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Hypercube diagrams for knots, links, and knotted tori

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HYPERCUBE DIAGRAMS FOR KNOTS, LINKS, AND KNOTTED TORI

A Dissertation

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Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
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in

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by

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Abstract

For a knot $K$ the cube number is a knot invariant defined to be the smallest $n$ for which there is a cube diagram of size $n$ for $K$. Examples of knots for which the cube number detects chirality are presented. There is also a Legendrian version of this invariant called the Legendrian cube number. We will show that the Legendrian cube number distinguishes the Legendrian left hand torus knots with maximal Thurston-Bennequin number and maximal rotation number from the Legendrian left hand torus knots with maximal Thurston-Bennequin number and minimal rotation number.

Finally, there is a generalization of cube diagrams, called hypercube diagrams. We use such diagrams, which represent immersed Lagrangian tori in $\mathbb{R}^4$ to study embedded Legendrian tori in the standard contact space. We then show how to compute one of the classical invariants, the rotation class, and discuss applications to contact homology.
Chapter 1
Introduction and Background

1.1 Knots in $\mathbb{R}^3$

A knot $K \subset \mathbb{R}^3$ is an embedding $f : S^1 \to \mathbb{R}^3$. This is a mathematical formalization of the intuitive concept of taking a string and tangling it up in some manner. Of course, if the ends of the string are left loose, then any knot can theoretically be untied. However, if we connect the ends to form a continuous loop, we find that some knots can not be untied. Two knots are said to be equivalent if there is an isotopy taking one to the other. Colloquially, this is analogous to manipulating the string without cutting it apart. This brings us to one of the classical problems in knot theory. That is,

**Question 1.1.** Can one distinguish non-trivially knotted loops from the unknot?

Modern techniques have answered this question in the affirmative using knot Floer homology and its ability to detect the genus of the knot (cf. [28]). However, in general, this is still quite difficult. The more general version of this question is the following:

**Question 1.2.** Given two knots $K_1$ and $K_2$ can one determine whether or not an isotopy between them exists?

To say the least, it is difficult to show that two knots are equivalent, since one must provide an explicit isotopy taking one knot to the other. However, any quantity associated to the knot that does not change under isotopy is called a knot invariant. When two knots have distinct invariants, we can say that the knots
are not isotopic. However, this too is imperfect since if two knots yield the same
invariants one cannot conclude that the knots are isotopic.

When working with knots in $\mathbb{R}^3$, one is faced with a dilemma: one may either
work directly with the embedding itself, which can be difficult, or work with a
projection, where certain information is lost. Most of the time it is convenient
to proceed by first looking at a knot diagram, or *regular projection* of the knot,
in which the only intersections are transverse double points, and crossing data is
shown as in Figure 1.1.

![Knot diagram for the trefoil.](image)

Knot diagrams have the advantage that they are easy to construct, and isotopy
may be completely described in terms of moves on the diagram, i.e. the Reidemeis-
ter moves. A special class of knot diagrams, called *grid diagrams* is discussed in
section 1.2.

Given a knot diagram representing a knot, $K$, one may construct a diagram for
an associated knot, called the *mirror image* of $K$ and denoted $mK$, by reversing
the crossings as shown in Figure 1.2

Any knot that is distinct from its mirror image is called *chiral*, while any knot
that is isotopic to its mirror image is called *amphichiral*. Many familiar knots are
chiral. For example, any torus knot, that is, any knot that can be embedded on a
torus, is chiral.
1.2 Grid Diagrams

**Definition 1.3.** A grid diagram $G$ is an $n \times n$ square grid together with a set of $X$ and $O$ markings placed in distinct squares so that each row and each column contains exactly one $X$ and one $O$. We obtain a knot (or link) diagram from $G$ by drawing a directed edge from $X$ to $O$ in each column and from $O$ to $X$ in each row. At each crossing we specify that the vertical segment crosses over the horizontal segment.

Grid diagrams were introduced by Brunn over 100 years ago (cf. [6]) and were further developed by Cromwell (cf. [10]) in the 90’s to study knots embedded in an open book. In recent years, grid diagrams have been used to develop a combinatorial version of knot Floer homology (cf. [19]) and to study Legendrian knots (cf. [27] and [25]). One nice aspect of grid diagrams is that there is a complete set of moves describing topological isotopy of knots called grid moves, allowing one
to prove invariants using grid diagrams. One of the moves, a *stabilization* move, involves the insertion of a new row and column somewhere in the grid, as well as a new $X$ and $O$ marking. Since such a move may be done on any grid diagram, it is easy to construct larger grid diagrams representing the same knot by repeatedly stabilizing. However, it is not easy to go in the other direction. In fact, one may define the following invariant:

**Definition 1.4.** The arc index of a knot $K$, denoted $\alpha(K)$, is the smallest $n$ for which there exists a grid diagram of size $n$ representing $K$.

The arc index is one of many invariants in knot theory which is easy to define, clearly a knot invariant, but is difficult to compute. However, one thing is clear, since grid diagrams are ultimately 2-dimensional, the arc index cannot distinguish between mirror images: any grid diagram for $K$ may be converted to a grid diagram for $mK$ by changing $X$ markings to $O$ markings, and vice versa, and rotating the grid $90^\circ$ counterclockwise. Next we introduce a 3-dimensional analogue of grid diagrams.

### 1.3 Definition of a Cube Diagram

Let $n \in \mathbb{Z}^+$ and $\Gamma$ an $n \times n \times n$ cube, thought of as a 3-dimensional Cartesian grid with integer-valued vertices. A flat of $\Gamma$ is any cuboid (a right rectangular prism) with integer vertices in $\Gamma$ such that there are two orthogonal edges of length $n$ with the remaining orthogonal edge of length 1. A flat with an edge of length 1 that is parallel to the $x$-axis, $y$-axis, or $z$-axis is called an $x$-flat, $y$-flat, or $z$-flat respectively. Note that the cube itself is canonically oriented by the standard orientation of $\mathbb{R}^3$ (right hand orientation).
A *marking* is a labeled half-integer point in $\Gamma$. We mark unit cubes of $\Gamma$ with either an $X$, $Y$, or $Z$ such that the following *marking conditions* hold:

- each flat has exactly one $X$, one $Y$, and one $Z$ marking;
- the markings in each flat form a right angle such that each segment is parallel to a coordinate axis;
- for each $x$-flat, $y$-flat, or $z$-flat, the marking that is the vertex of the right angle is an $X, Y,$ or $Z$ marking respectively.

We get an oriented link in $\Gamma$ by connecting pairs of markings with a line segment whenever two of their corresponding coordinates are the same. Each line segment is oriented to go from an $X$ to a $Y$, from a $Y$ to a $Z$, or from a $Z$ to an $X$. The markings in each flat define two perpendicular segments of the link $L$ joined at a vertex, call the union of these segments a *cube bend*. If a cube bend is contained in an $x$-flat, we call it an *$x$-cube bend*. Similarly, define *$y$-cube bends* and *$z$-cube bends*. 
Arrange the markings in $\Gamma$ so that at every intersection point of the $(x, y)$-projection (i.e., $\pi_z : \mathbb{R}^3 \to \mathbb{R}^3$ given by $\pi_z(x, y, z) = (x, y)$), the segment parallel to the $x$-axis has smaller $z$-coordinate than the segment parallel to the $y$-axis. Similarly, arrange so that in the $(y, z)$-projection, $z$-parallel segments cross over the $y$-parallel segments, and in the $(z, x)$-projection, the $x$-parallel segments cross over the $z$-parallel segments (see Figure 1.5).

A set of markings in $\Gamma$ satisfying the marking conditions and crossing conditions is called a **cube diagram** for the knot or link.

As is the case for grid diagrams, there is a set of **cube moves** that completely describe topological isotopy in terms of cube diagrams. Again, there is a stabilization move, that increases the size of the diagram by 1. Thus we may make the following definition:

**Definition 1.5.** The **cube number** of a knot $K$, denoted $c(K)$, is the smallest $n$ for which there exists a cube diagram of size $n$ for the knot.

As with arc index, this invariant is difficult to compute. However, since the projections of a cube diagram are, themselves, grid diagrams, we immediately obtain:

**Theorem 1.6** (Baldridge, Lowrance [3]). For any knot, $\alpha(K) \leq c(K)$.
1.4 Liftability of grid diagrams

Because cube diagrams project to grid diagrams, it is natural to think of a cube diagram as a lift of a grid diagram corresponding to, say, the \((x, y)\)-projection of the cube. However, such lifts do not always exist (c.f. [3] and [4]).

Before proceeding, we need to establish some terminology and facts about grid diagrams (for more details see [3]). A *bend* in a grid diagram, \(G\), is a pair of segments that meet at a common \(X\) or \(O\) marking. We will refer to the former pair of segments as an \(X\)-bend and the latter as an \(O\)-bend. There are two ways to decompose any link component in \(G\) into a set of non-overlapping bends, corresponding to a choice of \(X\)-bends or \(O\)-bends. In particular, for a knot there are only two ways to decompose \(G\) into non-overlapping bends, and such a choice will be called a *bend decomposition*.

Consider a grid diagram, \(G\), together with a choice of a bend decomposition. If possible we wish to lift \(G\) to a cube diagram where \(G\) is the \((x, y)\)-projection of the cube diagram and the bend decomposition of \(G\) determines the \(z\)-cube bends of the cube diagram. While \(G\) carries with it an orientation on the knot, so does the \((x, y)\)-projection of the cube diagram. In order that these orientations agree, the \(X\)-bend decompositon of \(G\) has to be chosen—\(O\)-bends cannot be lifted to \(z\)-cube bends. Furthermore, because of the symmetry between all three projections in a cube diagram, it is enough to work only with the \((x, y)\)-projection and lift \(X\)-bends to \(z\)-cube bends.

The crossings in a grid diagram sometimes generate a *partial order* on the \(X\)-bends. Let \(b_1\) and \(b_2\) be two \(X\)-bends. If \(b_1\) crosses over \(b_2\) in \(G\) we say that \(b_1 > b_2\). Thus in any lift of \(G\), the \(z\)-cube bend corresponding to \(b_1\) must have \(z\)-coordinate greater than that of the \(z\)-cube bend corresponding to \(b_2\).
Not every grid diagram has a partial order on the $X$-bends. A grid diagram for which there is no partial order on the $X$-bends may not even lift to a lattice knot that has well-defined knot projections in the other planes (Figure 5 of [3]). However, if there is a partial ordering on the $X$-bends of the grid diagram, it will lift to a lattice knot in which all projections are well-defined knot projections (c.f. [3]). Nevertheless, even a partial order doesn’t guarantee liftability to a cube diagram as the $(y, z)$- and $(z, x)$-projections may not be grid diagrams in such a lift (c.f [3] and [4]). Below, we will introduce some grid configurations that fail to lift, not because of a lack of partial ordering but due to crossings in the $(y, z)$- or $(z, x)$-projections that do not satisfy the crossing conditions for a cube diagram. In Figures 1.6 and 1.7, the shaded regions are determined by the corresponding $X$-bend and extend from the $X$-bend to the boundary of the grid diagram as indicated. Furthermore, a dotted edge represents a sequence of edges in the grid that remains in the shaded region. This condition guarantees that at least one $z$-parallel edge will introduce a crossing in either the $(y, z)$ or $(z, x)$-projection which does not follow the crossing condition (c.f. [21]).

**Theorem 1.7.** The Type 1 configurations shown in Figure 1.6 do not appear in the projection of a cube diagram.

![Figure 1.6. Type 1 configurations.](image-url)
Proof. We will prove the result for the center configuration. The remaining cases are similar. If we assume first that there is a partial order on the $X$-bends of the grid diagram, then the shaded region must contain an $O$ marking. If there is no $O$ marking in the shaded region, then $a$ and $b$ must end at the same $X$ mark, and thus there is no partial order on the $X$-bends, and no lift. In any lift of the grid to a lattice knot satisfying the marking conditions for a cube diagram at least one such $O$ marking must represent a vertical edge that passes through the flat containing the $X$-bend shown, since $a$ is below the bend and $b$ is above. Since this $z$-parallel edge is located in the shaded region, it must either cross over the $x$-parallel edge shown in the $(z, x)$-projection or behind the $y$-parallel edge shown in the $(y, z)$-projection, which breaks the crossing condition shown in Figure 1.5.

![FIGURE 1.7. Type 2 configurations.](image)

**Theorem 1.8.** The Type 2 configurations shown in Figure 1.7 do not appear in the projection of a cube diagram.

Proof. For the first configuration shown in Figure 1.7, if the edge ending in $a$ has $z$-coordinate less than the lower $X$-bend then the sequence of edges connecting $a$ to $b$ must contain an $O$-marking. That $O$-marking corresponds to a $z$-parallel edge that crosses behind that lower $X$-bend in the $(y, z)$-projection. Otherwise, the
edge ending in \( a \) is above that lower \( X \)-bend. In this case, the sequence of edges connecting \( a' \) to \( b' \) must contain an \( O \)-marking that corresponds to a \( z \)-parallel edge that crosses above the edge ending in \( a \) in the \((z,x)\)-projection. A similar argument will show that the other configuration fails to lift as well. \( \square \)
Chapter 2
Cube Number Can Detect Chirality

2.1 Cube number and chirality

Theorem 2.1. The cube number detects the chirality of the trefoil. That is, if $K_L$ is the left hand trefoil and $K_R$ is the right hand trefoil, then $c(K_L) < c(K_R)$.

Remark 2.2. The cube number of the left hand trefoil is 5 and the cube number of the right hand trefoil is 7 (cf. Figure 2.2).

Proof. Note first that $c(K_L) = \alpha(K_L)$ (see Figure 1.4). For the right hand trefoil, Figure 2.1 shows all minimal grid diagrams. For columns 1, 4, and 5 there is a Type 1 configuration present in the grid (shown in blue and marked by the $X$-bend). For columns 2 and 3 there is either no partial order on the bends or there is a Type 2 configuration (shown in blue and marked by the $X$-bend).

For all examples that have been computed, the cube number detects the chirality of the knot as in Theorem 2.3. For knots with arc index greater than 5 proofs of this nature become infeasible. However, a computer can do the same basic checks.
for Type 1 and Type 2 configurations. A program was written that generates all grid diagrams, sifts out those that contain Type 1 and 2 configurations, and then attempts to lift the remaining diagrams to cube diagrams. Upon finding a valid cube diagram, the Jones polynomial is computed to identify the knot type. This program has been successfully run up to size 9 diagrams, generating the following result:

**Theorem 2.3.** The knots $3_1$, $5_1$, $5_2$, $7_1$, $7_2$, $7_3$, $7_4$, $7_5$, $8_{19}$, $9_{49}$, $10_{124}$, $10_{139}$, $10_{145}$, $10_{161}$, and $12_{n591}$ are distinguished from their mirror images by cube number.

![FIGURE 2.2. Size 7 cube diagram for the right hand trefoil.](image)

For most knots mentioned in Theorem 2.3 one knot has cube number equal to arc index and all that is known is that the mirror image has cube number strictly greater than arc index. However, the calculation has also yielded the result that the cube number for the right hand trefoil is equal to 7 (Figure 2.2) and the cube number for the right hand version of $5_1$ is 10 since the program found no diagrams for the right hand version of $5_1$ of size 9 or less and an example of size 10 may be constructed in a manner similar to what is shown in Figure 2.2.
Chapter 3
Background in Contact Topology

A contact structure on a 3-manifold $M$ is a completely non-integrable plane field $\xi$ in the tangent bundle. Locally, such a plane field may be given as the kernel of a 1-form $\alpha$ (i.e. for each $x \in M$, we have $\xi_x = \ker(\alpha_x)$) such that $\alpha \wedge d\alpha \neq 0$ and defines the orientation on $M$. A Legendrian knot, $L$, in a contact 3-manifold is an embedded $S^1$ such that the tangent vector always lies in the contact plane, that is:

$$T_xL \subset \xi_x, \forall x \in L.$$  

The standard contact structure on $\mathbb{R}^3$, with coordinates $(x, y, z)$, is determined by the global 1-form $\alpha = dz - ydx$. Thus, the contact planes are given by:

$$\xi_{std} = \text{Span}\left\{ \frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} \right\}$$

Darboux’s Theorem says that every contact structure looks locally like this one. That is, given any point, $x$, in a contact manifold $(M, \xi)$, there is a neighborhood $U$ that is diffeomorphic to $(\mathbb{R}^3, \xi_{std})$, and the diffeomorphism takes the contact structure on $M$ to the contact structure on $\mathbb{R}^3$. What this means is that by studying Legendrian knots in $\mathbb{R}^3$, we are studying local Legendrian knots in any contact 3-manifold.

Given a parametrization of a Legendrian knot:

$$\phi : S^1 \to (\mathbb{R}^3, \xi) : \theta \mapsto (x(\theta), y(\theta), z(\theta))$$

the contact condition says that $\phi'(\theta) \in \xi_{\phi(\theta)}$ for all $\theta \in S^1$. Since $\xi = \ker(dz - ydx)$ we have:

$$z'(\theta) - y(\theta)x'(\theta) = 0$$  \hspace{1cm} (3.1)
There are two standard projections for studying Legendrian knots in $\mathbb{R}^3$: the front projection, and the Lagrangian projection. For the time being, we will work with the more commonly used front projection:

$$\pi_y : \mathbb{R}^3 \rightarrow \mathbb{R}^2 : (x, y, z) \mapsto (x, z)$$

The image of a Legendrian knot $L$ under this map is called the front projection of $L$. If $\phi$ is a parametrization of $L$, then we get a parametrization of the front projection of $L$ as follows:

$$\phi_{\pi_y} = \pi_y \circ \phi : S^1 \rightarrow \mathbb{R}^2 : \theta \mapsto (x(\theta), z(\theta))$$

Equation 3.1 guarantees that $z'(\theta) = y(\theta)x'(\theta)$. Hence, $z'(\theta)$ is 0 whenever $x'(\theta)$ is 0. This means that the front projection cannot have vertical tangencies, and hence $\phi_{\pi_y}$ cannot be an immersion. Since $y(\theta) = \frac{z'(\theta)}{x'(\theta)}$ whenever $x'(\theta) \neq 0$ we can recover the $y$ coordinate from the projection. If we take our knots to be sufficiently generic (cf. [14]) we can arrange so that whenever $x'(\theta_0) = 0$, we have:

$$y(\theta_0) = \lim_{\theta \to \theta_0} \frac{z'(\theta)}{x'(\theta)}$$

Thus we may assume that front projections may be parametrized by a map that is an immersion except at a finite collection of cusps, each of which has a well-defined tangent line (cf. Figure 3.1).

![Inserting a cusp](Figure 3.1)

Hence there are three properties that characterize front projections of Legendrian knots. Any knot diagram:
1. that has no vertical tangencies,

2. that has only cusps as non-smooth points, and,

3. such that at each crossing the slope of the overcrossing is less than that of the undercrossing,

represents the front projection of some Legendrian knot. The last condition is due to the fact that when we take the standard \((z, x)\) Cartesian plane, then the right hand rule says that we should orient the positive \(y\)-axis into the page.

![Figure 3.2. Twisting a crossing in a diagram.](image)

Any knot diagram may be converted to a Legendrian front. Replace any arc containing a vertical tangency with a cusp (as shown in Figure 3.1). Then replace any crossing that does not satisfy Condition 3 above, with the configuration shown in Figure 3.2.

One would like to classify Legendrian knots up to Legendrian isotopy, that is, smooth isotopy through Legendrian knots. Since any Legendrian isotopy is itself a smooth isotopy, it is clear that different topological knot types are also different as Legendrian knots. Since any topological knot type may be realized as a Legendrian knot, one might ask, does topological classification imply Legendrian classification? In other words, is it possible to find non Legendrian isotopic knots that are smoothly isotopic? Clearly any topological knot invariant is insufficient for the purposes of this finer classification.
3.1 Classical Legendrian Invariants

The first classical Legendrian invariant we will discuss is the *Thurston-Bennequin invariant* which measures the twisting of the contact structure around the knot. Take a non-zero vector field along $L$ transverse to $\xi$, and define $L'$ to be the push-off of $L$ along this vector field. Then define the Thurston-Bennequin number of $L$, denoted $tb(L)$ to be the linking number of $L$ and $L'$, that is, $tb(L) = lk(L, L')$. The fact that the Thurston-Bennequin number is a Legendrian invariant follows from the Legendrian isotopy extension theorem (cf. [16]).

In $\mathbb{R}^3$ we take the vector field $v = \frac{\partial}{\partial z}$, which for any Legendrian knot defines a vector field transverse to $\xi$ along $L$. Thus $L'$ is the knot obtained by shifting $L$ slightly in the positive $z$ direction. Thus, the linking number, which is just the signed intersection count between $L$ and $L'$ may be computed directly from the front projection. Each right hand crossing contributes $+1$ while each left hand crossing contributes $-1$ to the linking number (cf. Figures 3.3 and 3.4).

![Figure 3.3](image1.png)

**FIGURE 3.3.** Positive and negative crossings in a diagram. Note, one must also consider these pictures reflected horizontally and vertically.

![Figure 3.4](image2.png)

**FIGURE 3.4.** Vertical shift at a crossing.
Each cusp also introduces a left hand crossing between $L$ and $L'$ (cf. Figure 3.5). Hence, one may calculate the Thurston-Bennequin number as follows:

$$tb(L) = \omega(\pi_y(L)) - \frac{1}{2} \text{(number of cusps in } \pi_y(L)\text{)},$$

where $\omega$ is the writhe of the diagram.

![Figure 3.5. Vertical shift at a cusp.](image)

The next classical Legendrian invariant is the *rotation number*, sometimes called the *rotation class*. While the Thurston-Bennequin number is well-defined for any Legendrian knot in any contact manifold, the rotation number is only defined for null-homologous knots. However, in $\mathbb{R}^3$ all knots are nullhomologous, since all knots bound a Seifert surface. Let $L = \partial \Sigma$ where $\Sigma$ is a Seifert surface. Then since $\Sigma$ is a surface with boundary, $\xi|_{\Sigma}$ is a trivial two-plane bundle. This trivialization induces a trivialization of $\xi|_L = L \times \mathbb{R}^2$. Let $\nu$ be a vector field tangent to $L$ pointing in the direction of the orientation on $L$. Then, using the trivialization, we may think of $\nu$ as defining a non-zero path of vectors in $\mathbb{R}^2$. As such, it has a winding number. Define the rotation number, $r(L)$ to be the winding number of this path of vectors.

In $\mathbb{R}^3$ the role of the Seifert surface may be downplayed. As we move along the knot when the $y$-coordinate increases the contact planes twist counterclockwise (viewed from the perspective of the front projection), and as the $y$-coordinate decreases, the contact planes twist clockwise, never completing a full twist in any direction. Therefore, we can see immediately that $\xi|_L$ must be trivial. Moreover, we
can use the vector field \( w = \frac{\partial}{\partial y} \) to trivialize \( \xi|_L \) without needing to find a Seifert surface. In particular, we need to count with sign how many times the tangent vector \( \nu \) to \( L \) points in the same direction as \( w \). A value of +1 is assigned any time \( \nu \) passes \( w \) in the counter-clockwise direction and a value of −1 is assigned any time \( \nu \) passes \( w \) in the clockwise direction. In the front projection, \( \nu = \pm w \) at each cusp, and only at the cusps. At a downward oriented cusp, \( \nu \) passes \( \pm w \) in the counter-clockwise direction, while at an upward oriented cusp, \( \nu \) passes \( \pm w \) in the clockwise direction. Thus, the rotation number may be computed in the front projection as follows:

\[
r(L) = \frac{1}{2}(D - U)
\]

where \( D \) is the number of downward oriented cusps, and \( U \) is the number of upward oriented cusps.

### 3.2 Classification of Legendrian Knots

Figure 3.6 shows the local picture for a Legendrian stabilization in the front projection. For any arc in the front projection, one may replace it with the one of the configurations on the right of Figure 3.6. The first one adds two downward oriented cusps, and is called a positive stabilization. The second adds two upward oriented cusps and is called negative stabilization.

![FIGURE 3.6. Positive and Negative Legendrian Stabilization.](image)

It is clear from Equation 3.2 and Figure 3.6 that positive and negative stabilization both decrease the Thurston Bennequin number by one, while the rotation
number changes by ±1 (depending on the sign of the stabilization). Thus it is easy
to make the Thurston-Bennequin number arbitrarily negative by repeated stabi-
lization. However, increasing the Thurston-Bennequin number is more difficult. In
a tight contact 3-manifold, that is, any contact manifold in which there does not
exist an embedded disk tangent to the contact planes along the entire boundary,
Eliashberg proved the following inequality relating the Thurston-Bennequin and
rotation number.

**Theorem 3.1** (Eliashberg, [13]). Let \((M, \xi)\) be a tight contact 3-manifold. Let \(L\) be a Legendrian knot in \(M\) with Seifert surface \(\Sigma_L\). Then

\[
\text{tb}(L) + |r(L)| \leq -\chi(\Sigma_L)
\]

In particular, Theorem 3.1 provides an upper bound on the Thurston-Bennequin
number of a knot in any tight contact 3-manifold. Therefore we can make the
following definition.

**Definition 3.2.** The maximal Thurston-Bennequin number \(\bar{tb}(K)\) of a topological
knot \(K\) is defined to be:

\[
\bar{tb}(K) = \max\{\text{tb}(L)|L\text{ is a Legendrian knot representing } K\}
\]

Hence, maximal Thurston-Bennequin number is a topological knot invariant.
More importantly though, the Thurston-Bennequin and rotation numbers provide
a convenient scheme for an initial classification of Legendrian knots. While the
Thurston-Bennequin and rotation numbers do not completely classify Legendrian
knots (it is possible to find distinct Legendrian knots with the same classical invari-
ants), certain classes of knots are completely classified by the classical invariants.
Specifically,
**Theorem 3.3** (Etnyre, Honda [15]). *In any tight contact three manifold, Legendrian torus knots are determined up to Legendrian isotopy by their knot type, Thurston-Bennequin invariant and rotation number.*

In fact, in the case of torus knots, the maximal Thurston-Bennequin number is known, and, for knots realizing the maximal Thurston-Bennequin number, the possible rotation numbers are known as well.

**Theorem 3.4** (Etnyre, Honda, [15]). *If $T_{p,q}$ is a torus knot with $p,q > 0$, then*

$$\bar{tb}(T_{p,q}) = pq - p - q,$$

*and if $p < 0$ and $q > 0$ then*

$$\bar{tb}(T_{p,q}) = pq.$$

**Theorem 3.5** (Etnyre, Honda, [15]). *Let $L$ be a Legendrian torus knot $T_{p,q}$, with maximal Thurston-Bennequin number. If $p,q > 0$ then $r(L) = 0$. If $p < 0$ and $q > 0$ then*

$$r(L) \in \left\{ \pm(p - |q| - 2qk) : k \in \mathbb{Z}, 0 \leq k < \frac{|p| - |q|}{|q|} \right\}.$$
Chapter 4
Legendrian Cube Number

4.1 Legendrian Cube Number

Any grid diagram represents the front projection of a Legendrian knot by following this procedure. First smooth the northeast and southwest corners (cf. Figure 4.1) of the diagram. Then convert northwest and southeast corners to cusps and rotate the grid diagram 45 degrees counterclockwise. Alternatively, to obtain a Legendrian front projection for the mirror image of the knot represented by the given grid diagram, reverse all crossings, rotate the grid 45 degrees clockwise, convert northeast and southwest corners to cusps and smooth the remaining corners. While there is no similar construction to convert a cube diagram into a Legendrian knot, each of the projections of a cube diagram is a grid diagram, and hence, represents the Legendrian front projection of some knot. Therefore one can make the following definition:

Definition 4.1. The Legendrian cube number, $c_\ell(K)$, is the smallest $n$ such that there is a cube diagram for the knot $K$ of size $n$ where the $(x, y)$-projection of the cube diagram is a grid diagram representing the Legendrian knot $K$.

It is not immediately obvious that the Legendrian cube number is defined for all Legendrian knots. The construction given in [3] shows how to lift any grid diagram
(at the cost of stabilizing the grid) to a cubic lattice knot satisfying the marking conditions for a cube diagram. The same construction may be done using only stabilizations of the grid that preserve the Legendrian type of the front projection represented by the grid. Given a grid diagram $G$ representing a Legendrian knot $K$ one may perform grid stabilizations that preserve the corresponding Legendrian knot and lift the diagram to a cubic lattice knot satisfying the marking conditions of a cube diagram, and the crossing conditions of the $(x, y)$-projection. All that remains is to show that the crossing conditions of the other two projections may be corrected by a *twisted crossing*. Figures 4.2 and 4.3 show how to insert such a correction in the $(y, z)$-projection. The construction for the $(z, x)$-projection is similar. Note that Figure 4.2 is almost the same as the twisted crossing given in [3] but has been modified slightly so that the grid stabilizations in the $(x, y)$-projection preserve the corresponding Legendrian front.

**FIGURE 4.2.** Inserting a twisted crossing.

**FIGURE 4.3.** The grid stabilizations in the $(x, y)$-projection after the insertion of a twisted crossing.
Recall that the rotation number may be computed from the Legendrian front projection as follows:

\[ r(L) = \frac{1}{2}(D - U) \]

where \( D \) is the number of downward oriented cusps and \( U \) is the number of upward oriented cusps in the Legendrian front projection. Also, the Thurston-Bennequin number of a Legendrian knot may be computed as follows:

\[ tb(L) = \omega(L) - \frac{1}{2}C \]

where \( \omega(L) \) is the writhe of the front projection of \( L \) and \( C \) is the number of cusps in the front. Furthermore, according to [25], any minimal grid diagram for a left hand Legendrian torus knot, \( T_{p,q} \), must realize the maximal Thurston-Bennequin number. We prove this fact here with a series of lemmas.

**Lemma 4.2.** For any topological knot \( K \),

\[ -\alpha(K) \leq \bar{tb}(K) + \bar{tb}(mK), \]

where \( mK \) is the mirror of \( K \).

**Proof.** Let \( G \) be a grid diagram representing \( K \) and realizing the arc index \( \alpha(K) \). Then one may convert \( G \) into two different Legendrian fronts, \( L_G \) and \( \bar{L}_G \), representing knots with topological type \( K \) and \( mK \) respectively. Since \( \omega(G) = -\omega(\bar{G}) \), one sees that

\[ tb(L_G) + tb(L_G) = -\frac{1}{2}(C_{L_G} + C_{\bar{L}_G}) = -\alpha(K). \]

Maximizing the Thurston-Bennequin number for \( K \) and \( mK \) separately means that we obtain the desired inequality. \[ \square \]

**Lemma 4.3.** For a \((p,q)\) torus knot, the arc index is \( p + q \).
Proof. Assume \( p, q > 0 \). Then by Theorem 3.4 and Lemma 4.2 we see that:

\[
-\alpha(K) \leq pq - p - q - pq = -p - q
\]

Therefore, \( \alpha(K) \geq p + q \). To obtain equality, it observe that one may always construct a grid diagram for \( T_{p,q} \) of size \( p + q \) (cf. Figure 4.4 when \( p = 9 \) and \( q = 2 \)).

\[\square\]

**Lemma 4.4.** If a grid diagram \( G \) for \( T_{p,q} \) realizes the arc index \( p + q \), then the Legendrian knots determined by it realize the maximal Thurston-Bennequin number as well.

**Proof.** Let \( L_G \) be the Legendrian knot determined by \( G \) and \( L_{\bar{G}} \) be the Legendrian knot determined by the mirror image. By assumption, and Lemma 4.2 we have:

\[
tb(L_G) + tb(L_{\bar{G}}) = -p - q = \tilde{tb}(K) + \tilde{tb}(mK).
\]

If \( tb(L_G) < \tilde{tb}(K) \) then \( tb(L_{\bar{G}}) > \tilde{tb}(K) \) which is a contradiction. Since \( tb(L_G) \) cannot be greater than \( \tilde{tb}(K) \), we have \( tb(L_G) = \tilde{tb}(K) \). \[\square\]

According to Theorem 3.5, such torus knots with maximal Thurston Bennequin number must have rotation number satisfying:

\[
r(K) \in \left\{ \pm(p - 2 - 4k) : k \in \mathbb{Z}, 0 \leq k < \frac{p - 2}{2} \right\}.
\]

### 4.2 Cube Number Detects Legendrian Type

**Theorem 4.5.** Let \( p \geq 5 \), \( K_{\text{min}} \) be the left hand \((p,2)\)-torus knot with maximal Thurston-Bennequin number and rotation number, \( r(K_{\text{min}}) = 2 - p \) and \( K_{\text{max}} \) the \((p,2)\)-torus knot with maximal Thurston-Bennequin number and \( r(K_{\text{max}}) = p - 2 \). Then the Legendrian cube number distinguishes between \( K_{\text{min}} \) and \( K_{\text{max}} \).
The proof of Theorem 4.5 will begin with a series of lemmas. Given a front projection of a Legendrian knot with maximal Thurston-Bennequin number, we use Legendrian invariants to compute the number of maxima, minima and the number of downward and upward oriented cusps (Lemma 4.7). We then get upper and lower bounds on what the writhe of the diagram can be (Lemma 4.8). By Lemma 4.3 we have that $\alpha(K) = p + 2$. Using this, we show that $c_{\ell}(K_{\text{max}}) = \alpha(K) = p + 2$ for all $p$ (Lemma 4.9). Finally, we interpret what such a front would look like as a minimal grid diagram, and show that such a grid for $K_{\text{min}}$ will necessarily contain Type 1 configurations.

Before proceeding we will define a partial order on the lattice points of a grid which will prove useful when thinking of grid diagrams as Legendrian front projections.

**Definition 4.6.** Given two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, we say that $P_2 \preceq P_1$ if and only if $x_2 \leq x_1$ and $y_2 \leq y_1$. In this case we say that $P_2$ is below $P_1$, or alternatively that $P_1$ is above $P_2$ (see Figure 4.5).

Note that in Figure 4.5 $P_3$ is not comparable to $P_1$. Points that are comparable using this partial order may be connected by an arc that consists only of upward oriented cusps (thought of as coming from a grid diagram rotated to a Legendrian
front). Note that for a pair of points that are not comparable any path in the grid connecting them, will introduce a new local extremum.

Let $G$ be a minimal grid diagram for $K_{min}$. Denote the number of northeast $X$-bends in $G$ by $X_{NE}$. Similarly define $X_{SE}$, $X_{NW}$, $X_{SW}$, $O_{NE}$, $O_{SE}$, $O_{NW}$, and $O_{SW}$. When converting $G$ to a left hand Legendrian front projection (i.e. a left hand torus knot) the number of downward oriented cusps will be $D_L = X_{NW} + O_{SE}$, and the number of upward oriented cusps will be $U_L = O_{NW} + X_{SE}$. When converting $G$ to a right hand Legendrian front projection (i.e. a right hand torus knot) the number of downward oriented cusps will be $D_R = X_{NE} + O_{SW}$ and the number of upward oriented cusps will be $U_R = O_{NE} + X_{SW}$.

$D_L = 2 + \omega + p$  \hspace{0.5cm} $U_L = \omega + 3p - 2$  \hspace{0.5cm} $D_R = 2 - p - \omega$  \hspace{0.5cm} $U_R = 2 - p - \omega$

FIGURE 4.5. $P_2$ is below $P_1$.

FIGURE 4.6. The types of bends in $G$ with the number of each type that may occur where $\omega = \omega(G)$. 

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Lemma 4.7. For a minimal grid diagram, $G$, representing $K_{\text{min}}$, the number of bends is as follows: $D_L = 2 + \omega + p$, $U_L = \omega + 3p - 2$, and $D_R = U_R = 2 - p - \omega$. Furthermore, the number of maxima and minima in a Legendrian front corresponding to $G$ must be equal.

Proof. According to [15] any Legendrian front projection for a right hand torus knot with maximal Thurston-Bennequin number has rotation number equal to 0. Hence, the number of downward oriented cusps equals the number of upward oriented cusps in the right hand Legendrian front obtained from $G$. That is, $D_R = U_R$. Furthermore, according to [15] the maximal Thurston-Bennequin number of the left hand Legendrian front corresponding to $G$ is $-2p$. Hence, for a minimal grid diagram (which must have maximal Thurston-Bennequin number according to [25]) we have the following equation:

$$-2p = \omega(G) - \frac{1}{2}(D_L + U_L).$$

Also, since the minimal rotation number realizable in a minimal grid diagram is $2 - p$ and the minimum rotation number must equal $\frac{1}{2}(D_L - U_L)$ we have the following:

$$D_L - U_L = 4 - 2p.$$

Solving for $D_L$ and $U_L$ we obtain:

$$D_L = 2 + \omega(G) + p$$

$$U_L = \omega(G) + 3p - 2$$

where $\omega(G)$ is the writhe of the diagram. Also, the total number of bends in a minimal grid $G$ of any type is $2(p + 2) = D_R + U_R + D_L + U_L$. Since $D_R = U_R$ we find:

$$D_R = U_R = 2 - p - \omega(G)$$
For the last statement, since the Euler characteristic of $S^1$ is 0 the number of index 1 critical points (relative maxima) and the number of index 0 critical points (relative minima) in a Legendrian front must be equal.

Lemma 4.8. Given a minimal grid diagram for $K_{min}$ we have the following bound on the writhe: $-p - 2 \leq \omega(G) < 2 - p$.

Proof. Since any knot diagram must contain relative maxima and minima, $D_R > 0$ and hence by Lemma 4.7, $\omega(G) < 2 - p$. Also, since $D_L \geq 0$, Lemma 4.7 implies that $\omega(G) \geq -p - 2$.

Lemma 4.9. $c_\ell(K_{\text{max}}) = \alpha(T_{p,2}) = p + 2$.

Proof. Extend the construction shown in Figure 4.7 in the obvious way.

![Figure 4.7. A cube diagram for $K_{\text{max}}$ when $p = 5$.](image)

Proof of Theorem 4.5. The remainder of the proof breaks down into four cases based on the value of $\omega(G)$ (c.f. Lemma 4.8). For all but the first case, each case breaks down into several subcases based on the relative positions of the local extrema, and the upward oriented arcs.

Case 1: $\omega(G) = 1 - p$. 

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By Lemma 4.7, \( D_L = 3, U_L = 2p - 1 \) and \( D_R = U_R = 1 \). After converting \( G \) to a Legendrian front projection we obtain a knot diagram with exactly one maximum and one minimum. Since \( T_{p,2} \) is a two-bridge knot, any diagram must contain at least 2 maxima and minima, thus we obtain a contradiction.

**Case 2:** \( \omega(G) = -p \).

By Lemma 4.7, \( D_L = 2, U_L = 2p - 2 \), and \( D_R = U_R = 2 \). A Legendrian front for \( K_{\min} \) has exactly 2 relative maxima and 2 relative minima. There are three subcases to consider:

1. both relative maxima (and both relative minima) are marked by an \( X \),

2. both relative maxima (and both relative minima) are marked by an \( O \),

3. one relative maximum (respectively minimum) is marked by an \( O \) and one relative maximum (respectively minimum) is marked by an \( X \)

![Figure 4.8](image.png)

**FIGURE 4.8.** There are two possible ways to connect the upward oriented arcs.

Note that for Subcase 1 (and by symmetry Subcase 2) the labels on the maxima and minima must all be the same, lest we have too many bends of type \( D_R \) or \( U_R \). Therefore, each maximum must connect along a downward oriented arc to a minimum via a single downward oriented cusp (c.f. Figure 4.8). There are two
possibilities for how to connect the upward oriented arcs. Denote a connection between two endpoints with a colon.

1. $A_O : A_I$ and $B_O : B_I$.


Since the first possibility creates two components, we do not consider it. For the second, since there are only 2 downward oriented cusps the downward oriented arcs may either cross once, or not at all depending on how the two configurations shown in Figure 4.8 are situated. Note that since $A_O$ connects to $B_I$, $X_1 \preceq X_3$ and since $B_O$ connects to $A_I$, $X_4 \preceq X_2$ (see Figure 4.9). Either $X_3$ will be comparable to $X_2$ or not. If $X_3$ is not comparable to $X_2$ then we have one of the four configurations shown in Figure 4.10 corresponding to the position of $X_1$ relative to $X_4$. We dispense with these four configurations by observing that in each case, the insertion of the remaining upward oriented bends yields a diagram in which there must be a commutation move that reduces the crossing number of the diagram by 2. In each case, the resulting diagram will not represent $K_{min}$. We may then assume, without loss of generality, that $X_2 \preceq X_3$. Then, we may also assume that $X_4 \preceq X_1$, following the same line of reasoning that we used to show that $X_2$ and $X_3$ must be comparable. Since $X_2 \preceq X_3$ and $X_4 \preceq X_1$ the downward oriented arcs must cross once as shown in Figure 4.11.

Thus, the upward oriented twisting arcs must complete at least $p - 1$ half twists in order to construct $K_{min}$, requiring all $2p - 2$ upward oriented cusps. Since $p \geq 5$ there must be at least four half-twists which will necessarily create Type 1 configurations. See Figure 4.11 for an example of a Legendrian front for $K_{min}$.
FIGURE 4.9. $X_1 \preceq X_3$.

FIGURE 4.10. Cases where $X_3$ is not comparable to $X_2$.

FIGURE 4.11. One possibility for diagrams with 2 relative maxima, both labeled with $X$.

For the third subcase, when one maximum is labeled with an $X$ and one is labeled with an $O$, we again consider whether the marked points are comparable.
or not. If the maxima are not comparable, the minima are not comparable, and each downward oriented arc contains a single bend, then the diagram will match one of the diagrams Figure 4.12 or 4.13. For the diagram shown in Figure 4.12 the downward oriented arcs must be positioned relative to each other such that the maxima labeled by $X$ and $O$ lie above the dotted lines, lest the upward oriented arcs require the addition of a maximum to connect with $A_I$ and $B_I$. In such a case, there are not enough upward oriented bends to create $K_{min}$.

**FIGURE 4.12.** Non-comparable maxima and minima.

For the diagram shown in Figure 4.13, the maxima must be above the crossing of the dotted lines, in order for the upward oriented arcs to connect up with $A_I$ and $B_I$. In this case, the twisting of the upward arcs requires all $2p - 2$ bends available (see Figure 4.14) and since $p \geq 5$ introduces Type 1 configurations.

**FIGURE 4.13.** Crossings of upward arcs must occur above the crossing of the dotted lines.
If both downward oriented cusps lie on the same arc (as in Figure 4.15), then the twisting of the upward oriented arcs must occur above the crossing of the downward oriented arcs. In this case, at least one of the arcs will require the addition a maximum to connect with a relative maximum.

In the case where either the maxima or minima are comparable, as in Figure 4.16, connecting the endpoints via upward oriented bends will produce a diagram in which commutation moves will allow the grid diagram to be destabilized.
Case 3: $\omega(G) = -p - 1$.

In this case $D_L = 1$, $U_L = 2p - 3$, and $U_R = D_R = 3$. Because the Legendrian front contains a single downward oriented cusp, two of the relative maxima must connect to two relative minima by a single edge each, while the third relative maximum connects to the third relative minimum via a single downward oriented cusp (both cases shown in Figure 4.17). For the two pair of extrema connected by a single edge, if the two maxima are labeled with an $X$, then the two minima must be labeled with an $O$, and hence there would be 4 bends of type $D_R$. Thus, for these extrema there is one $X$ maximum and one $O$ maximum as shown in Figure 4.17. For the upward oriented arcs there are three possibilities for how to connect the labelled endpoints in Figure 4.17. Denoting a connection between two endpoints with a colon, the subcases are:

1. $B_O : A_I$, $C_O : B_I$, $A_O : C_I$,
2. $C_O : A_I$, $A_O : B_I$, $B_O : C_I$,

The first two possibilities lead to a single component, while the third produces more than one component. The following is for the diagrams shown on the left in Figures 4.17 and 4.18. The argument for the diagrams shown on the right in Figures 4.17 and 4.18 is similar.
For Subcase 1 refer to the arc connecting $A_O$ to $C_I$ by $\alpha$, the arc connecting $C_O$ to $B_I$ by $\beta$ and the arc connecting $B_O$ to $A_I$ by $\gamma$. To form $K_{\text{min}}$ two of the upward oriented arcs must twist. The $\beta$ and $\gamma$ arcs cannot twist since the entire $\beta$ arc must lie below the $O$ maximum and in order for the $\gamma$ arc to enter the region below the $O$ maximum, it would have to contain an additional relative maximum. For similar reasons the $\alpha$ and $\gamma$ arcs cannot twist either. Therefore any twisting that occurs must occur between the $\alpha$ and $\beta$ arcs. The twisting of $\alpha$ and $\beta$ also means that $B_I$ must lie above $A_O$ and $A_I$ must lie above $B_O$, meaning that the downward oriented arcs connected to these ends must cross as shown in Figure 4.17. In order to construct $K_{\text{min}}$, the $\alpha$ and $\beta$ arcs must twist $p$ times requiring $2p - 2$ bends. Since there are only $2p - 3$ available in a minimal diagram such a diagram of $K_{\text{min}}$ cannot be minimal.

For Subcase 2 refer to the arc connecting $A_O$ to $B_I$ by $\alpha$, the arc connecting $C_O$ to $A_I$ by $\beta$ and the arc connecting $B_O$ to $C_I$ by $\gamma$. Reasoning as before, we find that the $\beta$ and $\gamma$ arcs must twist. In addition, the endpoint labeled $A_I$ must be above the endpoint labeled $B_O$ and the endpoint labeled $B_I$ must be above the endpoint labeled $A_O$. Furthermore, one may construct $K_{\text{min}}$ so that the $\beta$ and $\gamma$ arcs complete $p - 2$ half-twists as shown in Figure 4.19. The upward arc connecting the two $O$ markings requires at least one cusp, while the twisting of $\beta$ and $\gamma$ requires
$2p - 4$ bends, thus using all available upward oriented bends. Since $p \geq 5$ the twisting of $\beta$ and $\gamma$ requires at least three half-twists, and hence, contains a Type 1 configuration. A similar argument to that given for the configurations shown in Figure 4.10 will show that indeed the $X$ extrema must be nested as shown in the top left diagram of Figure 4.18. The construction described above requires that the downward oriented cusp be placed between the relative maximum and minimum labeled with $O$-markings, leading to a twist as shown on the righthand side of Figure 4.19. Indeed, there are other possibilities for how this third downward arc (connecting a relative maximum and minimum labeled with $X$-markings via a single cusp) is placed in the diagram relative to the other maxima and minima. However, if it is not placed as shown in Figure 4.19, it will not produce a minimal diagram for $T_{(p,2)}$.

![Diagram](image)

FIGURE 4.18. $\omega(G) = -p - 1$. 

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Case 4: $\omega(G) = -p - 2$.

In this Case $D_L = 0$, $U_L = 2p - 4$, and $D_R = U_R = 4$. For such a Legendrian front projection, there will be 4 relative maxima and 4 relative minima. Since there are no downward oriented cusps, each relative maximum must connect to a relative minimum by a single edge. Therefore, for each relative maximum marked with an $X$ there must be a corresponding relative minimum marked with an $O$. Since there are 4 bends each of types $D_R$ and $U_R$ there must be two relative maxima marked with an $X$ and two marked with an $O$ (the same is true for relative minima). Since the outgoing edges (those with subscript $O$ in Figure 4.20) must connect
to incoming edges (those with subscript \( I \) in Figure 4.20) we find 9 subcases for how the free ends may be connected by upward oriented arcs. Denote a connection between two endpoints with a colon.

1. \( G_O : B_I, Y_O : R_I, B_O : G_I, R_O : Y_I \)
2. \( G_O : B_I, Y_O : R_I, R_O : G_I, B_O : Y_I \)
3. \( G_O : B_I, B_O : R_I, Y_O : G_I, R_O : Y_I \)
4. \( Y_O : B_I, G_O : R_I, B_O : G_I, R_O : Y_I \)
5. \( Y_O : B_I, G_O : R_I, R_O : G_I, B_O : Y_I \)
6. \( Y_O : B_I, B_O : R_I, R_O : G_I, G_O : Y_I \)
7. \( R_O : B_I, G_O : R_I, Y_O : G_I, B_O : Y_I \)
8. \( R_O : B_I, Y_O : R_I, B_O : G_I, G_O : Y_I \)
9. \( R_O : B_I, B_O : R_I, Y_O : G_I, G_O : Y_I \)

Of these 9 subcases only Subcases 2, 3, 4, 6, 7, and 8 represent knots. The remaining subcases have more than one component. Subcases 3, 6, 7, and 8 are all handled in the same way. We will show the result in Subcase 3. Each arc has one end directed upward. By choosing one of the arcs in Figure 4.20 and following the upward end, we connect it with one of the other three arcs in Figure 4.20. Then, the other pair of arcs in Figure 4.20 must be connected by an upward arc. Therefore to construct \( K_{\min} \) in this case we must choose one of the configurations shown in Figure 4.21 and pair it with one of the configurations shown in Figure 4.22. We outline the arguments for each pairing below, and summarize the results in Table 1.
ad: Since $B_O$ connects to $R_I$ via upward cusps $O_4$ must lie above $X_2$. Either the segment connecting $X_1$ to $O_1$, denoted $XO_1$, crosses the segment connecting $X_4$ to $O_4$, denoted $XO_4$, or not. If $XO_1$ and $XO_4$ do not cross (as in Figure 4.23) then $X_1$ cannot lie above $O_3$, but this is required to connect $Y_O$ to $G_I$. If $XO_1$ and $XO_4$ do cross (as in Figure 4.24) then we cannot construct $T_{(p,2)}$.

ae and bd: Since in either case $Y_O$ connects to $G_I$ via upward cusps $O_3$ must lie below $X_1$. This ensures that $X_2$ is not below $O_4$, hence $R_I$ cannot connect to $B_O$.

af: Since $B_O$ connects to $R_I$ via upward cusps, $O_4$ must lie above $X_2$. A similar argument shows that $O_3$ must lie below $X_1$. The resulting diagram cannot be arranged so as to represent a minimal $T_{(p,2)}$. 
FIGURE 4.23. Case ad where \( XO_1 \) and \( XO_4 \) do not cross.

FIGURE 4.24. Case ad where \( XO_1 \) and \( XO_4 \) do cross.

**be:** Since \( Y_O \) connects to \( G_I \) via upward cusps \( X_2 \) must lie above \( O_3 \). Since \( O_3 \preceq X_1 \preceq O_2 \) and \( O_4 \preceq X_3 \) in order for \( R_I \) to connect with \( B_O \) via upward cusps it must be that \( X_2 \preceq O_4 \) and hence segments \( XO_3 \) and \( XO_1 \) must cross, and we cannot construct \( T_{(p,2)} \).

**bf, cd and ce:** Since \( B_O \) connects to \( R_I \) via upward cusps \( O_4 \) must lie above \( X_2 \). Also, since \( Y_O \) connects to \( G_I \) via upward cusps, \( X_1 \) must lie above \( O_3 \). The resulting diagram cannot be arranged so as to represent a minimal \( T_{(p,2)} \).
**cf**: Since $B_O$ connects to $R_I$ via upward cusps $O_4$ must lie above $X_2$. Similarly, since $Y_O$ connects to $G_I$ via upward cusp, $X_1$ must lie above $O_3$. It is possible to construct $T_{(p,2)}$ as shown in Figure 4.25, but it cannot be minimal.

![Figure 4.25. $\omega(G) = -p - 2.$](image)

**TABLE 4.1. Summary of results**

<table>
<thead>
<tr>
<th>Case</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ad$</td>
<td>Cannot produce the right knot type.</td>
</tr>
<tr>
<td>$ae$</td>
<td>Configurations cannot be closed up to produce a knot.</td>
</tr>
<tr>
<td>$af$</td>
<td>Cannot produce the right knot type.</td>
</tr>
<tr>
<td>$bd$</td>
<td>Configurations cannot be closed up to produce a knot.</td>
</tr>
<tr>
<td>$be$</td>
<td>Cannot produce the right knot type.</td>
</tr>
<tr>
<td>$bf$</td>
<td>Cannot produce the right knot type.</td>
</tr>
<tr>
<td>$cd$</td>
<td>Cannot produce the right knot type.</td>
</tr>
<tr>
<td>$ce$</td>
<td>Cannot produce the right knot type.</td>
</tr>
<tr>
<td>$cf$</td>
<td>Produces a non-minimal diagram.</td>
</tr>
</tbody>
</table>

For Subcase 2 (Subcase 4 is similar) we choose one of the three configurations shown in Figure 4.21 and one of the three shown in Figure 4.26. Because the configurations in Figures 4.21 and 4.26 are the same up to labelling we can reduce the number of cases considered (e.g. the choice $ah$ is the same as $bg$). Table 2 summarizes the results.

**ag:** Since $R_O$ connects to $G_I$ via upward cusps $X_1$ must lie above $X_4$. However, this means that there is no way for $X_3$ to be placed above $X_2$, which is necessary in order for $B_O$ to connect with $Y_I$ via upward cusps.

**ah:** Since $B_O$ connects $Y_I$ via upward cusps $X_3$ must lie above $X_2$. Similarly, since $R_O$ connects to $G_I$ via upward cusps $X_1$ must lie above $X_4$. These two conditions force segments $XO_4$ and $XO_1$ to cross. In this configuration, it is not possible to produce a minimal diagram for $T_{(p,2)}$.

**ai and bi:** As above, $X_1$ must lie above $X_4$ and $X_3$ must lie above $X_2$. While it is possible to form a $(p,2)$ torus knot from this configuration, it will not be minimal since all twisting must occur on the right-most upward arcs.

**bh:** Since $R_O$ connects to $G_I$ via upward cusps, $X_1$ must lie above $X_4$. This means that $X_3$ cannot lie above $X_2$. However, $X_3$ must lie above $X_2$ if $B_O$ is to connect to $Y_I$ via upward cusps.

**ci:** An example of this configuration is shown in Figure 4.27. Note that following a similar argument as was given in the 2 maxima/minima case (c.f. Figure 4.10) we find that the maxima and minima in this case must be nested as shown in Figure 4.27. To form $K_{min}$ it is necessary for the pair of arcs on the left to twist $i$ times, and the pair of arcs on the right to twist $j$ times, where exactly one of $i, j$ is odd and the other is even, lest there be two components. Furthermore, $i + j$ must be at least $p - 2$. The $i$ half-twists will require $2i$ bends, while the $j$ half-twists will require $2j$
bends. Thus the total number of bends required will be 

\[ 2i + 2j = 2(p - 2) = 2p - 4. \]

Since at least one of \( i, j \) is greater than 1 (because \( p \geq 5 \)), at least one of the pairs of twisted arcs must introduce a Type 1 configuration.

Table 2 summarizes the above results. Note that completing each construction will lead to the wrong knot type or a non-minimal diagram, unless we choose case \( ci \), in which case, at least one Type 1 configuration will be present.

<table>
<thead>
<tr>
<th>Case</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>( ag )</td>
<td>Configurations cannot be closed up to produce a knot.</td>
</tr>
<tr>
<td>( ah )</td>
<td>Cannot produce the right knot type.</td>
</tr>
<tr>
<td>( ai )</td>
<td>Produces a non-minimal diagram.</td>
</tr>
<tr>
<td>( bh )</td>
<td>Configurations cannot be closed up to produce a knot.</td>
</tr>
<tr>
<td>( bi )</td>
<td>Cannot produce the right knot type.</td>
</tr>
<tr>
<td>( ci )</td>
<td>See above.</td>
</tr>
</tbody>
</table>

FIGURE 4.27. \( \omega(G) = -p - 2. \)

more or less Finally, since all grid diagrams for \( K_{min} \) fail to lift due to the appearance of Type 1 configurations, \( c_\ell(K_{min}) > p + 2. \)

In the preceding theorem it was required that \( p \geq 5 \). For \( p = 3 \) (the trefoil) Legendrian cube number does not distinguish between \( K_{min} \) and \( K_{max} \). The above
proof fails for the $p = 3$ case because introducing a single half-twist for the pair of arcs shown in either top diagram in Figure 4.18 is sufficient to build the trefoil, thus avoiding the introduction of Type 1 configurations.
Chapter 5
Generalizations to Higher Dimensions

Though defined in Chapter 3 for 3-manifolds, contact structures are defined in any odd dimension. Given $M^{2n+1}$, we define a contact structure to be a completely non-integrable hyperplane field $\xi$ in the tangent bundle locally defined by a 1-form $\alpha$ such that $\alpha \wedge d\alpha^n \neq 0$. Legendrian submanifolds are embedded $n$-manifolds that are everywhere tangent to the contact hyperplanes. As in dimension 3, Darboux’s Thoerem implies that every contact structure on $M^{2n+1}$ is locally contactomorphic to the standard one on $\mathbb{R}^{2n+1}$. From this point forward, we specialize to the case of $\mathbb{R}^5$ with $wxyz$ coordinates. In this case, the standard contact structure is the kernel of the 1-form $\alpha = dt - ydw - xdz$.

Compared to Legendrian knots in $\mathbb{R}^3$, little is known about knotted Legendrian surfaces in $\mathbb{R}^5$. One reason is that in higher dimensions there are no standard representations of embedded Legendrian submanifolds that enable one to study with the same facility as front projections or Lagrangian projections of Legendrian knots in $\mathbb{R}^3$. For example, one may easily compute the classical invariants of Thurston-Bennequin and rotation numbers by looking at the front projection of a knot in $\mathbb{R}^3$. Moreover, the classical invariants are quite effective at distinguishing many knots up to Legendrian isotopy.

While the Thurston-Bennequin number may be generalized to higher dimensions, it is not always as useful as it is for knots in dimension 3. In the case we study here, knotted Legendrian tori $L \in \mathbb{R}^5$, the Thurston-Bennequin invariant is well defined (cf. [32]), but uninteresting since it is always equal to zero. In fact, the Thurston-Bennequin number in $\mathbb{R}^{2n+1}$ equals $\frac{1}{2}\chi(L)$ when $n$ is even. Furthermore,
while topological knot type provides an additional invariant for Legendrian knots in $\mathbb{R}^3$, all knotted Legendrian surfaces in $\mathbb{R}^5$ are topologically equivalent provided they are of the same genus.

The rotation number, or class, is harder to generalize to higher dimensions. Unlike the Thurston-Bennequin number, which may be defined in terms of a linking number, the rotation number requires the computation of the homotopy class of a map from $L$ to the space of Lagrangians of $\mathbb{R}^4$ with symplectic structure induced by the contact form on $\mathbb{R}^5$. Since writing down this map is non-trivial this invariant is more difficult to compute in higher dimensions.

We overcome many of the difficulties involved in writing down examples of Legendrian tori by working directly with the Lagrangian projection:

$$\pi_t : \mathbb{R}^5 \to \mathbb{R}^4 : (w, x, y, z, t) \mapsto (w, x, y, z),$$

and building a Lagrangian torus that we can then lift to a Legendrian torus in $\mathbb{R}^5$. Lagrangian hypercube diagrams (cf. Section 5.3) provide a way to construct explicit embeddings of Legendrian tori in precisely this manner. Using the explicit map defined by a Lagrangian hypercube diagram we demonstrate that the rotation class may be calculated combinatorially as follows:

**Theorem 5.1.** Given a Lagrangian hypercube diagram

$$H\Gamma = (C, \{W, X, Y, Z\}, G_{zx}, G_{wy})$$

with Lagrangian grid diagram projections $G_{zx}$ and $G_{wy}$ in $\mathbb{R}^2$, and let $L \subset \mathbb{R}^5$ be the embedded Legendrian torus determined by the lift of the Lagrangian torus defined by $H\Gamma$. Let $H_1(L) = \langle \tilde{\gamma}_{zx}, \tilde{\gamma}_{wy} \rangle$ be generated by $\tilde{\gamma}_{zx}$ and $\tilde{\gamma}_{wy}$ as in Theorem 5.23. Then, the rotation class of $L$, $r(L)$, satisfies:

$$r(L) = (w(G_{zx}), w(G_{wy})).$$
where \( w(G_{zx}) \) is the winding number of the immersed curve determined by \( G_{zx} \).

In particular, the winding number can be computed combinatorically from the Lagrangian grid diagram projection:

\[
w(G) = \frac{1}{4}(\#(\text{counterclockwise corners of } G) - \#(\text{clockwise corners of } G)).
\]

**Example 5.2.** Let \( H \Gamma \) be the Lagrangian hypercube diagram constructed from the Lagrangian grid diagrams shown in Figure 5.1 (Theorem 5.30). The Lagrangian hypercube determines an immersed Lagrangian torus \( T \) (Theorem 5.22). The lift of the Lagrangian torus \( T \) is a knotted, embedded Legendrian torus \( L \) (Theorem 5.23).

By Theorem 5.1, the rotation class of the Legendrian torus \( L \) is \( r(L) = (1, 0) \).

**FIGURE 5.1.** Unknots with rotation number 1 and 0 respectively.

Recall that the Maslov index, as defined in [29] and [12], may be viewed as a map \( \mu : H_1(L) \to \mathbb{Z} \).

**Corollary 5.3.** For \((a, b) \in H_1(L) = \langle \bar{\gamma}_{zx}, \bar{\gamma}_{wy} \rangle\), the Maslov index is

\[
\mu(A) = 2aw(G_{zx}) + 2bw(G_{wy}).
\]

The Maslov number of the torus \( L \) is the smallest positive number that is the Maslov index of some nontrivial loop (cf. [12]). Thus Corollary 5.3 enables us to compute the Maslov number of \( L \) as follows:

**Corollary 5.4.** The Maslov number of \( L \) is the non-negative number

\[
2\gcd(w(G_{zx}), w(G_{wy})).
\]

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In [12], Ekholm, Etnyre, and Sullivan compute the classical invariants for Legendrian tori obtained by front-spinning, showing that, in particular, the rotation class of the surface so obtained, is determined by the rotation number of the front projection used in the construction. Thus, their construction leads to tori with rotation class of the form \((0, r)\). Not only are we able to construct Legendrian tori in which both factors of the torus are knotted, but we show that Legendrian tori constructed from hypercube diagrams realize every possible pair of integers under the isomorphism defined by \(H\Gamma\). In particular, we get examples where the rotation class is \((0, r)\) in the following theorem by taking one of the knots to be a trivial knot with rotation number zero:

**Theorem 5.5.** Let \((m, k) \in \mathbb{Z}^2\), and \(K_1, K_2\) be any two topological knots in \(\mathbb{R}^3\). Then there is a hypercube diagram, \(H\Gamma = (C, \{W, X, Y, Z\}, G_{zx}, G_{wy})\) such that \(G_{zx}\) and \(G_{wy}\) are Lagrangian grid diagrams representing Legendrian knots in \(\mathbb{R}^3\) with the same topological knot type as \(K_1\) and \(K_2\). The Legendrian torus \(L\) determined by the lift of the Lagrangian torus determined by \(H\Gamma\) satisfies \(r(L) = (m, k)\).

Theorem 5.5 is a statement about the existence of Lagrangian hypercube diagrams. The methods used in the proof to find Lagrangian hypercube diagrams lead in general to excessively large diagrams. In practice, however, Lagrangian hypercube diagrams are easy to build by hand. Knot theory benefited greatly because of the development of nice representations for the knots: braids, knot projections, grid diagrams, etc. Theorem 5.1 and 5.5 together can be viewed as our attempt to create similar useful representations of Legendrian tori in \(\mathbb{R}^5\). In fact, computers can be used to easily generate and compute examples (see Theorem 5.30).

The results presented here represent one of the first attempts to explicitly compute classical Legendrian invariants for a large class of knotted Legendrian sub-
manifolds in $\mathbb{R}^{2n+1}$ for $n \geq 2$ (cf. [12]). We see the potential for much more: we present here key elements involved in computing the gradings and dimensions of the moduli spaces used in computing the differential in contact homology. Our future work will be on how to use the representations and the calculations presented here to compute the contact homology algorithmically directly from Lagrangian hypercube diagrams.

In fact, we were particularly interested in studying the contact homology of embedded Legendrian tori in $\mathbb{R}^5$ (or $S^5$) because of their relationship to Special Lagrangian Cones used to study the String Theory Model in physics. Briefly, according to this model, our universe is a product of the standard Minkowsky space $\mathbb{R}^4$ with a Calabi-Yau 3-fold $X$. Based upon physical grounds, the SYZ-conjecture of Strominger, Yau, and Zaslov (cf. [31]) expects that this Calabi-Yau 3-fold can be given a fibration by Special Lagrangian 3-tori with possibly some singular fibers. To make this idea rigorous one needs control over the singularities, which are not understood well. One method used to study these singularities (cf. Haskins [17] and Joyce [18]) is to model them locally as special Lagrangian cones $C \subset \mathbb{C}^3$. A special Lagrangian cone can be characterized by its associated link $L = C \cap S^5$ (the link of the singularity), which turns out to be a minimal Legendrian surface. When the link type of $L$ is a sphere, then $C$ must be a special Lagrangian plane. The interesting tractable case appears to be when the link type is an embedded torus. Several authors (cf. Castro-Urbano [7], Haskins [17], Joyce [18]) have shown that there exist infinite families of nontrivial special Lagrangian cones arising from minimal embedded Legendrian tori. We see these results as possibly laying groundwork for developing tools to understand special Lagrangian cones through the lens of contact homology.
In Section 5.1 we present a definition for the rotation class in dimension 5 and prove that it is characterized by a pair of integers. Section 5.2 discusses Lagrangian grid diagrams, which enable us to define a Lagrangian hypercube diagram in Section 5.3. In Section 5.4 we prove that a Lagrangian hypercube diagram represents an immersed Lagrangian torus in dimension 4. This torus is shown in Section 5.5 to lift to a Legendrian torus in \( \mathbb{R}^5 \) with the standard contact structure. We then prove Theorem 5.1 (Section 5.6) and close with a proof of Theorem 5.5 and further examples (Section 5.7).

### 5.1 Rotation class for embedded Legendrian tori in \( \mathbb{R}^5 \)

In [12] the classical Legendrian invariants of Thurston-Bennequin number and rotation number are generalized for \( \mathbb{R}^{2n+1} \). We recall the definition of rotation class for \( \mathbb{R}^5 \) here. Let \( \mathbb{R}^5 \) be parametrized using \( wxyzt \)-coordinates. Then \( \alpha = dt - ydw - xdz \) is a contact 1-form representing the standard contact structure on \( \mathbb{R}^5 \). The contact hyperplanes are given by:

\[
\xi = \ker(\alpha) = \{\partial_x, \partial_y, \partial_w + y\partial_t, \partial_z + x\partial_t\}.
\]

Let \( f : L \to (\mathbb{R}^5, \xi) \) be a Legendrian immersion. Then the image of \( df_x : T_xL \to T_{f(x)}\mathbb{R}^5 \) is a Lagrangian subspace of the contact hyperplane \( \xi_{f(x)} \). Choose the complex structure \( J : \xi_{(w,x,y,z,t)} \to \xi_{(w,x,y,z,t)} \) such that \( J(\partial_w + y\partial_t) = \partial_y \), \( J(\partial_y) = -(\partial_w + y\partial_t) \), \( J(\partial_z + x\partial_t) = \partial_x \), and \( J(\partial_x) = -(\partial_z + x\partial_t) \). Then the complexification \( df_C : TL \otimes \mathbb{C} \to \xi \) is a fiberwise bundle isomorphism. The homotopy class of \( (f, df_C) \) is called the rotation class of \( L \). Note that the Lagrangian projection \( \pi_t : \mathbb{R}^5 \to \mathbb{C}^4 \) gives a complex isomorphism between \( (\xi, J) \) and the trivial bundle with fiber \( \mathbb{C}^2 \). Composing \( df_C \) with \( \pi_t \) we get a trivialization \( TL \otimes \mathbb{C} \to \mathbb{C}^2 \), which we identify with \( df_C \). Furthermore, we choose Hermitian metrics on \( TL \otimes \mathbb{C} \) and \( \mathbb{C}^2 \) so that \( df_C \) is unitary. Thus \( f \) gives rise to an element of \( U(TL \otimes \mathbb{C}, \mathbb{C}^2) \). The
group of continuous maps \( C(L, U(2)) \) acts freely and transitively on \( U(TL \otimes \mathbb{C}, \mathbb{C}^2) \) and hence \( \pi_0(U(TL \otimes \mathbb{C}, \mathbb{C}^2)) \) is in one to one correspondence with \([L, U(2)]\). From this point forward, we will consider \( r(L) \) as an element \([L, U(2)]\).

In general, if \( L \) is a genus \( g \) Legendrian surface in \( \mathbb{R}^5 \), then the rotation class is an element of \([\Sigma_g, U(2)]\). When \( g = 0 \), \([S^2, U(2)] \cong \pi_2(U(2))\), and hence, the rotation class is always trivial, and uninteresting (for spheres, neither classical invariant yields any useful information). However, when \( g \geq 1 \), the rotation class can be nontrivial. In fact,

**Theorem 5.6.** The rotation class for a Legendrian torus can be thought of as an element in \( \mathbb{Z} \times \mathbb{Z} \) via the isomorphism \([T, U(2)] \cong \pi_1(U(2)) \times \pi_1(U(2))\).

**Proof.** Given a map of the standard torus, \( i : T^2 \to \mathbb{R}^5 \), let \( a = i(1 \times S^1) \) and \( b = i(S^1 \times 1) \). For \( \pi_1(U(2)) \), choose basepoint \( 1 \in U(2) \). Define \( H : [T, U(2)] \to \pi_1(U(2)) \times \pi_1(U(2)) \) to be the map \( f \mapsto (f|_a, f|_b) \). \( H \) is surjective since \( H(fg)(p, q) = (fg|_a(p), fg|_b(q)) = (f(p), g(q)) \) for any pair \( f, g \in \pi_1(U(2)) \). The \( \ker(H) \) is the set of homotopy classes of maps \( f : T \to U(2) \) such that the \( f|_{a \cup b} \) is nullhomotopic. Since \( U(2) \) is aspherical, any map such that \( f|_{a \cup b} \) is nullhomotopic must itself be nullhomotopic. Hence, the kernel is trivial and \( H \) is an isomorphism.

The existence of the isomorphism in Theorem 5.6 is, by itself, not useful in general for calculations due to the fact that the isomorphism depends heavily upon the choice of loops on the torus used to define the map: a generic embedding \( i : T^2 \to \mathbb{R}^5 \) does not have a preferred basis for homology (one can precompose with any element of \( SL(2, \mathbb{Z}) \) for example). However, Lagrangian hypercube diagrams do provide natural, albeit not canonical, choices for these loops as the torus is embedded in \( \mathbb{R}^5 \) (cf. \( \tilde{\gamma}_{zz} \) and \( \tilde{\gamma}_{wy} \) in Theorem 5.23). It is these choices together with Theorem 5.6 that allows us to write down our “preferred” calculations of
rotation class and Maslov index for loops in the embedded Legendrian torus. The calculations are important to our future work in computing contact homology of knotted Legendrian tori algorithmically. While all of our calculations in computing the contact homology from a Lagrangian hypercube diagram will depend upon these choices, the contact homology calculation in the end will not.

Before moving on to the definition of a Lagrangian hypercube diagram, we begin with a discussion of Lagrangian grid diagrams.

### 5.2 Lagrangian Grid Diagrams

Let $\mathbb{R}^3$ be given $wyt$-coordinates. Then $\alpha = dt - ydw$ is a contact 1-form representing the standard contact structure on $\mathbb{R}^3$. The contact planes are given by:

$$\xi = \ker(\alpha) = \{\partial_y, \partial_w + y\partial_t\}.$$  

The Lagrangian projection is given by:

$$\pi_L := \pi_t \circ L : S^1 \to \mathbb{R}^2 : \theta \mapsto (w(\theta), y(\theta)).$$

In general, a given knot diagram will not represent the Lagrangian projection of a Legendrian knot. However, an immersion $\gamma : S^1 \to \mathbb{R}^2 : \theta \mapsto (w(\theta), y(\theta))$ will correspond to the Lagrangian projection of a Legendrian knot in $(\mathbb{R}^3, \xi)$ if the following hold:

$$\int_0^{2\pi} y(\theta)w'(\theta)d\theta = 0$$

(5.1)

$$\int_{\theta_0}^{\theta_1} y(\theta)w'(\theta)d\theta \neq 0 \text{ whenever } \theta_0 \neq \theta_1 \text{ and } \gamma(\theta_0) = \gamma(\theta_1).$$

(5.2)

We now translate 5.1 and 5.2 in the context of grid diagrams. First, we recall the main features of an oriented grid diagram here (cf. [2]). Usually, grid diagrams
do not refer to any coordinate axes, but instead refer to ”horizontal” and ”vertical” directions. An oriented grid diagram is a grid diagram whose orientation is determined by the plane in which it is embedded. This feature is useful for keeping track of horizontal and vertical conventions when using grid diagrams that come from projections of higher dimensional objects. For example, if one embeds a grid diagram as previously defined in the \(wy\)-plane, one obtains a \(wy\)-oriented grid diagram by changing \(O\) markings to \(W\) markings and \(X\) markings to \(Y\) markings. Then take the orientation on the knot itself so that \(w\)-parallel segments are oriented from the \(W\)-marking to the \(Y\)-marking, and \(y\)-parallel segments are oriented from the \(Y\)-marking to the \(W\)-marking.

Let \(\hat{G}\) be a \(wy\)-oriented grid diagram. Typically one assigns the \(y\)-parallel segments in \(\hat{G}\) to be the over-strands at any crossing. However, in the following definition we will ignore such crossing conditions, and think of \(\hat{G}\) as an immersed \(S^1\).

**Definition 5.7.** An immersed grid diagram is an oriented grid diagram \(G\) with no crossing data specified.

An immersed grid diagram \(G\) may be thought of as a mapping \(\gamma : S^1 \to \mathbb{R}^2 : \theta \mapsto (w(\theta), y(\theta))\). Since \(w'(\theta)\) is 0 along any segment in \(G\) parallel to the \(y\)-axis, and \(y(\theta)\) is constant along any segment parallel to the \(w\)-axis, Condition 5.1 translates into

\[
\int_0^{2\pi} y(\theta)w'(\theta)d\theta = \sum_{i=1}^{n} \sigma(a_i) \cdot y_i \cdot \text{length}(a_i) = 0,
\]

where \(\{a_i\}\) is the collection of segments of \(G\) parallel to the \(w\)-axis, \(y_i\) is the \(y\)-coordinate of \(a_i\), and \(\sigma(a_i)\) is +1 if \(a_i\) is oriented left to right and −1 otherwise.

Given a crossing in \(G\) (i.e. given \(\theta_0 < \theta_1\) such that \(\gamma(\theta_0) = \gamma(\theta_1)\)), Condition 5.2
becomes:
\[
\int_{\theta_0}^{\theta_1} y(\theta) w'(\theta) d\theta = \sum_{i=1}^{m} \sigma(a_i) \cdot y_i \cdot \text{length}(c_i) \neq 0,
\]
where \( \{c_i\} \) is the set of \( w \)-parallel segments in the loop beginning and ending at the given crossing and such that \( \gamma(\theta) \neq \gamma(\theta_0) \) for all \( \theta \in (\theta_0, \theta_1) \). Condition 5.1 guarantees that choosing the other loop \( (\theta_1, \theta_0) \in \mathbb{R}/2\pi\mathbb{Z} \) will give the same integral up to sign as the one chosen. Therefore any immersed grid diagram \( G \) satisfying Conditions (1) and (2) lifts to a piecewise linear Legendrian knot in \((\mathbb{R}^3, \xi)\) as follows: choose some \( \theta_0 \in S^1 \) and define the \( t \)-coordinate \( t_0 \) of \( \gamma(\theta_0) \) to be 0. Then define

\[
t_\theta = t_0 + \int_{\theta_0}^{\theta} y(u) w'(u) du. \tag{5.3}
\]

Condition 5.1 guarantees that in defining the \( t \)-coordinate this way, the lift will be a closed loop. Condition 5.2 guarantees that the vertical and horizontal segments at a crossing will have different \( t \)-coordinates.

**Definition 5.8.** A Lagrangian grid diagram is an immersed grid diagram \( G \) satisfying Conditions 5.1 and 5.2.

Given a Lagrangian projection of a Legendrian knot \( L \), one may compute the rotation number as follows. Just as in Section 3.1, use the vector field \( w = \frac{\partial}{\partial y} \) to trivialize \( \xi|_L \). Then the rotation number may be calculated to be the winding number of the tangent vector to \( L \) with respect to this trivialization:

\[
r(L) = w(\pi_L).
\]

For a Lagrangian grid, this is simply a signed count of the corners of \( G \). Let \( B \) be the collection of corners in \( G \). Then for a corner \( b \in B \) let \( \eta(b) \) be a function that assigns a value of +1 to any corner of type \( W : NE, Y : NW, W : SW \), and
Y : SE (i.e. a counterclockwise oriented corner), and a value of −1 to any corner of type W : NW, Y : NE, W : SE, and Y : SW (i.e. a clockwise oriented corner) following the same notation as in [27] and [26]. Figure 4.6 illustrates the types of corners. Thus we observe that:

**Lemma 5.9.** Given a Lagrangian grid diagram $G$ with Legendrian lift $L$, the rotation number satisfies:

$$r(L) = w(G) = \frac{1}{4} \sum_{b \in B} \eta(b).$$

Example 5.10. Observe that $\int_G ydw = \frac{3}{2} + \frac{7}{2} - 2\left(\frac{5}{2}\right) = 0$, and for a path connecting the crossing to itself, $\int_G ydw = \frac{7}{2} - \frac{5}{2} = 1$. Hence, the unknot shown in Figure 5.2 is a Lagrangian grid. Set the $t$-coordinate of the $w$-mark in column 1 to 0 and define the lift as in Equation 5.3. Then the front projection corresponding to the lift of $G$ is shown in Figure 5.2. The rotation number is easily computed from this projection since $G$ has 3 bends that are assigned a value of +1 and 3 that are assigned a value of −1. Hence, $r(G) = 0$. 

FIGURE 5.2. A $wy$ immersed grid diagram for the unknot and its corresponding front projection.
The Legendrian knots produced using the above method will be piecewise linear, not smooth. However, we can produce smoothly embedded knots as follows. Choose $0 < \epsilon << 1$. Delete an $\epsilon$ neighborhood of each vertex of $G$ and replace it with a smooth curve (cf. Figure 5.3). Such a smoothing may be accomplished so as to guarantee that the diagram is smooth at the boundary of the $\epsilon$ neighborhood as well. For example, the image of the map

$$E(t) = (w + \epsilon - \epsilon \cos(t/\epsilon), y + \epsilon - \epsilon \sin(t/\epsilon)).$$

allows one to replace a $W : SE$ corner with a smooth arc, but the resulting rounded corner will only be $C^1$ at the boundary of the $\epsilon$ neighborhood. Note that the smoothing may be done so that the resulting curve is symmetric about the line of slope $\pm 1$ through the vertex of the bend. Furthermore, given a choice of a smoothing at a corner such that the area enclosed by the smooth curve and the original bend is $A$, one may obtain a different smoothing so that the area enclosed is $rA$ where $r \in \mathbb{R}$ such that $0 < r \leq 1$.

![Figure 5.3. A smoothing of a corner.](image)

**Proposition 5.11.** Let $\gamma : S^1 \to \mathbb{R}^2$ be the piecewise linear immersion determined by the Lagrangian grid diagram, $G$. There exists a $\delta > 0$ such that for any $0 < \epsilon \leq \delta$ there is a choice of smoothing curves based upon $\epsilon$ such that the immersion determined by the smoothed grid, $\gamma_\epsilon : S^1 \to \mathbb{R}^2$ satisfies the following:

- the lift of $\gamma_\epsilon$ is $C^0$-close to the lift of $\gamma$, and
for any two $\epsilon, \epsilon' < \delta$ the Legendrian knots $K, K'$ are Legendrian isotopic.

**Proof.** Choose $\delta > 0$ such that $\delta^2 < \frac{1}{2n}$. Let $\epsilon < \delta/2$. Enumerate the corners $b_{i,j} \in B$ so that corner $b_{i,1}$ is the corner on the lefthand side of row $i$ and $b_{i,2}$ is the corner on the righthand side of row $i$. Let $A_{i,j}$ be the absolute value of the area of the region enclosed by the smoothed arc and the original corner of the corner $b_{i,j} \in B$. Construct each smoothing so that $|A_{i,j}| \leq \epsilon$. Denote by $r_i$ the horizontal edge in row $i$. Then we have the following:

$$
\int_G ydw = \sum_{i=1}^{n} \sigma(r_i) \cdot (i \cdot \text{length}(r_i) - \tau_1(i)A_{i,1} - \tau_2(i)A_{i,2}) =
$$

$$
- \sum_{i=1}^{n} \sigma(r_i) \cdot (\tau_1(i)A_{i,1} + \tau_2(i)A_{i,2})
$$

where $\sigma(r_i)$ is $+1$ if the edge is directed left to right and $-1$ otherwise, $\tau_j(i)$ is $+1$ if the smoothing lies above the horizontal edge, and $-1$ otherwise.

Since not all of $\sigma(r_i) \cdot \tau_1(i)$ will evaluate to $+1$ (respectively, all $-1$), we may choose the smoothings so that

$$
\sum_{i=1}^{n} \sigma(r_i) \cdot (\tau_1(i)A_{i,1} + \tau_2(i)A_{i,2}) = 0.
$$

Since the value of the integral in Equation 5.2 may only change from the piecewise linear calculation by an amount less than $\frac{1}{4}$, the smoothed diagram has the same crossing data as the original Lagrangian grid diagram. The second condition of the Lemma is clear.

Note if $A_{i,j} = A$ for all $i,j$, the above sum evaluates to $4Ar(G)$. Thus, if the rotation number is 0 then the same smoothing may be used for all vertices of $G$.

**Corollary 5.12.** Let $\gamma_\epsilon$ be parametrized by $\theta \mapsto (w(\theta), y(\theta))$. Then,

$$
\left| \int_{\theta_0}^{\theta_1} y(\theta)w'(\theta)d\theta - \int_{\theta_0}^{\theta_1} y_\epsilon(\theta)w_\epsilon'(\theta)d\theta \right| < \frac{1}{4}.
$$
Proposition 5.11 and Corollary 5.12 show that a Lagrangian grid diagram corresponds to a smoothly embedded Legendrian knot that does not depend on the choice of epsilon used in the smoothing. Hence we may refer to the Legendrian knot corresponding to a Lagrangian grid diagram.

**Example 5.13.** Since the rotation number of the unknot in Figure 5.2 is 0 we may choose to smooth all corners in the same way, thus obtaining a Lagrangian projection of a smoothly embedded Legendrian knot in \((\mathbb{R}^3, \xi)\).

![Figure 5.4](image)

**Example 5.14.** The unknot shown in Figure 5.4 may easily be seen to have rotation number 1. In order to smooth the diagram, we perform the following calculation. To simplify matters choose the smoothings so that the areas satisfy \(A_{i,1} = A_{i,2}, A_i = A_{i,j}\), and all are less than \(\frac{1}{100}\),

\[
\sum_{i=1}^{n} \sigma(r_i) \cdot (\tau_1(i)A_i + \tau_2(i)A_i) = 2A_1 + 2A_2 + 2A_3 - A_4 + A_4 - 2A_5
\]

Choose the \(A_i\) so that \(A_1 = A_2 = A_3\) and \(A_5 = 3A_1\). Then this sum will be 0 and the Lagrangian grid conditions will still be satisfied by the smoothed diagram, and the diagram will be the Lagrangian projection of a smoothly embedded Legendrian knot in \((\mathbb{R}^3, \xi)\).

The Legendrian lift of the smoothed Lagrangian grid diagram is unique up to Legendrian isotopy (Proposition 5.11). By Corollary 5.12 we can do integer calculations directly from the Lagrangian grid diagram instead of the smooth \(\gamma_\epsilon\) loop,
without worrying about changing the crossing information of the lift of the La-
grangian grid diagram. In particular, there is a correspondence of horizontal edges
with opposite orientation in each column that allows one to re-interpret the La-
grangian grid conditions as a signed area sum. That is:

**Corollary 5.15.** There is a set of rectangles (possibly overlapping) with horizontal
edges lying on the knot diagram whose signed areas sum to the same value as the
integral in Equation 5.3.

**Example 5.16.** For the grid diagram in Figure 5.5, we see by computing the signed
areas shown that the integral in Equation 5.3 evaluate to $-7$. Hence, it is not a
Lagrangian grid diagram.

![Figure 5.5. Decomposition of grid into rectangles.](image)

In practice, the area calculation described in the previous example may be carried
out by simply decomposing the grid into polygonal regions where the top-most
horizontal edges are all oriented left (resp. right) and the bottom-most horizontal
edges are all oriented right (resp. left). Then, the signed area of these polygonal
regions will correspond to the integrals defined in Conditions 5.1 and 5.2. For
convenience, in the proofs that follow, we will use this signed area calculation to
compute the integrals defined in Conditions 5.1 and 5.2.

**Theorem 5.17.** Any topological knot type with any rotation number may be real-
ized as a Lagrangian grid diagram.
Before proving the theorem, we introduce some definitions and lemmas that we will use only for the proofs in this paper.

**Definition 5.18.** An almost Lagrangian grid diagram is an immersed grid diagram such that:

- the top right corner has a marking,
- there is a parametrization $\gamma : I \to G \subset \mathbb{R}^2$ in which $\gamma(\theta)$ starts and ends at that marking point.
- $\int_{\theta_1}^{\theta_2} y(\theta)w'(\theta)d\theta \neq 0$ whenever $\theta_1, \theta_2 \in (0, 1)$, $\theta_1 \neq \theta_2$ and $\gamma(\theta_1) = \gamma(\theta_2)$.

Let the $t$-coordinate of $\gamma(0)$ be 0. Then define,

$$t_\theta = \int_0^\theta y(u)w'(u)du.$$

Thus the last condition of Definition 5.18 guarantees that an almost Lagrangian grid diagram gives rise to an embedded Legendrian arc. Since the endpoints of this arc project to the top right corner marking and differ only in their $t$-coordinates, an almost Lagrangian grid diagram still gives rise to a knot in $\mathbb{R}^3$ by attaching the endpoints by a segment parallel to the $t$-axis.

**Lemma 5.19.** An almost Lagrangian grid diagram can always be modified (using configurations listed in Table 5.1) to get a Lagrangian grid diagram with the same topological knot type and winding number as the knot given by the almost Lagrangian grid diagram.

**Proof.** An almost Lagrangian grid diagram represents a Legendrian arc whose endpoints have $t$-coordinates that differ by some $k \in \mathbb{Z}$. Attach one of the configurations shown in Table 5.1. Each time such a configuration is attached, the
resulting grid will again be an almost Lagrangian grid diagram, but the difference between the end points of the new Legendrian arc will be reduced by 1 or 2. Continue reducing this difference until the arc closes up to give a Lagrangian grid diagram.

Lemma 5.20. Let $k \in \mathbb{Z}$. Any Lagrangian grid diagram can be modified to obtain a Lagrangian grid diagram with rotation number $k$.

Proof. Let $k \in \mathbb{Z}$. If the Lagrangian grid diagram does not have a marking in the top right corner, modify it so that does by stabilizing in the righthand column and commuting the horizontal edge of length 1 to the top of the grid, to obtain an almost Lagrangian grid diagram. Then, at this top right corner, attach one of the configurations shown in Figure 5.6 to change the rotation number to $k$. This new object is an almost Lagrangian grid diagram. Apply Lemma 5.19 to obtain a Lagrangian grid diagram whose lift has the same topological knot type as the original Lagrangian grid diagram.

We now proceed with the proof of Theorem 5.17.

Proof. We use Lenhard Ng’s arguments, [24], as a guide to construct Lagrangian grid diagrams. Recall that a grid diagram (in the usual sense) may be thought of
as a front projection of a Legendrian knot. Given such a front projection, we may resolve the front to obtain the Lagrangian projection of a knot isotopic to the one determined by the front. This Lagrangian projection will have the same crossing data as the original grid, and, as a diagram, is isotopic to the original grid after adding loops at each southeast corner.

We follow a similar procedure, but modify it so that we obtain a Lagrangian grid diagram. Given a grid diagram (in the usual sense), stabilize at each southeast corner (without adding a crossing), and commute the horizontal edge of length 1 to the bottom of the grid to obtain a simple front (cf. [24]). By applying another stabilization in the right-most column, and then commutation moves, we may ensure that this grid has a marking in the top right corner. Then add a loop at each southeast corner, as is done in constructing the front resolution. By possibly inserting some number of empty rows and columns, we may adjust the enclosed areas so that we obtain a diagram whose lift represents the same knot in $\mathbb{R}^3$ as the grid diagram we started with. This diagram will, in general, not be a grid diagram, since it contains empty rows and columns. At the top right corner, attach a configuration as shown in Figure 5.7 to fill in any empty rows and columns, and thus obtain an almost Lagrangian grid diagram. Then, by applying Lemmas 5.19 and 5.20, we may obtain a Lagrangian grid diagram representing the same topological knot type as the original grid diagram, and having any rotation number $k$. □
5.3 Lagrangian hypercube diagrams in dimension 4

The definition of a Lagrangian hypercube diagram codifies a data structure that mimics that of hypercube diagrams, cube diagrams and grid diagrams. While the definition appears similar to that of 4-dimensional hypercube diagrams as defined in [2], they are not equivalent. Let $n$ be a positive integer and let the hypercube $C = [0, n] \times [0, n] \times [0, n] \times [0, n] \subset \mathbb{R}^4$ be thought of as a 4-dimensional Cartesian grid, i.e., a grid with integer valued vertices with axes $w, x, y, \text{and } z$. Orient $\mathbb{R}^4$ with the orientation $w \wedge x \wedge y \wedge z$.

A flat is any right rectangular 4-dimensional prism with integer valued vertices in the hypercube such that there are two orthogonal edges at a vertex of length $n$ and the remaining two orthogonal edges are of length 1. Name flats by the axes parallel to the two orthogonal edges of length $n$. For example, a $yz$-flat is a flat that has a face that is an $n \times n$ square that is parallel to the $yz$-plane.

Similarly, a cube is any right rectangular 4-dimensional prism with integer vertices in the hypercube such that there are three orthogonal edges of length $n$ at a vertex with the remaining orthogonal edge of length 1. Name cubes by the three edges of the cube of length $n$. See Figure 5.8 for examples.
A marking is a labeled point in $\mathbb{R}^4$ with half-integer coordinates. Mark unit hypercubes in the 4-dimensional Cartesian grid with either a $W$, $X$, $Y$, or $Z$ such that the following marking conditions hold:

- each cube has exactly one $W$, one $X$, one $Y$, and one $Z$ marking;

- each cube has exactly two flats containing exactly 3 markings in each;

- for each flat containing exactly 3 markings, the markings in that flat form a right angle such that each ray is parallel to a coordinate axis;

- for each flat containing exactly 3 markings, the marking that is the vertex of the right angle is $W$ if and only if the flat is a $zw$-flat, $X$ if and only if the flat is a $wx$-flat, $Y$ if and only if the flat is a $xy$-flat, and $Z$ if and only if the flat is a $yz$-flat.

The 4th condition rules out the possibility of either $wy$-flats or a $zx$-flats with three markings. As with oriented grid diagrams and cube diagrams, we obtain an oriented link from the markings by connecting each $W$ marking to an $X$ marking by a segment parallel to the $w$-axis, each $X$ marking to a $W$ marking by a segment parallel to the $x$-axis, and so on.

Let $\pi_{xz}, \pi_{wy} : \mathbb{R}^4 \to \mathbb{R}^2$ be the natural projections. Define $G_{wy} := \pi_{xz}(C)$ and $G_{zx} := \pi_{wy}(C)$ which are immersed grid diagrams. Let $\{c_i\}$ be the crossings in $G_{zx}$, and $\{c'_i\}$ be the crossings in $G_{wy}$. Then we say that the Lagrangian crossing conditions hold for the pair $G_{zx}$ and $G_{wy}$ if $|\Delta t(c_i)| \neq |\Delta t(c'_i)| \, \forall i, j$ where $\Delta t$ is the difference in the $t$-coordinates at each crossing determined by Equation 5.2.
Definition 5.21. If the markings $\{W, X, Y, Z\}$ in $C$ satisfy the marking conditions, and the immersed grid diagrams $G_{wy}$ and $G_{zx}$ are Lagrangian grid diagrams satisfying the Lagrangian crossing conditions, then we define a Lagrangian hypercube diagram to be $H\Gamma = (C, \{W, X, Y, Z\}, G_{zx}, G_{wy})$.

5.4 Building a torus from a Lagrangian hypercube diagram

A hypercube schematic (cf. Figure 5.9) conveniently displays the markings of a Lagrangian hypercube diagram so that the Lagrangian grid diagrams $G_{zx}$ and $G_{wy}$ may be read off of the diagrams directly. To see $G_{wy}$ treat each $n \times n$ $zx$-flat as a cell of $G_{wy}$ (i.e. consider the projection $\pi_x \circ \pi_z$). Each $zx$-flat containing a $W$ and $Z$ marking will project to a cell of $G_{wy}$ containing a $W$ marking and each $zx$-flat containing an $X$ and $Y$ marking will project to a cell of $G_{wy}$ containing a $Y$ marking. In Figure 5.9, the blue shading indicates the diagram associated to $G_{wy}$. To see $G_{zx}$ in the schematic, note that each pair of markings in a $zx$-flat on
the schematic corresponds to an edge of the Lagrangian grid diagram $G_{zx}$. Placing these segments on a single $n \times n$ grid will produce a copy of $G_{zx}$.

To produce an immersed torus from the Lagrangian hypercube diagram, place a copy of the immersed grid $G_{zx}$ at each $zx$-flat on the schematic that contains a pair of markings (shown in red on Figure 5.9). Doing so produces a schematic with two copies of $G_{zx}$ with the same $y$-coordinates and two with the same $w$-coordinates. For each pair of copies sharing the same $w$-coordinate, we may translate one parallel to the $w$-axis toward the other. Doing so traces out an immersed tube connecting these two copies of $G_{zx}$. Similarly, we may translate parallel to the $y$-axis to produce an immersed tube connecting two copies of $G_{zx}$ with the same $y$-coordinates. Since we are connecting copies of $G_{zx}$ in flats corresponding to the markings of $G_{wy}$, the tube will close to produce an immersed torus. Thus we obtain:

**Theorem 5.22.** A Lagrangian hypercube diagram determines an immersed Lagrangian torus $i : T \to \mathbb{R}^4$. Furthermore, the map determines a preferred set of loops, $\gamma_{zx} = S^1 \times 1$ and $\gamma_{wy} = 1 \times S^1$, that map to curves projecting to the Lagrangian grid diagrams $G_{zx}$ and $G_{wy}$.

Since the torus is formed by the translation of $x$ and $z$-parallel segments to the $w$ and $y$ axes, we see that only $wx$, $wz$, $yz$, and $xy$ rectangles are used in the construction of the torus. Since $wy$ and $zx$ rectangles are never used in the construction of the torus, it is Lagrangian with respect to the symplectic form $dw \wedge dy + dz \wedge dx$. Furthermore, just as in the case of Lagrangian grid diagrams, we obtained a smooth embedding by carefully smoothing corners, we may obtain a smooth embedding of the torus in $\mathbb{R}^5$ by first smoothing $G_{zx}$ and $G_{wy}$ as in Lemma 5.11.
Furthermore, the torus has only two types of singularities: double point circles and intersections of double point circles. Each crossing of $G_{zx}$ generates a double point circle as shown by the yellow dots in Figure 5.9. Similarly each crossing of $G_{wy}$ generates a double point circle, which is visible in the schematic as the $zx$-flat where a $w$-parallel tube passes through a $y$-parallel tube. In Figure 5.9 this is shown by the yellow diagram. The green dot in Figure 5.9 corresponds to an intersection of two double point circles.

5.5 Lifting the hypercube to $\mathbb{R}^5$

Let $i : T \to \mathbb{R}^4$ be the immersed torus obtained from a Lagrangian hypercube diagram as given by Theorem 5.22. Note that, $d\alpha|_{wxyz\text{-hyperplane}} = \omega = dw \wedge dy + dz \wedge dx$ is a symplectic form on $\mathbb{R}^5$. We will show that $H\Gamma$ represents the
Lagrangian projection of a Legendrian surface in \( \mathbb{R}^5 \) with respect to the standard contact structure \( \xi \).

In order to lift \( i(T) \) we begin by choosing some point \( p \in i(T) \) to have \( t \)-coordinate equal to some \( t_0 \in \mathbb{R} \). If we attempt to lift \( i(T) \) to a Legendrian surface with respect to \( \alpha \) we should choose to define the \( t \)-coordinate of \( p' \neq p \) to be:

\[
t = t_0 + \int_\gamma ydw + \int_\gamma xdz,
\]

where \( \gamma \) is a path from \( p \) to \( p' \). This integral will be independent of path precisely when the 1-form \( i^*(ydw + xdz) \) is 0 on \( H_1(T) \). Recall that \( H_1(T) \) is generated by \( \gamma_{zx} \) and \( \gamma_{wy} \).

In order check for path-independence of the integral in Equation 5.4, we evaluate the following:

\[
\begin{align*}
i^*(ydw + xdz)[i^*(\gamma_{zx})] &= \int_{i^*(\gamma_{zx})} i^*(ydw + xdz) = \int_{\gamma_{zx}} ydw + \int_{\gamma_{zx}} xdz = \int_{\gamma_{zx}} ydw. \\
i^*(ydw + xdz)[i^*(\gamma_{wy})] &= \int_{i^*(\gamma_{wy})} i^*(ydw + xdz) = \int_{\gamma_{wy}} ydw + \int_{\gamma_{wy}} xdz = \int_{\gamma_{wy}} xdz.
\end{align*}
\]

Since \( G_{zx} \) and \( G_{wy} \) are Lagrangian grid diagrams, these integrals will both evaluate to 0 and we get a well-defined lift to a Legendrian torus in \( \mathbb{R}^5 \) using Equation 5.4. Furthermore, the Lagrangian crossing conditions guarantee that the lift will be embedded. Let \( L \) be the lift of \( i(T) \) obtained from Equation 5.4. Define \( \pi_t : \mathbb{R}^5 \to \mathbb{R}^4 \) to be the projection \( (w,x,y,z,t) \mapsto (w,x,y,z) \). Then \( \pi_t(L) = i(T) \), i.e. the torus determined by \( H\Gamma \) is the Lagrangian projection of the Legendrian torus \( L \).

Thus we obtain the following:

**Theorem 5.23.** The torus determined by a Lagrangian hypercube diagram \( H\Gamma \) lifts to an embedded Legendrian torus \( L \subset (\mathbb{R}^5, \xi) \). Furthermore, the generators \( \gamma_{zx} \) and \( \gamma_{wy} \) lift to curves \( \tilde{\gamma}_{zx} \) and \( \tilde{\gamma}_{wy} \) that generate \( H_1(L) \).
Remark 5.24. If we omit the Lagrangian crossing conditions from the definition of a Lagrangian hypercube diagram, then the above procedure will still produce an immersed Legendrian torus in $\mathbb{R}^5$, but it will not, in general, be embedded.

Example 5.25. Figure 5.9 shows a schematic picture of a Lagrangian hypercube diagram where all grid-projections are unknots as in Example 5.10. By Lemma 5.23, the torus determined by this Lagrangian hypercube diagram lifts to a Legendrian torus in $(\mathbb{R}^5, \xi)$.

5.6 Proof of Theorem 5.1

With the rotation class understood to be an element of $[T, U(2)]$ we see from Theorem 5.6 that the class may be identified with a pair of integers corresponding to the elements of $\pi_1(U(2))$ determined by a meridian and longitude of the torus. Before proving Theorem 5.1 we identify an explicit generator of $\pi_1(U(2))$. Recall that $U(2)$ parametrizes framed Lagrangians of $(\mathbb{R}^2, \omega)$. Identify the $yx$, $xy$, $yz$, and $zy$ planes with the following matrices:

$$U_{xy} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad U_{yx} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad U_{yz} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad U_{zy} = \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}.$$  

Note that $U_{xy}$, $U_{yx}$, $U_{yz}$, and $U_{zy}$ correspond to unitary Lagrangian frames (cf. [23]):

$$U_{xy} \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U_{yx} \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad U_{yz} \mapsto \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad U_{zy} \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Note that as maps from $\mathbb{R}^2 \to \mathbb{R}^4$ these frames produce $xy$, $(-x)y$, $(-z)y$, and $zy$-planes respectively. Geometrically, this matches up with the fact that the La-
grangian planes along an \(xz\)-slice of the hypercube will be given by a positively or negatively oriented \(\partial_x\) or \(\partial_z\) vector paired with a positively oriented \(\partial_y\)-vector.

Choose \(U_{xy}\) to be the basepoint. We define a loop \(\gamma : [0, 1] \to U(2)\) that begins at \(U_{xy}\) and rotates through \(U_{yz}, U_{yx}\) and \(U_{zy}\). We will define \(\gamma\) in 4 pieces. First, define a map \(\hat{\gamma} : [0, 1] \to U(2)\) as follows:

\[
\hat{\gamma}(t) = \begin{pmatrix}
1 & 0 \\
0 & e^{\frac{\pi}{2}it}
\end{pmatrix}.
\]

Then, define \(\gamma_1(t) = \hat{\gamma}(t)U_{xy}, \gamma_2(t) = \hat{\gamma}(t)U_{yz}, \gamma_3(t) = \hat{\gamma}(t)U_{yx},\) and \(\gamma_4(t) = \hat{\gamma}(t)U_{zy}\). Finally, define \(\gamma(t) = \gamma_1 \ast \gamma_2 \ast \gamma_3 \ast \gamma_4\). Thus \(\gamma\) corresponds to a rotation of Lagrangian planes, beginning at an \(xy\)-plane, and rotating through \(yz, yx\), and \(zy\)-planes.

**Lemma 5.26.** The loop \(\gamma\) represents a generator of \(\pi_1(U(2))\).

**Proof.** Observe that the determinant, \(\text{det} : U(2) \to U(1)\) induces an isomorphism on \(\pi_1\) that takes \(\gamma\) to a generator of \(\pi_1(U(1))\). \(\square\)

The same argument will show that there is a generator for \(\pi_1(U(2))\) given by acting on matrices \(U_{xy}, \hat{U}_{yx}, U_{xw},\) and \(U_{wx}\) on the left by:

\[
\tilde{\gamma}(t) = \begin{pmatrix}
e^{\frac{\pi}{2}it} & 0 \\
0 & 1
\end{pmatrix}.
\]

Note that \(U_{yx} \neq \hat{U}_{yx}\) as matrices in \(U(2)\) but they give rise to the same Lagrangian planes, with the same orientation. While \(U_{yx}\) corresponds to a unitary Lagrangian frame giving rise to the Lagrangian plane \(\{-\partial_x, \partial_y\}\), \(\hat{U}_{yx}\) gives rise to the Lagrangian plane \(\{\partial_x, -\partial_y\}\).

Much of the content of the paper to this point has been building up toward presenting the following proof. Our discussion of Lagrangian grid diagrams in Section 5.2 enables us to define an immersed Lagrangian torus corresponding to a
Lagrangian hypercube diagram as in Theorem 5.22. Lemma 5.23 shows how to obtain a Legendrian torus from the Lagrangian hypercube diagram. Having determined easy methods for computing the rotation number of the Lagrangian grid diagrams (Lemma 5.9), we are ready to prove Theorem 5.1.

**Proof.** Lemma 5.23 guarantees that the lift, $L$, exists. We must see that the image of $r(L) \in [T, U(2)]$ under the isomorphism defined in Theorem 5.6 is $(w(G_{zx}), w(G_{wy}))$. $G_{zx}$ and $G_{wy}$ each correspond to one of the two factors of $T$. Let $[f_{zx}]$ and $[f_{wy}]$ be the elements of $\pi_1(U(2))$ determined by $G_{zx}$ and $G_{wy}$ (since $G_{zx}$ and $G_{wy}$ are constant, choice of base point is irrelevant). Then the isomorphism defined in Theorem 5.6 maps $r(L)$ to $([f_{zx}], [f_{wy}])$. We must show that $[f_{zx}] = w(G_{zx})[\gamma]$. Clearly, $w(G_{zx})$ computes how many times the tangent vector to the grid $G_{zx}$ wraps around the loop $\gamma$. By Lemma 5.26 $[\gamma]$ generates $\pi_1(U(2))$. A similar argument shows that $[f_{wy}] = w(G_{wy})[\gamma]$. \qed

**Corollary 5.3** Let $H_1(T_{HG})$ be generated by $i(\gamma_1)$ and $i(\gamma_2)$ (as in Theorem 5.22). The Maslov index, $\mu : H_1(T_{HG}) \to \mathbb{Z}$ can be computed directly. For $A = (a, b) \in H_1(T_{HG})$,

$$\mu(A) = 2aw(G_{zx}) + 2bw(G_{wy})$$

**Proof.** Given an embedded loop $\gamma : S^1 \to T_{HG}$ representing a primitive class $A \in H_1(T_{HG})$, for any $p \in S^1$, $T_{\gamma(p)}T_{HG}$ is a Lagrangian plane, $L_{\gamma(p)}$. Thus we obtain a map $S^1 \to \text{Lag}(\mathbb{C}^2)$ such that $p \mapsto L_{\gamma(p)}$. The isomorphism defined in the proof of Theorem 5.1 is valid here as well, once we identify planes that differ only in orientation, which produces a factor of 2. \qed

**Corollary 5.4** The Maslov number is $2\gcd(w(G_{zx}), w(G_{wy}))$. 

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Proof. Follows directly from the previous corollary and the fact that the Maslov number is the smallest positive number that is the Maslov index of a non-trivial loop in $H_1(T_{HR})$ and 0 if every non-trivial loop has Maslov index 0 (cf. [12]). □

5.7 Proof of Theorem 5.5 and Examples

Before proceeding with the proof of Theorem 5.5 we establish a few preliminary results. The construction of Theorem 5.30 can be used to produce a hypercube diagram (in the sense of [2]) given any pair of Lagrangian grid diagrams. However if the Lagrangian crossing conditions are not satisfied by the pair of Lagrangian grid diagrams, the resulting Legendrian torus will not be embedded (cf. Remark 5.24). Theorem 5.27, 5.28, and Corollary 5.29 show that for any pair of topological knots, and any rotation numbers, one may find a pair of Lagrangian grid diagrams such that the Lagrangian crossing conditions are satisfied and hence construct a Lagrangian hypercube diagram that lifts to an embedded Legendrian torus.

**Theorem 5.27.** Let $G$ be a Lagrangian grid diagram with an upper-right corner. Enumerate the crossings of $G$ by $\{c_i\}$. Then, for any $M > 0$ there is another Lagrangian grid diagram $G'$, representing the same topological knot and having the same rotation number as $G$, such that $|\Delta t(c'_i)| > M$ for all $i$.

Proof. Scale $G$ by $k \in \mathbb{Z}$ (each segment of the diagram of length $\ell$ becomes a segment of length $k\ell$). This produces a diagram satisfying the Lagrangian conditions (Equations 5.1 and 5.2), but, of course, it will not be a grid diagram, due to empty rows and columns. However, the area of each rectangle (as in Corollary 5.15) will be multiplied by $k^2$. Therefore, $|\Delta t(c_i)|$ may be made arbitrarily large for all $i$. We must then show that the empty rows and columns may be filled in, while preserving the Lagrangian grid conditions.
By following the techniques of Theorem 5.17 we may assume that the upper-right corner of $G$ (prior to scaling) has a horizontal and vertical edge of length 1 or 2. Begin by inserting one additional row and column at the upper-right corner. The additional area created by this will be either $2k+1$, $3k+1$, or $4k+1$ depending on the initial lengths of the horizontal and vertical edges of the upper-right corner. Then attach the configuration shown in Figure 5.10. The unshaded regions will be equal in area, but with opposite sign due to the symmetry between empty rows and columns after scaling the initial grid. The dark-grey regions will also be equal in magnitude but with opposite sign. Finally the light-grey region at the top right may be extended so that it is of area $2k+1$, $3k+1$, or $4k+1$ (an even or odd area may be achieved by placing an additional box as shown by the dotted lines at the upper-right corner of Figure 5.10).

Finally, observe that for all of the original crossings, $\Delta t$ has been scaled up by a factor of $k^2$. However, this procedure creates 4 additional crossings: $d_1$, $d_2$, $d_3$, and $d_4$. By choosing $k$ sufficiently large, and possibly making our initial grid diagram larger, we may ensure that $\min|\Delta t(d_i)| \geq jk + 1$ for $j = 2, 3, 4$.

We showed in the previous theorem that the minimum value of $|\Delta t(C_i)|$ may be made arbitrarily large for a Lagrangian grid diagram, the following theorem shows that we may make Lagrangian grid diagrams arbitrarily large, while keeping $\Delta t(c_i)$ small.

**Theorem 5.28.** Given a Lagrangian grid diagram $G$ of size $n$, there exists $m > n$ such that one may modify $G$ to obtain a Lagrangian grid diagram, $G'$ of size $n'$ for any $n' > m$, with the same topological type and rotation number as $G$. Moreover, if $\Delta_1$ is the maximum over $|\Delta t(c_i)|$ for $G$ and $\Delta_2$ is defined similarly for $G'$, then $\Delta_2 \leq \Delta_1 + |a| + 1$. 

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Proof. We may assume that $G$ has an upper-right corner. Let $k \in \mathbb{Z}$. At the top right corner of the grid, we stabilize and attach a configuration of size $2k$ as shown in Figure 5.11. Since we began with a Lagrangian grid diagram, each new crossing created in this procedure will have $|\Delta t|$ equal to either $a \pm 1$ or $a$, and at the new top right corner, the $t$-coordinates will differ by $a \pm 1$. We then apply Lemma 5.19 to obtain a Lagrangian grid diagram. By carefully choosing which configurations we use in applying Lemma 5.19, we may ensure that the Lagrangian grid diagram we obtain has even or odd size. The statement about the bound on $\Delta_2$ is clear from the construction. \qed
Corollary 5.29. Given two Lagrangian grid diagrams, $G$ and $G'$ of size $m$ and $n$, they may be stabilized to obtain Lagrangian grid diagrams representing the same two topological knots, without changing the rotation number, and such that if $c_i$ is the set of crossings in $G$ and $c'_j$ is the set of crossings in $G'$, $|\Delta t(c_i)| < |\Delta t(c'_j)|$ for all $i, j$.

**Proof.** Apply Theorem 5.27 to $G$, choosing $k$ sufficiently large to guarantee that $k^2 > 4k + 1$ and $2k + 1 > \max\{\Delta t(c'_i)\} + |a| + 1$ where $a$ is as shown in Figure 5.11. This guarantees that $\min\{\Delta t(c_i)\} > \max\{\Delta t(c'_i)\} + |a| + 1$. Then apply Theorem 5.28 to $G'$ so that both grids are the same size. \qed

Theorem 5.30. Let $G_{wy}$ and $G_{zx}$ be Lagrangian grid diagrams of the same size such that if $c_i$ is the set of crossings in $G_{wy}$ and $c'_j$ is the set of crossings in $G_{zx}$, then $|\Delta t(c_i)| \neq |\Delta t(c'_j)|$ for all $i, j$. Then, there is a Lagrangian hypercube diagram such that the wy and zx-projections are given by these grids.

**Proof.** Following the orientation of the diagram label the markings $W_0, Y_0, W_1, Y_1, \ldots$ etc. Do the same for $G_{zx}$. Denote the coordinates of $W_i$ by $(w_{w,i}, y_{w,i})$, $Y_i$ by $(w_{y,i}, y_{y,i})$ etc. Place $Z_i$ in the hypercube at position $(w_{w,i}, x_{z,i}, y_{w,i}, z_{z,i})$, $W_i$ at position $(w_{w,i}, x_{x,i}, y_{w,i}, z_{z,i})$, $X_i$ at position $(w_{y,i}, x_{x,i}, y_{y,i}, z_{x,i})$, and $Y_i$ at position $(w_{y,i}, x_{z,i} + 1, y_{y,i}, z_{z,i} + 1)$ where $i$ is taken modulo $n$. \qed

Having developed the results on Lagrangian grid diagrams in Section 5.2, and having shown in Theorems 5.30, 5.28, and Corollary 5.29 we now have the necessary framework to complete the proof of Theorem 5.5 below.

**Proof.** Given $(m, k) \in \mathbb{Z}^2$, and two knot types $K_1$ and $K_2$. Theorem 5.17 allows one to construct Lagrangian grid diagrams $G_1$ and $G_2$ representing $K_1$ and $K_2$ with rotation numbers $m$ and $k$ respectively. Corollary 5.29 allows one to find
Lagrangian grid diagrams, $G'_1$ and $G'_2$, of the same size representing the same topological knots and having the same rotation numbers as $G_1$ and $G_2$. Applying Theorem 5.30 enables us to construct a Lagrangian hypercube diagram such that $G_{zx} = G'_1$ and $G_{wy} = G'_2$.

**Example 5.31.** One may construct a Lagrangian grid diagram for the unknot with arbitrary rotation number by following the construction shown in Figure 5.12. To realize rotation number $r > 0$ construct the diagram as in Figure 5.12 using $r + 1$ horizontal bars of length $r$. The resulting diagram will have size $2r + 3$. Let $G_{zx}$ be such a grid diagram. Let $G_{wy}$ be the Lagrangian grid diagram for the unknot of size $2r + 3$ given by the construction shown in Figure 5.13. Then applying Theorem 5.30, Lemma 5.23 and Theorem 5.1 we obtain a Lagrangian hypercube diagram with rotation class $(r, 0)$. Figure 5.9 shows the construction for $r = 1$.

Note that if $r = 0$ one must first apply Corollary 5.29. However, for $r > 1$, $|\Delta t(c_i)|$ is never equal to $|\Delta t(c)|$ where $c$ is the unique crossing in $G_{zx}$ and $\{c_i\}$ is the set of crossings in $G_{wy}$.

![Figure 5.12. Construction of a Lagrangian unknots with rotation number 0, ±1, ±2.](image)

![Figure 5.13. Construction of a Lagrangian unknots with rotation number 0.](image)
Example 5.32. Figure 5.14 shows a Lagrangian hypercube diagram with $G_{zx}$ representing a trefoil, and $G_{wy}$ representing a $(5,2)$ torus knot. One may check that $G_{wy}$ has rotation number 0, $G_{zx}$ has rotation number 1, and hence, the Lagrangian hypercube diagram has rotation class $(1,0)$.
References


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Born in 1982, Ben McCarty grew up in the piney woods of East Texas, where he spent much of his time fishing, and playing the 5-string banjo. While at the University of Texas at Tyler he fell in love with mathematics and went on to earn his Bachelor of Science degree in 2005. From there, he went on to Louisiana State University where he was awarded a Master of Science degree in 2007. He is currently a PhD candidate at Louisiana State University where he will receive his degree in August 2012.