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Continuity of affine transformations of white noise test functionals and applications

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Translations and scalings defined on the Schwartz space of tempered distributions induce continuous transformations on the space of white noise test functionals [25]. Continuity of the induced transformations with respect to their parameters is proved. As a consequence one obtains a direct simple proof of the fact that the space of white noise test functionals is infinitely differentiable in Fréchet sense. Moreover, it is shown that the Wiener semigroup acts as a mollifier on the space of test functionals.

white noise analysis * affine transformation * Wiener semigroup * Fréchet derivative * Hida distribution * test functional

1. Introduction

Consider the probability space \((\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)\) of white noise: \(\mathcal{S}'(\mathbb{R})\) is the Schwartz space of tempered distributions, \(\mathcal{B}\) its weak Borel algebra, and \(\mu\) the centered Gaussian measure whose covariance is given by the scalar product of \(L^2(\mathbb{R})\) (with Lebesgue measure).

Let \((L^2)\) be the space of (complex-valued) square integrable random variables, \((\mathcal{S})\) the space of white noise test functionals, and \((\mathcal{S})^*\) the space of Hida distributions (see below). The Gel'fand triple

\[ (\mathcal{S})^* \supseteq (L^2) \supseteq (\mathcal{S}), \]

has been studied in a number of articles: we refer the reader to [9, 11, 13, 14, 15, 18, 19, 21, 23, 25] and the references quoted there. Hida distributions have applications in various domains: in quantum theory [1, 4, 10], in the theory of (anticipating)

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stochastic differential equations [17], in the theory of stochastic partial differential
equations [2, 20], and in other fields (cf., e.g., the articles in [8]).

Let $y \in \mathcal{S}'(\mathbb{R})$, $\lambda \in \mathbb{R}$. Consider the mappings $x \mapsto x + y$, $x \mapsto \lambda x$, $x \in \mathcal{S}'(\mathbb{R})$, on $\mathcal{S}'(\mathbb{R})$. In [25] it was shown that these mappings induce continuous transformations from $(\mathcal{S})$ into itself:

$$\tau_y : \varphi \mapsto \varphi(\cdot + y),$$

$$\sigma_\lambda : \varphi \mapsto \varphi(\lambda \cdot).$$

(1.1)

(1.2)

Here $\varphi(\cdot + y)$ and $\varphi(\lambda \cdot)$ stand for the $\mu$-classes of $\tilde{\varphi}(\cdot + y)$ and $\tilde{\varphi}(\lambda \cdot)$, where $\tilde{\varphi}$ is the pointwise defined, strongly continuous version of $\varphi \in (\mathcal{S})$ (cf. [1, 15]). Moreover, some interesting consequences of these continuities for those measures which are represented in $(\mathcal{S})^*$ (cf. [30] and [23, 24]) have been worked out in [25], and in [28].

In the present paper we establish the (strong) continuity of the mappings $y \mapsto \tau_y \varphi$, $\lambda \mapsto \sigma_\lambda \varphi$, where $\varphi \in (\mathcal{S})$. As a consequence we obtain the following two results:

1. the space $(\mathcal{S})$ is $C^\infty$ with respect to Fréchet differentiation, the Fréchet
derivative being given by the Hida derivative (Theorem 4.1);

2. the Wiener semigroup acts as a mollifier on $(\mathcal{S})$: it maps an element $\varphi \in (\mathcal{S})$ (which is a $\mu$-class) into a pointwise defined, smooth function on $\mathcal{S}'(\mathbb{R})$ and converges pointwise and uniformly on bounded sets to $\tilde{\varphi}$ as the variance parameter of the Wiener semigroup tends to zero (Theorem 5.3).

The organization of the article is as follows. In Section 2 we provide a quick review of the necessary ingredients of white noise analysis. Section 3 contains the proof of the above mentioned continuity results. In Section 4 we prove the Fréchet differentiability of the white noise test functionals, while Section 5 contains the proof of the mollifier property of the Wiener semigroup acting on $(\mathcal{S})$.

2. White noise analysis

Consider the Wiener–Itô decomposition of $(L^2)$ (e.g., [6]): every $\varphi \in (L^2)$ is in one-to-one correspondence with a sequence $(f^{(n)}, n \in \mathbb{N}_0)$ of elements $f^{(n)}$ in $\overline{L^2}(\mathbb{R}^\sigma)$, where $\overline{\cdot}$ means symmetrization, and

$$\|\varphi\|_2^2 = \sum_{n=0}^{\infty} n!\|f^{(n)}\|^2_{L^2(\mathbb{R}^\sigma)}.$$  

(2.1)

In order to give a concrete realization of $\varphi$ in terms of the sequence $(f^{(n)}, n \in \mathbb{N}_0)$, we shall now recall the definition of Wick-ordered distributions. Let $x \in \mathcal{S}'(\mathbb{R})$. Define by recursion the following distributions in $\mathcal{F}'(\mathbb{R}^\sigma)$, $n \in \mathbb{N}_0$:

$$x^{\otimes 0} := 1,$$  

(2.2a)

$$x^{\otimes 1} := x,$$  

(2.2b)

$$x^{\otimes (n+1)} := x^{\otimes n} \otimes x - n : x^{\otimes (n-1)} \otimes \text{Tr},$$  

(2.2c)
where $\text{Tr}$ is the following distribution in $\mathcal{S}'(\mathbb{R}^2)$,

$$
(\text{Tr}, f^{(2)}) = \int_{\mathbb{R}} f^{(2)}(t, t) \, dt, \quad f^{(2)} \in \mathcal{S}'(\mathbb{R}^2).
$$

Then we have that $\mu$-a.e.

$$
\varphi(x) = \sum_{n=0}^{\infty} \langle x^{\otimes n}, f^{(n)} \rangle, \quad x \in \mathcal{S}'(\mathbb{R}),
$$

(2.3)

where the dual pairings are to be understood in $(L^2)$-sense.

Later on we shall make use of the following transformation on $(L^2)$: for $\xi \in \mathcal{S}(\mathbb{R})$ we set

$$
\mathcal{F}\varphi(\xi) = \int \varphi(x + \xi) \, d\mu(x), \quad \varphi \in (L^2).
$$

(2.4)

By the translation formula for the Gaussian measure $\mu$ (e.g., [16]) we have

$$
\mathcal{F}\varphi(\xi) = \int \varphi(x) :e^{(x, \xi)}: \, d\mu(x),
$$

(2.5)

with the notation

$$
:e^{(x, \xi)}: = e^{(|\xi|^2/2)},
$$

where $|\cdot|_2$ is the norm of $L^2(\mathbb{R})$. It is easy to verify that for $\varphi \in (L^2)$ given as in (2.3) we have

$$
\mathcal{F}\varphi(\xi) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} f^{(n)}(u) \xi^{\otimes n}(u) \, du.
$$

(2.6)

Let $A$ be a closable operator on $L^2(\mathbb{R})$. Its second quantization $\Gamma(A)$ on $(L^2)$ can be defined (on a suitable domain) by

$$
\Gamma(A)\varphi(x) = \sum_{n=0}^{\infty} \langle x^{\otimes n}, A^\otimes f^{(n)} \rangle
$$

(cf., e.g., [3, 22]). In particular, if $A$ is self-adjoint with $\mathcal{S}(\mathbb{R}) \subset \mathcal{D}(A)$ then $\Gamma(A)$ is essentially self-adjoint on the algebra $\mathcal{P}$ generated by the random variables $X_\xi = \langle \cdot, \xi \rangle, \xi \in \mathcal{S}(\mathbb{R})$ (which forms a dense subspace of $(L^2)$). From now on we make the special choice that $A$ is the self-adjoint extension of the operator given on $\mathcal{S}(\mathbb{R})$ by $A = -\frac{d^2}{du^2} + 1 + u^2$, and we shall denote the self-adjoint extension of $\Gamma(A)$ on $(L^2)$ by the same symbol. Note that $\text{infspec} \Gamma(A) = 1$. Therefore we may consider the following family of norms $\| \cdot \|_{2,p}, p \in \mathbb{R}_+$, on $\mathcal{P}$:

$$
\| \varphi \|_{2,p} = \| \Gamma(A)^p \varphi \|_2, \quad \varphi \in \mathcal{P}.
$$

By $(\mathcal{F})_p, p \in \mathbb{R}_+$, we denote the completion of $\mathcal{P}$ under $\| \cdot \|_{2,p}$, by $(\mathcal{F})$ the projective limit of the family $\{(\mathcal{F})_p, p \in \mathbb{N}_0\}$. $(\mathcal{F})^*$ is by definition the dual of $(\mathcal{F})$. Elements in $(\mathcal{F})$ are called white noise test functionals, while those in $(\mathcal{F})^*$ are called Hida distributions.
The spaces \((\mathcal{F})\) and \((\mathcal{F})^*\) have been investigated and applied in a number of articles (see [18, 23, 24, 25, 27, 28, 29] and the references quoted there).

Let us recall that \((\mathcal{F})\) is a nuclear Fréchet algebra [10, 13, 14, 25, 30] whose elements \(\varphi\) admit a pointwise defined, strongly continuous version \(\tilde{\varphi}\) [1, 15]. Furthermore, the operators \(\tau_y, \sigma_\lambda, y \in \mathcal{F}'(\mathbb{R}), \lambda \in \mathbb{R}\), mentioned in Section 1 act continuously on \((\mathcal{F})[25]\).

In [12, 30] it was proved that positive Hida distributions are measures on \((\mathcal{F}'(\mathbb{R}), \mathcal{B})\). Therefore it is interesting to establish a criterion that implies that a measure on \((\mathcal{F}'(\mathbb{R}), \mathcal{B})\) has a representation by a positive Hida distribution. This has been done in [23] where \((\mathcal{F})^*\) has been characterized in terms of analytic properties of the \(\mathcal{F}\)- and the Fourier (-Gauss) transform of its elements (cf. also below). Concerning the question of representation of measures on \((\mathcal{F}'(\mathbb{R}), \mathcal{B})\) this result provides a criterion which is formulated in terms of the characteristic function of the measure in question. Roughly speaking, the characteristic function has to be real entire on \(\mathcal{F}(\mathbb{R})\), and of order 2, uniformly on the set \(\{ \xi \in \mathcal{F}(\mathbb{R}) : |A^p \xi|_2 \leq 1\}\) for some \(p \in \mathbb{N}_0\). The result of [23] has been sharpened in [27].

The space \((\mathcal{F})\) can be characterized similarly [11, 18, 27]. Moreover, there is another characterization of \((\mathcal{F})^*\) through analytic properties of the \(\mathcal{F}\)-transform given by Lee in [19].

We remark that for all \(z \in \mathbb{C}, \xi \in \mathcal{F}(\mathbb{R})\), the function (more precisely its \(\mu\)-class) \(\exp(z(\cdot, \xi))\) belongs to \((\mathcal{F})\). Therefore we may extend the \(\mathcal{F}\) transform to \((\mathcal{F})^*\) as follows:

\[
\mathcal{F}\Phi(\xi) = \langle \Phi, \exp(-\cdot, \xi) \rangle. \tag{2.7}
\]

In [21] it was shown that for all \(\Phi, \Psi \in (\mathcal{F})^*\) there is an element \(\Phi \circ \Psi \in (\mathcal{F})^*\) so that

\[
\mathcal{F}\Phi \circ \Psi = \mathcal{F}\Phi \cdot \mathcal{F}\Psi. \tag{2.8}
\]

\(\Phi \circ \Psi\) is called the Wick product of \(\Phi\) and \(\Psi\).

We conclude this section with a sketch of the differential calculus on \((\mathcal{F})\) and \((\mathcal{F})^*\) (cf. [25, 7]).

Let \(y \in \mathcal{F}'(\mathbb{R})\) and denote by \(D_y\) the Gâteaux derivative of functions on \(\mathcal{F}'(\mathbb{R})\) in direction \(y\). For example, all functions in \(\mathcal{P}\) are Gâteaux differentiable in any direction \(y \in \mathcal{F}'(\mathbb{R})\). In [25] this has been extended to \((\mathcal{F})\): for every \(y \in \mathcal{F}'(\mathbb{R})\) there is a continuous extension of \(D_y\) acting on \((\mathcal{F})\) (denoted by the same symbol). Therefore \((\mathcal{F})\) is \(C^\infty\) in every direction of \(\mathcal{F}'(\mathbb{R})\). Moreover, we want to mention that in [25] the Taylor formula was established for \(\tau_y \varphi\),

\[
\tau_y \varphi(x) = \exp D_y \varphi(x), \quad \mu\text{-a.e. } x \in \mathcal{F}'(\mathbb{R}).
\]

Let us consider the special choice \(y = \delta_t, \; t \in \mathbb{R}\), where \(\delta_t\) is the Dirac distribution concentrated at \(t \in \mathbb{R}\). In this case we denote \(D_{\delta_t} = \partial\), and call \(\partial\), Hida derivative. Furthermore, we define for \(\varphi \in (\mathcal{F})\),

\[
\nabla \varphi(t, x) = \partial \varphi(x), \quad t \in \mathbb{R}, \quad \mu\text{-a.e. } x \in \mathcal{F}'(\mathbb{R}).
\]
It is easy to check that $\nabla \varphi \in \mathcal{F}(\mathbb{R}) \otimes (\mathcal{F})$ and more generally $\nabla^k : (\mathcal{F}) \to \mathcal{F}(\mathbb{R}^k) \otimes (\mathcal{F})$ for every $k \in \mathbb{N}$. Moreover, we have the formula [25],

$$D_y \varphi = (y, \nabla \varphi),$$

where the pairing is the one between $\mathcal{F}'(\mathbb{R})$ and $\mathcal{F}(\mathbb{R})$. The last equation justifies that we call $\nabla \varphi$ the gradient of $\varphi$.

Let $D^*_y$, $y \in \mathcal{F}'(\mathbb{R})$, denote the dual of $D_y$. It is not hard to see that if $y \in \mathcal{F}(\mathbb{R})$, $D^*_y$ maps $\mathcal{F}$ into itself. In view of (2.9) this implies that we may consider $\nabla$ as a mapping from $(\mathcal{F})^*$ into $\mathcal{F}'(\mathbb{R}) \otimes (\mathcal{F})^*$, or more generally for every $k \in \mathbb{N}$, $\nabla^k : (\mathcal{F})^* \to \mathcal{F}'(\mathbb{R}^k) \otimes (\mathcal{F})^*$.

### 3. Continuity of affine transformations

In this section we establish the continuity of the following mappings of $\mathcal{F}'(\mathbb{R})$ and $\mathbb{R}$ into $(\mathcal{F})$. For fixed $\varphi \in (\mathcal{F})$,

$$y \mapsto \tau_y \varphi = \varphi(\cdot + y), \quad y \in \mathcal{F}'(\mathbb{R}), \quad (3.1)$$

$$\lambda \mapsto \sigma_\lambda \varphi = \varphi(\lambda \cdot), \quad \lambda \in \mathbb{R}. \quad (3.2)$$

Here and in the following we consider $\mathcal{F}'(\mathbb{R})$ as equipped with the strong topology.

First we recall two formulae which have been proved in [25].

**Lemma 3.1.** Let $x, y \in \mathcal{F}'(\mathbb{R})$, $\lambda \in \mathbb{R}$, $n \in \mathbb{N}$. Then

$$((x + y)^{\otimes n}) = \sum_{k=0}^{n} \binom{n}{k} x^{\otimes k} \otimes y^{\otimes (n-k)}, \quad (3.3)$$

$$((\lambda x)^{\otimes n}) = \lambda^n \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (2k-1)!! (1 - \lambda^{-2})^k (x^{\otimes (n-2k)} \otimes \text{Tr}^{\otimes k}). \quad (3.4)$$

From now on we shall denote by $| \cdot |_{2,r}$, $r \in \mathbb{R}$, the norm on $\mathcal{F}(\mathbb{R}^n)$, $n \in \mathbb{N}$, given by $| (A^{\otimes n})^r |_{2}$, and where the last norm is the one of $L^2(\mathbb{R}^n)$. Note that the family of Schwartz space norms is equivalent to the family $\{ | \cdot |_{2,p}, p \in \mathbb{N}_0 \}$, and that consequently $\mathcal{F}'(\mathbb{R}^n)$ (as a set) is the union of the spaces $\mathcal{F}_p(\mathbb{R}^n)$, $p \in \mathbb{N}_0$, where $\mathcal{F}_p(\mathbb{R}^n)$, $r \in \mathbb{R}$, denotes the completion of $\mathcal{F}(\mathbb{R}^n)$ under $| \cdot |_{2,r}$. Moreover, the strong topology on $\mathcal{F}'(\mathbb{R}^n)$ is equivalent to the inductive limit topology of the chain $\mathcal{F}_p(\mathbb{R}^n)$, $p \in \mathbb{N}_0$.

**Lemma 3.2.** Let $y, z \in \mathcal{F}'(\mathbb{R})$, $\varphi \in (\mathcal{F})$. For every $p \geq 0$ there exist $s, r \geq 0$ so that

$$\| \tau_y \varphi - \tau_z \varphi \|_{2,p} \leq |y - z|_{2,-q} \| \varphi \|_{2,q} (1 - [2^{-r}(1 + |y|_{2,-q} + |z|_{2,-q})]^2)^s, \quad (3.5)$$

where $q \geq p$ is such that $y, z \in \mathcal{F}_q(\mathbb{R})$. 
Proof. Denote by \((f^{(n)}; n \in \mathbb{N}_0)\) the chaos decomposition of \(\varphi\). Note that by construction of \((\mathcal{F})\), for every \(p \in \mathbb{R}\),

\[
\| \varphi \|_{2,p}^2 = \sum_{n=0}^{\infty} n! \| f^{(n)} \|_{2,p}^2 < \infty.
\]

By Lemma 3.1 we have

\[
\tau_\varphi(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \langle x^{\otimes k}; \langle y^{\otimes (n-k)}, f^{(n)} \rangle \rangle,
\]

so that

\[
\tau_\varphi(x) - \tau_\varphi(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \langle x^{\otimes k}; \langle y^{\otimes (n-k)} - z^{\otimes (n-k)}, f^{(n)} \rangle \rangle.
\]

Therefore

\[
\| \tau_\varphi - \tau_\varphi \|_{2,p} \leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \langle (A^{\otimes k})^p (y^{\otimes (n-k)} - z^{\otimes (n-k)}), f^{(n)} \rangle \|
\]

Now estimate as follows:

\[
\| (A^{\otimes k})^p (y^{\otimes (n-k)} - z^{\otimes (n-k)}), f^{(n)} \|_2 \leq \| (A^{-n}y)^{\otimes (n-k)} - (A^{-n}z)^{\otimes (n-k)} \|_2 \| f^{(n)} \|_{2,q}
\]

\[
\leq (n-k) \rho^{n-k-1} \| y - z \|_{2,-q} \| f^{(n)} \|_{2,q},
\]

where we have set \(\rho = \| y \|_{2,-q} \vee |z|_{2,-q}\). Thus by choosing \(s, r \geq 0\) large enough we obtain the following estimation:

\[
\| \tau_\varphi - \tau_\varphi \|_{2,p} \leq \| y - z \|_{2,-q} \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} \binom{n}{k} \langle (k!)^{1/2} (n-k) \rho^{n-k-1} \| f^{(n)} \|_{2,q} \|
\]

\[
\leq \| y - z \|_{2,-q} \sum_{n=1}^{\infty} (n!)^{1/2} \| f^{(n)} \|_{2,s} 2^{-m} (1 + \rho)^{(n-1)}
\]

\[
\leq \| y - z \|_{2,-q} \| \varphi \|_{2,s} \Bigg( \sum_{n=1}^{\infty} 2^{-2m} (1 + \rho)^{2(n-1)} \Bigg)^{1/2}.
\]

Here we made use of the fact that the operator norm of \(A^{-\alpha}, \alpha \geq 0\), is equal to \(2^{-\alpha}\). It is clear that we obtain (3.5) from the last inequality.

As an immediate consequence we have

**Theorem 3.3.** For every \(\varphi \in (\mathcal{F})\) the mapping \(y \mapsto \tau_\varphi, y \in \mathcal{F}'(\mathbb{R})\), is continuous from \(\mathcal{F}'(\mathbb{R})\) into \((\mathcal{F})\). \(\square\)

Next we want to present another proof of Theorem 3.3 which makes use of some formulae of [25]. As a preparation we quote the following result from [21].
Lemma 3.4. Let $\Phi, \Psi \in (\mathcal{S})^*$. Then we have for $\Phi \circ \Psi \in (\mathcal{S})^*$ the following inequality:
\[
\| \Phi \circ \Psi \|_{2,p} \leq \| \Phi \|_{2,p+1/\gamma} \| \Psi \|_{2,p+1/\gamma}, \quad p \in \mathbb{R}.
\] (3.6)

Lemma 3.5. Let $y, z \in \mathcal{S}'(\mathbb{R}), \varphi \in (\mathcal{S})$. Set $\rho = \|y\|_{2,-q} \vee \|z\|_{2,-q}$ for $q$ sufficiently large so that $y, z \in \mathcal{S}'_{-q}(\mathbb{R})$. Then for every $p \in \mathbb{R}$,
\[
\| \tau_y \varphi - \tau_z \varphi \|_{2,p} \leq \| y - z \|_{2,-q} \| \varphi \|_{2,p+q+1/2} e^{\rho^{1/2}}.
\] (3.7)

Proof. Note that by Corollary 2.8 in [25], we have that for every $\Phi \in (\mathcal{S})^*$ the following formula holds:
\[
\langle \Phi, \tau_y \varphi - \tau_z \varphi \rangle = \langle \Phi \circ (E_y - E_z), \varphi \rangle,
\]
where $E_x$, $x \in \mathcal{S}'(\mathbb{R})$, is determined by $\mathcal{S}E_x(\xi) = \exp((x, \xi))$, $\xi \in \mathcal{S}(\mathbb{R})$. Note that $x \in \mathcal{S}'(\mathbb{R})$ implies that $E_x \in (\mathcal{S})_p$, $p \in \mathbb{R}$.

Let $r = p \vee q + 1/2$. Then
\[
\| \tau_y \varphi - \tau_z \varphi \|_{2,p} \leq \sup_{\Phi \in (\mathcal{S})^*} \| \langle \Phi \circ (E_y - E_z), \varphi \rangle \|
\]
\[
= \sup_{\Phi} \| \Phi \circ (E_y - E_z) \|_{2,r} \| \varphi \|_{2,r}
\]
\[
\leq \sup_{\Phi} \| \Phi \|_{2,-r+1/2} \| E_y - E_z \|_{2,-r+1/2} \| \varphi \|_{2,r}
\]
\[
\leq \| E_y - E_z \|_{2,-q} \| \varphi \|_{2,r}.
\]
It remains to estimate
\[
\| E_y - E_z \|_{2,-q} = \sum_{n=1}^{\infty} \frac{1}{n!} \| y^\otimes n - z^\otimes n \|_{2,-q}
\]
\[
\leq \| y - z \|_{2,-q} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \rho^{2(n-1)}
\]
\[
= \| y - z \|_{2,-q} e^{\rho^{1/2}},
\]
to finish the proof. □

It is clear that Lemma 3.5 provides another proof of Theorem 3.3.

Next we turn our attention to the scaling transformation $\sigma_x$. First we prepare the following result whose proof is elementary and therefore omitted.

Lemma 3.6. Let $n, k \in \mathbb{N}_0$, $k \leq \lfloor \frac{1}{2} n \rfloor$, and $\lambda_1, \lambda_2 \in \mathbb{R}$. Then
\[
| \lambda_1^{(n-2k)}(\lambda_1^2 - 1)^k - \lambda_2^{(n-2k)}(\lambda_2^2 - 1)^k | \leq n | \lambda_1 - \lambda_2 | (1 + \lambda^2)^{(n-1)/2},
\]
where $\lambda = | \lambda_1 | \vee | \lambda_2 |$. □
Lemma 3.7. Let $\lambda_1, \lambda_2 \in \mathbb{R}$, $\varphi \in (\mathcal{S})$. For every $p \in \mathbb{R}$ there exists $q \in \mathbb{R}$ so that

$$\|\sigma_{\lambda_1} \varphi - \sigma_{\lambda_2} \varphi\|_{2,p} \leq K_\lambda |\lambda_1 - \lambda_2| \|\varphi\|_{2,q},$$

where $K_\lambda > 0$ is a constant depending only on $\lambda$.

Proof. Without loss of generality we may assume that $p > \frac{1}{4}$. By Lemma 3.1 we have for $\varphi \in (\mathcal{S})$ with chaos decomposition given by $(f^{(n)}): n \in \mathbb{N}_0$,

$$\sigma_{\lambda_1} \varphi(x) - \sigma_{\lambda_2} \varphi(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{\lceil n/2 \rceil} \binom{n}{2k} (2k-1)!! (\lambda_1^{n-2k} - 1)^k - (\lambda_2^{n-2k} - 1)^k \cdot \langle \text{Tr}^{\otimes k}, f^{(n)} \rangle.$$

The factor in $[\cdot]$ will be denoted by $\gamma_{n,k}(\lambda_1, \lambda_2)$. Then

$$\|\sigma_{\lambda_1} \varphi - \sigma_{\lambda_2} \varphi\|_{2,p} \leq \sum_{n=0}^{\infty} \sum_{k=0}^{\lceil n/2 \rceil} \binom{n}{2k} (2k-1)!! ((n-2k)!)^{1/2} \gamma_{n,k}(\lambda_1, \lambda_2).$$

Note that $((n-2k)!)^{1/2}(2k-1)!! \leq (n!)^{1/2}$. It is easy to see that $|\text{Tr}|_{2,-p} = \|A^{-2p}\|_{\text{HS}} < \infty$, for $p > \frac{1}{4}$. This implies the following estimate:

$$\|(A^{n-2k}) \langle \text{Tr}^{\otimes k}, f^{(n)} \rangle\|_{2} \leq \|f^{(n)}\|_{2,p} \|A^{-2p}\|_{\text{HS}}^k.$$

(cf. also [25, proof of Theorem 2.11], correcting a mistake there). Using Lemma 3.6 we find

$$\|\sigma_{\lambda_1} \varphi - \sigma_{\lambda_2} \varphi\|_{2,p} \leq |\lambda_1 - \lambda_2| \sum_{n=0}^{\infty} (n!)^{1/2} n (1 + \lambda^2)^{(n-1)/2} (1 + \|A^{-2p}\|_{\text{HS}}^{1/2})^n \|f^{(n)}\|_{2,p}.$$

Choose $q > p$ large enough. Then

$$\|\sigma_{\lambda_1} \varphi - \sigma_{\lambda_2} \varphi\|_{2,p} \leq K_\lambda |\lambda_1 - \lambda_2| \sum_{n=0}^{\infty} (n!)^{1/2} 2^{-n} \|f^{(n)}\|_{2,q} \leq K_\lambda |\lambda_1 - \lambda_2| \|\varphi\|_{2,q}.$$

This concludes the proof. □

We remark in passing that one can estimate $\|\sigma_{\lambda_1} \varphi - \sigma_{\lambda_2} \varphi\|_{2,p}$ also in a similar way as in our second proof of Theorem 3.2.

We have proved the following result.

Theorem 3.8. For every $\varphi \in (\mathcal{S})$ the mapping $\lambda \mapsto \sigma_\lambda \varphi$ is continuous from $\mathbb{R}$ into $(\mathcal{S})$. □

We conclude this section by indicating a consequence for measures on $(\mathcal{S}(\mathbb{R}), \mathcal{B})$ which are represented by a Hida distribution. Assume that $\nu$ is a measure on
Then there exists \( d\nu/d\mu \in \mathcal{S}^* \) so that for all \( \varphi \in \mathcal{S} \) we have
\[
\left\langle \frac{d\nu}{d\mu}, \varphi \right\rangle = \int \tilde{\varphi}(x) \, d\nu(x).
\]
Set \( \sigma_\lambda \nu(E) = \nu(\lambda^{-1} E), \lambda \neq 0, E \in \mathcal{B} \). By [25, Theorem 2.12], we have that \( d\nu_\lambda/d\mu \in \mathcal{S}^* \), too. Theorem 3.8 implies:

**Corollary 3.9.** The family of measures \( \sigma_\lambda \nu, \lambda \in \mathbb{R}, \lambda \neq 0 \), is weakly continuous on \( \mathcal{S} \).

We expect that Corollary 3.9 will be important for a formulation of the theory of large deviations within white noise analysis.

### 4. Fréchet differentiability of test functionals

In this section we apply the continuity result of Section 3 to establish the following theorem.

**Theorem 4.1.** Every element \( \varphi \in \mathcal{S} \) is \( \mu \)-a.e. infinitely often Fréchet differentiable, its Fréchet derivative of order \( k \) being given by \( \nabla^k \varphi \in \mathcal{S}(\mathbb{R}^k) \otimes \mathcal{S} \). In particular, the pointwise defined version \( \tilde{\varphi} \) of \( \varphi \) is everywhere \( C^\infty \) in Fréchet sense.

**Remark.** This theorem follows also from the results in [19]. We are going to give a direct proof below.

**Proof.** For notational simplicity, let us identify \( \varphi \in \mathcal{S} \) with its pointwise defined version. Fix \( x, y \in \mathcal{S}(\mathbb{R}) \), and let \( \lambda \in \mathbb{R} \). Set
\[
\phi_{x,y}(\lambda) := \varphi(x + \lambda y).
\]
Because \( \varphi \) is everywhere Gâteaux differentiable in direction \( y \), we see immediately that \( \phi_{x,y} \) is differentiable and that
\[
\phi_{x,y}'(\lambda) = (D_x \varphi)(x + \lambda y) = (\sigma_\lambda \tau_y D_x \varphi)(y).
\]
Note that the results in [25] imply that \( \tau_y D_x \varphi \) is the continuous version of an element in \( \mathcal{S} \). Therefore Theorem 3.8 implies that \( \phi_{x,y}' \) is continuous. Moreover, we have
\[
\varphi(x + y) - \varphi(x) = \int_0^1 \phi_{x,y}'(\lambda) \, d\lambda,
\]
and by (2.9),
\[
\varphi(x + y) - \varphi(x) - \langle y, \nabla \varphi(x) \rangle = \int_0^1 \langle y, \nabla \varphi(x + \lambda y) - \nabla \varphi(x) \rangle \, d\lambda.
\]
y is in $\mathcal{S}_p(\mathbb{R})$ for some $p \in \mathbb{N}_0$. Assume that it tends to zero in $\mathcal{S}_p(\mathbb{R})$ (so that it tends to zero in the strong topology of $\mathcal{S}'(\mathbb{R})$). Estimate as follows:

$$
|\varphi(x+y) - \varphi(x) - \langle y, \nabla \varphi(x) \rangle| \leq |y|_{2,-p} \int_0^1 |\nabla \varphi(x + \lambda y) - \nabla \varphi(x)|_{2,p} \, d\lambda
$$

$$
\leq |y|_{2,-p} \sup_{\lambda \in [0,1]} |\nabla \varphi(x + \lambda y) - \nabla \varphi(x)|_{2,p}.
$$

We write

$$
\|\nabla \varphi(x + \lambda y) - \nabla \varphi(x)\|_{2,p} = \sup_{z \in \mathcal{S}'(\mathbb{R}), |z|_{2,-p} = 1} |\langle z, \nabla \varphi(x + \lambda y) - \nabla \varphi(x) \rangle|
$$

$$
= \sup_z \left| (D_2 \varphi)(x + \lambda y) - (D_2 \varphi)(x) \right|
$$

$$
- \sup_z \left| \langle \delta_x, \tau_y D_2 \varphi - D_2 \varphi \rangle \right|
$$

where $\delta_x \in (\mathcal{S})^*$ is evaluation at $x \in \mathcal{S}'(\mathbb{R})$: $\langle \delta_x, \psi \rangle = \tilde{\psi}(x)$, $\psi \in (\mathcal{S})$. Note that $\delta_x \in (\mathcal{S})_-$, for some $r \in \mathbb{R}_+$ (e.g., [15]). Applying Lemma 3.5 we find the bound

$$
|\nabla \varphi(x + \lambda y) - \nabla \varphi(x)|_{2,p} \leq \lambda |y|_{2,-p} \|\delta_x\|_{2,-r} e^{\|y\|^2_{2,-r/2}} \sup_z \|D_2 \varphi\|_{2,r'},
$$

for some $r' \in \mathbb{R}_+$. Inequality (3.18) of [25] implies

$$
\|D_2 \varphi\|_{2,r'} \leq \text{const} \cdot \|z\|_{2,-p} \|\varphi\|_{2,r'}
$$

for some $r'' \in \mathbb{R}_+$. Therefore we have altogether proved that for $y \in \mathcal{S}'(\mathbb{R})$ with $|y|_{2,-p} \leq 1$,

$$
|\nabla \varphi(x + \lambda y) - \nabla \varphi(x)|_{2,p} \leq C_x |y|_{2,-p} \|\varphi\|_{2,r''},
$$

where $C_x$ is a constant depending only on $x \in \mathcal{S}'(\mathbb{R})$. Hence it follows that

$$
|\varphi(x+y) - \varphi(x) - \langle y, \nabla \varphi(x) \rangle| \leq C_x |y|_{2,-p} \|\varphi\|_{2,r''}.
$$

Thus we have proved that $\varphi$ has Fréchet derivative $\nabla \varphi$. Since $\nabla \varphi \in \mathcal{S}(\mathbb{R}) \otimes (\mathcal{S})$ [25] we may now use induction to obtain the statement of the theorem. \qed

5. The Wiener semigroup as a mollifier

In this section we consider the Wiener semigroup acting on $(\mathcal{S})$.

We begin with the following remark. For $t \geq 0$, $x \in \mathcal{S}'(\mathbb{R})$, $\varphi \in (\mathcal{S})$, we have

$$
\sigma_{\sqrt{t}} \tau_x \varphi \in (\mathcal{S}),
$$

and

$$
(\sigma_{\sqrt{t}} \tau_x \varphi)(y) = (\tau_y \varphi)(\sqrt{t} y) = \tilde{\varphi}(\sqrt{t} y + x).
$$

Thus we may define

$$
(P_t \varphi)(x) := \langle 1, \sigma_{\sqrt{t}} \tau_x \varphi \rangle, \quad (5.1)
$$
and it is clear that we obtain for \( t > 0 \),

\[
P_t \varphi(x) = \int (u_{it} \tau_x \varphi)(y) \, d\mu(y) - \int \tilde{\varphi}(\sqrt{t} y + x) \, d\mu(y)
= \int \tilde{\varphi}(y) \, d\mu_{t,x}(y),
\]

(5.2)

where \( \mu_{t,x} \) is the Gaussian measure on \((\mathcal{F}(\mathbb{R}), \mathcal{B})\) with mean \( x \) and covariance operator \( t \cdot \text{Id} \). The family \( \{ P_t : t \in \mathbb{R}_+ \} \) is called Wiener semigroup (cf. [5]). For a discussion of the semigroup property we also refer to the end of this section.

Using Theorem 2.6 and Theorem 2.11 in [25] we may also represent \( P_t \) as follows. \( \mu_{t,x} \) is represented by a Hida distribution \( d\mu_{t,x} / d\mu \in (\mathcal{F})^*, \) and we can write \((t > 0, \ x \in \mathcal{F}(\mathbb{R}))\)

\[
(P_t \varphi)(x) = \left( \frac{d\mu_{t,x}}{d\mu} \varphi \right).
\]

(5.3)

By Theorem 2.11 in [25], which states the continuity of \( \sigma \), as an operator on \( \mathcal{F} \), and by Theorem 3.3 and Theorem 3.8, we obtain the following result.

**Theorem 5.1.** For every \( \varphi \in \mathcal{F} \), the mapping

\[
(t, x) \mapsto P_t \varphi(x)
\]

is continuous from \([0, \infty) \times \mathcal{F}(\mathbb{R})\) into \( C \).

Let \( t > 0 \). We can say more about \( P_t \varphi(x) \) as a function of \( x \in \mathcal{F}(\mathbb{R}) \):

**Lemma 5.2.** For every \( \varphi \) in \( \mathcal{F} \), \( t > 0 \), the function \( x \mapsto P_t \varphi(x) \) on \( \mathcal{F}(\mathbb{R}) \) is \( C^\infty \) (in Gâteaux sense) in every direction of \( \mathcal{F}(\mathbb{R}) \).

**Proof.** Let \( \varphi \in \mathcal{F} \), \( \xi, \eta \in \mathcal{F}(\mathbb{R}) \). Consider

\[
\mathcal{F}\varphi(\xi) - \mathcal{F}\varphi(\eta) = (\mathcal{L}^{(t, \xi)}, -\mathcal{L}^{(t, \eta)}, \varphi)
= \int_0^1 \left[ \frac{d}{d\lambda} \left( \mathcal{L}^{(t, \lambda \xi + (1-\lambda) \eta)}, \varphi \right) \right] d\lambda
= \int_0^1 \left( \frac{d}{d\lambda} \mathcal{L}(\lambda \xi + (1-\lambda) \eta) \right) d\lambda.
\]

Denote \( \xi_\lambda = \lambda \xi + (1-\lambda) \eta \). It is straightforward to check that for every \( p \in \mathbb{N}_0 \),

\[
\left| \frac{d}{d\lambda} \mathcal{L}(\xi_\lambda) \right| \leq |\xi - \eta|_{L^{2-p}} e^{\|\xi\|_{L^p}^p + \|\eta\|_{L^p}^p} \|\varphi\|_{L^{2,p+1}}.
\]

Thus for every \( \varphi \in \mathcal{F} \), \( \mathcal{F}\varphi \) extends to a strongly continuous mapping on \( \mathcal{F}(\mathbb{R}) \). Moreover, it follows from the results in [18] that \( \mathcal{F}\varphi \) is ray entire, which in turn
implies that (the extended map) $\mathcal{S}\varphi$ is Gâteaux differentiable to every order in every direction (cf. also Proposition 2.3 in [23]). Now we may write ($t > 0$)

$$P_t\varphi(x) = \mathcal{S}(\sigma_{\sqrt{t}}\varphi)(t^{-1/2}x).$$

Thus $x \mapsto P_t\varphi(x)$ is infinitely often Gâteaux differentiable in every direction. \qed

Next we investigate the behaviour of $P_t\varphi$ as $t$ tends to zero.

By Theorem 3.8 we know that as $t \to 0$, $\sigma_{\sqrt{t}}\varphi$ converges in $\mathcal{S}'$ to $\sigma_0\varphi$, i.e., to the $\mu$-class $[\tilde{\psi}(0)]$ of $\tilde{\psi}(0)$. Therefore we have

$$\lim_{t \to 0} (1, \sigma_{\sqrt{t}}\tau_x\varphi) = (1, [\tilde{\varphi}(x)]),$$

and hence

$$\lim_{t \to 0} P_t\varphi(x) = \tilde{\varphi}(x),$$

the convergence being uniform on bounded sets of $\mathcal{S}'(\mathbb{R})$ (since $\tilde{\varphi}$ is uniformly continuous on bounded sets [15]).

**Theorem 5.3.** The Wiener semigroup maps $\varphi \in \mathcal{S}$ into a pointwise defined function $x \mapsto P_t\varphi(x)$ on $\mathcal{S}'(\mathbb{R})$, which is infinitely often differentiable (in Gâteaux sense) in every direction. As $t$ tends to zero, $P_t\varphi(x)$ converges pointwise (uniformly on bounded sets) to $\tilde{\varphi}(x)$.

Let us give a second simple proof of Theorem 5.3 for the case that $t$ tends to zero through a positive sequence.

**Second proof of Theorem 5.3** (sequential case). Let us show that for every $x \in \mathcal{S}'(\mathbb{R})$, $d\mu_{t,x}/d\mu \to \delta_x$ strongly in $(\mathcal{S})^*$ as $t$ tends to zero.

Compute the $\mathcal{S}$-transform of $d\mu_{t,x}/d\mu$: let $\xi \in \mathcal{S}(\mathbb{R})$ then

$$\mathcal{S}\left(\frac{d\mu_{t,x}}{d\mu}(\xi)\right) = e^{-\|x\|^2/2+(x,\xi)} \int e^{\mathcal{S}(\delta_x)(y)} d\mu(y) = e^{-\|x\|^2/2+(x,\xi)}.$$

Obviously, $\mathcal{S}(d\mu_{t,x}/d\mu)(\xi)$ converges to $\exp((x, \xi))$: as $t \to 0$. But this expression is the $\mathcal{S}$-transform of $\delta_x$. Theorem 2.7 in [23] implies then that $d\mu_{t,x}/d\mu \to \delta_x$ strongly in $(\mathcal{S})^*$ as $t \to 0$. \qed

We conclude this paper with a remark on the semigroup property of the family $\{P_t: t \in \mathbb{R}_+\}$.

It was shown in [29] that for $\varphi \in \mathcal{S}$, $P_t\varphi$ is a version of an element $[P_t\varphi]$ in $\mathcal{S}$, and because of Theorem 5.1 it is the (unique) pointwise defined, strongly continuous version of this class: $[P_t\varphi] = P_t\varphi$. We remark in passing that Theorem 4.1 implies that $P_t\varphi$ is infinitely often Fréchet differentiable.
We define

\[ P_sP_t \phi := P_s(P_t \phi), \quad s, t \in \mathbb{R}_+. \]

For \( s, t > 0 \) one obtains after a straightforward computation

\[ (P_sP_t \phi)(x) = \int \tilde{\phi}(y + x) d(\mu_{s,t,0} \ast \mu_{t,0})(y), \]

where \( \ast \) denotes convolution. It is well-known (and simple to prove) that \( \mu_{s,0} \ast \mu_{t,0} = \mu_{s+t,0} \). (For example, this follows directly from Theorem 4.1, (4.2) and (2.30) in [25].) Thus we have

\[ (P_sP_t \phi)(x) = \int \tilde{\phi}(y + x) d\mu_{s+t,0}(y) = P_{s+t} \phi(x), \]

which states the semigroup property of \( \{P_t : t \in \mathbb{R}_+\} \).

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References