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On well-filtered spaces and ordered sets

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A topological space is well-filtered if any filtered family of compact sets with intersection in an open set must have some member of the family contained in the open set. This well-known and important property automatically satisfied in Hausdorff spaces assumes a life of its own in the T_0-setting. Our main results focus on giving general sufficient conditions for a T_0-space to be well-filtered, particularly the important case of directed complete partially ordered sets equipped with the Scott topology.

1. Introduction

Driven by topologies arising in the spectral theory of rings and C^*-algebras, in the domain theory [2] of theoretical computer science, and in categorical topology, the theory of non-Hausdorff topological spaces has undergone substantial development, as the recent monograph by Goubault-Larrecq [1] documents. Spectral and domain theory are concerned with special classes of T_0-spaces, spaces in which any two points can be separated by an open set, and these will be the main concern in this paper. Many of the familiar concepts from general topology reappear in the T_0-setting, but frequently in an altered or nuanced manner. For example in the Hausdorff setting, compact subspaces satisfy the properties of being closed under finite intersections, being locally compact, and having nonempty intersections for filtered families of nonempty compact sets. The property of compactness alone turns out to be much weaker in the T_0-setting, and any of the three mentioned properties may fail for a compact set. Indeed the closest analog to a compact Hausdorff space in the T_0-setting is that of a stably compact space, a special type of compact space that also satisfies the other three conditions. One might say that topology in the T_0-setting can give deeper insight into the

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nature of topological properties by seeing what needs to be hypothesized to achieve these properties in this more general setting.

In this paper we uncover new insights into one of the extra properties needed to simulate in the T_0-setting the behavior of compact subsets in a Hausdorff space, namely the property of being well-filtered, which we define in the next section. The requirement that a space be Hausdorff, or more generally sober, are the only sufficient conditions that have been identified for a space to be well-filtered to the best of our knowledge. The main purpose of this paper is to identify other useful sufficient conditions that can be applied to existing and future examples and classes of examples, particularly examples outside the sphere of the intensively studied special class of dcpos consisting of continuous domains and quasidomains.

2. Monotone convergence spaces

Recall that a subset A of a topological space X is *saturated* if it is the intersection of all the open sets containing it. In a topological space X with at least T_1-separation every subset is saturated, so the notion is only interesting for those spaces that have singleton subsets that are not closed. A nonempty family of subsets of a set X is said to be *filtered* if any two members of the family contain some third member. A space is said to be *well-filtered* if for every filtered family $\mathcal{F}$ of compact saturated sets with intersection $\bigcap \mathcal{F}$ contained in some open set U, it follows that $F \subseteq U$ for some $F \in \mathcal{F}$ [2, Definition I-1.24.1], [1, Section 8.3.1]. This basic, well-known, and useful property of compact sets holds in any Hausdorff space and holds more generally for an important class of T_0-spaces, the sober spaces [2, Lemma II-1.19]. (Recall a T_0-space is sober if every closed set that is not the union of two smaller ones is the closure of a singleton set.) Conversely if X is locally compact, T_0, and well-filtered, then X is sober [2, Theorem II-1.21] [1, Proposition 8.3.8], but this is not true in general without local compactness.

The *order of specialization* for a T_0-space X is a partial order on X given by $x \leq y$ if $x \in \overline{\{y\}}$. For $A \subseteq X$ we set $\uparrow A = \{y \in X : \exists x \in A, x \leq y\}$; $\downarrow A$ is defined in an order dual fashion. A directed subset D of a partially ordered set P is a nonempty subset satisfying for every $d_1, d_2 \in D$, there exists $d_3 \in D$ such that $d_1, d_2 \leq d_3$ and a directed subset of a T_0-space is one that is directed with respect to the order of specialization. A directed subset is said to converge to some point of a T_0-space if it converges in the sense of a net, where the directed set is viewed as a net indexed by itself. A T_0-space is called a *monotone convergence space* [2, Definition II-3.12] if every directed set has a supremum to which it converges.

Proposition 2.1. *A well-filtered T_0-space X is a monotone convergence space.*

Proof. Let D be a nonempty subset of X that is directed in the order of specialization with supremum e, and let U be an open set around e. Since D is directed, the family $\{\uparrow d : d \in D\}$ is filtered and one easily verifies that $\bigcap\{\uparrow d : d \in D\}$ consists of all upper bounds of D, which is contained in $\uparrow e$. Since X is well-filtered, $\uparrow d_0 \subseteq U$ for some $d_0 \in D$, and hence $\uparrow d \subseteq \uparrow d_0 \subseteq U$ for all $d \geq d_0$. □

A *coherent* topological space is one in which the intersection of two compact saturated sets is compact (the intersection is automatically saturated).

Proposition 2.2. *For a T_0-space X the following are equivalent.*

(1) *X is compact, coherent, and well filtered.*
(2) *X is compact with respect to the patch topology, the topology with a closed subbasis consisting of the closed sets in the given topology together with the compact saturated sets.*

Proof. (1)⇒(2): By the Alexandroff Subbasis Theorem, it suffices to show that every subbasic open cover $\mathcal{U}$ has a finite subcover. Let W be the union of all members of $\mathcal{U}$ that are open in X. If $W = X$, then finitely many members of $\mathcal{U}$ cover X by the assumption that X is compact. If W is a proper subset, we consider the family of compact saturated sets $\mathcal{K} = \{K : X \setminus K \in \mathcal{U}\}$. Since $\mathcal{U}$ covers X, it follows that the sets $X \setminus K \in \mathcal{U}$ must cover $X \setminus W$, i.e., $\bigcap \mathcal{K} \subseteq W$. Since X is coherent, the family consisting of finite intersections of members of $\mathcal{K}$ is a filtered family of compact saturated sets with intersection contained in W. Using well-filteredness we conclude $\bigcap_{i=1}^n K_i \subseteq W$ for some finite family of $\mathcal{K}$. It follows that $\bigcap_{i=1}^n K_i$ is covered by members of $\mathcal{U}$ that are open in X, and hence finitely many of them, say $U_1, \ldots, U_m$ cover the compact set $\bigcap_{i=1}^n K_i$. The collection

$$U_1, \ldots, U_m, X \setminus K_1, \ldots, X \setminus K_n$$

is then a finite subcover of X coming from $\mathcal{U}$.

(2)⇒(1): The compactness of the patch topology implies the compactness of the given topology, since the latter is coarser. Let K, L be two compact saturated sets. Then K and L are closed in the patch topology (by definition) and hence their intersection $K \cap L$ is closed. Thus $K \cap L$ is a closed subset of the compact space X equipped with the patch topology, and hence $K \cap L$ is compact in the patch topology, thus compact in the weaker original topology of X.

Let $\mathcal{F}$ be a filtered family of compact saturated nonempty sets in X with intersection contained in an open set U. Then each $F \in \mathcal{F}$ is closed in (X, patch), a compact space, and hence the filtered family of closed sets $\mathcal{F}$ must have some member F with $F \subseteq U$, by a basic property of compact spaces. It follows that X is well-filtered. □

Remark 2.3. (1) It follows from Proposition 2.1 that a T_0-space satisfying either of the equivalent conditions of the preceding proposition is a monotone convergence space.

(2) The special case of the preceding proposition that X is a dcpo equipped with the Scott topology (see Section 3) was established in [4]. The main ideas of the proof appear already there.

Proposition 2.4. *Let X be a monotone convergence space with the property that $\downarrow(K \cap A)$ is closed whenever K is a compact saturated set and A is a closed set. Then X is well-filtered.*

Proof. Let $\mathcal{K}$ be a filtered collection of compact saturated sets and let U be an open set such that $\bigcap_{K \in \mathcal{K}} K \subseteq U$. Assume that no $K \in \mathcal{K}$ is contained in U. Then setting $A = X \setminus U$, we have $A \cap K \neq \emptyset$ for all $K \in \mathcal{K}$. Let $\mathcal{C}$ be the collection of all closed sets C that are subsets of A and have the property that $C \cap K \neq \emptyset$ for all $K \in \mathcal{K}$. Note that $A \in \mathcal{C}$ and with respect to the partial order of inclusion is the largest element. If $\mathcal{M}$ is a totally ordered subset of $\mathcal{C}$ and $M_0 = \bigcap \mathcal{M}$, then $M_0 \in \mathcal{C}$ since for any $K \in \mathcal{K}$,

$$M_0 \cap K = \left(\bigcap_{M \in \mathcal{M}} M \right) \cap K = \bigcap_{M \in \mathcal{M}} M \cap K \neq \emptyset,$$

where the nonemptiness follows from the fact the last intersection is the intersection of a chain of nonempty sets closed in the compact space K. We conclude from Zorn's Lemma that the family $\mathcal{C}$ has some minimal element C_0. Since C_0 is closed, it follows from the fact that X is a monotone convergence space that every chain in C_0 with respect to the order of specialization has a supremum to which it converges, and hence the supremum is also in C_0. It follows, again from Zorn's Lemma, that C_0 has at least one maximal element y. Since $\bigcap \mathcal{K} \subseteq U$ and $y \in C_0 \subseteq A$, there exists some $K \in \mathcal{K}$ such that $y \notin K$. From the maximality of y in C_0, we conclude that $y \notin \downarrow(K \cap C_0)$, and hence $\downarrow(K \cap C_0)$ is a proper subset of $\downarrow C_0 = C_0$. By hypothesis $\downarrow(K \cap C_0)$ is closed. For any $K_1 \in \mathcal{K}$, there exists $K_2 \in \mathcal{K}$ such that $K_2 \subseteq K_1 \cap K$. Since $C_0 \in \mathcal{C}$, $\emptyset \neq C_0 \cap K_2 \subseteq \downarrow(C_0 \cap K) \cap K_1$. We conclude that $\downarrow(C_0 \cap K)$ is a member of $\mathcal{C}$ strictly smaller than C_0, a contradiction. □

3. Directed complete posets

A directed complete partially ordered set (dcpo) is a partially ordered set P for which every directed set has a supremum. The *Scott topology* on a dcpo P has for open sets all upper sets U such that if $\sup D \in U$ for some directed set D, then some residual subset of D belongs to U, i.e., the directed set is eventually in U. The closed sets are the lower sets closed under taking directed suprema. The smallest closed set containing a point x is easily seen to be $\downarrow x$, so the original order on P agrees with the order of specialization for the Scott topology. It then follows readily that P endowed with the Scott topology is a monotone convergence space. Dcpos endowed with the Scott topology have been widely studied in domain theory [2] and have been formative for substantial portions of the theory of T_0-spaces.

The *Lawson topology* is a refinement of the Scott topology and is formed by taking the join of the Scott topology and the topology with subbasis of closed sets all $\uparrow x$, $x \in P$.

Proposition 3.1. *Let P be a dcpo for which every $\uparrow x$ is compact in the relative Lawson topology. Then with respect to the Scott topology $\sigma(P)$, P is a monotone convergence space satisfying $\downarrow(K \cap A)$ is Scott-closed whenever K is a Scott-compact saturated set and A is a Scott-closed set. Hence $(P, \sigma(P))$ is well-filtered.*

Proof. Let A be a nonempty Scott-closed subset of P and let K be a Scott-compact saturated set such that $K \cap A \neq \emptyset$. We present the proof in a series of steps.

Step 1: For each $x \in A$, $\downarrow(\uparrow x \cap A)$ is Scott-closed. Let $D \subseteq \downarrow(\uparrow x \cap A)$ be a directed set with $e = \sup D$. For each $d \in D$, $\uparrow d$ is Lawson-closed, $\uparrow x$ is also Lawson-closed, and hence $\uparrow d \cap \uparrow x \cap A$ is Lawson-closed. Fix some $d_0 \in D$. By hypothesis $\uparrow d_0$ is Lawson-compact, so its closed subset $\uparrow d_0 \cap A$ is also compact. Thus $\{\uparrow d \cap \uparrow x \cap A : d \in D, d_0 \leq d\}$ is a filtered family of nonempty Lawson-closed subsets contained in the Lawson-compact subset $\uparrow d_0 \cap A$, and hence has nonempty intersection. For y in this intersection, $y \geq d$ for all $d \in D$, and hence $\sup D \leq y$. Also by construction $y \in \uparrow x \cap A$, so $\sup D \in \downarrow(\uparrow x \cap A)$. We conclude that $\downarrow(\uparrow x \cap A)$ is Scott-closed.

Step 2: $\downarrow(K \cap A)$ is Scott-closed. Let D be a directed set contained in $\downarrow(K \cap A)$. Since $\downarrow(K \cap A) \subseteq A$, we conclude from step 1 that $A_d = \downarrow(\uparrow d \cap A)$ is Scott-closed for each $d \in D$. Now $\uparrow d$ meets $K \cap A$ for each $d \in D$ and $A_d \cap K$ contains this intersection, so $A_d \cap K$ is a filtered family of nonempty closed subsets in the relative Scott topology of the Scott-compact set K, and hence has a nonempty intersection. For any w in the intersection, $w \in K \cap A$ and $\uparrow w \cap (\uparrow d \cap A) \neq \emptyset$ for all $d \in D$. As d varies, the family $\{\uparrow d \cap \uparrow w \cap A\}$ forms a filtered family of nonempty Lawson-closed sets contained in the Lawson-compact set $\uparrow w$, and hence has a nonempty intersection. For any y in the intersection, $d \leq y$ for all $d \in D$, so $\sup D \leq y$, i.e., $\sup D \in \downarrow(\uparrow w \cap A) \subseteq \downarrow(K \cap A)$ (the inclusion follows from $w \in K \cap A$ implies $\uparrow w \subseteq \uparrow K = K$). Hence $\downarrow(K \cap A)$ is Scott-closed.

It follows from the remarks preceding this proposition that P equipped with the Scott topology is a monotone convergence space, so the conclusion of the proposition follows from step 2 and Proposition 2.4. □

Two basic examples of dcpos are complete lattices and the slightly more general *bounded complete* dcpos, those dcpos for which every nonempty set has a greatest lower bound. It is known that these examples are compact in the Lawson topology [2, Theorem III-1.9], and since each $\uparrow x$ is by definition closed in the Lawson topology, we have the following corollary.

Corollary 3.2. *Let L be a complete lattice or a bounded complete dcpo. Then L equipped with the Scott topology is well-filtered. More generally, any dcpo P with compact Lawson topology has a well-filtered Scott topology.*

We consider two well-known examples in light of our results. The first is an example of Peter Johnstone [5].

Example 3.3. (Johnstone space) The Johnstone space $\mathbb{N} \times (\mathbb{N} \cup \{\infty\})$ has partial order defined by $(j,k) \leq (m,n)$ if $j = m$ and $k \leq n$, or if $k \leq m$ and $n = \infty$. Johnstone showed that it is an example of a countable dcpo for which the Scott topology is not sober and hence that the sobriety of continuous domains does not extend to dcpos. It is significant for our purposes that it satisfies the stronger condition of not being well filtered; see [1, Exercise 8.3.9].

Later John Isbell supercharged the example of Johnstone to produce a complete lattice [3].

Example 3.4. (Isbell lattice) Isbell gave an example of a complete lattice that is not sober in the Scott topology. It was hitherto unknown whether this lattice was well-filtered. Corollary 3.2 provides an affirmative answer.

H. Kou gave an example in [6] that shows that a dcpo that is well-filtered with respect to the Scott topology need not be sober. In light of our results the Isbell lattice provides another such example.

4. Applications to coherence

A recent result of Jia, Jung, and Li [4] asserts the following.

Lemma 4.1. *Let P be a dcpo that is well-filtered with respect to the Scott topology. Then P is coherent if $\uparrow x \cap \uparrow y$ is compact for all $x, y \in P$.*

With the aid of our earlier results, we can improve their main result [4, Theorem 3.4] and establish significant links among the notions of compactness, coherence, and well-filteredness,

Theorem 4.2. *Let P be a dcpo equipped with the Scott topology. The following statements are equivalent:*

(1) *P is compact and well-filtered and $\uparrow x \cap \uparrow y$ is compact for all $x, y \in P$.*
(2) *P is compact, well-filtered, and coherent.*
(3) *P is compact with respect to the patch topology, the topology with a closed subbasis consisting of the Scott-closed sets and the compact saturated sets.*
(4) *P is compact in the Lawson topology.*

Proof. $(1 \Rightarrow 2)$ This follows directly from Lemma 4.1.

$(2 \Rightarrow 3)$ This is the implication $(5 \Rightarrow 1)$ in [4, Theorem 3.4]. It also follows from our more general Proposition 2.2.

$(3 \Rightarrow 4)$ Since each $\uparrow x$ is compact saturated, the Lawson topology is coarser than the patch topology, hence also compact.

$(4 \Rightarrow 1)$ Since the Scott topology is coarser than the Lawson topology, P is Scott-compact. Since $\uparrow x$ and $\uparrow y$ are Lawson closed, $\uparrow x \cap \uparrow y$ is also Lawson closed, hence Lawson compact, and thus Scott-compact. By Corollary 3.2 P is well-filtered. □

The following is a corollary of the preceding theorem and Corollary 3.2.

Corollary 4.3. *Let P be a complete lattice or bounded complete dcpo. Then P equipped with the Scott topology is well-filtered and coherent and both the Lawson and patch topologies are compact.*

It is perhaps worthwhile to note that in the preceding case the Lawson and patch topologies may or may not agree; see [2, Proposition VI-6.25].

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