e$ and $e.e = e$ then $1.e = e.e$, implies $1 \equiv_e e$. Claim $\equiv_e$ is the smallest one identifying 1 and e. Suppose $\sim$ is a left congruence identifying 1 and e and $x \equiv_e y$. By definition $x = x.1 \sim x.e = ye \sim y.1 = y \Longrightarrow \equiv_e \subseteq \sim$.

(b) Claim that no two distinct members of Te are related under $\equiv_e$. Note that $Tee = Te^2 = Te$, hence e acts as a left identity on Te. Suppose that $t_1e \neq t_2e$ such that $t_1e \equiv_e t_2e$, then $t_1ee = t_2ee$ implies that $t_1e = t_2e$, which is a contradiction. Thus no two members of Te are related $\equiv_e$.

(c) Suppose that $\equiv_e \subseteq \equiv_f$. Since $1 \equiv_e e$ then $1e = ee$. But $\equiv_e \subseteq \equiv_f$ implies $f = 1f = ef$. Conversely let $f = ef$. If $xe = ye$ then $xf = xef = yef = yf$. Thus $\equiv_e \subseteq \equiv_f$. Similarly we can show $\equiv_f \subseteq \equiv_e$ if and only if; $fe = e$. Hence $\equiv_e = \equiv_f$ if and only if; $ef = f$ and $fe = e$. To show that if $\equiv_e = \equiv_f$ then e and f are related under Green $\mathscr{H}$ relation, we note that Since $ft = eft$ $\forall t \in T$. Then $fT \subseteq eT$. Similarly since $et = fet \in fT$ then $eT \subseteq fT$. Hence $fT = eT$. $\square$

Remark 2.32. *The left congruences of the form $\equiv_f$ of lemma 2.27 are called principal left congruences.*

Theorem 2.33. *For each idempotent $e \in \beta S$, the S-subsystem $(\beta S + e, e)$ of βS is isomorphic to the S-system $(\beta S/\equiv_e, [\hat{0}])$. The isomorphism is given by assigning to any equivalence class its unique member in $\beta S + e$.*

Proof. We see in Lemma 2.31, each equivalence class of $\equiv_e$ contains exactly one member of $\beta S + e$. Consider the continuous onto quotient map $\phi : \beta S \longrightarrow \beta S/$

$\equiv_e$. Then, clearly, the restriction of ϕ to $\beta S + e$ is continuous and onto its image. Since each equivalence class contains exactly one member of $\beta S + e$, then $\phi|_{\beta S+e}$ is also one-to-one. To see that $(\phi|_{\beta S+e})^{-1}$ is a homeomorphism it suffices to show that $(\phi|_{\beta S+e})^{-1}$ is continuous. Note that since we have λ_e is continuous and βS is compact, then $\beta S + e$ is compact. Let F be a closed set in $\beta S + e$. Then F is compact, and thus $\phi(F)$ is a closed set. Since $(\phi^{-1})^{-1}(F) = \phi(F)$ is closed, ϕ^{-1} is closed map. Also $\phi|_{\beta S+e}$ is a homomorphism since $\phi|_{\beta S+e}((t + e) + (s + e) = [t + e + s + e] = [t + e] + [s + e] = \phi|_{\beta S+e}(t + e) + \phi|_{\beta S+e}(s + e)$ and since $[\hat{0}]$ is identified with e under $\equiv_e$, the result follows. $\square$

The previous Theorem 2.33, allows us to identify the S-system principal left congruences with the principal left ideals of βS which are generated by idempotents.

Lemma 2.34. *Let X be an S-system. The following are equivalent.*

(a) For any non-empty compact subset Y of X, $SY \subseteq Y$ implies $X = Y$.

(b) For any $x \in X$, the orbit closure $\Gamma_{(x)}$ is all of X.

Proof. Assume (a). For $x \in X$ since $S(Sx) \subseteq Sx$, then the orbit Sx is is invariant under S. We claim $S\Gamma_{(x)} \subseteq \Gamma_{(x)}$. Let $v \in \Gamma_{(x)}$ then there exist a net $v_n \in Sx$ such that v_n converges to v. Let $s \in S$. Since Sx is invariant then $sv_n \in Sx$. By separate continuity of the action , $sv_n \longrightarrow sv \in \Gamma_{(x)}$. Therefore $\Gamma_{(x)}$ is an invariant set. Since $\Gamma_{(x)}$ is also non empty compact subset of X, then by part (a) $\Gamma_{(x)} = X$.

Assume (b). Let Y be any non-empty compact S invariant subset of X. We claim that Y contains the orbit closure of any point in it. Let $x \in Y$. To see that $\Gamma_{(x)} \subseteq Y$, we first note that since X is a Hausdorff space, Y is closed. Let $y \in \Gamma_{(x)}$. Then there exist a net $s_n x \in Y$ such that $s_n x \longrightarrow y$. Hence since Y is is closed set, $y \in Y$. By part (b), $\Gamma_{(x)} = X \subseteq Y$. Therefore $Y = X$. $\square$

Definition 2.35. *An S-system X is called minimal if it satisfies either (and hence both) of the conditions of Lemma 2.34. A pointed S-system (X, p) is called minimal if X is minimal. Note that such a system must be point transitive since, in particular, the orbit closure of p is X.*

Theorem 2.36. *Let L be a minimal left ideal of βS, the universal right topological monoidal compactification of S. Then the S-system L of βS is minimal.*

Proof. Let (S, Y, π) be an S-subsystem of subsystem (S, L, π) where $Y \subseteq L$. To show $(S, Y, \pi) = (S, L, \pi)$. We have seen in Theorem 2.20, that we can extend the action π to $\hat{\pi}$ on βS to get the extended system $(\beta S, Y, \hat{\pi})$. Note that the extended action is $+$, then $\beta S + Y \subseteq Y$. Then Y is a left ideal of $(\beta S, +)$. However L is a minimal left ideal. Therefore $Y = L$. □

Theorem 2.37. *Let $f : X \longrightarrow Y$ be a homomorphism of S-system from X onto Y. If X is minimal, so is Y.*

Proof. Let M be a proper closed S-invariant subset of Y (i.e $SM \subseteq M$). Hence $f^{-1}(SM) \subseteq f^{-1}(M)$. Since f is onto and $M \subset Y$, implies $f^{-1}(M)$ is proper subset of X. Also since f is continuous then $f^{-1}(M)$ closed in the Hausdorff space X. Therefore $f^{-1}(M)$ is compact and a Hausdorff subset of X. Since f is a homomorphisim then $Sf^{-1}(M) \subseteq f^{-1}(M)$. Impling that $f^{-1}(M)$ is an S-system contained in X. But X is minimal, implies $X = f^{-1}(M)$. Since f is onto, then $M = Y$. □

Theorem 2.38. *Let (S, X, π) be an S-system, $x \in X$, and L a minimal left ideal of βS. Then the following are equivalent.*

(a) The orbit closure $\Gamma_{(x)}$ of x contains x and is a minimal S-system.

(b) For the extended action of βS on X, there exist $t \in M(\beta S)$, the minimal ideal, and $y \in X$ such that $x = ty$.

(c) $x \in Lx$.

(d) *There exists an idempotent* $e \in L$ *such that* $ex = x$.

(e) $\Gamma_{(x)} = Lx$.

Proof. $(a) \Rightarrow (c)$. We have seen in Theorem 2.20, that we can extend the action π to $\hat{\pi}$ on βS. Let L be a closed, minimal left ideal in βS. Since $\widehat{\pi}$ is continuous, βS is a compact and X is compact Hausdorff space, then $\widehat{\pi}(L, x) = L + x$ is a closed compact subset of L giving us the S-subsystem $(S, L, \hat{\pi})$ of X. Note that $x \in X$ and $Lx \subseteq \Gamma_{(x)}$. But since $\Gamma_{(x)}$ is minimal system , thus $x \in \Gamma_{(x)} = Lx$.

$(c) \Rightarrow (d)$. Since $x \in Lx$, there exists $t \in L$ such that $tx = x$. Since $(\beta S, +)$ is a compact right topological semigroup, then by Theorem 1.30 let H be a maximal subgroup with identity e in containing t. Then

$$\begin{aligned} ex &= e(tx) \\ &= (e+t)x \\ &= tx \\ &= x. \end{aligned} \tag{2.13}$$

$(d) \Rightarrow (b)$. By (d) there exist an idempotent $e \in L \subseteq M(\beta S)$ such that $ex = x$. Take $e = t$ and the proof follows immediately .

$(b) \Rightarrow (a)$. Consider the S-homomorphism of Theorem 2.23, $\hat{\pi} : (\beta S, \hat{0}) \longrightarrow (X, y)$ defined by $\hat{\pi}(p) = py$ where $x = ty$ for some $t \in M(\beta S)$. Since $M(\beta S)$ is the union of all the minimal left ideals, there exists a minimal left ideal J such that $t \in J$. The restriction of the homomorphism $\hat{\pi}$ to J gives an S-homomorphism $g : J \longrightarrow Jy$. By Theorem 2.36, the S-subsystem J of βS is minimal and by Theorem 2.37, Jy is a minimal S-system (which contains x since $x = ty$ and $t \in J$). Hence by Lemma 2.34, it must be the orbit of x.

$(c) \iff (e)$. Assume (c). By Theorem 2.36, the S-subsystem L is minimal and by Theorem 2.37, Lx is a minimal S-system. From the hypothesis it contains x. By Definition of minimality $Lx = \Gamma_{(x)}$. The converse is immediate since there is an identity adjoint to βS. $\square$

Corollary 2.39. *Let X be an S-system and let L be a minimal left ideal of the universal right topological monoidal compactification βS. Then*

(a) The set of points x for which $\Gamma_{(x)}$ is a minimal S-system containing x is equal to $\bigcup\{Ly : y \in X\}$.

(b) The collection $\{Ly : y \in X\}$ is a partition of closed sets in the sense that $Ly \bigcap Lz \neq \emptyset$ implies $Ly = Lz$.

In particular, if X is a union of minimal S-systems, then the collection partitions the space and the orbit closure of every point is a minimal S-system containing that point.

Proof. (a) Let $x \in \bigcup\{Ly : y \in X\}$ then $x \in Ly$ for some $y \in X$. Hence $x = ty$ for some $t \in L$. By Theorem 2.38 part (a), $\Gamma_{(x)}$ is a minimal S-system containing x. Conversely if $\Gamma_{(x)}$ is a minimal S-system containing x, then by Theorem 2.38 part (a) and (e), $\Gamma_{(x)} = Lx$.

(b) Since L is a minimal left ideal, by Theorem 2.36, $(L, X, \hat{\pi})$ is a minimal S-system. Let $y, z \in X$. Then Ly and Lz are a minimal S-system. To show $Ly \bigcap Lz \neq \emptyset$ implies $Ly = Lz$. Let $r \in Ly \bigcap Lz$, then $r = l_1 y = l_2 z$. But $Lr = Ly = Lz$. Hence $Ly = Lz$. The particular part follows from the preceding ones (a) and (b). $\square$

Part (b) of the following theorem can be found in [6] for the case that S is a discrete group. But here we consider the case where S is the semigroup $(S, +)$.

Theorem 2.40. *Let S be the semitopological semigroup with universal right topological monoidal compactification βS.*

(a) Let $\sim$ be a left congruence on βS. The S-system $(\beta S/\sim, [\hat{0}])$ is a minimal pointed system, if, and only if, $e \sim \hat{0}$ for some idempotent e in some minimal left ideal of βS, i.e., if and only if, $\equiv_e$ defined by $x \equiv_e y$ if and only if $xe = ye$ is contained in $\sim$ for some minimal idempotent e.

(b) Let L be a minimal left ideal in βS. For each idempotent $e \in L$, the minimal pointed S-system $(L = \beta S + e, e)$ is maximal among all the minimal pointed S-systems in the ordering of S-systems. As e is allowed to run over the idempotents of L, one obtains in this way unique representatives for the isomorphism classes of all the minimal pointed S-systems which are maximal as minimal pointed S-systems. Every minimal (pointed) S-system is the image of one of these minimal (pointed) systems.

Proof. (a) Suppose that $(\beta S/\sim, [\hat{0}])$ is a minimal S-system. By Theorem 2.38 part (d) (by considering $X = \beta S/\sim$ and $x = [\hat{0}]$) there exists an idempotent e in some minimal left ideal of βS such that $e = e + \hat{0} = \hat{0}$. Hence $e \sim \hat{0}$.

Conversely, suppose that e is an idempotent in a minimal left ideal L such that $e \sim \hat{0}$. Then $e[\hat{0}] = [e + \hat{0}] = [e] = [\hat{0}]$. Which implies that the condition (d) of Theorem 2.38 holds. Then by part (a) of Theorem 2.38, $(\beta S/\sim, [\hat{0}]$ is a minimal S-system. To show that $\equiv_e \subseteq \sim$. Let $x, y \in \beta S$ such that $x \equiv_e y$ then

$$\begin{aligned} &\Rightarrow x + e = y + e \\ x + e \sim y + \hat{0} &= x \quad \text{Since } e \sim \hat{0} \\ y + e \sim y + \hat{0} &= y \quad \text{Since } e \sim \hat{0} \\ \Rightarrow x \sim x + e &= y + e \sim y \\ &\Rightarrow x \sim y \end{aligned} \tag{2.14}$$

(b) Since L is a minimal left ideal then by Theorem 2.36, (L, e) is a minimal system. Let $f \in L$ be an idempotent and $f \neq e$. Since $L = L + e$ then $f = t + e$ for some $t \in L$. It follows then $f + e = t + (e + e) = t + e = f$. Similarly we can show $e + f = e$. So e and f act as right identities for each other. Note that if $e + f = f$ then since $f + e = e$, by Lemma 2.31, $e = f$ which is a contradiction. Thus e and f are not left identities for each other and hence $(L, , e)$ and (L, f) are distinct S-systems. Let (X, p) be minimal S-system. Then by Theorem 2.29, there exist a closed left coinsurance $\sim$ such that $(\beta S/\sim, [\hat{0}]) \cong (X, p)$. Hence $(\beta S/\sim, [\hat{0}])$ is a minimal system. We have shown in part (a) there exist an idempotent $e \in L$ such that $e \sim \hat{0}$ and thus $e \equiv_e \subseteq \hat{0}$. Hence by Theorem 2.26, there is a homomorphism $(L, e) \longrightarrow (X, p) \cong (\beta S/\sim, \hat{0})$. If (X, p) in turn admitted a homomorphism onto an (L, f) where $f \neq e$, then the composition would be a homomorphism $(L, e) \longrightarrow (L, f)$. This would imply that by Theorem 2.26 $\equiv_e \subseteq \equiv_f$. Thus by Lemma 2.31, $e + f = f$. Since $\beta S + f = L = \beta S + e$ and $e + f = e$, then $e = f$. That is a contradiction because $f \neq e$. Since we have homomorphism $\phi : (L, e) \longrightarrow (X, p)$ and a homomorphism $g : (X, p) \longrightarrow (L, e)$ and $g \circ \phi$ is the identity map from (L, f) to (L, e). Then f and g are inverse homeomorphism and hence (X, p) and (L, e) are isomorphic S-systems. It follows that if a pointed minimal system admits a homomorphism to one of the form (L, f), then it is isomorphic to (L, f), and that every pointed minimal system is the homomorphic image of some (L, e) for some idempotent $e \in L$. Thus these systems give a complete set of representative of the minimal pointed systems which are maximal as such. $\square$

Definition 2.41. *Let X be an S-system. A point $x \in X$ is called an almost periodic point if given any neighborhood U of x, there exists a compact subset K of S such that given $s \in S$, there exists $k \in K$ with $ksx \in U$.*

Definition 2.42. *A subset A of S is said to be Syndetic if there is a compact subset K of S such that $KS = \{ka \mid k \in K, a \in A\} \subseteq A$.*

The next theorem we will give us another representation of the notion of an almost periodic point.

Theorem 2.43. *Let (S, X) be a flow, and let $x \in X$. Then x is an almost periodic point if and only if for every neighborhood U of x there is a syndetic subset A of S such that $Ax \subseteq U$.*

Proof. Let U be a neighborhood of x and let A be syndetic subset of S such that $Ax \subseteq U$. Since A is syndetic then there is a compact subset K of S such that $SK = KS \subseteq A$. Hence $KSx \subseteq Ax \subseteq U$. Now, given $s \in S$ there exist $k \in K$ with $ksx \in U$..

Conversely, Let x be an almost periodic point. Then there exist a compact subset K of S and neighborhood U of x such that $Knx \subseteq U$. To show S has a syndetic set A such that $Ax \subseteq U$. Consider $KS = A$ and let $J = K$ be our compact set. This complete the proof. □

Remark 2.44. *Let x be an almost periodic point and $\Gamma_{(x)}$ is not a minimal set. Then for some $y \in \Gamma_{(x)}$, we get $y = n.x$ for some $n \in S$. Implies for some $a \in S$, $a.y = a.s.x \in \Gamma_{(x)}$ where $a.y \in \Gamma_{(x)}$. Therefore $\Gamma_{(y)} \subset \Gamma_{(x)}$. But $\Gamma_{(y)} \neq \Gamma_{(x)}$.*

We recall the following result from [5] and we present it here without proof.

Theorem 2.45. *Let X be an S-system for which the action $\pi : S \times X \longrightarrow X$ is continuous (not just separately continuous). Then $x \in X$ is almost periodic if and only if $\Gamma_{(x)}$ is a minimal S-system which contains x.*

Remark 2.46. *Note that in this case that for a large class of semigroups, separately continuous actions on compact spaces are automatically jointly continuous: locally compact groups, discrete semigroups, subsemigroups of groups containing* *the identity in the closure of their interiors. For such semigroups, Theorem 2.45 add to the string of equivalences of Theorem 2.38.*

The following corollary can be found in [5], but we will give simple proof for it.

Corollary 2.47. *If X is compact and if x is almost periodic, then each point of $\Gamma_{(x)}$ is an almost periodic point.*

Proof. Since x is almost periodic point , then by Theorem 2.45, $\Gamma_{(x)}$ is a minimal system. Thus by minimality every point $y \in \Gamma_{(x)}$ we have $\Gamma_{(x)} = \Gamma_{(y)}$. Hence y is an almost periodic point. $\square$

Definition 2.48. *Let X be an S-system. Two points $x, y \in X$ are said to be proximal if for any open cover Ω of X, there exists $s \in S$ and $U \in \Omega$ such that $sx, sy \in U$. Two points that are not proximal are said to be distal. A point of a dynamical system is said to be distal if it is proximal only to itself. The system X is distal if every two distinct points are distal.*

Theorem 2.49. *Let x, y be distinct points in the S-system X. Then x and y are proximal, if and only if, $tx = ty$ for some $t \in \beta S$ with respect to the extended action.*

$$\begin{tikzcd}
\beta S \times X \arrow[dr] & \\
S \times X \arrow[u, "j \times i"] \arrow[r] & X
\end{tikzcd}$$

Proof. Suppose that x and y are proximal. Then by definition of proximality for each open cover α of X, we can pick $n_\alpha \in S$ and $U_\alpha \in \alpha$ such that $n_\alpha x, n_\alpha y \in U_\alpha$. Since βS is compact, the net $\{j(n_\alpha\}$ has a convergent subnet to some $t \in \beta S$. By the right continuity of the extended action $n_\alpha x \longrightarrow tx$ and $n_\alpha y \longrightarrow ty$ for this subnet.

If we now take α to be the neighborhood basis of X, then $\alpha \times \alpha$ will be a basic neighborhood the diagonal Δ in $X \times X$. Note that the net $(n_\alpha x, n_\alpha y) \longrightarrow (tx, ty)$. It follows that the net $\{(n_\alpha x, n_\alpha y)\}$ must converge to some member of the diagonal, i.e.,the individual nets converge to the same point. Since $\{(n_\alpha x, n_\alpha y)\}$ also converge to (tx, ty), it follows that $tx = ty$.

Conversely suppose that $tx = ty$ for some $t \in \beta S$. Let Ω be an open cover of X, and let $U \in \Omega$ contain $tx = ty$. By right continuity there exists an open set V containing t such that $Vx \subseteq U$ and $Vy \subseteq U$. Since $j(S)$ is dense in βS, there exists $n \in S$ such that $j(n) \in V$. It follows that $nx, ny \in U$. □

Corollary 2.50. *Let x be a member of an S-system X for which the action is jointly continuous. Then there exists an almost periodic point $y \in X$ which is proximal to x.*

Proof. Consider the extended action of βS. Let e be an idempotent in a minimal left ideal L of βS. Let $y = ex$. Then $y \in Lx = Lex = Ly$. By Corollary 2.39, $\Gamma_{(x)}$ is a minimal S-system. Hence by Theorem 2.45, y is almost periodic. Since $ey = eex = ex$, it follows from Theorem 2.49, that y and x are proximal. □

The following lemma will help us prove Theorem 2.52 and allows us to use the structure of the minimal ideal of βS as a union of its minimal left ideals as well as the union of its right ideal

Lemma 2.51. *Let X be an S-system and R be a minimal right ideal in βS. If x and y are proximal then there exist an element $k \in L$ such that $kx = ky$.*

Proof. Since x and y are proximal then then by Theorem 2.49, there exist $r \in \beta(S)$ such that $rx = ry$. Let R be a minimal right ideal and let $t \in R$, then $trx = try$

where $tr \in R$. Let L be a minimal left ideal such that $tr \in L$. This complete the proof. $\square$

The following theorem can be found in [1] but we will give alternate and direct proof for it.

Theorem 2.52. *Let X be an S-system and $x, y \in X$. Then x and y are proximal if and only if $tx = ty$ for all $t \in L$, for some minimal left ideal $L \in \beta S$.*

Proof. By Lemma 2.51, if R is a minimal right ideal in βS, there exist $k \in R$ such that $kx = ky$. Let L be be a minimal left ideal in βS containing k. Let $F = \{t \in L : tx = ty\}$, $F \neq \emptyset$. Let $r \in \beta S$, $t \in F$ then $rtx = rty$, implies $\beta SF \subseteq F$. Thus F is a left ideal. Since $F \subseteq L$ and L is a minimal left ideal implies that $F = L$. The other direction it is straightforward from Theorem 2.49. $\square$

The following was posed as a question in [1]. Here we will use the machinery provided in this section to give a direct proof for it.

Theorem 2.53. *Let (S, X) be a flow and let $x, y \in X$. Show that if x and y are proximal and (x, y) is an almost periodic point of $(X \times X, S)$ then $x = y$.*

Proof. Let x and y be aproximal and let (x, y) be an almost periodic point in $(X \times X, S)$. By Theorem 2.45, $\Gamma_(x, y)$ is a minimal S-system and $(x, y) \in \Gamma_(x, y)$. We first claim this $\Gamma_(x, y) \cap \Delta \neq \emptyset$. If $\Gamma_(x, y) \cap \Delta = \emptyset$. Since (x, y) is proximal implies by Theorem 2.8, $tx = ty$ for some $t \in \beta(S)$. But in this case $(tx, ty) \in \Gamma_(x, y)$ and Δ which is a contradiction. Now since $\Gamma_(x, y) \cap \Delta \neq \emptyset$, let $(z, z) \in \Gamma_(x, y)$ and Δ. This then would imply that $\Gamma_(z, z) = \Gamma_(x, y)$. Note that $\Gamma_(z, z) = \{\overline{s(z, z)} : s \in S\} \subseteq \Delta$. Since Δ is closed . Hence $x = y$. $\square$

Theorem 2.54. *Let (S, X) be an S-system. The proximal relation $\simeq$ is an equivalence relation if and only if βS has a unique minimal left ideal.*

Proof. Suppose βS contains exactly one minimal left ideal. For the reflexive and symmetric part is trivial. Suppose (x, y) and (y, z) are proximal points, then by Theorem 2.52, there exist a minimal left ideal L_1 such that $tx = ty$ for some $t \in L_1$. Similarly Since (y, z) is proximal then by Theorem 2.52, there exist a minimal left ideal L_2 such that $ky = kz$ for some $k \in L_2$. Since we have unique minimal left ideal then $L_1 = L_2$. Thus we can pick $j \in L_1$ such that $jx = jz$. Hence the proximal relation $\simeq$ is an equivalence relation.

Conversely, suppose $\simeq$ is equivalence relation and let L_1 and L_2 be a minimal left ideals in βS. Let u_1 and u_2 be two idempotents in L_1 and L_2 respectively with $u_1 \sim u_2$ if and only if $u_1u_2 = u_2$ and $u_2u_1 = u_1$. Then (x, u_1x) and (x, u_2x) are proximal. Since $\simeq$ is an equivalence relation, it implies that (u_1x, u_2x) are aproximal. Now, we also we have $u_1(u_1x, u_2x) = (u_1u_1x, u_1u_2x) = (u_1x, u_1u_2x) = (u_1x, u_2x)$, so by Theorem 2.38, $\Gamma_(u_1x, u_2x)$ is a minimal S-system containing (u_1x, u_2x). Thus (u_1x, u_2x) is an almost periodic point of $(X \times X, S)$. It follows then from Theorem 2.53, $u_1x = u_2x$. Since $x \in X$ is an arbitrary $u_1 = u_2 \in L_1 \cap L_2$. since the minimal left ideal in βS are disjoint, therefore $L_1 = L_2$. □

The following example gives us a dynamical system where the proximal relation is an equivalence relation

Example 2.55. *Let $S := (R, +)$ act on the one-point compactificaion $X := R \cup \{\infty\}$ of R by $sx = s + x$, where $s + \infty$ is defined to be ∞ for all $s \in S$. Claim $X = R \cup \{\infty\}$ has unique minimal left ideal. By definition of left ideal, if L is a left ideal of X then $X + L \subseteq L$. Which implies that ∞ an element of L. Let $J = \{\infty\}$ then clearly J is a left ideal which is contained in L. Hence X has a unique minimal left ideal. Therefore by Theorem 2.54, the proximal relation on the system (S, X) is an equivalence relation.*

Remark 2.56. *In case* $(S = N, +)$*. Since* $(\beta(N), +)$ *has more than one minimal left ideal. Therefore the proximal relation is not an equivalence relation.*

The following corollary can be found in [1] where the acting semigroup is the Enveloping semigroup. Here we will adapt the procedure developed so far and prove the corollary in the case where βS is the acting semigroup.

Corollary 2.57. *Let* (S, X) *be an* S*-system and let* P *be a set of all proximal pairs which is closed in* $X \times X$*. Then it is an equivalence relation.*

Proof. Suppose L_1 and L_2 are two minimal left ideals in βS. Let u_1 and u_2 be two idempotents in L_1 and L_2 respectively with $u_1 \sim u_2$ if and only if $u_1 u_2 = u_2$ and $u_2 u_1 = u_1$. Let $x \in X$. Since $\sim$ is an equivalence relation, $(x, u_1 x)$ is a proximal pair. Also $(u_2 x, u_1 x) = (u_2 x, u_2 u_1 x) = lim\, s_j(x, u_1 x)$ where $\{s_j\}$ is a net in S such that $s_j \longrightarrow u_2$. Since $(x, u_1 x)$ is a proximal a pair, then there exist $t \in \beta S$ such that $tx = tu_1 x$ which implies $s_j tx = s_j tu_1 x$. But $s_j \in \Lambda(\beta(S))$ this would imply that $ts_j x = ts_j u_1 x$. Hence $(s_j x, s_j u_1 x)$ is a proximal pair. Since P is closed in $X \times X$ we have $(u_2 x, u_1 x) = lim\, s_j(x, u_1 x) \in P$. As we have noted in the proof of Theorem 2.54, $u_1(u_2 x, u_1 x) = (u_1 u_2 x, u_1 u_1 x) = (u_2 x, u_1 u_1 x) = (u_2 x, u_1 x)$, so by Theorem 2.38, $\Gamma_{(}u_1 x, u_2 x)$ is a minimal S-system containing $(u_2 x, u_1 x)$. Thus $(u_2 x, u_1 x)$ is an almost periodic point, so by Theorem 2.53, $u_2 x = u_1 x$. Since $x \in X$ is an arbitrary $u_1 = u_2 \in L_1 \cap L_2$. since the minimal left ideal in βS are disjoint, therefore $L_1 = L_2$. Therefore by Theorem 2.54, the proximal relation is an equivalence relation. □

Corollary 2.58. *If the acting semigroup is* $(\beta N, +)$ *then the set of all proximal pairs is not closed in* $X \times X$*.*

Proof. Assume that the set of all proximal pairs is a closed set. By Corollary 2.57, the proximal relation is an equivalence relation. Thus by Theorem 2.54, βS has a unique minimal left ideal. Which is a contradiction. □

Remark 2.59. *A flow (S, X) is said to be distal if for all $x, y \in X$ and $x \neq y$ the pair (x, y) is not proximal. Equivalently*

$$\triangle \bigcap \overline{\{(sx, sy) : s \in S\}} = \emptyset,$$

where $\triangle := \{(z, z) : z \in X\}$.

Theorem 2.60. *Let (S, X) be a flow. the following statements are equivalent:*

(a) (S, X) *is distal.*

(b) $\lim_\alpha n_\alpha x \neq \lim_\alpha n_\alpha y$ *for all $(x, y) \in (X \times X)|\triangle$ and all nets $\{n_\alpha\}$ in S for which both limits exist.*

(c) Σ *is a group whose identity is the identity function. where Σ is the enveloping semigroup.*

(d) Σ *is left simple and contains the identity function.*

Proof. The equivalence of (a) and (b) follows from the extended action definition on Theorem 2.17 we have $\lim_\alpha n_\alpha x = tx \neq ty \neq \lim_\alpha n_\alpha y$ for all $t \in \Sigma$. By Remark 2.59 (a) and (b) holds.

Assume that (b) holds. Then $tx \neq ty\ \forall t \in \Sigma$. Therefore every member of Σ is one-to-one and every member of Σ has an inverse. we claim that the only idempotent in Σ is the identity mapping. Suppose P is an idempotent in Σ and $Px \neq x$. Consider $P(Px) = Px$. Since P is one-to-one then $Px = x$ which is a contradiction. Thus $Px = x$. Therefore if P is an idempotent in Σ then P is an identity So every idempotent in Σ is an identity. Therefore there is a unique identity. Hence the smallest ideal $K(\Sigma)$ is an a group which contains the identity. Hence $\Sigma = K(\Sigma)$ is a group. Thus (b) implies (c).

clearly, (c) implies (d).

Assume that (d) holds and let $\xi \in \Sigma$. Since $\Sigma\xi$ is a left ideal and we have Σ is left simple then $\Sigma\xi = \Sigma$. Since Σ contains the identity function I, there exists a $\zeta \in \Sigma$ such that $\zeta\xi = I$. Hence $\zeta\xi(x) = x$ for all $x \in X$. Let $x \neq y$ then $Ix \neq Iy$. Suppose $tx = ty$ for some t, then there exist t^{-1} such that $t^{-1}tx = t^{-1}ty$. Hence $Ix = Iy$ which is a contradiction. Thus $x = y$. Thus (b) holds. □

Theorem 2.61. *The minimal sets of a flow (S, X) are precisely the sets $J(x) = \{\xi(x) : \xi \in J\}$, where $x \in X$ and J is a minimal left ideal of Σ. in particular, every closed invariant set contains a minimal set.*

Proof. Let J be any minimal left ideal of Σ and let $x \in X$. For any $\xi \in J$, where $\overline{(S\xi(x))} = (\Sigma\xi)(x) = J(x)$. Hence $J(x)$ is a minimal set. On the other hand, if Y is a nonempty, closed, invariant set and $x \in Y$, then $J(x) \subset Y$, with equality holding if Y is minimal. □

Corollary 2.62. *Let (S, X) be a distal flow.Then every orbit closure is a minimal set. Thus X is a disjoint union of minimal sets.*

Proof. Let $x \in X$. By Theorem 2.61, $\overline{Sx} = \Sigma(x)$ contains a minimal set, and by Theorem 2.60, $\overline{Sx} = \Sigma(x)$ is a minimal. Since two minimal sets are either disjoint or identical and since $x \in \Sigma(x)$, X is a disjoint union of minimal sets. □

Theorem 2.63. *Let X be a distal S-system, and let L be a minimal left ideal of the universal right topological monoidal compactification βS. Then for every idempotent $e \in L$, $ex = x$ for every $x \in X$. It follows that the orbit closure $\Gamma_{(x)}$ is minimal for every $x \in X$ and the enveloping semigroup Σ is a compact right topological group, which is the continuous image of the group $e\beta Se$ for any idempotent $e \in L$.*

Proof. All but the assertion of the last clause by Theorem 2.60 and corollary 2.62. If $\Gamma_{(x)}$ is the orbit closure of Sx, then by minimality $\Gamma_{(x)} = Lx$. Since e acts as a left identity on X then for $t \in L$, $tx = etex$. Hence $(e\beta Se)x = \Gamma_{(x)}$. $\square$

Corollary 2.64. *Let X be an S-system and let e be an idempotent in a minimal left ideal L of βS. Then X is distal, if, and only if, under the extended action $\Gamma_{(x)} \cup \{x\} = (e + \beta S + e)x$ for every $x \in X$.*

Proof. Suppose X is distal then $\Gamma_{(x)} \cup \{x\} = (e+\beta S+e)x$ for every $x \in X$ follows from Theorem 2.63. Conversely, suppose x, y are proximal. Then by Theorem 2.49 $nx = ny$ for some $n \in \beta S$. By hypothesis $x = etex$, so $ex = eetex = x$, so e acts on the left as an identity. Thus $enex = eney$. The element ene is in the group $e+\beta S+e$, so there exists q with $qene = e$. Then $x = ex = qenex = qeney = ey = y$. Therefore X is distal. $\square$

The next theorem and corollary can be found in [1] but we will give an alternate and direct proof by using the machinery used in this section.

Theorem 2.65. *Let X be an S-system and x be a distal point, and let L be a minimal left ideal of the universal right topological right monoidal compactification βS, then for every idempotent $e \in L$, $ex = x$.*

Proof. Suppose x a distal point, then by Corollary 2.50 there exists an almost periodic point $y \in X$ which is proximal to x.

Case 1: If $x \neq y$. Since x and y are proximal. Then x is not distal which is a contradiction.

Case 2: If $x = y$, then x is almost periodic point. By Theorem 2.45, $\Gamma_{(x)}$ is a minimal S-system which contains X (i.e $\Gamma_{(x)} = \Gamma_{(y)} = X$). By the Theorem 2.38, part (c) and (d) $\Gamma_{(x)} = Lx$, and for each $t \in L$, $tx = etex$. Hence $X = \Gamma_{(x)} = (e + \beta S + e)x$, so $x = etex$.Thus $ex = eetex = x$. □

Definition 2.66. *Let (X, T) be a flow with X is compact Hausdorff space. Then X is a (necessarly disjoint) union of minimal subsets if and only if every point of X is almost periodic. (In this case we say X is pointwise almost periodic).*

Corollary 2.67. *Let X be a flow. Then*

(a) If $x \in X$ is a distal point, it is an almost periodic point.

(b) If (S, X) is distal, it is pointwise almost periodic.

Proof. Consider the extended action by βS. Let L be a minimal left ideal of βS since x is distal , then by Theorem 2.65, there exist an idempotent $e \in L$ such that $ex = x$. By part (d) of Theorem 2.38, $\Gamma_{(x)}$ is a minimal S-system and this end proof part (a). Since X is distal , so our work in part (a) satisfy for every point. Hence (S, X) is point wise almost periodic. □

2.4 Decomposing Left Congruences on the Stone-Čech Compactification

This section is based on Auslander [1] and Lisan [15]. In this section we consider the congruences on the Stone-Čech Compactification and identify the distal and proximal flows by factoring congruences on it. We will be more interested on minimal left ideals L in βS. Throughout this section let L be a minimal left ideal of βS , and view L as a minimal S-flow. Let $e = e^2 \in L$ be a fixed idempotent in L. Let $\equiv_e$ denote the closed left congruence on βS (defined in Lemma 2.31) determined by e. If (X, x) is a transitive pointed flow, then it is a minimal flow for which $ex = x$ if and only if the associated left congruence $\approx_{(X,x)}$ contains $\equiv_e$ (see Theorem 2.40).

Lemma 2.68. *Let L be a minimal left ideal as described above. Let $x, y \in X$ and let $\pi : \beta S \times X \longrightarrow X$ be a natural projection. The following are equivalent:*

(a) $\pi^e(x) = \pi^e(y)$, *where* $\pi^e(x) = ex$ *and* $\pi^e(y) = ey$.

(b) $tx = ty$ *for some* $t \in L$.

(c) $tx = ty$ *for all* $t \in L$.

Proof. It easy to see $(c) \Rightarrow (a)$ by taking $t = e$. $(a) \Rightarrow (b)$ since $e \in L$. For $(b) \Rightarrow (c)$, since $t \in L$ and L is a minimal left ideal , $L + t = L$. For any ultrafilter $n \in L$, if $n = b + t$, then $nx = (b + t)x = (b + t)y = ny$. □

Definition 2.69. *For a minimal left ideal L of βS, we define the L-proximal relation P_L on an S-flow X by xP_Ly if any of the three equivalent conditions of Lemma 2.68 are satisfied.*

Theorem 2.70. *Two points x, y in an S-flow X are proximal if and only if xP_Ly for some minimal left ideal L. Thus the proximal relation P (This proximal relation P is defined on Definition 2.48), is equal to $\cup P_L$, where the union is taken over all minimal left ideals.*

Proof. Suppose x, y are proximal. By Theorem 2.49, there exists an ultrafilter $t \in \beta S$ such that $tx = ty$. Pick $u \in M(\beta S)$ where $M(\beta S)$ asmallest ideal in βS then $(u + t)x = (u + t)y$, and note that $u + t$ is in some minimal left ideal J of βS. Hence by Lemma 2.68, xP_Jy. Note that this implies $P \subseteq \cup P_L$.

Coversely, if xP_Ly for some minimal left ideal L, then $tx = ty$ for some ultrafilter $t \in L$ and then from Theorem 2.49, x and y are proximal (i.e any two points in one of these P_L it will be proximal). This also says $\cup P_L \subseteq P$, implies that $\cup P_L = P$. □

Remark 2.71. *The relation P_L for the minimal flow on L is called the strict proximal relation on L, and is denoted by Φ in this special case. It is characterized* *by any of the three conditions of Lemma 2.68 (in particular, for $x, y \in L$, $x\Phi y$ if and only if $ex = ey$).*

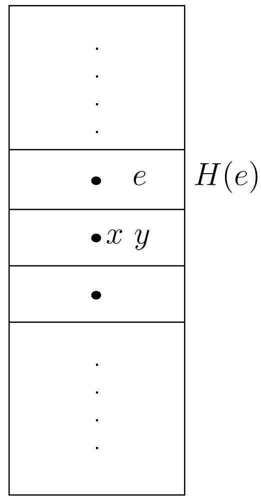
Theorem 2.72. *The relation P_L is a right congruence on βS. Its restriction to L, Φ, is the smallest right congruence on L which identifies the idempotents in L.*

Proof. Let e be an idempotent in L. By Lemma 2.68, we have xP_Ly if and only if $ex = ey$. This is clearly an equivalence relation. If $ex = ey$ and $t \in \beta S$, then $ext = eyt$. Thus xtP_Lyt, thus P_L is right congruence on βS. By definition of Φ, then Φ is also a right congruence on L. Since $L + f = L$ for each ultrafilter $f \in L$, the idempotents in L are right identities for L. Hence if f and g are idempotents in L , $e + f = e = e + g$, and thus $f\Phi g$. Suppose that Ω is a right congruence on L which identifies all the idempotents. Let $x, y \in L$ such that $ex = ey$. Hence x in some maximal subgroup of L. Let f be the identity of this maximal subgroup. Then $e\Omega f$. Since Ω is a right congruence, then $ex\Omega fx = x$. Similarly, y is also in some maximal subgroup of L with identity g. Then $e\Omega g$ and $ey\Omega gy = y$. Hence $x\Phi ex = ey\Phi y$, so by transitivity $x\Omega y$. Thus $\Phi \subseteq \Omega$. □

Remark 2.73. *We recall from definition 2.30, two elements n,t on a monoid T are related under Green relation $\mathscr{H}$ if they generate the same principal right ideals. When restricted to a minimal left ideal L in a compact right topological monoid, by Lemma 2.74, this relation can be characterized by $(n, t) \in \mathscr{H}$ if and only if $(n, t) \in H(e)$, the maximal subgroup containing the idempotent e for some $e \in L$. Thus the $\mathscr{H}$-equivalence classes are the maximal subgroups of L.*

We recall that the smallest ideal in the semigroup βS is the union of disjoint minimal left (or right) ideals. Equivalently it is the union of its maximal subgroups. (Let us now look at these subgroups as boxes. Then any element $x \in L$ could come from any of these boxes say $H(f)$ for some idempotent $f \in L$). Thus two elements

$x, y \in L$ are related under green relation if they both belong to the same box. We would now like to describe this relation between x and y in terms of some fixed idempotent $e \in H(e)$.



Lemma 2.74. *Let $(T, +)$ be a right topological semigroup and let L be a minimal left ideal. Then $(n, t) \in \mathscr{H}$ the green relation if and only if $n, t \in H(e)$ for some ultrafilters $n, t \in L$.*

Proof. Let $n, t \in H(e)$ for some $n, t \in L$. Since $n, t \in L$ then $T+n = T+t = L$. Note that $H(e) = L \cap R$, where R is some minimal right ideal. Then $n+T = t+T = R$.

Conversely, Suppose $(n, t) \in \mathscr{H}$ then $n + T = t + T$ and since $(n, t) \in L$ then $T + n = T + t$. Since $T + n = L$ is a minimal left ideal and $t + T$ by Theorem‘1.27, is a minimal right ideal and $t + n \in T + n \cap t + T$, it follows $t + n \in H(f)$ for some idempotent f. We want to show $n, t \in H(f)$ which is the same box containing $n + t$. Suppose $t \notin H(f)$ (this mean $t \neq f$). Note that $f + T$ is a minimal right ideal such that $L \cap f + T = G$ where G is some maximal group. Since $f \in L$ and $f \in f + T$ and f is an idempotent then $f \in G$. Hence $G = H(f)$. Note we have $t \in L$ and if $t \in f + T$ then $t \in H(f)$ which is a contradiction with the hypothesis $t \notin H(f)$. Thus $t \notin f + T$. Therefore $t + T$ and $f + T$ are two disjoint right ideals and $H(f) \subset fT$. Hence $t + T \cap H(f) = \emptyset$. But $t + n \in t + T$ and $t + n \in H(f)$, thus $t \in H(f)$ and similarly $n \in H(f)$. $\square$

Definition 2.75. *Let α is a closed left congruence on βS, we define the $\mathscr{H}$-trace $\alpha_{\mathscr{H}}$ of α on L to be $\mathscr{H} \cap \alpha \cap (L \times L)$. Let $H(e) = e + \beta S + e = e + L$ denote the maximal subgroup containing e in the minimal left ideal L. We define the $H(e)$-trace α_e to be the set $(H(e) \times H(e)) \cap \alpha$. Note that α_e is a closed left congruence on $H(e)$, and hence the equivalence class consist of left cosets of the subgroup $H_\alpha(e) := \{g \in H(e) : g\alpha e\}$. Finally, we define the Φ-trace α_Φ of α to be $\Phi \cap \alpha$.*

Lemma 2.76. *For $x, y, e = e^2 \in L$ and α a closed left congruence on βS, the following are equivalent:*

(a) $(x, y) \in \alpha_{\mathscr{H}}$

(b) $x, y \in H(f)$ *for some idempotent* $f \in L$ *and* $(ex, ey) \in \alpha_e$

(c) $x, y \in H(f)$ *for some idempotent* $f \in L$ *and* $ex \in eyH_\alpha(e)$

Thus the subgroup $H_\alpha(e)$ determines the left congruence $\alpha_{\mathscr{H}}$.

Proof. $(a) \Rightarrow (b)$: Let $(x, y) \in \alpha_{\mathscr{H}} = \mathscr{H} \cap \alpha \cap (L \times L)$ (i.e., $(x, y) \in \mathscr{H}$). We will show by Definition 2.75 $(x, y) \in H(f) \cap \alpha_e$. By Lemma 2.74, $(x, y) \in H(f)$ for some idempotent $f \in L$. We now claim that $e + x, e + y \in H(e) \times H(e) \cap \alpha$. Since α is a left congruence, $(e + x, e + y) \in \alpha$. Since $H(e) = e + L$, $e + x \in H(e)$ and $e + y \in H(e)$ therefore $(e + x, e + y) \in \alpha_e$.

$(b) \Rightarrow (c)$: Suppose that $x, y \in \alpha_e = (H(e) \times H(e)) \cap \alpha$. Then by left adding by the inverse of $e+y$ in $H(e)$, we obtain $((e+y)^{-1}+e+x, e) \in \alpha$, i.e.,$(e+y)^{-1}+e+x \in H_\alpha(e)$. Thus $e + x = e + y + (e + y)^{-1} + e + x \in (e + y) + H_\alpha(e)$.

$(c) \Rightarrow (a)$: Suppose that $x, y \in H(f)$ and $e + x \in (e + y) + H_\alpha(e)$, then $e + x = e+y+h$, where $(h, e) \in \alpha_e$. By adding on the left by $e+y$ we get $(e+x, e+y) \in \alpha$, and left adding again by f, we obtain $(f+x, f+y) = (f+e+x, f+e+y) = (x,y) \in \alpha$. Since $x, y \in H(f)$ then by Lemma 2.74, $(x,y) \in \mathscr{H}$. Hence $(x,y) \in \alpha_{\mathscr{H}}$. $\square$

Definition 2.77. *For the relation α and β on a set X, $\alpha \circ \beta := \{(x,z) : (x,y) \in \alpha, (y,z) \in \beta$ for some $y \in X\}$.*

Remark 2.78. *It is easy to show the relation $\alpha \circ \beta$ is left closed congruence relation. To show $\alpha \circ \beta$ is a left congruence. Let $(x,z) \in \alpha \circ \beta$ then by definition there exist an ultrafilter $y \in \beta(S)$ such that $(x,y) \in \alpha$ and $(y,z) \in \beta$. Since the relations α and β are left congruence relations then by definition for all $p \in \beta(S)$, $p+x = p+y$ and $p+y = p+z$. Thus $p+x = p+z$. Implies that $\alpha \circ \beta$ is left congruence. To show $\alpha \circ \beta$ is a closed relation. Let $(x_n, z_n) \in \alpha \circ \beta$ such that $\{x_n\} \longrightarrow x$, $\{z_n\} \longrightarrow z$. Then there exist a net $\{y_n\} \longrightarrow y$ such that $(x_n, y_n) \in \alpha$ and $(y_n, z_n) \in \beta$. Since the relations α and β are closed relations then $(x,y) \in \alpha$ and $(y,z) \in \beta$. Therefore $(x,z) \in \alpha \circ \beta$.*

The next theorem gives the a factorization of the restriction to L of a closed left congruence on βS.

Theorem 2.79. *Let α be a closed left congruence on βS. Then $\alpha \cap (L \times L) = \alpha_{\Phi} \circ \alpha_{\mathscr{H}} = \alpha_{\mathscr{H}} \circ \alpha_{\Phi}$.*

Proof. From the Definitions 2.75, we know that $\alpha_{\mathscr{H}} = \mathscr{H} \cap \alpha \cap (L \times L)$ and $\alpha_{\Phi} = \Phi \cap \alpha$. We also see from the definition of $\circ$ that $\alpha_{\mathscr{H}} \circ \alpha_{\Phi} \subseteq \alpha \circ \alpha \subseteq \alpha$. Thus $\alpha_{\mathscr{H}} \circ \alpha_{\Phi} \subseteq \alpha \cap L \times L$. Since the $\circ$ is commutative then $\alpha_{\Phi} \circ \alpha_{\mathscr{H}} \subseteq \alpha \cap L \times L$.

For the other direction, suppose that $x, y \in L$ are two ultrafilters and $x \alpha y$. Since $x \in L$, $x \in H(f)$, where $H(f)$ is the maximal subgroup box in L with the identity f such that $f + x = x$. Note that $f + x$ and $f + y$ are in the same subgroup box $H(f)$. Then since α is a left congruence, $(x = f + x)\alpha(f + y)$. By Lemma 2.74, $(x = f + x, f + y) \in \alpha_{\mathscr{H}}$. Also since $(f + y)\alpha(f + x = x)\alpha y$ (i.e.,

$f + y\alpha y$) and since $(f + y, y) \in \Phi$ implies $(f + y, y) \in \alpha_\Phi$. Thus $(x, y) \in \alpha_{\mathscr{H}} \circ \alpha_\Phi$. Hence $\alpha \cap (L \times L) \subseteq \alpha_{\mathscr{H}} \circ \alpha_\Phi$. Therefore $\alpha \cap (L \times L) = \alpha_{\mathscr{H}} \circ \alpha_\Phi$. Similarly $\alpha \cap (L \times L) = \alpha_\Phi \circ \alpha_{\mathscr{H}}$. Finaly since if $(x, y) \in \alpha$ implies $(y, x) \in \alpha$. Hence $\alpha_\Phi \circ \alpha_{\mathscr{H}} = \alpha_{\mathscr{H}} \circ \alpha_\Phi$. □

Theorem 2.80. *Let $e = e + e \in L$ and let α and β be closed left congruences on $\beta(S)$. Then $\alpha \cap L \times L \subseteq \beta$ if and only if $H_\alpha(e) \subseteq H_\beta(e)$ and $\alpha_\Phi \subseteq \beta_\Phi$. The restriction of a left congruence α to L is determined by the group $H_\alpha(e)$ and its Φ-trace α_Φ.*

Proof. Suppose that $\alpha \cap (L \times L) \subseteq \beta$. Let x, y be two ultrafilters in $L \times L$ and $(x, y) \in \alpha$. Then $(x, y) \in \beta$. Since Φ is restricted on L, then by definition $\alpha_\Phi \subseteq \beta_\Phi$. To show $H_\alpha(e) \subseteq H_\beta(e)$. Let $x \in H_\alpha(e)$. Then $x\alpha e$. Since $(x, e) \in L \times L$ then $(x, e) \in \alpha \cap \ (L \times L) \subseteq \beta$. Thus $x\beta e$ and therefore $x \in H_\beta(e)$. Hence $H_\alpha(e) \subseteq H_\beta(e)$. On the other direction, suppose that $H_\alpha(e) \subseteq H_\beta(e)$ and $\alpha_\Phi \subseteq \beta_\Phi$, and let $x, y \in H_\alpha(e) \subseteq H_\beta(e)$. By Lemma 2.74, $(x, y) \in \alpha_{\mathscr{H}} \subseteq \beta_{\mathscr{H}}$. By Theorem 2.79, we have $\alpha \cap (L \times L) = \alpha_\Phi \circ \alpha_{\mathscr{H}} \subseteq \beta_\Phi \circ \beta_{\mathscr{H}} \subseteq \beta$. It is clear from Lemma 2.76 that $H_\alpha(e)$ and α_Φ determine α. Since $\alpha \cap (L \times L) = \alpha_\Phi \circ \alpha_{\mathscr{H}}$. Again by Lemma 2.76, $H_\alpha(e)$ determine the left congruence $\alpha_{\mathscr{H}}$. Thus $H_\alpha(e)$ and α_Φ determine α. □

Chapter 3

The Case of βN and Proximal Homomorphisms

3.1 Distal and Proximal Homomorphisms on βS

In this section we will extend the work of [1] and [15] to investigate proximal and distal homomorphisms. By Theorem 2.38 if (X, x) is a minimal S-flow there is going to be a left closed congruence α such that L/α is isomorphic to X. We will then show the existence of closed left congruences $\alpha_{\mathscr{H}}$ and α_ϕ which are factors of α, and use these factors to analyze α and in turn study (X, x) by means of α_Φ and $\alpha_{\mathscr{H}}$.

Definition 3.1. *Let (X, x) be a minimal S-flow and L be a minimal left ideal on $\beta(S)$. Define the closed left congruence relation $\simeq_{(X,x)}$ by $(x, y) \in \simeq_{(X,x)}$ if and only if $nx = tx$. When the relation $\simeq_{(X,x)}$ restricted to L, we will denote it by α.*

Lemma 3.2. *Let (X, x) be a minimal S-flow. Let $\mathscr{G}(X, x) = \{g \in H(e) : gx = x\}$. Then $H_\alpha(e) = \mathscr{G}(X, x)$ where e is an idempotent in the minimal left ideal L.*

Proof. We need to verify that $H_\alpha(e) = \mathscr{G}(X, x)$. Note that

$$\begin{aligned} H_\alpha(e) =& \{g \in H(e) : g\alpha e\} \\ =& \{g \in H(e) : gx = ex\} \\ =& \{g \in H(e) : (e + g)x = ex\} \\ =& \{g \in H(e) : gx = x\} \end{aligned} \tag{3.1}$$

$$\text{Hence} \quad H_\alpha(e) \subseteq \mathscr{G}(X, x).$$

Let $g \in \mathscr{G}(X, x)$ then by definition of green relation $gx = x = ex$. Thus $g\alpha e$. Hence $\mathscr{G}(X, x) \subseteq H_\alpha(e)$. $\square$

Definition 3.3. *Let X and Y be an S-systems.*

(a) Two points $x, y \in X$ are said to be proximal if for any open cover Ω of X, there $s \in S$ and $U \in \Omega$ such that $sx, sy \in U$. Two points that are not proximal are said to be distal. A point of a dynamical system is said to be distal if it is proximal only to itself. The system X is distal if every two distinct points are distal.

(b) The S-flow homomorphism $\pi : X \longrightarrow Y$ is distal (respectively proximal) if $\pi(x) = \pi(y)$, $x \neq y$ imply that x, y are distal (respectively proximal).

Lemma 3.4. *Let X be a minimal S-flow, and let $x \in X$. If $f + f = f \in L$, where L is a minimal left ideal in $\beta(S)$. For distinct $g, h \in H(f)$, and $gx \neq hx$, then gx, hx are distal.*

Proof. Suppose that gx, hx are proximal. By Theorem 2.49, $(n+g)x = (n+h)x$ for some ultrafilter $n \in \beta S$. Since $g, h \in H(f)$, where f is the identity of this subgroup $H(f)$, then $f + g = g$ and $f + h = h$. Hence $(n + g)x = (f + n + f + g)x = (f + n + f + h)x$. By Theorem 1.26, $(f + n + f) \in f + \beta S + f = H(f)$, and so $f + n + f$ has an inverse k in the group $H(f)$. By left adding k, then

$$\begin{aligned} gx &= (f + g)x \\ &= (k + f + n + f + g)x \\ &= (k + f + n + f + h)x \\ &= (f + h)x \\ &= hx. \end{aligned} \tag{3.2}$$

This contradicts the hypothesis $gx \neq hx$. $\square$

We find the following result in [1] in the case where S is a group. We modify the result and give a proof to the case where $(S, +)$ is a semigroup.

Theorem 3.5. *Let S be a semitopological semigroup and let $L = L + u$ be a minimal left ideal in βS for some idempotent $u \in L$. Let $\pi : (X, x) \longrightarrow (Y, y)$ be a homomorphism of minimal pointed transitive flows where x is an almost periodic point in X with $ux = x$. Then $\mathscr{G}(X, x) = \mathscr{G}(Y, y)$ if and only if π is a proximal extension.*

Proof. Suppose π is a proximal homomorphism. Let $g \in \mathscr{G}(X, x)$, then

$$\begin{aligned} & gx = x \\ & \Rightarrow \pi(gx) = \pi(x) \\ & \Rightarrow g\pi(x) = \pi(x) \quad \text{(since } \pi \text{ is a homomorphism)} \\ & \Rightarrow gy = y. \end{aligned} \tag{3.3}$$

So $\mathscr{G}(X, x)$ is a subset of $\mathscr{G}(Y, y)$. Now we need to show $\mathscr{G}(Y, y)$ is a subset of $\mathscr{G}(X, x)$. Suppose $f \in \mathscr{G}(Y, y) \backslash \mathscr{G}(X, x)$, then $\pi(fx) = f\pi(x) = fy = y = \pi(x)$. Since $ux = x$ then u in the group $\mathscr{G}(X, x)$, implies u is the identity of $\mathscr{G}(X, x)$. Also since $uy = \pi(ux) = \pi(x) = y$ then u in the group $\mathscr{G}(Y, y)$, implies u is the identity element in $\mathscr{G}(Y, y)$. Hence if (fx, x) are aproximal then by Theorem 2.52,

$$\begin{aligned} & (u + f)x = ux \quad u \in L \\ & \Rightarrow (u + f)x = x \\ & \Rightarrow fx = x. \quad \text{Since } f \in \mathscr{G}(Y, y) \text{ then u+f=f} \end{aligned} \tag{3.4}$$

But $fx \neq x$. Therefore (x, fx) is not proximal. Hence π is not proximal which is a contradiction.

Conversely, Suppose that $\mathscr{G}(X, x) = \mathscr{G}(Y, y)$ and π is not proximal. Then there are distinct $x_1, x_2 \in X$ such that $\pi(x_1) = \pi(x_2)$ where (x_1, x_2) are not proximal. Hence $ux_1 \neq ux_2$. Since $\pi(x_1) = \pi(x_2)$, then $u\pi(x_1) = u\pi(x_2)$ implies that $\pi(ux_1) = \pi(ux_2)$. Since X is minimal flow, then $X = \Gamma_{(x)}$, where $\Gamma_{(x)}$ is the orbit closure of x. Then

$$\begin{aligned}
&\Rightarrow x_1 = kx \quad \text{for some } k \in \beta(S) \\
&\Rightarrow ux_1 = (u+k)x \\
&\qquad\quad = (u+k+u)x \quad \text{since ux=x} \\
&\Rightarrow u+k+u+u = u+k+u \in L+u.
\end{aligned} \tag{3.5}$$

Similarly, $ux_2 = (u+t+u)x$, where $(u+t+u) \in L+u$ for some $t \in \beta(S)$.

Let $\delta = u+k+u$, $\xi = u+t+u$ be two element in the subgroup $u+\beta(S)+u \subseteq L+x$ with $\delta x = ux_1$ and $\xi x = ux_2$. Then $\delta y = \delta\pi(x) = \pi(\delta x) = \pi(ux_1) = \pi(ux_2) = \pi(\xi x) = \xi y$. So $\xi^{-1}+\delta+y = y$ implies $\xi^{-1}+\delta \in \mathscr{G}(Y,y)$.

But in the same time since

$$\begin{aligned}
&ux_1 \neq ux_2 \\
&\Rightarrow \delta x \neq \xi x \\
&\Rightarrow (\xi^{-1}+\delta)x \neq x \\
&\Rightarrow \xi^{-1}\delta \notin \mathscr{G}(X,x)
\end{aligned} \tag{3.6}$$

Which is a contradiction since $\mathscr{G}(X,x) = \mathscr{G}(Y,y)$. Hence π is proximal. □

The relation α_Φ and $\alpha_{\mathscr{H}}$ are closely linked with notions of distality and proximality, as the next theorem indicates in [15] and we will give an alternate proof for it.

Theorem 3.6. *Let $\pi : (X,x) \longrightarrow (Y,y)$ be a homomorphism of pointed minimal flows, and $ex = x$ where e is the idempotent on the minimal left ideal L. Let α denote $\simeq_{(X,x)}$ restricted to L and β denote the restriction of $\simeq_{(Y,y)}$ to L (Recall $(n,t) \in \simeq_{(X,x)}$ if $nx = tx$).*

(a) The homomorphism π is distal if and only if $\alpha_\Phi = \beta_\Phi$.

(b) the following statement are equivalent:

(i) The homomorphism π is proximal.

(ii) $\mathscr{G}(X,x) = \mathscr{G}(Y,y)$.

(iii) $\alpha_{\mathscr{H}} = \beta_{\mathscr{H}}$.

Proof. (a): Assume that π is distal, and suppose that $(n,t) \in \beta_\Phi = \beta \cap \Phi$. Then $ny = ty$ and $e+n = e+t$. It follows that

$$\begin{aligned}\pi(nx) &= n\pi(x)\\ &= ny\\ &= ty\\ &= t\pi(x)\\ &= \pi(tx).\end{aligned} \tag{3.7}$$

and $e(nx) = (e+n)x = (e+t)x = e(tx)$. By Theorem 2.49, nx, tx are proximal. But since π it distal that implies $nx = tx$. We can then conclude that $(n,t) \in \alpha \cap \Phi = \alpha_\Phi$, and thus $\beta_\Phi \subseteq \alpha_\Phi$. For the other inclusion let $(n,t) \in \alpha$, then $nx = tx$. It follows that $\pi(nx) = n\pi(x) = ny$ and $\pi(tx) = t\pi(x) = ty$. Hence $(n,t) \in \beta$ and so $\alpha_\Phi \subseteq \beta_\Phi$.

Conversely, suppose that $\alpha_\Phi = \beta_\Phi$, and let p and q are distinct elements in X such that $\pi(p) = \pi(q)$. We want to show π is distal homomorphisim by showing p,q are distal points. Since $Lx = X$ by minimality of X, so there exists $t, s \in L$ with $tx = p$ and $q = sx$. By Lemma 2.34 and Theorem 2.38 part (d), there exists an idempotent $f+f = f \in L$ such that $q = fq$. Since $L = L+f$ which mean f is a right identity of L then $q = (s+f)x$. Thus $q = fq = (f+s+f)x$. Also $(t, f+t) \in \Phi$ (since $f+t = f+f+t = f+t$). Also $ty = t\pi(x) = \pi(tx) = \pi(p) = \pi(q)$ and so $(f+t)(y) = f\pi(q) = \pi(fq) = \pi(q)$. Since $ty = (f+t)y$, thus $(t, f+t) \in \beta$ implies that $(t, f+t) \in \beta_\Phi$. By hypothesis since $\alpha_\Phi = \beta_\Phi$ then $(t, f+t) \in \alpha$, and hence $p = tx = (f+t)x$. Since $(f+t)$ and $f+S+f$ are in the same subgroup $H(f)$ and since $(f+t)x = p \neq q = (f+S+f)x$. Then by Lemma 3.4, p and q are distal points.

(i) $\iff$ (ii): Note that by Theorem 3.5, π is proximal if and only if $\mathscr{G}(X,x) = \mathscr{G}(Y,y)$.

(ii) $\iff$ (iii) Suppose that $\mathscr{G}(X,x) = \mathscr{G}(Y,y)$ To show $\alpha_{\mathscr{H}} = \beta_{\mathscr{H}}$. Suppose that $\mathscr{G}(X,x) = \mathscr{G}(Y,y)$. To show $\alpha_{\mathscr{H}} \subseteq \beta_{\mathscr{H}}$ it is enough to show that $\alpha \subseteq \beta$. Let $(p,q) \in \alpha_{\mathscr{H}} = \mathscr{H} \cap \alpha \cap (L \times L)$ then $(p,q) \in \alpha$, implies $px = qx$

$$\begin{aligned} &\Rightarrow \pi(px) = \pi(qx) \\ &\Rightarrow p\pi(x) = q\pi(x) \\ &\Rightarrow py = qy \\ &\Rightarrow (p,q) \in \beta \\ &\alpha \subseteq \beta. \end{aligned} \tag{3.8}$$

To show $\beta_{\mathscr{H}} \subseteq \alpha_{\mathscr{H}}$ it is enough to show $\beta \subseteq \alpha$. Let $(p,q) \in \beta$, then $py = qy$. Since p, q in some subgroup $H(f)$ then by Lemma 2.76,

$$\begin{aligned} &\Rightarrow (q^{-1}+p)y = y \\ &\Rightarrow (q^{-1}+p) \in \mathscr{G}(Y,y) = \mathscr{G}(X,x) \\ &\Rightarrow (q^{-1}+p)x = x \\ &\Rightarrow q((q^{-1}+p)x) = qx \\ &\Rightarrow (q^{-1}+q^{-1}+p)x = qx \end{aligned} \tag{3.9}$$

Therefore we obtain

$$\begin{aligned} &\Rightarrow px = qx \\ &\Rightarrow (p,q) \in \alpha \\ &\Rightarrow \beta \subseteq \alpha. \end{aligned} \tag{3.10}$$

(iii) $\iff$ (i) Suppose that $\alpha_{\mathscr{H}} = \beta_{\mathscr{H}}$ to show that π is proximal. Let p and q are distinct elements in X such that $\pi(p) = \pi(q)$. We want to show π is proximal by showing $tp = tq$ for some $t \in \beta(S)$. Assume that $tp = tq$ for all $t \in L$. Since $LX = X$ by minimality of X, so there exist $r, s \in L$ such that $p = sx$ and $q = rx$. Hence $t(sx) = t(rs)$. But $\pi(p) = \pi(t(sx)) = \pi(t(rx)) = \pi(q)$

$$\begin{aligned} &\Rightarrow (t+s)\pi(x) = (t+r)\pi(x) \\ &\Rightarrow (t+s)y = (t+r)y \qquad (3.11) \\ &\text{Hence} \quad (t+s, t+r) \in \beta. \end{aligned}$$

Also since $t+s, t+r \in L \cap R$, where $R = t + \beta(S)$ is a minimal right ideal. Hence $t+s, t+r \in H(f) = \mathscr{H}$ for some idempotent $f \in L$. Therefore $(t+s, t+r) \in \alpha_{\mathscr{H}} = \beta_{\mathscr{H}}$. Thus $(t+s, t+r) \in \alpha$. Hence $(t+s)x = (r+t)x$ which is a contradiction. □

For a trivial single point flow $Y = \{y\}$, the associated left L-congruence (which is $\simeq_{(X,x)/L}$) is of course $L \times L$. A flow is distal (respectively proximal) if and only if the homomorphism to this trivial flow is distal (respectively proximal). The next corollary follows directly from Theorem 3.6, by considering the homomorphism to the trivial flow.

Corollary 3.7. *Let (X, x) be a pointed minimal flow with $ex = x$ and associated left L-congruence α. Then the flow is proximal if and only if $\alpha_{\mathscr{H}} = \mathscr{H} \cap (L \times L)$ if and only if $\mathscr{G}(X, x) = H(e) = e + \beta(S) + e$. It is distal if and only if $\alpha_\Phi = \Phi$.*

(a) The following statement are equivalent:

(i) (X, x) is proximal.

(ii) $\alpha_{\mathscr{H}} = \mathscr{H} \cap (L \times L)$.

(iii) $\mathscr{G}(X, x) = H(e) = e + \beta(S) + e$.

(b) (X, x) is distal if and only if $\alpha_\Phi = \Phi$.

Proof. (i) $\iff$ (ii) Suppose X is aproximal and we would like to show $\alpha_{\mathscr{H}} = \mathscr{H} \cap (L \times L)$. We will apply Theorem 3.6, and to do that, let $Y = \{y\}$ be a trivial single point flow space. Let us construct a map $\pi : (X, x) \longrightarrow Y = \{y\}$ defined by $\pi(x) = y$ which is a homomorphism since $\pi(nx) = y$ and $n\pi(x) = ny = y$. Since X is proximal then by the previous theorem then π is proximal. Note that for all $(p, q) \in (L \times L)$ then $py = qy = y$. This show that all the elements in $L \times L$ are related respect to β. Then

$$\begin{aligned} \beta_{\mathscr{H}} &= \mathscr{H} \cap \beta \cap (L \times L) \\ &= \mathscr{H} \cap (L \times L) \quad \text{ since } L \times L \in \beta \end{aligned} \tag{3.12}$$

Since π is proximal, by Theorem 3.6 part (b), $\alpha_{\mathscr{H}} = \beta_{\mathscr{H}} = \mathscr{H} \cap (L \times L)$. Now for the converse if $\alpha_{\mathscr{H}} = \mathscr{H} \cap (L \times L) = \beta_{\mathscr{H}}$, since $\beta_{\mathscr{H}}$ from above is also equal $\mathscr{H} \cap (L \times L)$ it is immediate that $\alpha_{\mathscr{H}} = \beta_{\mathscr{H}}$. Thus By Theorem 3.6 part (b), π is proximal. (i) $\Longleftrightarrow$ (iii) Assume (X, x) is proximal. Since π is proximal then $\mathscr{G}(X, x) = \mathscr{G}(Y, y)$. Note that $\mathscr{G}(Y, y) = \{g \in H(e) : gy = y\} = H(e)$. Therefore $\mathscr{G}(X, x) = \mathscr{G}(Y, y) = H(e)$. Conversely, if $\mathscr{G}(X, x) = H(e) = \mathscr{G}(Y, y)$ Then by Theorem 3.6 part (b), π is proximal homomorphism.

(b) Suppose that (X, x) is distal $\pi : (X, x) \longrightarrow Y = \{y\}$ is distal homomorphism.

By Theorem 3.6 part (a), then $\alpha_\phi = \beta_\phi$. Since $\beta_\phi = \Phi \cap \beta$ we get $\beta_\Phi \subseteq \Phi$. Thus $\alpha_\phi \subseteq \Phi$. Let $(t, n) \in \Phi$ then $ny = ty = y$, implies $(n, t) \in \beta$. Thus $\alpha \subseteq \Phi$. Hence $\beta_\Phi = \Phi \cap \beta = \Phi$. Therefore $\alpha_\Phi = \Phi$ as desired. conversely, if $\alpha_\Phi = \Phi$. Since as we show above $\beta_\Phi = \Phi$ then $\beta_\Phi = \beta_\Phi$. Thus by Theorem 3.6 part (a), π is distal and then (X, x) is distal. $\square$

The following result is in [15], but we will look at the special case when the acting of the semigroup is $\mathbb{N}$ and $\beta(N)$ will be taken as a universal extension of flows.

Theorem 3.8. *Let (X, x) be a pointed minimal $\mathbb{N}$-flow with $ex = x$ and let α be the associated left L-congruence for (X, x) (recall $(n, t) \in \alpha$ if $nx = tx$ for $n, t \in$*

L). There exist a closed left congruence γ such that $(\beta(N)/\gamma, [\hat{1}])$ is a universal proximal extension of (X, x) with $e[\hat{1}] = [\hat{1}]$. Given any minimal $\mathbb{N}$-flow (Y, y) with $ey = y$ and associated left L-congruence β, a necessary and sufficient condition for it to be a factor of the universal proximal extension is that $\mathscr{G}(X, x) \subseteq \mathscr{G}(Y, y)$.

Proof. We show in the proof of Theorem 2.40 that $\equiv_e \subseteq \simeq_{(X,x)}$ (where $(n, t) \in \equiv_e$ if $n + e = t + e$ and $(n, t) \in \simeq_{(X,x)}$ if $nx = tx$). Consider the set of all closed left congruence σ on $\beta(N)$ which contain $\equiv_e$ and $\alpha_{\mathscr{H}}$. Note that $\alpha_{\mathscr{H}} = \mathscr{H} \cap \alpha \cap (L \times L) \subseteq \alpha$, also if $(n, t) \in \equiv_e$, then we have $n + e = t + e$

$$\begin{aligned}
&\Rightarrow (n+e)x = (t+e)x \\
&\Rightarrow n(ex) = t(ex) \\
&\Rightarrow nx = tx \quad \text{since } ex = x \\
&\Rightarrow (n, t) \in \alpha.
\end{aligned} \tag{3.13}$$

Hence $\equiv_e \subseteq \alpha$. Since $\alpha_{\mathscr{H}}$ and $\equiv_e \subseteq \alpha$ therefore $\sigma \neq \emptyset$. Let γ be the intersection of these closed left congruences. Since X is compact space then $\gamma \neq \emptyset$. Since α is in the collection of congruence implies $\gamma \subseteq \alpha$, and so $\gamma_{\mathscr{H}} \subseteq \alpha_{\mathscr{H}}$. Also since $\alpha_{\mathscr{H}}$ contained in γ, thus $\alpha_{\mathscr{H}} \subseteq \gamma_{\mathscr{H}}$. Therefore $\alpha_{\mathscr{H}} = \gamma_{\mathscr{H}}$. Since γ is the intersection of all closed left congruence containing $\equiv_e$, $\alpha_{\mathscr{H}}$ implies that $\equiv_e \subseteq \gamma$. So by Theorem 2.40 part (a), by taking $\widetilde{X} = \beta(N)/\gamma$,then $(\beta(N)/\gamma, [\hat{1}])$ is a minimal $\mathbb{N}$-flow for which e fixes the base point $[\hat{1}] = [e]$. By Theorem 2.26, the inclusion $\gamma \subseteq \alpha$ implies a

$$\begin{tikzcd}
\beta(N)/\alpha \arrow[dr, "\psi"] & \\
\beta(N)/\gamma \arrow[u, "\varphi"] \arrow[r] & (X, x)
\end{tikzcd}$$

homomorphism $\varphi : (\beta(N)/\gamma, [\hat{1}]) \longrightarrow (\beta(N)/\alpha, [\hat{1}])$. Since α is associated left L-congruence there is an isomorphism $\psi : (\beta(N)/\alpha, [\hat{1}]) \longrightarrow (X, x)$, $[t] \longmapsto tx$. Note the composition $\psi \circ \varphi$ gives a homomorphism from $(\beta(N)/\gamma, [\hat{1}])$ to (X, x). Since $\gamma_{\mathscr{H}} = \alpha_{\mathscr{H}}$ by Theorem 3.6, we can conclude that homomorphism is proximal.

For the universality of $(\beta(N)/\gamma, [\hat{1}])$. Let (W, w) be a minimal flow with $ew = w$ and $\psi : (W, w) \longrightarrow (X, x)$ a proximal homomorphism. Let θ be the associated left congruence for the pointed flow (W, w). By Theorem 3.6, $\theta_{\mathscr{H}} = \alpha_{\mathscr{H}}$ and since $ew = w$ then $\equiv_e \subseteq \theta$. It follows that $\gamma \subseteq \theta$, and as in the preceding paragraph, by Theorem 2.26 and Theorem 3.6, there is aproximal homomorphism from $(\beta(N)/\gamma, [\hat{1}])$ onto (W, w).

Finally, suppose that (Y, y) is a minimal flow with $ey = y$, β is the associated left congruence for this pointed flow, and $\mathscr{G}(X, x) \subseteq \mathscr{G}(Y, y)$. Let $g \in H_\alpha(e)$ then $g\alpha e$

$$\begin{aligned} &\Rightarrow gx = ex = x \\ &\Rightarrow g \in \mathscr{G}(X, x) \\ &\Rightarrow H_\alpha(e) \subseteq \mathscr{G}(X, x). \end{aligned} \tag{3.14}$$

Now let $g \in \mathscr{G}(X, x)$, then $gx = x$

$$\begin{aligned} &\Rightarrow gx = ex \\ &\Rightarrow g\alpha e \\ &\Rightarrow g \in H_\alpha(e) \\ &\Rightarrow \mathscr{G}(X, x) \subseteq H_\alpha(e). \end{aligned} \tag{3.15}$$

Hence $\mathscr{G}(X, x) = H_\alpha(e)$, and similarly, $\mathscr{G}(Y, y) = H_\beta(e)$. Since $\mathscr{G}(X, x) \subseteq \mathscr{G}(Y, y)$ and hence $H_\alpha(e) \subseteq H_\beta(e)$. By Lemma 2.76 we can conclude that $\alpha_{\mathscr{H}} \subseteq \beta_{\mathscr{H}}$. Since $\equiv_e \subseteq \beta$ and $\alpha_{\mathscr{H}} \subseteq \beta_{\mathscr{H}} \subseteq \beta$, from the definition of γ, we have $\gamma \subseteq \beta$. By Theorem 2.26, there exist a homomorphism from $(\beta(S)/\gamma, [\hat{1}])$ onto (Y, y). Conversely, suppose we have a homomorphism from $(\beta(S)/\gamma, [\hat{1}])$ onto (Y, y). By Theorem 2.26, $\gamma \subseteq \beta$ and since we have shown $\alpha_{\mathscr{H}} = \gamma_{\mathscr{H}}$. Hence $\alpha_{\mathscr{H}} = \gamma_{\mathscr{H}} \subseteq \beta_{\mathscr{H}}$. By Theorem 3.6 part (b), $\mathscr{G}(X, x) \subseteq \mathscr{G}(Y, y)$. $\square$

The next theorem shows the existence of the universal distal extension of flows.

Theorem 3.9. *Let (X, x) be a pointed minimal S-flow with $ex = x$ and let α be the associated left L-congruence for (X, x). There exist a closed left congruence γ such that $(\beta(S)/\gamma, [\hat{1}])$ is a universal distal extension of (X, x) with $e[\hat{1}] = [\hat{1}]$. Given any minimal S-flow (Y, y) with $ey = y$ and associated left L-congruence β, a necessary and sufficient condition for it to be a factor of the universal distal extension is that $\alpha_\Phi \subseteq \beta_\Phi$.*

Proof. In Theorem 3.8, we show that $\equiv_e \subseteq \approx_{(X,x)}$. Let σ be the set of all closed left congruence on $\beta(S)$ which contain $\equiv_e$ and α_Φ. Note that $\alpha_\Phi = \alpha \cap (L \times L) \subseteq \alpha$, also if $(n, t) \in \equiv_e$ then $n + e = t + e$

$$\begin{aligned}
&\Rightarrow (n + e)x = (t + e)x \\
&\Rightarrow nx = tx \quad \text{(since } ex = x) \\
&\Rightarrow (n, t) \in \alpha \\
&\Rightarrow \equiv_e \subseteq \alpha.
\end{aligned} \tag{3.16}$$

Since α_Φ, $\equiv_e \subseteq \alpha$ therefore $\sigma \neq \emptyset$. Let γ be the intersection of these closed left congruences. Since the space is compact then $\gamma \neq \emptyset$. Since α is in the collection of congruence implies $\gamma \subseteq \alpha$,and so $\gamma_\Phi \subseteq \alpha_\Phi$. Also since α_Φ contained in γ, thus $\alpha_\Phi \subseteq \gamma_\Phi$. Therefore $\alpha_\Phi = \gamma_\Phi$. Since γ is the intersection of all closed left congruence containing $\equiv_e$, α_Φ implies that $\equiv_e \subseteq \gamma$. By Theorem 2.40 part (a), then $(\beta(S)/\gamma, [\hat{1}])$ is a minimal S-flow for which e fixes the base point $[\hat{1}] = [e]$.

$$\begin{tikzcd}
\beta(S)/\alpha \arrow[dr, "\psi"] & \\
\beta(S)/\gamma \arrow[u, "\varphi"] \arrow[r] & X
\end{tikzcd}$$

By Theorem 2.26 the inclusion $\gamma \subseteq \alpha$ implies a homomorphism $\varphi : (\beta(S)/\gamma, [\hat{1}]) \longrightarrow (\beta(S)/\alpha, \hat{1})$. Since α is associated left congruence there is an isomorphism $\psi : (\beta(S)/\alpha, \hat{1}) \longrightarrow (X, x)$, $[t] \longmapsto tx$. The composition $\psi \circ \varphi$ gives a homomorphism from $(\beta(S)/\gamma, [\hat{1}])$ to (X, x). Since $\gamma_\Phi = \alpha_\Phi$, by Theorem 3.6, we can conclude that homomorphism is distal.

For the universality of $(\beta(S)/\gamma, [\hat{1}])$. Let (W, w) be a minimal flow with $ew = w$ and $\psi : (W, w) \longrightarrow (X, x)$ be a distal homomorphism. Let θ be the associated left congruence for the pointed flow (W, w). By Theorem 3.6, $\theta_\Phi = \alpha_\Phi$. Since $ew = w$ then $\equiv_e \subseteq \theta$, also since $\theta_\Phi \subseteq \theta$ therefore $\gamma \subseteq \theta$. As in the preceding paragraph, by Theorem 2.26 and Theorem 3.6, there is a distal homomorphism from $(\beta(S)/\gamma, [1])$ onto (W, w).

Finally, suppose that (Y, y) be a minimal flow with $ey = y$ and β is the associated left congruence for this pointed flow, and $\alpha_\Phi \subseteq \beta_\Phi$. So $\alpha_\Phi \subseteq \beta$. Also if $(n, t) \in \equiv_e$, then $n + e = t + e$

$$
\begin{aligned}
&\Rightarrow n + e = t + e \\
&\Rightarrow (n + e)y = (t + e)y \quad \text{(since } ex = x\text{)} \\
&\Rightarrow ny = ty \\
&\Rightarrow (n, t) \in \beta \\
&\Rightarrow \equiv_e \subseteq \beta.
\end{aligned}
\tag{3.17}
$$

Then from the definition of γ we have $\gamma \subseteq \beta$. By Theorem 2.26, there exist a homomorphism from $(\beta(S)/\gamma, [\hat{1}])$ onto (Y, y). Conversely, suppose we have a homomorphism from $(\beta(S)/\gamma, [\hat{1}])$ onto (Y, y). By Theorem 2.26, $\gamma \subseteq \beta$ and since we have shown $\alpha_\Phi = \gamma_\Phi$. Hence $\alpha_\Phi = \gamma_\Phi \subseteq \beta_\Phi$. $\square$

The next theorem in [15] give quick and direct proof of ([1], Theorem 10.9) we include it here without proof to prove Theorem 3.13.

Theorem 3.10. *Let (X, x), (Y, y), and (Z, z) be minimal pointed S-flows with $ex = x$, $ey = y$, and $ez = z$ and associated left L-congruences α, β, γ respectively. Let $\pi : (X, x) \longrightarrow (Z, z)$ and $\psi : (Y, y) \longrightarrow (Z, z)$ be homomorphisms with ψ distal homomorphism. Then there is a homomorphism $\chi : (X, x) \longrightarrow (Y, y)$ such that $\pi = \psi \circ \chi$ if and only if $\mathscr{G}(X, x) \subseteq \mathscr{G}(Y, y)$.*

Theorem 3.11. *Let (X, x), (Y, y) be as in Theorem 3.10 with Y distal. Then there exist a homomorphism $\chi : (X, x) \longrightarrow (Y, y)$ if and only if $\mathscr{G}(X, x) \subseteq \mathscr{G}(Y, y)$. In particular, if both are distal then they are isomorphic if and only if $\mathscr{G}(X, x) = \mathscr{G}(Y, y)$.*

Proof. Let $Z = \{z\}$ be a trivial single point flow. Define $\psi : (Y, y) \longrightarrow Z$ by $\psi(y) = z$ which is a homomorphism since $\psi(ry) = z$ and $r\pi(y) = rz = z$. To show ψ is distal, let $p, q \in Y$ such that $p \neq q$ and $\psi(p) = \psi(q)$. By distality of Y , p and q are distal. Thus this would imply that ψ is distal homomorphism, which means that all the hypothesis of Theorem 3.10 are satisfied which completes the proof.

For the particular part, suppose χ is an isomorphism. From the first part we have $\mathscr{G}(X, x) \subseteq \mathscr{G}(Y, y)$. Since we have an isomorphism and (Y, y) is distal then similarly we can get $\mathscr{G}(Y, y) \subseteq \mathscr{G}(X, x)$. Conversely if $\mathscr{G}(X, x) = \mathscr{G}(Y, y)$, then by Theorem 3.10, there is a homomorphism $(X, x) \longrightarrow (Y, y)$ and a homomorphism $(Y, y) \longrightarrow (X, x)$. By Theorem 2.6. □

In the next theorem we will present the proof for the case where ψ is proximal in the statement of Theorem 3.10.

Theorem 3.12. *Let (X, x), (Y, y), and (Z, z) be minimal pointed S-flows with $ex = x$, $ey = y$, and $ez = z$ and associated left L-congruences α, β, γ respectively. Let $\pi : (X, x) \longrightarrow (Z, z)$ and $\psi : (Y, y) \longrightarrow (Z, z)$ be homomorphisms with ψ*

proximal. Then there is a homomorphism $\chi : (X, x) \longrightarrow (Y, y)$ such that $\pi = \psi \circ \chi$ if and only if $\alpha_\phi \subseteq \beta_\phi$.

Proof. First assume that ψ is a proximal homomorphism and χ exists. To show $\alpha_\Phi \subseteq \beta_\Phi$. By Theorem 2.26, $\alpha \subseteq \beta$, and hence $\alpha_\Phi \subseteq \beta_\Phi$.

Conversely, suppose that $\alpha_\Phi \subseteq \beta_\Phi$ is hold. Since ψ is a proximal homomorphism, then by Theorem 3.6 part (b), we have $\beta_{\mathscr{H}} = \gamma_{\mathscr{H}}$. Also since π is a homomorphism, by Theorem 2.26, $\alpha \subseteq \gamma$. Therefore $\alpha_{\mathscr{H}} \subseteq \beta_{\mathscr{H}} = \gamma_{\mathscr{H}}$. Note that $\alpha = \simeq_{(X,x)} \cap (l \times L)$ where ($\simeq_{(X,x)}$ is left closed congruence defined by $(n, t) \in \simeq_{(X,x)}$ iff $nx = tx$). Let $\alpha^* = \simeq_{(X,x)}$. Hence

$$\begin{aligned} \alpha &= \alpha^* \cap (L \times L) \\ &= \alpha_\Phi \circ \alpha_{\mathscr{H}} \quad \text{By Theorem 2.79} \\ &\subseteq \beta_\Phi \circ \beta_{\mathscr{H}} \\ &\subseteq \beta \circ \beta \\ &\subseteq \beta. \end{aligned} \tag{3.18}$$

Hence by Theorem 2.26, it follows there exist a homomorphism $(X, x) \longrightarrow (Y, y)$. It follows that there is induced a well- defined homomorphism $\chi : (X, x) \longrightarrow (Y, y)$. Since homomorphism are unique between pointed follows, we conclude $\pi = \psi \circ \chi$. □

The following theorem follows directly from Theorem 3.12.

Theorem 3.13. *Let (X, x), (Y, y) be as in Theorem 3.12 with Y proximal. Then there exist a homomorphism $\chi : (X, x) \longrightarrow (Y, y)$ if and only if $\alpha_\phi \subseteq \beta_\phi$. In particular, if both are proximal then they are isomorphic if and only if $\alpha_\phi = \beta_\phi$.*

Proof. Let $Z = \{z\}$ be a trivial single point flow. Define $\psi : (Y, y) \longrightarrow Z$ by $\psi(y) = z$ which is a homomorphism since $\psi(py) = z$ and $p\pi(y) = pz = z$. To show ψ

is proximal, let $p, q \in Y$ such that $p \neq q$ and $\psi(p) = \psi(q)$. By proximality of Y then p and q are proximal. This would imply that ψ is proximal homomorphism. Which means that all the hypothesis of Theorem 3.12, satisfied. Thus the conclusion of Theorem 3.12 holds.

For the particular part, suppose χ is an isomorphism. From the first part we have $\alpha_\phi \subseteq \beta_\phi$. Similarly since we have isomorphism and (Y, y) is proximal we get $\beta_\phi \subseteq \alpha_\phi$. Conversely if $\alpha_\phi = \beta_\phi$, then by Theorem 3.12, there is a homomorphism $(X, x) \longrightarrow (Y, y)$ and a homomorphism $(Y, y) \longrightarrow (X, x)$ where α and β are the kernel relation respectively. By Theorem 2.6, we have an isomorphism. □

3.2 The Case of $\beta\mathbb{N}$ and Ultrafilters as Acting Semigroups

In this section we will concentrate on the special case where the phase semigroup is the set of natural numbers $\mathbb{N}$ under addition. In so doing we will present results which relate notions of proximality and almost periodicity in a dynamical system to combinatorially rich central subsets of $\mathbb{N}$.

Definition 3.14. *Let X be a compact Hausdorff space. Given the dynamical system (N, X, π), $x \in X$, and $p \in \beta N$, we define the extended action of $\beta\mathbb{N}$ as follows: $px = \pi_p(x) = \cap_{A \in p}\, cl\{\pi_n(x) : n \in A\}$.*

Theorem 3.15. *Let $(\mathbb{N}, X, \pi)$be a dynamical system, $x, y \in X$, and $p \in \beta N$. Then the following are equivalent:*

(a) $y \in \pi_p(x)$.

(b) For each neighborhood G of y, $\{s \in \mathbb{N} : \pi_s(x) \in G\} \in p$.

(c) $\pi_p(x) = \{y\}$.

Proof. (a) implies (b). To show that $K = \{s \in \mathbb{N} : \pi_s(x) \in G\} \in p$, let $y \in \pi_p(x)$ and let G be neighborhood of y (i.e $y \in \pi_p(x) \cap G$). Since $y \in \pi_p(x)$, by Definition 3.14, for each $A \in p$, $y \in cl\{\pi_s(x) : s \in A\}$. Which implies that

$B = \{\pi_s(x) : s \in A\} \cap G \neq \emptyset$. Let $\pi_t(x) \in B$ for some $t \in A$. Note that $\pi_t(x)$ is in G and hence $M = \{A : A \in p\} \cup \{\{s \in N : \pi_s(x) \in G\}\}$ satisfies the finite intersection property. But since p is an ultrafilter which is maximal with respect to the finite intersection property, $M = p$. Therefore $\{s \in \mathbb{N} : \pi_s(x) \in G\} \in p$.

(b) implies (c). To show that $\pi_p(x) = \{y\}$ suppose that $y, z \in \pi_p(x)$ and $y \neq z$. Let G_1 and G_2 be two disjoint open sets such that $y \in G_1$ and $z \in G_2$. Consider the two sets $A = \{s \in \mathbb{N} : \pi_s(x) \in G_1\} \in p$ and $B = \{s \in \mathbb{N} : \pi_s(x) \in G_2\} \in p$. Note that p is an ultrafilter and hence $A \cap B \neq \emptyset$. Let $s_1 \in A \cap B$. Then $\pi_{s_1}(x) \in G_1 \cap G_2$, which is a contradiction.

(c) implies (a). Clearly $y \in \{y\} = \pi_p(x)$. □

From here on we will write $\pi_p(x) = \{y\} = y$.

As the special case where $X = \beta\mathbb{N}$ Definition 3.14 will allow us to extend the operation on $(\mathbb{N}, +)$ to $\beta\mathbb{N}$ as follows:

Definition 3.16. *Let (S,+) be a semigroup, for any $A \subseteq S$ and $x \in S$ we define $-x + A = \{s \in S : x + s \in A\}$. Given any two ultrafilter $p, q \in \beta S$ we define their sum by $p + q = \{A \subset S | \{x \in S | -x + A \in q\} \in p\}$.*

The following theorem shows the action of $\mathbb{N}$ can be extended to the action on $\beta\mathbb{N}$.

Theorem 3.17. *Let $(\mathbb{N}, X, \pi)$ be a dynamical system, $x, y \in X$, and $p, q \in \beta\mathbb{N}$. If $\pi_p(x) = y$ and $\pi_q(y) = z$, then $\pi_{q+p}(x) = z$, where $q + p$ is defined as above.*

Proof. To show that $\pi_{p+q}(x) = z$, let G be a neighborhood of z and let $A = \{s \in \mathbb{N} : \pi_s(x) \in G\}$ and let $B = \{s \in \mathbb{N} : \pi_s(y) \in G\}$. Since $\pi_q(y) = z$ then by Theorem 3.15, $B \in q$. If we can show that $B \subseteq \{s \in \mathbb{N} : -s + A \in p\}$ then $\{s \in \mathbb{N} : -s + A \in p\} \in q$. So that $A \in q + p$ as required. Let $s \in B$. Since π_s

is continuous and $\pi_s(y) \in G$, pick a neighborhood U of y with $\pi_s(U) \subseteq G$. Let $C = \{t \in \mathbb{N} : \pi_t(x) \in U\}$. Since $\pi_p(x) = y$ then by Theorem 3.15, $C \in p$ and given $t \in C$ implies $\pi_s(\pi_t(x)) = \pi_s(tx) = s(tx) = (s+t)x = \pi_{t+s}(x) \in G$. Hence $t + s \in A$. Thus $C \subseteq -s + A$. So $-s + A \in q$. $\square$

Theorem 3.18. *Let $(\mathbb{N}, X, \pi)$ be a dynamical system and X a compact Hausdorff space. Then*

(a) For each $p \in \beta\mathbb{N}$, π_p of definition 3.14 is a function from X to X.

(b) $\Sigma(X) = \{\pi_p : p \in \beta\mathbb{N}\}$.

Proof. (a) Let $x \in X$, and $p \in \beta\mathbb{N}$. Since X is a compact space, $\pi_p(x) \neq \emptyset$. Let $y \in \pi_p(x)$ then by Theorem 3.15, $\pi_p(x)$ is only a point in X. Hence π_p is well define.

(b) Given $p \in \beta\mathbb{N}$, take $\Theta = \{\{\pi_s :\ s \in A\} : A \in p\}$. We claim Θ is a filter in a set $\{\pi_s :\ s \in \mathbb{N}\}$:

(i) Since $\emptyset \notin p$ then $\Theta \neq \emptyset$. (ii) Let $A, B \in p$. Since p is an ultrafilter then $A \cap B \neq \emptyset$. Let $C_1 = \{\pi_s : s \in A\} \in \Theta$ and $C_2 = \{\pi_s : s \in B\} \in \Theta$. Then $C_1 \cap C_2 = \{\pi_s : s \in A \cap B\} \neq \emptyset$. (iii) Let $C_1 = \{\pi_s : s \in A\} \in \Theta$ and $C_2 = \{\pi_t : t \in B\}$ such that $C_1 \subseteq C_2$. Suppose $t \in A$ then $\pi_t \in C_1$ and since $C_1 \subseteq C_2$ then $\pi_t \in C_2$ implies that $t \in B$. Therefore $A \subseteq B$. Since p is unaltrafilter and $A \in p$ then $B \in p$. Hence $C_2 \in \Theta$.

Let $x \in X$. Then using the same proof above it is clear that $\Theta(x) = \{\{\pi_s(x) : s \in A\} : A \in p\}$ is a filter on X. By the definition of $\pi_p(x)$, then $\Theta(x)$ converges to $\pi_p(x)$. Thus $\pi_p \in cl\{\pi_s :\ s \in N\} = \Sigma(X)$. Hence $\{\pi_p : p \in \beta N\} \subseteq \Sigma(X)$.

Coversely, Given $f \in cl\{\pi_s :\ s \in N\} = \Sigma(X)$. Then there exists a filter Θ on a set $\{\pi_s :\ s \in N\}$ such that Θ converges to f. Take $K = \{\{s \in N,\ \pi_s \in A\} : A \in \Theta\}$, K is a filter on $\mathbb{N}$. Let $p \in \beta\mathbb{N}$ such that $K \subseteq p$. Let $x \in X$, then $\Theta(x)$ converges to $f(x)$. Then given any neighborhood G of $f(x)$, there is a set $A \in \Theta$ such that

$\{\pi_s(x) : \pi_s \in A\} \subseteq G$. Then $\{s \in \mathbb{N} : \pi_s \in A\} \subseteq \{s \in \mathbb{N} : \pi_s(x) \in G\} \in p$. By Theorem 3.15 $\pi_p(x) = f(x)$. Thus $f = \pi_p$. $\square$

Theorem 3.19. *Let $(\mathbb{N}, X, \pi)$ be a dynamical system and X a compact Hausdorff space. Define $\phi : N \longrightarrow \Sigma(X)$ by $\phi(s) = \pi_s$ where $s \in \mathbb{N}$, and let Φ be the continuous extension of ϕ to $\beta\mathbb{N}$. Then*

(a) For each $p \in \beta\mathbb{N}$, $\Phi(p) = \pi_p$.

(b) $\Phi(p + q) = \Phi(p) \circ \Phi(q)$ where $p, q \in \beta\mathbb{N}$.

$$\begin{tikzcd}
\beta(\mathbb{N}) \arrow[dr, "\Phi"] & \\
\mathbb{N} \arrow[u] \arrow[r, "\phi"] & \Sigma(X)
\end{tikzcd}$$

Proof. (a) Let $p \in \beta\mathbb{N}$. By Definition 3.14, $\pi_p(x) = \cap_{A\in p}\, cl\{\pi_s(x) : s \in A\}$, $x \in X$. Hence we have $\pi_p = \cap_{A\in p}\, cl\{\pi_s : s \in A\}$. Let $A \in p$. Then $\Phi(A) = \phi(A) = \{\pi_s : s \in A\}$. Since Φ is continuous, $\Phi(p) = \cap_{A\in p}\Phi\overline{(A)} \subseteq \cap_{A\in p}\overline{\Phi(A)} = \cap_{A\in p}\, cl\{\pi_s : s \in A\} = \pi_p$.

(b) By Theorem 3.17, $\Phi(p + q) = \pi_{p+q} = \pi_p \circ \pi_q = \Phi(p) \circ \Phi(q)$. $\square$

Definition 3.20. *Give a subsets A of $\mathbb{N}$ and given $s \in \mathbb{N}$, then $A + s = \{a + s \in \mathbb{N} : a \in A\}$*

Theorem 3.21. *On $(\beta\mathbb{N}, +)$. Define shift function $\phi_s : \beta\mathbb{N} \longrightarrow \beta\mathbb{N}$ by $\phi_s(p) = p + s = \{A + s : A \in p\}$ where $s \in S$ and $p \in \beta\mathbb{N}$. Then*

(a) $(\beta\mathbb{N}, \phi_s)$ is a dynamical system.

(b) For $p, q \in \beta\mathbb{N}$, $\phi_p(q) = p + q$

Proof. (a) Given $s, t \in \mathbb{N}$ and $p \in \beta\mathbb{N}$. Then $(\phi_s \circ \phi_t)(p) = \phi_s(\phi_t(p)) = \phi_s(p+t) = (p+t)+s = p+(t+s) = \pi_{t+s}(p)$. Hence $\phi_s \circ \phi_t = \phi_{t+s}$. Let $A \in p$. we claim that $\Phi(\overline{A}) \subseteq \overline{\phi(A)}$ and so ϕ_s is continuous as required. If $q \in \overline{A}$ then $A \in q$ implies $A + s \in q + s$. Hence $\phi_s(q) = q + s \in \overline{A+s} = \overline{\phi_s(A)}$.

(b) Given $p, q \in \beta N$ and $D \subseteq \mathbb{N}$. Then $D \in \phi_p(q) = \cap_{A \in p}\, cl\{\phi_s(q) : s \in A\}$ if and only if for each $B \in p$ there exists $s \in B$ such that $-s + D \in q$ if and only if $\{s \in \mathbb{N} : -s + D \in q\} \in p$. □

The following two Theorems are identical to that of Theorem 2.38 and Theorem 2.49 respectively and the proofs are not included here.

Theorem 3.22. *Let $(\mathbb{N}, X, \pi)$ be a dynamical system and X a compact Hausdorff space and suppose $x \in X$. Let L be a minimal left ideal of $(\beta\mathbb{N}, +)$. Then the following statements are equivalent:*

(a) x is almost periodic point.

(b) There exists $p \in L$ such that $\pi_p(x) = x$.

(c) There exists $p \in L$ with $p + p = p$ such that $\pi_p(x) = x$.

(d) There exists $p \in L$ with $p + p = p$, and $y \in X$ such that $\pi_p(y) = x$.

Proof. The proof is identical of Theorem 2.38. □

Theorem 3.23. *Let $(\mathbb{N}, X, \pi)$ be a dynamical system, $x, y \in X$. Then x and y are proximal if and only if there exist $p \in \beta\mathbb{N}$ and $z \in X$ such that $\pi_p(x) = \pi_p(y) = z$.*

Proof. The proof is identical of Theorem 2.49. □

Theorem 3.24. *Let $(\mathbb{N}, X, \pi)$ be a dynamical system. Let x and y be proximal in X with y almost periodic point. Then there is an idempotent u in the minimal left ideal of the semigroup $(\beta\mathbb{N}, +)$ such that $\pi_u(x) = y$.*

Proof. Let x and y be aproximal. By Lemma 2.52, there is a minimal left ideal L of $(\beta\mathbb{N}, +)$ such that whenever $t \in L$, $\pi_t(x) = \pi_t(y)$. Since y an almost periodic point. By Theorem 3.22 (c), there exists an idempotent $p \in L$ such that $\pi_p(y) = y$. Thus $\pi_p(x) = \pi_p(y) = y$. □

Definition 3.25. *Let $(N, +)$ be a semigroup . A set $A \subseteq \mathbb{N}$ is central in $\mathbb{N}$ if and only if there is an idempotent p in a minimal left ideal of $(\beta\mathbb{N}, +)$ such that $A \in p$*

The central sets developed by Bergelson and Hindman on Topological dynamics and the next theorem will give us a basic characterization about the central notion of being central in $(\mathbb{N}, +)$.

Theorem 3.26. *Let $(\mathbb{N}, +)$ be a semigroup with an identity e adjoined to it. A set $A \subseteq \mathbb{N}$ is central in $\mathbb{N}$ if and only if there exists a dynamical system $(\mathbb{N}, X, \pi)$ and there exist $x, y \in X$ and a neighborhood G of y such that x and y are proximal, y is almost periodic point, and $A = \{s \in \mathbb{N} : sx \in G\}$.*

Proof. Let A be a central subset of $\mathbb{N}$. Let $X = \prod_{n \in N}\{0, 1\}_n$, where X is the set of infinite sequence of "0"s and "1"s. Let $P_s : X \longrightarrow \{0, 1\}_n$ be the projection, where P_s assigns to the point $x = \{x(s)\}$, its *sth* coordinate $x(s) \in \{0, 1\}$ (i.e. $P_s(x) = x(s)$). For each $s \in N$, define $\pi_s : X \longrightarrow X$ by $\pi_s(x(t)) = x(s+t)$. Clearly π_s is continuous. Let $s \in \mathbb{N}$ and consider a subbasic set $P_t^{-1}(a)$ where $t \in \mathbb{N}$ and $a \in \{0, 1\}$. Then $\pi_s^{-1}(P_t^{-1}(a)) = P_{s+t}^{-1}(a)$. Also given $s, t \in N - \{e\}$, $r \in N$ and $x \in X$. We get $(\pi_s \circ \pi_t)(x)(r) = \pi_s(\pi_t(x))(r) = \pi_t(x)(s+r) = x(t+s+r) = \pi_{t+s}(x)(r)$. Therefore (N, X, π) is a dynamical system.

Let $x \in X$ such that $x(s) = 1$ when $s \in A$, and $x(s) = 0$ when $s \notin A$. Pick an idempotent u in the a minimal left ideal of $(\beta\mathbb{N}, +)$ with $A \in u$. Let $y = \pi_u(x)$. Then by Theorem 3.22, y is almost periodic point. Since $\pi_u(y) = \pi_u(\pi_u(x)) = \pi_{u+u}(x) = \pi_u(x)$. Thus by Theorem 3.23, x and y are proximal. Let

$G = P_e^{-1}(y(e)) = \{z \in X : z(e) = y(e) = y(e)\}$. Since $y = \pi_u(x)$, we have that $\{s \in N : \pi_s(x) \in G\} \in u$. So we may pick $t \in A$ with $\pi_t(x) \in G$. Since $t \in A$ hence $y(e) = \pi_t(x)(e) = x(t+e) = x(t) = 1$. Thus $G = \{z \in X : z(e) = 1\}$. Then given $s \in \mathbb{N}$, $\pi_s(x) \in G$ if and only if $\pi_s(x)(e) = 1$ if and only if $x(t) = 1$ if and only if $s \in A$. Hence $A = \{s \in N : \pi_s(x)\} \in X$ as required.

Coversly, since x and y are proximal, y is an almost periodic point, by Theorem 3.24 there exists an idempotent u in a minimal left ideal such that $y = \pi_u(x)$. Then $A \in u$, so that A is central. $\square$

3.3 Symbolic Dynamics: an Example

In this section we will apply the machinery in previous section for an example that illustrates some of the ideas that have been developed. This example is one of the most basic examples of an action of $\mathbb{N}$, because isomorphic copies of it arise in a wide variety of settings. Symbolic dynamics has been used in ergodic theory, topological dynamics, hyperbolic dynamics, information theory and complex dynamics. We will consider an $\mathbb{N}$ action where $\mathbb{N}$ is the set of natural number on the space $X = \prod_{i=1}^{\infty}\{0,1\}$, where X is the set of infinite sequence of "0"s and "1"s.

Lemma 3.27. *Let $T : X \longrightarrow X$ be a continuous map. The action of $\pi : N \times X \longrightarrow X$ defined by $\pi(n, x) = T^n(x)$ where $T^n(x)$ is a left shifting operator gives rise to an $\mathbb{N}$-system.*

Proof. Since any topology on the finite set $\{0, 1\}$ is compact and Hausdorff space, so by Tychonoff theorem X is compact and Hausdorff space.

We claim π is separately continuous map. Let $n \in \mathbb{N}$ be fixed, then $\pi(n, x) = T^n(x)$ is continuous since T is continues. Similarly if $x \in X$ is fixed. Since $\mathbb{N}$ is a discrete then π is continuous. Moreover since for all $n, t \in \mathbb{N}$ and $x \in X$,

$\pi(nt, x) = T^{nt}(x) = T^n(T^t(x)) = \pi(n, \pi(tx))$ we get an action. Therefore π is an $\mathbb{N}$-system. $\square$

Theorem 3.28. *The orbit closure of any tuple which contains every finite string of "0"s and "1"s is all of X. Therefore the system $(\mathbb{N}, X, \pi)$ is a point transitive $\mathbb{N}$-system.*

Proof. Given $x = (x_1, x_2, x_3, ..., x_n, x_{n+1}, ..) \in X$. Let p be a tuple in X which contains every finite string "0"s and "1"s. we need to show that for any neighborhood U of x there exists a point $k.p$ in the orbit, $\mathbb{N}.p$, of p such that $k.p \in U$ where $k \in \mathbb{N}$, i.e $U \cap \mathbb{N}.p \neq \emptyset$. Note for every basic neighborhood $U = \{x_1\} \times \{x_2\} \times ... \times \{x_n\} \times \prod_{i=1}^{\infty}\{0, 1\}$ of x it will contains the set $V_n = \{y \,/ y_i = x_i, 1 \leq i \leq n\}$ for some n,i.e.; V_n is the sequence whose entries agree with elements with U up to the n^{nt} entry. Give p be a point in X which contains every finite string "0"s and "1"s, i.e., $p = (p_1, p_2, ..., p_r, ..., p_k, x_1, x_2, ...x_n, p_s, p_{s+1}, ...)$. There exist some k such that $T^k(p) = x_1, x_2, ...x_n, p_s, p_{s+1}, ...) \in V_n \subseteq U$. Hence $\overline{k.p} = \overline{T^k(p)} = X$ and therefore X is p-transitive point. $\square$

The proof of the following Lemma is identical of the proof of Lemma 2.28, but we have it here as an alternate proof by using notions of ultrafilters.

Lemma 3.29. *Let $\alpha : \beta N = \beta N/ \simeq \longrightarrow \Sigma\beta\mathbb{N}$ be the homomorphism which is defined by $\alpha(p) = \tilde{p}$ where $\tilde{p}(x) = px$. Then the relation $\simeq$ defined by $p \simeq q$ if and only if $px = qx$ for all $x \in x$ is trivial and hence α is an homeomorphism.*

Proof. To show $\simeq$ is trivial on $\beta\mathbb{N}$ we need to show that for any distinct elements p and q in $\beta(N)$, p is not related to q. Since $\beta\mathbb{N}$ is Hausdorff space pick disjoint subset $A, B \subseteq \mathbb{N}$ such that $p \in \overline{A}$ and $q \in \overline{B}$ (implies $A \in p$ and $B \in q$). Define an element $w = (....., 1^{n^{th}}, ...) \in X$ if and only if $n \in A$. Therefore each

$n \in A$, $n.w = T^n(w) = (1, ...)$ has first coordinate is 1, and for each $n \in B$, $n.w = T^n(w) = (0, ...)$ has first coordinate is 0. Since $A \cap B = \emptyset$ and $A \in p$ and $B \in q$, then $p.w$ has first coordinate is 1 and $q.w$ has first coordinate is 0. It follows that $p \neq q$ as a function on X. Thus $\simeq$ is the trivial relation on $\beta\mathbb{N}$ which is the kernel relation on $\beta\mathbb{N}$ for the homomorphism $\alpha : \beta\mathbb{N} = \beta\mathbb{N}/\simeq \longrightarrow \Sigma\beta\mathbb{N}$. By 1^{st} isomorphism theorem there exist a continuous isomorphism $\widehat{\pi} : \beta\mathbb{N} \longrightarrow \Sigma\beta\mathbb{N}$. Since $\beta\mathbb{N}$ is compact, then this isomorphism is homeomorphism. $\square$

The following theorem tell us that our example has a proximal pair which is not an equivalence relation.

Theorem 3.30. *Let $(\beta\mathbb{N}, +)$ be a semigroup where $+$ is the extension of the usual addition. The dynamical system $(N, \beta\mathbb{N})$ has at least one proximal pair.*

Proof. Let $(\mathbb{N}, \beta\mathbb{N})$ is a distal. By Theorem 2.60, $\Sigma(\beta\mathbb{N})$ is a group. By Theorem 3.29, we have shown that $\Sigma(\beta\mathbb{N})$ and $\beta\mathbb{N}$ are isomorphic. It means that $(\beta\mathbb{N}, +)$ is a group which is a contradiction. Hence $(\mathbb{N}, \beta\mathbb{N})$ is not distal. Hence there exists at least one proximal pair. Also, since $(\beta\mathbb{N}, +)$ has more than one minimal left ideal, hence by Theorem 2.54, the proximal relation is not an equivalence relation. $\square$

Definition 3.31. *Let $n = (n_0, n_1, n_2, ...)$ and $t = (t_0, t_1, t_2, ...)$ be two elements in X. Let $d[n, t] = \Sigma_{k \in Z} \frac{|n_k - t_k|}{2^{|k|}}$.*

We will now show that this function $d[n, t]$ is a generalization of the usual concept of distance.

Theorem 3.32. *d is a metric on X.*

Proof. Clearly, $d[n, t] \geq 0$ for any $n, t \in X$, and $d[n, t] = 0$ if and only if $n_i = t_i$ for all i. Since $|n_i - t_i| = |t_i - n_i|$, it follows that $d[n, t] = d[t, n]$. Finally if r, n and $t \in X$, then $|r_i - n_i| + |n_i - t_i| \geq |r_i - t_i|$ from we deduce that $d[r, n] + d[n, t] \geq d[r, t]$. $\square$

The following Theorem says that two points are close to each other if their first few terms agree.

Theorem 3.33. *(The proximity theorem): Let $n, t \in X$.*

(a) If $n_i = t_i$ for $i = 0, 1, 2, ..., m$, then $d[n, t] \leq \frac{1}{2^m}$.

(b) If $d[n, t] \leq \frac{1}{2^m}$, then $n_i = t_i$ for $i = 0, 1, 2, ..., m$.

Proof. If $n_i = t_i$ for $i \leq n$,then

$d[n, t] = \sum_{i=0}^{m} \frac{|n_i - t_i|}{2^i} + \sum_{i=m+1}^{\infty} \frac{|n_i - t_i|}{2^i} \leq \sum_{i=m+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^n}$.

On the other hand, if $n_j \neq t_j$ for some $j \leq m$, then we must have $d[n, t] \geq \frac{1}{2^i} \geq \frac{1}{2^m}$. Therefore if $d[n, t] < \frac{1}{2^n}$, then $n_i = t_i$ for $i \leq m$. □

Theorem 3.34. *X has a dense set of periodic points.*

Proof. Let $T : X \longrightarrow X$ be the shift operator which is a continuous function. Let S be the set of periodic points. To show S is dense in X we need to produce a sequence of periodic points S_n which is convergent to an arbitrary point $x = (x_0, x_1 x_2, ...)$ in X. We define a sequence $S_n = (x_0, x_1, ...x_n, x_0, x_1, ...x_n, x_0, ...)$, this means S_n is the repeating sequence whose entries agree with x to the n^{th} entry. By Proximity Theorem, $d[x, s_n] < \frac{1}{2^n}$, so we have $S_n \longrightarrow x$. □

The induced topology by the metric d of Theorem 3.32 is equivalent to the product topology on X.

Lemma 3.35. *The metric d defined on X induced topology is equivalent to the product topology on X.*

Proof. Let $U = \prod_{n=0}^{\infty} O_n = \{t_0\} \times \{t_1\} \times ...\{t_n\} \times \prod_{i=n+1}^{\infty}\{0, 1\}$ be a basic open set in product topology. Let $x = (x_0, x_1, ...x_n, x_{n+1}...) \in U$, then $x_j \in O_j$ $\forall j$. But $O_j = \{t_j\}$ $j = 0, 1, ..., n$. So for any $x \in U$ we have $x = (t_1, t_2, ...t_n, x_{n+1}, ...)$. Take $r = \frac{1}{2^{n+1}}$ and $B_1(x_1, r), B_2(x_2, r), ...B_n(x_n, r)$. Since $d_i(x_i, y_i) = |y_i - x_j| = 0$ or 1, then we have $B_j(x_j, r) = \{t_j\}$ for $j = 0, ...n$. Therefore $\prod_{j=0}^{\infty} B_j(x_j, r) \times \prod\{0,1\} = U$. $\square$

Definition 3.36. *In the dynamical system (S, X, π) a point $x \in X$ is called periodic with period t if there exist $t > 0$ so that $\pi(t, x) = x$.*

Example 3.37. *In the set of sequences in our space $X = \prod_{i=1}^{\infty}\{0,1\}$ where the action is shifting a periodic point will be like $x = (....bbbb.....)$ for some finite block bbbb.*

Theorem 3.38. *Periodic points are almost periodic points in $(X, \mathbb{N})$.*

Proof. Let $x = (x_1, x_2, x_3, ..., x_n, x_1, x_2, x_3, ..., x_n, ...)$ be a periodic point and let $U = \{x_1\} \times \{x_2\} \times ... \times \{x_n\} \times \prod_{i=1}^{\infty}\{0,1\}$ be an open neighborhood of x. Let $K = \{1, 2, ..., n\}$ be a compact subset of $\mathbb{N}$. Suppose $m \in \mathbb{N}$, we need to show there exist $k \in K$ such that $mkx \in U$. If $m \in K$ then $n-m \in K$ and $m+(n-m)x = nx \in U$. If $m \notin K$, pick $j \in K$ such that $m \equiv j \pmod{n}$, then $m + (n - j) = rn$ for some $r \in \mathbb{N}$. Hence $(m + (n - j))x = rn.x \in U$. $\square$

The converse of Theorem 3.38 is not true, and we will provide a counter example.

Definition 3.39. *A finite subtuple U in a tuple w is "Syndetically often" iff there exist $k \in N$ (depending on U) such that U is a subtuple of every subtuple of length k (i.e U is a subtuple of $w(n), w(n+1), ..., w(n+k)$).*

Definition 3.40. *A set M in the space X is minimal if does not contain any other set A satisfying the same property as set M.*

Lemma 3.41. *A point $x \in X$, where the action is the shifting defined in Lemma 3.27, is almost periodic point iff every subtuple in x is syndetically often.*

Proof. Suppose $x = (x_1, x_2, x_3, ..., x_r, ...)$ is an almost periodic point. Let $w = (x_m, x_{m+1}, ..., x_{m+l})$ be any finite tuple of length l. We want to show that this finite tuple w is syndetically often by showing that by defenition there is k such that w is a subtuple of any finite subtuple from x of length k. Consider $U = \{x_1\} \times \{x_2\} \times ...\{x_{m+1}\} \times ... \times \{x_{m+l}\} \times \prod_{i=1}^{\infty}\{0,1\}$ be an open neighborhood of x. By definition of almost periodic point there exist a compact set $K = \{\alpha_1, \alpha_2, ..., \alpha_p\}$ of $\mathbb{N}$ such that given $n \in \mathbb{N}$, there exist $\alpha_k \in K$ with $(\alpha_k + n)x \in U$. That means the string $x_1, x_2, x_3, ..., x_{m+l}$ will be repeating in our sequence x and the gaps are of finite length. Let $\beta =$ max K. We claim that any subtuple of length $J = (m + l) + l + \beta$ contains w. Note that for the sequence

$x = (x_1, x_2, x_3, ..., x_m, x_{m+1}, ...x_{m+l}, x_{\beta_1}, x_{\beta_2}, ...x_1, x_2, x_3...)$ we have the following:

Case (1): Clearly any subtuple which starts with an initial points $\{x_i\}$ where $i = 1, 2, ..., m - 1$ and length J will contain w since $J > m + l$.

Case (2): If the subtuple of length J start from any point from the gap $x_{\beta_1}, x_{\beta_2}, ...$ Since x is almost periodic point we see from above the maximal length of this gap will be β and then $x_1, x_2, x_3, ..., x_m, x_{m+1}, ..$ will appear again. Therefore the subtuple of length J with an initial point from the gap contains w.

Case (3): Suppose the subtuple starts with an initial point $\{x_i\}$ where $i = m, m+1, ..., m+l$. Let $\gamma = x_i, x_{i+1}, x_{i+2}, ...x_{i+\alpha_i}, x_1, x_2, x_3, ..., x_m, x_{m+1}$ with length J. It is easy to see that γ contain the tuple w and $|\gamma| = \alpha_i + l + m + l \leq J$. Therefore w is syndetically often.

Conversely, let $x = (x_1, x_2, x_3, ..., x_r, ...) \in X$ such that every finite subtuple is syndetically often. Let $U = \{x_1\} \times \{x_2\} \times ...\{x_r\} \times \prod_{i=1}^{\infty}\{0,1\}$ be an open neighborhood of x. Then $\{x_1, x_2, ..., x_r\} \subseteq \{x_m, x_{m+1}, ..., x_{m+k}\}$ for some length $\alpha_k \in N$. Consider $K = \{1, 2, ..., k + 1\}$ be the compact set in $\mathbb{N}$. Consider $x_{n+1}, x_{n+2}, ..., x_{n+k}$. Note that this contains $x_1, x_2, x_3, ..., x_r$. Now let $x_{n+m} = x_1$ for some $m \in k$. Then it is easy to see that $(m + n)x \in U$. $\square$

Example 3.42. *Let* $X = \prod_{i=1}^{\infty}\{0,1\}$. *We will write* $\bar{0} = 1, \bar{1} = 0$ *and if* $a = a_1a_2...a_m$ *is a finite tuple, define* $\bar{a} = \bar{a_1}\bar{a_2}...\bar{a_m}$ *(for example if* $a = 01101$ *then* $\bar{a} = 10010$ *). We define inductively a sequence of finite tuples* a_n *where each* a_n *is a subtuple of* a_{n+1}*(i.e we have* $a_1 \subset a_2 \subset a_3 \subset ....$*). Let* $a_1 = 0$*, and if* a_n *has been defined, let* $a_{n+1} = a_n\bar{a_n}$*. Thus* $a_2 = 01, a_3 = 0110, a_4 = 01101001$ *etc , then* $\lim_{n\to\infty} a_n = w = 0110100110010110....$

Another way to construct this is as follows.

Definition 3.43. *If* b *is a finite tuple, let* b^* *denote the tuple of length twice the length of* b *obtained from* b *by the substitution* $0 \to 01$, $1 \to 10$*. (For example, if* $b = 01101$*, then* $b^* = 0110100110$*). Let* $b_1 = a_1 = 0$*, and inductively let* $b_{n+1} = b_n^*$*.*

Lemma 3.44. *Let* b *as on Definition 3.43 then:*

(a) $(\bar{b^*}) = (\bar{b})^*$*.*

(b) $(a_{n-1}\bar{a}_{n-1})^* = a^*_{n-1}\bar{a}^*_{n-1}$*.*

(c) $b_n = a_n$*.*

Proof. By using the definition it is easy to see that :

$(\bar{b^*}) = \bar{b\bar{b}} = \bar{b}b = (\bar{b})^*$ and $(a_{n-1}\bar{a}_{n-1})^* = (a_{n-1}\bar{a}_{n-1}\bar{a}_{n-1}a_{n-1}) = a^*_{n-1}\bar{a}^*_{n-1}$. For part (c) Suppose, inductively that $b_k = a_k$, for $k \leq n$. Then $b_n = a_n = a_{n-1}\bar{a}_{n-1}$, and $b_{n+1} = b_n^* = a_n^* = (a_{n-1}\bar{a}_{n-1})^* = a^*_{n-1}\bar{a}^*_{n-1} = a^*_{n-1}\bar{a^*}_{n-1} = b^*_{n-1}\bar{b^*}_{n-1} = b_n\bar{b_n} = a_n\bar{a_n} = a_{n+1}$. □

It follows from the construction above that the limit of a finite tuple w produces itself under the substitution $0 \to 01$, $1 \to 10$.

Theorem 3.45. *Let* w *be as defined in Example 3.42 then* w *is an almost periodic point but is not periodic point.*

Proof. We claim every finite subtuple in w is syndetically often. Note that for 0 it is syndetically often since the sequence made up of pairs 01 and 10. But the sequence reproduces itself under the substitution $0 \to 01$, $1 \to 10$. So 01 is also syndetically often. For the same reason for initial finite subtuple $0110, 01101001$ etc are syndetically often. But this sequence of initial finite subtuple (in our notation b_n) include all tuple of w as a subtuples, so all tuple w is syndetically often (If $r_1, r_2, ...r_k$ any subtuple in w then we can extend this subtuple from both side by starting with 01100 from the left side and we will get $a_k = (01100...r_1, r_2, ...r_k, m_1m_2....)$, for example for the subtuple 01011 we can extend this and we will get $a_5 = 0110100110010110$ which contains this subtuple)

To obtain an almost periodic point of X, we extend w by reflection. That is, if $n < 0$, we define $w(n) = w(-n-1)$, thus w is the two- sided infinite sequence (i.e $w = ...10010110\overset{\downarrow}{0}110100110010110....$ where the vertical arrows indicates the 0^{th} position $w(0)$. Then w is the "limit" of the tuple $\overleftarrow{a}_n a_n$ (that is if $a = a_1a_2...a_m$ then $\overleftarrow{a} = a_m a_{m-1}....a_1$) where the 0^{th} position is the initial finite tuple of a_n. We claim that w is an almost periodic point. Note we can easily to show by induction $\overleftarrow{a}_{2n+1} = a_{2n+1}$, for $n = 1, 2, ...$

$$\text{If } n = 1, \text{then } a_3 = (0110) = \overleftarrow{a}_3$$

$$\text{suppose its true for k then, } a_{2k+1} = \overleftarrow{a}_{2k+1}$$

for k+1

$$\begin{aligned} a_{2(k+1)+1} &= a_{2k+3} \\ &= a_{2k+2}\ \bar{a}_{2k+2} \\ &= a_{2k+1}\ \bar{a}_{2k+1}\ \overline{a_{2k+1}\ \bar{a}_{2k+1}} \\ &= a_{2k+1}\ \bar{a}_{2k+1}\ \bar{a}_{2k+1}\ a_{2k+1} \\ &= \overleftarrow{a}_{2k+1}\ \overleftarrow{\bar{a}}_{2k+1}\ \overleftarrow{\bar{a}}_{2k+1}\ \overleftarrow{a}_{2k+1} \end{aligned} \tag{3.19}$$

$$
\begin{aligned}
&= \overleftarrow{a}_{2k+2}\ \overleftarrow{\bar{a}}_{2k+2} \\
&= \overleftarrow{a}_{2k+3}
\end{aligned}
\tag{3.20}
$$

and the similar argument above for the one side sequence shows that $a_{2n+1}a_{2n}$ syndetically often in w. It follows as above that all finite subtuple occur syndetically often. Therefore w is an almost periodic point by Lemma 3.41. Its orbit closure, M_0 is minimal set by Theorem 2.45. We now claim w is not periodic . Let w_0 denote the one-side of the sequence w. Then $w = \overleftarrow{w}_0 w_0$. An argument similar to the one given above for w shows that $\xi = \overleftarrow{\bar{w}}_0 w_0$ is also an almost periodic point ($\xi = ...0110100\overset{\downarrow}{1}01101001....$). Then the orbit closure of ξ say M_1 is a minimal set by Theorem 2.45.

We note that $w(n) = \xi(n)$ for $n \geq 0$, it follow that $lim\ d(T^n(w), T^n(\xi)) = 0$ (For example if k=5 . since the right side of the two sequences are the same then $d(T^k(w), T^k(\xi)) = \sum_{n=-\infty}^{5} \frac{|w(n)-\xi(n)|}{2^{|n|}} = \sum_{n=5}^{\infty} \frac{|w(-n)-\xi(-n)|}{2^{|k|}} = \frac{1}{2^4}$, in general if n=m then $d(T^k(w), T^k(\xi)) = \sum_{n=m}^{\infty} \frac{|w(-n)-\xi(-n)|}{2^{|k|}} = \frac{1}{2^{m-1}}$ and as $k \longrightarrow \infty$ then $d(T^k(w), T^k(\xi)) \longrightarrow 0$) (We say that w and ξ are positively asymptotic). Since $T^n(w) \in M_0$, $T^n(\xi) \in M_1$, then $M_0 \cap M_1 \neq \emptyset$. Since the intersection of closed invariant sets is closed invariant, so $M_0 \cap M_1$ is closed invariant. But M_0 is a minimal set and so M_0 can not contain $M_0 \cap M_1$. Therefore $M_0 = M_1$ (Since for two minimal set M_0 and M_1 either $M_0 = M_1$ or $M_0 \bigcap M_1 = \emptyset$). We recall that a finite minimal set can not contains (distance and infinite) asymptotic points. Hence the minimal set M_0 is infinite, and w is not periodic. $\square$

References

[1] J.Auslander. *Minimal Flows and Their Extensions*, North-Holland, Amsterdam, 1988.

[2] J.E.Berglund, H.D.Junghenn, and P.Milens. *Analysis on Semigroups*, John Wiley and Sons, Inc., New York, 1989.

[3] J.F.Berglund and N.Hindman. *Sums of Idempotents in βN, Semigroup Forum*, 44 (1992), 107-111.

[4] J.H.Carruth, J.A Hildebrant and R.J.Koch. *The Theory of Topological Semigroups*, Marcel Dekker, New York, 1983.

[5] R.L.Devaney. *Chaotic Dynamical Systems, Second Edition* , Addition-Wesley, New York, 1989.

[6] R.Ellis. *Lectures on Topological Dynamic*, Benjamin, New Yourk, W.A. Benjamin, 1969.

[7] S.Glasner. *Proximal Flows, Springer Lecture Notes in Mathematics*, 517 Springer-Verlag, Heidelberg, 1976.

[8] W.H.Gottscchalk. *Almost Periodic Points with Respect to Transformation Semigroups*, Annals of Math. 47(1946), 762-766.

[9] N.Hindman and A.Lisan. *Does N^* Contain A topological and Algebraic Copy of βN? Topological and Its Applications*, 35(1990), 291-297.

[10] N.Hindman, J.Lawson and A.Lisan. *Separating Points of βN*, Can. J. Math. 46(4), (1994), 758-771.

[11] N.Hindman and I.Leader. *The Semigroup of Ultrafilter Near o Semigroup Forum*, 59 (1999), 33-55.

[12] N.Hindman and D.Strauss. *Algebra In The Stone-Cech Compactification Theory and Applications*, Second Edition, 2010.

[13] J.D.Lawson. *Flows and Compactifications*, J. London Math. Soc.(2). 46(1992), 349-363.

[14] J.D.Lawson and A.Lisan. *Transitive Flows: Asemigroup Approach*, Mathematika 38(1991), 348-361.

[15] J.D.Lawson and A.Lisan. *Flows, Congruences, and Factorizations, Topological and Its Applications*, 58(1994), 35-46.

[16] J.D.Lawson and A.Lisan. *Groups Associated with Minimal Flows*, Czechoslovak Mathematical. 55 (2005), 471-477.

[17] A.Lisan. *Ultrafilters on A discrete Set with Two Binary Operations*, Springer-Verlag New Yourk. 43 (1991), 77-81.

[18] A.Lisan. *Quasifactors, Proximal Extensions and Other Structures of Nondiscrete Minimal Flows*, Houston Jornal of Mathematics. 32 (2006), 1-7.

[19] J.R.Munkres. *Topology Second Edition*, Prentice Hall,Inc.,NJ, 2000.

Vita

Lieth Abdalateef Majed was born on September 07 1976, in Diyala City, Iraq. He finished his undergraduate studies at Al-Mustansiriya University May 1998. He earned a master of science degree in mathematics from Al-mustansiriya university in December 2001. He earned a master of science degree in mathematics from Western Mechigan University in December 2011. In January 2011 he came to Louisiana State University to pursue graduate studies in mathematics and earned a master of science degree in mathematics from Louisiana State University in December 2013. He is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in December 2015.