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Jimmie Lawson
Louisiana State University

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EXISTENCE AND UNIQUENESS OF THE KARCHER MEAN ON UNITAL C^* -ALGEBRAS

JIMMIE LAWSON

ABSTRACT. The Karcher mean on the cone Ω of invertible positive elements of the C^* -algebra $\mathcal{B}(E)$ of bounded operators on a Hilbert space E has recently been extended to a contractive barycentric map on the space of L^1 - probability measures on Ω . In this paper we first show that the barycenter satisfies the Karcher equation and then establish the uniqueness of the solution. Next we establish that the Karcher mean is real analytic in each of its coordinates, and use this fact to show that both the Karcher mean and Karcher barycenter map exist and are unique on any unital C^* -algebra. The proof depends crucially on a recent result of the author giving a converse of the inverse function theorem.

Subject Classification: Primary: 47B65; Secondary: 47L07, 46L99

Key words and phrases. Karcher mean and barycenter, Karcher equation, C^* -algebra, cone of positive elements, integrable measures, Wasserstein distance, Bochner integral, real analytic

1. INTRODUCTION

In the past 15 years there has been a rather intensive study of geometric matrix means, in particular, one that has been called the Fréchet mean, Cartan mean, least squares mean, or Karcher mean, and their extensions to barycentric maps on spaces of probability measures [12], [8]. H. Karcher [3] gave an equational characterization of this mean on Riemannian manifolds, and in [7] Y. Lim and the author showed that this equational characterization could be extended to the cone of positive invertible operators in the C^* -algebra of bounded linear maps on a Hilbert space. Later M. Palfia studied more general classes of means and barycenters and more general versions of the Karcher equation [11].

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The main purpose of this paper is to extend this theory of the Karcher mean to the cone of invertible positive elements in *any* unital C^* -algebra, since the methods of [7] do not extend beyond monotone complete C^* -algebras. Indeed that paper closed with the problem of whether the theory could be extended to all unital C^* -algebras, and it is quite desirable to extend the theory to this general setting. It should be remarked that this extension has also recently been achieved by Y. Lim and M. Pálfia [9] using more technical machinery loosely based on well-known work of Crandell and Liggett, while the proof given here in some sense squeezes the result out of earlier results on the Karcher mean. In addition, each paper contains significant results not contained in the other.

There are interesting side results that are derived in the course of the paper. In Section 3 we derive strong connections between the barycentric map for the arithmetic mean and the Bochner integral in the setting of probability measures. In Section 4 we note new and general continuity conditions holding for the contractive barycenter map extending the Karcher mean [8] and use them to establish that the barycentric map yields solutions of the Karcher equation. In Section 5 the uniqueness of the Karcher barycenter for integrable probability measures is established. In Section 6 we show that the Karcher mean is separately real analytic in each of its coordinates. In Section 7 we show using Section 6 and the Identity principle that the cone Ω of positive invertible elements in any unital C^* -algebra is closed under the Karcher mean operation (in a completely different fashion from [9]). The proof is tied up with showing via a recent converse of the inverse function theorem [5] that Karcher mean as a function of one of its variables (with the others fixed) is a real analytic diffeomorphism of Ω .

2. PRELIMINARIES

For a metric space X , let $\mathcal{B}(X)$ be the algebra of Borel sets, the smallest σ -algebra containing the open sets. A *Borel measure* μ is a countably additive (positive) measure defined on $\mathcal{B}(X)$. The support of μ consists of all points x for which $\mu(U) > 0$ for each open set U containing x . The support of μ is always a closed set. The *finitely supported probability measures* are those of the form $\sum_{i=1}^n r_i \delta_{x_i}$, where for each i , $r_i \geq 0$, $\sum_{i=1}^n r_i = 1$, and δ_{x_i} is the point measure of mass 1 at the point x_i .

We recall the *Prohorov metric* $\pi(\mu, \nu)$ defined for two Borel probability measures μ, ν on X as the infimum of all $\varepsilon > 0$ such that for all closed sets A ,

$$\mu(A) \leq \nu(A^\varepsilon) + \varepsilon, \quad \nu(A) \leq \mu(A^\varepsilon) + \varepsilon,$$

where $A^\varepsilon = \{x \in X : d(x, y) < \varepsilon \text{ for some } y \in A\}$. The following result appears in [4].

Proposition 2.1. *A Borel probability measure μ on a metric space (X, d) has separable support. Furthermore, the following are equivalent.*

- (1) *There exists a sequence $\{\mu_n\}$ of finitely supported measures (with rational coefficients) that converges to μ with respect to the Prohorov metric.*
- (2) *The support of μ has measure 1, i.e., μ is support-concentrated.*

Remark 2.2. One may replace (1) by a sequence of uniform probability measures of finite support, i.e., of the form $(1/n) \sum_{i=1}^n \delta_{x_i}$; see Lemma 3.1 of [8].

Let $\mathcal{P}(X)$ be the set of all support-concentrated Borel probability measures on $(X, \mathcal{B}(X))$ and $\mathcal{P}_0(X)$ the set of all uniform probability measures $\mu = \frac{1}{n} \sum_{j=1}^n \delta_{x_j}$ of finite support. Let $\mathcal{P}^1(X) \subseteq \mathcal{P}(X)$ be the set of probability measures with *finite first moment*: for some (and hence all) $y \in X$,

$$\int_X d(x, y) d\mu(x) < \infty.$$

The Borel probability measures with finite first moment are also called the *integrable* probability measures.

Let (X, \mathcal{M}) be a *measure space*, a set X equipped with a σ -algebra \mathcal{M} , and (Y, d) a metric space. A function $f : X \rightarrow Y$ is *measurable* if $f^{-1}(A) \in \mathcal{M}$ whenever $A \in \mathcal{B}(Y)$. For f to be measurable, it suffices that $f^{-1}(U) \in \mathcal{M}$ for each open subset U of Y . Hence continuous functions are measurable in the case X is a metrizable space and $\mathcal{M} = \mathcal{B}(X)$, the Borel algebra. A measurable map $f : X \rightarrow Y$ between metric spaces induces a *push-forward* map $f_* : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ defined by $f_*(\mu)(B) = \mu(f^{-1}(B))$ for $\mu \in \mathcal{P}(X)$ and $B \in \mathcal{B}(Y)$. Note for f continuous that $\text{supp}(f_*(\mu)) = f(\text{supp}(\mu))^-$, the closure of the image of the support of μ .

For X a metric space, we say that $\omega \in \mathcal{P}(X \times X)$ is a *coupling* for $\mu, \nu \in \mathcal{P}(X)$ and that μ, ν are *marginals* for ω if for all $B \in \mathcal{B}(X)$

$$\omega(B \times X) = \mu(B) \quad \text{and} \quad \omega(X \times B) = \nu(B).$$

Equivalently μ and ν are the push-forwards of ω under the projection maps π_1 and π_2 resp. We note that one such coupling is the product measure $\mu \times \nu$, and that for any coupling ω it must be the case that $\text{supp}(\omega) \subseteq \text{supp}(\mu) \times \text{supp}(\nu)$. We denote the set of all couplings for $\mu, \nu \in \mathcal{P}(X)$ by $\Pi(\mu, \nu)$.

The Wasserstein distance d_w (alternatively Kantorovich-Rubinstein distance) on $\mathcal{P}^1(X)$ is defined by

$$d_w(\mu_1, \mu_2) := \inf_{\pi \in \Pi(\mu_1, \mu_2)} \int_{X \times X} d(x, y) d\pi(x, y). \quad (1)$$

It is known that d_w is a metric on $\mathcal{P}^1(X)$, is complete resp. separable whenever d is complete resp. separable and that $\mathcal{P}_0(X)$ is d_w -dense in $\mathcal{P}^1(X)$ [1, 12].

For the case that $\mu = (1/n) \sum_{i=1}^n \delta_{x_i}$ and $\nu = (1/n) \sum_{i=1}^n \delta_{y_i}$, the equation (1) reduces to

$$d_w(\mu, \nu) = \min_{\sigma \in S^n} \max\{d(x_j, y_{\sigma(j)}) : 1 \leq j \leq n\}, \quad (2)$$

where S^n is the permutation group on $\{1, \dots, n\}$.

3. MEANS AND BARYCENTERS

We begin this section by recalling (Definition 3.1 through Proposition 3.5) several needed notions and results from Section 3 of [8].

Definition 3.1. (1) An n -mean G_n on a set X for $n \geq 2$ is a function $G_n : X^n \rightarrow X$ that is idempotent in the sense that $G_n(x, \dots, x) = x$ for all $x \in X$.

(2) An n -mean G_n is *symmetric* or *permutation invariant* if for each permutation σ of $\{1, \dots, n\}$, $G_n(\mathbf{x}_\sigma) = G_n(\mathbf{x})$, where $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{x}_\sigma = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$. A (symmetric) *mean* G on X is a sequence of means $\{G_n\}$, one (symmetric) mean for each $n \geq 2$.

(3) A *barycentric map* or *barycenter* on the set of finitely supported uniform measures $\mathcal{P}_0(X)$ is a map $\beta : \mathcal{P}_0(X) \rightarrow X$ satisfying $\beta(\delta_x) = x$ for each $x \in X$.

For $\mathbf{x} = (x_1, \dots, x_n) \in X^n$, we let

$$\mathbf{x}^k = (x_1, \dots, x_n, x_1, \dots, x_n, \dots, x_1, \dots, x_n) \in X^{nk}, \quad (3)$$

where the number of \mathbf{x} -blocks is k . We define the *carrier* $S(\mathbf{x})$ of \mathbf{x} to be the set of entries in \mathbf{x} , i.e., the smallest finite subset F such that $\mathbf{x} \in F^n$. We set $[\mathbf{x}]$ equal to the equivalence class of all n -tuples obtained by permuting the coordinates of $\mathbf{x} = (x_1, \dots, x_n)$. Note that the operation $[\mathbf{x}]^k = [\mathbf{x}^k]$ is well-defined and that all members of $[\mathbf{x}]$ all have the same carrier set $S(\mathbf{x})$.

A tuple $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ induces on $S(\mathbf{x})$ a uniform probability measure μ with finite support by $\mu = \sum_{i=1}^n (1/n)\delta_{x_i}$, where δ_{x_i} is the point measure of mass 1 at x_i . Since the tuple may contain repetitions of some of its entries, each singleton set $\{x\}$ for $x \in \{x_1, \dots, x_n\}$ will have measure k/n , where k is the number of times that it appears in the listing x_1, \dots, x_n . Note that every member of $[\mathbf{x}]$ induces the same finitely supported probability measure.

Lemma 3.2. *For each probability measure μ on X with finite support F for which $\mu(x) (= \mu(\{x\}))$ is rational for each $x \in F$, there exists a unique $[\mathbf{x}]$ inducing μ such that any $[\mathbf{y}]$ inducing μ is equal to $[\mathbf{x}]^k$ for some $k \geq 1$.*

Definition 3.3. A mean $G = \{G_n\}$ on X is said to be *iterative* if for all $n, k \geq 2$ and all $\mathbf{x} = (x_1, \dots, x_n) \in X^n$,

$$G_n(\mathbf{x}) = G_{nk}(\mathbf{x}^k).$$

If a mean is both iterative and symmetric, we say it is *intrinsic*.

We have the following corollary to Lemma 3.2.

Corollary 3.4. *Let G be an iterative mean. Then for any finitely supported probability measure μ with support F and taking on rational values, we may define $\beta_G(\mu) = G_n(\mathbf{x})$, for any $\mathbf{x} \in F^n$ that induces μ .*

Corollary 3.4 provides the basis for the following equivalence.

Proposition 3.5. *There is a one-to-one correspondence between the intrinsic means and the barycentric maps on $\mathcal{P}_0(X)$ given in one direction by assigning to an intrinsic mean G the barycentric map β_G and in the reverse direction assigning to a barycentric map β the mean $G_n(x_1, \dots, x_n) = \beta(\frac{1}{n} \sum_{i=1}^n \delta_{x_i})$.*

We specialize to means and barycenters in metric spaces.

Definition 3.6. An n -mean $G_n : X^n \rightarrow X$ is said to be *subadditive* if for all $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in X^n$,

$$d(G_n(\mathbf{x}), G_n(\mathbf{y})) \leq \frac{1}{n} \sum_{j=1}^n d(x_j, y_j).$$

A mean $G = \{G_n\}$ is said to be subadditive if each G_n is.

We recall the definition of a contractive barycentric map.

Definition 3.7. Let (X, d) be a metric space. A *contractive barycentric map* on $\mathcal{P}^1(X)$ is a map $\beta : \mathcal{P}^1(X) \rightarrow X$ satisfying $\beta(\delta_x) = x$ for all $x \in X$ and $d(\beta(\mu), \beta(\nu)) \leq d_w(\mu, \nu)$ for all $\mu, \nu \in \mathcal{P}^1(X)$.

The following is part of Proposition 2.7 of [8].

Proposition 3.8. A subadditive intrinsic mean G on a metric space X uniquely gives rise to a contractive barycentric map on $\mathcal{P}_0(X)$. If X is complete, the barycentric map uniquely extends to a contractive barycentric map $\beta_G : \mathcal{P}^1(X) \rightarrow X$.

Example 3.9. We consider the arithmetic mean $A = \{A_n\}$ on a Banach space E defined by $A_n(x_1, \dots, x_n) = (1/n) \sum_{i=1}^n x_i$. It is easily checked to be symmetric and iterative, i.e., intrinsic. To see that A is contractive we compute with the help of the triangle inequality for $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ in E^n ,

$$d(A_n(\mathbf{x}), A_n(\mathbf{y})) = \left\| \frac{1}{n} \sum_{i=1}^n x_i - \frac{1}{n} \sum_{i=1}^n y_i \right\| \leq \frac{1}{n} \sum_{i=1}^n \|x_i - y_i\| = \frac{1}{n} \sum_{i=1}^n d(x_i, y_i).$$

Hence A extends to a contractive barycentric map $\beta_E : \mathcal{P}^1(E) \rightarrow E$, called the *arithmetic barycentric map* of E . We note that if we weight x_i with weight $w_i > 0$ for $i = 1, \dots, n$, where $w_1 + \dots + w_n = 1$, then, as expected, β_E returns the weighted arithmetic mean $\sum_{i=1}^n w_i x_i$.

As long as one is dealing with probability measures, there is a close relationship between the arithmetic barycentric map and the Bochner integral, to which we now turn. We consider functions defined on a probability measure space (A, \mathcal{A}, μ) , where A is a set, \mathcal{A} is a σ -algebra of subsets of A and μ is a σ -additive probability measure

on \mathcal{A} . A μ -simple function with values in a Banach space E is a function from A into E of the form $\sum_{i=1}^n \chi_{A_i} x_i$, where $A_i \in \mathcal{A}$, $x_i \in E$ and $\chi_{A_i} x_i(a) = x_i$ if $a \in A_i$ and 0 otherwise. A function $f : A \rightarrow E$ is *strongly μ -measurable* if there exists a sequence $\{s_n\}$ of μ -simple functions converging pointwise to f μ -almost everywhere.

Definition 3.10. A function $f : A \rightarrow E$ is μ -Bochner integrable if there exists a sequence of μ -simple functions $s_n : A \rightarrow E$ such that the following two conditions are met:

- (1) $\lim_n s_n = f$ μ -almost everywhere as $n \rightarrow \infty$;
- (2) $\lim_n \int_A \|f - s_n\| d\mu = 0$.

We note that $f - s_n$ is strongly μ -measurable since f is by definition and s_n is trivially, and hence $\|f - s_n\|$ is strongly μ -measurable. Note that every μ -simple function is μ -Bochner integrable. For $s = \sum_{i=1}^n \chi_{A_i} x_i$ we set

$$\int_A s d\mu = \sum_{i=1}^n \mu(A_i) x_i.$$

It is routine to check that this definition is independent of the representation of s . If f is μ -Bochner integrable, the limit

$$\int_A f d\mu = \lim_{n \rightarrow \infty} \int_A s_n d\mu$$

exists in E and is called the *Bochner integral* of f with respect to μ . It is a standard result that this definition is independent of the approximating sequence $\{s_n\}_{n=1}^{\infty}$ of μ -simple functions.

We recall that given $f : A \rightarrow E$ that is μ -measurable, the *pushforward measure* $f_*\mu$ is the measure on E given by $f_*\mu(B) = \mu(f^{-1}(B))$ for any Borel set B . The following proposition relates the Bochner integral to the arithmetic barycentric map β_E of Example 3.9.

Proposition 3.11. *Let (A, \mathcal{A}, μ) be a probabilistic measure space and E a Banach space. A measurable map $f : A \rightarrow E$ is Bochner integrable if and only if $f_*\mu \in \mathcal{P}^1(E)$ and in this case $\int_A f d\mu = \beta_E(f_*\mu)$.*

Proof. A well-known necessary and sufficient condition for Bochner integrability is that $\int_A \|f\| d\mu < \infty$. By change of variables $\int_E d_E(x, 0) df_*\mu(x) = \int_A \|f(a)\| d\mu(a)$,

so $f * \mu$ is also integrable, i.e., is a member of $\mathcal{P}^1(E)$, if and only if f is Bochner integrable.

Assuming f is Bochner integrable, there exists a sequence of simple functions s_n that converges pointwise μ -almost everywhere to f and for which

$$\int_A f d\mu = \lim_{n \rightarrow \infty} \int_A s_n d\mu.$$

Since s_n is simple, the support of $(s_n)_*\mu$ is finite, and it is easily verified that $\int_A s_n d\mu$ is the weighted arithmetic mean of $(s_n)_*\mu$. Define $F : A \rightarrow E \times E$ by $F(a) = (f(a), s_n(a))$ and $\pi_n = F_*\mu$. It is easy to verify that π_n has marginals $f_*\mu$ and $(s_n)_*\mu$. We calculate via change of variables, noting that the distance $d_E(x, y) = \|x - y\|$.

$$\int_{E \times E} \|x - y\| dF_*\mu(x, y) = \int_A d_E(F(a)) d\mu(a) = \int_A \|f - s_n\| d\mu.$$

Since $d_w(f_*\mu, (s_n)_*\mu)$ is bounded above by the preceding integral and the integral on the right approaches 0 as $n \rightarrow \infty$, we conclude that $(s_n)_*\mu \rightarrow f_*\mu$ with respect to the Wasserstein distance. Since $\int_A s_n d\mu = \beta_E((s_n)_*\mu)$ for each n , their limits $\int_A f d\mu$ and $\beta_E(f_*\mu)$ are equal.

□

4. SOLUTIONS OF THE KARCHER EQUATION: EXISTENCE

In this section we work in the following setting. Let \mathbb{A} be a C^* -algebra with identity and let Ω denote the open cone of positive invertible elements in \mathbb{A} . Let $H = \mathcal{H}(\mathbb{A})$ denote the subspace of hermitian elements endowed with the norm metric and topology.

The closed cone $\overline{\Omega}$ induces a closed partial order on H defined by $x \leq y$ if $y - x \in \overline{\Omega}$. We equip Ω with the Thompson metric, which can be defined from the partial order by

$$d(x, y) = \log \max\{M(x/y), M(y/x)\} \text{ where } M(x/y) = \inf\{t > 0 : x \leq ty\}.$$

The Thompson metric is a complete metric on Ω and the metric topology agrees with the relative norm topology. The Thompson metric can be alternatively computed from the formula $d(x, y) = \|\log(x^{-1/2}yx^{1/2})\|$; see [7].

The exponential function $\exp : H \rightarrow \Omega$ defined by the usual exponential power series is a diffeomorphism with inverse denoted by \log . This exponential function is known to be expansive, so the logarithmic function is contractive. For $x \in \Omega$, the map $\theta_x(y) = x^{-1/2}yx^{1/2}$ is a linear order isomorphism on H carrying Ω to Ω , and restricted to Ω is an isometry for the Thompson metric (since it can be defined from the order relation). Since θ_x is an isometry on Ω and $\log : \Omega \rightarrow H$ is contractive, they induce an isometry $(\theta_x)_* : \mathcal{P}^1(\Omega) \rightarrow \mathcal{P}^1(\Omega)$ and a contractive map $\log_* : \mathcal{P}^1(\Omega) \rightarrow \mathcal{P}^1(H)$, where the induced maps carry a measure to its pushforward measure and the metrics are the appropriate Wasserstein metrics arising from the Thompson metric on Ω and the restricted norm metric on H .

Lemma 4.1. *For $x \in \Omega$, the map $K_x : \mathcal{P}^1(\Omega) \rightarrow H$ defined by*

$$K_x(\mu) = \int_{\Omega} \log(x^{-1/2}ax^{-1/2}) d\mu(a)$$

is contractive.

Proof. From the previous paragraphs the composite map $\log_*(\theta_x)_* : \mathcal{P}^1(\Omega) \rightarrow \mathcal{P}^1(H)$ is contractive with the image of μ being the pushforward measure under the composition $\log \circ \theta_x$. Since the pushforward measure belongs to $\mathcal{P}^1(H)$, we conclude from Proposition 3.11 that the Bochner integral $\int_{\Omega} \log(x^{-1/2}ax^{-1/2}) d\mu(a)$ exists and is equal to $\beta_H(\log_*(\theta_x)_*(\mu))$. Since the composition $\beta_H \log_*(\theta_x)_*$ is a composition of contractions, it is a contraction and the lemma follows. \square

Lemma 4.2. *The function $K : \Omega \times \mathcal{P}^1(\Omega) \rightarrow H$ is continuous, where*

$$K(x, \mu) = \int_{\Omega} \log(x^{-1/2}ax^{-1/2}) d\mu(a)$$

Proof. Choose convergent sequences $x_n \rightarrow x$ in Ω and $\mu_n \rightarrow \mu$ in $\mathcal{P}^1(\Omega)$. There exists M such that $d(x_n, x) \leq M$ for all n . Define $f_n, f : \Omega \rightarrow H$ by $f_n(a) = \log(x_n^{-1/2}ax_n^{-1/2})$ and $f(a) = \log(x^{-1/2}ax^{-1/2})$. By continuity of multiplication f_n converges pointwise

to f and for all n

$$\begin{aligned} \int_{\Omega} \|f_n(a)\| d\mu(a) &= \int_{\Omega} \|\log(x_n^{-1/2} a x_n^{-1/2})\| d\mu(a) \\ &= \int_{\Omega} d(x_n, a) d\mu(a) \leq \int_{\Omega} (M + d(x, a)) d\mu(a) \\ &\leq M + \int_{\Omega} d(x, a) d\mu(a) < \infty. \end{aligned}$$

where the last inequality follows from $\mu \in \mathcal{P}^1(\Omega)$. Thus the family $\{\|f_n\|\}$ is uniformly bounded by an integrable function, and hence by Lebesgue Bounded Convergence Theorem for the Bochner integral, we conclude that $\lim_n \int_{\Omega} f_n d\mu = \int_{\Omega} f d\mu$.

Let $\varepsilon > 0$ and pick N such that $n \geq N$ implies $\|\int_{\Omega} f d\mu - \int_{\Omega} f_n d\mu\| < \varepsilon/2$ and $d_w(\mu, \mu_n) < \varepsilon/2$. We then have for $n \geq N$

$$\begin{aligned} \|K(x, \mu) - K(x_n, \mu_n)\| &= \left\| \int_{\Omega} f d\mu - \int_{\Omega} f_n d\mu_n \right\| \\ &\leq \left\| \int_{\Omega} (f - f_n) d\mu \right\| + \left\| \int_{\Omega} f_n d\mu - \int_{\Omega} f_n d\mu_n \right\| \\ &\leq \varepsilon/2 + d_w(\mu, \mu_n) < \varepsilon, \end{aligned}$$

where we apply Lemma 4.1 to obtain the first inequality in the last line. \square

We say $x \in \Omega$ is a solution of the Karcher equation for $\mu \in \mathcal{P}^1(\Omega)$, or alternatively that (x, μ) is a solution of the Karcher equation, if

$$\int_{\Omega} \log(x^{-1/2} a x^{-1/2}) \mu(a) = 0. \quad (4)$$

For the case that $\mu = \sum_{i=1}^n \omega_i \delta_{a_i}$ is a finitely supported probability measure the Karcher equation becomes

$$\sum_{i=1}^n \omega_i \log(x^{-1/2} a_i x^{-1/2}) = 0. \quad (5)$$

For the special case of the C^* -algebra $\mathbb{B}(E)$ of bounded linear operators on a Hilbert space E , it was shown in [7, Theorem 6.5] (in the setting of means) that any finitely supported probability measure has a unique solution $\Lambda(\mu)$ for its Karcher equation, and that solution is called the *Karcher mean*. (There is an alternative characterization given in [7] of the Karcher mean as a limit of power means.)

In [8] the Karcher mean was extended from $\mathcal{P}_0(\Omega)$ to $\mathcal{P}^1(\Omega)$ by observing that $\Lambda : \mathcal{P}_0(\Omega) \rightarrow \Omega$ is contractive from the Wasserstein metric to the Thompson metric, and since $\mathcal{P}_0(\Omega)$ is dense in $\mathcal{P}^1(\Omega)$ and the Thompson metric is complete [13], Γ extends to a contractive barycentric map from $\mathcal{P}^1(\Omega)$ to Ω . It makes sense to call this extension the *Karcher barycentric map*. We can use Lemma 4.2 to show the desirable result that this extension still satisfies the Karcher equation.

Proposition 4.3. *Let Ω be the open cone (in the subspace $\mathcal{H}(E)$ of hermitian operators) of positive invertible elements, let $\Lambda : \mathcal{P}^1(\Omega) \rightarrow \Omega$ be the Karcher barycenter map, and let $\mu \in \mathcal{P}^1(\Omega)$. Then $x = \Gamma(\mu)$ satisfies the Karcher equation (4).*

Proof. Let $\{\mu_n\}$ be a sequence in $\mathcal{P}_0(\Omega)$ converging to μ , and let $\Lambda(\mu_n)$ be the Karcher mean for each μ_n . By continuity of Λ , $\lim_n \Lambda(\mu_n) = \Lambda(\mu)$. By Lemma 4.2

$$K(\Lambda(\mu), \mu) = \lim_n K(\Lambda(\mu_n), \mu_n) = 0,$$

which completes the proof. \square

5. SOLUTIONS OF THE KARCHER EQUATION: UNIQUENESS

In this section we work in the setting of the C^* -algebra $\mathbb{A} = \mathcal{B}(E)$ of bounded linear operators on a Hilbert space E with Ω denoting the cone of invertible positive operators and $H = \mathcal{H}(E)$ the Banach space of hermitian operators.

Lemma 5.1. *Let (X, d) be a metric space, let $\mu, \nu \in \mathcal{P}(X)$, and suppose $B_r(z) \cap \text{supp } \mu \neq \emptyset$, where $B_r(z)$ is the open ball of radius r around z . If $B_{2r}(z) \cap \text{supp } \nu = \emptyset$, then $d_w(\mu, \nu) \geq r\mu(B_r(z))$.*

Proof. Since $B = B_r(z)$ meets $\text{supp } \mu$, $\mu(B) > 0$. For any $\pi \in \Pi(\mu, \nu)$, $\text{supp}(\pi) \subseteq \text{supp}(\mu) \times \text{supp}(\nu)$, so

$$\begin{aligned} \int_{X \times X} d(x, y) d\pi(x, y) &= \int_{X \times \text{supp } \nu} d(x, y) d\pi \\ &= \int_{B \times \text{supp } \nu} d(x, y) d\pi + \int_{X \setminus B \times \text{supp } \nu} d(x, y) d\pi \\ &\geq \int_{B \times \text{supp } \nu} d(x, y) d\pi \geq \int_{B \times \text{supp } \nu} r d\pi \\ &= r \int_{B \times \text{supp } \nu} d\pi = r\pi(B \times X) = r\mu(B). \end{aligned}$$

Since $\pi \in \Pi(\mu, \nu)$ was arbitrary, the lemma follows. \square

Lemma 5.2. *If (x, μ) is a solution of the Karcher equation and $\varepsilon > 0$, then there exists $\eta \in \mathcal{P}_0(\Omega)$ such that $d_w(\mu, \eta) < \varepsilon$ and (x, η) is a solution of the Karcher equation.*

Proof. Since $\mathcal{P}_0(\Omega)$ is dense in $\mathcal{P}^1(\Omega)$, we may select a sequence $\{\mu_n\} \subseteq \mathcal{P}_0(\Omega)$ that converges to μ with respect to the Wasserstein metric on $\mathcal{P}^1(\Omega)$ arising from the Thompson metric on Ω . Without loss of generality we may assume that $\mu_n = (1/N) \sum_{i=1}^N \delta_{x_i}$ for some $N \geq n$, since otherwise we may uniformly divide each mass $(1/N)\delta_{x_i}$ into m equal parts, where $mN > n$ and rewrite the summation accordingly. Pick R large enough so that the open ball $B = B_R(1)$ of radius R around 1 meets $\text{supp}(\mu)$; thus $\mu(B) > 0$. For all sufficiently large n , $d_w(\mu, \mu_n) < R\mu(B)$, so by Lemma 5.1, $\text{supp}(\mu_n) \cap B_{2R}(1) \neq \emptyset$ for such n . We can thus relabel if necessary so that for $\mu_n = (1/N) \sum_{i=1}^N \delta_{x_i}$, we have $d(1, x_1) < 2R$.

By Lemma 4.1 the map $K_x(\nu) = \int_{\Omega} x^{-1/2} a x^{-1/2} d\nu(a)$ from $\mathcal{P}^1(\Omega)$ to H is contractive. Since by hypothesis $K_x(\mu) = 0$ we have

$$0 = \lim_{n \rightarrow \infty} K_x(\mu_n) = \lim_n \frac{1}{N} \sum_{i=1}^N \log(x^{-1/2} x_i x^{-1/2}),$$

where N and each x_i are functions of n . We note that $\lim_n (1/N) \log(x^{-1/2} x_1 x^{-1/2}) = 0$ since $N \geq n$ and

$$\begin{aligned} d_{\|\cdot\|}(0, \log(x^{-1/2} x_1 x^{-1/2})) &= d_{\|\cdot\|}(\log(1), \log(x^{-1/2} x_1 x^{-1/2})) \leq d(1, x^{-1/2} x_1 x^{-1/2}) \\ &\leq d(1, x^{-1}) + d(x^{-1/2}(1)x^{-1/2}, x^{-1/2} x_1 x^{-1/2}) \\ &= d(1, x^{-1}) + d(1, x_1) \leq d(1, x^{-1}) + 2R. \end{aligned}$$

The bottom equality follows since the map $x \mapsto axa$ on Ω is an isometry. We conclude that

$$\begin{aligned}
 0 &= \lim_n \frac{1}{N} \sum_{i=1}^N \log(x^{-1/2} x_i x^{-1/2}) \\
 &= \lim_n \left[\frac{1}{N} \log(x^{-1/2} x_1 x^{-1/2}) + \frac{1}{N} \sum_{i=2}^N \log(x^{-1/2} x_i x^{-1/2}) \right] \\
 &= \lim_n \frac{1}{N} \sum_{i=2}^N \log(x^{-1/2} x_i x^{-1/2}). \tag{6}
 \end{aligned}$$

We set $y_1 = x^{1/2} \exp(-\sum_{i=2}^N \log(x^{-1/2} x_i x^{-1/2})) x^{1/2}$ and define $\nu_n \in \mathcal{P}_0(\Omega)$ by $\nu_n = (1/N)(\delta_{y_1} + \sum_{i=2}^N \delta_{x_i})$. It then follows directly from the definition of y_1 that

$$\frac{1}{N} \log(x^{-1/2} y_1 x^{-1/2}) + \frac{1}{N} \sum_{i=2}^N \log(x^{-1/2} x_i x^{-1/2}) = 0,$$

and hence that (x, ν_n) is a solution of the Karcher equation.

To complete the proof we show $d_w(\mu_n, \nu_n) \rightarrow 0$ as $n \rightarrow \infty$, and hence that $\lim_n \nu_n = \mu$. To obtain ν_n to μ_n , we simply transport the mass of $(1/N)$ from x_1 to y_1 . Hence the Wasserstein distance $d_w(\mu_n, \nu_n)$ satisfies $d_w(\mu_n, \nu_n) \leq (1/N)d_w(x_1, y_1)$. We note that

$$\begin{aligned}
 d_w(x_1, y_1) &\leq d(x_1, 1) + d(1, x^{1/2} \exp(-\sum_{i=2}^N \log(x^{-1/2} x_i x^{-1/2})) x^{1/2}) \\
 &\leq 2R + d(1, x) + d(x, x^{1/2} \exp(-\sum_{i=2}^N \log(x^{-1/2} x_i x^{-1/2})) x^{1/2}) \\
 &= 2R + d(1, x) + d(1, \exp(-\sum_{i=2}^N \log(x^{-1/2} x_i x^{-1/2}))) \\
 &= 2R + d(1, x) + \left\| \sum_{i=2}^N \log(x^{-1/2} x_i x^{-1/2}) \right\|.
 \end{aligned}$$

Thus

$$d_w(\mu_n, \nu_n) \leq \frac{d(x_1, y_1)}{N} \leq \frac{2R + d(1, x) + \left\| \sum_{i=2}^N \log(x^{-1/2} x_i x^{-1/2}) \right\|}{N},$$

and we conclude from this inequality and (6) that $\lim_n d_w(\mu_n, \nu_n) = 0$. \square

Theorem 5.3. *Let Ω be the open cone (in the subspace $\mathcal{H}(E)$ of hermitian operators) of positive invertible elements, let $\Lambda : \mathcal{P}^1(\Omega) \rightarrow \Omega$ be the Karcher barycenter map, and let $\mu \in \mathcal{P}^1(\Omega)$. Then $x = \Gamma(\mu)$ is the unique solution of the Karcher equation (4) for $\mu \in \mathcal{P}^1(\Omega)$.*

Proof. Suppose that (x, μ) and (y, μ) are both solutions, $x \neq y$. Set $\varepsilon = (1/2)d(x, y)$. By Lemma 5.2 we can pick $\eta, \tau \in \mathcal{P}_0(\Omega)$ such that $d(\eta, \mu) < \varepsilon$, $d(\tau, \mu) < 0$ and (x, η) and (y, τ) are solutions of the Karcher equation. By uniqueness of the Karcher mean for members of $\mathcal{P}_0(\Omega)$ [7], $x = \Lambda(\eta)$ and $y = \Lambda(\tau)$. Since Λ is contractive [8],

$$d(x, y) = d(\Lambda(\eta), \Lambda(\tau)) \leq d_w(\eta, \tau) \leq d_w(\eta, \mu) + d_w(\mu, \tau) < 2\varepsilon = d(x, y),$$

a contradiction. □

6. ANALYTICITY

In this section we continue to work in the setting of $\mathbb{A} = \mathcal{B}(E)$, the C^* -algebra of bounded linear operators on a Hilbert space E with cone Ω of positive invertible elements. Let $\mathcal{H}(\mathbb{A})$ denote the closed subspace of hermitian operators and let 1 denote the identity of \mathbb{A} .

It is standard that the maps $\exp : \mathcal{H}(\mathbb{A}) \rightarrow \Omega$ and $\log : \Omega \rightarrow \mathcal{H}(\mathbb{A})$ are inverse real analytic diffeomorphisms. Moreover, they are regular in the sense that their derivatives at each point in the domain are invertible maps (see, for example, the proof of Theorem V.1 and Examples VI.6 and VI.7 of [10] or Theorem 9.1 and its proof in [6]).

As mentioned earlier, for $a_1, \dots, a_n, x \in \Omega$ the Karcher equation

$$\log(x^{-1/2}a_1x^{-1/2}) + \dots + \log(x^{-1/2}a_nx^{-1/2}) = 0. \quad (7)$$

has a unique solution $x = \Lambda(a_1, \dots, a_n) \in \Omega$, the Karcher mean.

Lemma 6.1. *For fixed $a_1, \dots, a_{n-1}, x \in \Omega$, there uniquely exists $a \in \Omega$ such that*

$$\log(x^{-1/2}a_1x^{-1/2}) + \dots + \log(x^{-1/2}a_{n-1}x^{-1/2}) + \log(x^{-1/2}ax^{-1/2}) = 0. \quad (8)$$

The solution is given by $a = x^{1/2} \exp \left(- \sum_{i=1}^{n-1} \log(x^{-1/2}a_i x^{-1/2}) \right) x^{1/2}$. The solution a is also uniquely determined by $x = \Lambda(a_1, \dots, a_{n-1}, a)$.

Proof. One verifies directly by substitution that the a defined in the lemma is indeed a solution to equation (8). If b were another solution, then it would follow that $\log(x^{-1/2}ax^{-1/2}) = \log(x^{-1/2}bx^{-1/2})$ and hence $a = b$. The fact that equation (8) holds and the uniqueness of the Karcher mean yield the last assertion. \square

Definition 6.2. For fixed $\mathbf{a} = (a_1, \dots, a_{n-1})$, the preceding lemma gives rise to a real analytic function $\gamma : \Omega \rightarrow \Omega$ defined by

$$\gamma(x) = \gamma_{\mathbf{a}}(x) = x^{1/2} \exp\left(-\sum_{i=1}^{n-1} \log(x^{-1/2}a_i x^{-1/2})\right)x^{1/2}.$$

We also define $\lambda : \Omega \rightarrow \Omega$ by $\lambda(a) = \Lambda(a_1, \dots, a_{n-1}, a)$.

Lemma 6.3. For fixed $a_1, \dots, a_{n-1} \in \Omega$, γ and λ are inverse functions.

Proof. That each of $\lambda\gamma$ and $\gamma\lambda$ is the identity on Ω follows directly from Lemma 6.1. \square

Lemma 6.4. The identity function from $(\Omega, \|\cdot\|)$ to (Ω, d) is locally Lipschitz, and hence $\lambda : (\Omega, \|\cdot\|) \rightarrow (\Omega, d)$ is locally Lipschitz.

Proof. Let $W_M = \{x \in \Omega : Mx \in 1 + \Omega\}$. Then W_M is an open set, and $\bigcup_M W_M = \Omega$. For $x, y \in W_M$, let $\beta = \|x - y\|$. Since $x - y \leq \|x - y\|1$, we obtain

$$x \leq y + \|x - y\|1 \leq y + M\|x - y\|y = (1 + M\|x - y\|)y$$

and interchanging the role of x and y yields $y \leq (1 + M\|x - y\|)x$. It follows that

$$d(x, y) \leq \log(1 + M\|x - y\|) \leq M\|x - y\|$$

since $\log(1 + Mt) \leq Mt$ for $t \geq 0$.

For the second assertion the contractivity of Λ [8] implies $\lambda : (\Omega, d) \rightarrow (\Omega, d)$ is contractive. So $\lambda : (\Omega, \|\cdot\|) \rightarrow (\Omega, d)$, which is the composition of this map and that of the first paragraph, is the composition of a contractive map and a locally Lipschitz map, hence locally Lipschitz. \square

The next recent result, a converse of the inverse function theorem and the main result of [5], is crucial for what follows.

Theorem 6.5. *Let $g : U \rightarrow V$ and $f : V \rightarrow U$ be inverse homeomorphisms between open subsets of Banach spaces. If g is differentiable of class C^p and f is locally Lipschitz, then the Fréchet derivative of g at each point of U is invertible and f must be differentiable of class C^p .*

Corollary 6.6. *Let Ω be the cone of invertible positive elements in the C^* -algebra $\mathcal{B}(E)$ of bounded operators on a Hilbert space E . For fixed $a_1, \dots, a_{n-1} \in \Omega$ the function $\lambda(a) = \Lambda(a_1, \dots, a_{n-1}, a)$ is an analytic diffeomorphism on Ω .*

Proof. By Lemma 6.3, λ and γ or inverse functions and by Lemma 6.4 λ is locally Lipschitz. From Definition 6.2 γ is a composition of real analytic functions, hence real analytic. Thus the corollary follows from the preceding theorem for the case $p = \omega$, the real analytic case. \square

7. THE KARCHER MEAN IN GENERAL C^* -ALGEBRAS

In this section we extend previous results to the setting of a general unital C^* -algebra \mathbb{A} . We begin with what is frequently called the Identity Theorem and at other times grouped with the principle of analytic continuation. For a proof of the version we quote, see [14, Theorems 1.11, 3.1].

Theorem 7.1. *Suppose that D is an open connected subset of a Banach space E and that $f, g : D \rightarrow F$ are real analytic maps into another Banach space. If there exists an open subset U of D such that $f|_U = g|_U$, then $f = g$.*

The next theorem is proved as part of Theorem 6.3 of [7]. The proof there is given in the context of a C^* -algebra of bounded operators on a Hilbert space, but works just as well in the context of a general unital C^* -algebra. Also we use the modification of the implicit function theorem used in the proof of that case to the real analytic case.

Theorem 7.2. *Let \mathbb{A} be a unital C^* -algebra. Then there exists an open set U containing the identity 1 such that the restriction of the Karcher mean $\Lambda : U^n \rightarrow \Omega$ is real analytic.*

By the Gelfand-Naimark theorem the C^* -algebra \mathbb{A} is isometrically $*$ -isomorphic to a closed C^* -subalgebra containing the identity of a C^* -algebra $\mathcal{B}(E)$ of bounded

operators on a Hilbert space. We fix some such $\mathcal{B}(E)$ and hence forth identify \mathbb{A} with the isomorphic subalgebra.

Lemma 7.3. *For any $a_1, \dots, a_n \in \Omega$, the cone of invertible positive elements of \mathbb{A} , we have $\Lambda(a_1, \dots, a_n) \in \Omega$.*

Proof. We consider the map Λ defined on the cone of positive elements in $\mathcal{B}(E)$ restricted to Ω , $\Lambda : \Omega^n \rightarrow \mathcal{B}(E)$. Suppose that $a_1, \dots, a_{n-1} \in U$, where U is given in Theorem 7.2, $b \in \Omega$, but $z = \Lambda(a_1, \dots, a_{n-1}, b) \notin \Omega$, hence not in \mathbb{A} (since an element of \mathbb{A} is a positive invertible in \mathbb{A} if and only if is in $\mathcal{B}(E)$). By the Hahn-Banach theorem there exists a continuous linear functional π defined on $\mathcal{B}(E)$ such that $\pi(\mathbb{A}) = 0$ and $\pi(z) \neq 0$. By Lemma 7.3, however, $\pi(\Lambda(a_1, \dots, a_{n-1}, a)) = 0$ for all $a \in U$. Thus the composition $a \mapsto \pi(\Lambda(a_1, \dots, a_{n-1}, a))$ is a real analytic function on Ω which is 0 on U . Hence by Theorem 7.1 it agrees with the 0 function everywhere, which contradicts $\pi(\Lambda(a_1, \dots, a_{n-1}, b)) \neq 0$.

We next consider the case $a_1, \dots, a_{n-2} \in U$, $a_n \in \Omega$, and $\Lambda(a_1, \dots, a_{n-2}, b, a_n) \notin \Omega$. We proceed along the lines of the preceding paragraph, noting from the previous paragraph that $\Lambda(a_1, \dots, a_{n-2}, a, a_n) \in \Omega$ for all $a \in U$, and again derive a contradiction. We can then proceed inductively through all the coordinates, and derive the assertion of the lemma. \square

With the aid of Lemma 7.3 we can restrict the results of the earlier section to \mathbb{A} and derive existence and uniqueness of the Karcher mean in an arbitrary unital C^* -algebra.

Theorem 7.4. *In any unital C^* -algebra \mathbb{A} the Karcher mean and the Karcher barycenter maps are defined on Ω , the cone of positive invertible elements. In each case the Karcher mean or barycenter is the unique solution of the corresponding Karcher equation.*

Proof. As remarked before the statement of the theorem, the Karcher mean case follows from our earlier results. The means may be reinterpreted as a contractive barycenter map on $\mathcal{P}_0(\Omega)$ (as in [8]), which extends uniquely to a contractive barycenter map on $\mathcal{P}^1(\Omega)$. By Proposition 4.3 the contractive barycentric map sends a member of $\mathcal{P}^1(\Omega)$ to its Karcher barycenter. Since the image of $\mathcal{P}_0(\Omega)$ is dense

in the image of $\mathcal{P}^1(\Omega)$, the latter image will be contained in the positive invertible elements of \mathbb{A} , that is, in Ω . The Karcher equation of \mathbb{A} is the restriction of the one for $\mathcal{B}(E)$, so existence and uniqueness follow from the earlier corresponding results for $\mathcal{B}(E)$.

□

These results on the existence and uniqueness of the Karcher mean have been obtained earlier by Lim and Palfia [9] by entirely different methods. They show based on Crandall-Liggett type arguments that the Karcher mean is the norm limit of the power means, which establishes the existence of the Karcher mean in an arbitrary unital C^* -algebra. The author views the approach given here as more elementary and computationally less intense.

We also have the following variant of Corollary 6.6.

Corollary 7.5. *Let Ω be the cone of invertible positive elements in a unital C^* -algebra \mathbb{A} . For fixed $a_1, \dots, a_{n-1} \in \Omega$ the function $\lambda(a) = \Lambda(a_1, \dots, a_{n-1}, a)$ is an analytic diffeomorphism on Ω .*

Proof. As previously we assume by the Gelfand-Naimark theorem the C^* -algebra \mathbb{A} is isometrically $*$ -isomorphic to a closed C^* -subalgebra containing the identity of a C^* -algebra $\mathcal{B}(E)$ of bounded operators on a Hilbert space. Directly from its definition (Definition 6.2) γ carries Ω into Ω and it follows from Lemma 7.3 it follows that λ also carries Ω into Ω . From Lemma 6.3 γ and λ are inverse maps on the cone of positive invertible operators in $\mathcal{B}(E)$, the same holds for their restrictions to Γ . The analyticity now follows from Corollary 6.6. □

8. PROPERTIES OF THE KARCHER MEAN AND BARYCENTER IN GENERAL C^* -ALGEBRAS

In this section we work in the general setting of a unital C^* -algebra \mathbb{A} . We denote the closed subspace $\mathcal{H}(\mathbb{A})$ of hermitian elements by H , and the cone of positive invertible elements by Ω , an open cone in H . For a *weight* $\omega = (w_1, \dots, w_n)$ with entries non-negative real numbers summing to 1 and $\mathbf{a} = (a_1, \dots, a_n) \in \Omega^n$, we

consider the corresponding Karcher equation on Ω

$$\sum_{i=1}^n w_i \log(x^{1/2} a_i^{-1} x^{1/2}) = 0, \quad (9)$$

with solution the Karcher mean $x = \Lambda(\omega, \mathbf{a})$. By Theorem 7.4 the Karcher equation has a unique solution in Ω .

In [7, Theorem 6.8] several basic properties of the Karcher mean were derived in the setting of $\mathbb{A} = \mathcal{B}(E)$, the C^* -algebra of bounded linear operators on a Hilbert space E . These properties can be carried over to a general unital C^* -algebra \mathbb{A} by first embedding \mathbb{A} into some $\mathcal{B}(E)$ as a closed C^* -subalgebra (by the Gelfand-Naimark theorem), noting that the Karcher mean Λ of $\mathcal{B}(E)$ restricted to the embedded \mathbb{A} is the Karcher mean of \mathbb{A} , and hence the equalities and inequalities of the following theorem that hold in $\mathcal{B}(E)$ remain valid in \mathbb{A} .

Theorem 8.1. *For a weight $\omega = (w_1, \dots, w_n)$ and $\mathbf{a} = (a_1, \dots, a_n) \in \Omega^n$, the following properties hold.*

- (P1) (*Consistency with scalars*) $\Lambda(\omega; \mathbf{a}) = a_1^{w_1} \cdots a_n^{w_n}$ if the a_i 's commute;
- (P2) (*Joint homogeneity*) $\Lambda(\omega; (t_1 a_1, \dots, t_n a_n)) = t_1^{w_1} \cdots t_n^{w_n} \Lambda(\omega; \mathbf{a})$, all $t_i > 0$;
- (P3) (*Permutation invariance*) $\Lambda(\omega_\sigma; \mathbf{a}_\sigma) = \Lambda(\omega; \mathbf{a})$, where $\omega_\sigma = (w_{\sigma(1)}, \dots, w_{\sigma(n)})$;
- (P4) (*Monotonicity*) If $b_i \leq a_i$ for all $1 \leq i \leq n$, then $\Lambda(\omega; \mathbf{b}) \leq \Lambda(\omega; \mathbf{a})$;
- (P5) (*Continuity*) Λ is contractive for the Thompson metric;
- (P6) (*Congruence invariance*) $\Lambda(\omega; m^* \mathbf{a} m) = m^* \Lambda(\omega; \mathbf{a}) m$ for any invertible m ;
- (P7) (*Joint concavity*) $\Lambda(\omega; \lambda \mathbf{a} + (1-\lambda) \mathbf{b}) \geq \lambda \Lambda(\omega; \mathbf{a}) + (1-\lambda) \Lambda(\omega; \mathbf{b})$ for $0 \leq \lambda \leq 1$;
- (P8) (*Self-duality*) $\Lambda(\omega; a_1^{-1}, \dots, a_n^{-1})^{-1} = \Lambda(\omega; a_1, \dots, a_n)$; and
- (P9) (*AGH weighted mean inequalities*) $(\sum_{i=1}^n w_i a_i^{-1})^{-1} \leq \Lambda(\omega; \mathbf{a}) \leq \sum_{i=1}^n w_i a_i$.

Some of these properties extend naturally to the Karcher barycenter map $\Lambda : \mathcal{P}^1(\Omega) \rightarrow \Omega$. The following items are key observations for these extensions.

Definition 8.2. A Lipschitz function $F : \Omega \rightarrow \Omega$ preserves the Karcher barycenter $\Lambda : \mathcal{P}^1(\Omega) \rightarrow \Omega$ if for all $\mu \in \mathcal{P}^1(\Omega)$, $F(\Lambda(\mu)) = \Lambda(F_* \mu)$, where $F_* \mu$ is the pushforward of μ . Similarly F preserves the Karcher mean if for any $n \geq 2$, $x_1, \dots, x_n \in \Omega$, $F(\Lambda(x_1, \dots, x_n)) = \Lambda(F(x_1), \dots, F(x_n))$.

Remark 8.3. It is easy to verify that F preserves the Karcher mean if and only if F preserves the Karcher barycenter for all $\mu \in \mathcal{P}_0(\Omega)$.

Lemma 8.4. *A Lipschitz map $F : \Omega \rightarrow \Omega$ preserves the Karcher mean if and only if it preserves the Karcher barycenter on $\mathcal{P}^1(\Omega)$.*

Proof. Suppose F preserves the Karcher mean. By Remark 8.3 F preserves the Karcher barycenter on $\mathcal{P}_0(\Omega)$. By the density of $\mathcal{P}_0(\Omega)$ in $\mathcal{P}^1(\Omega)$, there exists a sequence $\{\mu_n\} \subseteq \mathcal{P}_0(\Omega)$ converging to μ . Since F is Lipschitz, the pushforward map $F_* : \mathcal{P}^1(\Omega) \rightarrow \mathcal{P}^1(\Omega)$ is continuous, so $F_*(\mu_n) \rightarrow F_*(\mu)$. By continuity of the barycentric map Λ , $\Lambda(F_*(\mu_n)) \rightarrow \Lambda(F_*(\mu))$. By hypothesis and continuity of Λ and

$$F_*(\Lambda(\mu_n)) = \Lambda(F_*(\mu_n)) \rightarrow \Lambda(F_*(\mu)) \text{ and } F_*(\Lambda(\mu_n)) \rightarrow F_*(\Lambda(\mu)),$$

so $\Lambda(F_*(\mu)) = F_*(\Lambda(\mu))$.

In light of Remark 8.3, the converse is immediate. \square

Theorem 8.5. *Let $\mu \in \mathcal{P}^1(\Omega)$. The barycenter map $\Lambda : \mathcal{P}^1(\Omega) \rightarrow \Omega$ satisfies the following properties.*

- (P1) $\Lambda(t_*\mu) = t\Lambda(\mu)$, where $t_*\mu$ is the pushforward of μ under $a \mapsto ta$, $t > 0$;
- (P2) If $\mu(U) \leq \nu(U)$ for all open upper sets $U = \uparrow U = \{y : \exists x \in U, x \leq y\}$, then $\Lambda(\mu) \leq \Lambda(\nu)$;
- (P3) The barycentric map Λ is contractive from $\mathcal{P}^1(\Omega)$ with the Wasserstein metric to Ω with the Thompson metric;
- (P4) For $m \in \mathbb{A}$ invertible, $\Lambda(m^*\mu m) = m^*\Lambda(\mu)m$, where $m^*\mu m$ is the pushforward of μ under $a \mapsto m^*am$;
- (P5) $\Lambda(\lambda\mu + (1-\lambda)\nu) \geq \lambda\Lambda(\mu) + (1-\lambda)\Lambda(\nu)$ for $0 \leq \lambda \leq 1$;
- (P6) $\Lambda(J_*(\mu)) = (\Lambda(\mu))^{-1}$, where $J(a) = a^{-1}$;
- (P7) $\mathcal{H}(\mu) \leq \Lambda(\mu) \leq \mathcal{A}(\mu)$, where \mathcal{H} is the harmonic barycenter map and \mathcal{A} is the arithmetic barycenter map, and μ is chosen in the intersection of the three barycenter domains.

Proof. (P1) If we apply (P2) of Theorem 8.1 to $\nu = (1/n) \sum_{i=1}^n \delta_{a_i}$, identified as customary with $(\omega, (a_1, \dots, a_n))$ where $\omega_i = 1/n$ for $i = 1, \dots, n$, we obtain property (P1) for ν . Hence if we choose μ_n converging to μ with each $\mu_n \in \mathcal{P}_0(\Omega)$, we obtain (P1) for μ in the limit.

(P2) The first part of (P2) defines a closed order on $\mathcal{P}^1(\Omega)$, called the stochastic order. see [2, Proposition 5.4]. For $\mu \leq \nu$, there exists sequences $\{\mu_n\}, \{\nu_n\} \subseteq \mathcal{P}_0(\Omega)$ such that $\mu_n \rightarrow \mu$, $\nu_n \rightarrow \nu$ and $\mu_n \leq \nu_n$ for each n ; see Proposition 5.2 of [4], where it is also shown that the stochastic order restricted to $\mathcal{P}_0(\Omega)$ agrees with the order given in (P4) of Theorem 8.1 for some permutation of coordinates. So using (P4) we obtain in the limit $\Lambda(\mu) \leq \Lambda(\nu)$, since the stochastic order is closed.

(P3) This has already been alluded to in the previous material. See [8, Section 4] for the original treatment.

(P4) The map $F_m : \Omega \rightarrow \Omega$ defined by $F_m(a) = m^*am$ is an order isomorphism, hence an isometry for the Thompson metric. So (P4) follows from (P6) of Theorem 8.1 and Lemma 8.4.

(P5) This property follows from (P7) of Theorem 8.1, with the proof proceeding along the lines of the proof for (P1).

(P6) This property follows from (P8) with a proof similar to that for (P4). (In this case the *anti-isomorphism* of inversion induces an isometry of the Thompson metric.)

(P7) Let $\mu \in \mathcal{P}^1(\Omega, d) \cap \mathcal{P}^1(\Omega, \|\cdot\|)$. Pick a sequence $\{\mu_n\}$ of finitely supported measures such that $\lim_n \mu_n = \mu$ both in $\mu \in \mathcal{P}^1(\Omega, d)$ and in $\mathcal{P}^1(\Omega, \|\cdot\|)$; see appendix. It then follows from the continuity of the barycenter maps Λ and \mathcal{A} , the closeness of the stochastic order, and Theorem 8.1 (P9) that $\Lambda(\mu) \leq \mathcal{A}(\mu)$.

Note from (P9) of Theorem 8.1 that $\mathcal{H} = J\mathcal{A}J_*$, where J is the inversion map on Ω , for all finitely supported measures. By continuity of \mathcal{H} and density of the set of finitely supported measures the equality holds for all μ such that $J_*(\mu) \in \mathcal{P}^1(\Omega, \|\cot\|)$. If also $\mu \in \mathcal{P}^1(\Omega, d)$, we conclude from the previous paragraph that

$$(\mathcal{H}(\mu))^{-1} = (J\mathcal{A}J_*(\mu))^{-1} = \mathcal{A}J_*(\mu) \geq \Lambda(J_*\mu).$$

. Taking the inverses of the extremes yields $\mathcal{H}(\mu) \leq (\Lambda(J_*\mu))^{-1} = \Lambda(\mu)$ by (P6).

□

9. APPENDIX

In this section we develop a specific construction for approximating $\mu \in \mathcal{P}^1(X)$, where (X, ρ) is a metric space, by a probability measure with finite support. Our results are applied in the proof of (P7) of Theorem 8.5.

Let C be the support of μ , a closed separable set (Proposition 2.1). Fix $p \in X$. Then $\int_X \rho(a, p) d\mu(a) = \int_C \rho(a, p) d\mu(a) < \infty$.

Let \mathcal{E} be a countable collection of Borel sets, each of diameter less than ε , that cover C . We exclude from \mathcal{E} those that miss C , if any, and enumerate the remaining ones, obtaining $\{E_n\}$. We refine this into a pairwise disjoint sequence $\{B_n\}$ of Borel sets that cover C as follows. Set $B_1 = E_1 \cap C$. Inductively if B_1, \dots, B_n have been chosen, set $B_{n+1} = (E_{m+1} \cap C) \setminus \bigcup_{i=1}^n B_i$, where m is chosen as the least index for which this set is nonempty. In this way one obtains a covering sequence $\{B_n\}$ (possibly finite) of C consisting of pairwise disjoint nonempty Borel subsets of C each of diameter less than ε .

Let $A_n = \bigcup_{i=1}^n B_i$. The characteristic functions χ_n of A_n form an increasing sequence of nonnegative functions with supremum χ_C . By Lebesgue's dominated convergence theorem

$$\lim_n \int_X \rho(a, p) \chi_n(a) d\mu(a) = \int_X \rho(a, p) \chi_C(a) d\mu(a) = \int_X \rho(a, p) d\mu(a),$$

the last equality holding since C is the support of μ . Thus there exists N such that $\int_{X \setminus A_n} \rho(a, p) d\mu(a) = \int_X \rho(a, p) d\mu(a) - \int_X \rho(a, p) \chi_n(a) d\mu(a) < \varepsilon$ for $n \geq N$.

Pick $x_n \in B_n$ for each n . We define a probability measure μ_n with finite support as follows. Let $\tau_i = \mu(B_i)$ for $1 \leq i \leq n$. We define

$$\mu_n = \sum_{i=1}^n \tau_i \delta_{x_i} + \left(1 - \sum_{i=1}^n \tau_i\right) \delta_p.$$

Lemma 9.1. *For $n \geq N$, $d_w(\mu, \mu_n) < 2\varepsilon$.*

Proof. We define $f_n : X \rightarrow X$ by $f_n(x) = x_i$ if $x \in B_i$ for $1 \leq i \leq n$, and $f_n(x) = p$ otherwise. Note that f_n is Borel measurable. We define μ_n to be the pushforward measure of μ by f_n , $\mu_n = (f_n)_*(\mu)$ and note that it has support $\{x_1, \dots, x_n, p\}$. We define $F_n : X \rightarrow X \times X$ by $F_n(x) = (x, f_n(x))$ and we let $\pi_n = (F_n)_*(\mu)$ be the pushforward measure under F_n . We note π is a coupling of μ and μ_n . Using the change of variables formula we compute for $n \geq N$

$$\begin{aligned}
\int_{X \times X} \rho(x, y) d\pi(x, y) &= \int_{X \times X} \rho(x, y) d(F_n)_*(\mu)(x, y) \\
&= \int_X \rho(x, f_n(x)) d\mu(x) \\
&= \sum_{i=1}^n \int_{B_i} \rho(x, x_i) d\mu(x) + \int_{X \setminus A_n} \rho(x, p) d\mu(x) \\
&\leq \varepsilon \mu(A_n) + \varepsilon < 2\varepsilon.
\end{aligned}$$

Hence $d_w(\mu, \mu_n) < 2\varepsilon$. □

Corollary 9.2. *Let d and ρ be metrics on X that generate the same topology. Given $\mu \in \mathcal{P}^1(X, d) \cap \mathcal{P}^1(X, \rho)$, there exists a sequence of finitely supported measures that converges to μ in $\mathcal{P}^1(X, d)$ and $\mathcal{P}^1(X, \rho)$,*

Proof. Let $\varepsilon = 1/2n$. For each point $x \in C$, the support of μ , pick an open set U_x containing x such that U_x has diameter less than ε in both metrics. In either metric C is separable, so there exists a countable subcover \mathcal{E} made up of various U_x . Lemma 9.1 and the construction preceding it yield a finitely supported measure μ_n such that $d_w(\mu, \mu_n) < 2\varepsilon = 1/n$. This inequality holds both in $\mathcal{P}^1(X, d)$ and $\mathcal{P}^1(X, \rho)$ since members of the countable cover had diameter less than ε with respect to both metrics. The sequence $\{\mu_n\}$ attained in this fashion is the one desired. □

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DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LA 70803,
USA

E-mail address: lawson@math.lsu.edu