2009

The Segal-Bargmann transform on inductive limits of compact symmetric spaces

Keng Wiboonton
Louisiana State University and Agricultural and Mechanical College, kwibo01@math.lsu.edu

Follow this and additional works at: https://digitalcommons.lsu.edu/gradschool_dissertations

Part of the Applied Mathematics Commons

Recommended Citation
https://digitalcommons.lsu.edu/gradschool_dissertations/562

This Dissertation is brought to you for free and open access by the Graduate School at LSU Digital Commons. It has been accepted for inclusion in LSU Doctoral Dissertations by an authorized graduate school editor of LSU Digital Commons. For more information, please contact gradetd@lsu.edu.
THE SEGAL-BARGMANN TRANSFORM ON INDUCTIVE LIMITS OF
COMPACT SYMMETRIC SPACES

A Dissertation
Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in
The Department of Mathematics

by
Keng Wiboonton
B.S., Chulalongkorn University, Thailand, 2000
M.S., Chulalongkorn University, Thailand, 2002
M.S., University of Wisconsin-Madison, 2005
August 2009
Acknowledgments

This dissertation would not have been possible without the help and valuable contributions from several people.

First I would like to express my deep appreciation to my advisor, Professor Gestur Ólafsson, for his invaluable time, endless help, inspiration, patience, and supervision. He has taught me innumerable lessons and provided me with insights into academic research. Without him, it would have been impossible to finish this dissertation. He is not only a brilliant mathematician but also he is an outstanding teacher, showing an apparent concern towards an understanding of the material.

I would also like to thank the Development and Promotion for Science and Technology talent project of Thailand (DPST) for providing me financial support.

Finally, I am forever grateful to my parents, Terd and Rachanee Wiboonton for their love, support, and encouragement.
# Table of Contents

Acknowledgments .................................................. ii

Abstract .............................................................. v

Chapter 1: Introduction ............................................. 1

Chapter 2: The Segal-Bargmann Transform for $\mathbb{R}^n$ ............... 5
  2.1 Introduction .................................................. 5
  2.2 The Heat Equation on $\mathbb{R}^n$ .............................. 5
  2.3 The Segal-Bargmann Transform on $L^2(\mathbb{R}^n)$ ............ 9
  2.4 Hilbert Spaces of Holomorphic Functions ...................... 11
  2.5 The Fock Space as the Image of the Segal-Bargmann Transform . 16
  2.6 Another Proof of Unitarity of $H_t$ .......................... 26
  2.7 The Bargmann Transform and the Segal-Bargmann Transform on $L^2(\mathbb{R}^n, h_t(x) dx)$ .................................. 29
  2.8 Hermite Polynomials, Hermite Functions and the Fourier Transforms on $\mathcal{H}_t(\mathbb{C}^n)$ and on $\mathcal{F}_t(\mathbb{C}^n)$ ........... 31

Chapter 3: The Restriction Principle ................................ 39
  3.1 Some Facts from Functional Analysis ......................... 39
  3.2 The Restriction Principle .................................... 40
  3.3 Another Proof by the Restriction Principle ................... 43

Chapter 4: The Segal-Bargmann Transform for a Compact Symmetric Space $\mathcal{X} = U/K$ ........................................ 48
  4.1 Introduction .................................................. 48
  4.2 Harmonic Analysis on Compact Groups ......................... 48
  4.3 Riemannian Symmetric Spaces ................................ 52
  4.4 Harmonic Analysis on Compact Symmetric Spaces $\mathcal{X} = U/K$ .... 54
  4.5 Compact Symmetric Spaces $\mathcal{X} = U/K$ and Their Noncompact Duals $\mathcal{Y} = G/K$ ............................................ 61
  4.6 The Heat Equation on $\mathcal{X} = U/K$ .......................... 67
  4.7 The Fock Space $\mathcal{H}_t(\mathcal{X}_C)$ ........................... 77
  4.8 The Segal-Bargmann Transform on $L^2(\mathcal{X})$ ................ 79

Chapter 5: The Segal-Bargman Transform on the Direct Limits ........ 85
  5.1 Introduction .................................................. 85
  5.2 Basic Notations ............................................. 85
  5.3 Propagations of Lie Algebras ................................ 88
  5.4 Propagations of Compact Symmetric Spaces .................. 91
  5.5 The $L^2$-Theory ............................................. 94
5.6 The Segal-Bargman Transform on $L^2(M_n)^K_n$ . . . . . . . . . . 95
5.7 Direct Limits and Inverse Limits . . . . . . . . . . . . . . . . . 100
5.8 The Segal-Bargman Transform on the Direct Limits . . . . . . 105

References ................................................................. 109

Vita ............................................................ 114
Abstract

We construct the Segal-Bargmann transform on the direct limit of the Hilbert spaces \( \{L^2(M_n)^{K_n}\}_n \) where \( \{M_n = U_n/K_n\}_n \) is a propagating sequence of symmetric spaces of compact type with the assumption that \( U_n \) is simply connected for each \( n \). This map is obtained by taking the direct limit of the Segal-Bargmann transforms on \( L^2(M_n)^{K_n}, \ n = 1, 2, \ldots \). For each \( n \), let \( \hat{U}_n \) be the set of equivalence classes of irreducible unitary representations of \( U_n \) and let \( \hat{U}_n/K_n \subseteq \hat{U}_n \) be the set of \( K_n \)-spherical representations. The definition of the propagation gives a nice property allowing us to embed \( \hat{U}_n/K_n \) into \( \hat{U}_m/K_m \) for \( m \geq n \) in a natural way. With these embeddings, we can produce the unitary embeddings from \( L^2(M_n)^{K_n} \) into \( L^2(M_m)^{K_m} \) for \( m \geq n \). Hence, the direct limit of the Hilbert spaces \( \{L^2(M_n)^{K_n}\}_n \) is obtained in the category of Hilbert spaces and unitary embeddings and we can construct the Segal-Bargmann transform on the resulting limit in a canonical way.
Chapter 1
Introduction

It is well-known that a solution of the heat equation

$$\Delta_x u(x,t) = \frac{\partial u}{\partial t}(x,t), \quad (x,t) \in \mathbb{R}^n \times (0, \infty)$$

with the initial condition

$$\lim_{t \to 0^+} u(x,t) = f(x), \quad x \in \mathbb{R}^n$$

where $f$ is a function in $L^2(\mathbb{R}^n)$ is given by $f * h_t$. Here $h_t$ is the heat kernel which is a fundamental solution of the heat equation.

The Segal-Bargmann transform on $L^2(\mathbb{R}^n)$ is defined by sending a function $f \in L^2(\mathbb{R}^n)$ to the holomorphic extension to $\mathbb{C}^n$ of $f * h_t$. The original version of this transform was first studied by Bargmann in [5] where Bargmann considered (with a slightly different normalization) the map, which we refer as the Bargmann transform,

$$L^2(\mathbb{R}^n) \ni f \mapsto \text{the holomorphic extension of } ((f \sqrt{h_t}) * h_t).$$

The images of the Segal-Bargmann transform and the Bargmann transform are not the same. However, they are unitarily isomorphic. These images, which we call the Fock spaces, are the Hilbert spaces of holomorphic functions. The Segal-Bargmann transform and the Bargmann transform are unitary isomorphisms onto their images. There is also another version of the Segal-Bargmann transform where we consider the domain to be the space $L^2(\mathbb{R}^n, h_t(x)dx)$, i.e. the $L^2$-space with the heat kernel measure. The formula of this version is given by

$$L^2(\mathbb{R}^n, h_t(x)dx) \ni f \mapsto \text{the holomorphic extension of } (f * h_t).$$

The image of this transform is the same as the image of the Bargmann transform. In fact, all three versions of these transforms are related by a commutative diagram. The history of the Segal-Bargmann transforms in the classical case can be found in [28] and [29].

In [27], Hall gave the generalizations of the Segal-Bargmann transforms to the compact group case. In particular, Hall studied the Segal-Bargmann transform on $L^2(U)$, where $U$ is an arbitrary compact connected Lie group. The Segal-Bargmann transform of $f \in L^2(U)$ is the holomorphic extension of $f * h_t$ to the complexification $U_\mathbb{C}$ of $U$. Again $h_t$ is the heat kernel on $U$. It is well-known that $f * h_t$ is a solution of the heat equation on $U$ with the initial condition $f$. In [27], Hall showed that the Segal-Bargmann transform is a unitary isomorphism from $L^2(U)$ onto
$O(U_C) \cap L^2(U_C, \nu_t)$ where $O(U_C)$ denotes the space of holomorphic functions on $U_C$ and $\nu_t$ is the $U$-average heat kernel on $U_C$. Analogous results for compact symmetric spaces are given by Stenzel in [56] where he worked directly in the level of symmetric spaces.

Hall also considered in [27] the case of quotient spaces $U/K$ where $K$ is any closed, connected subgroup of $U$ by applying the Segal-Bargmann transform for $U$ and restricting to $K$-invariant functions. This consideration also gives the same transform for a compact symmetric space as the one given by Stenzel. The image of the Segal-Bargmann transform is a Hilbert space of holomorphic functions on the complexification $U_C/K_C$ of $U/K$. However, the ways of describing the measures on $U_C/K_C$ for the image of the Segal-Bargmann transform in Hall’s setting and Stenzel’s setting are different. Hall used the heat kernel measure on the symmetric space $U_C/U$ while Stenzel used the heat kernel measure on the non-compact dual $G/K$ of compact symmetric space $U/K$ to explain the image of the Segal-Bargmann transform. In fact, those measures coincide by using the method of M. Flensted-Jensen ([22]). For more details about this discussion, we refer to [29] and [61].

Another proof that the Segal-Bargmann transform for a compact symmetric space is unitary where the image is described by the heat kernel measure on the dual non-compact symmetric space was given by Faraut in [15]. Faraut used the Gutzmer’s formula and the Laurent series on the complexification $U_C/K_C$ of $U/K$, which were introduced by Lassalle in [40], to prove the unitarity of the Segal-Bargmann transform.

The restriction principle was first introduced by Ólafsson and Ørsted in [45] to give another proof of the unitarity of the Segal-Bargmann transform in the compact group case with the assumption that the explicit formula for the reproducing kernel of the Fock space were known.

In this thesis, we give another proof of the unitarity of the Segal-Bargmann transform for a compact symmetric space. Our approach employs the restriction principle and some integration formulas collected in [15]. In our method, we do not assume the explicit formula of the reproducing kernel for the Fock space. We do not use the Gutzmer’s formula in our proof either. In summary, we use the straightforward calculations to prove that the Segal-Bargmann transform is an isometry and then apply the restriction principle to show the surjectivity of the Segal-Bargmann transform.

We are also interested in the construction of the Segal-Bargmann transform in the infinite dimensional case. There are some works on constructing the heat kernel measure on the direct limit of some complex groups. In [26], Gordina constructed the Fock space on $SO(\infty, \mathbb{C})$, using the heat kernel measure determined by an inner product on the Lie algebra $so(\infty, \mathbb{C})$. There is also a work on constructing the heat kernel measure on the projective limits of symmetric spaces of non-compact type. In [52], Sinton developed the theory of a spherical Fourier transform for measures on certain projective limits of symmetric spaces of non-compact type and applied this theory to obtain a heat kernel measure on the limit space.
However, the infinite dimensional consideration in this thesis differs from the natural one. In stead of looking at the $L^2$-space of functions on $U_\infty/K_\infty$ where $U_\infty$ and $K_\infty$ are the direct limits of the sequences of compact groups $U_n$ and $K_n$ respectively in the category of Lie groups, we consider the direct limit of the Hilbert spaces $L^2(U_n/K_n)$, $n = 1, 2, \ldots$. By using the result of Wolf in [68] where we consider the nice sequence of symmetric spaces of compact type, we can construct the direct limit of $\{L^2(U_n/K_n)^{K_n}\}$ in the category of Hilbert spaces and unitary embeddings. This construction is valid because the highest weights behave appropriately between each level in the direct system and the harmonic analysis on $L^2(U_n/K_n)$ has the simple formula. Then we obtain the Segal-Bargmann transform on the direct limit.

Chapter 2 provides the self-contained discussion for the Segal-Bargmann transforms in the classical case. We give two difference proof of the unitarity of the Segal-Bargmann transform. The relations between the three versions of the Segal-Bargmann transforms are given in term of the commutative diagram. At the end of this chapter, we present an application of the Segal-Bargmann transforms allowing us to construct the Hermite polynomials and Hermite functions as the bases for the spaces $L^2(\mathbb{R}^n, h_{t}(x)dx)$ and $L^2(\mathbb{R}^n)$ respectively. We also define the Fourier transforms on the Fock spaces.

In Chapter 3, we give and prove a version of the restriction principle for general connected complex manifolds. Then we apply the restriction principle to give yet another proof of the unitarity of the Segal-Bargmann transform in the classical case. Chapter 4 contains another proof of the unitarity of the Segal-Bargmann transform for a compact symmetric space. We show how to obtain harmonic analysis of compact symmetric spaces from the harmonic analysis of compact groups. Then we give the formulas for the Segal-Bargmann transform in term of the holomorphic extension of a solution of the heat equation on a compact symmetric space and in term of the series of the holomorphic functions. Using the integration formulas in [15], some results in [57], and the restriction principle, we prove that the Segal-Bargmann transform is a unitary isomorphism at the end.

Our main results are presented in Chapter 5. We first consider a propagating sequence $M_n = U_n/K_n$ of symmetric spaces of compact type defined in [46]. Then use the Cartan-Helgason theorem to get the parametrizations of the $K_n$-spherical representations of the symmetric spaces $M_n = U_n/K_n$. These parametrizations behave nicely on each level in the propagating sequence. This allows us to define the unitary embedding on each level in order to get the direct system of $\{L^2(U_n/K_n)^{K_n}\}_n$ and hence obtain the direct limit in the category of Hilbert spaces and unitary embedding. We then define the Segal-Bargmann on the resulting direct limit.

Finally, we mention that the materials in Chapter 2 and 3 can also be found in the forthcoming book “The Segal-Bargmann Transform On Euclidean Space And Generalizations, An Introduction to Harmonic Analysis and Hilbert Spaces of Holomorphic Functions” by G. Ólafsson [44]. This book contains several aspects of the Segal-Bargmann transforms and their generalizations especially the connection to representation theory and infinite dimensional analysis. There are also
discussions on the heat equation and the Segal-Bargmann transform on Riemannian symmetric spaces and the more general framework of the Heckmann-Opdam theory of hypergeometric functions associated to root systems.
Chapter 2
The Segal-Bargmann Transform for $\mathbb{R}^n$

2.1 Introduction

The materials of this chapter can also be found in the forthcoming book [44]. This book is the main reference of this chapter. We first give the discussion of the heat equation on $\mathbb{R}^n$ in Section 2.2. Then define the Segal-Bargmann transform on $L^2(\mathbb{R}^n)$ in Section 2.3. Section 2.4 give the notion of the Hilbert space of holomorphic functions. We also prove some properties of the Hilbert space of holomorphic functions in this section. Then we use these results to define the Fock spaces in Section 2.5. The formulas of the reproducing kernels for the Fock spaces are given explicitly and then we prove the unitarity of the Segal-Bargmann transform. Section 2.6 present another proof the unitarity of the Segal-Bargmann transform. We define the other two versions of the Segal-Bargmann transform and prove their relations in term of the commutative diagram in Section 2.7. Finally in Section 2.8, we obtain the Hermite polynomials and the Hermite functions from the Segal-Bargmann transforms. We define the Fourier transforms on the Fock spaces at the end of this chapter. We use the notation $\mathbb{Z}^+$ for the set $\{0, 1, 2, \ldots\}$.

2.2 The Heat Equation on $\mathbb{R}^n$

Let $\Delta_x = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ be the Laplacian on $\mathbb{R}^n$. The heat equation on $\mathbb{R}^n$ is the following Cauchy problem:

$$\Delta_x u(x, t) = \frac{\partial u}{\partial t}(x, t), \quad (x, t) \in \mathbb{R}^n \times (0, \infty) \quad (2.1)$$

$$\lim_{t \to 0^+} u(x, t) = f(x), \quad x \in \mathbb{R}^n \quad \text{(the initial condition)}$$

where $f$ is a function in $L^2(\mathbb{R}^n)$.

To solve the heat equation, we formally apply the Fourier transform in the variable $x$. We use the formula for the Fourier transform of a function $g \in L^1(\mathbb{R}^n)$ as the following:

$$\hat{g}(\lambda) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} g(x)e^{-ix\cdot\lambda}dx.$$ 

Then we have the Fourier inversion formula: if $f \in L^1(\mathbb{R}^n)$ and $\hat{f} \in L^1(\mathbb{R}^n)$, then $f$ equals to a continuous function $f_0$ almost everywhere and

$$\hat{(\hat{f})} = (f^\vee) = f_0.$$
where we define
\[ g^\vee(x) = \hat{g}(-x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} g(\lambda)e^{i\lambda \cdot x} d\lambda \]
for any \( g \in L^1(\mathbb{R}^n) \). Therefore, if \( f \in L^1(\mathbb{R}^n) \) and \( \hat{f} = 0 \), then \( f = 0 \) almost everywhere. It follows that \( \sim \) is one-to-one on \( L^1(\mathbb{R}^n) \).

Moreover, by the Plancherel’s theorem, we have
\[ \sim : L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \]
and the map \( \sim \mid_{L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)} \) extends uniquely to a unitary isomorphism \( \mathcal{F} \) on \( L^2(\mathbb{R}^n) \). We call \( \mathcal{F} \) the Fourier transform on \( L^2(\mathbb{R}^n) \). For the theory of Fourier analysis on \( \mathbb{R}^n \), we refer to Chapter 8 in [21].

Now we formally take the Fourier transform in the \( x \)-variable in (2.1) and differentiate under the integral sign formally to get
\[ -|\lambda|^2 \hat{u}(\lambda,t) = \left( \sum_{j=1}^{n} i^2 \lambda_j^2 \right) \hat{u}(\lambda,t) = \hat{\Delta_x} u(\lambda,t) = \frac{\partial \hat{u}}{\partial t}(\lambda,t) = \frac{\partial \hat{u}}{\partial t}(\lambda,t). \]

For simplicity, we assume from now on that \( f \) is also in \( L^1(\mathbb{R}^n) \). Next we take the Fourier transform in \( x \)-variable in the initial condition to get \( \hat{u}(\lambda,0) = \hat{f}(\lambda) \). So the Fourier transformation leads to the ordinary differential equation (in \( t \)-variable)
\[ \frac{\partial \hat{u}}{\partial t}(\lambda,t) = -|\lambda|^2 \hat{u}(\lambda,t) \]
with the initial condition \( \hat{u}(\lambda,0) = \hat{f}(\lambda) \). The solution of this differential equation is
\[ \hat{u}(\lambda,t) = \hat{f}(\lambda)e^{-t|\lambda|^2}. \]  
(2.2)

Since \( \hat{f}(\lambda)e^{-t|\lambda|^2} \in L^1(\mathbb{R}^n) \) by Hölder’s inequality, if we assume that \( u(\cdot,t) \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n) \) for each \( t \in (0,\infty) \), then by the Fourier inversion formula we get
\[ u(x,t) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{f}(\lambda)e^{-|\lambda|^2t}e^{i\lambda \cdot x} d\lambda. \]  
(2.3)

Another way to obtain \( u(x,t) \) from (2.2) is to use the following properties of Fourier transform on \( L^1(\mathbb{R}^n) \):

1. If \( g(x) = e^{-|x|^2/2} \), then \( \hat{g} = g \).
2. For \( a > 0 \) and \( g \in L^1(\mathbb{R}^n) \), if \( g(x) = h(ax) \), then \( \hat{g}(\lambda) = a^{-n} \hat{h}(\lambda/a) \).
3. \( \hat{g} \ast \hat{h} = (2\pi)^{n/2} \hat{g} \hat{h} \) for \( g, h \in L^1(\mathbb{R}^n) \).
We have
\[ \hat{u}(\lambda, t) = \hat{f}(\lambda) e^{-t|\lambda|^2} = \hat{f}(\lambda) e^{-|\sqrt{2t}\lambda|^2/2} = \hat{f}(\lambda) (\sqrt{2t})^{-n} (e^{-|\lambda|/\sqrt{2t}})^n (\lambda) = (2\pi)^{n/2} \left\{ \hat{f}(\lambda) (4\pi t)^{-n/2} e^{-|\lambda|^2/4t} (\lambda) \right\}. \]

By the above Property 3 and the one-to-one property of \( \hat{\cdot} \), we have
\[ u(x, t) = (f * h_t)(x) = (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(y) e^{-|x-y|^2/4t} dy. \]

where
\[ h_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}, \]

the heat kernel on \( \mathbb{R}^n \).

By a direct calculation, the heat kernel \( h_t(x) \) satisfies equation (2.1), that is \( (\Delta - \frac{\partial}{\partial t}) h_t = 0 \). In fact, the heat kernel \( h_t \) is the solution to the heat equation with \( f = \delta_0 \) and \( h_t \) can be written as (cf. (4.17))
\[ h_t(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-|\lambda|^2 t} e^{i\lambda \cdot x} d\lambda. \] (2.4)

Now we can drop the assumption \( f \in L^1(\mathbb{R}^n) \) since the function \( f * h_t \) still makes sense for \( f \in L^2(\mathbb{R}^n) \). Note that by Young’s inequality, \( f * h_t \in L^2(\mathbb{R}^n) \). By a straightforward computation, we see that
\[ \Delta_x (h_t(x - y)) = \frac{(x - y)^2 e^{-|x-y|^2/4t}}{4t^2 (4\pi t)^{\frac{n}{2}}} - \frac{ne^{-|x-y|^2/4t}}{2t (4\pi t)^{\frac{n}{2}}} \]
\[ = \frac{\partial}{\partial t} (h_t(x - y)), \]

where we write
\[ x^2 = x_1^2 + x_2^2 + \ldots + x_n^2 \]
for any \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \).

To show that for \( f \in L^2(\mathbb{R}^n) \), \( f * h_t \) satisfies the heat equation, we want to pass the Laplacian \( \Delta_x \) and the partial derivative \( \frac{\partial}{\partial t} \) inside the integral sign of \( f * h_t \). A way of justification in this passing is to find some upper bound \( L^1 \)-functions for \( \Delta_x (h_t(x - y)) \) uniformly in \( x \) and for \( \frac{\partial}{\partial t} (h_t(x - y)) \) uniformly in \( t \). Then, by using the dominated convergence theorem and the mean value theorem, we can pass \( \Delta_x \) and the \( \frac{\partial}{\partial t} \) inside the integral sign of \( f * h_t \). The following rough estimates work for our propose when \( f \in L^2(\mathbb{R}^n) \) : for \( 0 < \epsilon < t < \delta \) where \( 0 < \epsilon < \delta \) are arbitrary,
\[ |\partial_t (f(y) h_t(x - y))| \leq |f(y)| (\frac{(x - y)^2 e^{-|x-y|^2/4t}}{4e^2 (4\pi \epsilon)^{\frac{n}{2}}} + |f(y)| \frac{ne^{-|x-y|^2/4t}}{2e (4\pi \epsilon)^{\frac{n}{2}}}, \]
and for $|x| \leq \rho$, where $\rho > 0$ is arbitrary,

$$
|\Delta_x (f(y) h_t(x-y))| \leq \begin{cases} 
|f(y)| \frac{\rho^2}{4t^2(4\pi t)^n} + |f(y)| \frac{ne^{-\frac{(|y| - \rho)^2}{4t}}}{2t(4\pi t)^{\frac{n}{2}}} & , |y| \leq \rho, \\
|f(y)| \frac{|y| + \rho^2}{4t^2(4\pi t)^n} & , |y| > \rho.
\end{cases}
$$

The bound for the first estimate is integrable by Hölder’s inequality. The first part of the bound for the second estimate is integrable on $|y| \leq \rho$ since $|\Delta_x (h_t(x-y))|$ is uniformly bounded on $|x| \leq \rho$, $|y| \leq \rho$ and $f$ is integrable on $|y| \leq \rho$ (by Hölder’s inequality). The second part of the above estimation is also in $L^2(\mathbb{R}^n)$ by Hölder’s inequality. Therefore,

$$
\Delta_x (f * h_t) = \Delta_x \left( \int_{\mathbb{R}^n} f(y) h_t(x-y) dy \right)
\begin{align*}
&= \int_{\mathbb{R}^n} f(y) \Delta_x (h_t(x-y)) dy \\
&= \int_{\mathbb{R}^n} f(y) \frac{\partial}{\partial t} (h_t(x-y)) dy \\
&= \frac{\partial}{\partial t} \int_{\mathbb{R}^n} f(y) h_t(x-y) dy \\
&= \frac{\partial}{\partial t} (f * h_t).
\end{align*}
$$

So $f * h_t$ satisfies equation (2.1). The heat kernel $h_t$ is a good kernel. In fact, it can be used as an approximate identity as it satisfies the initial condition $\lim_{t \to 0^+} f * h_t = f \in L^2(\mathbb{R}^n)$; this equality holds a.e. since we regard $f \in L^2(\mathbb{R}^n)$; in fact the initial condition above holds by using the fact from approximate identity, see [21], Theorem 8.15. Therefore, $u(x,t) = (f * h_t)(x)$ is indeed a solution of the heat equation with the initial function $f \in L^2(\mathbb{R}^n)$.

**Remark.** We note that the solution $f * h_t$ is not the only solution of the heat equation with initial value $f$. For example, if $n = 1$ and $f = 0$, then the function

$$
u(x,t) = \frac{x}{t} h_t(x)
$$

satisfies the heat equation for $t > 0$ with $\lim_{t \to 0^+} u(x,t) = 0$ for every $x$ but $u$ is not continuous at the origin $(0,0)$. This example illustrates non-uniqueness of the solution of the heat equation with initial value 0. In fact the uniqueness is obtained if we put some continuity assumption and a certain growth condition to $u$. We refer to [10] on page 58 and [53] on pages 164, 171-173 for more information about the uniqueness of the solution.

Another useful property of $h_t$ is that if $g \in C_c(\mathbb{R})$, then $\lim_{t \to 0^+} g * h_t = g$ uniformly on $\mathbb{R}$. This fact can be used to prove the Weierstrass approximation theorem. (It
is just a matter of classical approximate identity kernels and it shows up naturally when solving the heat equation and it was used by Weierstrass in proving his approximation theorem.)

We introduce the following notation. For \( z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \), we define
\[
z^2 = z_1^2 + z_2^2 + \cdots + z_n^2.
\]
Then for \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \),
\[
x^2 = x_1^2 + x_2^2 + \cdots + x_n^2 = |x|^2.
\]
These notation for \( x^2 \) and \( z^2 \) have an advantage that they reflex a fact that the heat kernel
\[
h_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} = (4\pi t)^{-\frac{n}{2}} e^{-\frac{(x_1^2 + x_2^2 + \cdots + x_n^2)}{4t}} = (4\pi t)^{-\frac{n}{2}} e^{-\frac{x^2}{4t}}
\]
where \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) has a holomorphic extension
\[
\tilde{h}_t(z) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{z^2}{4t}}
\]
on \( \mathbb{C}^n \). We note that the function
\[
(4\pi t)^{-\frac{n}{2}} e^{-\frac{|z|^2}{4t}} = (4\pi t)^{-\frac{n}{2}} e^{-\frac{z^2}{4t}}
\]
is not a holomorphic continuation of \( h_t(x) \). In the next section, we will use this holomorphic extension \( \tilde{h}_t \) to construct the holomorphic extension of a solution of the heat equation \( u(x, t) = (f \ast h_t)(x) \).

### 2.3 The Segal-Bargmann Transform on \( L^2(\mathbb{R}^n) \)

From now on, the variable \( t \) is always positive throughout all chapters.

**Lemma 2.1.** Let \( \Omega \) be a non-empty open subset of \( \mathbb{R}^n \). Assume that \( U \) is a non-empty open set in \( \mathbb{C}^n \) and \( f : U \times \Omega \rightarrow \mathbb{C} \) has the following properties:

1. for each \( z \in U \), the function \( x \mapsto f(z, x) \) is in \( L^1(\Omega) \),
2. for each \( x \in \Omega \), the function \( z \mapsto f(z, x) \) is holomorphic on \( U \), and
3. for every \( z_0 \in U \), there exist a neighborhood \( W \) of \( z_0 \) with \( W \subseteq U \) and a non-negative function \( g \in L^1(\Omega) \) such that for all \( z \in W \), \( |f(z, \cdot)| \leq g(\cdot) \) on \( \Omega \).

Then the function
\[
F(z) = \int_{\Omega} f(z, x) \, dx
\]
is holomorphic on \( U \).
Proof. By Hartogs’ Theorem on separate holomorphicity, it suffices to show that the function $F(z)$ is holomorphic in each variable $z_j$ when the other coordinates $z_k$ for $k \neq j$ are fixed. Let $a = (a_1, \ldots, a_n) \in U$ and let $j \in \{1, 2, \ldots, n\}$. Define

$$U_j := \{ z \in \mathbb{C} | (a_1, \ldots, a_{j-1}, z, a_{j+1}, \ldots, a_n) \},$$

and $F|_j : U_j \to \mathbb{C}$ by $z \mapsto F(a_1, \ldots, a_{j-1}, z, a_{j+1}, \ldots, a_n)$. Then $U_j$ is open in $\mathbb{C}$.

We show that $F|_j$ is holomorphic on $U_j$. Let $(a_1, \ldots, a_{j-1}, w, a_{j+1}, \ldots, a_n) \in U_j$. Then by assumption (3), there is an $r > 0$ and $g \in L^1(\Omega)$ such that

$$|f(a_1, \ldots, a_{j-1}, z, a_{j+1}, \ldots, a_n, x)| \leq g(x)$$

for all $|z - w| \leq r$ and all $x \in \Omega$. For simplicity, we let

$$h(z, x) = f(a_1, \ldots, w + z, \ldots, a_n, x)$$

for $|z| \leq r$ and $x \in \Omega$. By the one-dimensional Cauchy integral formula we get for $|z| < r/2$ and $x \in \Omega$,

$$\left| f(a_1, \ldots, w + z, \ldots, a_n, x) - f(a_1, \ldots, w, \ldots, a_n, x) \right| = \left| h(z, x) - h(0, x) \right|$$

and

$$\left| h(z, x) - h(0, x) \right| = \left| \frac{1}{2\pi i} \int_{|\xi| = r} \left( \frac{h(\xi)}{\xi - z} - \frac{h(\xi, x)}{\xi} \right) d\xi \right|$$

$$\leq \frac{1}{2\pi} \int_{|\xi| = r} \frac{|f(a_1, \ldots, w + \xi, \ldots, a_n, x)|}{|\xi| |\xi - z|} |d\xi|$$

$$\leq \frac{1}{2\pi} \int_{|\xi| = r} \frac{g(x)}{r(r/2)} |d\xi|$$

$$= \frac{2g(x)}{r}.$$

Therefore, by the dominated convergence theorem, we can interchange the limit and the integral to get

$$\lim_{z \to 0} \frac{(F|_j)(w + z) - (F|_j)(w)}{z} = \lim_{z \to 0} \int_{\Omega} \frac{f(a_1, \ldots, w + z, \ldots, a_n, x) - f(a_1, \ldots, w, \ldots, a_n, x)}{z} dx$$

$$= \int_{\Omega} \lim_{z \to 0} \frac{f(a_1, \ldots, w + z, \ldots, a_n, x) - f(a_1, \ldots, w, \ldots, a_n, x)}{z} dx$$

$$= \int_{\Omega} \partial_j f(a_1, \ldots, a_{j-1}, w, a_{j+1}, \ldots, a_n, x) dx.$$

This completes the proof. \qed
Theorem 2.2. For every $f \in L^2(\mathbb{R}^n)$, the function $\widetilde{f \ast h_t} : \mathbb{C}^n \rightarrow \mathbb{C}^n$ defined by

$$\widetilde{(f \ast h_t)}(z) = \int_{\mathbb{R}^n} f(y) \tilde{h}_t(z - y) dy = (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(y) e^{-(z - y)^2/4t} dy$$

is the analytic continuation to $\mathbb{C}^n$ of $f \ast h_t$.

Proof. Let $R > 0$ and $z = x + iy$ with $x, y \in \mathbb{R}^n$ and $|x|, |y| \leq R$. Then

$$|e^{-(z-v)^2/4t}| = e^{-(|x|^2 - |y|^2 + |v|^2 - 2x \cdot v)/4t} \leq e^{-(|x|^2 - |y|^2 + |v|^2 - 2R|v|)/4t} \leq e^{(|y|^2 - |x|^2 + R^2)/4t} \cdot e^{-(|v| - R)^2/4t} \leq e^{3R^2/4t} \cdot e^{-(|v| - R)^2/4t}.$$ 

Hence,

$$|f(v)e^{-(z-v)^2/4t}| \leq e^{3R^2/4t} \cdot e^{-(|v| - R)^2/4t} |f(v)|.$$ 

The function on the right hand side is integrable (by Hölder’s inequality) and independent of $z$. Since $R$ is arbitrary and the function $z \mapsto f(v)e^{-(z-v)^2/4t}$ is holomorphic, by the previous lemma, it follows that the map

$$z \mapsto (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(y)e^{-(z-y)^2/4t} dy$$

is holomorphic on $\mathbb{R}^n$. \hfill $\square$

We define the Segal-Bargmann transform $H_t$ on $L^2(\mathbb{R}^n)$ by

$$H_t(f) = \widetilde{f \ast h_t}.$$ 

Then $H_t : L^2(\mathbb{R}^n) \rightarrow \mathcal{O}(\mathbb{C}^n)$. We would like to find the image of the Segal-Bargmann transform, $H_t(L^2(\mathbb{R}^n))$.

2.4 Hilbert Spaces of Holomorphic Functions

Let $D$ be a domain (open and connected) in a complex manifold $M$. The space $\mathcal{O}(D)$ of holomorphic functions on $D$ is a topological vector spaces equipped with the topology of uniform convergence on compact subsets which is defined by the seminorms

$$p_Q(F) = \sup_{x \in Q} |F(z)|$$

as $Q$ ranges over compact subsets of $D$.

A Hilbert space of holomorphic functions on $D$ is a subspace $\mathcal{H}$ of $\mathcal{O}(D)$ which is equipped with the structure of a Hilbert space such that the embedding

$$\mathcal{H} \hookrightarrow \mathcal{O}(D)$$
is continuous, which means that: for every compact set $Q \subset D$ there is a constant $C_Q$ such that
\[ \forall F \in \mathcal{H}, \forall z \in Q, \quad |F(z)| \leq C_Q \|F\|_{\mathcal{H}}. \]
Therefore, if $F_n \to F$ in $\mathcal{H}$, then $F_n \to F$ uniformly on compact subsets of $D$.

Let $\mathcal{H} \subseteq \mathcal{O}(D)$ be a Hilbert space of holomorphic functions on $D$. Then for every $w \in D$, the evaluation map $ev_w : \mathcal{H} \to \mathbb{C}$ defined by
\[ ev_w(F) = F(w), \quad F \in \mathcal{H} \]
is a bounded linear functional on $\mathcal{H}$. Therefore, by the Riesz representation theorem, for each $w \in D$ there exists a unique function $K_w \in \mathcal{H}$ such that
\[ F(w) = \langle F, K_w \rangle_{\mathcal{H}}, \quad F \in \mathcal{H}. \]
The kernel $K, K(z, w) = K_w(z)$, is called the reproducing kernel of $\mathcal{H}$. The reproducing kernel $K$ has the properties
\[ K(z, w) = \langle K_w, K_z \rangle_{\mathcal{H}} = \|K_z\|_{\mathcal{H}}^2 = K(z, w), \]
for all $z, w \in D$. Thus, $K(z, w)$ is holomorphic in $z$, antiholomorphic in $w$. By Hartogs’s theorem, it follows that $K(z, \overline{w})$ is holomorphic on $D \times D$. In particular, the kernel $K$ is continuous on $D \times D$.

**Proposition 2.3.** Let $M$ be a connected complex manifold. Suppose that there is a continuous strictly positive measure $\rho$ on $M$, i.e. in each chart, $\rho$ has a continuous strictly positive density with respect to the Lebesgue measure. Then, $\mathcal{O}(M) \cap L^2(M, \rho)$ is a Hilbert space of holomorphic functions.

**Remark.** We define the set $\mathcal{O}(M) \cap L^2(M, \rho)$ as follows:
\[ \mathcal{O}(M) \cap L^2(M, \rho) = \left\{ F \in \mathcal{O}(M) : \|F\|_2^2 = \int_M |F(z)|^2 d\rho(z) < \infty \right\}. \]
Note that since $\rho$ is a continuous strictly positive measure, if $F$ and $G$ are in $\mathcal{O}(M) \cap L^2(M, \rho)$ and $F = G$ a.e., then we must have $F = G$ everywhere. Therefore, we can consider the space $\mathcal{O}(M) \cap L^2(M, \rho)$ as a subspace of $L^2(M, \rho)$ and the notation $\mathcal{O}(M) \cap L^2(M, \rho)$ is somewhat legitimate to use.

**Proof.** Suppose that the complex manifold $M$ has dimension $n$. Let $Q$ be a compact subset of $M$. For each $z \in Q$, let $(\varphi_z, \mathcal{U}_z)$ be a chart such that $z \in \mathcal{U}_z$. We can assume that $\varphi_z(z) = 0$ for every $z \in Q$. For each $z \in Q$, let $r_z = ((r_{z,1}), \ldots, (r_{z,n})) \in (\mathbb{R}^+)^n$ be such that $P_{r_z}(0) \subseteq \varphi_z(\mathcal{U}_z)$ where $P_{r_z}(0)$ is a polydisc $\{ w \in \mathbb{C}^n : |w_j| < (r_z)_j, \quad j = 1, \ldots, n \}$. Let $V_z = \varphi_z^{-1}(P_{r_z}(0)) \subseteq \mathcal{U}_z$ for each $z \in Q$. Then $V_z$ is open in $M$ for all $z \in Q$. Then $\bigcup_{z \in Q} V_z$ is an open cover of $Q$. Since $Q$ is
compact, $Q \subseteq \bigcup_{j=1}^{l} V_{z_j}$ for some $z_j \in Q$, $j = 1, \ldots, l$. Moreover, $Q \subseteq \bigcup_{j=1}^{l} V_{z_j}$. Note that each $W_j := V_{z_j}$ is compact since $V_{z_j} = (\varphi_z)^{-1}(P_{2r_{z_j}}(0))$ and $P_{2r_{z_j}}(0)$ is compact in $\mathbb{C}^n$. Therefore, the compact set $Q$ can be covered by a finite union of compact sets homeomorphic to $P_{2r_{z_j}}(0)$. It is therefore enough to show that for each $j \in \{1, \ldots, l\}$, there exists a constant $C_j > 0$ such that 

$$|F(\varphi_j(w))| \leq C_j |F|$$

for all $F \in \mathcal{O}(M) \cap L^2(M, \rho)$ and for all $w \in W_j = V_{z_j}$. For simplicity, we write $\varphi_j = \varphi_{z_j}$, $U_j = U_{z_j}$ and $r(j) = r_{z_j}$ for all $j = 1, \ldots, l$.

Pick an $\varepsilon > 0$ such that $\varepsilon < \min \{\frac{1}{2r(j)} : j = 1, \ldots, l, k = 1, \ldots, n\}$. For simplicity, we write just $\varepsilon = (\varepsilon, \ldots, \varepsilon)$. Fix $j \in \{1, \ldots, l\}$. Let $F \in \mathcal{O}(M) \cap L^2(M, \rho)$ and $w \in W_j$. Note that $P_{\varepsilon}(\varphi_j(w)) \subset P_{\frac{1}{2r(j)}+\varepsilon}(0) \subset \varphi_j(U_j)$. Since $F \circ \varphi_j$ is holomorphic, by the Cauchy integral formula if $t = (t_1, \ldots, t_n)$ with $0 < t_k \leq \varepsilon$ for all $k = 1, \ldots, n$, then

$$\frac{(F \circ \varphi_j)(\varphi_j(w))}{(2\pi i)^n} = \frac{1}{(2\pi i)^n} \int_{\partial(P_{\varepsilon}(\varphi_j(w)))} \frac{(F \circ \varphi_j)(\xi)}{(\xi - \varphi_j(w))} d\xi \int_{\partial(P_{\varepsilon}(\varphi_j(w)))} \frac{(F \circ \varphi_j)(\xi)}{(\xi - \varphi_j(w))} d\xi = \frac{1}{(2\pi)^n} \int_{0}^{2\pi} \cdots \int_{0}^{2\pi} (F \circ \varphi_j)(t_1 e^{i\theta_1}, \ldots, t_n e^{i\theta_n}) d\theta_1 \cdots d\theta_n.
$$

Thus, using polar coordinates on each of the discs $\{\xi_k : |\xi_k - \varphi_j(w)_k| \leq \varepsilon\}$, we get

$$\int_{P_{\varepsilon}(\varphi_j(w))} (F \circ \varphi_j^{-1})(\xi)d\xi = \int_{[0,\varepsilon]^n} \int_{[0,2\pi]^n} (F \circ \varphi_j^{-1})(t_1 e^{i\theta_1}, \ldots, t_n e^{i\theta}) t_1 \cdots t_n d\theta_1 \cdots d\theta_n dt_1 \cdots dt_n = (2\pi)^n \int_{[0,\varepsilon]^n} (F \circ \varphi_j^{-1})(\varphi_j(w)) t_1 \cdots t_n dt_1 \cdots dt_n = (\pi \varepsilon^2)^n (F \circ \varphi_j^{-1})(\varphi_j(w)).$$
By the assumption that in each chart, \( \rho \) has a continuous strictly positive density with respect to the Lebesgue measure, it follows that

\[
|F(w)| = |(F \circ \varphi_j^{-1})(\varphi_j(w))| \\
\leq \frac{1}{(\pi \epsilon^2)^n} \int_{P_\epsilon(\varphi_j(w))} |(F \circ \varphi_j^{-1})(\xi)| d\xi \\
= \frac{1}{(\pi \epsilon^2)^n} \int_{P_\epsilon(\varphi_j(w))} |(F \circ \varphi_j^{-1})(\xi)|(D_{\varphi_j}(\xi))^{-1/2}(D_{\varphi_j}(\xi))^{1/2} d\xi \\
\leq \frac{1}{(\pi \epsilon^2)^n} \left( \int_{P_\epsilon(\varphi_j(w))} (D_{\varphi_j}(\xi))^{-1} d\xi \right)^{1/2} \left( \int_{P_\epsilon(\varphi_j(w))} |(F \circ \varphi_j^{-1})(\xi)|^2(D_{\varphi_j}(\xi)) d\xi \right)^{1/2} \\
= \frac{1}{(\pi \epsilon^2)^n} \left( \int_{P_\epsilon(\varphi_j(w))} (D_{\varphi_j}(\xi))^{-1} d\xi \right)^{1/2} \left( \int_{\varphi_j^{-1}(P_\epsilon(\varphi_j(w)))} |F(z)|^2 d\rho(z) \right)^{1/2} \\
\leq \frac{1}{(\pi \epsilon^2)^n} \left( \int_{\varphi_j^{-1}(P_\epsilon(\varphi_j(w)))} (D_{\varphi_j}(\xi))^{-1} d\xi \right)^{1/2} |F|_2 \\
\leq C_j |F|_2,
\]

where \( D_{\varphi_j} \) is the continuous strictly positive density with respect to the chart \( (\varphi_j, U_j) \) and

\[
C_j = \frac{1}{(\pi \epsilon^2)^n} \left[ \text{Vol}(\varphi_j^{-1}(P_{\epsilon/2}(j))) \cdot \max \left\{ (D_{\varphi_j}(\xi))^{-1} : \xi \in \varphi_j^{-1}(P_{\epsilon/2}(j)) \right\} \right]^{1/2} < \infty.
\]

Therefore, if we let \( C_Q = \max\{C_j : j = 1, \ldots, l\} \), then \( |F(z)| \leq C_Q |F|_2 \) for all \( z \in Q \) and all \( F \in \mathcal{O}(M) \cap L^2(M, \rho) \). Since \( Q \) is arbitrary, this statement holds for every compact subset of \( M \). From this it follows that if \( (F_n) \) is a sequence in \( \mathcal{O}(M) \cap L^2(M, \rho) \) such that \( F_n \to F \) in \( L^2(M, \rho) \), then \( F_n \to F \) uniformly on compact subsets of \( M \) which implies that \( F \in \mathcal{O}(M) \cap L^2(M, \rho) \). Thus, \( \mathcal{O}(M) \cap L^2(M, \rho) \) is a closed subspace of \( L^2(M, \rho) \). Hence, \( \mathcal{O}(M) \cap L^2(M, \rho) \) is a Hilbert space of holomorphic functions.

If we can find a countable orthonormal basis for a Hilbert space of holomorphic functions, then the next proposition gives us a formula for the reproducing kernel. First, we prove the following lemma which is known as Dini’s theorem.

**Lemma 2.4 (Dini’s Theorem).** Let \( X \) be a compact topological space and \( (f_n)_{n=1}^\infty \) an increasing sequence of continuous functions \( f_n : X \to \mathbb{R}^n \), such that \( f_n \) converges pointwise to a continuous function on \( X \). Then \( f_n \) converges uniformly to \( f \) on \( X \).

**Proof.** Let \( \epsilon > 0 \). For each \( n \), let \( U_n = \{ x \in X \mid |f_n(x) - f(x)| < \epsilon \} \). Since each \( f_n \) and \( f \) are continuous, \( U_n \) is open for each \( n \). Moreover, \( \bigcup_{n=1}^\infty U_n = X \)
because \( f_n \rightarrow f \) on \( X \). Therefore, \( \{U_n\} \) is an open cover for \( X \). But \( X \) is compact, some finite collection of \( U_n \)'s suffices to cover \( X \). Since \( f_n \) is increasing to \( f \) and 
\[
 f - f_{n+1} = (f - f_n) + (f_n - f_{n+1}), \quad U_n \subseteq U_{n+1} \quad \text{for all} \quad n. 
\]
Thus, we have \( X = U_N \) for some \( N \). Therefore, if \( n \geq N \) and \( x \in X \), then
\[
 |f_n(x) - f(x)| = (f(x) - f_N(x)) + (f_N(x) - f_n(x)) < \epsilon + (f_N(x) - f_n(x)) \leq \epsilon. 
\]
Hence, \( f_n \) converges uniformly to \( f \) on \( X \).

**Proposition 2.5.** Let \( D \) be a domain in a complex manifold \( M \) and \( \mathcal{H} \subseteq \mathcal{O}(D) \) be a Hilbert space of holomorphic functions on \( D \). Suppose that \( \mathcal{H} \) has a countable orthonormal basis \( \{\varphi_n\}_{n=1}^\infty \). Then
\[
 K(z, w) = \sum_{n=1}^\infty \varphi_n(z)\overline{\varphi_n(w)}. 
\]
The series converges absolutely and uniformly on compact subsets of \( D \times D \).

**Proof.** Since \( \{\varphi_n\}_{n=1}^\infty \) is an orthonormal basis of \( \mathcal{H} \), for each \( w \in D \)
\[
 K_w = \sum_{m=1}^\infty (\langle K_w, \varphi_n \rangle_{\mathcal{H}}) \varphi_n = \sum_{n=1}^\infty \varphi_n(w)\overline{\varphi_n}. 
\]
The series converges for the topology of \( \mathcal{H} \), hence for the topology of \( \mathcal{O}(D) \). Therefore, for each \( w \in D \),
\[
 K_w(z) = \sum_{n=1}^\infty \varphi_n(w)\overline{\varphi_n(z)} 
\]
where the series converges uniformly on compact subsets of \( D \). In particular, we obtain the pointwise convergence on \( D \times D \): for every \( z, w \in D \),
\[
 K(z, w) = \sum_{n=1}^\infty \varphi_n(z)\overline{\varphi_n(w)}. 
\]
It remains to show that the convergence is uniform on compact subsets of \( D \times D \).
From the reproducing property of the kernel \( K \) it follows that
\[
 \langle \varphi_n, K_w \rangle_{\mathcal{H}} = \varphi_n(w),
\]
and hence by the Parseval’s identity we obtain
\[
 \sum_{n=1}^\infty |\varphi_n(w)|^2 = \sum_{n=1}^\infty |\langle \varphi_n, K_w \rangle_{\mathcal{H}}|^2 = \|K_w\|^2_{\mathcal{H}} = K(w, w).
\]
Since the map \( w \mapsto K(w, w) \) is continuous on \( D \times D \), by Dini’s theorem the convergence
\[
 \sum_{n=1}^\infty |\varphi_n(w)|^2 = K(w, w)
\]
is uniform on every compact subset of $D$. Now let $A$ be any compact subset of $D \times D$ and let $\epsilon > 0$ be given. Then $\pi_1(A)$ and $\pi_2(A)$ are compact subsets of $D$ where $\pi_1$ and $\pi_2$ are the canonical projections to the first coordinate and to the second coordinate respectively. Therefore, there is a positive integer $L$ such that for all $M \geq L$ and $N \geq L$ with $M > N$,

$$\sum_{n=N}^{M} |\varphi_n(z)|^2 < \epsilon \quad \text{and} \quad \sum_{n=0}^{M} |\varphi_n(w)|^2 < \epsilon$$

for all $z \in \pi_1(A)$ and for all $w \in \pi_2(A)$. Thus, for all $M \geq L$ and $N \geq L$ with $M > N$ and for all $(z, w) \in A$,

$$\left| \sum_{n=N}^{M} \varphi_n(z) \overline{\varphi_n(w)} \right| \leq \sum_{n=N}^{M} |\varphi_n(z)||\varphi_n(w)|$$

$$\leq \left( \sum_{n=N}^{M} |\varphi_n(z)|^2 \right)^{1/2} \left( \sum_{n=N}^{M} |\varphi_n(w)|^2 \right)^{1/2}$$

$$< (\epsilon^{1/2} \epsilon)^{1/2}$$

$$= \epsilon.$$

Hence, $\left( \sum_{n=1}^{N} \varphi_n(z) \overline{\varphi_n(w)} \right)$ and $\left( \sum_{n=1}^{N} |\varphi_n(z)||\varphi_n(w)| \right)$ are uniformly Cauchy on $A$. This implies that the series

$$\sum_{n=1}^{\infty} \varphi_n(z) \overline{\varphi_n(w)}$$

converges absolutely and uniformly on compact subsets of $D \times D$. This completes the proof.

\[\square\]

### 2.5 The Fock Space as the Image of the Segal-Bargmann Transform

We define the Fock space, $\mathcal{H}_t(\mathbb{C}^n)$, on $\mathbb{C}^n$ as follows:

$$\mathcal{H}_t(\mathbb{C}^n) := \left\{ F \in \mathcal{O}(\mathbb{C}^n) : |F|_{\mathcal{H}_t}^2 := \int_{\mathbb{C}^n} |F(x + iy)|^2 h_+(y) dx dy < \infty \right\}.$$

We also define the classical Fock space, $\mathcal{F}_t(\mathbb{C}^n)$, on $\mathbb{C}^n$ by

$$\mathcal{F}_t(\mathbb{C}^n) := \left\{ F \in \mathcal{O}(\mathbb{C}^n) : |F|_{\mathcal{F}_t}^2 := \int_{\mathbb{C}^n} |F(x + iy)|^2 h_+(x)h_+(y) dx dy < \infty \right\}.$$
Note that
\[ h_\frac{1}{2}(y) dxdy = h_\frac{1}{2}(y) dx + i dy = (2\pi t)^{-n/2} e^{-|\alpha|^2/2t} dz \]
and
\[ h_\frac{1}{2}(x) h_\frac{1}{2}(y) dxdy = h_\frac{1}{2}(x) h_\frac{1}{2}(y) dx + i dy = (2\pi t)^{-n} e^{-|\alpha|^2/2t} dz. \]
Therefore,
\[ \mathcal{H}_t(\mathbb{C}^n) = \mathcal{O}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n, (2\pi t)^{-n/2} e^{-|\alpha|^2/2t} dz) \]
and
\[ \mathcal{F}_t(\mathbb{C}^n) = \mathcal{O}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n, (2\pi t)^{-n} e^{-|\alpha|^2/2t} dz). \]
Thus, these spaces are Hilbert spaces of holomorphic functions by Proposition 2.3. We will prove that the image of the Segal-Bargmann transform on \( L^2(\mathbb{R}^n) \) is the Fock space \( \mathcal{H}_t(\mathbb{C}^n) \) and the image of the Segal-Bargmann transform on \( L^2(\mathbb{R}^n, h_t(x) dx) \), which we will define in the next section, is the classical Fock space \( \mathcal{F}_t(\mathbb{C}^n) \). The main step to prove these facts is to find the reproducing kernel for the spaces \( \mathcal{H}_t(\mathbb{C}^n) \) and \( \mathcal{F}_t(\mathbb{C}^n) \). We will give an explicit unitary isomorphism between these two Hilbert spaces so that we obtain the reproducing kernel of one from the reproducing kernel of another. It turns out that we can find an orthonormal basis for the space \( \mathcal{F}_t(\mathbb{C}^n) \) and hence obtain the reproducing kernel for \( \mathcal{F}_t(\mathbb{C}^n) \).

**Lemma 2.6.** For \( \alpha \in (\mathbb{Z}^+)^n \) and \( z \in \mathbb{C}^n \), let
\[ \zeta_{\alpha,t}(z) = \frac{1}{(2\pi t)^{|\alpha|/2} \sqrt{\alpha!}} z^\alpha. \]
Then \( \{\zeta_{\alpha,t}\}_{\alpha \in (\mathbb{Z}^+)^n} \) is an orthonormal basis for \( \mathcal{F}_t(\mathbb{C}^n) \).

**Proof.** First of all, we note that for \( a, b \in \mathbb{Z}^+ \),
\[
\int_\mathbb{C} |z^a||z^b| e^{-|z|^2/2t} \frac{dz}{2\pi t} = \frac{1}{2\pi t} \int_0^{2\pi} \int_0^\infty r^{a+b} e^{-r^2/2t} r dr d\theta \quad (2.5)
\]
\[
= \frac{1}{t} \int_0^\infty r^{a+b+1} e^{-r^2/2t} dr \quad (2.6)
\]
\[
= \frac{1}{t} \int_0^\infty (2tu)^{a+b+1} e^{-u t} du \quad (2.7)
\]
\[
= (2t)^{(a+b)/2} \int_0^\infty u^{a+b+2} e^{-u} du \quad (2.8)
\]
\[
= (2t)^{(a+b)/2} \Gamma \left( \frac{a+b+2}{2} \right) \quad (2.9)
\]
\[
< \infty. \quad (2.10)
\]
Therefore,
\[
\int_\mathbb{C} \cdots \int_\mathbb{C} |z^a||z^b| e^{-|z|^2/2t} \frac{dz_1 \cdots dz_n}{(2\pi t)^n} = \prod_{j=1}^n \int_\mathbb{C} |z_j|^{\alpha_j} |\overline{z}_j|^{\beta_j} e^{-|z_j|^2/2t} \frac{dz_j}{2\pi t} < \infty,
\]
for \( \alpha = (\alpha_1, ..., \alpha_n) \), \( \beta = (\beta_1, ..., \beta_n) \in (\mathbb{Z}^+)^n \). Thus by Fubini-Tonelli theorem and by using the polar coordinates, for \( \alpha, \beta \in (\mathbb{Z}^+)^n \) such that \( \alpha \neq \beta \) (say \( \alpha_k \neq \beta_k \) for some \( k \in \{1, ..., n\} \)),

\[
\langle z^\alpha, z^\beta \rangle_{\mathcal{F}_t} = \int_{\mathbb{C}^n} z^\alpha \overline{z^\beta} e^{-|z|^2/2t} \frac{dz}{(2\pi t)^n}
= \int_{\mathbb{C}} ... \int_{\mathbb{C}} (z_1^{\alpha_1} ... z_n^{\alpha_n})(\overline{z_1^{\beta_1} ... \overline{z_n^{\beta_n}}}) e^{-|z_1|^2/2t} ... e^{-|z_n|^2/2t} dz_1 ... dz_n
= \prod_{j=1}^n \int_{\mathbb{C}} z_j^{\alpha_j} \overline{z_j^{\beta_j}} e^{-|z_j|^2/2t} \frac{dz_j}{2\pi t}
= \prod_{j=1}^n \int_0^{2\pi} \int_0^\infty r_j^{\alpha_j+\beta_j} e^{-r_j^2/2t} r_j d\theta e^{i(\alpha_j-\beta_j)\theta} d\theta
= 0,
\]
since \( \int_0^{2\pi} e^{i(\alpha_k-\beta_k)\theta} d\theta = 0 \). If \( \alpha = \beta \in (\mathbb{Z}^+)^n \), then by the previous equations and the same calculations as in those from equations (2.5) to (2.8), we get

\[
\langle z^\alpha, z^\alpha \rangle_{\mathcal{F}_t} = \prod_{j=1}^n 2\pi \int_0^\infty r_j^{2\alpha_j} e^{-r_j^2/2t} \frac{dr_j}{2\pi t}
= \prod_{j=1}^n \int_0^\infty r_j^{2\alpha_j+1} e^{-r_j^2/2t} dr_j
= \prod_{j=1}^n (2t)^{\alpha_j} \Gamma(\alpha_j + 1)
= \prod_{j=1}^n (2t)^{\alpha_j} (\alpha_j)!
= (2t)^{|\alpha|} |\alpha|!.
\]

Therefore, we have proven that \( \{\zeta_{\alpha,t}\}_{\alpha \in (\mathbb{Z}^+)^n} \) is an orthonormal set. It remains to show that it is a complete orthonormal set. To prove this, we let \( F \in \mathcal{F}_t(\mathbb{C}^n) \) such that \( \langle F, \zeta_{\alpha,t} \rangle = 0 \) for all \( \alpha \in (\mathbb{Z}^+)^n \). Since \( F \) is holomorphic on \( \mathbb{C}^n \), we can write for \( z \in \mathbb{C}^n \)

\[
F(z) = \sum_{\alpha \in (\mathbb{Z}^+)^n} a_\alpha z^\alpha
\]

where \( a_\alpha \in \mathbb{C} \) and the series converges uniformly on compact subsets of \( \mathbb{C}^n \).

For each \( k \in \mathbb{Z}^+ \), let \( \underline{k} = (k, ..., k) \in (\mathbb{Z}^+)^n \) and define

\[
P_{\underline{k}}(0) = \{z = (z_1, ..., z_n) \in \mathbb{C}^n \mid |z_j| \leq k \text{ for all } j = 1, ..., n\}.
\]

Observe that for each \( \alpha \in (\mathbb{Z}^+)^n \),

\[
|F(z)|^2 e^{-|z|^2/2t} \chi_{P_{\underline{k}}(0)}(z) \leq |F(z)|^2 |z^\alpha| e^{-|z|^2/2t} = \left(|F(z)| e^{-|z|^2/4t}\right) \left(z^\alpha e^{-|z|^2/4t}\right),
\]

18
which belongs to $L^1(\mathbb{C}^n, dz)$ because $|F(z)|e^{-|z|^2/4t}$, $|z^\alpha|e^{-|z|^2/4t} \in L^2(\mathbb{C}^n, dz)$. Similarly, for each $\alpha \in (\mathbb{Z}^+)^n$

$$|z^{2\alpha}e^{-|z|^2/2t}x_{P_k(0)}(z)| \leq |z^{2\alpha}|e^{-|z|^2/2t} \in L^1(\mathbb{C}^n, dz).$$

Finally, we also note that with the same calculation as above we have

$$\int_{P_k(0)} z^{2\alpha}e^{-|z|^2/2t} \frac{dz}{(2\pi t)^n} = 0$$

for all $k \in \mathbb{Z}$ provided that $\alpha \neq \beta \in (\mathbb{Z}^+)^n$. Now by the Lebesgue dominated convergence theorem, the uniform convergence of the power series of $F$ on compact subsets in $\mathbb{C}^n$ and the Lebesgue dominated convergence theorem again, we have for each $\alpha \in (\mathbb{Z}^+)^n$,

$$0 = \langle F, \zeta_{\alpha,t} \rangle = \int_{\mathbb{C}^n} F(z)\overline{z^{\alpha}}e^{-|z|^2/2t} \frac{dz}{(2\pi t)^n} = \lim_{k \to \infty} \int_{P_k(0)} F(z)\overline{z^{\alpha}}e^{-|z|^2/2t} \frac{dz}{(2\pi t)^n} = \lim_{k \to \infty} \sum_{\beta \in (\mathbb{Z}^+)^n} a_\beta \left( \sum_{\beta \in (\mathbb{Z}^+)^n} a_\beta \overline{z^{\beta}} \right) \frac{dz}{(2\pi t)^n} = \lim_{k \to \infty} \sum_{\beta \in (\mathbb{Z}^+)^n} a_\beta \left( \int_{P_k(0)} z^{2\alpha}e^{-|z|^2/2t} \frac{dz}{(2\pi t)^n} \right) = \lim_{k \to \infty} a_\alpha \int_{P_k(0)} |z^{2\alpha}|e^{-|z|^2/2t} \frac{dz}{(2\pi t)^n} = a_\alpha \int_{\mathbb{C}^n} |z^{2\alpha}|e^{-|z|^2/2t} \frac{dz}{(2\pi t)^n} = a_\alpha \langle z^\alpha, z^\alpha \rangle_{F_t} = a_\alpha (2t)^{|\alpha|/|\alpha|!}.

This implies that $a_\alpha = 0$ for all $\alpha \in (\mathbb{Z}^+)^n$. Hence, $F = 0$ and the proof is now complete.

**Lemma 2.7.** The reproducing kernel of $\mathcal{F}_t(\mathbb{C}^n)$ is given by

$$K_{\mathcal{F}_t}(z, w) = e^{z\bar{w}/2t}.$$
Proof. By Proposition 2.5 and Lemma 2.6, it follows that the reproducing kernel for $F \in \mathcal{F}_t(\mathbb{C}^n)$ is

$$K_{\mathcal{F}_t}(z, w) = \sum_{\alpha \in (\mathbb{Z}^+)^n} \frac{1}{(2t)^{||\alpha||_1} \alpha!} z^\alpha \overline{w^\alpha}$$

$$= \sum_{\alpha \in (\mathbb{Z}^+)^n} \frac{1}{(2t)^{||\alpha||_1} \alpha!} \prod_{j=1}^n (z_j \overline{w_j})^{\alpha_j}$$

$$= \sum_{n=0}^\infty \frac{1}{n!} \sum_{|\alpha| = n} \frac{n!}{\alpha!} \prod_{j=1}^n \left( \frac{z_j}{\sqrt{2t}} \frac{\overline{w_j}}{\sqrt{2t}} \right)^{\alpha_j}$$

$$= \sum_{n=0}^\infty \frac{1}{n!} \left( \frac{z_1}{\sqrt{2t}} \frac{\overline{w_1}}{\sqrt{2t}} + \cdots + \frac{z_n}{\sqrt{2t}} \frac{\overline{w_n}}{\sqrt{2t}} \right)^n$$

$$= e^{\overline{w}/2t}.$$

\[ \square \]

**Lemma 2.8.** For each $F \in \mathcal{F}_t(\mathbb{C}^n)$, we define

$$(\Psi_t(F))(z) = \frac{e^{-z^2/16t}}{(16\pi t)^{n/4}} F(z/2).$$

Then $\Psi_t : \mathcal{F}_t(\mathbb{C}^n) \rightarrow \mathcal{H}_{2t}(\mathbb{C}^n)$ is a unitary isomorphism with the inverse

$$(\Psi_t^{-1}(G))(z) = (16\pi t)^{n/4} e^{z^2/4t} G(2z)$$

for $G \in \mathcal{H}_{2t}(\mathbb{C}^n)$.

**Proof.** First, it is easy to see that $\Psi_t$ is linear. Next we show that $\Psi_t$ is an isometry. Let $F \in \mathcal{F}_t(\mathbb{C}^n)$. Then for $z = x + iy \in \mathbb{C}^n$, $x, y \in \mathbb{R}^n$,

$$|\Psi_t(F)(z)|^2 = \frac{|e^{-z^2/8t}|}{(16\pi t)^{n/2}} |F(z/2)|^2$$

$$= \frac{e^{\Re(-z^2/8t)}}{(16\pi t)^{n/2}} |F(z/2)|^2$$

$$= \frac{e^{-(x^2 - y^2)/8t}}{(16\pi t)^{n/2}} |F(z/2)|^2.$$
Therefore, we have
\[
\|\Psi_t(F)\|_{H_{2t}}^2 = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\Psi_t(F)(z)|^2 e^{-y^2/4t} \, dxdy
\]
\[
= \frac{(16\pi t)^{-n/2}}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |F(z/2)|^2 e^{-(x^2+y^2)/(8t)} e^{-y^2/4t} \, dxdy
\]
\[
= \frac{2^{-n}}{(4\pi t)^n} \int_{\mathbb{C}^n} |F(z/2)|^2 e^{-|z|^2/8t} \, dz
\]
\[
= \frac{2^{-n}}{(4\pi t)^n} 2^n \int_{\mathbb{C}^n} |F(z)|^2 e^{-|z|^2/2t} \, dz \quad (z \mapsto 2z)
\]
\[
= \frac{1}{(2\pi t)^n} \int_{\mathbb{C}^n} |F(z)|^2 e^{-|z|^2/2t} \, dz
\]
\[
= \|F\|_{\mathcal{F}_r}^2.
\]

Let \( G \in H_{2t}(\mathbb{C}^n) \). Define \((\Phi_t(G))(z) = (16\pi t)^{n/4} e^{z^2/4t} G(2z)\). Then by a simple calculation \(\Psi_t(\Phi_t(G)) = G\). Moreover,
\[
|\Phi_t(G)|_{\mathcal{F}_{t'}}^2 = \frac{(16\pi t)^{n/2}}{(2\pi t)^n} \int_{\mathbb{C}^n} |G(2z)|^2 e^{z^2/2t} e^{-|z|^2/2t} \, dz
\]
\[
= \frac{2^{2n}}{(4\pi t)^{n/2}} \int_{\mathbb{C}^n} |G(z)|^2 e^{-\Im(z)/2t} \, dz
\]
\[
= \frac{2^{2n}}{(4\pi t)^{n/2}} 2^{-2n} \int_{\mathbb{C}^n} |G(z)|^2 e^{-\Im(z)/4t} \, dz \quad (z \mapsto z/2)
\]
\[
= \|G\|_{H_{2t}}^2.
\]

Thus, \(\Phi_t(G) \in \mathcal{F}_t(\mathbb{C}^n)\). Hence, \(\Psi_t^{-1}(G) = \Phi_t(G)\) and this completes the proof.

**Corollary 2.9.** For \(\alpha \in (\mathbb{Z}^+)^n\) and \(z \in \mathbb{C}^n\), let
\[
\varphi_{\alpha,t}(z) = e^{-z^2/8t} (\alpha/2) \alpha ! (8\pi t)^{n/4} t^{\alpha/2} \sqrt{\alpha} !.
\]

Then \(\{\varphi_{\alpha,t}\}_{\alpha \in (\mathbb{Z}^+)^n}\) is an orthonormal basis for \(H_t(\mathbb{C}^n)\). Furthermore, the reproducing kernel for \(H_t(\mathbb{C}^n)\) is
\[
K_{H_t}(z, w) = \frac{e^{-(z-w)/8t}}{(8\pi t)^{n/2}} \tilde{h}_{2t}(z - w).
\]

**Proof.** By a direct calculation, we see that \(\Psi_t(\zeta_{\alpha,t}) = \varphi_{\alpha,t}\). Thus, by Lemma 2.6 and 2.8, \(\{\varphi_{\alpha}\}_{\alpha \in (\mathbb{Z}^+)^n}\) is an orthonormal basis for \(H_t(\mathbb{C}^n)\). By Proposition 2.5, the
reproducing kernel for $\mathcal{H}_t(\mathbb{C}^n)$ is (where we calculate as in the proof of Lemma 2.7)

$$K_{\mathcal{H}_t}(z,w) = \sum_{\alpha \in (\mathbb{Z}^+)^n} \varphi_\alpha(z)\overline{\varphi_\alpha(w)}$$

$$= e^{-\frac{(z^2 + \overline{w^2})}{8t}} \sum_{\alpha \in (\mathbb{Z}^+)^n} \frac{(z/2)^\alpha (\overline{w/2})^\alpha}{t^{\mid\alpha\mid} \alpha!}$$

$$= e^{-\frac{(z^2 + \overline{w^2})}{8t}} e^{\frac{z\overline{w}}{4t}}$$

$$= e^{-\frac{(z-w)^2}{8t}} = \tilde{h}_{2t}(z - \overline{w}).$$

Recall that $H_t(L^2(\mathbb{R}^n)) \subset \mathcal{O}(\mathbb{C}^n)$. We define the norm $\|\cdot\|_t$ on $H_t(L^2(\mathbb{R}^n))$ by

$$\|F\|_t = \|H_t(f)\|_t = \|f\|_2$$

for $F = H_t(f) \in H_t(L^2(\mathbb{R}^n))$.

This definition is well-defined because the map $H_t : L^2(\mathbb{R}^n) \longrightarrow \mathcal{O}(\mathbb{C}^n)$ is one-to-one: if $H_t(f) = H_t(g)$ for $f, g \in L^2(\mathbb{R}^n)$, then $f \ast h_t = g \ast h_t$ which implies that $f \ast h_t = g \ast h_t$ and hence

$$0 = (f - g) \ast h_t = (2\pi)^{n/2} (\mathcal{F}(f - g) \tilde{h}_t)^\vee = (\mathcal{F}(f - g) e^{-t(\cdot)^2})^\vee,$$

here we use the formula

$$f \ast g = (2\pi)^{n/2} (\mathcal{F}(f) \mathcal{F}(g))^\vee$$

for any $f, g \in L^2(\mathbb{R}^n)$; this formula is obtained by writing the convolution as an $L^2$ inner product of certain functions and then applying the unitarity of $\mathcal{F}$, see the first part of the proof of Proposition 2.15; thus, by the Fourier inversion formula (note that $\mathcal{F}(f - g) e^{-t(\cdot)^2} \in L^1(\mathbb{R}^n)$ by Hölder’s inequality), we have the following almost everywhere equations

$$e^{-t(\cdot)^2} \mathcal{F}(f - g) = ((\mathcal{F}(f - g) e^{-t(\cdot)^2})^\vee)^\vee = 0$$

and whence $\mathcal{F}(f - g) = 0$ in $L^2(\mathbb{R}^n)$; so $f = g$ a.e. by the injectivity of the Fourier transform $\mathcal{F}$. We see that the norm $\|\cdot\|_t$ forces the map $H_t$ to be a unitary isomorphism from $L^2(\mathbb{R}^n)$ onto $H_t(L^2(\mathbb{R}^n))$. Therefore, $(H_t(L^2(\mathbb{R}^n)), \|\cdot\|_t)$ is a Hilbert space.
Lemma 2.10. For \( z \in \mathbb{C}^n \),
\[
\int_{\mathbb{R}^n} e^{-(z-y)^2/4t} dy = (4\pi t)^{n/2}.
\]

Proof. Let \( F(z) = \int_{\mathbb{R}^n} e^{-(z-y)^2/4t} dy \). We first show that \( F \) is holomorphic on \( \mathbb{C}^n \). As in the proof of Theorem 2.2, for every \( R > 0 \), we obtain the inequality
\[
|e^{-(z-y)^2/4t}| \leq e^{3R^2/4t} e^{-3R^2/4t} \leq e^{3R^2/4t} \in L^1(\mathbb{R}^n)
\]
where \( |\Re(z)|, |\Im(z)| \leq R \). Since \( e^{3R^2/4t} e^{-3R^2/4t} \in L^1(\mathbb{R}^n) \) for all \( R > 0 \), by Lemma 2.1, \( F \) is holomorphic on \( \mathbb{C}^n \). But we know that for all \( x \in \mathbb{R}^n \),
\[
\int_{\mathbb{R}^n} e^{-(x-y)^2/4t} dy = (4\pi t)^{n/2}.
\]
Thus, \( F \) is equal to a constant \((4\pi t)^{n/2}\) on \( \mathbb{R}^n \) and hence on \( \mathbb{C}^n \).

Proposition 2.11. \((H_t(L^2(\mathbb{R}^n)), \| \cdot \|_t)\) is a Hilbert space of holomorphic functions.

Proof. Let \( Q \) be any compact subset of \( \mathbb{C}^n \). Then there exists an \( R > 0 \) such that \( |\Re(z)|, |\Im(z)| \leq R \) for all \( z \in Q \). Let \( z \in Q \) and \( F \in H_t(L^2(\mathbb{R}^n)) \). Then there is a unique \( f \in L^2(\mathbb{R}^n) \) such that \( F = f \ast \widetilde{h}_t \). Therefore, using the estimate as in the proof of the previous lemma, we have
\[
|F(z)| = |(H_tF)(z)|
= |(f \ast \widetilde{h}_t)(z)|
= \left| \int_{\mathbb{R}^n} f(y) \widetilde{h}_t(z-y) dy \right|
\leq (4\pi t)^{-n/2} \left( \int_{\mathbb{R}^n} |e^{-(z-y)^2/4t} dy| \right)^{1/2} \|f\|_2
\leq (4\pi t)^{-n/2} \left( \int_{\mathbb{R}^n} e^{3R^2/2t} e^{-3R^2/2t} dy \right)^{1/2} \|F\|_t
\leq C_{t,R} \|F\|_t,
\]
for some finite constant \( C_{t,R} \). Hence, \((H_t(L^2(\mathbb{R}^n)), \| \cdot \|_t)\) is a Hilbert space of holomorphic functions.

Lemma 2.12. The reproducing kernel for the Hilbert space \((H_t(L^2(\mathbb{R}^n)), \| \cdot \|_t)\) is
\[
K(z, w) = \widetilde{h}_t(z - w) = e^{-(z-w)/(8t)} / (8\pi t)^{n/2}.
\]
Therefore, the Hilbert spaces \((H_t(L^2(\mathbb{R}^n)), \| \cdot \|_t)\) and \((H_t(\mathbb{C}^n), \| \cdot \|_{H_t})\) have the same reproducing kernel.
Proof. Let $K : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be the reproducing kernel of $(H_t(L^2(\mathbb{R}^n)), \| \cdot \|_t)$. Then for $F = H_t(f) \in H_t(L^2(\mathbb{R}^n))$ and $w \in \mathbb{C}^n$,

$$
\langle F, K_w \rangle_t = F(w) = H_t f(w) = \int_{\mathbb{R}^n} f(x) h_t(x - w) dx = \langle f, \tau_w h_t \rangle_t = \langle H_t f, H_t(\tau_w h_t) \rangle_t = \langle F, H_t(\tau_w h_t) \rangle_t
$$

where $\tau_w h_t(x) = h_t(x - w)$. Thus,

$$
K(z, w) = K_w(z) = H_t(\tau_w h_t)(z) = ((\tau_w h_t) \ast h_t)(z) = \int_{\mathbb{R}^n} h_t(x - w) h_t(z - x) dx = \int_{\mathbb{R}^n} h_t(x) h_t(z - (x + w)) dx \quad (x \mapsto x + w) = (h_t \ast h_t)(z - w) = h_{2t}(z - w).
$$

\[\square\]

**Theorem 2.13.** $H_t(L^2(\mathbb{R}^n)) = \mathcal{H}_t(\mathbb{C}^n)$ with $\| \cdot \|_t = \| \cdot \|_{\mathcal{H}_t}$. That is as Hilbert spaces, $H_t(L^2(\mathbb{R}^n))$ and $\mathcal{H}_t(\mathbb{C}^n)$ are identical.

Proof. Let $K_t(z, w) = h_{2t}(z - w)$. Then $K_t$ is the reproducing kernel for both Hilbert spaces $(H_t(L^2(\mathbb{R}^n)), \| \cdot \|_t)$ and $(\mathcal{H}_t(\mathbb{C}^n), \| \cdot \|_{\mathcal{H}_t})$. Let $A$ be the space of finite linear combinations $\sum_j \alpha_j K_{z_j}$, $\alpha_j \in \mathbb{C}$ and $z_j \in \mathbb{C}^n$. Then $A$ is a dense subspace of $(H_t(L^2(\mathbb{R}^n)), \| \cdot \|_t)$ and $(\mathcal{H}_t(\mathbb{C}^n), \| \cdot \|_{\mathcal{H}_t})$. We digest the proof into the following steps.

Step 1: $\langle K_z, K_w \rangle_t = \langle K_z, K_w \rangle_{\mathcal{H}_t}$ for all $z, w \in \mathbb{C}^n$.

Let $z, w \in \mathbb{C}^n$. Then $\langle K_z, K_w \rangle_t = K(w, z) = \langle K_z, K_w \rangle_{\mathcal{H}_t}$.

Step 2: $\| F \|_t = \| F \|_{\mathcal{H}_t}$ for all $F \in A$. 

24
Let $F = \sum_j \alpha_j K_{z_j} \in A$. Then by Step 1, we have

\[
\|F\|_t^2 = \left\langle \sum_j \alpha_j K_{z_j}, \sum_k \alpha_j K_{z_k} \right\rangle_t = \sum_{j,k} \alpha_j \overline{\alpha}_k \langle K_{z_j}, K_{z_k} \rangle_t = \sum_{j,k} \alpha_j \overline{\alpha}_k \langle K_{z_j}, K_{z_k} \rangle_{\mathcal{H}_t} = \left\langle \sum_j \alpha_j K_{z_j}, \sum_k \alpha_j K_{z_k} \right\rangle_{\mathcal{H}_t} = \|F\|_{\mathcal{H}_t}^2.
\]

Step 3: There is a unitary isomorphism $T : (\mathcal{H}_t(L^2(\mathbb{R}^n)), \| \cdot \|_t) \longrightarrow (\mathcal{H}_t(\mathbb{C}^n), \| \cdot \|_{\mathcal{H}_t})$ such that $T(F) = F$ for all $F \in A$.

By Step 2, the identity map $\text{id} : (A, \| \cdot \|_t) \longrightarrow (\mathcal{H}_t(\mathbb{C}^n), \| \cdot \|_{\mathcal{H}_t})$ is an isometry. Since $A$ is dense in both $(\mathcal{H}_t(L^2(\mathbb{R}^n)), \| \cdot \|_t)$ and $(\mathcal{H}_t(\mathbb{C}^n), \| \cdot \|_{\mathcal{H}_t})$, $\text{id}$ extends to a unique isometry $T : (\mathcal{H}_t(L^2(\mathbb{R}^n)), \| \cdot \|_t) \longrightarrow (\mathcal{H}_t(\mathbb{C}^n), \| \cdot \|_{\mathcal{H}_t})$. The map $T$ is defined by

\[
T(F) = \lim_{n \to \infty} T(F_n)
\]

where $F \in \mathcal{H}_t(L^2(\mathbb{R}^n))$ and $(F_n)$ is a sequence in $A$ such that $F_n \to F$ in $(\mathcal{H}_t(L^2(\mathbb{R}^n)), \| \cdot \|_t)$. The map $T$ is well-defined, i.e. the limit exists and that it is independent of the sequence $(F_n)$. We also have

\[
\|T(F)\|_{\mathcal{H}_t} = \| \lim_{n \to \infty} T(F_n)\|_{\mathcal{H}_t} = \lim_{n \to \infty} \|T(F_n)\|_{\mathcal{H}_t} = \lim_{n \to \infty} \|F_n\|_t.
\]

Step 4: $T$ is the identity map onto $(\mathcal{H}_t(\mathbb{C}^n), \| \cdot \|_{\mathcal{H}_t})$.

Let $F \in \mathcal{H}_t(L^2(\mathbb{R}^n))$. Then there is a sequence $(F_n) \in A$ such that $(F_n)$ converges to $F$ in $(\mathcal{H}_t(L^2(\mathbb{R}^n)), \| \cdot \|_t)$. We show that $T(F) = F$. Let $z \in \mathbb{C}^n$ and $Q$ a compact subset of $\mathbb{C}^n$ containing $z$. Since $(\mathcal{H}_t(L^2(\mathbb{R}^n)), \| \cdot \|_t)$ and $(\mathcal{H}_t(\mathbb{C}^n), \| \cdot \|_{\mathcal{H}_t})$ are Hilbert spaces of holomorphic functions, there are constants $M_Q$ and $N_Q$ such that for all $n$,

\[
|F(z) - (T(F))(z)| \leq |F(z) - F_n(z)| + |F_n(z) - (T(F))(z)| = |F(z) - F_n(z)| + |(T(F_n))(z) - (T(F))(z)| \leq M_Q\|F - F_n\|_t + \|T(F_n) - T(F)\|_{\mathcal{H}_t}.
\]

Letting $n$ go to $\infty$, we obtain $(T(F))(z) = F(z)$. Since $z$ is arbitrary, $T(F) = F$ as desired. The map $T$ is onto $(\mathcal{H}_t(\mathbb{C}^n), \| \cdot \|_{\mathcal{H}_t})$ because $T$ is isometric (hence $T(\mathcal{H}_t(L^2(\mathbb{R}^n)))$ is closed in $\mathcal{H}_t(\mathbb{C}^n)$ and $T(A) = A$ is dense in $\mathcal{H}_t(\mathbb{C}^n)$) which imply that $T(\mathcal{H}_t(L^2(\mathbb{R}^n))) = T(\mathcal{H}_t(L^2(\mathbb{R}^n))) \supset T(A) = \overline{A} = \mathcal{H}_t(\mathbb{C}^n)$. Therefore, as Hilbert spaces, $\mathcal{H}_t(L^2(\mathbb{R}^n))$ and $\mathcal{H}_t(\mathbb{C}^n)$ are identical. \qed
Remark. In Step 2 of the proof, we used the following fact about the extension of a bounded linear operator on a Hilbert space: Let $H_1$ and $H_2$ denote Hilbert spaces with norms $\| \cdot \|_1$ and $\| \cdot \|_2$, respectively. Suppose $S$ is a dense subspace of $H_1$ and $T_0 : S \rightarrow H_2$ a linear transformation that satisfies $\| T_0(f) \|_2 = \| f \|_1$ whenever $f \in S$. Then $T_0$ extends to a unique linear transformation $T : H_1 \rightarrow H_2$ that satisfies $\| T(f) \|_2 = \| f \|_1$ for all $f \in H_1$. Moreover, $T(H_1) = \overline{T_0(S)}$. The proof of this statement can be found in [55].

Corollary 2.14. The Segal-Bargmann transform $H_t : L^2(\mathbb{R}^n) \rightarrow \mathcal{H}_t(\mathbb{C}^n)$ is a unitary isomorphism.

Proof. This follows directly from the previous theorem. □

2.6 Another Proof of Unitarity of $H_t$

We can prove the above corollary without applying Theorem 2.13. Only Corollary 2.9, which tell us what the reproducing kernel for $\mathcal{H}_t(\mathbb{C}^n)$ looks like, is used. Now we give another proof of the above corollary. We show that $H_t : L^2(\mathbb{R}^n) \rightarrow \mathcal{H}_t(\mathbb{C}^n)$ is an isometry by using another formula for $H_tf$. First we prove the following proposition.

Proposition 2.15. Let $f \in L^2(\mathbb{R}^n)$. Then for $x \in \mathbb{R}^n$ and $z \in \mathbb{C}^n$

$$ (f * h_t)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} (\mathcal{F}f)(\lambda)e^{-t\lambda^2}e^{i\lambda \cdot x}d\lambda, $$

and

$$ (H_tf)(z) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} (\mathcal{F}f)(\lambda)e^{-t\lambda^2}e^{i\lambda \cdot z}d\lambda. $$

Proof. Since the Fourier transform $\mathcal{F}$ is a unitary isomorphism on $L^2(\mathbb{R}^n)$, for $x \in \mathbb{R}^n$,

$$ (f * h_t)(x) = \langle f, \tau_x h_t \rangle $$

$$ = \langle \mathcal{F}f, \tau_x \mathcal{F}h_t \rangle $$

$$ = \langle \mathcal{F}f, e^{-ix(\cdot)} \mathcal{F}h_t \rangle $$

$$ = \langle \mathcal{F}f, e^{-i(x-(\cdot))} e^{-t(\cdot)^2} \rangle $$

$$ = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} (\mathcal{F}f)(\lambda)e^{-t\lambda^2}e^{i\lambda \cdot x}d\lambda $$

where we define $(\tau_x g)(y) = g(y - x)$ for any function $g$ on $\mathbb{R}^n$ and $x, y \in \mathbb{R}^n$. So we have proved the first equation in the lemma. To prove the second equation, it suffices to show that the right hand side represents a holomorphic function on $\mathbb{C}^n$. 

26
Let \( R > 0 \) and \( z = x + iy \) with \( x, y \in \mathbb{R}^n \) and \( |y| \leq R \). Then
\[
|e^{-t\lambda^2 + i\lambda z}| = e^{-t\lambda^2 - \lambda y} \\
\leq e^{-t\lambda^2 + |\lambda|R} \\
= e^{-\left(\sqrt{t}\lambda - \frac{R}{2\sqrt{t}}\right)^2} \cdot e^{\frac{R^2}{4t}}.
\]
Thus, we have
\[
|{(\mathcal{F}f)(\lambda)}e^{-t\lambda^2} e^{i\lambda z}| = |{(\mathcal{F}f)(\lambda)}|e^{-t\lambda^2 - \lambda y} \tag{2.11}
\leq e^{\frac{R^2}{4t}} |{(\mathcal{F}f)(\lambda)}|e^{-\left(\sqrt{t}\lambda - \frac{R}{2\sqrt{t}}\right)^2} \tag{2.12}.
\]
The last function is integrable (by Hölder’s inequality) and independent of \( z \) as long as \( \Im(z) \leq R \). Since \( R \) is arbitrary and the function \( z \mapsto (\mathcal{F}f)(\lambda)e^{-t\lambda^2} e^{i\lambda z} \) is holomorphic on \( \mathbb{C}^n \), it follows by Lemma 2.1 that
\[
z \mapsto (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} (\mathcal{F}f)(\lambda)e^{-t\lambda^2} e^{i\lambda z} d\lambda
\]
is holomorphic on \( \mathbb{C}^n \). \(\square\)

For any function \( f \) on \( \mathbb{R}^n \), we define a new function \( Jf \) by \( (Jf)(x) = f(-x) \). It is clear that \( J : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \) is a unitary isomorphism. Let \( \mathcal{S}(\mathbb{R}^n) \) be the space of rapidly decreasing functions (the Schwartz space). That is
\[
\mathcal{S}(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} |x^\alpha (\partial^\beta f)(x)| < \infty \text{ for all } \alpha, \beta \in \mathbb{Z}^+ \right\}.
\]
Then \( \mathcal{S}(\mathbb{R}^n) \subset \{ f \in L^1(\mathbb{R}^n) \mid  \hat{f} \in L^1(\mathbb{R}^n) \} \). Therefore, by the Fourier inversion formula, \( J\hat{f} = (\hat{f})^\vee = f \) and hence \( \hat{f} = J^2 \hat{f} = Jf \) for all \( f \in \mathcal{S}(\mathbb{R}^n) \). We note that \( \mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) so we have \( \mathcal{F}^2 = J \) on \( \mathcal{S}(\mathbb{R}^n) \). Since \( J \) is unitary on \( L^2(\mathbb{R}^n) \) and \( \mathcal{S}(\mathbb{R}^n) \) is dense in \( L^2(\mathbb{R}^n) \), we also have \( \mathcal{F}^2 = J \) on \( L^2(\mathbb{R}^n) \).

**Lemma 2.16.** If \( f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \), then \( \mathcal{F}^{-1} f = f^\vee \).

**Proof.** Let \( f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \). Then we have \( Jf \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \). Therefore, \( \mathcal{F}(Jf) = \widehat{Jf} \). Moreover, by the above discussion \( \mathcal{F}^2(Jf) = J(\mathcal{F}f) = f \). Thus,
\[
f = \mathcal{F}^2(Jf) = \mathcal{F}(\mathcal{F}(Jf)) = \mathcal{F}(\widehat{Jf}).
\]
Hence, \( \mathcal{F}^{-1} f = \widehat{Jf} \). Since \( f^\vee = J\hat{f} = \widehat{Jf} \), \( \mathcal{F}^{-1} f = f^\vee \) as desired. \(\square\)
Lemma 2.17. Let $f \in L^2(\mathbb{R}^n)$. For each $y \in \mathbb{R}^n$, we define

$$(Hf)_y(x) = (H_t f)(x + iy)$$

for all $x \in \mathbb{R}^n$. Then

$$(H_t f)_y = \mathcal{F}^{-1} \left( \mathcal{F}(f) e^{-t \cdot \cdot^2} e^{-y \cdot \cdot} \right).$$

Proof. Fix $y \in \mathbb{R}^n$. Then as in the proof of Proposition 2.15 (see the Equations (2.10) and (2.11)), we have

$$|\mathcal{F}(f)(\lambda)| e^{-t \lambda^2 - \lambda \cdot y} \leq e^{\frac{|y|^2}{4t}} |\mathcal{F}(f)(\lambda)| e^{-\left(\sqrt{\lambda^2 - \frac{|y|^2}{2t}}\right)^2}$$

for all $\lambda \in \mathbb{R}^n$. Then $\mathcal{F}(f) e^{-t \cdot \cdot^2} e^{-y \cdot \cdot} \in L^2(\mathbb{R}^n)$ because $\mathcal{F} f \in L^2(\mathbb{R}^n)$ and $e^{-2\left(\sqrt{\lambda^2 - \frac{|y|^2}{2t}}\right)^2}$ is bounded on $\mathbb{R}^n$. Moreover, $\mathcal{F}(f) e^{-t \cdot \cdot^2} e^{-y \cdot \cdot} \in L^1(\mathbb{R}^n)$ by Hölder’s inequality. By the previous lemma, we get for $x \in \mathbb{R}^n$

$$\mathcal{F}^{-1} \left( \mathcal{F}(f) e^{-t \cdot \cdot^2} e^{-y \cdot \cdot} \right)(x) = \left( \mathcal{F}(f) e^{-t \cdot \cdot^2} e^{-y \cdot \cdot} \right)^\vee(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} (\mathcal{F} f)(\lambda) e^{-t\lambda^2} e^{-y \cdot \lambda} e^{i\lambda \cdot x} d\lambda.$$

Also, we know from Proposition 2.15 that

$$(H_t f)_y = (H_t f)(x + iy) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} (\mathcal{F} f)(\lambda) e^{-t\lambda^2} e^{i\lambda \cdot (x + iy)} d\lambda$$

and

$$(H_t f)_y = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} (\mathcal{F} f)(\lambda) e^{-t\lambda^2} e^{-y \cdot \lambda} e^{i\lambda \cdot x} d\lambda$$

for all $x \in \mathbb{R}^n$. Hence,

$$(H_t f)_y = \mathcal{F}^{-1} \left( \mathcal{F}(f) e^{-t \cdot \cdot^2} e^{-y \cdot \cdot} \right).$$

This completes the proof.

\[\square\]

Theorem 2.18. The Segal-Bargmann transform $H_t : L^2(\mathbb{R}^n) \longrightarrow \mathcal{H}_t(\mathbb{C}^n)$ is a unitary isomorphism.

Proof. Let $f \in \mathbb{R}^n$. We first show that $H_t$ is an isometry. Let

$$c = (2\pi t)^{-\frac{n}{2}} = \left( \int_{\mathbb{R}^n} e^{-\frac{y^2}{2t}} dy \right)^{-1}.$$
Then by the isometric property of $\mathcal{F}$, the previous lemma and the Tonelli’s theorem we have

$$\|H_tf\|_{H^t}^2 = c \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |(H_tf)(x + iy)|^2 \, dx \, e^{-\frac{y^2}{4t}} \, dy$$

$$= c \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |(H_tf)(x)|^2 \, dx \, e^{-\frac{y^2}{4t}} \, dy$$

$$= c \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\mathcal{F}((H_tf)(\lambda))|^2 \, d\lambda \, e^{-\frac{|\lambda|^2}{4t}} \, dy$$

$$= c \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |(\mathcal{F}f)(\lambda)|^2 e^{-\frac{2t\lambda^2}{4} - \frac{y^2}{4}} \, d\lambda \, dy$$

$$= \int_{\mathbb{R}^n} |(\mathcal{F}f)(\lambda)|^2 \left( c \int_{\mathbb{R}^n} e^{-\frac{(y + 2t\lambda)^2}{4t}} \, dy \right) \, d\lambda$$

$$= \int_{\mathbb{R}^n} |(\mathcal{F}f)(\lambda)|^2 \, d\lambda$$

$$= \int_{\mathbb{R}^n} |F(\lambda)|^2 \, d\lambda$$.
Theorem 2.19 (The Segal-Bargmann transform on $L^2(\mathbb{R}^n, h_t(x)dx)$). For $f \in L^2(\mathbb{R}^n, h_t(x)dx)$, define

$$(H_h,f)(z) = \tilde{f} * h_t = (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(y)e^{-(z-y)^2/4t}dy.$$ 

Then $H_{ht} : L^2(\mathbb{R}^n, h_t(x)dx) \rightarrow \mathcal{F}_t(\mathbb{C}^n)$ is a unitary isomorphism from $L^2(\mathbb{R}^n, h_t(x)dx)$ onto $\mathcal{F}_t(\mathbb{C}^n)$.

Proof. It is easy to see that the map $\Lambda : L^2(\mathbb{R}^n, h_t(x)dx) \rightarrow L^2(\mathbb{R}^n)$ defined by $\Lambda_t(f) = \sqrt{ht}f$ is a unitary isomorphism. Furthermore, by a direct computation, we have for each $f \in L^2(\mathbb{R}^n, h_t(x)dx)$,

$$\Psi_t^{-1}(H_{2t}(\Lambda_t f)) = H_{ht}(f).$$

That is $H_{ht} = \Psi_t^{-1} \circ H_{2t} \circ \Lambda$. Since $\Psi_t^{-1}$, $H_{2t}$ and $\Lambda_t$ are unitary isomorphisms, $H_{ht}$ is a unitary isomorphism.

We summarize the relation between all three versions of the Segal-Bargmann transform by the following commutative diagrams:

**Theorem 2.20** (The Bargmann transform). For $f \in L^2(\mathbb{R}^n)$, define

$$(B_t f)(z) = \frac{1}{(4\pi t)^{n/4}} \int_{\mathbb{R}^n} f(x)e^{-\frac{1}{4t}(x^2 - 4xz + 2z^2)}. $$

Then $B_t : L^2(\mathbb{R}^n) \rightarrow \mathcal{F}_t(\mathbb{C}^n)$ is a unitary isomorphism.

Proof. By the direct computation, for $f \in L^2(\mathbb{R}^n)$, $\Psi_t^{-1}(H_{2t}(f)) = B_t(f)$. Therefore, $B_t = \Psi_t^{-1} \circ H_{2t}$. Hence, $B_t$ is a unitary isomorphism.

We summarize the relation between all three versions of the Segal-Bargmann transform by the following commutative diagrams:
2.8 Hermite Polynomials, Hermite Functions and the Fourier Transforms on \( H_t(\mathbb{C}^n) \) and on \( \mathcal{F}_t(\mathbb{C}^n) \)

In this section, we will apply the unitarity of the Segal-Bargmann transforms \( H_{ht} \) and \( H_t \) to obtain nice bases for \( L^2(\mathbb{R}^n, h_t(x) \, dx) \) and \( L^2(\mathbb{R}^n) \). We have proven that \( \{\zeta_{\alpha, t}\}_{\alpha \in (\mathbb{Z}^+)^n} \) are orthonormal bases for \( \mathcal{F}_t(\mathbb{C}^n) \) and \( H_t(\mathbb{C}^n) \) respectively. Therefore, \( \{H_{ht}^{-1}(\zeta_{\alpha, t})\}_{\alpha \in (\mathbb{Z}^+)^n} \) and \( \{H_t^{-1}(\varphi_{\alpha, t})\}_{\alpha \in (\mathbb{Z}^+)^n} \) will be orthonormal bases for \( L^2(\mathbb{R}^n, h_t(x) \, dx) \) and \( L^2(\mathbb{R}^n) \) respectively. Now we define for each \( \alpha \in (\mathbb{Z}^+)^n \),

\[
p_{\alpha, t} := H_{ht}^{-1}(\zeta_{\alpha, t}) \quad \text{and} \quad q_{\alpha, t} := H_t^{-1}(\varphi_{\alpha, t}) .
\]

It turns out that \( p_{\alpha, t} \) and \( q_{\alpha, t} \) are the well-known Hermite polynomials and Hermite functions form a basis for \( L^2(\mathbb{R}^n) \), then we can use this fact to give another proof of surjectivity in the second part of the proof of Theorem 2.18 without using the explicit formula of the reproducing kernel of \( H_t(\mathbb{C}^n) \). But for our approach, we prefer to use the reproducing kernel of \( H_t(\mathbb{C}^n) \) to prove the unitarity of the Segal-Bargmann transform \( H_t \) and then obtain the Hermite functions as a basis for \( L^2(\mathbb{R}^n) \) which is a consequence of the unitarity of the Segal-Bargmann transform \( H_t \).

For each \( j = 1, 2, \ldots, n \), let

\[
D_{z_j} = \frac{\partial}{\partial z_j} : \mathcal{O}(\mathbb{C}^n) \longrightarrow \mathcal{O}(\mathbb{C}^n),
\]

\[
D_{x_j} = \frac{\partial}{\partial x_j} : C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)
\]

and define the multiplicative operators \( M_{x_j} \) and \( M_{z_j} \) by

\[
(M_{x_j}f)(x) = x_j f(x)
\]

and

\[
(M_{z_j}g)(z) = z_j g(z)
\]

for \( f \in C^\infty(\mathbb{R}^n) \), \( g \in \mathcal{O}(\mathbb{C}^n) \).

**Lemma 2.21.** The following diagrams commutes:

\[
\begin{array}{ccc}
L^2(\mathbb{R}^n, h_t(x) \, dx) & \xrightarrow{H_{ht}} & \mathcal{F}_t(\mathbb{C}^n) \\
\downarrow{D_{x_j}} & & \downarrow{D_{z_j}} \\
L^2(\mathbb{R}^n, h_t(x) \, dx) & \xrightarrow{H_{ht}} & \mathcal{F}_t(\mathbb{C}^n)
\end{array}
\]
and
\[
\begin{align*}
L^2(\mathbb{R}^n, h_t(x)dx) & \xrightarrow{H_{ht}} \mathcal{F}(\mathbb{C}^n) \\
M_{x_j}^{-2tD_{x_j}} & \xrightarrow{M_{x_j}} \\
L^2(\mathbb{R}^n, h_t(x)dx) & \xrightarrow{H_{ht}} \mathcal{F}(\mathbb{C}^n)
\end{align*}
\]

Here, \( j = 1, 2, \ldots, n \) and the vertical operators are densely defined.

**Proof.** Since the Schwartz space \( \mathcal{S}(\mathbb{R}^n) \) is dense in \( L^2(\mathbb{R}^n) \), \( \Lambda^{-1}_t(\mathcal{S}(\mathbb{R}^n)) \) is dense in \( L^2(\mathbb{R}^n, h_t(x)dx) \). We note that \( \Lambda^{-1}_t(\mathcal{S}(\mathbb{R}^n)) \subseteq \mathcal{S}(\mathbb{R}^n) \). Let \( f \in \Lambda^{-1}_t(\mathcal{S}(\mathbb{R}^n)) \). Then we can differentiate under the integral sign in the following computations:

\[
(D_{z_j}(H_{ht}f))(z) = (4\pi t)^{-\frac{n}{2}} D_{z_j} \int_{\mathbb{R}^n} f(x) e^{-(x-z)^2/4t} dx
\]

\[
= (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) D_{z_j} e^{-(x-z)^2/4t} dx
\]

\[
= - (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) D_{x_j} e^{-(x-z)^2/4t} dx
\]

\[
= (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} (D_{x_j} f(x)) (x) e^{-(x-z)^2/4t} dx
\]

\[
= (H_{ht}(D_{x_j} f))(z).
\]

Thus, \( D_{z} \circ H_{ht} = H_{ht} \circ D_{x_j} \) on \( \Lambda^{-1}_t(\mathcal{S}(\mathbb{R}^n)) \). For the second diagram, we have, as we reverse the last three equations above and evaluate the integral directly,

\[
(H_{ht}(D_{x_j} f))(z) = - (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) D_{x_j} e^{-(x-z)^2/4t} dx
\]

\[
= \frac{1}{2t (4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x) (x_j - z_j) e^{-(x-z)^2/4t} dx
\]

\[
= \frac{1}{2t} \left( (H_{ht}(M_{x_j} f))(z) - (M_{x_j}(H_{ht} f))(z) \right).
\]

This implies that

\[
M_{x_j}(H_{ht} f) = H_{ht}(M_{x_j} f - 2t D_{x_j} f).
\]

Hence, we have

\[
M_{z} \circ H_{ht} = H_{ht} \circ (M_{x_j} - 2t D_{x_j})
\]

on \( \Lambda^{-1}_t(\mathcal{S}(\mathbb{R}^n)) \).

**Theorem 2.22.** Let \( \alpha \in (\mathbb{Z}^+)^n \). Then we have

(1) \( p_{\alpha,t} \) is a polynomial of degree \( |\alpha| \) and \( p_{0,t} = 1 \),

\[
32
\]
(2) \( D_x p_{\alpha,t} = \sqrt{\frac{\alpha_j}{2t}} p_{\alpha-e_j,t} \) for \( \alpha \neq 0 \),

(3) \( p_{\alpha+e_j,t}(x) = \frac{1}{\sqrt{2t}} \frac{1}{\sqrt{\alpha_j+1}} (x_j - 2t D_j x_{j}) p_{\alpha,t}(x) \), and

(4) \( p_{\alpha,t}(x) = \frac{1}{(2t)^{|\alpha|/2} \sqrt{\alpha!}} \left( (-2t)^{|\alpha|} e^{x^2/4t} D_{x}^{\alpha} e^{-x^2/4t} \right) \).

Proof. By Lemma 2.10, \( H_{h_t} (1) (z) = 1 \) for all \( z \in \mathbb{C}^n \). Therefore, \( p_{0,t} = H_{h_t}^{-1}(\zeta_0,t) = H_{h_t}^{-1}(1) = 1 \). Then (1) follows from (2) by induction on \(|\alpha|\). Next, we prove (2).

We see that

\[
(D_{z_j} \zeta_{\alpha,t}) (z) = \alpha_j \frac{z^{\alpha-e_j}}{(2t)^{|\alpha|/2} \sqrt{\alpha!}} = \alpha_j \cdot \frac{(2t)^{|\alpha|-1}/2 \sqrt{(\alpha-e_j)!}}{(2t)^{|\alpha|/2} \sqrt{\alpha!}} \zeta_{\alpha-e_j,t} = \sqrt{\frac{\alpha_j}{2t}} \zeta_{\alpha-e_j,t}.
\]

Now, by the previous lemma, we get

\[
H_{h_t} (D_x p_{\alpha,t}) = D_{z_j} (H_{h_t} (p_{\alpha,t})) = D_{z_j} (\zeta_{\alpha,t}) = \sqrt{\frac{\alpha_j}{2t}} \zeta_{\alpha-e_j,t}
\]

and hence

\[
D_x p_{\alpha,t} = H_{h_t}^{-1} \left( \sqrt{\frac{\alpha_j}{2t}} \zeta_{\alpha-e_j,t} \right) = \sqrt{\frac{\alpha_j}{2t}} H_{h_t}^{-1} (\zeta_{\alpha-e_j,t}) = \sqrt{\frac{\alpha_j}{2t}} p_{\alpha-e_j,t}.
\]

This implies (2). Now, we show (3).

We note that

\[
M_{z_j} (\zeta_{\alpha,t}) (z) = z_j \frac{z^{\alpha}}{(2t)^{|\alpha|/2} \sqrt{\alpha!}} = \frac{(2t)^{|\alpha+e_j|/2} \sqrt{(\alpha+e_j)!}}{(2t)^{|\alpha|/2} \sqrt{\alpha!}} \frac{z^{\alpha+e_j}}{(2t)^{|\alpha+e_j|/2} \sqrt{(\alpha+e_j)!}} = \sqrt{2t} \sqrt{\alpha_j + 1} \zeta_{\alpha+e_j,t}.
\]

Again, by the previous lemma, we have \( M_{z_j} (\zeta_{\alpha,t}) = M_{z_j} (H_{h_t} (p_{\alpha,t})) = H_{h_t} \circ (M_{x_j} - 2t D_{x_j}) (p_{\alpha,t}) \), and thus

\[
(M_{x_j} - 2t D_{x_j}) p_{\alpha,t} = H_{h_t}^{-1} (M_{z_j} (\zeta_{\alpha,t})) = \sqrt{2t} \sqrt{\alpha_j + 1} H_{h_t}^{-1} (\zeta_{\alpha+e_j,t}) = \sqrt{2t} \sqrt{\alpha_j + 1} p_{\alpha+e_j,t}.
\]
This proves (3). Finally, to prove (4), we let
\[ \tilde{p}_{\alpha,t}(x) = \frac{1}{(2t)^{\alpha/2} \sqrt{\alpha!}} (-2t)^{\alpha} e^{x^2/4t} D_x^\alpha e^{-x^2/4t}. \]
Then \( \tilde{p}_{\alpha,t} = 1 \) and
\[
(D_x \tilde{p}_{\alpha,t})(x) = \left( \frac{x}{2t} \tilde{p}_{\alpha,t} + \frac{1}{(2t)^{\alpha/2} \sqrt{\alpha!}} (-2t)^{\alpha} e^{x^2/4t} D_x^{\alpha+1} e^{-x^2/4t} \right).
\]
\[
= \left( \frac{x}{2t} \tilde{p}_{\alpha,t} + \frac{\sqrt{2t} \sqrt{\alpha_j + 1}}{(-2t)^{\alpha+1} \sqrt{(\alpha + e_j)!}} \right) \cdot \frac{(-2t)^{\alpha+e_j} e^{x^2/4t} D_x^{\alpha+1} e^{-x^2/4t}}{(2t)^{\alpha+e_j/2} \sqrt{(\alpha + e_j)!}}
\]
\[
= \left( \frac{x}{2t} \tilde{p}_{\alpha,t} - \frac{\sqrt{2t} \sqrt{\alpha_j + 1}}{2t} \tilde{p}_{\alpha+e_j,t} \right).
\]
Hence, \( \tilde{p}_{\alpha,t} \) satisfies the recursion formula:
\[ \tilde{p}_{\alpha+e_j,t} = \frac{1}{\sqrt{2t} \sqrt{\alpha_j + 1}} \left( x_j - 2t D_{x_j} \right) (\tilde{p}_{\alpha,t}). \]

Since \( p_{0,t} = 1 = \tilde{p}_{0,t} \) and by (3), \( p_{0,t} \) and \( \tilde{p}_{0,t} \) satisfy the same recursion relation, \( p_{0,t} = \tilde{p}_{0,t} \) for all \( \alpha \in (\mathbb{Z}^+)^n \). This completes the proof. \( \square \)

**Corollary 2.23.** \( \{p_{0,t}\}_{\alpha \in (\mathbb{Z}^+)^n} \) and \( \{p_{0,t} \sqrt{h_t}\}_{\alpha \in (\mathbb{Z}^+)^n} \) are orthonormal bases for \( L^2(\mathbb{R}^n, h_t(x) \, dx) \) and \( L^2(\mathbb{R}^n) \) respectively. Moreover, we have
\[ q_{0,2t} = H_{2t}^{-1}(\varphi_{0,t}) = \Lambda_t(p_{0,t}) = p_{0,t} \sqrt{h_t} \quad \text{for all } \alpha \in (\mathbb{Z}^+)^n. \]

**Proof.** Since \( p_{0,t} = H_{2t}^{-1}(\varphi_{0,t}) \), \( H_{h_t} : L^2(\mathbb{R}^n, h_t(x) \, dx) \to \mathcal{F}_t(\mathbb{C}^n) \) is a unitary isomorphism and \( \{\zeta_{0,t}\}_{\alpha} \) is an orthormal basis for \( \mathcal{F}_t(\mathbb{C}^n) \), \( \{p_{0,t}\}_{\alpha} \) is an orthonormal basis for \( L^2(\mathbb{R}^n, h_t(x) \, dx) \). We know that \( \Lambda_t : L^2(\mathbb{R}^n, h_t(x) \, dx) \to L^2(\mathbb{R}^n) \) is a unitary isomorphism so \( \{\Lambda_t(p_{0,t}) = p_{0,t} \sqrt{h_t}\}_{\alpha} \) is an orthonormal basis for \( L^2(\mathbb{R}^n) \). Recall that \( \Psi_t(\zeta_{0,t}) = \varphi_{0,t} \) and \( H_{2t} \circ \Lambda_t = \Psi_t \circ h_t \). Then
\[ p_{0,t} \sqrt{h_t} = \Lambda_t(p_{0,t}) = H_{2t}^{-1}(H_{h_t}(p_{0,t})) = H_{2t}^{-1}(\Psi_t(\zeta_{0,t})) = H_{2t}^{-1}(\varphi_{0,t}). \] \( \square \)

We recall one of the definitions of the normalized Hermite polynomials:
\[ \frac{(-1)^{\alpha}}{\sqrt{\alpha!}} e^{x^2/2} D_x^\alpha e^{-x^2/2}. \]
We see that this is exactly $p_{\alpha,\frac{1}{2}}$. So we obtain $p_{\alpha,\frac{1}{2}}$ as the usual normalized Hermite polynomials. Also, $p_{\alpha,\frac{1}{2}} \sqrt{h_{\alpha}}$ are the usual Hermite functions for $L^2(\mathbb{R}^n)$.

Next, we will define the Fourier transform for the space $H_{\frac{1}{2}}(\mathbb{C}^n)$ by using the unitary map $H_{\frac{1}{2}}$. First, we prove the following Lemma.

**Lemma 2.24.** Let $\alpha \in (\mathbb{Z}^+)^n$. Then

$$\mathcal{F}\left(q_{\alpha,\frac{1}{2}}\right) = (-i)^{\left|\alpha\right|}q_{\alpha,\frac{1}{2}}.$$  

**Proof.** Suppose first that $\alpha = 0$. Then

$$q_{0,\frac{1}{2}} = p_{0,\frac{1}{4}} \sqrt{h_{\frac{1}{4}}} \pi^{-\frac{n}{4}} e^{-x^2/2}.$$  

By the properties of the Fourier transform we quoted in Section 2.2,

$$\mathcal{F}\left(q_{0,\frac{1}{2}}\right) = \mathcal{F}\left(\pi^{-\frac{n}{4}} e^{-x^2/2}\right) = \pi^{-\frac{n}{4}} e^{-x^2/2} = q_{0,\frac{1}{2}}.$$  

Next, we do the induction on $|\alpha|$. Assume the claim holds for $\alpha$. We show it holds for $\alpha + e_j$. First, we note that

$$\left(\frac{1}{2} x_j - \frac{1}{2} D_{x_j}\right) p_{0,\frac{1}{4}}(x) \sqrt{h_{\frac{1}{4}}(x)}$$

$$= \frac{1}{2} x_j p_{0,\frac{1}{4}} \pi^{-\frac{n}{4}} e^{-x^2/2} - \frac{1}{2} D_{x_j} p_{0,\frac{1}{4}} \pi^{-\frac{n}{4}} e^{-x^2/2}$$

$$= \frac{1}{2} x_j p_{0,\frac{1}{4}} \pi^{-\frac{n}{4}} e^{-x^2/2} - \frac{1}{2} \pi^{-\frac{n}{4}} e^{-x^2/2} D_{x_j} p_{0,\frac{1}{4}} + \frac{1}{2} x_j \pi^{-\frac{n}{4}} e^{-x^2/2} p_{0,\frac{1}{4}}$$

$$= \pi^{-\frac{n}{2}} e^{-x^2/2}\left(x_j p_{0,\frac{1}{4}}(x) - \frac{1}{2} D_{x_j} p_{0,\frac{1}{4}}(x)\right)$$

$$= \frac{\sqrt{\alpha_j + 1}}{\sqrt{2}} p_{\alpha + e_j,\frac{1}{4}}(x) \sqrt{h_{\frac{1}{4}}(x)}$$

where the last equation holds by (3) of Theorem 2.22. Therefore,

$$\sqrt{\frac{2}{\sqrt{\alpha_j + 1}}} \left(\frac{1}{2} x_j - \frac{1}{2} D_{x_j}\right) q_{\alpha,\frac{1}{2}}(x) = q_{\alpha + e_j,\frac{1}{2}}. \quad (2.13)$$

We recall some properties of the Fourier transform: for any $f \in \mathcal{S}(\mathbb{R}^n)$,

$$(\hat{D^\alpha f})(\lambda) = i^{\left|\alpha\right|} \lambda^\alpha \hat{f}(\lambda) \quad \text{and} \quad (x^\alpha \hat{f})(\lambda) = i^{\left|\alpha\right|} D^\alpha \hat{f}(\lambda).$$
Now by (2.13), the above properties, induction hypothesis, and (2.13) again, we have

\[ \mathcal{F}(q_{\alpha + e_j, \frac{1}{2}}) = \frac{\sqrt{2}}{\sqrt{\alpha_j + 1}} \mathcal{F}\left( \left( \frac{1}{2}x_j - \frac{1}{2}Dx_j \right) q_{\alpha, \frac{1}{2}}(x) \right) \]

\[ = \frac{\sqrt{2}}{\sqrt{\alpha_j + 1}} (-i) \left( \frac{1}{2}x_j - \frac{1}{2}Dx_j \right) \mathcal{F}(q_{\alpha, \frac{1}{2}}(x)) \]

\[ = \frac{\sqrt{2}}{\sqrt{\alpha_j + 1}} (-i)^{|\alpha|} \sqrt{\alpha_j + 1} q_{\alpha, \frac{1}{2}}(x) \]

\[ = (-i)^{|\alpha| + 1} \frac{1}{\sqrt{2}} \cdot q_{\alpha + e_j, \frac{1}{2}}(x). \]

This completes the proof. \(\square\)

**Theorem 2.25.** Define the map \( G : \mathcal{H}_{\frac{1}{2}}(\mathbb{C}^n) \rightarrow \mathcal{H}_{\frac{1}{2}}(\mathbb{C}^n) \) by

\[ G = H_t \circ \mathcal{F} \circ H_t^{-1}. \]

That is, \( G \) satisfies the commutative diagram

\[ L^2(\mathbb{R}^n) \xrightarrow{\mathcal{F}} L^2(\mathbb{R}^n) \]

\[ \begin{array}{ccc}
L^2(\mathbb{R}^n) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{R}^n) \\
\mathcal{H}_{\frac{1}{2}} & \xrightarrow{H_t} & \mathcal{H}_{\frac{1}{2}} \\
\mathcal{H}_{\frac{1}{2}}(\mathbb{C}^n) & \xrightarrow{G} & \mathcal{H}_{\frac{1}{2}}(\mathbb{C}^n) \\
\end{array} \]

We call \( G \), the **Fourier transform** for \( \mathcal{H}_{\frac{1}{2}}(\mathbb{C}^n) \). Then \( G \) is a unitary isomorphism and

\[ G(\varphi_{\alpha, \frac{1}{2}}) = (-i)^{|\alpha|} \varphi_{\alpha, \frac{1}{2}} \text{ for all } \alpha \in (\mathbb{Z}^+)^n. \]

**Proof.** Clearly, \( G \) is a unitary isomorphism. Moreover, by the previous lemma,

\[ G(\varphi_{\alpha, \frac{1}{2}}) = H_t \left( \mathcal{F}(H_t^{-1}(\varphi_{\alpha, \frac{1}{2}})) \right) \]

\[ = H_t \left( \mathcal{F}(q_{\alpha, \frac{1}{2}}) \right) \]

\[ = H_t \left( (-i)^{|\alpha|} q_{\alpha, \frac{1}{2}} \right) \]

\[ = (-i)^{|\alpha|} \varphi_{\alpha, \frac{1}{2}}. \]

\(\square\)
Theorem 2.26 (The Fourier transform on $\mathcal{F}_\frac{1}{4}(\mathbb{C}^n)$). Define the **Fourier transform** on $\mathcal{F}_\frac{1}{4}(\mathbb{C}^n)$ by the formula

$$\tilde{G} = B_{\frac{1}{4}} \circ \mathcal{F} \circ B_{\frac{1}{4}}^{-1}.$$ 

That is, $\tilde{G}$ satisfies the commutative diagram:

![Commutative Diagram](image)

Then $\tilde{G}$ is a unitary isomorphism and

$$\left(\tilde{G}F\right)(z) = F(-iz) \text{ for all } F \in \mathcal{F}_\frac{1}{4}(\mathbb{C}^n) \text{ and } z \in \mathbb{C}^n.$$ 

**Proof.** Clearly, $\tilde{G}$ is a unitary isomorphism. To show that $\left(\tilde{G}F\right)(z) = F(-iz)$ for all $F \in \mathcal{F}_\frac{1}{4}(\mathbb{C}^n)$ and $z \in \mathbb{C}^n$, it is enough to show that the formula holds for all $\{\zeta_{\alpha,\frac{1}{4}}\}_{\alpha \in (\mathbb{Z})^n}$ since the mapping $F(z) \mapsto F(-iz)$ is bounded linear on $\mathcal{F}_\frac{1}{4}(\mathbb{C}^n)$ and $\{\zeta_{\alpha,\frac{1}{4}}\}_{\alpha \in (\mathbb{Z})^n}$ is an orthonormal basis (and hence is dense) for $\mathcal{F}_\frac{1}{4}(\mathbb{C}^n)$. Let $\alpha \in (\mathbb{Z})^n$. Then

$$\tilde{G}\left(\zeta_{\alpha,\frac{1}{4}}\right) = B_{\frac{1}{4}} \left( \mathcal{F} \left( \Lambda_{\frac{1}{4}} \left( H_{\frac{1}{4}}^{-1} \left( \zeta_{\alpha,\frac{1}{4}} \right) \right) \right) \right)$$

$$= B_{\frac{1}{4}} \left( \mathcal{F} \left( r_{\alpha,\frac{1}{4}} \sqrt{h_{\frac{1}{4}}} \right) \right)$$

$$= B_{\frac{1}{4}} \left( \mathcal{F} \left( q_{\alpha,\frac{1}{2}} \right) \right)$$

$$= (-i)^{|\alpha|} B_{\frac{1}{4}} \left( q_{\alpha,\frac{1}{2}} \right)$$

$$= (-i)^{|\alpha|} H_{ht} \left( \Lambda_{\frac{1}{4}}^{-1} \left( q_{\alpha,\frac{1}{2}} \right) \right)$$

$$= (-i)^{|\alpha|} H_{ht} \left( p_{\alpha,\frac{1}{4}} \right)$$

$$= (-i)^{|\alpha|} \zeta_{\alpha,\frac{1}{4}}.$$
Hence, for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$,

\[
\tilde{\zeta} \left( \zeta_{\alpha, \frac{1}{\pi}} \right) (z) = (-i)^{|\alpha|} \zeta_{\alpha, \frac{1}{\pi}} (z)
\]

\[
= \frac{(-i)^{\alpha_1 + \alpha_2 + \ldots + \alpha_n} \cdot z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}}{(\frac{1}{2})^{|\alpha|/2} \sqrt{\alpha!}}
\]

\[
= \frac{(-iz)^\alpha}{(\frac{1}{2})^{|\alpha|/2} \sqrt{\alpha!}}
\]

\[
= \zeta_{\alpha, \frac{1}{\pi}} (-iz).
\]

\[\square\]
Chapter 3
The Restriction Principle

3.1 Some Facts from Functional Analysis

In this section, $\mathcal{H}$ and $\mathcal{K}$ will always be complex Hilbert spaces. A linear operator $T$ from $\mathcal{H}$ into $\mathcal{K}$ is, by definition, a linear mapping of a subspace $\mathcal{D}(T)$ of $\mathcal{H}$ into $\mathcal{K}$. The subspace $\mathcal{D}(T)$ is called the domain of $T$. We use the notation $T : \mathcal{D}(T) \subseteq \mathcal{H} \longrightarrow \mathcal{K}$ for a mapping $T$ from $\mathcal{H}$ into $\mathcal{K}$. The image $\mathcal{R}(T) = \{T(x) : x \in \mathcal{D}(T)\}$ of $T$ is called the range of $T$. The subset $\mathcal{N}(T) = \{x \in \mathcal{D}(T) : T(x) = 0\}$ is called the kernel of $T$. The set of those bounded operators from $\mathcal{H}$ into $\mathcal{K}$, whose domain is $\mathcal{H}$, will be denoted by $\mathcal{B}(\mathcal{H}, \mathcal{K})$. In the case when $\mathcal{H} = \mathcal{K}$, we denote $\mathcal{B}(\mathcal{H}, \mathcal{K})$ simply by $\mathcal{B}(\mathcal{H})$.

**Proposition 3.1.** Let $T : \mathcal{D}(T) \subseteq \mathcal{H} \longrightarrow \mathcal{K}$ be a densely defined linear operator. Then, we have

1. $T^*$ is closed;
2. $T$ is bounded iff $T^* \in \mathcal{B}(\mathcal{K}, \mathcal{H})$;
3. if $T^*$ is also densely defined, then $T^{**}$ is an extension of $T$;
4. $T^*$ is densely defined iff $T$ is closable; we then have $\overline{T} = T^{**}$; in particular, if $T$ is closed, then $T = T^{**}$;
5. $\mathcal{N}(T^*) = (\mathcal{R}(T))^\perp$;
6. if $\mathcal{R}(T)$ is dense, then $T^*$ is also injective, and we have $(T^*)^{-1} = (T^{-1})^*$.

**Proposition 3.2.** Let $T : \mathcal{D}(T) \subseteq \mathcal{H} \longrightarrow \mathcal{K}$ be a linear operator. Then the following statements are equivalent:

1. $T$ is closed and $\mathcal{D}(T)$ is closed;
2. $T$ is bounded and $\mathcal{D}(T)$ is closed;
3. $T$ is bounded and closed.

**Proposition 3.3.** Let $T : \mathcal{D}(T) \subseteq \mathcal{H} \longrightarrow \mathcal{K}$ be a densely defined closed operator. If $T$ is bounded, then $T$ is defined everywhere (i.e., $\mathcal{D}(T) = \mathcal{H}$) and hence $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$.

If $A \in \mathcal{B}(\mathcal{H})$ is a positive operator, i.e., $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$, then there exists a unique positive operator $B \in \mathcal{B}(\mathcal{H})$ with $B^2 = A$. In this case, we call the operator $S$, the positive square root of $A$ and denote the operator $B$ by $\sqrt{A}$. The proof of this fact can be found in [50]. We note that if $T \in \mathcal{B}(\mathcal{H})$, then $T^*T \in \mathcal{B}(\mathcal{H})$ is positive since $\langle T^*Tx, x \rangle = \langle Tx, Tx \rangle = \|Tx\|^2 \geq 0$ for all $x \in \mathcal{H}$ and hence $\sqrt{T^*T}$...
exists as the positive square root of $T^* T$. Finally, we note that if $T \in \mathcal{B}(\mathcal{H})$ is positive, then $T^* = T$. Again we refer to [50] for the proof of this statement. Next, we state the polar decomposition theorem the version we will use and its proof.

**Theorem 3.4 (Polar Decomposition).** Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Suppose that $T$ is injective and has dense range. Then

$$T = U \sqrt{T^* T}$$

where $U : \mathcal{H} \longrightarrow \mathcal{K}$ is a unitary isomorphism.

**Proof.** By the above discussion, $T^* T$ has a unique positive square root, $\sqrt{T^* T} \in \mathcal{B}(\mathcal{H})$. Let $P = \sqrt{T^* T}$. Then $P$ is positive and hence $P^* = P$. We first show that $\mathcal{N}(T) = \mathcal{N}(P)$. Observe that, for any $x \in \mathcal{H}$, we have

$$|Tx|^2 = \langle Tx, Tx \rangle = \langle T^* Tx, x \rangle = \langle Px, Px \rangle = |Px|^2,$$

whence it follows that $\mathcal{N}(T) = \mathcal{N}(P)$. But $T$ is injective, $\mathcal{N}(P) = \mathcal{N}(T) = \{0\}$. Therefore,

$$\mathcal{H} = \{0\}^\perp = \mathcal{N}(P)^\perp = \overline{\mathcal{R}(P^*)} = \overline{\mathcal{P}(\mathcal{H})} = \mathcal{P}(\mathcal{H}).$$

Now we define $U_0 : P(\mathcal{H}) \longrightarrow \mathcal{K}$ by setting $U_0(Px) = Tx$ for $x \in \mathcal{H}$. This is well defined because if $Px = Py$, then

$$T^* T(x - y) = P^2(x - y) = P^2 x - P^2 y = 0,$$

and we have

$$|Tx - Ty|^2 = \langle T(x - y), T(x - y) \rangle = \langle T^* T(x - y), x - y \rangle = 0,$$

which implies that $U_0(Px) = Tx = Ty = U_0(Py)$. So $U_0$ is well defined on $P(\mathcal{H})$. Furthermore, for $x \in \mathcal{H}$,

$$\langle U_0(Px), U_0(Px) \rangle = \langle Tx, Tx \rangle = \langle T^* Tx, x \rangle = \langle P^2 x, x \rangle = \langle Px, Px \rangle.$$

Thus $U_0$ is an isometry of $P(\mathcal{H})$ onto $\overline{\mathcal{R}(T)} = T(\mathcal{H})$. By the remark after Theorem 2.13, $U_0$ thus has a unique linear extension to an isometry $U$ of $\overline{P(\mathcal{H})} = \mathcal{H}$ onto $T(\mathcal{H})$. Since $T$ has dense range by the assumption, $T(\mathcal{H}) = \mathcal{K}$. Hence, $U : \mathcal{H} \longrightarrow \mathcal{K}$ is a unitary isomorphism. Finally, $U(Px) = U_0(Px) = Tx$ for all $x \in \mathcal{H}$. Hence, $T = U \sqrt{T^* T}$. This completes the proof. 

### 3.2 The Restriction Principle

In this section, we state and prove the restriction principle the version we will apply to prove the unitarity of the Segal-Bargmann transform $H_t$. 

---

40
Theorem 3.5 (The Restriction Principle). Let $M_C$ be a connected complex manifold and let $M \subseteq M_C$ be a totally real submanifold. Suppose that $M$ is $\sigma$-compact and there is a Radon measure $\mu$ on $M$. Let $\mathcal{H}$ be a Hilbert space of holomorphic functions on $M_C$ with reproducing kernel $K(z,w)$. Assume that there is a holomorphic function $\phi : M_C \to \mathbb{C}$ such that $\phi(z) \neq 0$ for all $z \in M_C$ and that the restriction map $R(F) = (\phi \cdot F)|_M$ is a densely defined linear operator from $\mathcal{H}$ into $L^2(M, d\mu)$. That is $R : \mathcal{D}(R) \subseteq \mathcal{H} \to L^2(M, d\mu)$ and $\mathcal{D}(R)$ is dense in $\mathcal{H}$. We note that in general $R : \mathcal{H} \to C^\infty(M)$. However, we will consider $R$ as $R : \mathcal{D}(R) \subseteq \mathcal{H} \to L^2(M, d\mu)$. Then we have

(1) the linear span of $\{K_x | x \in M\}$ is dense in $\mathcal{H}$;

(2) $R$ is linear, injective and closed (thus $R^*$ is densely defined from $L^2(M, d\mu)$ into $\mathcal{H}$);

(3) if, in addition, $R$ has dense range, $\mathcal{R}(R^*) \subseteq \mathcal{D}(R)$ and $RR^*$ is bounded on $\mathcal{D}(R^*)$, then

(i) $R$ and $R^*$ are, in fact, everywhere defined and continuous, i.e. $R \in \mathcal{B}(\mathcal{H}, L^2(M, d\mu))$ and $R^* \in \mathcal{B}(L^2(M, d\mu), \mathcal{H})$;

(ii) for $f \in L^2(M, d\mu)$ and $x \in M$, we have

$$(RR^* f)(x) = \int_M f(y)\phi(x)\overline{\phi(y)}K(y, x)d\mu(y);$$

(iii) there exists a unitary isomorphism $U : L^2(M, d\mu) \to \mathcal{H}$ such that for $f \in L^2(M, d\mu)$ and $z \in M_C$,

$$(Uf)(z) = ((1/\phi)|_M \cdot \sqrt{RR^* f})^\sim(z)$$

where $((1/\phi)|_M \cdot (\sqrt{RR^* f})^\sim$ is the analytic continuation of $(1/\phi)|_M \cdot (\sqrt{RR^* f})$ to $M_C$.

Proof. (1) To show that the linear span of $\{K_x | x \in M\}$, span$\{K_x | x \in M\}$, is dense in $\mathcal{H}$, it suffices to show that $(\text{span} \{K_x | x \in M\})^\perp = \{0\}$. Suppose that $F \in (\text{span} \{K_x | x \in M\})^\perp$. Then

$$F(x) = \langle F, K_x \rangle = 0,$$

for all $x \in M$. Thus, $F|_M = 0$. Since $M$ is a totally real submanifold of $M_C$, $F = 0$ on $M_C$.

(2) Clearly, $R$ is linear. Since $M$ is a totally real submanifold of $M_C$ and $\phi(x) \neq 0$ for all $x \in M$, $R$ is injective. To show that $R$ is closed, let $(F_n)$ be a sequence in $\mathcal{D}(R)$ such that $(F_n, R(F_n))$ converges to $(F, f)$ in $\mathcal{H} \times L^2(M, d\mu)$. We want to show that $(F, f) \in \mathcal{G}(R)$, that is we have to show that $f = (\phi F)|_M = R(F)$ in $L^2(M, d\mu)$. Since $F_n \to F$ uniformly on compact subsets of $D$. Moreover, as $\phi$ is bounded on compact subsets of $M_C$, $\phi \cdot F_n \to \phi \cdot F$ uniformly on compact subsets of $M_C$. In particular, $\langle (\phi \cdot F_n)|_M \rangle \to \langle (\phi F)|_M \rangle$ uniformly on compact subsets of $M$. Now, since $M$ is $\sigma$-compact, we can write $M = \bigcup_{j=1}^\infty K_j$ where each $K_j$ is a compact subset of $M$. Then by the previous argument, for each $j$, $(\phi F_n)|_{K_j} \to (\phi F)|_{K_j}$ uniformly on $K_j$. Then
which implies that \((\phi F_n)|_{K_j} \to (\phi F)|_{K_j}\) in \(L^2(K_j, d\mu)\) because \(K_j\) is compact.
Furthermore, \(\phi F_n = R(F_n) \to f\) in \(L^2(M, d\mu)\) implies \((\phi F_n)|_{K_j} \to f|_{K_j}\) in \(L^2(K_j, d\mu)\) for every \(j\). Therefore, for each \(j\), \((\phi F)|_{K_j} = f|_{K_j}\) almost everywhere. Suppose that for each, \((\phi F)_{K_j\setminus N_j} = f|_{K_j\setminus N_j}\) where \(\mu(N_j) = 0\). Then \(\phi(x)F(x) = f(x)\) for all \(x \in X\setminus(\bigcup_{j=1}^{\infty} N_j)\). As \(\mu(\bigcup_{j=1}^{\infty} N_j) = 0\), it follows that \(\phi F = f\) almost everywhere on \(M\). Therefore, \(f = (\phi F)|_M = R(F)\) in \(L^2(M, d\mu)\). Thus, \((F, f) \in \mathcal{G}(R)\).
Hence, the graph \(\mathcal{G}(R)\) is closed in \(\mathcal{H} \times L^2(M, d\mu)\). So \(R\) is closed.

(3) Suppose that \(R\) has dense range, \(\mathcal{R}(R^*) \subseteq \mathcal{D}(R)\) and \(RR^*\) is bounded on \(\mathcal{D}(R^*)\). We first show that \(R^*\) is bounded in \(\mathcal{D}(R^*)\). For each \(f \in \mathcal{D}(R^*) \subseteq L^2(M, d\mu)\),
\[
\|R^* f\|^2 = \langle R^* f, R^* f \rangle = \langle RR^* f, f \rangle \leq \|RR^*\| \|f\| \leq \|RR^*\| \|f\|^2.
\]
Then \(R^*\) is bounded. By part (1), \(R\) is closed. Thus \(R^*\) is densely defined by Proposition 3.1(4).
Furthermore, \(R^*\) is also closed by Proposition 3.1(1). Therefore, \(\mathcal{D}(R^*)\) is in fact the full space \(L^2(M, d\mu)\). That is, \(R^* \in \mathcal{B}(L^2(M, d\mu), \mathcal{H})\). So for any \(F \in \mathcal{D}(R)\), we have
\[
\|RF\|^2 = \langle RF, RF \rangle_2 = \langle R^*(RF), F \rangle_\mathcal{H} \leq \|R^*\| \|RF\|_2 \|F\|_\mathcal{H}.
\]
This implies that \(R\) is bounded. Again, being a closed, densely-defined bounded operator, \(R\) is, in fact, defined on all of \(\mathcal{H}\). Hence, \(R \in \mathcal{B}(\mathcal{H}, L^2(M, d\mu))\). So we have shown (i).

To prove (ii), let \(f \in L^2(M, d\mu)\) and \(x \in M\). Then for \(z \in M_C\),
\[
(R^* f)(z) = \langle R^* f, K_z \rangle_\mathcal{H} = \langle f, RK_z \rangle_2 = \int_M f(y)\overline{\phi(y)}K(z, y)dy
\]
as \(\overline{K(y, z)} = K(z, y)\). It follows that
\[
(RR^* f)(x) = \phi(x) \int_M f(y)\overline{\phi(y)}K(x, y)dy = \int_M f(y)\phi(x)\overline{\phi(y)}K(x, y)dy.
\]

Next, we prove (iii). Since \(R^* \in \mathcal{B}(L^2(M, d\mu), \mathcal{H})\), \(R^*\) is injective (by Proposition 3.1(6)), and \(R^*\) has dense range (because \(R\) is injective), by Theorem 3.4, we have a polar decomposition
\[
R^* = U \sqrt{(R^*)^*R^*} = U \sqrt{RR^*},
\]
where \(U : L^2(M, d\mu) \to \mathcal{H}\) is a unitary isomorphism. Note that
\[
R^* = (R^*)^* = (U \sqrt{RR^*})^* = (\sqrt{RR^*})^*U^* = \sqrt{RR^*}U^{-1}
\]
which implies that $RU = \sqrt{RR^*}$. Let $f \in L^2(M, d\mu)$. Then for any $x \in M$,
\[
\phi(x)(Uf)(x) = (R(Uf))(x) = (\sqrt{RR^*}f)(x)
\]
and hence,
\[
(Uf)(x) = \frac{1}{\phi(x)} \cdot (\sqrt{RR^*}f)(x).
\]
Since $Uf$ is a holomorphic function on $M_C$ and $M$ is a totally real submanifold of $M_C$, $Uf$ is the analytic continuation of $(1/\phi)|_M \cdot (\sqrt{RR^*}f)$. This completes the proof.

3.3 Another Proof by the Restriction Principle

We now use the restriction principle to give another proof of the unitarity of the map $H_t$, provided that we have already known that the reproducing kernel for $H_t(C^n)$ is given by $K(z, w) = h_{2t}(z - \overline{w})$ (Corollary 2.9). Before using the restriction principle, we prove the following results.

**Theorem 3.6.** We define the heat transform $u_t$ on $L^2(\mathbb{R}^n)$ by setting
\[
u_t(f) = f \ast h_t = h_t \ast f,
\]
for $f \in L^2(\mathbb{R}^n)$. Then $u_t : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is bounded linear, injective, self adjoint and has dense range. Moreover, $\{h_t\}_{t > 0}$ is a convolution semigroup, i.e., $h_s \ast h_t = h_{s+t}$ for all $s, t > 0$ which implies that $u_{s+t} = u_s u_t$ for all $s, t > 0$.

**Proof.** By Young’s inequality, for any $f \in L^2(\mathbb{R}^n)$,
\[
\|u_t(f)\|_2 = \|h_t \ast f\|_2 \leq \|h_t\|_1 \|f\|_2 < \infty.
\]
Therefore, $u_t : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is bounded. Clearly, $u_t$ is linear. With the same arguments as in the discussion after the proof of Corollary 2.9, we can show that $u_t$ is injective.

To show that $u_t$ is self adjoint, we let $f, g \in L^2(\mathbb{R}^n)$. Since by the Tonell’s theorem and the Hölder inequality,
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(y)h_t(x - y)\overline{g(x)}| \, dy \, dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x - y)h_t(y)\overline{g(x)}| \, dy \, dx
\]
\[
= \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |f(x - y)\overline{g(x)}| \, dx \right) h_t(y) \, dy
\]
\[
\leq \|f\|_2 \|\overline{g}\|_2 \int_{\mathbb{R}^n} h_t(y) \, dy
\]
\[
< \infty,
\]

43
we can apply the Fubini’s theorem in the following and use the fact that \( h_t \) is real and even to get

\[
\langle u_t f, g \rangle_2 = \int_{\mathbb{R}^n} (f * h_t)(x) \overline{g(x)} \, dx \\
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(y) h_t(x-y) \overline{g(x)} \, dy \, dx \\
= \int_{\mathbb{R}^n} f(y) \int_{\mathbb{R}^n} h_t(y-x) \overline{g(x)} \, dx \, dy \\
= \langle f, u_t g \rangle_2.
\]

Next, we prove that \( u_t \) has dense range. Assume that \( g \in L^2(\mathbb{R}^n) \) is perpendicular to the range of \( u_t \). Note that \( \mathcal{F}(f)e^{-t(\cdot)^2} \in L^2(\mathbb{R}^n) \) since \( e^{-t(\cdot)^2} \) is bounded on \( \mathbb{R}^n \). Also, \( \mathcal{F}(f)e^{-t(\cdot)^2} \in L^1(\mathbb{R}^n) \) by Hölder’s inequality. Then for all \( f \in L^2(\mathbb{R}^n) \), we have

\[
0 = \langle u_t(f), g \rangle_2 \\
= \langle f * h_t, g \rangle_2 \\
= \langle (2\pi)^{n/2} \mathcal{F}(f) \mathcal{F}(h_t)^\vee, g \rangle_2 \\
= \langle (\mathcal{F}(f)e^{-t(\cdot)^2})^\vee, g \rangle_2 \\
= \langle \mathcal{F}^{-1}(\mathcal{F}(f)e^{-t(\cdot)^2}), g \rangle_2 \quad \text{by Lemma 2.16} \\
= \langle \mathcal{F}(f)e^{-t(\cdot)^2}, \mathcal{F}(g) \rangle_2.
\]

Taking \( f = g \), we then get

\[
0 = \int_{\mathbb{R}^n} e^{-t\lambda^2} |\mathcal{F}(g)(\lambda)|^2 d\lambda = \int_{\mathbb{R}^n} |e^{-t\lambda^2/2} \mathcal{F}(g)(\lambda)|^2 d\lambda = \|e^{-t(\cdot)^2/2} \mathcal{F}(g)\|^2.
\]

It follows that \( e^{-t\lambda^2/2} \mathcal{F}(g)(\lambda) = 0 \) a.e. Thus, \( \mathcal{F}(g) = 0 \) in \( L^2(\mathbb{R}^n) \) and so \( g = 0 \) in \( L^2(\mathbb{R}^n) \). Hence, we have proved that \( (\mathcal{R}(u_t))^\perp = \{0\} \). Therefore, \( u_t \) has dense range.

Finally, we prove that \( \{h_t\}_{t>0} \) is a convolution semigroup. Let \( s, t > 0 \). Then we have

\[
\hat{h_s * h_t}(\lambda) = (2\pi)^{n/2} \hat{h_s}(\lambda) \hat{h_t}(\lambda) \\
= (2\pi)^{n/2} (2\pi)^{-n/2} e^{-s\lambda^2/2} (2\pi)^{-n/2} e^{-t\lambda^2} \\
= (2\pi)^{-n/2} e^{-(s+t)\lambda^2} \\
= \hat{h_{s+t}}(\lambda).
\]

This implies that \( h_s * h_t = h_{s+t} \) and hence, for \( f \in L^2(\mathbb{R}^n) \),

\[
u_{s+t} f = u_{t+s} f = f * h_{t+s} = f * (h_t * h_s) = (f * h_t) * h_s = u_s(u_t f).
\]

Therefore, \( u_{s+t} = u_s u_t \). \( \square \)
Corollary 3.7. The linear span of \( \{ L_x(h_t)|x \in \mathbb{R}^n \} \) is dense in \( L^2(\mathbb{R}^n) \). Here we define \( (L_x(f))(y) := f(y - x) \).

Proof. We first note that \( \langle f, L_x(h_t) \rangle = (f * h_t)(x) \) as \( h_t \) is real and even. Therefore, if \( L^2(\mathbb{R}^n) \ni f \perp \operatorname{span}\{L_x(h_t)|x \in \mathbb{R}^n\} \), then \( 0 = \langle f, L_x(h_t) \rangle = (f * h_t)(x) \) for all \( x \in \mathbb{R}^n \) and thus \( u_t(f) = f * h_t = 0 \) which implies that \( f = 0 \) in \( L^2(\mathbb{R}^n) \) by the preceding lemma.

Now we are ready to give another proof of unitarity of \( H_t \) by using the restriction principle.

Theorem 3.8. The Segal-Bargmann transform \( H_t : L^2(\mathbb{R}^n) \rightarrow \mathcal{H}_t(\mathbb{C}^n) \) is a unitary isomorphism.

Proof. Clearly, \( \mathbb{R}^n \) is a totally real submanifold of \( \mathbb{C}^n \). Recall that the reproducing kernel of \( \mathcal{H}_t(\mathbb{C}^n) \) is \( K(z,w) = \tilde{h}_{2t}(z - \overline{w}) = L_{\overline{w}}h_{2t}(z) \). Define \( R : \mathcal{H}_t(\mathbb{C}^n) \rightarrow C^\infty(\mathbb{R}^n) \) by

\[
R(F) = F|_{\mathbb{R}^n}.
\]

We see that, for any \( y \in \mathbb{R}^n \),

\[
R(K_y) = L_yh_{2t} \in L^2(\mathbb{R}^n).
\]

By Theorem 3.5(1), the linear span of \( \{K_y|y \in \mathbb{R}^n\} \) is dense in \( \mathcal{H}_t(\mathbb{C}^n) \). Therefore, we may and will consider \( R \) as a densely defined linear operator from \( \mathcal{H}_t(\mathbb{C}^n) \) into \( L^2(\mathbb{R}^n) \). Moreover, by Corollary 3.7, \( R \) has dense range. By Theorem 3.5(2), \( R \) is injective and closed and hence \( R \) has a densely-defined adjoint

\[
R^* : \mathcal{D}(R^*) \subseteq L^2(\mathbb{R}^n) \rightarrow \mathcal{H}_t(\mathbb{C}^n).
\]

For \( f \in \mathcal{D}(R^*) \), we have, for \( z \in \mathbb{C}^n \),

\[
(R^*f)(z) = \langle R^*f, K_z \rangle_{\mathcal{H}_t} = \langle f, RK_z \rangle_{\mathcal{H}_t} = \int_{\mathbb{R}^n} f(y)\tilde{h}_{2t}(y - \overline{z})dy = \int_{\mathbb{R}^n} f(y)\tilde{h}_{2t}(z - y)dy = H_{2t}f(z).
\]

Applying the map \( R : \mathcal{H}_t(\mathbb{C}^n) \rightarrow C^\infty(\mathbb{R}^n) \), we have

\[
RR^*f = f * h_{2t} = u_{2t}f \in L^2(\mathbb{R}^n),
\]

for every \( f \in \mathcal{D}(R^*) \). It follows that \( \mathcal{R}(R^*) \subseteq \mathcal{D}(R) \) and \( RR^* \) is bounded on \( \mathcal{R}(R^*) \). Moreover, since \( u_{2t} = RR^* \), \( u_{2t} \) is a positive operator for each \( t > 0 \). Thus, \( u_t \) is a positive operator for any \( t > 0 \). As \( \{h_t\}_{t>0} \) is a convolution semigroup, it follows
that $\sqrt{RR^*}(f) = f \ast h_t = u_t(f)$. Now, by the restriction principle, the mapping $U$ which has the formula for $f \in L^2(\mathbb{R}^n)$ as

$$Uf = (\sqrt{RR^*}(f)) = f \ast h_t = H_t f$$

is a unitary isomorphism of $L^2(\mathbb{R}^n)$ onto $\mathcal{H}_t(\mathbb{C}^n)$. This implies that $H_t = U$ is a unitary isomorphism of $L^2(\mathbb{R}^n)$ onto $\mathcal{H}_t(\mathbb{C}^n)$. \[\square\]

**Corollary 3.9.** The map $u_t : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is a positive operator, i.e., $\langle u_t f, f \rangle_2 \geq 0$ for all $f \in L^2(\mathbb{R}^n)$. In particular, $u_t$ is self adjoint.

**Remark.** (1). We can prove that the restriction map $R$ in the above proof has dense range by using the fact that the Hermite functions are dense in $L^2(\mathbb{R}^n)$. Then showing that the Hermite functions are images under $R$ of polynomials weighted by $\sqrt{h_t}$, we thus have that $R$ has dense range. However, with the help of Corollary 3.7, we can prove that (as we did in the above proof) the restriction map $R$ has dense range without using the fact that the Hermite functions are dense in $L^2(\mathbb{R}^n)$.

(2) By using the semigroup property of $u_t$ and the fact that $u_t$ is self adjoint (Theorem 3.6), we can directly show that $u_t$ is a positive operator for every $t > 0$: for each $t > 0$ and for any $f \in L^2(\mathbb{R}^n)$,

$$\langle u_t f, f \rangle_2 = \langle u_{t/2}^2 f, f \rangle_2 = \langle u_{t/2}^2 u_{t/2} f, f \rangle_2 = \langle u_{t/2} f, u_{t/2} f \rangle_2 = \|u_{t/2} f\|^2 \geq 0.$$

(3) There is yet another proof of positivity of the operator $u_t$ by employing the Bochner’s theorem on positive definite functions which states that among the continuous functions on $\mathbb{R}^n$, the positive definite functions are those functions which are the Fourier transforms of nonnegative Borel measures. Since

$$(2t)^n h_t((2t)\cdot)(\lambda) = \hat{h}_t(\lambda/2t) = (2\pi)^{-n/2} e^{-t(\lambda/2t)^2} = (2t)^{n/2} h_t(\lambda),$$

$h_t$ is the Fourier transform of the positive Borel measure $(2t^2/\pi)^2 h_t(2tx)dx$. Therefore, by Bochner’s theorem, $h_t$ is positive definite. That is,

$$\sum_{j,k=1}^{N} c_j \overline{c_k} h_t(\xi_j - \xi_k) \geq 0$$

46
for every finite set of elements \( \xi_1, \ldots, \xi_N \) in \( \mathbb{R}^n \) and every finite set of complex numbers \( c_1, \ldots, c_N \). It follows that, for every \( \phi \in C_c(\mathbb{R}^n) \),

\[
\langle u_t \phi, \phi \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} h_t(x - y) \phi(y) \phi(x) \, dy \, dx
\]

\[
= \int_{\text{supp}(\phi)} \int_{\text{supp}(\phi)} h_t(x - y) \phi(y) \phi(x) \, dy \, dx
\]

\[
\geq 0.
\]

The last double integrals is non-negative because it is the limit of non-negative double Riemann sums

\[
\sum_{j=1}^{N} \sum_{k=1}^{N} h_t(\xi_j - \xi_k) \phi(\xi_k) \overline{\phi(\xi_j)} \Delta(y_k) \Delta(x_j).
\]

Now, let \( f \in L^2(\mathbb{R}^n) \). Since \( C_c(\mathbb{R}^n) \) is dense in \( L^2(\mathbb{R}^n) \), there is a sequence \( (\phi_n) \) in \( C_c(\mathbb{R}^n) \) such that \( \phi_n \rightharpoonup f \) in \( L^2(\mathbb{R}^n) \). Thus, by the continuity of \( u_t \) and of \( \langle \cdot , \cdot \rangle_2 \) on \( L^2(\mathbb{R}^n) \), we have

\[
\langle u_t f, f \rangle_2 = \left\langle u_t \left( \lim_{n \to \infty} \phi_n \right), \lim_{n \to \infty} \phi_n \right\rangle_2
\]

\[
= \left\langle \lim_{n \to \infty} u_t (\phi_n), \lim_{n \to \infty} \phi_n \right\rangle_2
\]

\[
= \lim_{n \to \infty} \langle u_t (\phi_n), \phi_n \rangle
\]

\[
\geq 0.
\]
Chapter 4

The Segal-Bargmann Transform for a Compact Symmetric Space \( \mathcal{X} = U/K \)

4.1 Introduction

We recall some facts from harmonic analysis on compact groups in Section 4.2. Section 4.3 gives a very short review on Riemannian symmetric spaces. Then we derive the harmonic analysis on compact symmetric spaces from the harmonic analysis on compact groups in Section 4.4. An explicit formula of the Fourier series of an \( L^2 \)-function on a compact symmetric space is given in this section. Section 4.5 presents basic notions of compact symmetric spaces and their noncompact duals. We also give the parametrizations of the unitary dual of a compact group and the parametrizations of spherical representations. We discuss the heat equation on a compact symmetric space in Section 4.6. We give the series formula for the solution of the heat equation with the initial \( L^2 \)-function. The Fock space is defined in Section 4.7. Lastly, in Section 4.8, we define the Segal-Bargmann transform for a compact symmetric space and prove by using the restriction principle that it is a unitary isomorphism onto the Fock space. The main references of this chapter are [57] and [15]. We use the notation \( \mathbb{Z}^+ \) for the set \{0, 1, 2, ...\}.

4.2 Harmonic Analysis on Compact Groups

In this section, we review some well-known facts for harmonic analysis on compact groups. Most of the materials can be found in [11], [17], [19], [58], and [67].

**Proposition 4.1.** Let \( \pi \) be a representation of a compact group \( U \) on a finite dimensional vector space \( V \). There exists on \( V \) a Euclidean inner product for which \( \pi \) is unitary.

**Proof.** See the proof of Proposition 6.1.1 in [17].

**Theorem 4.2** (Schur’s Lemma). Let \( \mathbb{K} \) be an algebraically closed field. Let \( V \) be a finite dimensional vector space over \( \mathbb{K} \) and let \( \Phi \) be any irreducible family of operators on \( V \) (the only invariant subspaces, relatively to all operators of \( \Phi \), are \{0\} and \( V \)). Then, if an operator \( A \) commutes with all operators of \( \Phi \), \( A \) is a multiple of the identity operator (i.e., \( A \) is a scalar operator).

**Proof.** See the proof of Theorem 6.1.3 in [17].
**Theorem 4.3** (Schur’s Orthogonal Relations). Let \( \pi \) be an irreducible unitary \( \mathbb{C} \)-linear representation of a compact group \( U \) on a complex Euclidean vector space \( \mathcal{H} \) with dimension \( d(\pi) \). Then, for \( v, w \in \mathcal{H} \),

\[
\int_U |\langle \pi(u)v, w \rangle|^2du = \frac{1}{d(\pi)}\|v\|^2\|w\|^2,
\]

and, by polarization, for \( v, w, v', w' \in \mathcal{H} \),

\[
\int_U \langle \pi(u)v, w \rangle \langle \pi(u)v', w' \rangle du = \frac{1}{d(\pi)}\langle v, v' \rangle \langle w, w' \rangle.
\]

**Proof.** See the proof of Theorem 6.3.3 in [17].

**Theorem 4.4** (Schur’s Orthogonal Relations). Let \((\pi, \mathcal{H})\) and \((\pi', \mathcal{H}')\) be two irreducible unitary representations of a compact group \( U \) which are not equivalent. Then \( \mathcal{M}_\pi \) and \( \mathcal{M}_{\pi'} \) are two orthogonal subspaces of \( L^2(U) \):

\[
\int_U \langle \pi(u)v, w \rangle \overline{\langle \pi'(u)v', w' \rangle} du = 0 \quad (v, w \in \mathcal{H}, v', w' \in \mathcal{H}').
\]

**Proof.** See the proof of Theorem 6.3.4 in [17].

Let \( U \) be a compact group. Fix \((\pi, \mathcal{H})\), a unitary irreducible representation of \( U \). We denote by \( \mathcal{M}_\pi \), the subspace of \( L^2(U) \) generated by the matrix coefficients of \((\pi, \mathcal{H})\), that is by the functions of the following form:

\[
\pi_{v,w}(u) \mapsto \langle \pi(u)v, w \rangle, \quad v, w \in \mathcal{H}.
\]

Let \( \{e_1, ..., e_{d(\pi)}\} \) be an orthonormal basis of \( \mathcal{H} \). For \( i, j \in \{1, 2, ..., d(\pi)\} \), we define the matrix coefficient \( \pi_{ij} \) by

\[
\pi_{ij}(u) = \langle \pi(u)e_j, e_i \rangle, \quad u \in U.
\]

Then the matrix coefficients \( \{\pi_{ij}\}_{i,j \in \{1,2,\ldots,d(\pi)\}} \) span the subspace \( \mathcal{M}_\pi \). For each \( j \in \{1, 2, ..., d(\pi)\} \), let \( \mathcal{M}_\pi^{(j)} \) be the subspace of \( \mathcal{M}_\pi \) spanned by the entries of the \( j^{th} \) row, that is by the functions \( \pi_{jk} \), for \( k = 1, ..., d(\pi) \). We have the following theorem.

**Theorem 4.5.** Let \( \hat{U} \) be the set of equivalence classes of irreducible unitary representations of the compact group \( U \). Here, we use the notations in the above discussion. Let \( R \) denote the right regular representation of \( U \) on \( L^2(U) \):

\[
(R(u)f)(x) = f(xu), \quad f \in L^2(U), \quad u, x \in U.
\]
Then for each $\pi \in \hat{U}$ and each $j \in \{1, 2, ..., d(\pi)\}$, $\mathcal{M}_\pi^{(j)}$ is an invariant subspace of $L^2(U)$ under the representation $R$ of $U$ and the restriction of $R$ to $\mathcal{M}_\pi^{(j)}$ is equivalent with $\pi$. Moreover, we have the Hilbert space direct sum decomposition:

$$L^2(G) = \bigoplus_{\pi \in \hat{U}} \mathcal{M}_\pi$$

$$= \bigoplus_{\pi \in \hat{U}} \left( \bigoplus_{j=1}^{d(\pi)} \mathcal{M}_\pi^{(j)} \right),$$

and the direct sum decomposition of the unitary representation $R$ into irreducible representations:

$$L \cong \bigoplus_{\pi \in \hat{U}} \left( \bigoplus_{j=1}^{d(\pi)} \pi \right) \cong \bigoplus_{\pi \in \hat{U}} d(\pi)\pi.$$

Here, the notation

$$\bigoplus_{\pi \in \hat{U}} \mathcal{M}_\pi$$

denote the closure in $L^2(U)$ of the algebraic direct sum

$$\bigoplus_{\pi \in \hat{U}} \mathcal{M}_\pi,$$

which is the space of finite linear combinations of matrix coefficients of irreducible representations of $U$.

By considering the columns instead of the rows one get the same statement with respect to the contragredient representations $\pi^*$ of $\pi$:

$$\pi^*_{ij}(u) = \pi_{ji}(g) = \pi_{ji}(u^{-1}),$$

and the left regular representation

$$(L(u)f)(x) = f(u^{-1}x), \quad f \in L^2(U), \quad u, x \in U.$$

Proof. See the proof of Theorem 6.4.1 in [17]. \qed

Let $\mathcal{H}$ be a finite dimensional Hilbert space and $A \in \mathcal{B}(\mathcal{H})$. The Hilbert-schmidt norm on $A$ is defined by

$$\|A\|^2 = \text{Tr}(AA^*).$$

For each $\pi \in \hat{U}$, we choose a representative $(\pi, \mathcal{H}_\pi)$. Let $d(\pi)$ denote the dimension of $\mathcal{H}_\pi$. If $f \in L^1(U)$, then we define its Fourier coefficient $\pi(f)$ to be the operator on $\mathcal{H}_\pi$ such that

$$\langle \pi(f)v, w \rangle = \int_U f(u)\langle \pi(u)v, w \rangle \, du \quad \text{for all } v, w \in \mathcal{H}_\pi.$$
The map \( \pi(f) \) is well-defined for every \( f \in L^1(U) \) because for \( f \in L^1(U) \) if we set \( B_f(v, w) = \int_U f(u) \langle \pi(u)v, w \rangle du \), then \( B_f(v, w) \) is linear in \( v \) and conjugate linear in \( w \), and

\[
|B_f(v, w)| \leq \int_U |f(u)||\pi(u)v||w| \, du \\
\leq \|f\|_1||v||w
\]

which implies that \( B_f \) is a bounded sesquilinear functional and hence the consequence of the Riesz representation theorem implies that there is a unique bounded linear operator \( \pi(f) \) on \( H_\pi \) such that \( B_f(v, w) = \langle \pi(f)v, w \rangle \) for all \( v, w \in H_\pi \), see Theorem 21.1 in [4]. A straightforward computation shows that

\[
\pi(f \ast g) = \pi(f)\pi(g)
\]

for all \( f, g \in L^1(U) \). Note that \( L^2(U) \subseteq L^1(U) \) by the Hölder inequality and compactness of \( U \). Thus, \( \pi(f) \) is defined when \( f \in L^2(U) \). The following theorem follows directly from the Peter-Weyl Theorem and from Schur’s orthogonality relations.

**Theorem 4.6 (Plancherel’s Theorem).** Let \( \hat{U} \) be the set of equivalence classes of irreducible unitary representations of the compact group \( U \) and \( f \in L^2(U) \). Then \( f \) is equal to the sum of its Fourier series (in the \( L^2 \)-sense):

\[
f(u) = \sum_{\pi \in \hat{U}} d(\pi) \text{Tr}(\pi(u^{-1})\pi(f)) \\
= \sum_{\pi \in \hat{U}} d(\pi) \sum_{j=1}^{d(\pi)} \langle \pi(u^{-1})\pi(f) e_j, e_j \rangle
\]

where for each \( \pi \in \hat{U} \), \( \{e_1, \ldots, e_{d(\pi)}\} \) is an orthonormal basis for \( V_\pi \) and we also obtain the Plancherel formula

\[
\|f\|^2_2 = \sum_{\pi \in \hat{U}} d(\pi)\|\pi(f)\| \\
= \sum_{\pi \in \hat{U}} d(\pi) \text{Tr}(\pi(f)^*\pi(f)) \\
= \sum_{\pi \in \hat{U}} d(\pi) \sum_{j=1}^{d(\pi)} \|\pi(f) e_j\|^2.
\]
Proposition 4.7. Let $U$ be a compact group and $(\pi, \mathcal{H})$ be a unitary representation of $U$ on a Hilbert space $\mathcal{H}$. Let

$$\mathcal{H} = \bigoplus_{\alpha \in A} \mathcal{H}_\alpha$$

be a decomposition of $\pi$ into a Hilbert space direct sum of irreducible subrepresentations $(\pi, \mathcal{H}_\alpha)_{\alpha \in A}$. Suppose that

$$\mathcal{H} = \bigoplus_{\alpha \in B} V_\alpha$$

is another irreducible decomposition of $(\pi, \mathcal{H})$. Then there is a bijection $f$ from $A$ onto $B$ such that $\mathcal{H}_\alpha$ is unitarily equivalent to $V_{f(\alpha)}$ for every $\alpha \in A$.

Let $\widehat{U}$ be the set of unitary equivalence classes of irreducible unitary representations of $U$. Therefore, we can write

$$\pi \simeq \sum_{\psi \in \widehat{U}} m(\psi, \pi) \psi$$

where

$$m(\psi, \pi) \psi = \psi \oplus \cdots \oplus \psi \quad (m(\psi, \pi) \text{ summands})$$

such that the **multiplicity** $m(\psi, \pi)$ are well-defined, independent of choice of decomposition of $\pi$ as a Hilbert space direct sum of irreducible unitary representations. We have $0 \leq m(\psi, \pi) \leq \infty$ and if all $m(\psi, \pi) = 1$, we say that the irreducible decomposition of $(\pi, \mathcal{H})$ has **multiplicity 1**.

**Proof.** See the proof of Theorem 3.8, Chapter I in [58].

4.3 Riemannian Symmetric Spaces

We refer to [32] for the complete discussion of the Riemannian symmetric spaces. All the proofs of this section can be found in [32]. Recall that a **homogeneous space** is a manifold $M$ with a transitive action of a Lie group $G$. Equivalently, it is a manifold of the form $G/H$ where $G$ is a Lie group and $H$ a closed subgroup of $G$.

For a Riemannian manifold $M$, we denote $I(M)$ the set of all isometries of $M$. Recall that an isometry of $M$ is a diffeomorphism that preserves the metric of $M$. $I(M)$ forms a group under composition of functions. We call $I(M)$, the **isometry group** of $M$. We shall always consider $I(M)$ with the compact open topology.

**Theorem 4.8 (Myers-Steenrod).** The isometry group of a Riemannian manifold is a Lie group.
**Definition 4.9.** A Riemannian homogeneous space is a Riemannian manifold $M$ on which $I(M)$ acts transitively.

**Proposition 4.10.** Let $M$ be a Riemannian homogeneous space. Then the isotropy subgroup of a given point is a compact subgroup of $I(M)$. Moreover, $I(M)$ is compact if and only if $M$ is compact.

Hence, a Riemannian homogeneous space $M$ is diffeomorphic to a homogeneous space $G/K$, where $G = I(M)$ and $K$ is the isotropy subgroup of a point.

**Definition 4.11.** A Riemannian manifold $M$ is called a Riemannian symmetric space, if for any $p \in M$, there exists an involutive isometry $s_p$ of $M$ such that $p$ is isolated fixed point of $s_p$. In such a case, we call $s_p$ the symmetry of $M$ at $p$.

**Proposition 4.12.** Let $M$ be a Riemannian symmetric space. Then $I(M)$ acts transitively on $M$, and hence $M$ is a Riemannian homogeneous space. In fact, the identity component $I_0(M)$ of $I(M)$ also acts transitively on $M$.

To each rie mannian symmetric space, we can associate a Riemannian symmetric pair. We first recall the definition of a Riemannian symmetric pair.

**Definition 4.13.** Let $G$ be a connected Lie group and $H$ a closed subgroup. The pair $(G, H)$ is called a symmetric pair if there exists an involutive automorphism $\theta$ of $G$ such that $(G^\theta)_0 \subseteq H \subseteq G^\theta$. If, in addition, the image $\text{Ad}_G(H)$ under the map $\text{Ad}_G : G \rightarrow GL(g)$, where $g$ is the Lie algebra of $G$, is compact, $(G, H)$ is called a Riemannian symmetric pair.

**Proposition 4.14.** Let $(G, K)$ be a Riemannian symmetric pair with corresponding involutive automorphism $\theta$. Let $g$ be the Lie algebra of $G$. Then

1. $(d\theta)_e$ is an involutive automorphism of $g$.
2. $\mathfrak{k} = \{X \in g : (d\theta)_e(X) = X\}$ is the Lie algebra of $K$.
3. $g = \mathfrak{k} \oplus \mathfrak{m}$, a direct sum of vector spaces, where $\mathfrak{m} = \{X \in g : (d\theta)_e(X) = -X\}$.
4. $\text{Ad}_G(k)(\mathfrak{m}) = \mathfrak{m}$ for all $k \in K$.
5. $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{k}$, $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}$.

Suppose $(G, K)$ is a Riemannian symmetric pair. Then $\text{Ad}_G(K)$ is compact and $\text{Ad}_G(k)(\mathfrak{m}) = \mathfrak{m}$ for all $k \in K$. It follows that we can find an $\text{Ad}_G(K)$-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}$, namely

$$\langle X, Y \rangle = \int_{\text{Ad}_G(K)} \langle a(X), a(Y) \rangle \, da, \quad \text{for } X, Y \in \mathfrak{m}$$
where $\langle \cdot, \cdot \rangle$ is a fixed inner product on $\mathfrak{m}$. Let $M = G/K$ be the quotient manifold and $o = eK$ the origin. We define an action of $a \in G$ on $M$ by the left translation $\tau_a(xK) = axK$. Then $\tau_a$ gives a diffeomorphism of $M$. Further, for the canonical projection $\pi : G \to M = G/K$, its differential $d\pi$ gives an identification between $\mathfrak{m}$ and the tangent space $T_o(M)$. Then we transfer the $\text{Ad}_G(K)$-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}$ to the $\text{Ad}_G(K)$-invariant inner product $\langle \cdot, \cdot \rangle_o$ on $T_o(M)$. Now, the $\text{Ad}_G(K)$-invariant inner product $\langle \cdot, \cdot \rangle_o$ on $T_o(M)$ induces a $G$-invariant Riemannian metric $g$ on the coset space $M = G/K$:

$$g_{a,o}(X,Y) = \langle (d\tau_{a^{-1}})_a(X), (d\tau_{a^{-1}})_a(Y) \rangle_o,$$

for $a \cdot o \in M$ and $X, Y \in T_{a,o}(M)$. This is well-defined, and elements of $G$ act on $(M,g)$ as isometries. The map $s : xK \mapsto \theta(x)K$, where $\theta$ is the involutive automorphism for the pair $(G, K)$, is the symmetry of $M$ at the origin $o$. And at any $a \cdot o \in M$, the map $s_{a,o} = \tau_a s \tau_{a^{-1}}$ is the symmetry of $M$ at $a \cdot o$. Thus, $M = G/K$ is a Riemannian symmetric space.

Conversely, suppose $M$ is a Riemannian symmetric space. Let $G = I_o(M)$ be the identity component of $I(M)$. Let $p$ be any point in $M$ and $s_p$ the symmetry of $M$ at $p$. Let $K$ be the isotropy subgroup of $G$ at $p$. Then the map $s : x \mapsto s_p x s_p$ is an involutive automorphism of $G$ with $(G^*)_0 \subseteq K \subseteq G^*$. $\text{Ad}_G(K)$ is compact since $K$ is compact and $\text{Ad}_G$ is continuous. In this way, the Riemannian symmetric space $M$ determines a Riemannian symmetric pair $(G,K)$.

So, the above arguments suggest that there is a certain correspondence between Riemannian symmetric spaces and Riemannian symmetric pairs. We state this result as the following theorem. For the detailed proof of the theorem, we refer to [32].

**Theorem 4.15.** (1). Let $(G, K)$ be a Riemannian symmetric pair. Then there is a $G$-invariant metric $g$ on $M = G/K$ which makes $(M,g)$ a Riemannian symmetric space.

(2). Let $M$ be a Riemannian symmetric space and let $p$ any point in $M$. Let $G = I_0(M)$, $K$ the isotropy group of $G$ at $p$, and $s_p$ the symmetry of $M$ at $p$. Then $(I_0(M), K)$ is a Riemannian symmetric pair with the involutive automorphism given by $\theta : x \mapsto s_p x s_p$.

### 4.4 Harmonic Analysis on Compact Symmetric Spaces $\mathcal{X} = U/K$

**Definition 4.16.** Let $U$ be a compact group and $K$ a closed subgroup of $U$. We denote $\hat{U}$ the set of all equivalence classes of irreducible unitary representations of $U$. For each $\pi \in \hat{U}$ we always choose in the equivalence class a concrete representation $\pi$ on a complex vector space $V_\pi$ and denote $d(\pi)$ the dimension of $V_\pi$. We
define $V^K_\pi$ the space of $K$-fixed vectors in $V_\pi$ by

$$V^K_\pi = \{ v \in V_\pi \mid \forall k \in K, \pi(k)v = v \}.$$ 

Denote $c_\pi$ for the dimension of $V^K_\pi$. Define

$$\hat{(U/K)} = \{ \pi \in \hat{U} : V^K_\pi \neq \{0\} \} = \{ \pi \in \hat{U} : V^K_\pi \neq 0 \}.$$ 

We call a representation $\pi \in \hat{(U/K)}$ a $K$-spherical representation. The pair $(U, K)$ is called a compact Gelfand pair if the convolution algebra

$$L^1(K\backslash U/K) \cong \{ f \in L^1(U) : f(k_1uk_2) = f(u) \ \forall k_1, k_2 \in K \text{ and a.e } u \in U \}$$

is commutative.

Let $\mathcal{K}$ be a compact Riemannian symmetric space. Let $U = I_0(\mathcal{K})$ and $K$ the isotropy subgroup of $U$ at a point in $\mathcal{K}$. Then the pair $(U, K)$ is a Riemannian symmetric pair and $\mathcal{K} = U/K$. Note that by Proposition 4.10, $U$ is compact. We call $(U, K)$ a compact Riemannian symmetric pair.

**Theorem 4.17.** The compact Riemannian symmetric pair $(U, K)$ is a Gelfand pair.

**Proof.** See the proof of Corollary 1.5.6. in [24] or the proof of Corollary 8.1.4. in [67].

**Theorem 4.18.** For the compact Riemannian symmetric pair $(U, K)$, if $\pi \in \hat{(U/K)}$, then $c_\pi = \dim V^K_\pi = 1$.

**Proof.** Let $\pi \in \hat{(U/K)}$. Suppose that $c_\pi > 1$. Pick nonzero orthonormal vectors $v_1$ and $v_2$ in $V^K_\pi$. Note that both $\pi_{v_1,v_2}$ and $\pi_{v_2,v_1}$ are in $L^1(K\backslash U/K)$. By the Schur’s orthogonality relations, we see that

$$(\pi_{v_1,v_2} \ast \pi_{v_2,v_1})(u) = \int_U \langle \pi(a)v_2, v_1 \rangle \langle \pi(a^{-1}u)v_1, v_2 \rangle \ da$$

$$= \int_U \langle \pi(a)v_2, v_1 \rangle \langle \pi(u)v_1, \pi(a)v_2 \rangle \ da$$

$$= \int_U \langle \pi(a)v_2, v_1 \rangle \overline{\langle \pi(a)v_2, \pi(u)v_1 \rangle} \ da$$

$$= \frac{1}{d(\pi)} \langle v_2, v_2 \rangle \langle v_1, \pi(u)v_1 \rangle$$

$$= \frac{1}{d(\pi)} \pi_{v_1,v_1}(u).$$
Similarly, $\pi_{v_2,v_1} \circ \pi_{v_1,v_2} = \frac{1}{d(\pi)} \pi_{v_2,v_1}$. Again, using the Schur’s orthogonal relations, $\pi_{v_1,v_1}$ and $\pi_{v_2,v_2}$ are nonzero two orthogonal vectors in $L^1(K\backslash U/K)$, and hence they are not equal. This implies that $\pi_{v_1,v_2} \circ \pi_{v_2,v_1} \neq \pi_{v_2,v_1} \circ \pi_{v_1,v_2}$. Thus, $L^1(K\backslash U/K)$ is not commutative. This is a contradiction since the previous theorem implies that $(U,K)$ is a Gelfand pair. This completes the proof. \hfill \Box

**Remark.** We can give another proof of the above theorem by considering $\pi$ as a representation of the algebra $L^1(K\backslash U/K)$. It can be shown that $V^K_\pi$ is irreducible under the action $\pi$ of the commutative algebra $L^1(K\backslash U/K)$, and so is one-dimensional by the Schur’s lemma.

Now we will derive the harmonic analysis on $\mathcal{X} = U/K$ from the harmonic analysis on $U$. We define the action of $U$ on $L^2(\mathcal{X}) = L^2(U/K)$ by

$$(u \cdot f)(x) = f(u^{-1} \cdot x),$$

where we always consider $\mathcal{X} = U/K$ as the set of right cosets $uK$, $u \in U$ and the action of $U$ on $U/K$ is given by

$$u \cdot (xK) = u x K.$$

Let $L^2(U)^K$ be the right $K$-invariant functions in $L^2(U)$. Then $L^2(U)^K$ is invariant under the left regular representation $L$ of $U$ on $L^2(U)$. The map

$$f \mapsto (u \mapsto f(uK)) : L^2(\mathcal{X}) \to L^2(U)^K$$

is an isomorphism of Hilbert spaces and it interwines the action of $U$ on $L^2(\mathcal{X})$ and the left regular representation $L$ of $U$ on $L^2(U)^K$. Therefore, we can identify the $L^2$-functions on $U/K$ with the right $K$-invariant $L^2$-functions on $U$ and the harmonic analysis on $L^2(\mathcal{X})$ is equivalent to the harmonic analysis on $L^2(U)^K$. By the same token, the harmonic analysis on $L^2(\mathcal{X})^K$ is equivalent to the harmonic analysis on $K L^2(U)^K$. Here, $L^2(\mathcal{X})^K$ denotes the set of left $K$-invariant functions in $L^2(\mathcal{X})$ and $K L^2(U)^K$ denotes the set of $K$-biinvariant functions in $L^2(U)$. We recall that if we let $d(uK)$ be the push forward measure on $U/K$ of the Haar measure $du$ on $U$ via the canonical projection

$$u \in U \mapsto uK \in U/K,$$

then we have the integration formula for $f \in L^1(U/K)$:

$$\int_{U/K} f(uK) \, d(uK) = \int_U f(uK) \, du.$$

Recall that by Plancherel’s theorem, for $f \in L^2(U)$, we have the $L^2$-expansion:

$$f(u) = \sum_{\pi \in \hat{U}} d(\pi) \, \text{Tr}(\pi(u^{-1})\pi(f)) \quad (4.1)$$

$$= \sum_{\pi \in \hat{U}} d(\pi) \sum_{j=1}^{d(\pi)} \langle \pi(u^{-1})\pi(f)e_j, e_j \rangle \quad (4.2)$$

56
where for each $\pi \in \widehat{G}$, $\{e_1, \ldots, e_{d(\pi)}\}$ is an orthonormal basis for $V_\pi$ and we also obtain the Plancherel formula

$$\|f\|_2^2 = \sum_{\pi \in \widehat{U}} d(\pi) \|\pi(f)\|$$

$$= \sum_{\pi \in \widehat{U}} d(\pi) \text{Tr}(\pi(f)^* \pi(f))$$

$$= \sum_{\pi \in \widehat{U}} d(\pi) \sum_{j=1}^{d(\pi)} \|\pi(f)e_j\|^2.$$  \hfill (4.5)

(a). Assume that $f \in L^2(U)$ and $f$ is right $K$-invariant so that we can view $f$ as a function on $L^2(U/K)$. Fix $\pi \in \widehat{G}$. Then for each $v \in V_\pi$, we formally get

$$\pi(f)v = \int_\mathcal{U} f(x)\pi(x)v \, dx$$

$$= \int_\mathcal{U} f(xk^{-1})\pi(x)v \, dx$$

$$= \int_\mathcal{U} f(x)\pi(x)\pi(k)v \, dx$$

for all $k \in K$. Integrating over $K$, we formally have

$$\pi(f)v = \int_K \int_\mathcal{U} f(x)\pi(x)\pi(k)v \, dx \, dk$$

$$= \int_\mathcal{U} f(x)\pi(x) \left( \int_K \pi(k)v \, dk \right) \, dx.$$  \hfill (4.6)

We can verify the validity of the above equations by taking the inner product with $w \in V_\pi$ as we defined the definition of $\pi(f)$ in the first section, i.e. the above calculations are justified by the computations in the weak sense. Recall that we define the space of $K$-fixed vectors by

$$V^K_\pi = \{v \in V_\pi \mid \forall k \in K, \pi(k)v = v\}.$$  \hfill (4.7)

Next, for $v \in V_\pi$, we set

$$P_{V^K_\pi}(v) = \int_K \pi(k)v \, dk.$$  \hfill (4.8)

Then $P_{V^K_\pi} : V_\pi \rightarrow V^K_\pi$ is an orthogonal projection of $V_\pi$ on $V^K_\pi$ and

$$\pi(f)v = \pi(f)P_{V^K_\pi}(v).$$

In fact, for every $l \in K$, we formally obtain

$$\pi(l) \left( \int_K \pi(k)v \, dk \right) = \int_K \pi(l)\pi(k)v \, dk = \int_K \pi(lk)v \, dk = \int_K \pi(k)v \, dk.$$
and then for \( v \in V_\pi \),

\[
P^2_{V_\pi^K}(v) = \int_U \pi(l) \left( \int_K \pi(k)v \, dk \right) \, dl
= \int_K \int_K \pi(k)v \, dk \, dl
= \int_K \pi(k)v \, dk
= P_{V_\pi^K}(v);
\]

moreover, for every \( v, w \in V_\pi \),

\[
(P_{V_\pi^K}(v), w) = \int_K \langle \pi(k)v, w \rangle \, dk
= \int_K \langle v, \pi(k^{-1})w \rangle \, dk
= \int_K \langle v, \pi(k)w \rangle \, dk
= \langle v, P_{V_\pi^K}(w) \rangle
= \langle P^* V_\pi^K(v), w \rangle.
\]

Therefore, \( P_{V_\pi^K} \) is idempotent and self adjoint. It follows that \( P_{V_\pi^K} \) is an orthogonal projection (see Theorem 22.1 in [4]).

If \( V_\pi^K = \{0\} \), then \( \pi(f)v = 0 \) for all \( v \in V_\pi \). If \( V_\pi^K \neq \{0\} \), then \( \dim V_\pi^K = 1 \) by Theorem 4.18. Pick a vector \( e_\pi \in V_\pi^K \) such that \( \|e_\pi\| = 1 \). Let \( \{e_1 = e_\pi, e_2, \ldots, e_{d(\pi)}\} \) be an orthonormal basis for \( V_\pi \). Then (4.2) becomes

\[
f(u) = \sum_{\pi \in \hat{U}/K} d(\pi) \langle \pi(u)\pi^{-1}f, e_\pi \rangle,
\]

and (4.5) is reduced to

\[
\|f\|_2^2 = \sum_{\pi \in \hat{U}/K} d(\pi) \|\pi(f)e_\pi\|^2.
\]

Thus, (4.7) and (4.8) are the formulas for harmonic analysis on \( L^2(U/K) \). Recall Theorem 4.5. The above discussion implies that we have an orthogonal direct sum decomposition of the left regular representation \( L \) of \( U \) on \( L^2(U)^K \) as

\[
L^2(U)^K = \bigoplus_{\pi \in \hat{U}/K} \left( \bigoplus_{j=1}^{c(\pi)} \mathcal{M}_{\pi}^j \right)
= \bigoplus_{\pi \in \hat{U}/K} \mathcal{M}_{\pi}^{c(\pi)} \quad \text{(since } c(\pi) = 1 \text{)}
\]

58
where each $M^j_\pi$ is the subspace of $M$ spanned by the entries of the $j^{th}$ column. Therefore, the irreducible decomposition of the left regular representation $L$ of $U$ on $L^2(U^K)$ has multiplicity 1. We set the notation $C_\pi(X)$ for the space $M^1_\pi$.

(b). We suppose that $f \in L^2(U)$ and $f$ is $K$-biinvariant so that we can view $f$ as a function on $L^2(U/K)^K$. So we have all the results of part (a). It turns out that $\pi(f)e_\pi \in V^K_\pi$. Indeed, for every $k \in K$, we formally have

$$\pi(k)\pi(f)e_\pi = \pi(k) \left( \int_U f(x)\pi(k)(x)e_\pi \, dx \right)$$

$$= \int_U f(x)\pi(k)(x)e_\pi \, dx$$

$$= \int_U f(x)\pi(kx)e_\pi \, dx$$

$$= \int_U f(k^{-1}x)\pi(x)e_\pi \, dx$$

$$= \int_U f(x)\pi(x)e_\pi \, dx$$

$$= \pi(f)e_\pi.$$

But $\{e_\pi\}$ is a basis of $V^K_\pi$, there is a scalar $\hat{f}(\pi) \in \mathbb{C}$ such that

$$\pi(f)e_\pi = \hat{f}(\pi)e_\pi \quad (4.9)$$

Therefore, (4.7) and (4.8) become

$$f(u) = \sum_{\pi \in (U/K)} d(\pi)\hat{f}(\pi)\langle e_\pi, \pi(u)e_\pi \rangle \quad (4.10)$$

and

$$|f|^2 = \sum_{\pi \in (U/K)} d(\pi)|\hat{f}(\pi)|^2. \quad (4.11)$$

We can compute $\hat{f}(\pi)$ as the following:

$$\hat{f}(\pi) = \hat{f}(\pi)\|e_\pi\|^2$$

$$= \langle \hat{f}(\pi)e_\pi, e_\pi \rangle$$

$$= \langle \pi(f)e_\pi, e_\pi \rangle$$

$$= \int_U f(x)\langle \pi(x)e_\pi, e_\pi \rangle \, dx$$

$$= \int_U f(x)\overline{\langle e_\pi, \pi(x)e_\pi \rangle} \, dx.$$

In fact, the set $\{\sqrt{d(\pi)}\varphi_\pi \mid \pi \in \hat{G}, V^K_\pi \neq \{0\}\}$ is an orthonormal basis for the Hilbert space $L^2(U/K)^K$. Here $\varphi_\pi$ is defined by the formula

$$\varphi_\pi(u) = \langle e_\pi, \pi(u)e_\pi \rangle, \quad u \in U.$$
We call $\varphi_\pi$ the spherical function on $U/K$ associated with $\pi$. Note that $\varphi_\pi$ is $K$-biinvariant. Also, observe that

$$\hat{f}(\pi) = \int_U f(u)\overline{\varphi_\pi(u)} \, du = \langle f, \varphi_\pi \rangle_2.$$ 

Finally, we conclude that the harmonic analysis on $L^2(U/K)$ contains the following formulas:

$$f(x) = \sum_{\pi \in \hat{U}/\hat{K}} d(\pi)\hat{f}(\pi)\varphi_\pi(x)$$

and

$$||f||_2^2 = \sum_{\pi \in \hat{U}/\hat{K}} d(\pi)|\hat{f}(\pi)|^2,$$

where we define the scalar Fourier transform $\hat{f}(\pi)$ by

$$\hat{f}(\pi) = \langle f, \varphi_\pi \rangle_2 = \int_{\mathfrak{X}} f(x)\overline{\varphi_\pi(x)} \, dx = \int_U f(uK)\overline{\varphi_\pi(uK)} \, du.$$

**Remark.** We consider the situation in (a). We will rewrite the equations (4.7) and (4.8) in the different forms. First, we observe that

$$(\pi(u^{-1})\pi(f)e_\pi, e_\pi) = \langle \pi(f)e_\pi, \pi(u)e_\pi \rangle$$

$$= \int_{\mathfrak{X}} f(x)\langle \pi(x)e_\pi, \pi(u)e_\pi \rangle \, dx$$

$$= \int_U f(x) \sum_{j=1}^{d(\pi)} \langle \pi(x)e_\pi, e_j \rangle \langle e_j, \pi(u)e_\pi \rangle \, dx$$

$$= \sum_{j=1}^{d(\pi)} \left( \int_U f(x)e_j \overline{\langle \pi(x)e_\pi \rangle} \, dx \right) \cdot \langle e_j, \pi(u)e_\pi \rangle$$

$$= \sum_{j=1}^{d(\pi)} \hat{f}_j(\pi)e_j^2(u)$$
\[
\|\pi(f)e_\pi\|^2 = \langle \pi(f)e_\pi, \pi(f)e_\pi \rangle \\
= \sum_{j=1}^{d(\pi)} \langle \pi(f)e_\pi, e_j \rangle \langle e_j, \pi(f)e_\pi \rangle \\
= \sum_{j=1}^{d(\pi)} \int_U f(x) \langle \pi(x)e_\pi, e_j \rangle \, dx \, \overline{\langle \pi(f)e_\pi, e_j \rangle} \\
= \sum_{j=1}^{d(\pi)} \hat{f}_j(\pi) \overline{\hat{f}_j(\pi)} \\
= \sum_{j=1}^{d(\pi)} |\hat{f}_j(\pi)|^2,
\]

where for each \( j = 1, \ldots, d(\pi) \), we define
\[
\varphi^j_\pi(u) = \langle e_j, \pi(u)e_\pi \rangle, \quad u \in U
\]
and
\[
\hat{f}_j(\pi) = \int_U f(x) \overline{\varphi^j_\pi(x)} \, dx.
\]

We also see that
\[
\hat{f}_j(\pi) = \langle \pi(f)e_\pi, e_j \rangle.
\] (4.12)

Note that each \( \varphi^j_\pi \) is right \( K \)-invariant and \( \varphi^1_\pi = \varphi_\pi \) is \( K \)-biinvariant. Now (4.7) and (4.8) can be rewritten as
\[
f(u) = \sum_{\pi \in (U/K)} d(\pi) \left( \sum_{j=1}^{d(\pi)} \hat{f}_j(\pi) \varphi^j_\pi(u) \right)
\] (4.13)
and
\[
\|f\|_2^2 = \sum_{\pi \in (U/K)} d(\pi) \left( \sum_{j=1}^{d(\pi)} |\hat{f}_j(\pi)|^2 \right).
\] (4.14)

4.5 **Compact Symmetric Spaces** \( \mathcal{X} = U/K \) and Their Noncompact Duals \( \mathcal{Y} = G/K \)

Let \( \mathcal{X} \) be a compact Riemannian symmetric space associated with the compact Riemannian symmetric pair \((U, K)\) as in the previous section. Let \( \theta \) be the associated involutive automorphism of \( U \). Then \( \mathcal{X} = U/K \) with \( U \) a compact connected
Lie group and $K$ a compact subgroup of $U$ with the property $(G^\theta)_0 \subseteq K \subseteq G^\theta$. We denote $o$ as the base point $eK \in \mathcal{X}$. We have the canonical decomposition

$$u = \mathfrak{k} \oplus \mathfrak{p},$$

where $u$ is the Lie algebra of $U$, $\mathfrak{k} = \{ X \in u : (d\theta)_e(X) = X \}$ is the Lie algebra of $K$ and $\mathfrak{p} = \{ X \in u : (d\theta)_e(X) = -X \}$.

Since $U$ is compact, $U$ admits a faithful finite dimensional unitary representation. Therefore, if $N$ is the dimension of this representation, we may assume that $U$ is a closed subgroup of the unitary group $U(N)$. We can show that the set

$$U_C = \{ g = u \exp(iX) : u \in U, \ X \in u \}$$

is a closed subgroup of $GL(N, \mathbb{C})$. For simplicity, we suppose that $U$ is a closed subgroup of the unitary group $U(N)$ in this section. We have the corresponding involution which we also denote by $\theta$ and we have the same decomposition as above. Note that for $g \in U_C$, there exist a unique $u \in U$ and a unique $X \in u$ such that

$$g = u \exp(iX).$$

The Lie algebra $u_C$ of $U_C$ is the complex Lie algebra

$$u_C = u \oplus iu,$$

and $U_C$ is a complex submanifold of $GL(N, \mathbb{C})$. So $U_C$ is the complexification of compact Lie group $U$. (Indeed, $u_C = u \oplus iu$ is a Cartan decomposition with respect to the anti-complex-linear extension of $d\theta$.) We define the inner product on the Lie algebra $u_C \subset M(N, \mathbb{C})$ by

$$\langle X, Y \rangle = \text{Tr}(XY^*),$$

and we define the corresponding inner product on the dual space $(u_C)^*$ in the canonical way. The involution $\theta$ can be extended to a holomorphic involution on $U_C$, which we also denote by $\theta$. If $g = u \exp(iX)$ for $u \in U, \ X \in u$, then

$$\theta(g) = \theta(u)\theta(\exp(iX)) = \theta(u)\exp(d\theta(iX)) = \theta(u)\exp(id\theta(X))$$

since $d\theta$ is complex linear. Note that each element $u$ in $U$ can be written as

$$u = k \exp X,$$

for some $k \in K, \ X \in \mathfrak{p}$. This decomposition is in general not unique. A Cartan subspace $\mathfrak{a}$ for the Riemannian symmetric pair $(U, K)$ is a maximal abelian (as a subalgebra of $u$) subspace of $\mathfrak{p}$. It is known that Cartan subalgebras are conjugate under the adjoint action of $K$ on $\mathfrak{p}$ (see [32]). Therefore, the Cartan subalgebras have the same dimension. We call the dimension of a cartan subalgebra the rank of the Riemannian symmetric pair $(U, K)$. The set $A = \exp \mathfrak{a}$ is a connected abelian closed subgroup of $U$. We call $A$ the corresponding Cartan subgroup of $(U, K)$. Note
that \( K \cap A \) is a finite subgroup and \( K \cap A = \exp \Gamma_{U/K} \), where \( \Gamma_{U/K} \) is the lattice of the pair \((U, K)\) contained in \( a \) and is defined by
\[
\Gamma_{U/K} = \{ H \in a : \exp H \in K \}.
\]
We can show that
\[
K = K_0 \exp \Gamma_{U/K},
\]
where \( K_0 \) is the identity component of \( K \). If \( K = U^\theta \),
\[
\Gamma_{U/K} = \{ \frac{1}{2} H : H \in a, \exp H = e \}.
\]
The maximal torus of the Riemannian symmetric space \( X = U/K \) is defined by
\[
A_0 = \{ \exp H \circ : H \in a \} \simeq a/\Gamma_{U/K}.
\]
The quotient
\[
W = W(U, K) = N_K(A)/Z_K(A) = N_K(a)/Z_K(a)
\]
is a finite group which we call the Weyl group for the Riemannian symmetric pair \((U, K)\).

Let \( G = K \exp i p \). Then \( G \) is a closed subgroup of \( U_C \) with the Lie algebra \( g = \mathfrak{k} \oplus i \mathfrak{p} \).

The identity component \( G_0 \) of \( G \) is
\[
G_0 = K_0 \exp i p.
\]
We note that \( d\theta(g) \subseteq g \) with \( \theta(X + i Y) = X - i Y \) for \( X \in u, Y \in p \). Then, after a conjugation by an element in \( Z_G(K) \) if necessary, we have for \( g \in G \),
\[
\theta(g) = (g^*)^{-1}.
\]
Here, * denote the conjugate transpose.

The pair \((G, K)\) is a noncompact Riemannian symmetric pair associated to the involution \( \theta|_G \). We call \((G, K)\) the noncompact dual of \((U, K)\). The group \( G \) is closed and is stable by * , so \( G \) is a linear reductive group. The set \( i a \) is a Cartan subspace for the Riemannian symmetric pair \((G, K)\). We denote by \( \mathfrak{A} = \exp i a \subseteq G \), the corresponding Cartan subgroup.

If \( \alpha \in (i a)^* \), we define
\[
\mathfrak{g}^\alpha = \{ X \in \mathfrak{g} \mid \forall H \in i a, [H, X] = \alpha(H)X \}.
\]
If \( \mathfrak{g}^\alpha \neq \{0\} \), \( \alpha \) is called the restricted root. Set \( m_\alpha = \dim \mathfrak{g}^\alpha \). We let \( \Delta = \Delta(\mathfrak{g}, i a) = \Delta(u_C, a_C) \) be the restricted root system. Let \( t \) be a Cartan subalgebra of \( u \) containing \( a \). The system \( \Delta \) is the set of restrictions to \( a_C \) of the root system \( \Sigma(u_C, t_C) \) whose restrictions to \( a_C \) are not zero (cf. in [32], pp. 263-264).
If $U$ is simply connected, then $U^\theta$ is connected and thus
\[ \Gamma = \{ H \in \mathfrak{a} \mid \forall \alpha \in \Delta, \, \alpha(H) \in 2\pi i\mathbb{Z} \} . \]

We choose the positive system $\Delta^+$ and we set
\[ n = \sum_{\alpha \in \Delta^+} g^\alpha, \, N = \exp n . \]

We denote $(i\mathfrak{a})_+$ the associated positive Weyl chamber:
\[ (i\mathfrak{a})_+ = \{ H \in i\mathfrak{a} \mid \forall \alpha \in \Delta^+, \, \alpha(H) > 0 \} . \]

The group $G$ and its Lie algebra $\mathfrak{g}$ admit the Iwasawa decompositions:
\[ \mathfrak{g} = \mathfrak{k} + i\mathfrak{a} + n, \, G = KAN . \]

The Iwasawa decomposition of an element $g \in G$ is given by
\[ g = k \exp Hn, \]
where $k \in K, H \in i\mathfrak{a}, n \in N$. We set $H = \mathcal{H}(g)$.

Next we give the integration formula on $X = U/K$. From now on, we always use the same notation $o$ for the base point in all of the homogeneous space, for example we denote $o = eK$ in $U/K$. Let $m_0$ be a $U$-invariant measure on $X$. Then we have the following integration formula.

**Theorem 4.19.** Let $f$ be an integrable function on $X$. Then
\[ \int_{X} f(x)dm_0(x) = c_0 \int_{K} \int_{a/\Gamma U/K} f(k \exp H \cdot o) J_0(H) dH dk, \]
where
\[ J_0(H) = \left| \prod_{\alpha \in \Delta^+} (\sin \langle \alpha, iH \rangle)^{m_\alpha} \right| , \]
and $c_0$ is a positive constant.

**Proof.** See the proof of Theorem IV.1.1 in [15]. \[ \square \]

The complexification $K_\mathbb{C}$ of the compact group $K$ is a closed subgroup of $U_\mathbb{C}$ and $(U_\mathbb{C}, K_\mathbb{C})$ is non-Riemannian symmetris pair with respect to the involution $\theta$. The space $X_\mathbb{C} = U_\mathbb{C}/K_\mathbb{C}$ is a complex manifold and is the complexification of the symmetric space $X = U/K$.
Theorem 4.20. Each point \( z \in \mathcal{X}_C \) can be written as
\[ z = u \exp(H) \cdot o, \]
where \( u \in U \) and \( H \in i\mathfrak{a} \). If
\[ u_1 \exp(H_1) \cdot o = u_2 \exp(H_2) \cdot o, \]
then there exists \( w \in W \) such that \( H_2 = w \cdot H_1 \). If we choose \( H \in (i\mathfrak{a})_+ \), then \( H \) is unique.

Proof. See the proof of Theorem IV.2.1 in [15]. \[ \square \]

Let \( m \) be a \( U_C \)-invariant measure on \( \mathcal{X}_C \). Then we have the following integration formula for \( \mathcal{X}_C \).

Theorem 4.21. Let \( f \) be an integrable function on \( \mathcal{X}_C \). Then
\[ \int_{\mathcal{X}_C} f(z) dm(z) = c \int_{U} \int_{(i\mathfrak{a})_+} f(u \exp(H) \cdot o) J(H) dH du, \]
where
\[ J(H) = \prod_{\alpha \in \Delta^+} \sinh 2 \langle \alpha, H \rangle, \]
and \( c \) is a positive constant.

Proof. See the proof of Theorem IV.2.4 in [15]. \[ \square \]

Finally, we give an integration formula for the noncompact dual \( \mathcal{Y} = G/K \). Let \( m_1 \) be a \( G \)-invariant measure on \( \mathcal{Y} \). Then we have the following theorem.

Theorem 4.22. Let \( f \) be an integrable function on \( \mathcal{Y} \). Then
\[ \int_{\mathcal{Y}} f(x) dm_1(x) = c_1 \int_{K} \int_{i\mathfrak{a}} f(k \exp(H) \cdot o) J_1(H) dH dk, \]
where
\[ J_1(H) = \prod_{\alpha \in \Delta^+} \sinh \langle \alpha, H \rangle, \]
and \( c_1 \) is a positive constant. Note that \( J_1(2H) = J(H) \).

In the following, we give a parametrization for \((U/K)^{\wedge}\), the equivalence classes of the irreducible \( K \)-spherical representation of \( U \). Let \((\pi, V)\) be a finite dimensional irreducible unitary representation of \( U \). Then \( \pi \) extends to a holomorphic representation \( \pi_C \) of \( U_C \) such that
\[ \pi_C(g^*) = \pi_C(g^*). \]
Recall that the derived representation $d\pi_C$ is defined by

$$d\pi_C(X) = \frac{d}{dt}\pi_C(\exp tX)|_{t=0},$$

for $X \in u_C$. This is a Lie algebra representation of the complex Lie algebra $u_C$.

Let $t$ be a Cartan subalgebra of $u$ containing $a$. Then $t = (t \cap k) \oplus a$. Let $\Sigma(u_C, t_C)$ be the root system of the pair $(u_C, t_C)$. Let $\Sigma^+$ be a positive root system such that each root $\alpha \in \Delta^+$ is the restriction to $a_C$ of a root in $\Sigma^+$.

For $\gamma \in \Sigma$, let $u^\gamma_C$ be the corresponding eigenspace:

$$u^\gamma_C = \{X \in u_C : [H, X] = \gamma(H)X, \ \forall H \in t_C\}.$$

A highest weight vector of the representation $(\pi, V)$ is a nonzero vector $v$ in $V$ such that for $H \in t_C$,

$$d\pi_C(H)v = \lambda(H)v,$$

where $\lambda \in t^*_C$ and if

$$X \in u^+_C := \bigoplus_{\gamma \in \Sigma^+} u^\gamma_C,$$

then $d\pi_C(X)v = 0$. The corresponding linear functional $\lambda$ is called the highest weight of the representation $(\pi, V)$. Note that by $\mathbb{C}$-linearity, $\lambda \in (t_C)^*$ is completely determined by its restriction to either $t$ or $it$, thus we can view $\lambda \in (t_C)^*$ as an element of $(it)^*$ (real-valued) or $t^*$ (purely imaginary-valued). The subspace

$$V^\lambda = \{v \in V : d\pi_C(H)v = \lambda(H)v, \ \forall H \in t_C\}$$

is of dimension 1. Recall that we define the set of $K$-fixed vectors by

$$V^K_\pi = \{v \in V_\pi \mid \forall k \in K, \ \pi(k)v = v\}.$$

By Theorem 4.18, $V^K_\pi$ is of dimension 1.

**Theorem 4.23.** Let $(\pi, V)$ be an irreducible unitary representation with the highest weight $\lambda$. Let $v$ be a highest weight vector. Then the followings are equivalent:

(i) The spherical $\pi$ is $K$-spherical.

(ii) The highest vector $v$ is invariant under $K$, i.e. $v \in V^K_\pi$.

(iii) For $H \in t \cap k$, $\lambda(H) = 0$ (so $\lambda \in (ia)^*$) and for $H \in \Gamma_{U/K}$, $\lambda(H) \in 2i\pi\mathbb{Z}$.

**Proof.** See the proof of Theorem IV.4.2. in [15].

Recall that (see Theorem 4.28 in [37]) we have a one-to-one correspondence between $\hat{U}$ and the set $D^+$ of dominant analytically integral weights with respect to $t$ (the correspondence being that each $\pi \in \hat{U}$ sends to its highest weight $\lambda_\pi$): $\lambda$ is a dominant analytically integral weight if and only if $\lambda \in (it)^*$ such that

$$\lambda(i\Gamma_U) \subset 2\pi\mathbb{Z} \quad \text{(analytic integral condition)}$$
where $\Gamma_U = \{ H \in \mathfrak{t} : \exp H = e \}$ is the unit lattice of $U$, and (the dominant condition)

$$\langle \lambda, \alpha \rangle \geq 0, \quad \forall \alpha \in \Sigma^+. $$

Using this fact and the above theorem, we now obtain the following theorem on parametrization of $(\overline{U/K})$:

**Theorem 4.24.** We have the one-to-one correspondence between the set $(\overline{U/K})$ of equivalence classes of irreducible spherical representation and the set $P^+ \subset D^+$ of dominant restricted analytically integral weights: $\mu \in P^+$ if and only if $\mu \in (i\mathfrak{a})^*$ such that 

$$\mu(i\Gamma_{U/K}) \subset 2\pi \mathbb{Z},$$

and

$$\langle \mu, \alpha \rangle \geq 0, \quad \forall \alpha \in \Delta^+. $$

Next, we define

$$\rho' = \sum_{\alpha \in \Sigma^+} \alpha$$

and

$$\rho = \sum_{\alpha \in \Delta^+} m_\alpha \alpha.$$

**Remark.** We remark here that if $\mu \in (i\mathfrak{t})^*$ with $\mu|_{(i\mathfrak{t}^\theta)} = 0$ (so $\mu \in (i\mathfrak{a})^*$), then

$$\langle \rho', \mu \rangle = \left\langle \rho + \left( \sum_{\alpha \in \Sigma^+, \alpha|_{i\mathfrak{a}} = 0} \alpha \right), \mu \right\rangle = \langle \rho, \mu \rangle. \quad (4.15)$$

In fact, we have $\rho'|_{(i\mathfrak{a})^*} = \rho. \#$

**4.6 The Heat Equation on $\mathcal{X} = U/K$**

In this section we consider again a compact Riemannian symmetric space $\mathcal{X}$ associated with the compact Riemannian symmetric pair $(U, K)$ and the involution $\theta$. We do not suppose that $U$ is a closed subgroup of some unitary group as we did in the last section. However, almost everything we defined in the previous section still makes sense for the general connected compact group $U$. A few parts of the previous section will be defined in more general settings. First, the complexification of $U_{\mathbb{C}}$. 

67
Let \( \iota : U \to U_\mathbb{C} \) be the universal complexification of \( U \), see Chapter 27 in [9]. This means that if \( L \) is a complex Lie group and \( f : U \to L \) is a Lie group homomorphism, then there is a unique holomorphic homomorphism \( f_\mathbb{C} : U_\mathbb{C} \to L \) such that

\[
f_\mathbb{C} \circ \iota = f.
\]

As \( U \) is compact, it follows that there exists a faithful representation \( \pi : U \to \text{GL}(N, \mathbb{C}) \) for some \( N \). Applying the above to \( \pi \), we conclude that \( \iota \) has to be injective. Thus, we can assume that \( U \) is a subgroup of \( U_\mathbb{C} \). Moreover, since \( U \) is compact, \( U \) is closed in \( U_\mathbb{C} \). Using the universal property of \( U_\mathbb{C} \), we have the following result.

**Proposition 4.25.** Let \( \mu \in P^+ \). Then the function \( \varphi^j_\mu \) extends to a holomorphic function \( \tilde{\varphi}^j_\mu \) on \( U_\mathbb{C} \). The extension is given by

\[
\tilde{\varphi}^j_\mu(g) = \langle e_\mu, (\pi_\mu)_\mathbb{C}(g)e_\mu \rangle, \quad g \in U_\mathbb{C},
\]

where we denote \((\pi_\mu)_\mathbb{C}\) by \( \tilde{\pi}_\mu \). Moreover, \( \tilde{\varphi}^j_\mu \) is \( K_\mathbb{C} \)-invariant, and hence we can consider \( \tilde{\varphi}^j_\mu \) as a function on \( \mathcal{X}_\mathbb{C} = U_\mathbb{C}/K_\mathbb{C} \).

**Proof.** For the second statement, see the proof of Theorem 5.10. \( \square \)

The involution \( \theta \) has the holomorphic extension to \( U_\mathbb{C} \), which we also denote by \( \theta \). On the other hand, consider the complex conjugation on \( u_\mathbb{C} \):

\[
X + iY \mapsto X - iY, \quad X, Y \in u.
\]

Then the decomposition

\[
u_\mathbb{C} = u \oplus iu
\]

is a Cartan decomposition with respect to the complex conjugation on \( u_\mathbb{C} \) and hence we obtain the global Cartan decomposition

\[
U_\mathbb{C} = U \exp iu.
\]

Thus, for \( g \in U_\mathbb{C} \), there exist a unique \( u \in U \) and a unique \( X \in u \) such that

\[
g = u \exp(iX).
\]

In fact, \((U_\mathbb{C}, U)\) is a noncompact Riemannain symmetric pair with respect to the antiholomorphic involution \( \sigma \) whose fixed point set is \( U \) and whose differential at \( e \) is the complex conjugation. We also obtain the noncompact dual \((G, K)\), with respect to the involution \( \theta|_G \), and its Iwasawa decomposition as in the previous section. The results thereafter in the previous section will be available also.
Proposition 4.26. Let $\mu \in P^+$. Then for $g \in U_C$,

$$\tilde{\pi}_\mu(g)^* = \tilde{\pi}_\mu(g^*),$$

where we set $g^* = \sigma(g)^{-1}$.

Proof. See the proof of Lemma A.2.3 in [34].

Next, we define an action of $u$ on the space $C^\infty(U)$ of smooth functions on $U$ by

$$(X \cdot f)(u) = \frac{d}{dt} f(u \exp tX) \bigg|_{t=0}$$

for $X \in u$ and $f \in C^\infty(U)$. It is a representation of $u$ on $C^\infty(U)$. In fact, it is the derived representation of the right regular representation of $U$ on $C^\infty(U)$. This representation of $u$ on $C^\infty(U)$ can be extended to a representation of the universal enveloping algebra $U(u)$ on $C^\infty(U)$.

We define the Ad($U$)-invariant inner product on $u$ by

$$\langle Z_1 + X_1, Z_2 + X_2 \rangle := \langle Z_1, Z_2 \rangle_0 - B(X_1, X_2)$$

for $Z_1, Z_2 \in z(u)$, $X_1, X_2 \in u'$ in the decomposition $u = z(u) \oplus u'$ where $z(u)$ is the center of $u$ and $u'$ is the ideal of $u$ spanned by $[u, u]$ ($u$ is reductive since $u$ is compact, see Theorem 5.18 in [51]); here $\langle \cdot, \cdot \rangle_0$ is a fixed inner product on $z(u)$ and $B(\cdot, \cdot)$ is the Killing form which is negative definite on the compact semisimple Lie algebra $u'$. (We note that if we assume that $U$ is a closed subgroup of $GL(N, \mathbb{C})$, then the inner product we defined on $u \subseteq u(N)$ in the last section:

$$\langle X, Y \rangle = \text{Tr}(XY^*) = -\text{Tr}(XY),$$

is invariant under the adjoint representation of $U$.)

Let $\{X_j\}_{j=1}^n$ be an orthonormal basis of $u$. Then the Casimir operator of $U$ associated with the inner product $\langle \cdot, \cdot \rangle$ is defined by

$$C_U = \sum_{j=1}^n X_j^2.$$

The Casimir element $\Omega_u$ is independent of the choice of the chosen orthonormal basis $\{X_j\}_{j=1}^n$ and is in the center of the universal enveloping algebra $U(u)$. Recall that we have an orthogonal decomposition $u = \mathfrak{k} \oplus \mathfrak{q}$. Since $\langle \cdot, \cdot \rangle$ is Ad($U$)-invariant and Ad($k$)$\mathfrak{q} = \mathfrak{q}$, we can use $\langle \cdot, \cdot \rangle$ as an inner product on $\mathfrak{q}$ and it is Ad($K$)-invariant. Let $\{Y_j\}$ be an orthonormal basis of $\mathfrak{q}$. We then define the Casimir operator of $U/K$ associated with the inner product $\langle \cdot, \cdot \rangle$ by

$$C_{U/K} = \sum_j Y_j^2.$$
Let $g$ be the $U$-invariant Riemannian metric of $U/K$ defined by $\langle \cdot, \cdot \rangle$. Such $g$ exists because $\text{Ad}(K)$ is compact. Then the Laplace-Beltrami operator $\Delta_{U/K}$ on $U/K$ defined by $\langle \cdot, \cdot \rangle$ is identical with the Casimir operator $C_{U/K}$, see Theorem 5.2 in [60]. Moreover, if we identify the space of smooth functions $C^\infty(U/K)$ by $C^\infty(U)^K$, the space of right-$K$-invariant smooth functions on $U$ (which we always do), then $C_{U/K} = C_{U|C^\infty(U)^K}$.

Therefore, the Cauchy problem of the heat equation on $X = U/K$:

$$\Delta_X a(x,t) = \frac{\partial a}{\partial t}(x,t), \quad (x,t) \in X \times (0, \infty)$$

$$\lim_{t \to 0^+} a(x,t) = f(x), \quad f \in L^2(X) \quad \text{(the initial condition)}$$

is the same as the equation:

$$C_{U/K} a(x,t) = \frac{\partial a}{\partial t}(x,t), \quad (x,t) \in U/K \times (0, \infty)$$

$$\lim_{t \to 0^+} a(x,t) = f(x), \quad f \in L^2(U/K) \quad \text{(the initial condition)}$$

which then can be reduced to the equation:

$$C_U a(u,t) = \frac{\partial a}{\partial t}(u,t), \quad (u,t) \in U \times (0, \infty)$$

$$\lim_{t \to 0^+} a(x,t) = f(u), \quad f \in L^2(U)^K \quad \text{(the initial condition)}$$

where the function $a(u,t)$ is right-$K$-invariant in the first variable.

Before solving the above Cauchy problem, we present the following lemma.

**Lemma 4.27.** Let $\mu \in D^+$. Then we have

$$C_U(\varphi^{ij}_\mu) = -\langle \mu + 2\rho', \mu \rangle \varphi^{ij}_\mu.$$  

That is, the matrix coefficients $\varphi^{ij}_\mu$ are eigenfunctions for the Casimir operator $C_U$. If $f \in C^2(U)$, then we have

$$\pi_\mu(C_U f) = -\langle \mu + 2\rho', \mu \rangle \pi_\mu(f).$$

**Proof.** See the Proof of Lemmas 1.1 and 1.2 in [57].

**Corollary 4.28.** Let $\mu \in P^+$. then

$$C_U(\varphi^i_\mu) = -\langle \mu + 2\rho, \mu \rangle \varphi^i_\mu.$$  

If $f \in C^2(U)^K$, then we have

$$\pi_\mu(C_U f) = -\langle \mu + 2\rho, \mu \rangle \pi_\mu(f).$$
Proof. This is an immediate consequence of the previous theorem and the remark at the end of the previous section.

To solve the heat equation, we formally apply the $\mu$-th Fourier coefficient, $\mu \in P^+$, to (4.16) to get (by using the preceding lemma)

$$- \langle \mu + 2\rho, \mu \rangle \pi_{\mu} (a (\cdot, t)) = \pi_{\mu} (C_U (a (\cdot, t))) = \pi_{\mu} (\partial_t a (\cdot, t))$$

and

$$\pi_{\mu} (a (\cdot, 0)) = \pi_{\mu} (f).$$

In particular, (recall the Equation (4.12))

$$- \langle \mu + 2\rho, \mu \rangle \pi_{\mu} (\widehat{a} (\cdot, t)_{j} (\mu)) = - \langle \mu + 2\rho, \mu \rangle \pi_{\mu} (a (\cdot, t))_{j}$$

$$= \pi_{\mu} (\partial_t a (\cdot, t))_{j}$$

$$= \partial_t \left( \pi_{\mu} (a (\cdot, t))_{j} (\mu) \right)$$

$$= \partial_t \left( \widehat{a} (\cdot, t)_{j} (\mu) \right)$$

and

$$\widehat{a} (\cdot, 0)_{j} (\mu) = \pi_{\mu} (a (\cdot, 0))_{j} = \pi_{\mu} (f)_{j} = \hat{f}_{j} (\mu).$$

So we have a differential equation in $t$ variable with an initial condition. The solution of this differential equation is

$$\widehat{a} (\cdot, t)_{j} (\mu) = \hat{f}_{j} (\mu) \ e^{-t(\mu+2\rho, \mu)}. $$
Thus, we obtain the Fourier series expansion of $a(u, t)$ as

$$a(u, t) = \sum_{\mu \in P^+} d(\mu) \sum_{j=1}^{d(\mu)} e^{-t(\mu+2\rho, \mu)} \hat{f}_j(\mu) \varphi^j_{\mu}(u)$$

$$= \sum_{\mu \in P^+} d(\mu) e^{-t(\mu+2\rho, \mu)} \sum_{j=1}^{d(\mu)} \hat{f}_j(\mu) \varphi^j_{\mu}(u)$$

$$= \sum_{\mu \in P^+} d(\mu) e^{-t(\mu+2\rho, \mu)} \sum_{j=1}^{d(\mu)} \left( \int_U f(g) \overline{\varphi^j_{\mu}(g)} dg \right) \varphi^j_{\mu}(u)$$

$$= \sum_{\mu \in P^+} d(\mu) e^{-t(\mu+2\rho, \mu)} \int_U f(g) \sum_{j=1}^{d(\mu)} \langle \pi_{\mu}(g) e_{\mu}, e_j \rangle \langle e_j, \pi_{\mu}(u) e_{\mu} \rangle dg$$

$$= \sum_{\mu \in P^+} d(\mu) e^{-t(\mu+2\rho, \mu)} \int_U f(g) \langle e_{\mu}, \pi_{\mu}(g^{-1}u) e_{\mu} \rangle dg$$

$$= \sum_{\mu \in P^+} d(\mu) e^{-t(\mu+2\rho, \mu)} \int_U f(g) \varphi_{\mu}(g^{-1}u) \, dg$$

$$= \int_U f(g) \left( \sum_{\mu \in P^+} d(\mu) e^{-t(\mu+2\rho, \mu)} \varphi_{\mu}(g^{-1}u) \right) \, dg$$

where we formally interchange the integral and the summation in the last equation.

Now, we define the heat kernel $h_t$ on $U/K$ by

$$h_t(x) = \sum_{\mu \in P^+} d(\mu) e^{-t(\mu+2\rho, \mu)} \varphi_{\mu}(u), \quad x = u \cdot o.$$ 

We note that we can view $h_t$ as a $K$-invariant function on $U$. We formally write $a(u, t)$ as the convolution $f \ast h_t$. We show that the convolution $f \ast h_t$ is a solution of the above Cauchy problem of the heat equation. First, we note that for $\mu \in P^+$

$$|\varphi_{\mu}(u)| = |\langle e_{\mu}, \pi_{\mu}(u) e_{\mu} \rangle| \leq \|e_{\mu}\| \|\pi_{\mu}(u)\| e_{\mu} \| \leq 1.$$ 

Since $\sum_{\mu \in P^+} d(\mu) e^{-t(\mu+2\rho, \mu)} < \infty$ (recall that by the Weyl’s dimension formula, there are positive constants $m$ and $c$ such that $d(\mu) \leq c(\|\mu\| + \|\rho\|)^m$), the sum

$$\sum_{\mu \in P^+} d(\mu) e^{-t(\mu+2\rho, \mu)} \varphi_{\mu}(u)$$

72
converges uniformly on $U$ by the Weierstrass $M$-test and hence $h_t$ is continuous on $U$. Note that according to the formula of $h_t$, we see that for $\mu \in P^+$,
\[ \hat{h}_t(\mu) = e^{-t(\langle \mu + 2\rho, \mu \rangle)}. \]

**Lemma 4.29.** Let $f$ be a continuous function on $\mathcal{X}$. Then $f$ is smooth on $\mathcal{X}$ if and only if the Fourier coefficients $\pi_\mu(f)$ is rapidly decreasing, that is,
\[ \lim_{|\mu| \to \infty} \| \mu \|^k \| \pi_\mu(f) \| = 0 \]
for every non negative integer $k$. In this case, the Fourier series of $f$ converges to $f$ absolutely and uniformly on $U$.

**Proof.** See the proof of Theorem 1 and Theorem 3 in [57].

**Proposition 4.30.** The heat kernel $h_t$ is smooth on $U$ and satisfies the heat equation, i.e. $C_U(h_t) = \partial_t(h_t)$.

**Proof.** Let $\{e_1 := e_\mu, e_\mu, \ldots, e_{d(\mu)}\}$ be an orthonormal basis of $V_\mu$ such that $\{e_\mu\}$ is an orthonormal basis for $V^K_\mu$. Since $h_t$ is $K$-biinvariant, by (4.6) and (4.9),
\[ \| \pi_\mu(h_t) \|^2 = \sum_{j=1}^{d(\mu)} \| \pi_\mu(h_t)e_j \|^2 = \| \hat{h}_t(\mu)e_\mu \|^2 = \| \hat{h}_t(\mu) \|^2 = e^{-2t(\langle \mu + 2\rho, \mu \rangle)}. \]

Therefore, $h_t$ is smooth by the previous lemma (see the last part in the proof of the next proposition).

By Corollary (4.28), we see that
\[ \sum_{\mu \in P^+} d(\mu)e^{-t(\langle \mu + 2\rho, \mu \rangle)}C_\mu(\varphi_\mu(u)) = \sum_{\mu \in P^+} d(\mu)e^{-t(\langle \mu + 2\rho, \mu \rangle)}(-\langle \mu + 2\rho, \mu \rangle)\varphi_\mu(u). \]

The sum converges uniformly on $U$ since $|\varphi_\mu(u)| \leq 1$ for all $u \in U$ and
\[ \sum_{\mu \in P^+} d(\mu)e^{-t(\langle \mu + 2\rho, \mu \rangle)}\langle \mu + 2\rho, \mu \rangle \leq \sum_{\mu \in P^+} d(\mu)2|\rho||\mu|^2e^{-t||\mu||^2} < \infty, \]
(also see the last part in the proof of the next proposition). We see that
\[ \partial_t(e^{-t(\langle \mu + 2\rho, \mu \rangle)}) = (-\langle \mu + 2\rho, \mu \rangle)e^{-t(\langle \mu + 2\rho, \mu \rangle)}. \]

Since for every $0 < t_0 < t_1 < \infty$, $|e^{-t(\langle \mu + 2\rho, \mu \rangle)}| \leq e^{-t_0||\mu||^2}$ for all $t \in [t_0, t_1]$, as above the sum
\[ \sum_{\mu \in P^+} d(\mu)(-\langle \mu + 2\rho, \mu \rangle)e^{-t(\langle \mu + 2\rho, \mu \rangle)}\varphi_\mu(u) \]

73
converges uniformly on compact subsets of \( \{ t : t > 0 \} \). Finally, we have

\[
C_U(h_t(u)) = C_U \left( \sum_{\mu \in P^+} d(\mu) e^{-t(\mu + 2\rho, \mu)} \varphi_\mu(u) \right) \\
= \sum_{\mu \in P^+} d(\mu) e^{-t(\mu + 2\rho, \mu)} C_\mu(\varphi_\mu(u)) \\
= \sum_{\mu \in P^+} d(\mu) e^{-t(\mu + 2\rho, \mu)} (-\langle \mu + 2\rho, \mu \rangle) \varphi_\mu(u) \\
= \sum_{\mu \in P^+} d(\mu) \varphi_\mu(u) \partial_t(e^{-t(\mu + 2\rho, \mu)}) \\
= \partial_t \left( \sum_{\mu \in P^+} d(\mu) e^{-t(\mu + 2\rho, \mu)} \varphi_\mu(u) \right) \\
= \partial_t(h_t(u)).
\]

\[\square\]

**Proposition 4.31.** For every \( f \in L^2(\mathcal{S}) \), the function \( f * h_t \) is smooth on \( \mathcal{S} \).

**Proof.** We use the previous lemma. Let \( \mu \in P^+ \) and \( \{ e_1 := e_\mu, e_\mu, ..., e_{d(\mu)} \} \) be an orthonormal basis of \( V_\mu \) such that \( \{ e_\mu \} \) is an orthonormal basis for \( V^K_\mu \). First, we compute \( \| \pi_\mu(f * h_t) \| \). We see that

\[
\| \pi_\mu(f * h_t) \|^2 = \sum_{j=1}^{d(\mu)} \| \pi_\mu(f * h_t)e_j \|^2 \\
= \sum_{j=1}^{d(\mu)} \| \pi_\mu(f)\pi_\mu(h_t)e_j \|^2 \\
= \| \pi_\mu(f)\pi_\mu(h_t)e_\mu \|^2 \\
= \| \pi_\mu(f)\hat{h}_t(\mu)e_\mu \|^2 \\
= e^{-2t(\mu + 2\rho, \mu)} \| \pi_\mu(f)e_\mu \|^2 \\
\leq e^{-2t(\mu + 2\rho, \mu)} \| f \|_2^2.
\]

The last inequality holds because \( \| f \|_2 = \sum_{\mu} d(\mu) \| \pi_\mu(f)e_\mu \|^2 \). Now we have for \( k \in \mathbb{Z}^+ \),

\[
\| \mu \|^k \| \pi_\mu(f * h_t) \|^2 \leq \| f \|_2 \left( \| \mu \|^k e^{-t(\mu + 2\rho, \mu)} \right) \\
= \| f \|_2 \left( \| \mu \|^k e^{-t(\mu, \mu) - 2t(\rho, \mu)} \right) \\
\leq \| f \|_2 \left( \| \mu \|^k e^{-2t|\mu|^2} \cdot e^{-2t(\rho, \mu)} \right) \\
\leq \| f \|_2 \left( \| \mu \|^k e^{-t|\mu|^2} \right) \\
\rightarrow 0,
\]

74
as $\|\mu\| \to \infty$. The last inequality holds because the condition (see Proposition 4.14 in [37])

$$\frac{2\langle \mu, \rho \rangle}{\|\rho\|^2} \in \mathbb{Z}^+$$

implies that $-2t\langle \rho, \mu \rangle \leq 0$. Hence, Lemma 4.29, $f \ast h_t$ is smooth on $\mathcal{X}^\ast$. □

**Proposition 4.32.** For every $f \in L^2(\mathcal{X}^\ast)$, we have

$$(f \ast h_t)(x) = \sum_{\mu \in P^+} d(\mu) e^{-t\langle \mu+2\rho, \mu \rangle} \sum_{j=1}^{d(\mu)} \hat{f}_j(\mu) \varphi^j_\mu(x)$$

where the sum converges uniformly on $\mathcal{X}^\ast$.

*Proof.* Let $f \in L^2(\mathcal{X}^\ast)$. Then $f \in L^1(\mathcal{X}^\ast)$. Since for $\mu \in P^+$,

$$\pi_\mu(f \ast h_t) = \pi_\mu(f) \pi_\mu(h_t) \quad \text{and} \quad \pi_\mu(h_t)e_\mu = \hat{h}_t(\mu)e_\mu = e^{-t\langle \mu+2\rho, \mu \rangle}e_\mu,$$

we can write $f \ast h_t$ in terms of its the Fourier series as

$$(f \ast h_t)(x) = \sum_{\mu \in P^+} d(\mu) \langle \pi_\mu(u^{-1}) \pi_\mu(f \ast h_t)e_\mu, e_\mu \rangle$$

$$= \sum_{\mu \in P^+} d(\mu) \langle \pi_\mu(u^{-1}) \pi_\mu(f)e_\mu, e_\mu \rangle$$

$$= \sum_{\mu \in P^+} d(\mu) e^{-t\langle \mu+2\rho, \mu \rangle} \langle \pi_\mu(u^{-1}) \pi_\mu(f)e_\mu, e_\mu \rangle$$

$$= \sum_{\mu \in P^+} d(\mu) e^{-t\langle \mu+2\rho, \mu \rangle} \langle \pi_\mu(u^{-1}) \pi_\mu(f)e_\mu, e_\mu \rangle$$

$$= \sum_{\mu \in P^+} d(\mu) e^{-t\langle \mu+2\rho, \mu \rangle} \sum_{j=1}^{d(\mu)} \hat{f}_j(\mu) \varphi^j_\mu(x),$$

where $x = u \cdot o \in \mathcal{X}^\ast$. By the previous proposition, we know that $f \ast h_t$ is smooth on $\mathcal{X}^\ast$. Therefore, by Theorem 4.29, the Fourier series of $f \ast h_t$ converges uniformly to $f \ast h_t$ on $\mathcal{X}^\ast$. This completes the proof. □

**Proposition 4.33.** Let $f \in L^2(\mathcal{X}^\ast)$. Then $f \ast h_t$ solves the Cauchy problem of the heat equation, i.e. $C_U (f \ast h_t) = \partial_t (f \ast h_t)$ with $\lim_{t \to 0^+} (f \ast h_t) = f$.

*Proof.* By the preceding proposition, we have

$$(f \ast h_t)(x) = \sum_{\mu \in P^+} d(\mu) e^{-t\langle \mu+2\rho, \mu \rangle} \sum_{j=1}^{d(\mu)} \hat{f}_j(\mu) \varphi^j_\mu(x)$$

75
where the sum converges uniformly on \( \mathcal{X} \). Note that for \( \mu \in \mathcal{P}^+ \),

\[
\left| \frac{d(\mu)}{d(\mu)} \sum_{j=1}^{d(\mu)} \hat{f}_j(\mu) \varphi^j_\mu(x) \right| \leq \sum_{j=1}^{d(\mu)} |\hat{f}_j(\mu)| ||\varphi^j_\mu(x)|| \\
\leq \sum_{j=1}^{d(\mu)} \int_U |f(u)||\varphi^j_\mu(u)| \, du \\
\leq \sum_{j=1}^{d(\mu)} \int_U |f(u)| \, dx \\
\leq d(\mu) \|f\|_2^2.
\]

Therefore, by using the same estimates as in the proof of Proposition 4.30, we have

\[
C_U((f * h_t)(x)) = C_U \left( \sum_{\mu \in \mathcal{P}^+} d(\mu) e^{-t(\mu+2\rho, \mu)} \sum_{j=1}^{d(\mu)} \hat{f}_j(\mu) \varphi^j_\mu(x) \right) \\
= \sum_{\mu \in \mathcal{P}^+} d(\mu) e^{-t(\mu+2\rho, \mu)} \sum_{j=1}^{d(\mu)} \hat{f}_j(\mu) C_U(\varphi^j_\mu(x)) \\
= \sum_{\mu \in \mathcal{P}^+} d(\mu) e^{-t(\mu+2\rho, \mu)} (\mu + 2\rho) \sum_{j=1}^{d(\mu)} \hat{f}_j(\mu) \varphi^j_\mu(x) \\
= \sum_{\mu \in \mathcal{P}^+} d(\mu) \partial_t (e^{-t(\mu+2\rho, \mu)} \sum_{j=1}^{d(\mu)} \hat{f}_j(\mu) \varphi^j_\mu(x)) \\
= \partial_t \left( \sum_{\mu \in \mathcal{P}^+} d(\mu) e^{-t(\mu+2\rho, \mu)} \sum_{j=1}^{d(\mu)} \hat{f}_j(\mu) \varphi^j_\mu(x) \right) \\
= \partial_t ((f * h_t)(x))
\]
\[
\lim_{t \to 0^+} (f * h_t)(x) = \lim_{t \to 0^+} \sum_{\mu \in P^+} d(\mu) e^{-t(\mu + 2\rho, \mu)} \sum_{j=1}^{d(\mu)} \hat{f}_j(\mu) \varphi^j_{\mu}(x) \\
= \sum_{\mu \in P^+} d(\mu) \lim_{t \to 0^+} (e^{-t(\mu + 2\rho, \mu)} \sum_{j=1}^{d(\mu)} \hat{f}_j(\mu) \varphi^j_{\mu}(x) \\
= \sum_{\mu \in P^+} d(\mu) e^{-t(\mu + 2\rho, \mu)} \sum_{j=1}^{d(\mu)} \hat{f}_j(\mu) \varphi^j_{\mu}(x) \\
= f(x). 
\]

\[\square\]

### 4.7 The Fock Space \( \mathcal{H}_t(\mathfrak{K}_\mathbb{C}) \)

The discussion of the heat kernel on the noncompact symmetric space can be found in [23] and [2]. Let \( h^1_t \) be the heat kernel associated to the Laplace-Beltrami operator \( \Delta_G \) on the noncompact symmetric space \( \mathfrak{Y} = G/K \). Then \( h^1_t \) is given by (cf. (2.4))

\[
h^1_t(g) = \int_{(ia)^\ast} e^{-t(||\mu||^2 + ||\rho||^2)} \psi_{\mu}(g) \frac{d\mu}{c(\mu)^2}, \quad (4.17)
\]

where \( c(\cdot) \) is the Harish-Chandra \( c \)-function for \( G/K \) and \( \psi_{\mu} \) is the spherical function of the pair \((G, K)\) defined by

\[
\psi_{\mu}(g) = \int_K e^{\langle \mu - \rho, \mathfrak{X}(g^{-1}k) \rangle} dk
\]

for \( \mu \in a^\ast_\mathbb{C} \). We recall the following well known fact.

**Lemma 4.34.** Let \( \mu \in P^+ \). Then we have

\[
\varphi_{\mu}|_G = \psi_{-i(\mu + \rho)}. 
\]

**Proof.** See the proof of Lemma 2.5 in [8]. \[\square\]

Note that

\[
\int_{\mathfrak{Y}} h^1_t(x) \psi_{-\mu}(x) dm_1(x) = e^{-t(||\mu||^2 + ||\rho||^2)} , \ \mu \in (ia)^\ast. 
\]
Recall that by Theorem 4.22, we have the integration formula:
\[
\int_Y f(x) dm_1(x) = c_1 \int_K \int_{ia} f(k \exp H \cdot o) J_1(H) dH dk .
\]

Now we suppose that the measures \( m \) and \( m_1 \) are normalized so that \( c = |w| c_1 \). Therefore,
\[
c \frac{1}{|W|} \int_{ia} h_1^1(\exp H) \psi_{-\mu}(\exp H) J_1(H) dH = e^{-t(\|\mu\|^2 + \|\rho\|^2)} , \quad \mu \in (ia)^*.
\]

Note that the left-hand side is a holomorphic function of \( \mu \) and the right-hand side also admits a holomorphic extension as a function of \( \mu \in a^* \subset \mathbb{C} \), the extension being given by \( e^{-t(\sum \nu_j^2 + \|\rho\|^2)} \). Thus, the above equation holds for all \( \mu \in a^* \).

\[
(4.18)
\]

For \( z = u \exp H \cdot o \in \mathcal{X}_C \) (\( u \in U, H \in ia \)), we define
\[
p_t(z) = 2^r h_2^1(\exp(2H) \cdot o),
\]
where \( r = \dim a \).

We define the Fock space \( \mathcal{H}_t(\mathcal{X}_C) \) as follows:
\[
\mathcal{H}_t(\mathcal{X}_C) = \left\{ F \in \mathcal{O}(\mathbb{C}) : \|F\|^2_{\mathcal{H}_t} := \int_{\mathcal{X}_C} |F(z)|^2 p_t(z) dz < \infty \right\} .
\]

That is \( \mathcal{H}_t(\mathcal{X}_C) = \mathcal{O}(\mathbb{C}) \cap L^2(\mathcal{X}_C, p_t(z) dz) \). Note that the measure \( dz \) on \( \mathcal{X}_C = U_C/K_C \) is the push forward measure of a left Harr measure on the (connected) complex Lie group (hence a Lie group) \( U_C \) which, in each chart, has a continuous strictly positive density with respect to a Lebesgue measure. Therefore, the measure \( p_t(z) dz \) is also a continuous strictly positive measure on \( \mathcal{X}_C \). By Proposition 2.3, \( \mathcal{H}_t(\mathcal{X}_C) \) is a Hilbert spaces of holomorphic functions.

**Proposition 4.35.** The left regular representation of \( U \) on \( L^2(\mathcal{X}_C, p_t(z) dz) \) which is defined by
\[
(u \cdot F)(z) = F(u^{-1} \cdot z) \quad \text{for} \quad F \in L^2(\mathcal{X}_C, p_t(z) dz), \quad z \in \mathcal{X}_C
\]
is a unitary representation. Moreover, \( \mathcal{H}_t(\mathcal{X}_C) \) is a \( U \)-invariant closed subspace of \( L^2(\mathcal{X}_C, p_t(z) dz) \) and hence the left regular representation of \( U \) on \( \mathcal{H}_t(\mathcal{X}_C) \) is a unitary representation.
Lemma 4.36. For all \( \mu \in \mathcal{L} \) let

\[
\|u \cdot F\|_{\mathcal{H}_t}^2 = \int_{\mathcal{X}_c} |(u \cdot F)(z)|^2 p_t(z) dz
\]

\[
= \int_{\mathcal{X}_c} |F(u^{-1} \cdot z)|^2 p_t(z) dz
\]

\[
= c \int_U \int_{(i)a_+} |F(u^{-1} \cdot u \exp H \cdot o)|^2 2^r h_{2t}(\exp(2H) \cdot o) J(H) dH \, du
\]

\[
= c \int_U \int_{(i)a_+} |F(u \exp H \cdot o)|^2 2^r h_{2t}(\exp(2H) \cdot o) J(H) dH \, du
\]

\[
= \int_{\mathcal{X}_c} |F(z)|^2 p_t(z) dz
\]

\[
= \|F\|_{\mathcal{H}_t}.
\]

Therefore, \( U \) acts unitarily on \( L^2(\mathcal{X}_c, p_t(z) dz) \). Note that if \( F \in \mathcal{O}(\mathbb{C}) \), then \( u \cdot F \in \mathcal{O}(\mathbb{C}) \) for every \( u \in U \). So \( \mathcal{H}_t(\mathcal{X}_c) \) is a \( U \)-subrepresentation of \( L^2(\mathcal{X}_c, p_t(z) dz) \). Thus, the left regular representation of \( U \) on \( \mathcal{H}_t(\mathcal{X}_c) \) is a unitary representation.

\[\square\]

4.8 The Segal-Bargmann Transform on \( L^2(\mathcal{X}_c) \)

Lemma 4.36. For \( H \in ia \), \( (\exp H)^* = \exp H \).

Proof. Let \( H \in ia \). Then \( H = iS \) for some \( S \in a \). Since \( d\sigma \) is complex antilinear, \( d\sigma(H) = d\sigma(iS) = -i d\sigma(S) = -i S = -H \). Therefore,

\[
(\exp H)^* = \sigma(\exp H)^{-1} = \exp(d\sigma(H))^{-1} = \exp(-H)^{-1} = \exp(H).
\]

\[\square\]

Lemma 4.37. The holomorphic matrix coefficients \( \tilde{\varphi}^j_{\mu} \) are in the space \( \mathcal{H}_t(\mathcal{X}_c) \) for all \( \mu \in P^+, \ j = 1, \ldots, d(\mu) \) with the norms

\[
\|\tilde{\varphi}^j_{\mu}\|_{\mathcal{H}_t} = \frac{e^{t(\mu+2p_1, \mu)}}{\sqrt{d(\mu)}}.
\]

Proof. Let \( \mu \in P^+ \) and \( j \in \{1, \ldots, d(\mu)\} \). Then by Theorem 4.21,

\[
\|\tilde{\varphi}^j_{\mu}\|_{\mathcal{H}_t}^2 = \int_{\mathcal{X}_c} |\tilde{\varphi}^j_{\mu}(z)|^2 p_t(z) dz
\]

\[
= c \int_U \int_{(i)a_+} |\tilde{\varphi}^j_{\mu}(u \exp H \cdot o)|^2 2^r h_{2t}(\exp(2H) \cdot o) J(H) dH \, du
\]

\[
= c 2^r \int_{(i)a_+} h_{2t}(\exp(2H) \cdot o) \left( \int_U |\tilde{\varphi}^j_{\mu}(u \exp H \cdot o)|^2 du \right) J(H) dH.
\]

79
For $H \in (i\mathbf{a})_+$, we have

$$\int_U |\tilde{\varphi}_\mu(u \exp H \cdot o)|^2 du = \int_U |\langle v^j_{i\mu}, \tilde{\pi}_\mu(u \exp H) v_\mu \rangle|^2 du$$

$$= \int_U |\langle \pi_\mu(u^{-1}) v^j_{i\mu}, \tilde{\pi}_\mu(\exp H) v_\mu \rangle|^2 du$$

$$= \int_U |\langle \pi_\mu(u) v^j_{i\mu}, \tilde{\pi}_\mu(\exp H) v_\mu \rangle|^2 du$$

$$= \frac{1}{d(\mu)} \| v^j_{i\mu} \| \| \pi_\mu(\exp H) v_\mu \|^2$$

$$= \frac{1}{d(\mu)} \langle \pi_\mu(\exp H) v_\mu, \tilde{\pi}_\mu(\exp H) v_\mu \rangle$$

$$= \frac{1}{d(\mu)} \langle v_\mu, \tilde{\pi}_\mu((\exp H)^*) \pi_\mu(\exp H) v_\mu \rangle$$

$$= \frac{1}{d(\mu)} \langle v_\mu, \tilde{\pi}_\mu(\exp H \cdot \exp H) v_\mu \rangle$$

$$= \frac{1}{d(\mu)} \langle v_\mu, \tilde{\pi}_\mu(\exp 2H) v_\mu \rangle$$

$$= \frac{1}{d(\mu)} \langle v_\mu, \tilde{\pi}_\mu(\exp(2H) \cdot o) \rangle$$

where we use Lemma 4.36 in the eighth line. Therefore, we have

$$\| \tilde{\varphi}_\mu \|^2_{Ht} = \frac{c^2}{d(\mu)} \int_{(i\mathbf{a})_+} h_{2t}(\exp(2H) \cdot o) \tilde{\varphi}_\mu(\exp(2H) \cdot o) J(H) dH$$

$$= \frac{c^2}{d(\mu)|W|} \int_{i\mathbf{a}} h_{2t}(\exp(2H) \cdot o) \tilde{\varphi}_\mu(\exp(2H) \cdot o) J(H) dH$$

$$= \frac{c}{d(\mu)|W|} \int_{i\mathbf{a}} h_{2t}(\exp H \cdot o) \tilde{\varphi}_\mu(\exp H \cdot o) J_1(H) dH$$

$$= \frac{c}{d(\mu)|W|} \int_{i\mathbf{a}} h_{2t}(\exp H \cdot o) \psi_{-i(\mu+\rho)}(\exp H \cdot o) J_1(H) dH$$

where we use the fact that $J(H/2) = J_1(H)$ and we apply Lemma 4.34 to the last equation. Finally, using (4.18), we obtain

$$\| \tilde{\varphi}_\mu \|^2_{Ht} = \frac{c}{d(\mu)|W|} \int_{i\mathbf{a}} h_{2t}(\exp H \cdot o) \psi_{-i(\mu+\rho)}(\exp H \cdot o) J_1(H) dH$$

$$= \frac{1}{d(\mu)} e^{2t(\mu+2\rho) \cdot \mu}.$$

$\square$
Lemma 4.38 (Orthogonality Relations). We have the following orthogonality relations.

(1) If $\mu \in P^+$, then
\[
\langle \varphi^j_{\mu}, \varphi^k_{\mu} \rangle_{H_t} = 0
\]
for all $j \neq k \in \{1, \ldots, d(\mu)\}$.

(2) If $\mu \neq \delta \in P^+$, then
\[
\langle \varphi^j_{\mu}, \varphi^k_{\delta} \rangle_{H_t} = 0
\]
for all $j \in \{1, \ldots, d(\mu)\}$ and $k \in \{1, \ldots, d(\delta)\}$.

Proof. We first prove part (a). Let $\mu \in P^+$ and $j \neq k \in \{1, \ldots, d(\mu)\}$. Then
\[
\langle \varphi^j_{\mu}, \varphi^k_{\mu} \rangle_{H_t} = \int_{\mathbb{R}_C} \overline{\varphi^j_{\mu}(z)} \varphi^k_{\mu}(z) p_t(z) dz
\]
\[
= c \int_U \int_{(i_a)^+} \overline{\varphi^j_{\mu}(u \exp H)} \varphi^k_{\mu}(u \exp H) 2^r h_{2t}^1(\exp(2H)) J(H) dH du
\]
\[
= c2^r \int_{(i_a)^+} h_{2t}^1(\exp(2H)) \left( \int_U \overline{\varphi^j_{\mu}(u \exp H)} \varphi^k_{\mu}(u \exp H) du \right) J(H) dH.
\]

For $H \in (i_a)^+$, if $f(H) = \int_U \overline{\varphi^j_{\mu}(u \exp H)} \varphi^k_{\mu}(u \exp H) du$, then we have
\[
f(H) = \int_U \langle \nu^j_{\mu}, \pi^k_{\mu}(u \exp H) \nu^k_{\mu}(u \exp H) \nu_{\mu} \rangle du
\]
\[
= \int_U \langle \pi^j_{\mu}(u^{-1}) \nu^j_{\mu}, \pi^k_{\mu}(u \exp H) \nu^k_{\mu} \rangle du
\]
\[
= \int_U \langle \pi^j_{\mu}(u) \nu^j_{\mu}, \overline{\pi^k_{\mu}(u \exp H) \nu^k_{\mu}} \rangle du
\]
\[
= \frac{1}{d(\mu)} \langle \nu^j_{\mu}, \nu^k_{\mu} \rangle 
\]
by Theorem 4.3. Therefore, $\langle \varphi^j_{\mu}, \varphi^k_{\mu} \rangle_{H_t} = 0$.

Now we prove part (b). Let $\mu \neq \delta \in P^+$, $j \in \{1, \ldots, d(\mu)\}$ and $k \in \{1, \ldots, d(\delta)\}$. Then $\pi^\mu$ and $\pi^\delta$ are not equivalent. We see that
\[
\langle \varphi^j_{\mu}, \varphi^k_{\delta} \rangle_{H_t} = \int_{\mathbb{R}_C} \overline{\varphi^j_{\mu}(z)} \varphi^k_{\delta}(z) p_t(z) dz
\]
\[
= c \int_U \int_{(i_a)^+} \overline{\varphi^j_{\mu}(u \exp H)} \varphi^k_{\delta}(u \exp H) 2^r h_{2t}^1(\exp(2H)) J(H) dH du
\]
\[
= c2^r \int_{(i_a)^+} h_{2t}^1(\exp(2H)) \left( \int_U \overline{\varphi^j_{\mu}(u \exp H)} \varphi^k_{\delta}(u \exp H) du \right) J(H) dH.
\]
For $H \in \mathcal{L}(\mathcal{X})$, if $g(H) = \int_U \overline{\varphi}_\mu^j(u \exp H) \overline{\varphi}_\delta^k(u \exp H) du$, then we have
\[
g(H) = \int_U \langle v_\mu^j, \overline{\pi}_\mu(u \exp H)v_\mu \rangle \langle v_\delta^k, \overline{\pi}_\delta(u \exp H)v_\delta \rangle du
\]
\[
= \int_U \langle \pi_\mu(u^{-1})v_\mu^j, \overline{\pi}_\mu(\exp H)v_\mu \rangle \langle \pi_\delta(u^{-1})v_\delta^k, \overline{\pi}_\delta(\exp H)v_\delta \rangle du
\]
\[
= \int_U \langle \pi_\mu(u)v_\mu^j, \overline{\pi}_\mu(\exp H)v_\mu \rangle \langle \pi_\delta(u)v_\delta^k, \overline{\pi}_\delta(\exp H)v_\delta \rangle du
\]
\[
= 0,
\]
by Schur’s orthogonality relations. Hence, $\langle \overline{\varphi}_\mu^j, \overline{\varphi}_\delta^k \rangle_{\mathcal{H}_t} = 0$. This completes the proof. 

**Theorem 4.39.** Let $f \in L^2(\mathcal{X})$. Then for $z \in \mathcal{X}$,
\[
\overline{f \ast h_t}(z) = \sum_{\mu \in \mathcal{P}^+} d(\mu) e^{-t(\mu + 2\rho, \mu)} \sum_{j=1}^{d(\mu)} \overline{\tilde{f}_j(\mu)} \overline{\varphi}_\mu^j(z)
\]
where the sum converges both in $\mathcal{H}_t(\mathcal{X})$ and uniformly on compact subsets of $\mathcal{X}$. Here, $\overline{f \ast h_t}$ is the holomorphic extension to $\mathcal{X}$ of $f \ast h_t$. Moreover, we have
\[
\| \overline{f \ast h_t} \|_{\mathcal{H}_t} = \| f \|_2.
\]

**Proof.** By the previous Lemma, all $\{\overline{\varphi}_\mu^j\}_\mu$ are orthogonal in $\mathcal{H}_t(\mathcal{X})$. Therefore by Lemma 4.37, we see that
\[
\sum_{\mu \in \mathcal{P}^+} \frac{d(\mu)^2}{e^{2t(\mu + 2\rho, \mu)}} \sum_{j=1}^{d(\mu)} \overline{\tilde{f}_j(\mu)}^2 \| \overline{\varphi}_\mu^j \|^2_{\mathcal{H}_t} = \sum_{\mu \in \mathcal{P}^+} \frac{d(\mu)^2}{e^{2t(\mu + 2\rho, \mu)}} \sum_{j=1}^{d(\mu)} \overline{\tilde{f}_j(\mu)}^2 \frac{e^{2t(\mu + 2\rho, \mu)}}{d(\mu)}
\]
\[
= \sum_{\mu \in \mathcal{P}^+} d(\mu) \sum_{j=1}^{d(\mu)} |\overline{\tilde{f}_j(\mu)}|^2
\]
\[
< \infty.
\]
Therefore, $\sum_{\mu \in \mathcal{P}^+} d(\mu) e^{-t(\mu + 2\rho, \mu)} \sum_{j=1}^{d(\mu)} \overline{\tilde{f}_j(\mu)} \overline{\varphi}_\mu^j$ converges to a holomorphic function $F$ in $\mathcal{H}_t(\mathcal{X})$.

Let $Q$ be any compact subset of $\mathcal{X}$. Let $\{\mu_1, \mu_2, ..., \mu_n, ...\} = \mathcal{P}^+$ be any order of the set $\mathcal{P}^+$ and let
\[
F_n(z) = \sum_{k=1}^{n} d(\mu_k) e^{-t(\mu_k + 2\rho, \mu_k)} \sum_{j=1}^{d(\mu_k)} \overline{\tilde{f}_j(\mu_k)} \overline{\varphi}_{\mu_k}^j(z),
\]
\[
82.
\]
for all \( n \in \mathbb{Z}^+ \) and \( z \in \mathcal{X}_\mathbb{C} \). Since \( \mathcal{H}_t(\mathcal{X}_\mathbb{C}) \) is a Hilbert space of holomorphic functions, there is a constant \( M_Q \) such that for all \( n \in \mathbb{Z}^+ \),

\[
\sup_{z \in K} |F_n(z) - F(z)| \leq M_Q \|F_n - F\|_{\mathcal{H}_t}.
\]

But \( F_n \) converges to \( F \) in \( \mathcal{H}_t(\mathcal{X}_\mathbb{C}) \),

\[
\lim_{n \to \infty} \sup_{z \in K} |F_n(z) - F(z)| = 0.
\]

Therefore, \( F_n \) converges to \( F \) uniformly on \( K \). Hence,

\[
F(z) = \sum_{\mu \in P^+} d(\mu) e^{-t(\mu + 2\rho, \mu)} \sum_{j=1}^{d(\mu)} \tilde{f}_j(\mu) \tilde{\varphi}_\mu^j(z)
\]

where the sum converges uniformly on compact subsets of \( \mathcal{X}_\mathbb{C} \). Recall that by Proposition 4.32, we have

\[
(f \ast h_t)(x) = \sum_{\mu \in P^+} d(\mu) e^{-t(\mu + 2\rho, \mu)} \sum_{j=1}^{d(\mu)} \tilde{f}_j(\mu) \varphi_\mu^j(x)
\]

pointwisely for all \( x \in \mathcal{X} \). Thus, \( F \) is the holomorphic extension of \( f \ast h_t \) to \( \mathcal{X}_\mathbb{C} \). It follows that

\[
\widehat{f} \ast h_t = F.
\]

The above calculation shows that \( \|f \ast h_t\|_{\mathcal{H}_t} = \|f\|_2 \). This completes the proof. \( \square \)

Now we define the Segal-Bargmann transform for \( \mathcal{X} \), \( H_t : L^2(\mathcal{X}) \longrightarrow \mathcal{H}_t(\mathcal{X}_\mathbb{C}) \) by

\[
(H_t f)(z) = \widehat{f} \ast h_t(z) = \sum_{\mu \in P^+} d(\mu) e^{-t(\mu + 2\rho, \mu)} \sum_{j=1}^{d(\mu)} \tilde{f}_j(\mu) \tilde{\varphi}_\mu^j(z),
\]

for \( f \in L^2(\mathcal{X}) \). Clearly, \( H_t \) is linear.

**Corollary 4.40.** The Segal-Bargmann transform \( H_t : L^2(\mathcal{X}) \longrightarrow \mathcal{H}_t(\mathcal{X}_\mathbb{C}) \) is an isometry.

**Proof.** This is an immediate consequence of the previous theorem. \( \square \)

**Theorem 4.41.** The Segal-Bargmann transform \( H_t : L^2(\mathcal{X}) \longrightarrow \mathcal{H}_t(\mathcal{X}_\mathbb{C}) \) is a unitary isomorphism.
Proof. By Corollary 4.40, we know that $H_t$ is an isometry. Therefore, it remains to show that $H_t$ is onto $H_t(\mathcal{X}_C)$. To prove this, we use the restriction principle. Let $R : H_t(\mathcal{X}_C) \to L^2(\mathcal{X})$ be the restriction map defined by

$$R(F) = F|_\mathcal{X}$$

for $F \in H_t(\mathcal{X}_C)$. Note that $R$ is defined everywhere (i.e., $R(H_t(\mathcal{X}_C)) \subseteq L^2(\mathcal{X})$) because $\mathcal{X}$ is compact. In fact, $R(H_t(\mathcal{X}_C)) \subseteq C(\mathcal{X}) \subseteq L^2(\mathcal{X})$. Furthermore, since we know that the analytic continuations $\{ \tilde{\varphi}_j \}$ of the matrix coefficients $\{ \varphi_j \}$ are in $H_t(\mathcal{X}_C)$, $R$ has dense range. We next observe that since $H_t(\mathcal{X}_C)$ is a Hilbert space of holomorphic functions on $\mathcal{X}_C$ and $\mathcal{X} \subseteq \mathcal{X}_C$ is compact, there is a constant $M$ such that for every $F \in H_t(\mathcal{X}_C)$

$$\|F|_\mathcal{X}\|^2 = \int_\mathcal{X} |F|_\mathcal{X}(x)|^2 \, dx \leq \sup_{x \in \mathcal{X}} |F|_\mathcal{X}(x)|^2 \leq M^2 \|F\|_{H_t}^2.$$

Thus, $R$ is bounded. Moreover, it is easy to see that $R$ is an intertwining operator. Again, we consider here $H_t(\mathcal{X}_C)$ and $L^2(\mathcal{X})$ as the left regular representations of $U$. Therefore, by Theorem 3.5 (the restriction principle), there is a unitary intertwining isomorphism $T : L^2(\mathcal{X}) \to H_t(\mathcal{X}_C)$. Since we have the Hilbert space direct sum $L^2(\mathcal{X}) = \bigoplus_{\mu \in P^+} C_\mu(\mathcal{X})$ with multiplicity 1 (see the discussion in Section 4.4), we also have the Hilbert space irreducible decomposition $H_t(\mathcal{X}_C) = \bigoplus_{\mu \in P^+} T(C_\mu(\mathcal{X}))$ with multiplicity 1.

Suppose that $H_t$ is not onto. Then $H_t(L^2(\mathcal{X})) \neq \{0\}$. Now consider the left regular representation of $U$ on $H_t(L^2(\mathcal{X}))$. Recall that $H_t$ is a unitary intertwining operator from $L^2(\mathcal{X})$ onto $H_t(L^2(\mathcal{X}))$. So we have another irreducible decomposition of $H_t(\mathcal{X}_C)$ induced from the map $H_t$ and the left regular representation of $U$ on $H_t(L^2(\mathcal{X}))$. Let $V$ be an irreducible subrepresentation of the left regular representation of $U$ on $H_t(L^2(\mathcal{X}))$. Thus, by Proposition 4.7, $A$ is unitarily equivalent to $T(C_\delta(\mathcal{X}))$ for some $\delta \in P^+$. Let $S = H_t|_{C_\delta(\mathcal{X})} \circ T^*|_{T(C_\delta(\mathcal{X}))}$. Then we have $S : T(C_\delta(\mathcal{X})) \to H_t(C_\delta(\mathcal{X}))$ is a unitary intertwining operator because $T^*|_{T(C_\delta(\mathcal{X}))}$ and $H_t|_{C_\delta(\mathcal{X})}$ are unitary intertwining operators. Hence, $V$ is unitarily equivalent to $H_t(C_\delta(\mathcal{X}))$. Therefore, we have a Hilbert space irreducible decomposition of $H_t(\mathcal{X}_C) = H_t(L^2(\mathcal{X})) \oplus H_t(L^2(\mathcal{X}))$ for the left regular representation of $U$ such that the decomposition contains a component that has multiplicity greater than 1. This contradicts the fact that the irreducible decomposition of $H_t(\mathcal{X}_C)$ has multiplicity 1. Therefore, $H_t$ is onto. \(\square\)
Chapter 5
The Segal-Bargman Transform on the Direct Limits

5.1 Introduction

In this final chapter, we construct the Segal-Bargmann transform on the direct limit of the Hilbert spaces \( \{L^2(M_n)^{K_n}\}_n \) where \( M_n = U_n/K_n \) is a special sequence of symmetric spaces of compact type. We begin in Section 5.2 with the basic notations we use throughout the chapter. We also give a nice parametrization of the unitary dual of \( U_n/K_n \) under the assumption that \( U_n \) is simply connected. Section 5.3 and 5.4 give the definitions of the propagations for the Lie algebras and symmetric spaces which are introduced in [46]. We are interested in the propagating sequence of symmetric spaces of compact type, \( M_n = U_n/K_n \). This sequence has a nice property allowing us to embed \( \widetilde{U_n/K_n} \) into \( \widetilde{U_m/K_m} \) for \( m \geq n \) in a natural way. This embeddings are given in Section 5.4. Section 5.5 presents the \( L^2 \)-theory of \( L^2(M_n)^{K_n} \). The formula for the Segal-Bargmann transform on \( L^2(M_n)^{K_n} \) is given in Section 5.6. We then give a short overview of the direct limits and inverse limits in the category of Hilbert spaces and unitary embeddings. Finally, in Section 5.7, we give a construction of the Segal-Bargmann transform on the direct limits of \( \{L^2(M_n)^{K_n}\}_n \). We use the notation \( \mathbb{Z}^+ \) for the set \( \{0, 1, 2, ...\} \).

5.2 Basic Notations

Throughout this chapter, all compact symmetric spaces and all notations will be as the following.

Let \( M = U/K \) be a symmetric space of compact type, i.e. \( U \) is a connected semisimple Lie group with an involution \( \theta \) such that

\[
U^\theta_0 \subseteq K \subseteq U^\theta.
\]

Here, \( U^\theta \) denotes the subgroup of \( \theta \)-fixed points,

\[
U^\theta = \{u \in U : \theta(u) = u\},
\]

and \( U^\theta_0 \) is the connected component of \( U \) containing the identity. The list of irreducible symmetric spaces of compact type is given by the following table.
We further assume that $U$ is simply connected. Then $U^\theta$ is connected (see Lemma 2, p. 103 in [60]) and hence $K$ is connected. Then we can basically follow the notations and the constructions as we did in the last chapter in order to get the Segal-Bargmann transform on $M = U/K$. However, we can give a new parametrization of $\hat{U}/K$ because of the simple connectedness and the semisimplicity of $U$. Before giving this new parametrization, we repeat some notations we need here.

We denote by $u$ and $k$ the Lie algebra of $U$ and $K$ respectively. Then $k = \{ X \in u : d\theta(X) = X \}$ and we have the Cartan decomposition $u = k \oplus p$ where $p = \{ X \in U : d\theta(X) = -X \}$. The base point $eK \in X$ will be denoted by $o$. $p$ can be identified with the tangent space $T_oX$. Since $u$ is compact and semisimple, the Killing form $B(X,Y)$ on $u$ is negative definite. We define the inner product on $u$ by

$$\langle X,Y \rangle = -B(X,Y).$$

Then $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal subspaces. We assume that the Riemannian metric $g$ of $X$ is normalized such that it agrees with $\langle \cdot,\cdot \rangle$ on $\mathfrak{p} = T_oX$.

The inner product on $u$ determines an inner product on the dual space $u^*$ in a canonical way. Furthermore, these inner products have complex bilinear extensions to the complexifications $u_C$ and $u_C^*$. All these bilinear forms are denoted by the same symbol $\langle \cdot,\cdot \rangle$.

Let $\mathfrak{a} \subseteq \mathfrak{p}$ be a cartan subspace of $\mathfrak{p}$, i.e. $\mathfrak{a}$ be a maximal abelian subspace, $\mathfrak{a}^*$ its dual space, and $\mathfrak{a}_C^*$ the complexified dual space. For $\alpha \in \mathfrak{a}_C^*$, let

$$u_{C,\alpha} = \{ X \in u_C \mid [H,X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}_C \}.$$
If $u_{C, \alpha} \neq \{0\}$, then $\alpha$ is called a restricted root. Let $\Sigma = \Sigma(u, a)$ be the set of restricted roots. Since $M$ is of compact type, the restricted roots are purely imaginary on $a$ and $\Sigma(u, a) \subseteq i a^*$. Let $h \subseteq u$ be a Cartan subalgebra containing $a$. Then $h = (h \cap t) \oplus a$. Let $\Delta = \Delta(u_{C}, h_{C})$ denote the roots of $u$ with respect to $h$. Then $\Sigma$ is exactly the set of non-zero restrictions of $a$ of elements of $\Delta$. We fix a set $\Sigma^+ \subseteq \Sigma$ of positive restricted roots, and a compatible set $\Delta^+ \subseteq \Delta$ of positive roots. Note that since $u$ is compact, all elements of $\Delta$ take purely imaginary values on $h$.

We first recall the following fact (see Theorem 4.28 in [37]): Since $U$ is semisimple and simply connected, there is a one-to-one correspondence between $\hat{U}$ and the set of dominant algebraically integral weights on $i h^*$:

$$\Lambda^+(U) = \left\{ \mu \in i h^* : 2 \frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}^+ \text{ for all } \alpha \in \Delta^+ \right\}.$$ 

Recall the discussion before Theorem 4.24 that for any general (not necessarily semisimple nor simply connected) compact group we have the one-to-one correspondence between its unitary dual and $D^+$, the set of dominant analytically integral weights. In fact, for a general compact group, we have $D^+ \subseteq \Lambda^+(U)$. But if a compact group is semisimple and simply connected, then we have $D^+ = \Lambda^+(U)$. So in our case we have $\Lambda^+ (U)$ as a parametrization of $\hat{U}$.

Recall that for $\mu \in \Lambda^+(U)$, we define

$$V^K_{\mu} = \left\{ v \in V_{\mu} : \pi_{\mu}(k)v = v \text{ for all } k \in K \right\}.$$ 

Also recall Theorem 4.23. So we identify $i a^*$ with $\{ \mu \in i h^* : \mu|_{h \cap t} = 0 \}$. With this identification in mind, we set

$$\Lambda^+(U, K) = \left\{ \mu \in i a^* : \frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}^+ \text{ for all } \alpha \in \Sigma^+ \right\}.$$ 

The following theorem gives the parametrization of $\hat{U}/K$.

**Theorem 5.1** (Cartan-Helgason). The following are equivalent.

1. $\mu \in \Lambda^+(U, K)$,
2. $V^K_{\mu} \neq 0$,
3. $\pi_{\mu}$ is a subrepresentation of the representation of $U$ on $L^2(M)$.

When those conditions hold, $\dim V^K_{\mu} = 1$ and $\pi_{\mu}$ occurs with multiplicity 1 in the representation of $U$ on $L^2(M)$.

**Proof.** See the proof of Theorem 4.1, p. 535 in [33].
5.3 Propagations of Lie Algebras

In this section, we follow the definitions and the discussion in [46].

**Definition 5.2.** Let $\mathfrak{g}_n$ be a simple Lie algebra of classical type and let $\mathfrak{h}_n \subset \mathfrak{g}$ be a Cartan subalgebra. Let $\Delta_n = \Delta(\mathfrak{g}_n, \mathfrak{h}_n)$ be the set of roots of $\mathfrak{h}_n$ in $\mathfrak{g}_n$, $C$ and $\Psi_n = \Psi(\mathfrak{g}_n, \mathfrak{h}_n)$ a set of simple roots. We label the corresponding Dynkin diagram so that $\alpha_1$ is the right endpoint. We say that $\mathfrak{g}_m$ propagates $\mathfrak{g}_n$, if $\Psi_m$ is constructed from $\Psi_n$ by adding simple roots to the left end of the Dynkin diagram.

**Definition 5.3.** Let $\mathfrak{g}$ and $\mathfrak{k} \subset \mathfrak{g}$ be semisimple Lie algebras. Then $\mathfrak{g}$ propagates $\mathfrak{k}$ if we can number the simple ideals $\mathfrak{g}_j$, $j = 1, 2, ..., r$ in $\mathfrak{g}$ and $\mathfrak{k}_j$, $j = 1, 2, ..., s \leq r$ in $\mathfrak{k}$ such that $\mathfrak{g}_j$ propagates $\mathfrak{k}_j$ for $j = 1, 2, ..., s$.

\[
\begin{array}{|c|c|}
\hline
\Psi_n = A_n & \alpha_n \alpha_{n-1} \alpha_{n-2} \cdots \alpha_1 \quad n \geq 1 \\
\hline
\Psi_k = A_k & \alpha_k \alpha_{n-1} \alpha_{n-2} \cdots \alpha_1 \quad k \geq n \\
\hline
\Psi_n = B_n & \alpha_n \alpha_{n-1} \cdots \alpha_2 \alpha_1 \quad n \geq 2 \\
\hline
\Psi_k = B_k & \alpha_k \alpha_{n-1} \alpha_{n-2} \cdots \alpha_1 \quad k \geq n \\
\hline
\Psi_n = C_n & \alpha_n \alpha_{n-1} \cdots \alpha_2 \alpha_1 \quad n \geq 3 \\
\hline
\Psi_k = C_k & \alpha_k \alpha_{n-1} \alpha_{n-2} \cdots \alpha_1 \quad k \geq n \\
\hline
\Psi_n = D_n & \alpha_n \alpha_{n-1} \alpha_3 \alpha_2 \alpha_1 \quad n \geq 4 \\
\hline
\Psi_k = D_k & \alpha_k \alpha_{n-1} \alpha_3 \alpha_2 \alpha_1 \quad k \geq n \\
\hline
\end{array}
\]

When $\mathfrak{g}_m$ propagates $\mathfrak{g}_n$ as above, they have Cartan subalgebra $\mathfrak{h}_m$ and $\mathfrak{h}_n$ such that $\mathfrak{h}_n \subset \mathfrak{h}_m$, and we have choices of root order such that

if $\alpha \in \Psi_n$ then $\exists! \alpha' \in \Psi_m$ such that $\alpha'|_{\mathfrak{h}_n} = \alpha$.

This implies that

$$\Delta_n \subseteq \{ \alpha|_{\mathfrak{h}_n} \mid \alpha \in \Delta_m \text{ and } \alpha|_{\mathfrak{h}_n} \neq 0 \}.$$ 

For a Cartan subalgebra $\mathfrak{h}_\mathbb{C}$ in a simple complex Lie algebra $\mathfrak{g}_\mathbb{C}$ denote by $\mathfrak{h}_\mathbb{R}$ the Euclidean vector space

$$\mathfrak{h}_\mathbb{R} = \{ X \in \mathfrak{h}_\mathbb{C} \mid \alpha(X) \in \mathbb{R} \text{ for all } \alpha \in \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}) \}.$$
For $k \in \mathbb{N}$, we identify $\mathbb{R}^k$ with its dual. Let $f_1 = (0,0,\ldots,0,1)$, $\ldots$, $f_k = (1,0,\ldots,0,0)$ be the standard basis for $\mathbb{R}^k$ numbered in order opposite to the usual one. We write

$$x = x_1 f_1 + \ldots + x_k f_k = (x_k, \ldots, x_1)$$

to indicate that in the following we will be adding zero to the left to adjoint for our numbering in the Dynkin diagrams. We use the discussion in [62] as a reference for the realization of the classical Lie algebras.

**The case $A_k$, where $\mathfrak{g}_k = \mathfrak{sl}(k+1, \mathbb{C})$.**

In this case, we have

$$\mathfrak{h}_{m,\mathbb{R}} = \{ (x_{m+1}, \ldots, x_1) \in \mathbb{R}^{m+1} \mid x_1 + \ldots + x_{m+1} = 0 \},$$

where $x \in \mathbb{R}^{m+1}$ corresponds to the diagonal matrix

$$x \leftrightarrow \text{diag}(x) := \begin{pmatrix} x_{k+1} & 0 & \cdots & 0 \\ 0 & x_k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_1 \end{pmatrix}.$$ 

Then $\Delta_m = \{ f_i - f_j \mid 1 \leq i \neq j \leq m+1 \}$ where $f_i$ maps diagonal matrix to its $i^{th}$ diagonal elements.

We take

$$\Psi(\mathfrak{g}_m, \mathfrak{h}_m) = \{ f_j - f_{j-1} \mid j = 2, \ldots, k+1 \}.$$ 

The analogous notation will be used for $A_n$. In particular, denoting the zero vector of length $j$ by $0_j$, we have

$$\mathfrak{h}_{n,\mathbb{R}} = \left\{ (0_{m-n}, x_{n+1}, \ldots, x_1) \mid x \in \mathbb{R} \text{ and } \sum_{j=1}^{n+1} x_j = 0 \right\}$$

This corresponds to the embedding

$$\mathfrak{sl}(n, \mathbb{C}) \hookrightarrow \mathfrak{sl}(m, \mathbb{C}), \; X \mapsto \begin{pmatrix} 0_{k-n,k-n} & 0 \\ 0 & X \end{pmatrix}.$$ 

**The case $B_k$, where $\mathfrak{g}_k = \mathfrak{so}(2k+1, \mathbb{C})$.**

In this case $\mathfrak{h}_{m,\mathbb{R}} = \mathbb{R}^m$ where $\mathbb{R}^m$ is embedded into $\mathfrak{so}(2m+1, \mathbb{C})$ by

$$x \mapsto \begin{pmatrix} 0_{11} & 0 & 0 \\ 0 & \text{diag}(x) & 0 \\ 0 & 0 & -\text{diag}(x) \end{pmatrix}.$$
Then $\Delta_m = \{\pm (f_i \pm f_j) \mid 1 \leq j < i \leq m\} \cup \{\pm f_1, \ldots, \pm f_m\}$. Take

$$\Delta^+_m = \{f_i \pm f_j \mid 1 \leq j < i \leq m\} \cup \{f_1, \ldots, f_m\}$$

as a positive system. Then the simple root system is $\Psi_m = \Psi(g_m, h_m) = \{\alpha_1, \ldots, \alpha_m\}$ where

the simple root $\alpha_1 = f_1$, and $\alpha_j = f_j - f_{j-1}$ for $2 \leq j \leq k$.

We use the similar notation for $h_{n, \mathbb{R}}$. Our embedding $\mathfrak{h}_{n, \mathbb{R}} \hookrightarrow \mathfrak{h}_{m, \mathbb{R}}$ corresponds to the (non-standard) embedding of $\mathfrak{so}(2n + 1, \mathbb{C})$ into $\mathfrak{so}(2m + 1, \mathbb{C})$ given by

$$\begin{pmatrix} 0 & a & b \\ -b^t & A & B \\ -a^t & C & -A^t \end{pmatrix} \mapsto -\begin{pmatrix} 0 & 0_{k-n} & a & 0_{k-n} & b \\ 0_{k-n} & 0 & 0 & 0 & 0 \\ -b^t & 0 & A & 0 & B \\ 0_{k-n} & 0 & 0 & 0 & 0 \\ -a^t & 0 & C & 0 & -A^t \end{pmatrix}$$

where the zeros stands for the zero matrix of the correct size and we use the realization from [62], p. 303.

**The case $C_n$, where $g_k = \mathfrak{sp}(k, \mathbb{C})$.**

In this case, $\mathfrak{h}_{n, \mathbb{R}} = \mathbb{R}^m$ embedding in $\mathfrak{sp}(m, \mathbb{C})$ by

$$x \mapsto \begin{pmatrix} \text{diag}(x) & 0 \\ 0 & -\text{diag}(x) \end{pmatrix}.$$  

We have $\Delta_m = \{\pm (f_i \pm f_j) \mid 1 \leq j < i \leq m\} \cup \{\pm 2f_1, \ldots, \pm 2f_m\}$. We take

$$\Delta^+_m = \{f_i \pm f_j \mid 1 \leq j < i \leq m\} \cup \{2f_1, \ldots, 2f_m\}$$

as a positive system. Then the simple root system

$$\Psi_m = \Psi(g_m, h_m) = \{\alpha_1, \ldots, \alpha_m\}$$

is given by

the simple root $\alpha_1 = 2f_1$, and $\alpha_j = f_j - f_{j-1}$ for $2 \leq j \leq m$.

We embed $\mathfrak{sp}(n, \mathbb{C})$ into $\mathfrak{sp}(m, \mathbb{C})$ by

$$\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mapsto \begin{pmatrix} 0_{k-n,k-n} & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & 0_{k-n,k-n} & 0 \\ 0 & C & 0 & -A^t \end{pmatrix}$$

where 0 stands for the zero matrix of the correct size.

**The case $D_k$, where $g_k = \mathfrak{so}(2k, \mathbb{C})$.**

We take $\mathfrak{h}_{m, \mathbb{R}} = \mathbb{R}^m$ embedded in $\mathfrak{so}(2m, \mathbb{C})$ by

$$x \mapsto \begin{pmatrix} \text{diag}(x) & 0 \\ 0 & -\text{diag}(x) \end{pmatrix}.$$
Then $\Delta_m = \{ \pm (f_i \pm f_j) \mid 1 \leq j < i \leq m \}$ and we use the simple root system

$$\Psi_m = \Psi(g_m, h_m) = \{ \alpha_1, \ldots, \alpha_m \}$$

in the same manner as before. This corresponds to

$$
\begin{pmatrix}
A & B \\
C & -A^t
\end{pmatrix} \mapsto
\begin{pmatrix}
0_{k-n,k-n} & 0 & 0_{k-n,k-n} & 0 \\
0 & A & 0 & B \\
0_{k-n,k-n} & 0 & 0 & 0 \\
0 & C & 0 & -A^t
\end{pmatrix}.
$$

We conclude this section by giving the following consequence of the definition of propagation. It is implicit in the diagrams following that definition.

**Lemma 5.4.** Assume that $g_m$ propagates $g_n$. Let $h_m$ be a Cartan subalgebra of $g_k$ such that $h_n = h_m \cap g_n$. Choose a positive systems $\Delta^+(g_m, h_m) \subseteq \Delta(g_m, h_m)$ and $\Delta^+(g_n, h_n) \subseteq \Delta(g_n, h_n)$ such that $\Delta^+(g_n, h_n) \subseteq \Delta^+(g_m, h_m)|_{h_n}$. Then we can number the simple roots such that

$$\alpha_{n,j} = \alpha_{m,j}|_{h_n}$$

for $j = 1, \ldots, \dim h_n$.

### 5.4 Propagations of Compact Symmetric Spaces

In this section, $m$ is always bigger than $n$. We follow the discussion in [46]. Let $M_n = U_n/K_n$ be a sequence of symmetric spaces of compact type such that $U_n \subseteq U_m$, $\theta_m|_{u_n} = \theta_n$ where $\theta_m$ and $\theta_n$ are the corresponding involutions, and $K_n = K_m \cap U_n$. We emphasize here again that $U_n$ is assumed to be simply connected for every $n$. The eigenspace decompositions

$$u_m = \mathfrak{t}_m \oplus p_m, \quad u_n = \mathfrak{t}_n \oplus p_n$$

gives

$$\mathfrak{t}_n = \mathfrak{t}_m \cap u_n \quad \text{and} \quad p_n = u_n \cap p_m.$$  

We recursively choose maximal commutative subspaces $a_m \subseteq p_m$ such that $a_n \subseteq a_m$ for all $m \geq n$. For each $n$, let $r_n = \dim a_n$, the rank of $M_n$.

As in section 5.2, we let $\Sigma_n = \Sigma_n(u_n, a_n)$ denote the system of restricted roots of $a_{n,C}$ in $u_{n,C}$. Let $h_n$ be a $\theta_n$-stable Cartan subalgebra such that $h_n \cap p_n = a_n$. Let $\Delta_n = \Delta(u_{n,C}, h_{n,C})$. Note that $\Sigma_n \subseteq \mathfrak{a}_{n,C}^*$. We choose positive subsystems $\Delta_n^+$ and $\Sigma_n^+$ so that $\Sigma_n^+ \subseteq \Delta_n^+|_{a_n}$, $\Delta_n^+ \subseteq \Delta_n^+|_{h_{n,C}}$, and $\Sigma_n^+ \subseteq \Sigma_n^+|_{a_n}$. Let

$$\Sigma_{1/2,n} = \left\{ \alpha \in \Sigma_n \mid \frac{1}{2} \alpha \notin \Sigma_n \right\},$$

$$\mathfrak{a}_{n,C}^* = \left\{ \alpha \in \mathfrak{a}_{n,C}^* \mid \frac{1}{2} \alpha \notin \mathfrak{a}_{n,C}^* \right\},$$

$$\Delta_n^+ = \left\{ \alpha \in \Delta_n \mid \frac{1}{2} \alpha \notin \Delta_n \right\},$$

$$\Sigma_n^+ = \left\{ \alpha \in \Sigma_n \mid \frac{1}{2} \alpha \notin \Sigma_n \right\}.$$
and

$$\Sigma_{2,n} = \{ \alpha \in \Sigma_n \mid 2\alpha \not\in \Sigma_n \}.$$  

Then $\Sigma_{1/2,n}$ and $\Sigma_{2,n}$ are reduced root systems (see Lemma 3.2, p. 456 in [32]). Consider the positive systems $\Sigma_{1/2,n}^+ := \Sigma_{1/2,n} \cap \Sigma_n^+$ and $\Sigma_{2,n}^+ := \Sigma_{2,n} \cap \Sigma_n^+$. Let

$$\Psi_{1/2,n} = \Psi_{1/2,n}(u_n, a_n) \quad \text{and} \quad \Psi_{2,n} = \Psi_{2,n}(u_n, a_n)$$

denote the sets of simple roots for $\Sigma_{1/2,n}^+$ and $\Sigma_{2,n}^+$ respectively.

Suppose for a moment that $M_n$ is an irreducible symmetric space for every $n$. If $\Sigma_{1/2,n} \neq \Sigma_n$, there is exactly one simple root $\alpha$ with $2\alpha \in \Sigma_n$ and this simple root is at the right end of the Dynkin diagram for $\Psi_{1/2,n}$ (see the discussion in Section 3 of [46]). Also, either $\Psi_{1/2,n} = \{\alpha\}$ contains one simple root or $\Psi_{1/2,n}$ is of type $B_{2n}$. Note that if $\Sigma_{1/2,n}$ is of type $B$, the root system $\Sigma_{2,n}$ will be of type $C$. We say that $M_m$ propagates $M_n$ if we only add simple roots to the left end of the Dynkin diagram for $\Psi_{1/2,n}$ to obtain the Dynkin diagram for $\Psi_{1/2,m}$. In particular, $\Psi_{1/2,n}$ and $\Psi_{1/2,m}$ are of the same type. In general, if $M_m$ and $M_n$ are not irreducible, with universal covering $\tilde{M}_m$ respectively $\tilde{M}_n$, then $M_m$ propagates $M_n$ if we can enumerate the irreducible factors of $\tilde{M}_m = M_m^1 \times \ldots \times M_m^s$ and $\tilde{M}_n = M_n^1 \times \ldots \times M_n^t$, $i \leq j$ such that $M_m^s$ propagates $M_n^i$ for $s = 1, 2, \ldots, i$.

From now on, we assume that $M_m$ propagates $M_n$ for all $m \geq n$. We call the sequence $\{M_n = U_n/K_n\}$, the propagating sequence of symmetric spaces of compact type.

Some examples of propagating sequences of symmetric spaces of compact type are $\{\text{SU}(n)/\text{SO}(n)\}_{n=1}^{\infty}$ and $\{\text{Sp}(n)/\text{U}(n)\}_{n=1}^{\infty}$. Also a propagating sequence could contain an inclusion like $\text{SU}(n)/\text{SO}(n) \subset (\text{SU}(n) \times \text{SU}(n))/\text{diag}(\text{SU}(n) \times \text{SU}(n))$.

Now, let $\Psi_{2,n} = \{\alpha_{n,1}, \ldots, \alpha_{n,r_n}\}$. We note the following facts which follow from the explicit realization (5.2) of the root systems discussed in Section 5.3.

**Lemma 5.5.** Suppose that the $M_n$ are irreducible for all $n$. Number the simple root systems $\Psi_{2,n}$ as in (5.2). If $j \leq r_n$ then $\alpha_{m,j}$ is the unique element of $\Psi_{2,m}$ whose restriction to $a_n$ is $\alpha_{n,j}$.

**Remark.** Since $M_m$ propagates $M_n$, each irreducible factor of $M_m$ contains just one irreducible factor of $M_n$. In particular if $M_n$ is not irreducible, then $M_m$ is not irreducible, but we still can number the simple roots so that the lemma above applies.

Now recall the parametrization of $\overline{U_n/K_n}$ which can be given by

$$\Lambda_n^+ := \Lambda^+(U_n, K_n) = \left\{ \mu \in i\mathfrak{a}_n^* : \frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}^+ \quad \text{for all } \alpha \in \Sigma_n^+ \right\}.$$  

Define linear functionals $\xi_{n,j} \in i\mathfrak{a}_n^*$ for $1 \leq j \leq r_n$ by

$$\frac{\langle \xi_{n,j}, \alpha_{n,i} \rangle}{\langle \alpha_{n,i}, \alpha_{n,i} \rangle} = \delta_{j,i} \quad \text{for } 1 \leq i \leq r_n .$$
Lemma 5.6. $\xi_{n,j} \in \Lambda_n^+$ for $1 \leq j \leq r_n$.

Proof. Let $1 \leq j \leq r_n$. Let $\alpha \in \Sigma_n^+$. Then if $\alpha \in \Sigma_{2,n}^+$,
\[
\frac{\langle \xi_{n,j}, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}^+.
\]
If $\alpha \in \Sigma_n^+ \setminus \Sigma_{2,n}^+$, then $2\alpha \in \Sigma_n^+$ and hence
\[
\frac{\langle \xi_{n,j}, \alpha \rangle}{\langle \alpha, \alpha \rangle} = 2\frac{\langle \xi_{n,j}, 2\alpha \rangle}{\langle 2\alpha, 2\alpha \rangle} \in \mathbb{Z}^+.
\]

The weights $\xi_{n,j}$ are the class-1 fundamental weights for $(u_n, \mathfrak{k}_n)$. We set
\[
\Xi_n = \{\xi_{1,1}, \ldots, \xi_{n,r_n}\}.
\]
For $I = (k_1, \ldots, k_{r_n}) \in (\mathbb{Z}^+)^{r_n}$, define
\[
\mu_I := \mu(I) = k_1\xi_{n,1} + \ldots + k_{r_n}\xi_{n,r_n}.
\]

Lemma 5.7. If $\mu \in i\mathfrak{a}_n^*$, then $\mu \in \Lambda_n^+$ if and only if $\mu_I = \mu(I)$ for some $I \in (\mathbb{Z}^+)^{r_n}$.

Proof. This follows from the definition of $\xi_{n,j}$.

Lemma 5.8. Let $I_m = (k_1, \ldots, k_{r_m}) \in (\mathbb{Z}^+)^{r_m}$ and $\mu = \mu_{I_m}$. Then $\mu|_{\mathfrak{a}_n} \in \Lambda_n^+$. In particular, $\xi_{m,j}|_{\mathfrak{a}_n} \in \Lambda_n^+$ for $j = 1, \ldots, r_m$.

Proof. Let $v_\mu \in V_\mu$ be a nonzero highest weight vector and $e_\mu \in V_\mu$ a $K_m$-fixed unit vector. Denote by $W = \langle \pi_\mu(G_n)v_\mu \rangle$ the cyclic $G_n$-module generated by $v_\mu$ and let $\mu_n = \mu|_{\mathfrak{a}_n}$.

Write $W = \bigoplus W_j$ with $W_j$ irreducible. If $W_j$ has highest weight $v_j \neq \mu$ then $v_\mu \perp W_j$ so $\langle \pi_\mu(G_n)v_\mu \rangle \perp W_j$, contradicting $W_j \subseteq W = \bigoplus W_j$. Thus, each $W_j$ has highest weight $\mu$. Write $v_\mu = v_1 + \ldots + v_s$ with $0 \neq v_j \in W_j$. Since $\langle v_\mu, e_\mu \rangle \neq 0$ for some $k \in \{1, \ldots, s\}$. But then the projection of $e_\mu$ onto $W_j$ is a non-zero $K_n$-fixed vector in $W_j^{K_n} \neq 0$ and hence $\mu|_{\mathfrak{a}_n} \in \Lambda_n^+$.

Lemma 5.9 ([67], Lemma 6). Recall the root ordering of (5.2). If $1 \leq j \leq r_n$ then $\xi_{m,j}$ is the unique element of $\Xi_m$ whose restriction of $\mathfrak{a}_n$ is $\xi_{n,j}$.

Proof. This is clear when $\mathfrak{a}_m = \mathfrak{a}_n$. If $r_n < r_m$, it follows from the explicit construction of the fundamental weights for classical root system; see in [25], p. 102.
Therefore, by the above lemma, we can embed $\Lambda_n$ inside $\Lambda_m$ as follows: For $I_n = (k_1, \ldots, k_{r_n}) \in (\mathbb{Z}^+)^{r_n}$, we let

$$\mu_{I_n} = \sum_{j=1}^{r_n} k_j \xi_{n,j} \in \Lambda_n^+,$$

and send $\mu_{I_n}$ to the element

$$\mu((I_n,0_{r_m-r_n})) = \sum_{j=1}^{r_n} k_j \xi_{m,j} \in \Lambda_m^+.$$ 

This corresponds to the embedding $I_n \hookrightarrow I_m$

$$(k_1, \ldots, k_{r_n}) \hookrightarrow (k_1, \ldots, k_{r_n}, 0_{r_m-r_n})$$

where $I_n = (\mathbb{Z}^+)^{r_n}$ and $I_m = (\mathbb{Z}^+)^{r_m}$.

### 5.5 The $L^2$-Theory

We first recall now the basic facts about harmonic analysis on $L^2(M_n)^{K_n}$, see the detailed discussion in Chapter 4. Let $\mu \in \Lambda_n^+$ and $(\pi_\mu, V_\mu)$ the corresponding irreducible unitary representation in the decomposition of $L^2(M_n)$. Fix a $K_n$-invariant vector $e_\mu = e_{\mu,n} \in V_\mu^{K_n}$ of length one. The spherical function on $M_n = U_n/K_n$ associated with $\mu$ is the matrix coefficient

$$\varphi_\mu(u) = \langle e_\mu, \pi_\mu(u)e_\mu \rangle, \quad u \in U_n.$$ 

It is $K_n$-biinvariant, and it is independent of the choice of the unit vector $e_\mu$. So we can view $\varphi_\mu$ as a $K_n$-invariant function on $M_n$. Let $d(\mu)$ be the dimension of $V_\mu$. Let

$$\ell_2^d(\Lambda_n^+) = \left\{ (a_\mu)_{\mu \in \Lambda_n^+} \mid a_\mu \in \mathbb{C} \text{ and } \sum_{\mu \in \Lambda_n^+} d(\mu)|a_\mu|^2 < \infty \right\}.$$

Then $\ell_2^d(\Lambda_n^+)$ is a Hilbert space. The spherical Fourier transform

$$\hat{\cdot} : C(M_n)^{K_n} \longrightarrow \ell_2^d(\Lambda_n^+)$$

is defined by

$$\hat{f}(\mu) = \int_{M_n} f(x)\overline{\varphi_\mu(x)}dx = \langle f, \varphi_\mu \rangle_2, \quad f \in C(M_n)^{K_n}.$$ 

It extends to a unitary isomorphism $\hat{\cdot} : L^2(M_n)^{K_n} \longrightarrow \ell_2^d(\Lambda_n^+)$ with inverse

$$f = \sum_{\mu \in \Lambda_n^+} d(\mu)\hat{f}(\mu)\varphi_\mu.$$
The sum is to be interpreted as $L^2$-limit. In fact, $L^2(M_n)^{K_n}$ has an orthonormal basis
$$\{\sqrt{d(\mu)}\varphi_\mu|\mu \in \Lambda^+_n\}.$$ 

Using the notations in the previous section, we can also write $f \in L^2(M_n)^{K_n}$ as
$$f = \sum_{\mu \in \Lambda^+_n} d(\mu_\mu)\hat{f}(\mu_\mu)\varphi_{\mu_\mu}.$$ 

### 5.6 The Segal-Bargman Transform on $L^2(M_n)^{K_n}$

In this section, we denote by $\cdot : U_n \to U_n = U_n/K_n$ the canonical projection, i.e. $\cdot = uK_n$, and $du$ is the push forward measure of $du$, the normalized Haar measure on $U_n$, via the canonical map $\cdot$. We recall the Segal-Bargmann transform $H_t : L^2(M_n) \to \mathcal{H}_{t,n}(M_n^\mathbb{C})$ which is defined by
$$\langle H_t f(z) \rangle = \hat{f} \ast \tilde{h}_{t,n}(z) = \sum_{\mu \in \Lambda^+_n} d(\mu)e^{-t(\mu + 2\rho_\mu)} \sum_{j=1}^{d(\mu)} \hat{f}_j(\mu)\tilde{\varphi}_j^\mu(z),$$ 

for $f \in L^2(M_n)$.

**Lemma 5.10.** If $\tilde{f} : M_n^\mathbb{C} \to \mathbb{C}$ is the holomorphic extension of $f : M_n \to \mathbb{C}$ and $f$ is $K_n$-invariant on $M_n$, then $\tilde{f}$ is $K_n^\mathbb{C}$-invariant on $M_n^\mathbb{C}$.

**Proof.** Fix any $x \in M_n^\mathbb{C}$. Define the map $g : K_n \to \mathbb{C}$ by $g(k) = \tilde{f}(k \cdot x) - \tilde{f}(x)$. Then $g \equiv 0$ on $K_n$. Next we define the map $\tilde{g}(k) = \tilde{f}(\tilde{k} \cdot x) - \tilde{f}(x)$ for every $\tilde{k} \in K_n^\mathbb{C}$. Therefore $\tilde{g} : K_n^\mathbb{C} \to \mathbb{C}$ is the analytic continuation of $g$. This implies that $\tilde{g} \equiv 0$ on $K_n^\mathbb{C}$. Since $x$ is arbitrary, $\tilde{f}$ is $K_n^\mathbb{C}$-invariant on $M_n^\mathbb{C}$.

**Theorem 5.11.** We have $H_t,n \big(L^2(M_n)^{K_n}\big) = \mathcal{H}_t(M_n^\mathbb{C})^{K_n^\mathbb{C}}$. In other words, the map
$$H_t,n : L^2(M_n)^{K_n} \to \mathcal{H}_t(M_n^\mathbb{C})^{K_n^\mathbb{C}}$$ 

is a unitary isomorphism.

**Proof.** If $f \in L^2(M_n)^{K_n}$, then for $k \in K_n$ and $x \in M_n$,
$$(f \ast h_{t,n})(k \cdot x) = \int_{U_n} f(\tilde{u}) h_{t,n}(u^{-1}k \cdot x) \, du$$
$$= \int_{U_n} f(\tilde{u}) h_{t,n}((k^{-1}u)^{-1} \cdot x) \, du$$
$$= \int_{U_n} f(k \cdot \tilde{u}) h_{t,n}(u^{-1} \cdot x) \, du$$
$$= \int_{U_n} f(\tilde{u}) h_{t,n}(u^{-1} \cdot x) \, du$$
$$= (f \ast h_{t,n})(x).$$
So if \( f \in L^2(M_n)^K_n \), then \( f \ast h_{t,n} \) is \( K_n \)-invariant. But \( H_{t,n}f \) is the holomorphic extension of \( f \ast h_{t,n} \), \( H_{t,n}f \) is \( K_n^C \)-invariant for \( f \in L^2(M_n)^K_n \) by the previous lemma. Therefore, \( H_{t,n}(L^2(M_n)^K_n) \subseteq \mathcal{H}_t(M_n^C)^{K_n^C} \).

Conversely, let \( F \in \mathcal{H}_t(M_n^C)^{K_n^C} \). Then by Theorem 4.41, \( F = H_{t,n}f \) for some \( f \in L^2(M_n) \). Since \( H_{t,n}f = F \) is \( K_n^C \)-invariant, \( f \ast h_{t,n} \) is \( K_n \)-invariant. Thus, for each \( k \in K_n \) and \( x \in U_n/K_n \),

\[
(f \ast h_{t,n})(x) = (f \ast h_{t,n})(k \cdot x) = \int_{U_n} f(\hat{u}) \ h_{t,n}(u^{-1}k \cdot x) \ du = \int_{U_n} f(\hat{u}) \ h_{t,n}((k^{-1}u)^{-1} \cdot x) \ du = \int_{U_n} f(k \cdot \hat{u}) \ h_{t,n}(u^{-1} \cdot x) \ du = ((k \cdot f) \ast h_{t,n})(x).
\]

Therefore, \((k \cdot f) \ast h_{t,n} = f \ast h_{t,n}\) for all \( k \in K_n \). Hence, \( H_{t,n}f = H_{t,n}(k \cdot f) \) for all \( k \in K_n \). Since the map \( H_{t,n} \) is one to one, \( k \cdot f = f \) for every \( k \in K_n \). Thus, \( f \in L^2(M_n)^K_n \). Hence, we have \( H_{t,n}(L^2(M_n)^K_n) \supseteq \mathcal{H}_t(M_n^C)^{K_n^C} \). This finishes the proof. \( \square \)

Recall that by Theorem 4.39, we have the series formula for the Segal-Bargmann transform \( H_{t,n} \):

\[
H_{t,n}f = \sum_{\mu \in \Lambda_n^+} d(\mu) e^{-t(\mu + 2\rho, \mu)} \sum_{j=1}^{d(\mu)} \widehat{f}_j(\mu) \hat{\varphi}_\mu^j,
\]

for \( f \in L^2(M_n) \) with

\[
f = \sum_{\mu \in \Lambda_n^+} d(\mu) \left( \sum_{j=1}^{d(\mu)} \widehat{f}_j(\mu) \hat{\varphi}_\mu^j \right).
\]

Then for \( f \in L^2(M_n)^K_n \) with

\[
f = \sum_{\mu \in \Lambda_n^+} d(\mu) \widehat{f}(\mu) \varphi_\mu,
\]

we have

\[
H_{t,n}f = \sum_{\mu \in \Lambda_n^+} d(\mu) e^{-t(\mu + 2\rho, \mu)} \widehat{f}(\mu) \tilde{\varphi}_\mu, \tag{5.3}
\]

where the sum converges in \( \mathcal{H}_t(M_n^C)^{K_n^C} \) and hence uniformly on compact subsets of \( M_n^C \). Note that since \( \tilde{\varphi}_\mu \) is \( K_n^C \)-invariant for every \( \mu \in \Lambda_n^+ \), the sum (in \( \mathcal{H}_t(M_n^C) \))

\[
\sum_{\mu \in \Lambda_n^+} d(\mu) e^{-t(\mu + 2\rho, \mu)} \widehat{f}(\mu) \tilde{\varphi}_\mu
\]
is also $K_n^C$-invariant. By using the above formula for $H_{t,n}f$ when $f \in L^2(M_n)^{K_n}$, we can give the series description of the Hilbert space $\mathcal{H}_t(M_n^C)^{K_n^C}$ in the following theorem.

**Corollary 5.12.** The set

$$\left\{ \sqrt{d(\mu)} e^{-t(\mu+2\rho)} \mu \hat{\varphi}_\mu : \mu \in \Lambda_n^+ \right\}$$

is an orthonormal basis for $\mathcal{H}_t(M_n^C)^{K_n^C}$ and the Hilbert space $\mathcal{H}_t(M_n^C)^{K_n^C}$ equals to

$$\left\{ \sum_{\mu \in \Lambda_n^+} d(\mu)a_t(\mu) \hat{\varphi}_\mu : a_t : \Lambda_n^+ \to \mathbb{C} \text{ and } \sum_{\mu \in \Lambda_n^+} d(\mu)|a_t(\mu)|^2 e^{2t(\mu+2\rho)} < \infty \right\},$$

where the sum

$$F = \sum_{\mu \in \Lambda_n^+} d(\mu)a_t(\mu) \hat{\varphi}_\mu$$

converges in $\mathcal{H}_t(M_n^C)$ and hence uniformly on compact subsets of $M_n^C$ with

$$\|F\|_{\mathcal{H}_t}^2 = \sum_{\mu \in \Lambda_n^+} d(\mu)|a_t(\mu)|^2 e^{2t(\mu+2\rho)} < \infty.$$

**Proof.** The first statement follows directly from (5.3), the previous theorem, Lemma 4.37 and Lemma 4.38. Next, assume that $F \in \mathcal{H}_t(M_n^C)^{K_n^C}$. Then $F = H_{t,n}(f)$ for some $f \in L^2(M_n)^{K_n}$. Thus

$$F = \sum_{\mu \in \Lambda_n^+} d(\mu)e^{-t(\mu+2\rho)} \mu \hat{f}(\mu) \hat{\varphi}_\mu,$$

where the sum is in $\mathcal{H}_t(M_n^C)^{K_n^C}$. We have

$$\sum_{\mu \in \Lambda_n^+} d(\mu)|e^{-t(\mu+2\rho)} \mu \hat{f}(\mu)|^2 e^{2t(\mu+2\rho)} = \sum_{\mu \in \Lambda_n^+} d(\mu)|\hat{f}(\mu)|^2$$

$$= \|F\|_{\mathcal{H}_t}^2 < \infty.$$

Conversely, suppose that

$$F = \sum_{\mu \in \Lambda_n^+} d(\mu)a_t(\mu) \hat{\varphi}_\mu,$$

where the sum converges in $\mathcal{H}_t(M_n^C)^{K_n^C}$ and hence uniformly on compact subsets of $M_n^C$ with $a_t : \Lambda_n^+ \to \mathbb{C}$ such that

$$\sum_{\mu \in \Lambda_n^+} d(\mu)|a_t(\mu)|^2 e^{2t(\mu+2\rho)} < \infty.$$
Then $F \in \mathcal{O}(M_n^\mathbb{C})$ and is $K_n^\mathbb{C}$-invariant with
\[
\|F\|_{\mathcal{H}_t}^2 = \sum_{\mu \in \Lambda_n^+} |d(\mu)a_t(\mu)|^2 \|\varphi_\mu\|_{\mathcal{H}_t}^2
= \sum_{\mu \in \Lambda_n^+} d(\mu)^2 |a_t(\mu)|^2 \cdot \frac{e^{2t(\mu+2\rho, \mu)}}{d(\mu)}
= \sum_{\mu \in \Lambda_n^+} d(\mu) |a_t(\mu)|^2 e^{2t(\mu+2\rho, \mu)}
< \infty.
\]
Hence, $F \in \mathcal{H}_t(M_n^\mathbb{C})^{K_n^\mathbb{C}}$. This completes the proof of theorem.

We end this section by discussing the reproducing kernels of the Hilbert spaces $\mathcal{H}_t(M_n^\mathbb{C})$ and $\mathcal{H}_t(M_n^\mathbb{C})^{K_n^\mathbb{C}}$, a closed subspace of $\mathcal{H}_t(M_n^\mathbb{C})$. By the previous proof, we see that the Fock space $\mathcal{H}_t(M_n^\mathbb{C})^{K_n^\mathbb{C}}$ has an orthonormal basis
\[
\{ \sqrt{d(\mu)} e^{-t(\mu+2\rho, \mu)} \varphi_\mu : \mu \in \Lambda_n^+ \}.
\]
Therefore, we have the formula for the reproducing kernel of $\mathcal{H}_t(M_n^\mathbb{C})^{K_n^\mathbb{C}}$ as
\[
K_t(z, w) = \sum_{\mu \in \Lambda_n^+} d(\mu) e^{-2t(\mu+2\rho, \mu)} \varphi_\mu(z) \overline{\varphi_\mu(w)}
= \sum_{\mu \in \Lambda_n^+} d(\mu) e^{-2t(\mu+2\rho, \mu)} \varphi_\mu(z) \overline{\varphi_\mu(w^*)}
\]
where the series converges absolutely and uniformly on compact subsets of $M_n^\mathbb{C} \times M_n^\mathbb{C}$.

Here, we interpret the value of $\varphi_\mu(z^*)$ for $z \in M_n^\mathbb{C}$ in the following way. Let $\sigma_n : U_n^\mathbb{C} \longrightarrow U_n^\mathbb{C}$ is an antiholomorphic involution integrated from the complex conjugation of $u_n^\mathbb{C} = u_n + \bar{i}u_n$ with respect to $u_n$. We will denote this complex conjugation on $u_n^\mathbb{C}$ by the same symbol $\sigma_n$. The map $\sigma_n$ is the Cartan involution of the Cartan decomposition $u_n^\mathbb{C} = u_n + \bar{i}u_n$. Note that $\sigma_n(t_n^\mathbb{C}) = \bar{t}_n^\mathbb{C}$. We assume that $K_n^\mathbb{C}$ is connected. Then $\sigma_n(K_n^\mathbb{C}) = K_n^\mathbb{C}$ and the involution $\sigma_n : U_n^\mathbb{C} \longrightarrow U_n^\mathbb{C}$ induces the well-defined map on $M_n^\mathbb{C}$ by $g \cdot x_0 \longmapsto \sigma_n(g) \cdot x_0$ and we also denote this involution by $\sigma_n$. Here, $x_0 = eK_n^\mathbb{C}$ is the base point. Finally, for $g \cdot x_0 \in M_n^\mathbb{C}$, we define
\[
(g \cdot x_0)^* = \sigma_n(g \cdot x_0)^{-1} = \sigma_n(g)^{-1} \cdot x_0.
\]
Then for $gk \cdot x_0 = g \cdot x_0 \in M_n^\mathbb{C}$ where $g \in U_n^\mathbb{C}$ and $k \in K_n^\mathbb{C}$,
\[
(gk \cdot x_0)^* = \sigma_n(gk)^{-1} \cdot x_0 = \sigma_n(k^{-1}) \sigma_n(g)^{-1} \cdot x_0.
\]
Since $\varphi_\mu$ is $K_n^\mathbb{C}$-invariant by Lemma 4.7, $\varphi_\mu(kg \cdot x_0) = \varphi_\mu(g \cdot x_0)$ for all $g \in U_n^\mathbb{C}$ and $k \in K_n^\mathbb{C}$. Therefore, $\varphi_\mu(z^*)$ is just one value for every representative of $z \in M_n^\mathbb{C}$. That is, $\varphi_\mu(z^*)$ makes sense despite the fact that $z^*$ is not well-defined.
We can also find the reproducing kernel $K^t(z, w)$ for the Hilbert space $\mathcal{H}_t(M^C_n)$ as follows. For $F = H_t, f \in \mathcal{H}_t(\mathcal{F}_C)$, we have

\[
F(w) = (f \ast h_{t,n})(w) \quad (5.4)
\]

\[
= \langle f, L_{\sigma_n(w)}h_{t,n} \rangle \quad (5.5)
\]

\[
= \langle H_t, f, H_t(n(L_{\sigma_n(w)}h_{t,n})) \rangle \quad (5.6)
\]

where we define $L_{\sigma_n(w)}h_{t,n}(z) = \tilde{h}_{t,n}(\sigma_n(w)^{-1}z)$. Note that $\tilde{h}_{t,n}$ is $K^C_n$-invariant. So $\tilde{h}_{t,n}(\sigma_n(w)^{-1}z)$ is well-defined. Now, we have

\[
K^t(z, w) = K^t(z) \quad (5.7)
\]

\[
= H_t, (L_{\sigma_n(w)}h_{t,n})(z) \quad (5.8)
\]

\[
= (L_{\sigma_n(w)}h_{t,n}) \ast h_{t,n}(z) \quad (5.9)
\]

\[
= \tilde{h}_{2t,n}(\sigma_n(w)^{-1}z) \quad (5.10)
\]

\[
= \tilde{h}_{2t,n}(w \ast z). \quad (5.11)
\]

We note that

\[
\int_{K_n} K^t(k \cdot z, k \cdot w) \, dk = \int_{K_n} \tilde{h}_{2t,n}((kw)^*kz) \, dk
\]

\[
= \int_{K_n} \tilde{h}_{2t,n}(w^*kz) \, dk
\]

\[
= \sum_{\mu \in \Lambda_n^+} d(\mu) e^{-2t(\mu + 2\rho, \mu)} \tilde{\varphi}_\mu(w^*kz) \, dk
\]

\[
= \sum_{\mu \in \Lambda_n^+} d(\mu) e^{-2t(\mu + 2\rho, \mu)} \tilde{\varphi}_\mu(w^*) \tilde{\varphi}_\mu(z)
\]

Thus, we have the relation between $\mathcal{K}^t(z, w)$ and $K^t(z, w)$ as

\[
\int_{K_n} K^t(k \cdot z, k \cdot w) \, dk = \mathcal{K}^t(z, w).
\]

In other words, $\mathcal{K}^t(z, w)$ is the $K_n$-average version of $K^t(z, w)$.

**Remark.** In (5.5), we use the fact that

\[
H_t, f = f \ast h_t = f \ast \tilde{h}_t,
\]

where $\tilde{h}_t$ the a holomorphic extension to $M^C_n$ of $h_t$. This fact can be verified as follows.
Recall that we have the Fourier series expansion of \( h_t \) as

\[
h_t(x) = \sum_{\mu \in \Lambda_n^+} d(\mu) e^{-t(\mu + 2\rho, \mu)} \phi_{\mu}(u), \quad x = u \cdot o,
\]

where the sum converges uniformly on \( U \). By Proposition V.2.3. in [15], we have the estimate

\[
\widetilde{\varphi}_{\mu}(\exp(H)) \leq e^{\mu(H)}
\]

for \( H \in (\mathbf{i}a)_+, \mu \in P^+ \). It follows that for \( z = u \exp H \in \mathcal{X}_C, H \in (\mathbf{i}a)_+ \) (see Theorem 4.20),

\[
\left| \widetilde{\varphi}_{\mu}(z) \right|^2 = \left| \langle e_{\mu}, \widetilde{\pi}_{\mu}(u \exp H)e_{\mu} \rangle \right|^2
\]

\[
= \left| \langle \pi_{\mu}(u^{-1})\widetilde{\pi}_{\mu}((\exp H)^*)e_{\mu}, e_{\mu} \rangle \right|^2
\]

\[
\leq \| \widetilde{\pi}_{\mu}(\exp H)e_{\mu} \|^2
\]

\[
= \langle e_{\mu}, \widetilde{\pi}_{\mu}((\exp H)^*)\widetilde{\pi}_{\mu}(\exp H)e_{\mu} \rangle
\]

\[
= \langle e_{\mu}, \widetilde{\pi}_{\mu}(\exp H)\widetilde{\pi}_{\mu}(\exp H)e_{\mu} \rangle \quad \text{(by Lemma 4.36)}
\]

\[
= \langle e_{\mu}, \widetilde{\pi}_{\mu}(\exp 2H)e_{\mu} \rangle
\]

\[
\leq e^{\mu(2H)} \leq e^{2\|H\|\|\mu\|}.
\]

Therefore, the series

\[
\sum_{\mu \in \Lambda_n^+} d(\mu) e^{-t(\mu + 2\rho, \mu)} \widetilde{\varphi}_{\mu}
\]

converges uniformly on compact subsets of \( \mathcal{X}_C \). Hence, it represents a holomorphic function on \( \mathcal{X}_C \). Moreover, it is the holomorphic extension to \( \mathcal{X}_C \) of \( h_t \). Since the above series converges uniformly on compact subsets of \( \mathcal{X}_C \), by the same calculations as those on page 72, we have

\[
\widetilde{f} \ast h_t = f \ast \widetilde{h}_t.
\]

### 5.7 Direct Limits and Inverse Limits

The main reference of this section is in the appendixes A and B of [43]. We give the definitions of the direct and inverse limits by using the universal mapping properties.

**Definition 5.13.** Let \( C \) be a category and \( A \) a directed set. A family \( \{s_\alpha, \phi_{\beta, \alpha}\}_{\alpha, \beta \in A} \) is called a direct system in \( C \) indexed by \( A \) if

1. \( S_\alpha \) is an object in \( C \) for each \( \alpha \in A \) and
(ii) for $\alpha \leq \beta \leq \gamma$ in $A$, $\phi_{\beta,\alpha} : S_{\alpha} \to S_{\beta}$ are morphisms of $C$ such that

(a) $\phi_{\alpha,\alpha}$ are the identity morphisms for all $\alpha \in A$ and
(b) for $\alpha \leq \beta \leq \gamma$ in $A$, $\phi_{\gamma,\beta} \circ \phi_{\beta,\alpha} = \phi_{\gamma,\alpha}$.

**Definition 5.14.** Let $\{s_{\alpha}, \phi_{\beta,\alpha}\}_{\alpha,\beta \in A}$ be a direct system in $C$. Let $T$ be an object of $C$. Fix an index $\delta \in A$ and a family $\{f_{\alpha}\}_{\alpha \in A, \delta \leq \alpha}$, where $f_{\alpha} : S_{\alpha} \to T$ is a morphism in $C$ for each $\alpha \geq \delta$. Then the family $\{f_{\alpha}\}_{\alpha \in A}$ is called compatible if

$f_{\beta} \circ \phi_{\beta,\alpha} = f_{\alpha}$ for $\delta \leq \alpha \leq \beta$ in $A$.

**Definition 5.15.** Let $\{s_{\alpha}, \phi_{\beta,\alpha}\}_{\alpha,\beta \in A}$ be a direct system in $C$. Then its direct limit in $C$ is a pair $(S, \{\phi_{\alpha}\}_{\alpha \in A})$ where $S$ is an object of $C$ and $\phi_{\alpha} : S_{\alpha} \to S$ is a morphism in $C$ such that

$\phi_{\beta} \circ \phi_{\beta,\alpha} = \phi_{\alpha}$ for $\alpha \leq \beta$ in $A$.

and

for every object $T$ of $C$ and every compatible family of morphisms $f_{\alpha} : S_{\alpha} \to T$ in $C$, there is a unique morphism $f : S \to T$ in $C$ such that $f \circ \phi_{\alpha} = f_{\alpha}$ for all $\alpha \geq \delta \in A$.

The direct limit of $\{S_{\alpha}, \phi_{\beta,\alpha}\}_{\alpha,\beta \in A}$ in $C$ is denoted by

$\lim_{\longrightarrow} \{S_{\alpha}, \phi_{\beta,\alpha}\}_{\alpha,\beta \in A}$ or simply $S = \lim_{\longrightarrow} S_{\alpha}$.

The morphism $f$ is called the direct limit of $\{f_{\alpha}\}_{\alpha \in A}$; denoted by $f = \lim_{\longrightarrow} f_{\alpha}$.
The solid arrows of the above commutative diagram show a direct system and its direct limit, while the dashed arrow denotes the direct limit of the compatible family \( \{ f_\alpha \} \).

The concept of inverse system and inverse limit are dual to the concepts of direct system and direct limit: the inverse system and inverse limit are obtained by turning backwards all arrows in the definition of direct system and direct limit.

The following commutative diagram features the concept of inverse system and inverse limit.

![Diagram](image)

In the above commutative diagram, the solid arrows designate an inverse an inverse system and its inverse limit, the dotted arrows denote the compatible family of morphisms and the dashed arrow denotes the inverse limit of the compatible family \( \{ g_\alpha \}_{\alpha \in A} \), i.e. the unique morphism whose existence is guaranteed by the universal mapping property of the inverse system.

In the category of vector spaces, the direct limit \((V, \{ \phi_\alpha \}_{\alpha \in A})\) of a direct system \(\{V_\alpha, \phi_{\beta,\alpha}\}_{\alpha,\beta} \in A\) is constructed as follows. An element of the vector space \(V\) is an equivalence class \(\{v_\alpha\}\) of sets \(\{v_\alpha\}\) where each \(v_\alpha \in V_\alpha\) and, for some \(\beta \in A\), if \(\beta \leq \gamma\) then \(v_\gamma = \phi_{\gamma,\beta}(v_\beta)\). The equivalence relation is defined by the eventual behavior:

\[ \{v_\alpha\} \sim \{v'_\alpha\} \iff \exists \beta \in A, v_\gamma = v'_\gamma \text{ whenever } \beta \leq \gamma. \]

The vector operations of \(V\) are given by

\[ [v_\alpha] + [v'_\alpha] = [v_\alpha + v'_\alpha] \quad \text{and} \quad c[v_\alpha] = [cv_\alpha]. \]

The linear maps \(\phi_\beta : V_\beta \to V\) are defined by

\[ \phi_\beta(x) = [v_\gamma] \quad \text{where} \quad v_\gamma = \phi_{\gamma,\beta}(x) \text{ if } \beta \leq \gamma; \quad \text{and} \quad v_\gamma = 0_{V_\gamma} \text{ otherwise}. \]
Now we discuss the direct system and direct limit for the category of Hilbert spaces.

**Proposition 5.16.** Let \( \{H_\alpha, \eta_{\beta,\alpha}\}_{\alpha, \beta \in A} \) be a direct system of Hilbert spaces and unitary embeddings. Then the direct limit

\[
(H, \{\eta_\alpha\}_{\alpha \in A}) = \lim_{\alpha \to \beta} \{H_\alpha, \eta_{\beta,\alpha}\}_{\alpha, \beta \in A}
\]

exists in the category of Hilbert spaces and unitary embeddings.

*Proof.* Let \( V \) denote the direct limit \( (H, \{\eta_\alpha\}_{\alpha \in A}) = \lim_{\alpha \to \beta} \{H_\alpha, \eta_{\beta,\alpha}\}_{\alpha, \beta \in A} \) in the category of vector spaces. The norms of the spaces \( H_\alpha \) are compatible, i.e., whenever \( \alpha \leq \beta \) in \( H \) we have \( \|v\|_{H_\alpha} = \|\eta_{\beta,\alpha}(v)\|_{H_\beta} \) for all \( v \in H_\alpha \). Thus we can define a norm \( \|\cdot\| \) on \( V \) by \( \|v\| = \|\eta_\alpha^{-1}(v)\|_\alpha \) where \( \alpha \) is any index large enough that \( \eta_\alpha^{-1}(v) \neq \emptyset \). This is independent of the choice of \( \alpha \). Equipped with this norm, \( V \) is a pre-Hilbert space. Let \( H \) be its Hilbert completion.

If \( X \) is any Hilbert space and \( \{f_\alpha : H_\alpha \to X\}_{\alpha \in A} \) is compatible family of unitary injections, then the map \( f = \lim_{\alpha \to \beta} f_\alpha : H \to X \) is defined as follows. If \( v \in V \), then \( f(v) = f_\alpha(\eta_\alpha^{-1}(v)) \), where \( \alpha \) is any index large enough that \( \eta_\alpha^{-1}(v) \) is not empty. The map \( f \) is then extended uniquely by continuity to all of \( H \). \( f \) is well defined because the family \( \{f_\alpha\} \) is compatible. Clearly, \( f \) is linear. The definition of \( f \) implies that \( f \circ \eta_\alpha = f_\alpha \) for every index \( \alpha \). It remains to show that \( f \) is also a unitary embedding. This is obvious because for \( v \in V \),

\[
\|f(v)\| = \|f_\alpha(\eta_\alpha^{-1}(v))\| = \|\eta_\alpha^{-1}(v)\| = \|v\|,
\]

and hence the above also holds for all \( v \in H \) by the continuity extension. \( \square \)

**Proposition 5.17.** Let \( \{H^*_\alpha, \eta^*_{\beta,\alpha}\}_{\alpha, \beta \in A} \) be an inverse system of Hilbert spaces and surjective partial isometries. Then the inverse limit

\[
(H^*, \{\eta^*_\alpha\}_{\alpha \in A}) = \lim_{\alpha \to \beta} \{H^*_\alpha, \eta^*_{\beta,\alpha}\}_{\alpha, \beta \in A}
\]

exists in the category of Hilbert spaces and surjective partial isometries. In fact,

\[
(H, \{\eta_\alpha\}_{\alpha \in A}) = \lim_{\alpha \to \beta} \{H_\alpha, \eta_{\beta,\alpha}\}_{\alpha, \beta \in A} \iff (H^*, \{\eta^*_\alpha\}_{\alpha \in A}) = \lim_{\alpha \to \beta} \{H^*_\alpha, \eta^*_{\beta,\alpha}\}_{\alpha, \beta \in A}.
\]

*Proof.* First, we recall that every Hilbert space is self-dual and note that if \( T \) is a linear map between two Hilbert spaces, then \( T \) is a unitary embedding if and only if \( T \) is a surjective partial isometry.

The dual spaces \( H_\alpha = H^*_{\alpha^{**}} \) to the \( H^*_\alpha \) together with the adjoint maps \( \eta_{\beta,\alpha} = \eta^*_{\beta,\alpha} \) to the \( \eta_{\beta,\alpha} \), constitute the direct system \( \{H_\alpha, \eta_{\beta,\alpha}\}_{\alpha, \beta \in A} \) dual to \( \{H^*_\alpha, \eta^*_{\beta,\alpha}\}_{\alpha, \beta \in A} \). This direct system is a direct system of Hilbert spaces and unitary embeddings. By Proposition 5.16, it has a direct limit \( (H, \{\eta_\alpha\}_{\alpha \in A}) \).

103
Let $X$ be a Hilbert space and $\{f_\alpha : X \to H_\alpha^*\}$ a compatible family of surjective partial isometries. Then the first diagram is commutative. We will show that there exists a surjective partial isometry $f : X \to H_\alpha^*$ such that $\eta_\alpha^* \circ f = f_\alpha$ for all $\alpha \in A$, in other words such that the first diagram remains commutative when $f : X \to H_\alpha^*$ is adjoined to it.

Taking the adjoints of all maps in the first diagram we obtain

By Proposition 5.16, there is a unique unitary embedding $g : H \to X^*$ such that $\eta_\alpha \circ g = f_\alpha^*$ for each $\alpha \in A$. In other words, adjoining the arrow $g : H \to X^*$ to the second diagram results in a commutative diagram.

The map $f = g^*$ is a surjective partial isometry and has the required properties. This proves that $H^* = \lim \left\{ H_\alpha^*, \eta_{\beta,\alpha}^* \right\}_{\beta, \alpha \in A}$ in our category. Dualize the argument to prove the remaining statement.

\[ \square \]
5.8 The Segal-Bargman Transform on the Direct Limits

In this section, we will construct the Segal-Bargmann transform on the direct limit of the Hilbert spaces \( \{L^2(M_n)^{K_n}\}_n \). First for \( m > n \), we construct the unitary embeddings

\[
\eta_{m,n} : L^2(M_n)^{K_n} \longrightarrow L^2(M_m)^{K_m}
\]

by

\[
\eta_{m,n} \left( \sum_{I_n \in \mathcal{I}_n} d(\mu_{I_n}) \hat{f}(\mu_{I_n}) \varphi_{\mu_{I_n}} \right) = \sum_{(I_n,0) \in \mathcal{I}_m, I_n \in \mathcal{I}_n} d(\mu_{(I_n,0)}) \sqrt{\frac{d(\mu_{I_n})}{d(\mu_{(I_n,0)})}} \hat{f}(\mu_{I_n}) \varphi_{\mu_{(I_n,0)}},
\]

for \( f \in L^2(M_n)^{K_n} \). All the summations are in the \( L^2 \)-sense. We use the notations as in Section 5.5 for the above construction. By Lemma 5.9, the map \( \eta_{m,n} \) is well-defined. It is easy to see that it is linear. Moreover,

\[
\| \eta_{m,n}(f) \|_{L^2(M_m)^{K_m}}^2 = \sum_{(I_n,0) \in \mathcal{I}_m, I_n \in \mathcal{I}_n} d(\mu_{(I_n,0)})^2 \frac{d(\mu_{I_n})}{d(\mu_{(I_n,0)})} |\hat{f}(\mu_{I_n})|^2 \| \varphi_{\mu_{(I_n,0)}} \|_{L^2(M_m)^{K_m}}^2
\]

\[
= \sum_{(I_n,0) \in \mathcal{I}_m, I_n \in \mathcal{I}_n} d(\mu_{I_n}) |\hat{f}(\mu_{I_n})|^2
\]

\[
= \sum_{I_n \in \mathcal{I}_n} d(\mu_{I_n}) |\hat{f}(\mu_{I_n})|^2
\]

\[
= \| f \|_{L^2(M_n)^{K_n}}^2.
\]

Therefore, \( \eta_{m,n} \) are unitary embeddings for \( m > n \). Furthermore, it is not hard to see that if \( n \leq m \leq p \), then

\[
\eta_{p,n} = \eta_{p,m} \circ \eta_{m,n}.
\]

This allows us to have a direct system of Hilbert spaces \( \{L^2(M_n)^{K_n}, \eta_{m,n}\} \). By Proposition 5.16, we can construct the direct limit

\[
L^2_{\infty} := \lim_{\longrightarrow} \{L^2(M_n)^{K_n}, \eta_{m,n}\}
\]

in the category of Hilbert spaces and unitary embeddings from the above direct system. The associated direct limit maps

\[
\eta_n : L^2(M_n)^{K_n} \longrightarrow \lim_{\longrightarrow} \{L^2(M_n)^{K_n}, \eta_{m,n}\}
\]

define the direct limit

\[
L^2_{\infty} := \lim_{\longrightarrow \text{unitary embedding}} \{L^2(M_n)^{K_n}, \eta_{m,n}\} = \left( \bigcup_n \eta_n(L^2(M_n)^{K_n}) \right)_{\text{completion}}.
\]

We emphasize here again that the direct limit space \( L^2_{\infty} \) is not the same as the space of \( K^\infty \)-invariant \( L^2 \)-functions on \( M^\infty \), where \( M^\infty \) and \( K^\infty \) are the direct limits of
$M_n$ and $K_n$ in the categories of symmetric spaces and Lie groups respectively. That is any element in $L^2_\infty$ is an equivalent class (as defined in the construction of the direct limit) not an $L^2$-function on $M^\infty$. Next, we want to define the unitary embeddings in the level of the Fock spaces.

**Notation.** First, for simplicity of the formulas, for $m \geq n$ we define

\[
e(t, \mu_{I_n}) := e^{t(\mu_{I_n} + 2\rho_n, \mu_{I_n})}
\]

and

\[
e(t, \mu(I_n,0)) := e^{t(\mu(I_n,0) + 2\rho_m, \mu(I_n,0))}.
\]

We denote the summation \(\sum\) simply by \(\sum\).

Now for $m \geq n$, we define the map

\[
\phi_{m,n} : \mathcal{H}_t(M_n^C)^{K_n^C} \longrightarrow \mathcal{H}_t(M_m^C)^{K_m^C}
\]

by sending

\[
\sum_{I_n \in \mathcal{I}_n} d(\mu_{I_n}) a_t(\mu_{I_n}) \varphi_{\mu_{I_n}} \longrightarrow \sum_{I_n \in \mathcal{I}_n} d(\mu(I_n,0)) \sqrt{\frac{d(\mu_{I_n})}{d(\mu(I_n,0))}} a_t(\mu_{I_n}) \frac{e(t, \mu_{I_n})}{e(t, \mu(I_n,0))} \varphi_{\mu(I_n,0)}.
\]

Then for $F := \sum_{I_n \in \mathcal{I}_n} d(\mu_{I_n}) a_t(\mu_{I_n}) \varphi_{\mu_{I_n}} \in \mathcal{H}_t(M_n^C)^{K_n^C}$, we have

\[
\|\phi_{m,n}(F)\|_{2,t}^2 = \sum_{I_n \in \mathcal{I}_n} d(\mu(I_n,0)) \left| \frac{\sqrt{d(\mu_{I_n})}}{d(\mu(I_n,0))} a_t(\mu_{I_n}) \frac{e(t, \mu_{I_n})}{e(t, \mu(I_n,0))} \right|^2 e(t, \mu(I_n,0))^2
\]

\[
= \sum_{I_n \in \mathcal{I}_n} d(\mu_{I_n}) |a_t(\mu_{I_n}) e(t, \mu_{I_n})|^2
\]

\[
= \sum_{I_n \in \mathcal{I}_n} d(\mu_{I_n}) |a_t(\mu_{I_n}) e^{t(\mu_{I_n} + 2\rho_n, \mu_{I_n})}|^2
\]

\[
= \|F\|_{2,t}^2.
\]

Therefore, for $m \geq n$, the map $\phi_{m,n}$ are unitary embeddings. Furthermore, it is easy to verify that if $n \leq m \leq p$, then

\[
\phi_{p,n} = \phi_{p,m} \circ \phi_{m,n}.
\]

Thus, we obtain the direct limit $\mathcal{H}_t^\infty := \lim_{\longrightarrow} \{\mathcal{H}_t(M_n^C)^{K_n^C}, \phi_{m,n}\}$ in the category of Hilbert spaces and unitary embeddings. Next, we prove the following lemma.
Lemma 5.18. For $m \geq n$, the following diagram is commutative.

\[
\begin{array}{ccc}
L^2(M_n)^{K_n} & \xrightarrow{\eta_{m,n}} & L^2(M_m)^{K_m} \\
\downarrow \quad H_{t,n} & & \downarrow \quad H_{t,m} \\
\mathcal{H}_t(M_n)^{K_n} & \xrightarrow{\phi_{m,n}} & \mathcal{H}_t(M_m)^{K_m}
\end{array}
\]

Proof. Let $f \in L^2(M_n)^{K_n}$. First we find $\phi_{m,n}(H_{t,n}(f))$. Writing

\[f = \sum_{I_n \in \mathcal{I}_n} d(\mu_{I_n}) \hat{f}(\mu_{I_n}) \varphi_{\mu_{I_n}},\]

we see that

\[H_{t,n}(f) = \sum_{I_n \in \mathcal{I}_n} d(\mu_{I_n}) \hat{f}(\mu_{I_n}) e(t, \mu_{I_n})^{-1} \varphi_{\mu_{I_n}}.\]

Therefore, we have

\[
\phi_{m,n}(H_{t,n}(f)) = \sum_{I_n \leftarrow I_m} d(\mu_{I_n,0}) \sqrt{\frac{d(\mu_{I_n})}{d(\mu_{I_n,0})}} \hat{f}(\mu_{I_n}) e(t, \mu_{I_n})^{-1} \varphi_{\mu_{I_n,0}}.
\]

On the other hand,

\[
\eta_{m,n}(f) = \sum_{I_n \leftarrow I_m} d(\mu_{I_n,0}) \sqrt{\frac{d(\mu_{I_n})}{d(\mu_{I_n,0})}} \hat{f}(\mu_{I_n}) \varphi_{\mu_{I_n,0}}
\]

implies that

\[
H_{t,m}(\eta_{m,n}(f)) = \sum_{I_n \leftarrow I_m} d(\mu_{I_n,0}) \sqrt{\frac{d(\mu_{I_n})}{d(\mu_{I_n,0})}} \hat{f}(\mu_{I_n}) e(t, \mu_{I_n,0})^{-1} \varphi_{\mu_{I_n,0}}.
\]

Thus, we see that $\phi_{m,n}(H_{t,n}(f)) = H_{t,m}(\eta_{m,n}(f))$. Hence, the diagram is commutative.

Using this lemma, we obtain the following chain of commutative diagrams:
Theorem 5.19. Let $\mathcal{L}_\infty^2 := \lim\{L^2(M_n)^{K_n}, \eta_{m,n}\}$ and $\mathcal{H}_t^\infty := \lim\{\mathcal{H}_t(M_n^C)^{K_n^C}, \phi_{m,n}\}$ in the category of Hilbert spaces and unitary embeddings. For each positive integer $n$, define $f_n := \phi_n \circ H_{t,n} : L^2(M_n)^{K_n} \to \mathcal{H}_t^\infty$. Then there is a unique unitary isomorphism $H_{t,\infty} := \lim f_n : \mathcal{L}_\infty^2 \to \mathcal{H}_t^\infty$ such that, $H_{t,\infty} \circ \eta_n = f_n$ for all positive integers $n$.

Proof. For $v \in \bigcup_n \eta_n(L^2(M_n)^{K_n})$, we define

$$H_{t,\infty}(v) = f_n(\eta^{-1}(v)),$$

where $n$ is large enough so that $\eta^{-1}(v)$ is not empty. The map $H_{t,\infty}$ is well-defined because the above chain of diagrams is commutative. Clearly $H_{t,\infty}$ is linear. The map $H_{t,\infty}$ is then uniquely extended by continuity to all of $\mathcal{L}_\infty^2 = \left(\bigcup_n \eta_n(L^2(M_n)^{K_n})\right)^\text{completion}$.

Since $f_n = \phi_n \circ H_{t,n}$ is a unitary embedding for each $n$, $H_{t,\infty}$ is a unitary embedding. The definition of $H_{t,\infty}$ implies that $H_{t,\infty} \circ \eta_n = f_n$ for all positive integers $n$.

To show that $H_{t,\infty}$ is onto, we use the fact that $H_{t,n} : L^2(M_n)^{K_n} \to \mathcal{H}_t(M_n^C)^{K_n^C}$ is a unitary isomorphism. For each $n$, let

$$g_n := \eta_n \circ H_{t,n}^{-1}.$$ 

Next we define

$$g(w) = g_n(\phi^{-1}(w)) \quad \text{for} \quad w \in \bigcup_n \phi_n(\mathcal{H}_t(M_n^C)^{K_n^C})$$

where $n$ is large enough so that $\phi^{-1}(v)$ is not empty. We can uniquely extend $g$ by continuity to all of $\mathcal{H}_t^\infty$. It is easy to see that

$$H_{t,\infty} \circ g = I \quad \text{on} \quad \bigcup_n \phi_n(\mathcal{H}_t(M_n^C)^{K_n^C})$$

and

$$g \circ H_{t,\infty} = I \quad \text{on} \quad \bigcup_n \eta_n(L^2(M_n)^{K_n}).$$

Therefore, $H_{t,\infty} \circ g = I$ on $\mathcal{H}_t^\infty$ and $g \circ H_{t,\infty} = I$ on $\mathcal{L}_\infty^2$. So $g = H_{t,\infty}^{-1}$. Therefore, $H_{t,\infty}$ is onto. Hence, $H_{t,\infty}$ is a unitary isomorphism.
References


Vita

Keng Wiboonton was born in November 1978, in Chonburi, Thailand. He received, from Chulalongkorn University in Thailand, his bachelor of science degree in mathematics in March 2000, and master of science degree in mathematics in August 2002. He earned a master of science degree in mathematics from the University of Wisconsin-Madison in May 2005. In August 2005, he began his doctorate in mathematics at Louisiana State University. He is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August 2009.