Subgroups of the Torelli group

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SUBGROUPS OF THE TORELLI GROUP

A Dissertation

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Leah Childers
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For my husband
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Abstract

Let $\text{Mod}(S_g)$ be the mapping class group of an orientable surface of genus $g$, $S_g$. The action of $\text{Mod}(S_g)$ on the homology of $S_g$ induces the well-known symplectic representation:

$$\text{Mod}(S_g) \longrightarrow \text{Sp}(2g, \mathbb{Z}).$$

The kernel of this representation is called the Torelli group, $\mathcal{I}(S_g)$.

We will study two subgroups of $\mathcal{I}(S_g)$. First we will look at the subgroup generated by all SIP-maps, $\text{SIP}(S_g)$. We will show $\text{SIP}(S_g)$ is not $\mathcal{I}(S_g)$ and is in fact an infinite index subgroup of $\mathcal{I}(S_g)$. We will also classify which SIP-maps are in the kernel of the Johnson homomorphism and Birman-Craggs-Johnson homomorphism.

Then we will look at the symmetric Torelli group, $\mathcal{SI}(S_g)$. More specifically, we will investigate the group generated by Dehn twists about symmetric separating curves denoted $\mathcal{H}(S_g)$. We will show the well-known Birman-Craggs-Johnson homomorphism is not able to distinguish among $\mathcal{SI}(S_g)$, $\mathcal{H}(S_g)$, or $\mathcal{K}(S_g)$, where $\mathcal{K}(S_g)$ is the subgroup generated by Dehn twists about separating curves. Elements of $\mathcal{H}(S_g)$ act naturally on the symmetric separating curve complex, $C_{\mathcal{H}}(S)$. We will show that when $g \geq 5$

$$\text{Aut}(C_{\mathcal{H}}(S_g)) \cong \text{SMod}^\pm(S_g)/\langle \iota \rangle.$$
where $\text{SMod}(S_g)$ is the symmetric mapping class group and $\iota$ is a fixed hyperelliptic involution. Lastly we will give an algebraic characterization of Dehn twists about symmetric separating curves.
Chapter 1

Introduction and Background

The goal of this dissertation is to study subgroups of the mapping class group. More specifically we will consider two subgroups of the Torelli group, a subgroup of the mapping class group that is not well understood. First we will look at the subgroup generated by SIP-maps and then we will consider the symmetric Torelli group.

Let $S_{g,b,n}$ be an oriented surface of genus $g$ with $b$ boundary components and $n$ punctures. Our convention is that boundary components are always fixed pointwise. Further we will often omit an index if it is 0 (sometimes 1 depending when noted as such). We define the mapping class group of $S_g$, Mod($S_g$), to be:

$$\text{Mod}(S_g) := \text{Homeo}^+(S_g)/\text{Homeo}_0^+(S_g)$$

where $\text{Homeo}^+(S_g)$ is the group of of orientation preserving homeomorphisms of $S_g$ and $\text{Homeo}_0^+(S_g)$ is the normal subgroup consisting of elements isotopic to the identity. Thus Mod($S_g$) is the group of isotopy classes of orientation preserving self-homeomorphisms of a surface. See [2], [15], and [23] for background information. The extended mapping class
group, $\text{Mod}^{\pm}(S_g)$, includes both orientation preserving and reversing self-homeomorphisms of the surface. A subgroup of the mapping class group of primary importance is the Torelli group, $I(S_g)$, the kernel of the well-known symplectic representation of the mapping class group. Mapping classes act naturally on the first homology of the surface and preserve the intersection form, giving rise to a surjective map to $\text{Sp}(2g, \mathbb{Z})$ (see Chapter 7 of [15]). This action is known as the symplectic representation.

$$1 \to I(S_g) \to \text{Mod}(S_g) \to \text{Sp}(2g, \mathbb{Z}) \to 1$$

Equivalently, $I(S_g)$ is the subgroup of $\text{Mod}(S_g)$ acting trivially on the homology of the surface.

### 1.1 SIP-maps

There are three types of elements that naturally arise in studying $I(S_g)$: bounding pair maps (BP-maps), Dehn twists about separating curves, and simply intersecting pair maps (SIP-maps). Historically the first two types of elements have been the focus of the literature on $I(S_g)$. For example, in [27], [25], and [28] Johnson showed that when $g \geq 3$ BP-maps generate $I(S)$ and further that Dehn twists about separating curves generate an infinite index subgroup of $I(S_g)$, called the Johnson kernel, $K(S)$. However, SIP-maps have been brought to the forefront due to an infinite presentation of $I(S_g)$ introduced by Putman that uses all three types of elements [34]. Note that every SIP-map is supported on a lantern, that is, a sphere with four boundary components, $S_{0,4}$. We will prove some basic facts about SIP-maps in Section 2.1.

Next we consider the group generated by SIP-maps, which we call $\text{SIP}(S_g)$. It is natural
to ask: Is $\text{SIP}(S_g) = \mathcal{I}(S_g)$? If not, what is the index of $\text{SIP}(S_g)$ in $\mathcal{I}(S_g)$? We will answer these questions in Section 2.2.

We begin by looking at the image of SIP-maps under the *Johnson homomorphism*, $\tau: \mathcal{I}(S_{g,1}) \to \wedge^3 H$ where $H = H_1(S_{g,1}, \mathbb{Z})$. For the most part we will simply think of $H$ as an abelian group. Note that the Johnson homomorphism is one of the classical abelian quotients of $\mathcal{I}(S_{g,1})$. In order to do this calculation we first rewrite SIP-maps as the product of five BP-maps. Then we show the image under $\tau$ of an SIP-map that is naturally embedded in a lantern with boundary components $w, x, y, z$ is $\pm x \wedge y \wedge z$. Further we are able to characterize which SIP-maps are in $\ker \tau = \mathcal{K}(S)$.

**Proposition 1.1.1.** An SIP-map $f$ which is embedded in a lantern $L$, is an element of $\mathcal{K}(S)$ if and only if one of the boundary components of $L$ is null-homologous or if two components are homologous.

From these calculations, we are also able to deduce that $\text{SIP}(S_g) \neq \mathcal{I}(S_g)$ by noting that SIP-maps are in the kernel of the so-called “contraction map.” Further, the contraction map shows:

**Proposition 1.1.2.** The group $\text{SIP}(S_g)$ is an infinite index subgroup of $\mathcal{I}(S_g)$.

In proving this, we study the group $\text{SIP}(S_g) \cup \mathcal{K}(S)$ and discuss several interpretations of this group in Section 2.2. Next in Section 2.4 we look at the image of SIP-maps under the *Birman-Craggs-Johnson homomorphism*, $\sigma: \mathcal{I} \to B_3$, where $B_3$ is a $\mathbb{Z}_2$-vector space of Boolean (square free) polynomials with generators corresponding to non-zero elements of $H_1(S, \mathbb{Z})$ [26]. We show that

**Lemma 1.1.3.** Let $\sigma$ be the Birman-Craggs-Johnson homomorphism. If $f$ is an SIP-map which is embedded in a lantern, $L$, then $\sigma(f)$ is a cubic polynomial dependent on the boundary.
components of $L$. Further $\sigma(f) = 0$ if and only if one of the boundary components of $L$ is null-homologous.

Since $\text{SIP}(S_g)$ is not $\mathcal{I}(S_g)$, it is natural to ask: What is the precise structure of $\text{SIP}(S_g)$? For instance:

**Problem 1.1.4.** Is $\mathcal{I}(S_g)/\text{SIP}(S_g)$ abelian?

Building on work of Johnson, to answer Problem 1.1.4 it will suffice to establish if the intersection of the Johnson kernel and the Birman-Craggs-Johnson kernel lies in $\text{SIP}(S_g)$ [29]. We have characterized which SIP-maps are in the Johnson kernel, $\mathcal{K}(S)$, and which are in the Birman-Craggs-Johnson kernel. It remains to investigate the converse:

**Problem 1.1.5.** Which elements of the Johnson kernel, $\mathcal{K}(S_g)$, and Birman-Craggs-Johnson kernel lie in $\text{SIP}(S_g)$?

While Johnson has given a completely algebraic characterization of the Birman-Craggs-Johnson kernel [29], this kernel is still not well understood in terms of BP-maps, separating twists, and SIP-maps, all of which have a natural topological structure.

### 1.2 The Symmetric Torelli Group

Another subgroup of $\mathcal{I}(S_g)$ is the *symmetric Torelli group*, $\mathcal{SI}(S_g)$. Let the *symmetric mapping class group*, $\text{SMod}(S_g)$, be the subgroup of $\text{Mod}(S_g)$ that commutes with a fixed involution of the surface, see Chapter 3. Then

$$\mathcal{SI}(S_g) := \text{SMod}(S_g) \cap \mathcal{I}(S_g)$$
Birman-Hilden showed in [4] that the symmetric mapping class group modulo this fixed involution is actually isomorphic to the mapping class group of a $2g + 2$ punctured sphere, $\text{Mod}(S_{0,0,2g+2})$.

Hain conjectured in [17] the following:

**Conjecture 1.2.1.** The symmetric Torelli group, $\mathcal{SI}(S_g)$, is generated by Dehn twists about symmetric separating curves.

His conjecture is motivated by algebraic geometry as this result implies that the branch locus of the period map, from Torelli space to the Siegel upper half plane, has simply connected components. Brendle-Margalit have factored all known elements in $\mathcal{SI}(S_g)$ into products of Hain’s generators, thus giving strong evidence for Hain’s conjecture [6].

For convenience, let $\mathcal{H}(S_g)$ be the subgroup of $\mathcal{SI}(S_g)$ generated by Hain’s proposed generating set. In this notation, Hain’s conjecture says $\mathcal{H}(S_g) = \mathcal{SI}(S_g)$. If Hain’s conjecture proves false, it will nevertheless be interesting to compare $\mathcal{H}(S_g)$ and $\mathcal{SI}(S_g)$. For example, in Section 3.1 we will show the Birman-Craggs-Johnson homomorphism, $\sigma$, cannot determine the difference among $\mathcal{H}(S), \mathcal{SI}(S), \mathcal{K}(S)$ via direct calculations:

**Theorem 1.2.2.** Let $\mathcal{H}((S_g))$ be the subgroup of $\mathcal{SI}(S_g)$ generated by Hain’s generating set.

$$\sigma(\mathcal{H}(S_g)) = \sigma(\mathcal{SI}(S_g)) = \sigma(\mathcal{K}(S_g)) = B_2$$

We know $\mathcal{SI}(S_g) \preceq \mathcal{K}(S_g)$ when $g \geq 3$, but perhaps studying the interplay between $\mathcal{H}(S_g)$ and $\mathcal{SI}(S_g)$ would be another possible vehicle to understanding the kernel of the Birman-Craggs-Johnson homomorphism.

One technique frequently used in geometric group theory to understand a group is to find a simplicial complex that the group acts naturally on. For $\mathcal{H}(S_g)$, the natural simplicial
complex is the *symmetric separating curve complex*, \( C_{\mathcal{H}}(S_g) \). We show the following result about this complex.

**Main Theorem.** Let \( C_{\mathcal{H}}(S_g) \) be the symmetric separating curve complex and \( g \geq 5 \). Then

\[
\text{Aut}(C_{\mathcal{H}}(S_g)) \cong \text{Mod}^\pm(S_{0,2g+2}) \cong \text{SMod}^\pm(S_g)/\langle \iota \rangle.
\]

The last congruence is a result of Birman-Hilden.

**Outline of the Argument.** One way is easy. Every element in \( \text{SMod}^\pm(S) \) restricts to an element of \( \text{Aut}(\mathcal{H}(S_g)) \) as follows. If \( f \in \text{SMod}^\pm(S_g) \), then \( f \mapsto \phi \), where \( \phi(h) = fhf^{-1} \) for \( h \in \mathcal{H}(S_g) \) and it is clear \( \iota \) is in the kernel of this map. Now let \( \phi \in \text{Aut}(C_{\mathcal{H}}(S_g)) \).

**Step 1:** We extend \( \phi \) to a map on all symmetric curves in Section 3.3. Any non-separating symmetric curve \( c \) maps to an arc, \( \bar{c} \), connecting two marked points in \( S_{0,2g+2} \). Note that \( \bar{c} \) is uniquely determined up to isotopy by any two genus one symmetric separating curves which both contain the marked endpoints of \( \bar{c} \) in \( S_{0,2g+2} \). Since we know where \( \phi \) maps symmetric separating curves, this determines \( \phi(c) \) up to isotopy. Further this extension preserves disjointness between symmetric curves, making \( \phi \) a simplicial map.

**Step 2:** We extend \( \phi \) to “presymmetric curves” in Section 3.4 by showing that presym-metric curves are a boundary component of a regular neighborhood of a chain of symmetric curves. Thus it is clear where presymmetric curves go in \( S_{0,2g+2} \) and disjointness properties are preserved in \( S_{0,2g+2} \).

**Step 3:** Thus using a result of Korkmaz [31] and Birman-Hilden [4] we show \( \phi \) induces a map in \( \text{SMod}^\pm(S_g)/\langle \iota \rangle \).

Our work on curve complexes was motivated by trying to prove the following:
Conjecture 1.2.3. For $S$ a surface with genus $g \geq 5$, we have:

$$\text{Aut}(\mathcal{H}(S_g)) \cong \text{SMod}^\pm(S_g)/\langle \iota \rangle$$

Analogous results have been proven for $\mathcal{I}(S_g)$ and $\mathcal{K}(S_g)$ by Farb-Ivanov [14] and Brendle-Margalit [7]. The main ingredient necessary was to consider the appropriate simplicial complex. In addition, they used a purely algebraic characterization of Dehn twists about separating curves and BP-maps to show that the given natural maps were isomorphisms.

In our situation, we have a natural map

$$\text{SMod}^\pm(S_g)/\langle \iota \rangle \longrightarrow \text{Aut}(\mathcal{H}(S_g)).$$

In Section 3.5 we give an algebraic characterization of Dehn twists about symmetric separating curves.

Theorem 1.2.4. Let $S_g$ be a closed, oriented surface of genus $g \geq 3$. For nontrivial $f \in \mathcal{H}(S_g)$, $f$ is a power of a Dehn twist about a symmetric separating curve if and only if

1. $Z(C_H(f)) = \mathbb{Z}$
2. $Z(C_I(f)) = \mathbb{Z}$
3. $C_H(f) \neq \mathbb{Z}$
4. $\max_{a,b} H(f) = g - 1$.

We also show that this characterization is not sufficient to conclude that the natural map $\text{SMod}^\pm(S_g)/\langle \iota \rangle \longrightarrow \text{Aut}(\mathcal{H}(S_g))$ is an isomorphism.
1.3 Background

**Basic Definitions.** We will refer to a simple closed curve as a *curve* unless stated otherwise and we will often not distinguish between a curve and its isotopy class unless needed.

The simplest infinite order element in $\text{Mod}(S)$ is a (right) *Dehn twist* about a simple closed curve $c$, denoted $T_c$. One can think of this map as cutting the surface along $c$ and twisting a neighborhood of one of the boundary components $360^\circ$, and then gluing the surface back together along $c$. For example, in Figure 1.1, we see the image of the curve $d$ under the mapping class $T_c$.

![Figure 1.1: An example of the image of a curve under a Dehn twist.](image)

For completeness, note that Dehn twists are basic elements of the mapping class group in the following sense.

**Theorem 1.3.1** (Dehn, [12]). *The mapping class group, $\text{Mod}(S)$, is generated by finitely many Dehn twists about simple closed curves.*

The *algebraic intersection number* of a pair of transverse, oriented curves $\alpha$ and $\beta$ on a surface, denoted $\hat{i}(\alpha, \beta)$, is the sum of the indices of the intersection points of $\alpha$ and $\beta$, where an intersection point has index $+1$ if the orientation of the intersection agrees with the orientation of the surface, and $-1$ otherwise.

The *geometric intersection number* of a pair of curves $\alpha$ and $\beta$ is defined as

$$i(\alpha, \beta) = \min_{a \in \alpha, b \in \beta} |a \cap b|.$$
Note this is well-defined on isotopy classes of curves (Chapter 1, [15]).

**Relations in Mod(S).** We will discuss several well-known relations in Mod(S) that will be used throughout this paper, most notably the so-called lantern relation. Proofs for all these can be found in Chapter 2 of [15].

**Lemma 1.3.2.** Let \( f \in \text{Mod}(S) \) and \( a \) be a curve on \( S \). Then \( fT_a f^{-1} = T_{f(a)} \).

**Lemma 1.3.3.** Let \( a \) and \( b \) be curves on \( S \). Then \( T_a = T_b \) if and only if \( a \) is isotopic to \( b \).

**Lantern Relation in Mod(S).** The lantern relation is a relation in Mod(S) among 7 Dehn twists all supported on a subsurface of \( S \) homeomorphic to a sphere with 4 boundary components (otherwise known as a lantern). This relation was known to Dehn [12], and later rediscovered by Johnson [30]. The lantern relation will be particularly important in Lemma 2.2.2 when writing an SIP-map as the product of BP-maps. Given curves \( a, b, c, \) and \( d \), so that \( a, b, c, \) and \( d \) bound a lantern, then the following relation holds where the curves are as in Figure 1.2.

\[
T_a T_b T_c T_d = T_x T_y T_z.
\]

![Figure 1.2](image)

**Figure 1.2:** The curves in the lantern relation: \( T_a T_b T_c T_d = T_x T_y T_z \)

**The Torelli Group.** A subgroup of the mapping class group of primary importance is the Torelli group, \( \mathcal{I}(S_g) \), the kernel of the well-known symplectic representation of the
mapping class group.

\[ 1 \to \mathcal{I}(S_g) \to \text{Mod}(S_g) \to \text{Sp}(2g, \mathbb{Z}) \to 1 \]

Equivalently, \( \mathcal{I}(S_g) \) is the subgroup of \( \text{Mod}(S_g) \) acting trivially on the homology of the surface. Note that we will often refer to \( H_1(S, \mathbb{Z}) \) simply as \( H \). Further, because the symplectic group, \( \text{Sp}(2g, \mathbb{Z}) \) is well understood, \( \mathcal{I}(S) \) is often thought of as the “mysterious” part of \( \text{Mod}(S) \). Note that when \( g = 1 \) the symplectic representation is faithful, so \( \mathcal{I}(S) = 1 \). There are three types of elements that naturally arise in studying \( \mathcal{I}(S) \):

1. **Bounding Pair Maps.** Given two disjoint, non-separating, homologous simple closed curves \( c \) and \( d \), a **bounding pair map** (BP-map) is the product \( T_cT_d^{-1} \). If \( S = S_{g,1} \), then we say a BP-map has **genus** \( k \) if the subsurface whose boundary is \( c \cup d \) has genus \( k \).

   ![Figure 1.3: A genus k bounding pair.](image)

2. **Separating Twists.** A simple closed curve \( c \) is called **separating** if \( S - c \) is not connected. A **separating twist** is a Dehn twist about a separating curve. If \( S = S_{g,1} \), then we say a separating twist has **genus** \( k \) if the subsurface whose boundary is \( c \) has genus \( k \). As a side note, when \( g = 2 \), \( \mathcal{I}(S_2) = \mathcal{K}(S_2) \), the subgroup generated by separating twists, as there are no BP-maps in \( \mathcal{I}(S_2) \) [32].

3. **Simply Intersecting Pair Maps.** Let \( c \) and \( d \) be simple closed curves so that \( \hat{i}(c,d) = 0 \) and \( i(c,d) = 2 \). Then a **simply intersecting pair map** (SIP-map) is the
commutator of the Dehn twists about the two curves, that is \([T_c, T_d] = T_c T_d T_c^{-1} T_d^{-1}\).

The curves \(c\) and \(T_d(c)\) are homologous because \([T_d(c)] = [c] + \hat{i}(c, d)[d] = [c]\). Since twists about homologous curves have the same image under the symplectic representation (see Chapter 7 of [15] for further details), we can conclude that \(T_c T_d^{-1} \in \mathcal{I}(S)\). Note that in essence, SIP-maps are a natural generalization of BP-maps and could further be generalized by considering commutators of Dehn twists about curves with higher geometric intersection number which still have algebraic intersection number 0.

While the first two types of elements have been the focus of the literature on \(\mathcal{I}(S)\), SIP-
maps have been brought to the forefront due to an infinite presentation of $\mathcal{I}(S_g)$ introduced by Putman that uses all three types of elements [34]. Let $\text{SIP}(S_g)$ be the group generated by SIP-maps.
Chapter 2

SIP-maps

In this chapter we will prove basic facts about SIP-maps as well as the group they generate. We will also look at the image of SIP-maps under well-known representations of $\mathcal{I}(S)$.

2.1 Basic Facts About SIP-maps

In this section we will further investigate the structure of SIP-maps. We begin by showing they are pseudo-Anosov elements on a lantern.

Classification of Mapping Classes. Mapping classes are often classified according to whether or not they fix any curves in the surface, as follows. A curve, $c$, is called a reducing curve for a mapping class $f$, if $f^n(c) = c$ for some $n$.

Nielsen-Thurston Trichotomy. We are able to classify any mapping class, $f$, into one of the following categories:

1. The mapping class, $f$, is a finite order element; that is, there exists an $n$ such that
\[ f^n = \text{id} \]

2. The mapping class, \( f \), is reducible; that is it fixed a collection of pairwise disjoint curves, or

3. The mapping class, \( f \), is pseudo-Anosov if it is not finite order or reducible.

Note that there is an equivalent, somewhat more standard and more technical, definition of a pseudo-Anosov mapping class given in terms of measured foliations. We will not need to use this definition or the machinery of measured foliations explicitly in this work.

There is non-trivial overlap between the finite order and reducible elements. In order to make this a true trichotomy, we can replace the condition of having a reducing curve with that of having an essential reducing curve: a reducing curve \( c \) is essential for a mapping class \( h \) if for each simple close curve \( b \) on the surface such that \( i(c, b) \neq 0 \), and for each integer \( m \neq 0 \), the classes \( h^m(b) \) and \( b \) are distinct.

**Theorem 2.1.1** (Birman-Lubotzky-McCarthy, [5]). *For every mapping class \( h \) there exists a system (possibly empty) of essential reducing curves. Moreover, the system is unique up to isotopy, and there is an \( n \) such that cutting along the system, the restriction of \( h^n \) to each component of the cut-open surface is either pseudo-Anosov, finite order, or reducible.*

Note that a fixed curve of a finite order mapping class is never essential, because there is always an \( n \) such that \( h^n = \text{id} \) after cutting open along all the other curves.

The canonical reduction system for a mapping class, \( f \), is the collection of all essential reducing curves for \( f \). This classification, as well as the canonical reduction system, will be used throughout this paper.

Using work of Atalan-Korkmaz we will classify SIP-maps on a lantern, \( S_{0,4} \). They make the following characterizations of reducible elements on the lantern.
Lemma 2.1.2 (Atalan-Korkmaz, Lemma 3.4, [1]). The reducible elements of $\text{Mod}(S_{0,4})$ consist of conjugates of nonzero powers of $T_a$, $T_b$ and $T_aT_b$.

Thus we are able to deduce the following.

Corollary 2.1.3. Let $a$ and $b$ be two curves with $i(a,b) = 2$ and $\hat{i}(a,b) = 0$. Then the SIP-map $f = [T_a, T_b]$ is pseudo-Anosov on a regular neighborhood of $a$ and $b$; that is, on a lantern, $S_{0,4}$.

Proof. It is clear $T_aT_bT_a^{-1}T_b^{-1}$ is a cyclicly reduced word in the free group generated by $T_a$ and $T_b$. Thus $f$ is not conjugate to a power of $T_a$, $T_b$ or $T_aT_b$ and must be pseudo-Anosov. □

Further, we consider how many SIP-maps are supported on a given lantern.

Lemma 2.1.4. Consider the curves $x$, $y$, and $z$ as in Figure 1.2. Then the SIP-maps, $[T_x, T_y]$, $[T_y, T_z]$, and $[T_x, T_z]$, are all distinct, as well as their inverses.

Proof. Consider the lantern in Figure 1.2 with boundary components $a$, $b$, $c$, and $d$ and the lantern relation: $T_yT_yT_z = T_aT_bT_cT_d$. We consider the SIP-maps $[T_x, T_y]$ and $[T_y, T_z]$. Suppose $[T_x, T_y] = [T_y, T_z]$. Using the lantern relation and Lemmas 1.3.2 and 1.3.3 we have:

$$[T_x, T_y] = [T_y, T_z]$$

$$\iff T_xT_yT^{-1}yT^{-1} = T_yT_zT^{-1}zT^{-1}$$

$$\iff T_aT_bT_cT_dT^{-1}zT^{-1}xT^{-1}yT^{-1} = T_aT_bT_cT_dT^{-1}yT^{-1}zT^{-1}x$$

$$\iff T_zT^{-1}zT^{-1}xT^{-1}yT^{-1}zT^{-1} = T_yT^{-1}zT^{-1}y$$

$$\iff T^{-1}T_{T(z)} = T^{-1}T_{T_{T(y)}(z)}$$

$$\iff T_{T(z)} = T_{T_{T(y)}(z)}$$
A simple calculation shows these are not the same curve. Thus \([T_x, T_y] \neq [T_y, T_z]\). Similar arguments show that the remaining SIP-maps are also distinct. \(\square\)

Note that distinct pairs of curves can define the same SIP-map. For example, consider the SIP-maps \([T_z, T_x]\) and \([T_{T_y^{-1}(x)}, T_y]\) where \(x, y,\) and \(z\) are as in Figure 1.2. Using the lantern relation, Lemma 1.3.2, and the fact that Dehn twists about disjoint curves commute, we see

\[
[T_{T_y^{-1}(x)}, T_y] = T_{T_y^{-1}(x)} T_y T_{T_y^{-1}(x)} T_y^{-1}
= T_y^{-1} T_x T_y T_y^{-1} T_x^{-1} T_y T_y^{-1}
= T_y^{-1} T_x T_y T_x^{-1}
= (T_a^{-1} T_b^{-1} T_c^{-1} T_d^{-1} T_z T_x) T_x (T_a T_b T_c T_d T_x^{-1} T_z^{-1}) T_x^{-1}
= T_z T_x T_x^{-1}
= [T_z, T_x]
\]

In the next section instead of looking at individual SIP-maps, we will look at the group generated by SIP-maps and compare it to well known subgroups of \(I(S)\).

### 2.2 The \(SIP(S_g)\)-group

The goal of this section is to prove some basic results about the group generated by all SIP-maps in \(\text{Mod}(S)\), which we will denote as \(SIP(S)\). We will do this by looking at the image of \(SIP(S)\) under well-known representations of \(I(S)\) as well as classifying which SIP-maps are in the kernel of these representatives. Recall that we do not distinguish between a curve and its isotopy class. Similarly, we will frequently not distinguish between a curve and its
homology class. There is an issue regarding the orientation of a curve, and we will deal with this issue when necessary.

**Johnson Homomorphism.** Johnson defined a surjective homomorphism, \( \tau : \mathcal{I}(S_{g,1}) \longrightarrow \wedge^3 H \) in [25] that measures the action of \( f \in \mathcal{I}(S_{g,1}) \) on \( \pi_1(S) \). Johnson showed that separating twists are in \( \ker \tau \). Further he showed separating twists generate \( \ker \tau \). We call this subgroup the *Johnson kernel*, \( \mathcal{K}(S) \).

**Theorem 2.2.1** (Johnson, [25] and [28]). The group \( \ker \tau \) is generated by Dehn twists about separating curves.

In addition, Johnson showed how to calculate the image of a BP-map under \( \tau \) by first choosing a symplectic basis \( \{a_1, \ldots, a_k, b_1, \ldots, b_k\} \) for the homology of the subsurface bounded by \( c \) and \( d \). With the chosen basis he showed

\[
\tau(T_c T_d^{-1}) = \sum_{i=1}^{k} (a_i \wedge b_i) \wedge c
\]

Note that the orientation of \( c \) is chosen so that the subsurface not containing the boundary component is on the left. Johnson also showed that the image is independent of the choice of symplectic basis. For our purposes we will take this as the definition of \( \tau \), since BP-maps generate \( \mathcal{I}(S_{g,1}) \) when \( g \geq 3 \). We will usually use the standard symplectic basis for \( H \) shown in Figure 2.1.

![Figure 2.1: A collection of oriented curves that form a symplectic homology basis for \( H_1(S, \mathbb{Z}) \).](image-url)
It is natural to ask what the image of an SIP-map is under $\tau$. One way to calculate this is by factoring the SIP-map into BP-maps. Consider the SIP-map $[T_a, T_b]$ as shown in Figure 2.2.

Figure 2.2: A collection of curves needed to rewrite the SIP-map, $[T_a, T_b]$ in terms of BP-maps.

**Lemma 2.2.2.** Given the SIP-map $[T_a, T_b]$ as shown in Figure 2.2. Then

$$[T_a, T_b] = (T_x T_u^{-1})(T_w T_v^{-1})(T_f T_c^{-1})(T_d T_a^{-1})(T_e T_b^{-1}).$$

**Proof.** We will need to use the lantern relation twice to rewrite this SIP-map in terms of BP-maps.

**From the Top Lantern:** $T_a T_b T_c = T_x T_y T_z T_w$

**From the Bottom Lantern:** $T_f T_d T_e = T_y T_z T_v T_u$

Then using the above facts and disjointness, we see

$$[T_a, T_b] = (T_a T_b) T_a^{-1} T_b^{-1}$$
$$= T_x (T_y T_z) T_w T_c^{-1} T_a^{-1} T_b^{-1}$$
$$= T_x T_v^{-1} T_u^{-1} T_f T_d T_e T_w T_c^{-1} T_a^{-1} T_b^{-1}$$
$$= (T_x T_u^{-1})(T_w T_v^{-1})(T_f T_c^{-1})(T_d T_a^{-1})(T_e T_b^{-1})$$

$\square$
Now we are ready to compute the image of an SIP-map under \( \tau \). We will rely on a common principle used in the study of mapping class groups called the *change of coordinates principle*. The idea is that to prove a topological statement about a certain configuration of curves, if suffices to show the result on our “favorite” example of curves satisfying the condition. For example to show a result about a non-separating curve, up to homeomorphism, it suffices to show the result for any non-separating curve. See Section 1.3 of [15] for further details. We will make use of this principle in proving many of our main results.

**Proposition 2.2.3.** Let \( f \) be an SIP-map embedded in a lantern with boundary components \( w, x, y, \) and \( z \). Then \( \tau(f) = \pm [x] \land [y] \land [z] \).

**Proof.** Let \( f = [T_a, T_b] \). Then by the change of coordinates principle, showing the result for \( f \) will suffice to prove the general result.

\[
\tau([T_a, T_b]) = \tau((T_xT_u^{-1})(T_aT_v^{-1})(T_fT_c^{-1})(T_dT_a^{-1})(T_eT_b^{-1}))
\]
\[
= \tau(T_xT_u^{-1}) + \tau(T_wT_v^{-1}) + \tau(T_fT_c^{-1}) + \tau(T_dT_a^{-1}) + \tau(T_eT_b^{-1})
\]
\[
= (a_1 \land b_1) \land [x] + (a_1 \land b_1 + a_2 \land b_2 + a_3 \land b_3) \land [w] +
\]
\[
(a_1 \land b_1 + a_2 \land (b_2 - a_3 + b_3)) \land [f] + (a_1 \land b_1 + a_2 \land b_2) \land [d] +
\]
\[
((-a_2 + a_3) \land b_3) \land [e]
\]
\[
= (a_1 \land b_1) \land (-a_2) + (a_1 \land b_1 + a_2 \land b_2 + a_3 \land b_3) \land (-a_4) +
\]
\[
(a_1 \land b_1 + a_2 \land (b_2 - a_3 + b_3)) \land (a_2 - a_3 + a_4) +
\]
\[
(a_1 \land b_1 + a_2 \land b_2) \land (a_3) + ((-a_2 + a_3) \land b_3) \land (-a_2 + a_4)
\]
\[
= -a_2 \land a_3 \land a_4
\]
\[
= \pm [x] \land [y] \land [z]
\]

Note that every SIP-map is naturally embedded in a lantern with boundary components...
\(w, x, y,\) and \(z,\) hence we see the image of an SIP-map is \(\tau([T_a, T_b]) = \pm x \wedge y \wedge z\) where the orientations of \(w, x, y\) and \(z\) are so that the lantern is on the left. The sign is dependent on the ordering of the boundary components with respect to \(a\) and \(b.\)

Recently Putman [35] and independently Church [11] also calculated the image of a SIP-map under \(\tau\) directly, that is without using the above factorization.

**Theorem 2.2.4.** The subgroup \(\operatorname{SIP}(S_{g,1})\), is a proper subgroup of \(\mathcal{I}(S_{g,1})\).

**Proof.** To show this we will make use of the contraction map \(C\) which Johnson introduces in [25]. The contraction map \(C: \wedge^3 H \longrightarrow H\) is defined by

\[
a \wedge b \wedge c \longmapsto 2(\hat{i}(b,c)a + \hat{i}(a,c)b + \hat{i}(a,b)c).
\]

Hence using Proposition 2.2.3 it is easy to see that SIP-maps are in the kernel of \((C \circ \tau)\) since the boundary components of a lantern are disjoint.

\[
(C \circ \tau)([T_c, T_d]) = C(\pm w \wedge x \wedge y) = 0.
\]

Further, Johnson shows that \(C\) actually maps \(\mathcal{I}(S_{g,1})\) onto \(2H\). From this, we are able to deduce that \(\mathcal{I}(S_{g,1}) \neq \operatorname{SIP}(S_{g,1})\).

The following corollaries are immediate consequences of the proof of Theorem 2.2.4.

**Corollary 2.2.5.** The group \(\operatorname{SIP}(S_{g,1}) \nsubseteq \mathcal{K}(S_{g,1})\).

**Corollary 2.2.6.** The group, \(\operatorname{SIP}(S_{g,1})\), is an infinite index subgroup of \(\mathcal{I}(S_{g,1})\).

SIP-maps in \(\mathcal{K}(S)\). Note that we can now characterize which SIP-maps are in \(\mathcal{K}(S) = \ker \tau\).
Corollary 2.2.7. An SIP-map, $f$, which is embedded in a lantern, $L$, is an element of $\mathcal{K}(S)$ if and only if one of the boundary components of $L$ is null-homologous or if two components are homologous.

Proof. This follows directly from the calculation given in Proposition 2.2.3 of $\tau(f) = \pm [x] \wedge [y] \wedge [z]$. \hfill \blacksquare

See Figure 2.3 for examples of each type of SIP-map in $\mathcal{K}(S)$.

![Figure 2.3: Examples of SIP-maps in $\mathcal{K}(S)$](image-url)

The Subgroup $\text{SIP}(S_{g,1}) \cup \mathcal{K}(S_{g,1})$. The group $\text{SIP}(S_{g,1}) \cup \mathcal{K}(S_{g,1})$ has appeared in the literature before this, but has never been recognized in terms of SIP-maps. We will define basic terminology regarding winding numbers and the Chillingworth subgroup. Then we show how $\text{SIP}(S_{g,1}) \cup \mathcal{K}(S_{g,1})$ can be viewed in four different ways.

More can be said about the structure of $\wedge^3 H$, and it can be applied to our current situation. According to Sakasai, $\wedge^3 H$ has two irreducible components as Sp-modules (Section 2.3, [36]):

$$\wedge^3 H = H \oplus U$$

where $U$ is the kernel of $C$, the contraction map. It follows from irreducibility and normality that $\tau(\text{SIP}(S_{g,1})) = U$. From the following commutative diagram we see that

$$\text{SIP}(S_{g,1})/(\mathcal{K}(S_{g,1}) \cap \text{SIP}(S_{g,1})) \cong U.$$
Further, ker \((C \circ \tau)\) = SIP\((S_{g,1})\)\(\cup\)\(\mathcal{K}(S_{g,1})\). From this we can conclude \(\mathcal{I}(S_{g,1})/(\text{SIP}(S_{g,1}) \cup \mathcal{K}(S_{g,1})) \cong 2H\). Because \(2H\) is an infinite group and SIP\((S_{g,1}) \subset \text{SIP}(S_{g,1}) \cup \mathcal{K}(S_{g,1})\), it also follows that SIP\((S_{g,1})\) is of infinite index in \(\mathcal{I}(S_{g,1})\).

**Winding Number.** For a surface \(S\) with nowhere zero vector field \(X\) on \(S\), Chillingworth defines the concept of *winding number* with respect to \(X\) of an oriented regular curve, \(c\), to be the number of times its tangent rotates with respect to the framing induced by \(X\) \([9, 10]\), denoted as \(\omega_X(c)\). If \(f \in \mathcal{I}(S)\), we have the function

\[ e_{f,X}(c) = \omega_X(f(c)) - \omega_X(c). \]

This function measures the change in winding number induced by \(f\). Johnson showed this function is independent of the choice of vector field \(X\) \([25]\), hence we will write \(e_f\). Note that Johnson also showed \(e_f\) is a function on homology classes. We can then dualize the class \(e_f\) to a homology class \(t_f\) where \(c \cdot t_f = e_f(c)\). We call \(t_f\) the *Chillingworth class* of \(f\). Johnson showed that \(t_f = (C \circ \tau)(f)\). Thus we have shown \(\text{SIP}(S_{g,1}) \cup \mathcal{K}(S_{g,1})\) is the kernel of \(t\). The kernel of \(t\) is also called the *Chillingworth subgroup*.

**Chillingworth Subgroup.** Trapp showed in \([37]\) that the Chillingworth subgroup is characterized as:

\[ \langle T_{\gamma_1}T_{-\delta_1}T_{\gamma_2}T_{-\delta_2} \cdots T_{\gamma_n}T_{-\delta_n} \rangle \]

where \(T_{\gamma_i}T_{-\delta_i}^{-1}\) is a genus one BP-map and \(\sum_{i=1}^{k} 2[\gamma_i] = 0\) in \(H\). This follows from a calculation
done by Trapp and Johnson that if $T_\gamma T_\delta^{-1}$ is a genus one BP-map then $t(T_\gamma T_\delta^{-1}) = 2[\gamma]$. Further, we can extend this presentation to include BP-maps of genus $g$ in the following way.

If the BP-map $T_\gamma T_\delta^{-1}$ has genus $g(\gamma, \delta)$ with $a_i, b_i$ as a symplectic basis for the corresponding subsurface, then

$$t(T_\gamma T_\delta^{-1}) = C\left( \sum_{i=1}^{g(\gamma, \delta)} a_i \wedge b_i \wedge \gamma \right)$$

$$= C\left( \sum_{i=1}^{g(\gamma, \delta)} a_i \wedge b_i \wedge \gamma \right)$$

$$= \sum_{i=1}^{g(\gamma, \delta)} C(a_i \wedge b_i \wedge \gamma)$$

$$= \sum_{i=1}^{g(\gamma, \delta)} 2[\gamma]$$

$$= 2g(\gamma, \delta)[\gamma]$$

So we could write the Chillingworth subgroup as:

$$\langle T_{\gamma_1} T_{\delta_1}^{-1} T_{\gamma_2} T_{\delta_2}^{-1} \ldots T_{\gamma_n} T_{\delta_n}^{-1} \rangle$$

where $T_{\gamma_i} T_{\delta_i}^{-1}$ is a BP-map and $\sum_{i=1}^{k} 2g(\gamma_i, \delta_i)[\gamma_i] = 0$ in $H$. Equivalently, we can include separating twists, $T_\gamma$, because $t(T_\gamma) = 0$. Hence the Chillingworth subgroup is:

$$\langle f_1, f_2, \ldots, f_n \rangle$$

where $f_i = T_{\gamma_i}$ and $\gamma_i$ is a separating curve or $f_i = T_{\gamma_i} T_{\delta_i}^{-1}$ is a BP-map and $\sum_{i=1}^{k} 2g(\gamma_i, \delta_i)[\gamma_i] = 0$ in $H$, with $g(\gamma_i, \delta_i) := 0$ if $\gamma_i$ is separating.

Now we have the following equivalence:
Corollary 2.2.8. The following are equivalent definitions of the group $\text{SIP}(S_g, 1) \cup \mathcal{K}(S_g, 1)$.

1. The group $\text{SIP}(S_g, 1) \cup \mathcal{K}(S_g, 1)$ is the group generated by all separating twists and SIP-maps.

2. The group $\text{SIP}(S_g, 1) \cup \mathcal{K}(S_g, 1)$ is the kernel of $\mathcal{C} \circ \tau$.

3. The group $\text{SIP}(S_g, 1) \cup \mathcal{K}(S_g, 1)$ is the group of all elements in $\mathcal{I}(S_g, 1)$ with winding number zero.

4. The group $\text{SIP}(S_g, 1) \cup \mathcal{K}(S_g, 1)$ is the following group:

$$\langle T_{\gamma_1} T_{\delta_1}^{-1} T_{\gamma_2} T_{\delta_2}^{-1} \ldots T_{\gamma_n} T_{\delta_n}^{-1} \rangle$$

where $T_{\gamma_i} T_{\delta_i}^{-1}$ is a genus one BP-map and $\sum_{i=1}^{k} 2|\gamma_i| = 0$ in $H$.

2.3 Reinterpreting Relations

A potential application of studying SIP-maps is to find a better generating set for $\mathcal{I}(S)$. While Johnson found a finite generating set for $\mathcal{I}(S_g)$ when $g \geq 3$ it is extremely large [27]. Johnson conjectured that this generating set could be reduced to a more manageable size. Johnson’s main technique was to employ several relations he discovered among BP-maps. We will use SIP-maps to reinterpret one of Johnson’s relations in $\mathcal{I}(S)$. Perhaps similar techniques could be used to rewrite the other Johnson relations.
Independently, Putman showed how to factor an SIP-map into the product of two BP-maps [34].

**Lemma 2.3.1** (Putman, Fact F.5, [34]). Let curves $a, b,$ and $e$ be as in Figure 2.2. Then

\[ [T_a, T_b] = (T_{T_a(b)}T_e^{-1})(T_eT_b^{-1}). \]

Combining Lemmas 2.2.2 and 2.3.1 yields a new proof of the following relation in $\mathcal{I}(S)$ discovered by Johnson [27].

![Figure 2.4: The curves needed for Johnson’s relation.](image)

**Lemma 2.3.2** (Johnson, Lemma 10, [27]). Let curves $a, a', b, b', c, c', c_1, c_3, d, e, e', f, \text{ and } f'$ be as defined in Figure 2.4. Then

\[ T_aT_{a'}^{-1}T_bT_{b'}^{-1}T_dT_{c'}^{-1} = T_cT_{e'}^{-1}T_fT_{f'}^{-1}. \]

**Proof.** Using the factoring of Putman in Lemma 2.3.1 we see:

\[ [T_a, T_c] = T_dT_{c'}^{-1}T_cT_{c'}^{-1}, \text{ where } d = T_a(c). \]

Using the techniques and factoring in Lemma 2.2.2 we have the following:

\[ [T_a, T_c] = T_{b'}T_b^{-1}T_{a'}T_a^{-1}T_{c'}T_c^{-1}T_fT_{f'}^{-1}. \]
Combining these two equations and rearranging we have the desired result.

\[
T_dT_c^{-1}T_c'T_c^{-1} = T_b'T_a'T_a'T_c'T_c^{-1}T_fT_f^{-1} \\
T_aT_a'T_bT_b'T_dT_c'^{-1} = T_c'T_c'^{-1}T_fT_f^{-1}
\]

Further study is needed to determine whether other relations among Johnson’s generators in \(\mathcal{I}(S)\) can be realized by SIP-maps.

### 2.4 Birman-Craggs-Johnson Homomorphism

In order to define the Birman-Craggs-Johnson homomorphism, one of the most well known representations of the Torelli group, we will need first to consider the Birman-Craggs homomorphisms.

**Birman-Craggs Homomorphisms.** In [3] Birman and Craggs introduced a finite collection of homomorphisms from \(\mathcal{I}(S_{g,1})\) to \(\mathbb{Z}_2\) based on the Rochlin invariant.

**Rochlin Invariant.** Let \(W\) be a homology sphere and \(X\) be a simply connected parallelizable 4-manifold, so \(W = \partial X\). We know such a manifold \(X\) always exists, and the signature(\(X\)) is divisible by 8. Further, \(\frac{\text{signature}(X)}{8} \mod 2\) is independent of \(X\); hence, an invariant of \(W\) called the *Rochlin invariant* denoted by \(\mu\). A good reference for this material is [16].

The *Birman-Craggs homomorphisms* is a collection of homomorphisms

\[\rho_h : \mathcal{I} \to \mathbb{Z}_2\]
defined by fixing an embedding \( h : S \to S^3 \) and identifying \( S \) with \( h(S) \). For \( f \in \mathcal{I} \), split \( S^3 \) along \( S \) and reglue the two pieces using \( f \), creating a closed 3-manifold, \( W(h,f) \). Since \( f \) acts trivially on \( H_1(S,\mathbb{Z}) \), the 3-manifold \( W(h,f) \) is a homology sphere. Thus the Rochlin invariant \( \mu(h,f) \in \mathbb{Z}_2 \) is defined. Hence for a fixed embedding \( h \),

\[
\rho_h(f) := \mu(h,f)
\]

is the Birman-Craggs homomorphism. In addition, Johnson showed these homomorphisms correspond to the mod 2 self-linking forms associated with \( S \), hence there are only finitely many [26].

**Birman-Craggs-Johnson Homomorphism.** In [26], Johnson combined all the Birman-Craggs homomorphisms into one homomorphism \( \sigma \), in the sense that the kernel of \( \sigma \) is equal to the intersection of the kernels of all the Birman-Craggs homomorphisms. In order to describe this homomorphism, we first need to define boolean polynomials.

We construct from \( H_1(S,\mathbb{Z}_2) \) a \( \mathbb{Z}_2 \)-algebra \( B \) such that:

1. \( B \) is commutative with unity
2. \( B \) is generated by the abstract elements \( \bar{a} \) where \( a \) is nonzero in \( H_1(S,\mathbb{Z}_2) \),
3. \( \bar{a}^2 = \bar{a} \) for all \( a \neq 0 \) in \( H_1(S,\mathbb{Z}_2) \). (Sometimes this is referred to as a “square-free” algebra.)
4. \( (a + b) = \bar{a} + \bar{b} + a \cdot b \) where \( a \cdot b \in \mathbb{Z}_2 \subset B \) is the algebraic intersection of \( a \) and \( b \) modulo 2.

Elements of \( B \) are thought of as polynomials in the generators. The degree of an element is well defined, and we let \( B_3 \) equal the vector space of all elements in \( B \) of degree less than
or equal to 3.

Then the Birman-Craggs-Johnson homomorphism is a surjective homomorphism

\[ \sigma : \mathcal{I}(S_{g,1}) \rightarrow B_3. \]

Johnson also calculated the image of BP-maps and separating twists under \( \sigma \). Again since BP-maps generate \( \mathcal{I}(S_g) \) when \( g \geq 3 \), for our purposes we will take this as the definition of the map \( \sigma \). Thus,

1. A genus \( k \) separating curve, \( c \):
   - Choose a symplectic basis \( \{a_1, \ldots, a_k, b_1, \ldots, b_k\} \) for the subsurface bounded by \( c \).
   - \( \sigma(T_c) = \sum_{i=1}^{k} \bar{a}_i \bar{b}_i \)

2. A genus \( k \) BP-map \( T_cT_d^{-1} \):
   - Choose a symplectic basis \( \{a_1, \ldots, a_k, b_1, \ldots, b_k\} \) for the subsurface bounded by \( c \) and \( d \).
   - \( \sigma(T_cT_d^{-1}) = \sum_{i=1}^{k} \bar{a}_i \bar{b}_i (1 - \bar{c}) \)

Johnson showed both these calculations are independent of the choice of symplectic basis.

Given the rewriting of an SIP-map in terms of BP-maps in Lemma 2.2.2, it is not hard to determine the image of an SIP-map under \( \sigma \).

**Lemma 2.4.1.** Consider the SIP-map, \([T_a, T_b]\), as shown in Figure 2.2, which is naturally embedded in a lantern with boundary components \( w, x, y, \) and \( z \). Then \( \sigma([T_a, T_b]) = \bar{x}\bar{y}\bar{z} \).
Proof. The proof consists of the following calculation using Lemma 2.2.2 and the change of coordinates principle. We will be using the standard symplectic basis shown in Figure 2.1 for this calculation.

\[ [T_a, T_b] = (T_x T_u^{-1})(T_w T_v^{-1})(T_f T_c^{-1})(T_d T_a^{-1})(T_e T_b^{-1}) \]

\[ = \sigma(T_x T_u^{-1}) + \sigma(T_w T_v^{-1}) + \sigma(T_f T_c^{-1}) + \sigma(T_d T_a^{-1}) + \sigma(T_e T_b^{-1}) \]

\[ = \bar{a}_1 \bar{b}_1 (1 - \bar{a}_2) + (\bar{a}_1 \bar{b}_1 + \bar{a}_2 \bar{b}_2 + \bar{a}_3 \bar{b}_3)(1 - \bar{a}_4) \]

\[ + (\bar{a}_1 \bar{b}_1 + \bar{a}_2 \bar{b}_2)(1 + \bar{a}_3) + ((-a_2 + a_3 - a_4)) \]

\[ = \bar{a}_2 \bar{a}_4 + \bar{a}_2 \bar{a}_3 \bar{a}_4 \]

\[ = \bar{a}_2 (-a_2 + a_4) (-a_3 + a_4) \]

\[ = \bar{x} \bar{y} \bar{z} \]

Note that since \( w, x, y, \) and \( z \) bound a subsurface, the result is equivalent to that using any three of the four bounding curves. For example, we consider \( \bar{x} \bar{y} \bar{w} \).

\[ \bar{x} \bar{y} \bar{w} = \bar{x} \bar{y} (\bar{x} + \bar{y} + \bar{z}) \]

\[ = \bar{x} \bar{y} (\bar{z} + \bar{z} + \bar{z}) \]

\[ = \bar{x} \bar{y} + \bar{x} \bar{y} + \bar{x} \bar{y} \bar{z} \]

\[ = \bar{x} \bar{y} \bar{z} \]

\[ \square \]

**Corollary 2.4.2.** An SIP-map, \([T_a, T_b]\), which is naturally embedded in a lantern, \( L \), is an element of \( \text{ker} \ \sigma \) if and only if one of the boundary components of \( L \) is null-homologous.

Note that an SIP-map, \([T_a, T_b]\), where \( a \) or \( b \) is a separating curve, is always in \( K(S) \) and
sometimes in the kernel of \( \sigma \). We call these \textit{separating SIP-maps}. Let \( \text{SSIP}(S) \) be subgroup generated by separating SIP-maps. We will compute the image of \( \text{SSIP}(S) \) under \( \sigma \) and deduce that \( \text{SSIP}(S) \) is a proper subgroup of \( K(S) \).

**Proposition 2.4.3.** The image of the subgroup generated by separating SIP-maps, that is \( \text{SSIP}(S_{g,1}) \), under the Birman-Craggs-Johnson homomorphism, \( \sigma \), is \( \langle 1, \bar{a}_i, \bar{b}_i, \bar{a}_i \bar{b}_j, \bar{a}_i \bar{b}_i + \bar{a}_j \bar{b}_j | 1 \leq i,j \leq g, i \neq j \rangle \).

An immediate consequence of this result is the following:

**Corollary 2.4.4.** Let \( S \) be a surface with genus, \( g \geq 3 \). Then the group generated by separating SIP-maps, \( \text{SSIP}(S_{g,1}) \), is a proper subgroup of \( K(S_{g,1}) \).

**Proof of Proposition 2.4.3:** There are four basic types of generators of \( B_2 \):

1. \( \bar{a}_i \bar{b}_i \)
2. \( \bar{a}_i \bar{b}_j \) with \( i \neq j \) (this also includes \( \bar{a}_i \bar{a}_j, \bar{a}_i \bar{b}_j, \) and \( \bar{b}_i \bar{b}_j \))
3. \( \bar{a}_i (\bar{b}_i) \)
4. \( 1 \)

Elements of type (1) we will refer to as \textit{symplectic terms} and elements of type (2) will be \textit{nonsymplectic}. We will show that elements of type (2), (3) and (4) are in \( \sigma(\text{SSIP}(S) \cap K(S)) \).

Then we will show which elements generated by type (1) terms are in the image. Recall all coefficients in \( B_2 \) are in \( \mathbb{Z}/2\mathbb{Z} \).

Note that we will only consider separating SIP-maps, that is, SIP-maps where at least one of the defining curves is separating as in Figure 2.5.
Figure 2.5: An SIP-map, $[T_c, T_d]$ with $c$ a separating curve.

**Type (2):** Suppose $c$ and $d$ are as shown. Then

\[
\sigma([T_c, T_d]) = \sigma(T_c T_d T_c^{-1} T_d^{-1}) = \sigma(T_c) + \sigma(T_d) = \sigma(T_c) + \sigma(T_d(c))
\]

It is not hard to see that

\[
\sigma(T_c) = \bar{a}_1 \bar{b}_1.
\]

Further $T_d(c)$ is shown below with symplectic basis consisting of $a_1 + b_2$, and $b_1$ (where $a_i$ and $b_i$ are from the standard symplectic basis as shown in Figure 2.1.

Figure 2.6: $T_d(c)$ with symplectic basis $a_1 + b_2$, and $b_1$

Hence we see that $\sigma(T_{d(c)}) = (a_1 + b_2)\bar{b}_1 = \bar{a}_1 \bar{b}_1 + \bar{b}_1 \bar{b}_2$. Therefore $\sigma([T_c, T_d]) = \bar{b}_1 \bar{b}_2$. By change of coordinates we can get all elements of Type (2).

**Type (3):** Suppose $c$ and $d$ are as shown in Figure 2.7. We want to find $\sigma([T_c, T_d]) = \sigma(T_c) + \sigma(T_{d(c)})$. As shown in Figure 2.8, $c$ has symplectic basis $b_2$ and $b_1 + a_2 + b_2$. 

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Thus
\[ \sigma(T_c) = \bar{b}_2(b_1 + a_2 + b_2) = \bar{b}_2(b_1 + a_2 + b_2 + 1) = \bar{b}_1\bar{b}_2 + \bar{a}_2\bar{b}_2 \]

Further \( T_{d}(c) \) is shown in Figure 2.9 with symplectic basis consisting of \( b_2 \) and \( a_1 + b_1 + a_2 + b_2 \).

Hence we see that
\[ \sigma(T_{T_{d}(c)}) = (\bar{b}_2)(\bar{a}_1 + b_1 + a_2 + b_2) = \bar{a}_1\bar{b}_2 + \bar{b}_1\bar{b}_2 + \bar{a}_2\bar{b}_2 + \bar{b}_2. \]

Therefore
\[ \sigma([T_c, T_{d}]) = \bar{a}_1\bar{b}_2 + \bar{b}_2. \]
Since we can get all Type (2) elements by themselves, composing with the appropriate SIP-maps and using change of coordinates we can get all elements of Type (3).

**Type (4):** Suppose $c$ and $d$ are as shown in Figure 2.10. We want to find $\sigma([T_c, T_d]) = \sigma(T_c) + \sigma(T_{d(c)})$. Clearly $c$ has symplectic basis $a_1$ and $b_1$, hence $\sigma(T_c) = \bar{a}_1 \bar{b}_1$. Further $T_d(c)$ is shown in Figure 2.11 with symplectic basis consisting of $a_1 + a_2 + b_2$ and $b_1 + a_2 + b_2$.

![Figure 2.10: SIP-map for Type (4).](image)

Figure 2.10: SIP-map for Type (4).

![Figure 2.11: Curve $T_{d(c)}$ with symplectic basis $a_1 + a_2 + b_2$ and $b_1 + a_2 + b_2$.](image)

Figure 2.11: Curve $T_{d(c)}$ with symplectic basis $a_1 + a_2 + b_2$ and $b_1 + a_2 + b_2$.

Hence

$$\sigma(T_{d(c)}) = (a_1 + a_2 + b_2)(b_1 + a_2 + b_2)$$

$$= (\bar{a}_1 + \bar{a}_2 + \bar{b}_2 + 1)(\bar{b}_1 + \bar{a}_2 + \bar{b}_2 + 1)$$

$$= \bar{a}_1 \bar{b}_1 + \bar{a}_1 \bar{a}_2 + \bar{a}_1 \bar{b}_2 + \bar{a}_2 \bar{b}_1 + \bar{b}_1 \bar{b}_2 + \bar{b}_1 + \bar{a}_2 + \bar{b}_2 + 1$$

Therefore

$$\sigma([T_c, T_d]) = \bar{a}_1 \bar{a}_2 + \bar{a}_1 \bar{b}_2 + \bar{a}_2 \bar{b}_1 + \bar{b}_1 \bar{b}_2 + \bar{b}_1 + \bar{a}_2 + \bar{b}_2 + 1.$$
compose by the appropriate SIP-maps to get all Type (4) elements.

**Type (1):**  Now let us consider which elements of Type (1) are in $\sigma(\text{SSIP}(S))$. Let $c$ be any separating curve. $\sigma(T_c)$ must have a symplectic term (that is, a term of the form $\bar{a}_i\bar{b}_i$) and possibly other terms. We know

$$
\sigma([T_c,T_d]) = \sigma(T_cT_dT_c^{-1}T_d^{-1}) \\
= \sigma(T_c) + \sigma(T_dT_c^{-1}T_d^{-1}) \\
= \sigma(T_c) + T_d\sigma(T_c^{-1}) \\
= \sigma(T_c) + T_d\sigma(T_c)
$$

Let $[d] = \alpha_1a_1 + \cdots + \alpha_ga_g + \beta_1b_1 + \cdots + \beta_gb_g$. Without loss of generality, suppose $\bar{a}_1\bar{b}_1$ is a term in $\sigma(T_c)$.

Then using the fact that $[T^k_b(a)] = [a] + k\hat{i}(a,b)[b]$, we see

$$
T_d(\bar{a}_1\bar{b}_1) = T_d(\bar{a}_1)T_d(\bar{b}_1) \\
= (\overline{T_d(\bar{a}_1)})\overline{T_d(\bar{b}_1)} \\
= \overline{(\bar{a}_1 + \beta_1[d])(\bar{b}_1 + \alpha_1[d])}
$$

Hence the $\mathbb{Z}_2$ coefficients of the symplectic terms are:

$$
\bar{a}_1\bar{b}_1 : \alpha_1^2\beta_1^2 + (1 + \beta_1\alpha_1)(1 + \alpha_1\beta_1) = 1
$$

$$
\bar{a}_i\bar{b}_i : (\beta_1\alpha_i)(\alpha_1\beta_i) + (\beta_1\beta_i)(\alpha_1\alpha_i) = 0, \forall i : 1 < i \leq g
$$

So in $\sigma([T_c,T_d])$ the $\bar{a}_1\bar{b}_1$ terms will cancel out. Now suppose $\bar{a}_1\bar{b}_2$ is a term in $\sigma(T_c)$, similarly
for any other nonsymplectic term. Then

\[ T_d(\bar{a} \bar{b}_2) = (T_d(a_1))(T_d(b_2)) = (a_1 + \beta_1[d])(b_2 + \alpha_2[d]) \]

Again, we only need to consider the symplectic coefficients.

\[ \bar{a}_1\bar{b}_1 : (1 + \beta_1\alpha_1)(\alpha_2\beta_1) + \beta_1^2(\alpha_2\alpha_1) = \alpha_2\beta_1 \]

\[ \bar{a}_2\bar{b}_2 : (\beta_1\alpha_2)(1 + \alpha_2\beta_2) + (\beta_1\beta_2)(\alpha_2^2) = \alpha_2\beta_1 \]

\[ \bar{a}_i\bar{b}_i : (\beta_1\alpha_i)(\alpha_2\beta_i) + (\beta_1\beta_i)(\alpha_2\alpha_i) = 0, \forall i : 2 < i \leq g \]

Notice the \( \bar{a}_1\bar{b}_1 \) and \( \bar{a}_2\bar{b}_2 \) coefficients are the same and the other symplectic terms have coefficient 0. Hence we get a sum of two symplectic terms in \( \sigma([T_c, T_d]) \). This is the case for any nonsymplectic term; the only way an \( \bar{a}_i\bar{b}_i \) term will appear in \( \sigma([T_c, T_d]) \) is in a pair.

Suppose \( \bar{a}_1 \), or any other linear term, is in \( \sigma(T_c) \). Then \( T_d(\bar{a}_1) = (a_1 + \beta_1[d]) \) which has no terms of degree two. Similarly if 1 is in \( \sigma(T_c) \), then \( T_d(1) = 1 \) because the action of \( \text{Mod}(S) \) on \( B_2 \) is a linear isomorphism. Thus all symplectic terms in \( \sigma(\text{SSIP}(S)) \) are in \( \langle \bar{a}_i\bar{b}_i + \bar{a}_j\bar{b}_j | 1 \leq i, j \leq g, i \neq j \rangle \).

\[ \square \]

**Corollary 2.4.5.** Consider the subgroup generated by separating SIP-maps, \( \text{SSIP}(S) \). Then if \( g \geq 3 \), then \( \mathcal{K}(S) \nsubseteq \text{SSIP}(S) \).
Chapter 3

The Symmetric Torelli Group

3.1 Introduction

Let \( \iota \) be a fixed hyperelliptic involution of \( S_g \). That is, \( \iota \) is a homeomorphism of order two that acts by \(-I\) on the homology of \( S_g \), or equivalently has \( 2g + 2 \) fixed points [6]. The hyperelliptic involution is unique up to conjugacy.

Figure 3.1: A hyperelliptic involution of the surface.

The symmetric mapping class group, \( \text{SMod}(S_g) \), is the centralizer in \( \text{Mod}(S_g) \) of \( \iota \), \( C_{\text{Mod}(S_g)}(\iota) \).

Recall from the introduction that the symmetric Torelli group is

\[
\SI(S_g) := \text{SMod}(S_g) \cap \mathcal{I}(S_g)
\]

Further, recall Hain’s conjecture that \( \SI(S) \) is generated by symmetric separating twists.
For convenience, let $\mathcal{H}(S_g)$ be the subgroup of $\mathcal{SI}(S_g)$ generated by Hain’s proposed generating set, that is, the subgroup generated by Dehn twists about symmetric separating curves.

In this notation, Hain’s conjecture says the following:

**Conjecture 3.1.1** (Hain, Conjecture 1, [17]). *Let $\mathcal{H}(S_g)$ be the subgroup of $\mathcal{SI}(S_g)$ generated by Dehn twists about symmetric separating curves. Then $\mathcal{H}(S_g) = \mathcal{SI}(S_g)$.***

In support of Hain’s conjecture is a result of Perron, [33], showing that $\mathcal{SI}(S_g)$ is a subgroup of $\mathcal{K}(S)$.

**Lemma 3.1.2** (Perron, Lemma 0.5, [33]). *Let $\mathcal{K}(S)$ be the kernel of the Johnson homomorphism, then $\mathcal{SI}(S) \leq \mathcal{K}(S)$.***

**Proof.** This follows directly from properties of the Johnson homomorphism, $\tau : \mathcal{I}(S_g) \to \wedge^3 H$, [25]. Suppose $f \in \mathcal{SI}(S_g)$. Then by definition, $f_\iota f^{-1}_\iota^{-1} = 1$. Hence by the naturality of $\tau$ we have the following:

$$\tau(f_\iota f^{-1}_\iota^{-1}) = \tau(f) + \iota(\tau(f^{-1})) = 0.$$  

and

$$\iota(\tau(f^{-1})) = -\tau(f^{-1}) = \tau(f).$$

Thus we can conclude $\tau(f) = 0$. \hfill \qed

If Hain’s conjecture proves false, it will nevertheless be interesting to compare $\mathcal{H}(S_g)$ and $\mathcal{SI}(S_g)$. For example, the following theorem shows that the Birman-Craggs-Johnson homomorphism $\sigma$ is not able to distinguish the two groups.

**Theorem 3.1.3.** *Let $\mathcal{H}((S_g))$ be the subgroup of $\mathcal{SI}(S_g)$ generated by Dehn twists about...*
symmetric separating curves, then

\[ \sigma(\mathcal{H}(S_g)) = \sigma(\mathcal{SI}(S_g)) = \sigma(\mathcal{K}(S_g)) = B_2 \]

In order to show this result we will make use of the Birman-Hilden projection and classification of curves.

**Classification of Curves.** A curve is symmetric if it is fixed by the hyperelliptic involution \( \iota \). We say an isotopy class of curves is symmetric if it has a symmetric representative.

We call a curve \( c \) presymmetric if \( c \) and \( \iota(c) \) are disjoint. We say an isotopy class of curves is presymmetric if it has a presymmetric representative.

Birman-Hilden showed the following relating the symmetric mapping class group to the mapping class group of a \( 2g + 2 \) punctured sphere.

**Theorem 3.1.4** (Birman-Hilden, Theorem 1, [4]). Let \( S_g \) be a surface of genus \( g \), then \( \text{SMod}(S_g)/\langle \iota \rangle \cong \text{Mod}(S_{0,0,2g+2}) \).

Birman-Hilden use the 2-fold branched cover of \( S_g \) with \( 2g + 2 \) cone (or marked) points to classify curves in \( S_g \) by looking at their projection in \( S_{0,0,2g+2} \). We call a curve odd (or even) if its projection in \( S_{0,0,2g+2} \) partitions the marked points into an odd (or even) collection of points. Birman-Hilden created the following dictionary relating curves in \( S_g \) to their projection in \( S_{0,0,2g+2} \).

<table>
<thead>
<tr>
<th>Curve in ( S_g )</th>
<th>Curve/arc in ( S_{0,0,2g+2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symmetric non-separating curve</td>
<td>Arc between marked points</td>
</tr>
<tr>
<td>Symmetric separating curve</td>
<td>Odd curve</td>
</tr>
<tr>
<td>Pre-symmetric (non-separating) curve</td>
<td>Even curve</td>
</tr>
</tbody>
</table>
Proof of Theorem 3.1.3: The rightmost equality was shown by Johnson in [26]. We will show the result for $g = 3$. The methods can easily be extended for higher genus using the same types of curves.

Symplectic Terms: $\bar{a}_i \bar{b}_i$. In order to get all symplectic terms consider the following calculations viewed as projections in $S_{0,0,2g+2}$ Note the choice of symplectic basis is shown in most of the figures as arcs is $S_{0,0,2g+2}$.

Combining these terms we are able to get all the symplectic terms.

Terms: $\bar{a}_i \bar{b}_{i+1}$ or $\bar{a}_{i+1} \bar{b}_i$. Consider the images of twists about the following curves.
Combining the image of twists about these curves with the symplectic terms, we can get all terms of the type $\bar{a}_i \bar{b}_{i+1}$ or $\bar{a}_{i+1} \bar{b}_i$.

**Remaining Terms.**

- (k) $\bar{a}_1 \bar{a}_2 + \bar{a}_3 \bar{b}_1 + \bar{a}_2 \bar{b}_1 + \bar{a}_2$
- (l) $\bar{b}_1 \bar{b}_2 + \bar{a}_1 \bar{b}_2 + \bar{a}_2 \bar{b}_2 + \bar{b}_2$
- (m) $\bar{a}_1 \bar{a}_2 + \bar{a}_1 \bar{a}_3 + \bar{a}_2 \bar{a}_3 + \bar{a}_2 \bar{b}_2 + \bar{a}_3 \bar{b}_2 + \bar{a}_3$
- (n) $\bar{b}_2 \bar{b}_3 + \bar{a}_2 \bar{b}_3 + \bar{a}_3 \bar{b}_3 + \bar{b}_3$

Further, combining the image of twists about the above curves with previous ones, we get the following terms in $B_2$:

- $\bar{a}_1 \bar{a}_2 + \bar{a}_2$
- $\bar{b}_1 \bar{b}_2 + \bar{b}_2$
- $\bar{a}_1 \bar{a}_2 + \bar{a}_1 \bar{a}_3 + \bar{a}_2 \bar{a}_3 + \bar{a}_3$
- $\bar{b}_2 \bar{b}_3 + \bar{b}_3$

Using the image of the above curves, we map onto the following terms:

- $\bar{a}_1 \bar{a}_2$
• $\bar{a}_2$

• $\bar{b}_1\bar{b}_2 + \bar{b}_1$

• $\bar{b}_1 + \bar{b}_3$

• $\bar{b}_2\bar{b}_3 + \bar{b}_2$

• $\bar{b}_2 + \bar{b}_3$

• $\bar{b}_1 + \bar{b}_3$

Now we can map onto:

• $\bar{a}_1 + \bar{b}_2 + 1$

• $\bar{a}_3\bar{b}_1$

• $\bar{a}_1\bar{b}_3$

• $\bar{a}_1\bar{a}_3 + \bar{a}_3$

• $\bar{a}_2\bar{a}_3$

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\[ b_1b_3 + b_3 \]

- \[ a_1a_3 + a_1 \]

Using the above terms we are now able to map onto a generating set for \( B_2 \) as desired. Combining the images of twists about these last two curves with the previous ones, we are able to map onto the following terms:

- \[ b_1b_2 \]
- \[ b_1, b_2, \text{ and } b_3 \]
- \[ b_2b_3 \]
- \[ b_1b_3 \]
- \[ a_1 \text{ and } a_3 \]
- \[ a_1a_3 \]
- \[ 1 \]

**Rank.** The *rank* of a group \( G \), \( \text{rk}G \), is the rank of a largest maximal abelian subgroup contained in \( G \). We will find the rank of \( \mathcal{H}(S) \). This will be a key fact used later to classify twists about symmetric separating curves. Note that it will be convenient to denote the projection of a curve, \( c \), in \( S_{0,0,2g+2} \) by \( \bar{c} \).
Proposition 3.1.5. For any surface $S$ with genus $g \geq 3$ and $b$ boundary components (where $b = 0$ or 1), then $rk \mathcal{H}(S_g) = g - 1 + b$.

Proof. When $b = 0$ (or $b = 1$), it suffices to show that the maximal number of disjoint symmetric separating curves in $S_g$ is $g - 1$ (or $g$). Suppose $g = 3$ and $b = 0$. It is clear a maximal collection of symmetric separating curves contains at least 2 curves as shown in $S_{0,0,8}$ here:

Figure 3.2: The projection of disjoint symmetric separating curves is $S_{0,0,2g+2}$ when $g = 3$ and $b = 0$.

It is also clear that there cannot be a distinct third such curve else the curves would no longer be disjoint. Further if $g = 3$ and $b = 1$, it is clear a maximal collection of symmetric separating curves contains 3 curves as shown below in $S_{0,0,8}$.

Figure 3.3: The projection of disjoint symmetric separating curves is $S_{0,0,2g+2}$ when $g = 3$ and $b = 1$.

Assume the proposition is true for $g \leq n$ (when $b = 0$ or 1). Suppose $g = n + 1$ and $b = 0$. Consider a maximal collection of disjoint symmetric separating curves and their projections in $S_{0,0,2n+4}$. We know this collection contains at least $n$ curves because below is a collection of $n$ such curves projected in $S_{0,0,2n+4}$ and we know our collection is maximal.

What remains to be shown is that this collection contains no more than $n$ curves. Let $c$ be a curve in this collection whose projection in $S_{0,0,2n+4}$ partitions the marked points into two regions of say $x$ and $y$ marked points so that $|x - y|$ is maximal. Without loss of generality, assume $x > y$ and $x = 2k + 1$ for some $k$. 
Figure 3.4: The projection of disjoint symmetric separating curves is $S_{0,0,2g+2}$ when $g = n+1$ and $b = 0$.

By the inductive hypothesis, the side of $\bar{c}$ that has $x$ marked points has at most the projection of $k$ curves (including $c$) The remaining number of marked points (those on the side of $\bar{c}$ with $y$ marked points) is $2(n-k)+3$, and those contain the projection of at most $n-k$ curves. Thus the collection has at most $n$ symmetric separating curves.

If $g = n+1$ and $b = 1$ then again let $c$ be a curve in the collection whose projection in $S_{0,0,2n+4}$ partitions the marked points into two regions of say $x$ and $y$ marked points so that $|x - y|$ is maximal. Without loss of generality, we assume $x > y$ and $x = 2k + 1$ for some $k$. If $k < n+1$ the argument above shows the result. If $k = n+1$, then let $d$ be a curve whose projection is on the $x$ side of $\bar{c}$ so that $\bar{d}$ partitions the $x$ marked points into two regions of say $w$ and $z$ marked points and $|w - z|$ is maximal. Without loss of generality we can assume $w > z$ and $w = 2l + 1$ for some $l < k$. So by the inductive hypothesis there are at most $l$ curves on the $w$ side of $\bar{d}$ and at most $n-l-1$ curves on the $x$ side of $\bar{c}$ and the $z$ side of $\bar{d}$. Note there are no curves on the $y$ side of $\bar{c}$. Hence there are a total of at most $n+1$ curves as desired.

\[ \Box \]

**Curve Complexes.** When studying $\text{Mod}(S)$ and subgroups of $\text{Mod}(S)$ it is natural to try to find a simplicial complex on which the group acts nicely on in order to understand the group further. For $\text{Mod}(S)$, Harvey introduced the *curve complex* $C(S_g)$ in [19]. It is the simplicial flag complex with vertices corresponding to simple closed curves on $S_g$ and edges between vertices which can be realized as disjoint curves in $S_g$. 
**Symmetric Separating Curve Complex.** For $\mathcal{H}(S)$, the natural subcomplex of $C(S_g)$ to consider is the one spanned by symmetric separating curves, called the *symmetric separating curve complex*, $C_{\mathcal{H}}(S_g)$.

**Lemma 3.1.6.** If $g \geq 3$, then $C_{\mathcal{H}}(S_g)$ is connected.

*Proof.* Given a symmetric separating curve, $c$, on $S$, we know $\bar{c}$ is an odd curve, so $\bar{c}$ will partition the marked points in $S_{0,0,2g+2}$ into two regions, one of which will have more than three marked points. Thus we can find a genus 1 symmetric separating curve disjoint from $c$. Now it suffices to show that the subcomplex of all genus 1 symmetric separating curves is connected.

Consider two intersecting symmetric genus 1 separating curves $a$ and $b$. Then $\bar{a}$ partitions the marked points into two regions one of which has three marked points. We call this collection of marked points $A$. Similarly we define $B$. Then $\bar{a}$ and $\bar{b}$ partition the marked points into four regions:

1. The marked points in $A - B$,
2. The marked points in $B - A$,
3. The marked points in $A \cap B$, and
4. The marked points not in $A$ and not in $B$.

The union of the first three regions contains at most five marked points, hence the last region must contain at least three marked points. Thus there is a symmetric genus 1 separating curve, $d$, disjoint from $a$ and $b$, namely any curve enclosing three of the marked points not in $A$ and not in $B$. 

\[\square\]
Note that if $X$ is a flag complex, then $\text{Aut}(X) = \text{Aut}(X^{(1)})$, where $X^{(1)}$ is the 1-skeleton of $X$. Hence we only need to focus on vertices and edges when looking at automorphisms of $C(S)$ and $C_{\mathcal{H}}(S)$.

This leads to our main theorem regarding automorphisms of the symmetric separating curve complex.

**Main Theorem.** Let $C_{\mathcal{H}}(S_g)$ be the symmetric separating curve complex and $g \geq 5$. Then $\text{Aut}(C_{\mathcal{H}}(S_g)) \cong \text{SMod}^{\pm}(S_g)/\langle \iota \rangle$.

In order to prove our main theorem we will use the result of Birman-Hilden in Theorem 3.1.4 and the following result of Korkmaz.

**Theorem 3.1.7** (Korkmaz, Theorem 1, [31]). Let $C(S_{0,0,2g+2})$ be the curve complex associated to $S_{0,0,2g+2}$ with $g > 1$. Then $\text{Aut}(C(S_{0,0,2g+2})) \cong \text{Mod}^{\pm}(S_{0,0,2g+2})$.

By Birman-Hilden and Korkmaz, our strategy is to take an element $\phi \in \text{Aut}(C_{\mathcal{H}}(S))$ and map it to an element in $\text{Aut}(C(S_{0,0,2g+2}))$. Hence we will have constructed a map from $\text{Aut}(C_{\mathcal{H}}(S)) \to \text{Aut}(C(S_{0,0,2g+2}))$ which will be the inverse of the obvious map in the other direction. For the remainder of this paper let $\phi \in \text{Aut}(C_{\mathcal{H}}(S_g))$. We will proceed to show that $\phi$ induces an element of $\text{SMod}^{\pm}(S_g)/\langle \iota \rangle$ by extending $\phi$ to a simplicial map on all symmetric and presymmetric curves.

### 3.2 Basic Topology

In this section we will see in what ways $\phi \in \text{Aut}(C_{\mathcal{H}}(S))$ is able to detect the topological properties of curves on $S$. This section has the same results as those shown by Brendle-Margalit in [7] with the added condition that all curves are symmetric. We show their
proofs can be realized symmetrically except for that of the genus result which differs quite significantly. The following are two facts which will be necessary to show the key lemmas.

**Fact 3.2.1** (see Chapter 2 of [15]). Let $f$ and $h$ be Dehn twists about separating curves. Then we have $[f^j, h^k] = 1$ if and only if the intersection number between the corresponding curves is zero.

**Fact 3.2.2.** For any surface $S$ with genus $g \geq 3$ and no boundary, a maximal collection of disjoint symmetric genus 1 separating curves contains $\left\lfloor \frac{2g+2}{3} \right\rfloor$ curves.

The following lemma is a direct consequence of Fact 3.2.1 and the fact that the simplicial map $\phi$ is injective.

**Lemma 3.2.3.** (Disjointness) If $a$ and $b$ are symmetric separating curves in $S$, then $i(a, b) \neq 0$ if and only if $i(\phi(a), \phi(b)) \neq 0$.

We define a side of a separating curve, $z$, to be one of the components of $S - z$.

**Lemma 3.2.4.** (Sides) If $a$ and $b$ are symmetric separating curves on the same side of a symmetric separating curve, $z$, then $\phi(a)$ and $\phi(b)$ are symmetric separating curves on the same side of $\phi(z)$.

*Proof.* Symmetric separating curves $a$ and $b$ are on the same side of $z$ if and only if there exists a symmetric separating curve $c$ such that $i(a, c) \neq 0, i(b, c) \neq 0,$ and $i(z, c) = 0$. Thus by Lemma 3.2.3 we can conclude that $\phi(a)$ and $\phi(b)$ are symmetric separating curves on the same side of $\phi(z)$. 

**Proposition 3.2.5.** (Genus) Suppose $S$ is a surface with genus $g \geq 5$. If $z$ is a genus $m$ symmetric separating curve, then $\phi(z)$ is a genus $m$ symmetric separating curve. Further, if $a$ is on a genus $m$ side of $z$, then $\phi(a)$ is on a genus $m$ side of $\phi(z)$.
Proof. Suppose $z$ is a genus $m$ symmetric separating curve, then by Proposition 3.1.5 any maximal collection of disjoint symmetric separating curves in $S$ which contain $z$ is of the form:

$$\{a_1, \ldots, a_{m-1}, z, b_1, \ldots, b_{(g-m)-1}\}$$

where the $a_i's$ are disjoint symmetric separating curves on one side of $z$ and the $b_i's$ are disjoint symmetric separating curves on the other. By disjointness, Proposition 3.1.5, and the fact that $\phi \in \text{Aut}(C_\mathcal{H}(S_g))$ we have that the set

$$\{\phi(a_1), \ldots, \phi(a_{m-1}), \phi(z), \phi(b_1), \ldots, \phi(b_{(g-m)-1})\}$$

is a maximal collections of mutually disjoint symmetric separating curves on $S$. By Lemma 3.2.4 and Proposition 3.1.5, we have that either $\phi(z)$ is a genus 1 symmetric separating curve and that $\phi(a_i)$ and $\phi(b_i)$ are on the same side of $\phi(z)$, or that $\phi(z)$ is a genus $m$ curve with $\phi(a_i)$ on one side of $\phi(z)$ and $\phi(b_i)$ on the other.

When $m = 1$, then both cases are the same. Hence $\phi$ maps genus 1 symmetric separating curves to genus 1 symmetric separating curves. Now we will show if $m \geq 2$, then $\phi(z)$ cannot be a genus 1 curve, which will prove that $\phi(z)$ is a genus $m$ symmetric separating curve as desired.

Suppose $\phi(z)$ is a genus 1 curve. Here is where this proof differs extensively from [7] because in a maximal collection of disjoint symmetric separating curves there is not a fixed number of genus 1 curves. For example here are two maximal collections (viewed in $S_{0,0,2g+2}$ with a different number of genus 1 curves).

Figure 3.5: A maximal collection of disjoint symmetric separating curves with four genus 1 curves.
Choose a maximal collection of disjoint genus 1 symmetric separating curves on each side of \( z \). Note that there will be \( \left\lfloor \frac{2m+1}{3} \right\rfloor \) such curves on the “inside” of \( z \) and \( \left\lfloor \frac{2(g-m)+1}{3} \right\rfloor \) on the “outside” of \( z \).

If \( \left\lfloor \frac{2m+1}{3} \right\rfloor + \left\lfloor \frac{2(g-m)+1}{3} \right\rfloor = \left\lfloor \frac{2g+2}{3} \right\rfloor \) then by Fact 3.2.2 the union of the maximal collections of genus 1 symmetric separating curves on each side of \( z \) is actually a maximal disjoint collection of genus 1 symmetric separating curves for \( S \), which maps to a maximal collection of disjoint genus 1 curves. Hence \( \phi(z) \) cannot be a genus 1 curve.

If \( \left\lfloor \frac{2m+1}{3} \right\rfloor + \left\lfloor \frac{2(g-m)+1}{3} \right\rfloor \neq \left\lfloor \frac{2g+2}{3} \right\rfloor \) then we will choose a second maximal collection of disjoint symmetric genus 1 curves and sometimes one additional genus 1 curve on each side of \( z \) which will force \( \phi(z) \) to be a genus \( m \) curve.

Note the following argument requires that there are more than 5 marked points contained on at least one side of \( \bar{z} \) forcing the genus requirement, \( g \geq 5 \). For now we will focus on one side of \( z \), so without loss of generality suppose the inside of \( \bar{z} \) has at least 5 marked points; that is, \( 2m + 1 > 5 \).

If \( 2m + 1 \equiv 2 \mod 3 \), then let \( \{c_1, \ldots, c_k\} \) for some \( k \in \mathbb{Z} \), be one collection of maximal genus 1 curves on the inside of \( z \). Then choose a second disjoint maximal collection of symmetric genus 1 curves \( \{d_1, \ldots, d_k\} \) so that \( i(c_j, d_i) = 0 \) if and only if \( j \neq 1 \) or \( i + 1 \). We will choose one additional genus 1 symmetric separating curve \( e \) so that:

- \( i(e, c_j) = 0 \) if and only if \( j \neq 1 \)
\bullet i(e, d_i) = 0 \text{ if and only if } i \neq k \\
\bullet i(e, z) = 0 

Now we will consider the image of these curves under \( \phi \). That is,

\[
\{ \phi(c_i), \phi(d_j), \phi(e), \phi(z) \}
\]

First note that no \( \overline{\phi(c_i)} \) and \( \overline{\phi(d_j)} \) can contain the same three marked points. If they did then \( \overline{\phi(c_i)} \cup \overline{\phi(d_j)} \) would separate \( S_{0,0,2g+2} \) which would not allow for disjointness among the curves to be preserved by \( \phi \). Further \( \{ \overline{\phi(c_i)}, \overline{\phi(d_j)}, \overline{\phi(e)} \} \) must “contain” the same number of marked points as \( \{ c_i, d_j, e \} \) because reducing the number of marked points would force two of the curves to contain the same three marked points.

Note if \( 2m + 1 \equiv 1 \mod 3 \) then the above argument works except there is no need for the curve \( e \). Further the case \( 2m + 1 \equiv 0 \mod 3 \) was shown when \( \lfloor \frac{2m+1}{3} \rfloor + \lfloor \frac{2(g-m)+1}{3} \rfloor = \lfloor \frac{2g+2}{3} \rfloor \).

Similarly, if the outside of \( \overline{z} \) contains more than 5 marked points, that is \( 2(g-m)+1 > 5 \), then the same argument works showing the image of the two maximal disjoint collection of genus 1 curves must contain the same number of marked points. This means the genus of \( z \) must be \( m \) else \( \phi(z) \) would be a genus 1 symmetric separating curve, which means \( \overline{\phi(z)} \) contains 3 additional marked points which because \( \phi \) preserves disjointness does not leave enough room for the maximal collections on each of side of \( z \).

Now if one side of \( \overline{z} \) contains exactly 5 marked points, then the above argument gives the desired result by choosing one genus 1 symmetric separating curve on the side of \( \overline{z} \) with 5 marked points, call this curve \( f \). Then by disjointness \( \overline{\phi(z)} \) and \( \overline{\phi(f)} \) must contain 6 marked points. (Recall we are assuming \( \phi(z) \) is a genus 1 curve and want to reach a contradiction) We use the previous methods to obtain a maximal collection of curves on the other side of \( \overline{z} \)
whose images contain the same number of marked points, but this cannot happen; we would need one additional marked point.

3.3 Symmetric Curves

We show in this section how to extend $\phi \in \text{Aut}(C_H(S))$ to a map on all symmetric curves. In order to do this, we use the idea of “sharing pairs,” previously defined by Brendle-Margalit in [7], with the added condition that all curves are symmetric. Many of the proofs in [7] hold with the added symmetric condition, but we will show the symmetric condition restricts the number of moves between spines of sharing pairs. We will also construct a different surface $F$ to prove well-definedness.

Sharing Pairs. A non-separating symmetric curve $\beta$ is uniquely determined by a pair of distinct genus 1 symmetric separating curves, which bound subsurfaces that intersect in an annulus, with the condition that $\beta$ lies on both of the corresponding genus 1 subsurfaces.

Let $a$ and $b$ be two genus 1 symmetric separating curves bounding genus 1 subsurfaces $S_a$ and $S_b$ of $S$ respectively. We say $a$ and $b$ share a symmetric non-separating curve $\beta$ if $S_a \cap S_b$ is an annulus containing $\beta$ as its core and $S - (S_a \cap S_b)$ is connected. We say that $a$ and $b$ form a sharing pair for $\beta$. See Figure 3.7 for an example of a sharing pair.

![Figure 3.7: A sharing pair in the surface $S$.](image)

For much of this paper it will be useful to consider the projection of sharing pairs in $S_{0,0.2g+2}$. Figure 3.8 shows the projection of Figure 3.7.
Hence we see a sharing pair viewed in \( S_{0,0,2g+2} \) is simply two 3-curves which “share” two marked points, or equivalently an arc.

The extension of the map \( \phi \) is defined on symmetric non-separating curves as follows. If \( \mathcal{P}(\beta) = \{a, b\} \) is a sharing pair for a symmetric non-separating curve \( \beta \), then \( \phi(\beta) \) is the curve shared by \( \phi(\mathcal{P}(\beta)) \), that is, the curve shared by \( \{\phi(a), \phi(b)\} \). In order to show that this extension of \( \phi \) is well-defined on symmetric non-separating curves, we need to show that \( \phi(\mathcal{P}(\beta)) \) is a sharing pair and that \( \phi(\beta) \) is independent of the choice of \( \mathcal{P}(\beta) \).

There is a useful characterization of sharing pairs introduced by Brendle-Margalit that we will show can also be realized symmetrically.

**Lemma 3.3.1** (Brendle-Margalit, Lemma 4.1, [7]). *Let \( a \) and \( b \) be genus 1 separating curves in \( S \). Then \( a \) and \( b \) are a sharing pair if and only if there exist separating curves \( w, x, y, \) and \( z \) in \( S \) with the following properties:

- \( z \) is a genus 2 curve bounding a genus 2 subsurface \( S_z \).
- \( a \) and \( b \) are in \( S_z \) so that \( i(a, b) \neq 0 \).
- \( x \) and \( y \) are disjoint.
- \( w \) intersects \( z \), but not \( a \) and not \( b \).
- \( x \) intersects \( a \) and \( z \), but not \( b \).
- \( y \) intersects \( b \) and \( z \), but not \( a \).

We show this configuration of curves can be realized symmetrically in Figure 3.9.
The following lemma is a special case of Proposition 4.2 in [7]. The proof is included here for completeness.

**Lemma 3.3.2.** If two genus one symmetric curves $a$ and $b$ in $S$ form a sharing pair, then so do $\phi(a)$ and $\phi(b)$.

**Proof.** Since $a$ and $b$ share a curve, there are characterizing curves $w, x, y,$ and $z$ as in Lemma 3.3.1. Each property of this collection of curves (disjointness, sides, genus) is preserved by $\phi$, by Lemmas 3.2.3, 3.2.4, and 3.2.5, thus Lemma 3.3.1 implies $\phi(a)$ and $\phi(b)$ share a curve.  

We now have a function from $\text{Aut}(\mathcal{C}_n(S))$ to the set of functions on symmetric curves. In order to show that this function is well-defined with respect to the choice of sharing pairs, we will consider a sharing pair via its spine.

**Spines.** Given two symmetric non-separating curves $\alpha$ and $\beta$ with $i(\alpha, \beta) = 1$, we define $\mathcal{B}(\alpha, \beta)$ to be the genus 1 symmetric separating curve which is the boundary of a regular neighborhood of $\alpha \cup \beta$. An ordered collection of three distinct symmetric non-separating curves $\{\alpha, \beta, \gamma\}$ forms a spine of a sharing pair $\{a, b\}$ if:

- $i(\alpha, \beta) = i(\beta, \gamma) = 1$
- $i(\alpha, \gamma) \leq 1$. 

Figure 3.9: The projection of symmetric curves characterizing a sharing pair in $S_{0,0,2g+2}$. 


• \( \mathcal{B}(\alpha, \beta) = a \) and \( \mathcal{B}(\beta, \gamma) = b \)

• \( S - (\alpha \cup \beta \cup \gamma) \) is connected

We denote this spine by \( \alpha - \beta - \gamma \). Note we can always choose a spine for a sharing pair, but this choice is not unique.

**Moves.** We define a *move* between spines to be a change of the following form:

\[
\alpha - \beta - \gamma \mapsto \alpha - \beta - \gamma'
\]

where \( \gamma - \beta - \gamma' \) is also a spine. Note that \( \mathcal{B}(\alpha, \beta), \mathcal{B}(\beta, \gamma), \) and \( \mathcal{B}(\beta, \gamma') \) form three sharing pairs for \( \beta \).

A move is characterized by whether \( i(\alpha, \gamma), i(\alpha, \gamma'), \) and \( i(\gamma, \gamma') \) are 1 or 0. In [7], Lemma 4.3, Brendle-Margalit note that by the non-separating property of spines there are topologically 4 possible moves among spines as shown in Figure 3.10.

![Figure 3.10: Moves on spines.](image)

Note that a move is of type (a) when \( i(\alpha, \gamma) = i(\alpha, \gamma') = i(\gamma, \gamma') = 0 \). A move is of type (b) when exactly one of \( i(\alpha, \gamma), i(\alpha, \gamma'), i(\gamma, \gamma') \) is one, type (c) when exactly two are one, and type (d) when all three are one.
The added condition in our case, that each curve in the spine is symmetric, eliminates two of the possibilities. The only permissible moves are type (b) and (d), as shown in Figure 3.11.

![Figure 3.11: Moves on symmetric spines.](image)

**Lemma 3.3.3.** *Any move is topologically equivalent to one of the two moves pictured in Figure 3.11.*

*Proof.* Notice that if two symmetric curves intersect exactly once, then their projections in $S_{0,2g+2}$ must intersect at a marked point. A move of type (a) cannot be achieved in the symmetric case because $\alpha$ and $\gamma$ intersect $\beta$ once, but do not intersect each other. Hence $\alpha$ would intersect $\beta$ at one marked point and $\gamma$ would intersect $\beta$ at the other. Now $\gamma'$ must intersect $\beta$ as well, but this would require $\gamma'$ to intersect either $\alpha$ or $\gamma$.

Further a move of type (c) cannot be achieved in the symmetric case because such a configuration would look like that in Figure 3.12.

![Figure 3.12: A symmetric move of type (c) in $S_{0,2g+2}$.](image)

Clearly this would violate the condition that $\alpha - \beta - \gamma, \alpha - \beta - \gamma', \text{ and } \gamma - \beta - \gamma'$ are all spines because the non separating condition of a spine cannot be achieved in all three cases. This proves the lemma. \(\square\)
Lemma 3.3.4. Let $\beta$ be a symmetric non-separating curve in $S$. If $\{a, b\}$ and $\{a, b'\}$ are sharing pairs of $\beta$ which have spines differing by a move, then $\{\phi(a), \phi(b)\}$ and $\{\phi(a), \phi(b')\}$ share the same curve.

Proof. The construction of the proof by Brendle-Margalit in [7] (Lemma 4.4) can be realized symmetrically. We will repeat their argument here for completeness. Suppose that $\alpha - \beta - \gamma$ and $\alpha - \beta - \gamma'$ differ by a move, where $a = B(\alpha, \beta)$ and $b = B(\beta, \gamma)$, and $b' = B(\beta, \gamma')$. Since $a, b,$ and $b'$ pairwise share a common curve, Lemma 3.3.2 implies that $\phi(a), \phi(b)$, and $\phi(b')$ are pairwise sharing.

One can always find a symmetric separating curve $z$ which intersects $\gamma'$ but not any of $\alpha, \beta,$ or $\gamma$ (by Lemma 3.3.3). It follows that $z$ intersects $b'$ but not $a$ or $b$, and then by Lemma 3.2.3, we have that $\phi(z)$ intersects $\phi(b')$ but not $\phi(a)$ or $\phi(b)$.

Suppose that $\phi(a), \phi(b)$, and $\phi(b')$ do not all share the same symmetric non-separating curve. Let $\pi - \sigma - \tau$ be a spine for $\{\phi(a), \phi(b)\}$. By the assumption, $\phi(b')$ does not share $\sigma$ with $\phi(a)$ and $\phi(b)$, and hence it shares curves $\omega$ and $\nu$ with $\phi(a)$ and $\phi(b)$, respectively. Note that $\sigma, \omega$, and $\nu$ must all be distinct, because otherwise it follows that $\phi(b')$ is equal to either $\phi(a)$ or $\phi(b)$.

We will now argue that there is no curve which intersects $\phi(b')$ and is disjoint from both $\phi(a)$ and $\phi(b)$. This will contradict our earlier statement about $\phi(z)$. Indeed, any curve $c$ which intersects $\phi(b')$ must also intersect at least one of $\omega$ or $\nu$, say $\omega$. Since $\omega$ lies on the genus 1 subsurface bounded by $\phi(a)$, it follows that $c$ must also intersect $\phi(a)$.

By Lemma 3.3.4, well-definedness of $\phi$ is reduced to showing the following proposition which will follow from work of Harer [18].

Proposition 3.3.5. Any two spines $\alpha - \beta - \gamma$ and $\delta - \beta - \epsilon$ differ by a finite sequence of
moves.

**Harer’s Complex.** Let $F$ be a surface with boundary, $P$ be a finite collection of points on the boundary, and $P_0$ a subset of $P$. Harer defines an abstract simplicial complex $X = X(F, P, P_0)$ with:

**Vertices.** Isotopy classes of arcs in $F$ connecting a point in $P_0$ to a point in $P - P_0$.

**Edges.** Two vertices in $X$ are connected by an edge if the corresponding arcs are disjoint except possibly at the endpoints, and if the two arcs do not bound a subsurface of $F$.

Further, a $k$-simplex of $X$ is defined to be a collection of $k$ pairwise connected edges in $X$ with the property that the union of the corresponding arcs does not separate $F$. Harer in [18] proves:

**Theorem 3.3.6** (Harer, Theorem 1.4, [18]). $X$ is spherical of dimension $2g - 2 + r'$, where $r'$ is the number of boundary components of $F$ containing points in $P$.

Note that in Harer’s work our complex $X = BX(\Delta, \Delta^0)$. By further work of Hatcher [20] we also have:

**Theorem 3.3.7** (Hatcher, [20]). If $X$ is not a disk with $P$ contained in $\partial X$ or an annulus with $P$ contained in one component of $\partial X$, then $X$ is chain connected. That is, any two maximal simplices are connected by a finite sequence of maximal simplices, where consecutive simplices in the sequence share a simplex of codimension 1.

We now apply Harer’s work to prove the proposition. While the following argument is similar to Brendle-Margalit’s Proposition 4.5 in [7], we construct a different complex in order to prove our result.
Proof of Proposition 3.3.5. Let $\beta$ be a symmetric nonseparating curve. We view the projection of $\beta$ in $S_{0,0,2g+2}$, and we cut $S_{0,0,2g+2}$ along this curve, $\bar{\beta}$, and call the resulting surface $F$. Let $P$ be the set of $2g+2$ marked points in $S_{0,0,2g+2}$ and let $P_0$ be the set containing the two marked points which intersect $\bar{\beta}$. In order to use Harer’s result, $P$ needs to be on the boundary of $F$, so we blow up each marked point in $F$ and we pick a point on each new boundary component forming $P$. From now on when we refer to $F$, we mean this “blown up” version. Now we have the complex $X = X(F, P, P_0)$. Vertices of $X$ correspond to symmetric curves which intersect $\beta$ once and edges of $X$ correspond to a spine of a sharing pair for $\beta$. Note that a pair of vertices fail to have an edge between them only when the corresponding arcs intersect (not at a endpoint) or share both endpoints (in which case the arcs necessarily separate $F$). Thus a move between spines is achieved by changing an edge to a new edge which lies in a triangle with the original edge.

Let $\alpha - \beta - \gamma$ and $\delta - \beta - \epsilon$ be two spines for the symmetric nonseparating curve $\beta$. Think of spines as edges in $X$. Let $M$ and $N$ be two maximal simplices which contain these edges. Theorem 3.3.7 says there is a sequence of maximal simplices of $X$:

$$M = M_0, M_1, \ldots, M_k = N$$

where $M_i$ and $M_{i+1}$ share a codimension 1 face. Using this sequence we will construct our desired sequence of moves. Let $e_0$ be the edge corresponding to $\alpha - \beta - \gamma$. Let $e_i$ be an edge in $M_i$ connecting $v$ to $w$. Inductively define $e_{i+1}$ in the following way:

- If $v$ and $w$ are vertices of $M_{i+1}$, then $e_{i+1} := e_i$.

- If $w$ is not a vertex of $M_{i+1}$ (note this forces $v$ to be in $M_{i+1}$), then define $e_{i+1}$ to be the span of $v$ with any other $w'$ of $M_i \cap M_{i+1}$.
Since $v, w,$ and $w'$ all lie in $M_i$, they form a triangle and hence the edges $vw$ and $vw'$ differ by a move. Finally, $e_k$ differs from the edge corresponding to $\delta - \beta - \epsilon$ by at most two moves, since they both lie in the simplex $N$.

Now that we have extended $\phi \in \text{Aut}(C_{\mathcal{H}}(S))$ to a function on all symmetric curves, we observe that $\phi$ preserves disjointness between symmetric curves.

**Lemma 3.3.8.** Suppose $a$ and $b$ are symmetric curves in $S$. Then $i(\phi(a), \phi(b)) = 0$ if and only if $i(a, b) = 0$.

**Proof.** This proof is, in essence, a restriction of the result by Brendle-Margalit in [7] (Section 4.3). Though a gap in their argument was pointed out by Kida, Brendle-Margalit’s argument still holds in our situation and is outlined here. The argument breaks down into three cases:

1. If $a$ and $b$ are both separating the result is Lemma 3.2.3.

2. If $a$ and $b$ are both nonseparating, then the result follows from that fact that $a$ and $b$ are disjoint if and only if there are disjoint sharing pairs representing $a$ and $b$.

3. If $a$ is separating, and $b$ is nonseparating, then $a$ and $b$ are disjoint if and only if either $a$ is a part of a sharing pair for $b$ or $b$ has a sharing pair whose curves are disjoint from $a$.

An immediate consequence of Lemma 3.3.8 is that $\phi$ is a simplicial map on all symmetric curves.
3.4 Presymmetric Curves

Recall that we started with $\phi \in Aut(C_n(S))$. In Section 3.3 we extended $\phi$ to a function on all symmetric curves and showed it was simplicial and injective. The goal of this section is to extend $\phi$ to a function including presymmetric curves. We will use the following result of Ivanov to show that $\phi$ preserves certain topological properties of symmetric curves.

**Lemma 3.4.1** (Ivanov, Lemma 8.2A, [23]). Suppose that the genus of $S$ is at least 2. Let $\alpha_1$ and $\alpha_2$ be isotopy classes of two nontrivial curves on $S$. Then the geometric intersection number $i(\alpha_1, \alpha_2) = 1$ if and only if there exist isotopy classes $\alpha_3, \alpha_4,$ and $\alpha_5$ of nontrivial curves having the following two properties:

1. $i(\alpha_i, \alpha_j) = 0$ if and only if the $i$-th and $j$-th curves in Figure 3.13 are disjoint.
2. If $\alpha_4$ is the isotopy class of a curve $C_4$, then $C_4$ divides $S$ into two parts, one of which is a torus with one hole containing some representatives of the isotopy classes $\alpha_1$ and $\alpha_2$.

![Figure 3.13: Curves characterizing geometric intersection 1.](image)

**Lemma 3.4.2.** Suppose $a$ and $b$ are symmetric curves in $S$, then $i(\phi(a), \phi(b)) = 1$ if and only if $i(a, b) = 1$.

**Proof.** This follows from Lemma 3.4.1 since this characterization can be realized symmetrically (see Figure 3.14) and only depends on preserving disjointness and genus 1 symmetric...
We will extend $\phi$ to presymmetric curves via a “symmetric spine.”

**Symmetric Spines.** Let $a$ be a presymmetric curve in $S$, so that $\bar{a}$ is an even curve with $2k$ marked points on one side of $\bar{a}$ in $S_{0,0,2g+2}$. A *symmetric spine* is a collection of symmetric curves $\{c_1, \ldots, c_{2k-1}\}$ on $S_g$, so that $i(c_i, c_j) = 1$ if $j = i + 1$ and 0 otherwise. Because $\{c_i\}$ is a collection of an odd number of curves, the boundary of a regular neighborhood of $\bigcup c_i$ will have two components. We say $\{c_i\}$ is a *symmetric spine for* $a$, if $a$ is one of the boundary components of a regular neighborhood of $\bigcup c_i$.

**Lemma 3.4.3.** If $\{c_i\}$ is a symmetric spine, then $\{\phi(c_i)\}$ is a symmetric spine.

*Proof.* Follows directly from Lemma 3.3.8 and Lemma 3.4.2.

In order to extend $\phi$ to a presymmetric curve $a$, we will choose a symmetric spine for $a$, and then we will set $\phi(a)$ equal to the boundary component of a regular neighborhood of $\bigcup \phi(c_i)$ in $S_{0,0,2g+2}$.

Next we will need to show this extension of $\phi$ does not depend on the choice of symmetric spine of $a$. Let $\{c_i\}$ and $\{d_i\}$ be two symmetric spines for a presymmetric curve $a$ in $S$. By Lemma 3.4.3 we know $\{\phi(c_i)\}$ and $\{\phi(d_i)\}$ are also symmetric spines. It suffices to show regular neighborhoods of $\bigcup \phi(c_i)$ and $\bigcup \phi(d_i)$ in $S_{0,0,2g+2}$ are isotopic.

First consider $\{\bar{c}_i\}$ and $\{\bar{d}_i\}$ in $S_{0,0,2g+2}$. Clearly the boundary of a regular neighborhood of both $\bigcup \bar{c}_i$ and $\bigcup \bar{d}_i$ is $\bar{a}$. The proof reduces to two cases.
Case 1: Suppose $\bigcup \bar{c}_i$ and $\bigcup \bar{d}_i$ share the same marked points in $S_{0,0,2g+2}$.

Lemma 3.4.4. There exist symmetric curves $e$ and $f$ so that $\bar{e} \cup \bar{f}$ separate $S_{0,0,2g+2}$ into two subsurfaces where $\bigcup \bar{c}_i$ and $\bigcup \bar{d}_i$ are contained in the same subsurface, $S'$, of $S_{0,0,2g+2}$, and $S'$ only contains marked points that intersect $\bigcup \bar{c}_i$.

![Figure 3.15: An example of the construction used in Lemma 3.4.4.](image)

Proof. Choose two marked points not used by $\bigcup \bar{c}_i$ or $\bigcup \bar{d}_i$. Then connect the marked points by two arcs, $\bar{e}$ and $\bar{f}$, so that the subsurface $S'$ is as desired. This construction can be done because $\{c_i\}$ and $\{d_i\}$ are symmetric spines for $a$, see Figure 3.15 for an example of this construction.

Clearly $\phi(e)$ and $\phi(f)$ are defined because they are symmetric curves. Moreover, $\phi(e)$ and $\phi(f)$ intersect at least twice, specifically at their endpoints because of Lemma 3.3.8 and 3.4.2.

By Lemma 3.3.8 it is clear that $\phi(e)$ and $\phi(f)$ separate $S_{0,0,2g+2}$. In addition, $\bar{\phi}(c_i)$ and $\bar{\phi}(d_i)$ are in the same subsurface of $S_{0,0,2g+2} - (\phi(e) \cup \phi(f))$; we will denote this subsurface of $S_{0,0,2g+2}$ as $S''$.

Lemma 3.4.5. The subsurface $S''$ contains only marked points used in $\{\phi(c_i)\}$.

Proof. Suppose $S''$ contains a marked point, $x$, that is not used in $\{\phi(c_i)\}$. Pick any marked point used in $\{\phi(c_i)\}$, we will call is $y$. Then there exists an arc, $k$, from $x$ to $y$ contained in $S''$, further this means $i(k, \phi(e)) = i(k, \phi(f)) = 0$. By Lemma 3.3.8, we can conclude $i(k, \bar{e}) = i(k, \bar{f}) = 0$. But this cannot happen by the choice of $e$ and $f$. 

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Now we are ready to show that the boundary of regular neighborhoods of $\cup \phi(c_i)$ and $\cup \phi(d_i)$ are isotopic, thus showing the extension of $\phi$ to all presymmetric curves is well-defined.

**Proposition 3.4.6.** Let $\{c_i\}$ and $\{d_i\}$ be two symmetric spines for a presymmetric curve $a$ in $S$, then regular neighborhoods of $\cup \phi(c_i)$ and $\cup \phi(d_i)$ have an isotopic boundary component.

**Proof.** If we consider the subsurface obtained by cutting $S_{0,0,2g+2}$ along $\cup \phi(c_i)$, $\phi(e)$, and $\phi(f)$, we will have an annulus. Up to homotopy there is only one curve on the annulus and it is isotopic to the boundary of $\cup \phi(c_i)$. Similarly if we consider the subsurface obtained by cutting $S_{0,0,2g+2}$ along $\cup \phi(d_i)$, $\phi(e)$, and $\phi(f)$, we will have another annulus, containing one curve homotopic to the boundary of $\cup \phi(d_i)$. But because both of these annuli also share one boundary component, namely the one obtaining from $\phi(e)$ and $\phi(f)$, hence these two curves are isotopic.

**Case 2:** Suppose $\cup \overline{c_i}$ and $\cup \overline{d_i}$ are disjoint, implying they do not share any of the same marked points in $S_{0,0,2g+2}$.

In this situation, $\cup \overline{c_i}$ and $\cup \overline{d_i}$ use all $2g + 2$ marked points. By Lemma 3.3.8 $\{\phi(c_i)\}$ and $\{\phi(d_i)\}$ are disjoint, hence $\cup \overline{\phi(c_i)}$ and $\cup \overline{\phi(d_i)}$ are disjoint. Thus, if we cut $S_{0,0,2g+2}$ along $\cup \overline{\phi(c_i)}$ and $\cup \overline{\phi(d_i)}$ we will get an annulus, hence the boundary components of regular neighborhoods of $\cup \overline{\phi(c_i)}$ and $\cup \overline{\phi(d_i)}$ are isotopic.

**Lemma 3.4.7.** Let $a$ and $b$ be presymmetric curves in $S$. Then $i(\overline{\phi(a)}, \overline{\phi(b)}) = 0$ if and only if $i(\overline{a}, \overline{b}) = 0$.

**Proof.** The projections of the presymmetric curves $a$ and $b$, $\overline{a}$ and $\overline{b}$, are disjoint if and only if they have disjoint symmetric spines. Hence by Lemma 3.3.8, the result is shown.

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Thus \( \phi \) induces a map in \( \text{Aut}(C(S_{0,0,2g+2})) \). Hence we have shown the main theorem.

## 3.5 Applications of the Main Theorem

The main motivation for our result regarding the symmetric separating curve complex comes from the following theorems regarding \( \mathcal{I}(S) \) and \( \mathcal{K}(S) \).

**Theorem 3.5.1** (Farb-Ivanov, Theorem 6, [14]). For \( S \) a surface with genus \( g \geq 5 \), we have:

\[
\text{Aut}(\mathcal{I}(S_{g})) \cong \text{Mod}^{\pm}(S_{g})
\]

**Theorem 3.5.2** (Brendle-Margalit, Main Theorem 1, [7] and [8]). For \( S \) a surface with genus \( g \geq 3 \), we have:

\[
\text{Aut}(\mathcal{K}(S_{g})) \cong \text{Mod}^{\pm}(S_{g})
\]

The key ingredient to both proofs was to find an appropriate complex for the group and consider the automorphism group of that complex. For \( \mathcal{I}(S) \), Farb-Ivanov used the so-called *Torelli geometry* and for \( \mathcal{K}(S) \), Brendle-Margalit used the separating curve complex. Thus it is natural to conjecture that our main result could be used to show the following:

**Conjecture 3.5.3.** For \( S \) a surface with genus \( g \geq 5 \), we have:

\[
\text{Aut}(\mathcal{H}(S_{g})) \cong \text{SMod}^{\pm}(S_{g})/\langle \iota \rangle
\]

Note that every element in \( \text{SMod}^{\pm}(S_{g}) \) restricts to an element of \( \text{Aut}(\mathcal{H}(S_{g})) \) as follows. If \( f \in \text{SMod}^{\pm}(S_{g}) \), then \( f \mapsto \phi \), where \( \phi(h) = fhf^{-1} \) for \( h \in \mathcal{H}(S_{g}) \), then we get the
following natural map which has \( \iota \) in its kernel:

\[
\text{SMod}^\pm(S_g) \longrightarrow \text{Aut}(\mathcal{H}(S_g))
\]

Both Farb-Ivanov and Brendle-Margalit were able to argue that all automorphisms of \( \mathcal{I}(S) \) and \( \mathcal{K}(S) \) were of this type, thus an automorphism of \( \mathcal{I}(S) \) and \( \mathcal{K}(S) \) induces an automorphisms of the appropriate curve complex. They both used an algebraic characterization of separating twists and BP-maps, and showed that each automorphism of \( \mathcal{I}(S) \) and \( \mathcal{K}(S) \) preserved this algebraic characterization.

In the following, we give an algebraic characterization of Dehn twists about symmetric separating curves. Unfortunately one of the algebraic conditions is a “global” condition with respect to \( \mathcal{I}(S) \) in contrast to a purely “local” condition with respect to just \( \mathcal{H}(S) \). This global condition prevents us from being able to argue directly from our main theorem that all automorphisms of \( \mathcal{H}(S) \) are of the desired type. It may be possible to replace the global condition by additional local conditions which might allow us to show all automorphisms are of the desired type, hence showing an automorphism of \( \mathcal{H}(S) \) induces an automorphism of \( C_{\mathcal{H}(S)} \).

**Symmetric Separating Curves.** We will give an algebraic characterization that classifies when a mapping class is a Dehn twist about a symmetric separating curve. The argument will follow that of Farb-Ivanov in [13]. First we will need a few definitions and lemmas about the structure of elements in \( \mathcal{I}(S) \). Recall the definitions for finite order, reducible and pseudo-Anosov mapping classes are given is Section 2.4.

We call a mapping class \( f \) *pure* if \( f \) contains a homeomorphism \( f' \) which satisfies the following conditions on some closed one-dimensional submanifold \( C \) of \( S \):
1. The components of $C$ are nontrivial and are pairwise disjoint.

2. The homeomorphism $f'$ is fixed on $C$ and does not rearrange the components of $S - C$.

3. Lastly, $f'$ induces the identity or a pseudo-Anosov homeomorphism on each component of $S - C$.

Note that it is well known that all elements of $I(S)$ are pure (Ivanov, Theorem 3, [22]).

A curve system $C$ is a one-dimensional submanifold of $S$, where no component of $C$ is homotopically trivial or boundary parallel. A curve system, $C$, determines a surface $S - C$ which is obtained from $S$ by cutting along $C$. Note that each component of $C$ determines two boundary of the surface $S - C$. Any homeomorphism $f : S \rightarrow S$ with $f(C) = C$ induces a homeomorphism $f_C : S - C \rightarrow S - C$. If $Q$ is a component of $S - C$ and $f_C(Q) = Q$ then $f_C$ induces a homeomorphism $f_Q : Q \rightarrow Q$. We will need the following result of Ivanov-McCarthy in order to complete the proof of our characterization.

**Lemma 3.5.4** (Ivanov-McCarthy, Lemma 5.6, [24]). Let $\Gamma$ be any subgroup of finite index in $\text{Mod}(S)$ consisting entirely of pure elements. Let $f, h \in \Gamma$ and let $C_f$ denoted a representative on $S$ of a canonical reduction system for $f$. Then $h \in C_\Gamma(f)$ if and only if $h_Q$ commutes with $f_Q$ for every component $Q$ of $S - C_f$.

Note that $h_Q$ makes sense in this context because the canonical reduction system for $h$, $C_h$, equals $C_f$ because Lemma 2.6 of [5] says $h(C_f) = C_{h_f h^{-1}}$.

**Characterizing Dehn Twists About Symmetric Separating Curves.** Ivanov in Section 2 of [21] characterized Dehn twists about non-separating curves based on purely algebraic properties. In a similar fashion, Farb-Ivanov characterized powers of a Dehn twist about a separating curve or a BP-map.
**Proposition 3.5.5** (Farb-Ivanov, Proposition 8, [14]). *Let $S$ be a closed, oriented surface of genus $g \geq 3$ and $f \in \mathcal{I}(S)$ is nontrivial. Then $f$ is a power of a Dehn twist about a separating curve or a power of a BP-map if and only if*

1. $Z(C_{\mathcal{I}}(f)) = \mathbb{Z}$
2. $C_{\mathcal{I}}(f) \neq \mathbb{Z}$
3. $\max_{ab}I(f) = 2g - 3$.

Modifying the arguments of Farb-Ivanov [13], we characterize Dehn twists about symmetric separating curves. In the case of Ivanov and Farb-Ivanov all the conditions are stated in terms of the group being studied, namely $\mathcal{I}(S)$. We will not be able to state all our conditions in terms of the group $\mathcal{H}(S)$; namely condition (2) will be stated in terms of $\mathcal{I}(S)$. We will give an example showing the necessity of this condition after the proof of the theorem. Note that for the statement of Theorem 3.5.6 it might be useful to recall Proposition 3.1.5 and the definitions given in Section 2.4.

**Theorem 3.5.6.** *Let $S$ be a closed, oriented surface of genus $g \geq 3$. For nontrivial $f \in \mathcal{H}(S)$, then $f$ is a power of a Dehn twist about a symmetric separating curve if and only if*

1. $Z(C_{\mathcal{H}}(f)) = \mathbb{Z}$
2. $Z(C_{\mathcal{I}}(f)) = \mathbb{Z}$
3. $C_{\mathcal{H}}(f) \neq \mathbb{Z}$
4. $\max_{ab}H(f) = g - 1$.

*Proof.* Suppose $f = T_{\gamma}^k$ where $\gamma$ is a symmetric separating curve. Let $Q_1$ and $Q_2$ be the two components of $S - \gamma$ and let $\mathcal{H}(Q_1)$ and $\mathcal{H}(Q_2)$ denote the subgroups of $\mathcal{H}(S)$ supported
on $Q_1$ and $Q_2$ respectively, similarly $I(Q_i)$. The group $H(Q_i)$ is the subgroup of the group generated by twists about symmetric separating curves on $Q_i$ containing elements which fix the boundary of $Q_i$ pointwise. Note that if $H'(Q_i)$ is the corresponding group where the boundary need not be fixed pointwise, we have the exact sequence:

$$1 \longrightarrow \mathbb{Z} \longrightarrow H(Q_i) \longrightarrow H'(Q_i) \longrightarrow 1$$

Since $g \geq 3$ one of the $Q_i$'s, say $Q_1$, has genus at least 2. Further $H(Q_1)$ contains two independent pseudo-Anosov maps $u_1$ and $v_1$. See Chapter 13 of [15] for further details about constructing pseudo-Anosov maps.

If the genus of $Q_2 = 1$, then $H(Q_2) = 1$ and we have the following exact sequence:

$$1 \longrightarrow \langle T_\gamma \rangle \longrightarrow C_{H(S)}(T_\gamma) \longrightarrow H(Q_1) \longrightarrow 1$$

Hence we can deduce that

$$Z(C_{H(S)}(T_\gamma)) = Z(H(Q_1)) = \mathbb{Z}$$

where $\mathbb{Z}$ is generated by $T_\gamma$. We use Lemma 3.5.4 and the fact that no nontrivial mapping class commutes with two independent pseudo-Anosov maps, in our case $u_1$ and $v_1$.

If the genus of $Q_2 \geq 2$, then $H(Q_2)$ contains two independent pseudo-Anosov maps $u_2$ and $v_2$. If $h \in Z(C_{H(S)}(T_\gamma))$, then $h_{Q_i}$ must commute with both $u_i$ and $v_i$ for $i = 1$ or 2. Since $u_i$ and $v_i$ are independent, it is clear that $h_{Q_i} = Id$, $i = 1$ or 2. Thus $Z(C_{H(S)}(T_\gamma))$ can only contain powers of $T_\gamma$, and hence is infinite cyclic.

Farb-Ivanov’s characterization of separating twists and BP-maps in 3.5.5 gives condition
(2) above. Because there exists a symmetric separating curve disjoint from \( \gamma \), \( C_{\mathcal{H}}(f) \) contains a \( \mathbb{Z}^2 \) subgroup showing (3) is true. Lastly, condition (4) follows from Proposition 3.1.5.

Suppose \( f \in \mathcal{H} \) satisfies conditions (1), (2), (3) and (4). We know \( f \in \mathcal{H} \subset \mathcal{I} \) is pure (see [22]), so if \( f \) leaves a system \( C \) of mutually disjoint, non-homotopic essential curves invariant, then \( f \) leaves each component of \( C \) invariant. Let \( E_f \) be the canonical reduction system for \( f \). Let \( d \) denote the maximal rank of an abelian subgroup in \( \mathcal{I}(S) \) generated by Dehn twists about separating curves or bounding pairs in \( E_f \). We will look at cases according to \( d \) and argue that \( d = 1 \).

Suppose \( d \geq 2 \). Then \( E_f \) must contain two distinct elements \( T_\alpha \) and \( T_\beta \) where each of \( \alpha \) and \( \beta \) is a separating curve or a bounding pair. Note that any \( h \in C_{\mathcal{I}(S)}(f) \) leaves \( E_f \) invariant, ([5], Lemma 2.6: \( \sigma(E_f) = E_{\sigma f \sigma^{-1}} \)) hence \( T_\alpha \) and \( T_\beta \) commute with any such \( h \). Thus

\[
Z(C_{\mathcal{I}(S)}(f)) \supseteq \langle T_\alpha, T_\beta \rangle \cong \mathbb{Z}^2.
\]

This contradicts condition (2), so we can assume \( d \leq 1 \).

We can also assume \( E_f \) is non-empty otherwise \( f \) is a pseudo-Anosov element contradicting condition (3).

Next, as a step to proving that \( f \) is a power of a Dehn twist about a symmetric separating curve, we show that none of the maps \( f_Q \), where \( Q \) is a component of \( S - E \), is a pseudo-Anosov homeomorphism. Because \( f \) is pure, we know every such map, \( f_Q \), is either pseudo-Anosov or the identity.

Now we will consider what a component \( Q \) of \( S - E \) on which \( f_Q \) is pseudo-Anosov must look like. Since \( Q \) is a proper subsurface of \( S \) it is homeomorphic to \( \Sigma_{g,r} \) for some \( g \geq 0 \) and \( r \geq 1 \).
Case 1: If \( g = 0 \) then for \( Q \) to admit any pseudo-Anosov element it must be that \( r \geq 4 \).

Case 2: If \( g = 1 \) and \( r = 1 \), then \( \partial Q \) is a genus 1 separating curve and so \( f_Q \in \mathcal{I}(Q) = \mathcal{I}(\Sigma_{1,1}) \). But \( \mathcal{I}(\Sigma_{1,1}) \) is generated by the twist about its boundary curve, hence does not contain a pseudo-Anosov element.

Case 3: Suppose \( g \geq 2 \) and \( r = 1 \) or 2, or that \( g = 1 \) and \( r = 2 \). Since \( Q \) is a component of \( S - E \), it must be that \( \partial Q = \alpha \) where

1. \( \alpha \) is a separating curve or
2. \( \alpha \) is a bounding pair.

Otherwise \( \partial Q \) would consist of a separating curve and another curve (which must be separating too for \( Q \) to be a component of \( S - E \)), but this contradicts the fact that \( d \leq 1 \). Now let \( \hat{f}_Q \) be the extension of \( f_Q \) to the whole surface where it is the identity on \( S - Q \). Then we have

\[
Z(C_{\mathcal{I}(S)}(f)) \supseteq \langle \hat{f}_Q, T_{\alpha} \rangle \cong \mathbb{Z}^2
\]

which contradicts condition (2).

Remaining Cases: We are left with the cases \( g = 0 \) and \( r \geq 4 \), or \( g \geq 1 \) and \( r \geq 3 \).

Claim: There exists a \( k \geq 3 \) so that there is a union \( C = \beta_1 \cup \cdots \cup \beta_k \) of components of \( \partial Q \) so that \( C \) bounds in \( S \) but no subcollection of the \( \beta_i \)'s bound.

Proof of Claim: Since \( r \geq 3 \), if \( \partial Q \) has no bounding pairs or separating curves, then we can take \( C \) to be any minimal subset of components of \( \partial Q \) which separate \( S \).

Suppose \( \partial Q \) contains a bounding pair \( \alpha \). Now \( r \neq 3 \) and \( r \neq 4 \) for otherwise \( \partial Q \setminus \alpha \) being homologous to \( \alpha \), would be separating (or a bounding pair) in \( S \) contradicting the fact that
$d \leq 1$. Hence we have $r \geq 5$. Since $d = 1$ in this case, a minimal subcollection of components of $\partial Q$ which bound and which has at least 3 elements exists.

Suppose $\partial Q$ contains a separating curve $\alpha$. Now $r \neq 3$ for otherwise $\partial Q \setminus \alpha$ being homologous to $\alpha$, would be a bounding pair in $S$ contradicting the fact that $d \leq 1$. Hence we have $r \geq 4$, so we can conclude there is a minimal bounding subcollection with at least 3 elements. This completes the proof of the claim.

Now choose such a collection $C = \beta_1 \cup \cdots \cup \beta_k$. Let $\gamma$ be a separating curve in $S$ which lies in $S \setminus Q$ and which together with $C$ bounds a genus 0 subsurface of $S$.

By condition (4) we know $\max_{ab} H(f) = g - 1$, so there exists some free abelian group $A < H(S)$ of rank $g - 1$ which contains $f$. In the following arguments we will argue that certain situations cannot happen by contradicting the maximality of $A$. Note that any $h \in I(S)$ which leaves the components of $Q$ invariant must also leave $\gamma$ invariant.

Let $U_1$ be the component of $S_\gamma$ which does not contain $Q$ and let $U_2 = S_\gamma \setminus U_1$.

Since every $a \in A$ commutes with $f$, it leaves $E_f$, the canonical reduction system of $f$, invariant. Hence $a(Q) = Q$ and so $a(\partial Q) = \partial Q$, and so $a(C) = C$ and $a(\gamma) = \gamma$. Let $A_i$ with $i = 1, 2$ be the image of $A$ under the reduction homomorphism $\pi_{U_i} : H(S_\gamma) \to H(U_i)$. Since $\gamma$ is a separating curve, the natural inclusion $U_i \to S$ induces an injection

$$H(U_i) \to P_\gamma := \{ p \in H(S) : p(\gamma) = \gamma \}$$

and we have a homomorphism

$$\psi : H(U_1) \times H(U_2) \to P_\gamma$$
sending \(x_1\) and \(x_2\) to the mapping class with \(x_i\) on \(U_i\). Note that this is well-defined since elements of \(\mathcal{H}(U_i)\) fix \(\partial U_i\) pointwise.

**If \(\gamma\) is a symmetric separating curve** then let \(\gamma_i\) with \(i = 1, 2\) be a curve in \(U_i\) isotopic to \(\gamma\). Then it is easy to see that the kernel of \(\psi\) is generated by \(T_{\gamma_1}T_{\gamma_2}^{-1}\). Thus we have the following exact sequence:

\[
1 \to \mathbb{Z} \to \mathcal{H}(U_1) \times \mathcal{H}(U_2) \to P_{\gamma} \to 1
\]

Restricting \(\psi\) to \(A\) and noting that by maximality \(T_{\gamma} \in A\), we have the following restriction:

\[
1 \to \mathbb{Z} \to A_1 \times A_2 \to A \to 1
\]

So we have the following:

\[
\text{rank}(A) \leq \text{rank}(A_1) + \text{rank}(A_2) - 1
\]

\[
\leq \maxab(\mathcal{H}(U_1)) + \maxab(\mathcal{H}(U_2)) - 1
\]

\[
\leq g_1 + g_2 - 1
\]

Since \(r \geq 3\) we have that

\[
g_1 + g_2 \leq \text{genus}(S) - 2
\]

so

\[
\text{rank}(A) \leq (g_1 + g_2) - 1
\]

\[
\leq \text{genus}(S) - 3
\]

which contradicts the fact that \(A\) is maximal.
If $\gamma$ is not a symmetric separating curve then the kernel of $\psi$ is empty. Thus we have

$$\mathcal{H}(U_1) \times \mathcal{H}(U_2) \cong P_\gamma$$

Restricting $\psi$ to $A$, we have the following:

$$A_1 \times A_2 \cong A$$

Hence we have the following:

$$\text{rank}(A) = \text{rank}(A_1) + \text{rank}(A_2) \leq \text{maxab}(\mathcal{H}(U_1)) + \text{maxab}(\mathcal{H}(U_2)) \leq g_1 + g_2$$

Since $r \geq 3$ we have that

$$g_1 + g_2 \leq \text{genus}(S) - 2$$

so

$$\text{rank}(A) \leq (g_1 + g_2) \leq \text{genus}(S) - 2$$

which contradicts the fact that $A$ is maximal. Hence we have proven that $f$ acts by the identity on every component of $S - E$.

We now have that $f$ must be a multitwist about curves in $E_f$. Since $f \in \mathcal{I}(S)$, $f$ must be a multitwist about a union of separating curves and bounding pairs. Since $d \leq 1$ there is only one such curve or pair. Note, $d \neq 0$, otherwise $f$ would not be in $\mathcal{I}$. Further, since
$f \in \mathcal{H}(S)$, $f$ must be a power of a Dehn twist about a symmetric separating curve.

Example. In order to show the necessity of condition (2), consider the following example. Let $f = T_aT_b^{-1}$, where $a$ and $b$ are as shown.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{curve_example.png}
\caption{Curves $a, b, c, d$ and $C_f$.}
\end{figure}

Condition (1): $Z(C_H(f)) = \mathbb{Z}$. We know $f \in Z(C_H(f))$. We claim only powers of $f$ are in $Z(C_H(f))$. First note that the canonical reduction system for $f$, $C_f$, is the following collection of curves:

$$C_f = \{c, d, e\}.$$

Because $f$ is a pure mapping class, we know the restriction of $f$ to any component of $S - C_f$ is either the identity or a pseudo-Anosov element. We will let $Q_1$ be the component of $S - C_f$ that contains the curves $a$ and $b$ and $Q_2$ be the other component. Suppose there exists a mapping class $h \in Z(C_H(f))$. Thus $h \in C_H(f)$, thus by Lemma 3.5.4 $h_Q$ must commute with $f_Q$ for every component $Q$ of $S - C_f$. Because no two independent pseudo-Anosov elements commute and $h$ is a pure mapping class, we know $h_{Q_1}$ is either the identity or a power of $f$. Further $h_{Q_2}$ must be the identity, otherwise $h_{Q_2}$ would be a pseudo-Anosov element and we know there exist other mapping classes in $C_H(f)$ which are independent pseudo-Anosov elements when restricted to $Q_2$. Hence $h \notin Z(C_H(f))$. Now the only type of mapping class $h$ can be, besides a power of $f$, is a multitwist about curves in $C_f$. But no such multitwist is in $\mathcal{H}(S)$.

Condition (2): $Z(C_T(f)) \neq \mathbb{Z}$. Both $f$ and the element $T_eT_d^{-1}$, where $e$ and $d$ are as
in Figure 3.16, are in $Z(C_2(f))$, hence $Z(C_2(f))$ has a $\mathbb{Z}^2$ subgroup.

**Condition (3):** $C_\mathcal{H}(f) \neq \mathbb{Z}$. This follows from the fact that there clearly exists a symmetric separating curve disjoint from $a$ and $b$.

**Condition (4):** $\text{max}_{ab} H(f) = g - 1$. Such a group is generated by $f$ and Dehn twists about the curves pictured below.

![Figure 3.17: A collection of disjoint symmetric separating curves that with $f$ form a basis for $\text{max}_{ab} H(f)$](image)

Figure 3.17: A collection of disjoint symmetric separating curves that with $f$ form a basis for $\text{max}_{ab} H(f)$. 

Chapter 4

A Summary

In this dissertation we have done a preliminary investigation of SIP-maps and the group they generate, SIP(S). We have factored SIP-maps into the product of 5 BP-maps, shown the image of SIP-maps under well-known representations of the Torelli group, and characterized which SIP-maps are in $\mathcal{K}(S)$ and the kernel of the Birman-Craggs-Johnson homomorphism. Further we have shown SIP(S) $\neq$ $\mathcal{I}(S)$ and is in fact an infinite index subgroup when $g \geq 3$. We have also given several equivalent descriptions of the group SIP(S) $\cup$ $\mathcal{K}(S)$. We have also found a new interpretation in terms of SIP-maps of a relation among Johnson generators of $\mathcal{I}(S)$.

This work leads to many questions about SIP-maps as well as the structure of SIP(S) that deserve further investigation. For example:

- Is $\mathcal{I}(S)/$ SIP(S) abelian?
- Can other relations in $\mathcal{I}(S)$ be reinterpreted in terms of SIP-maps?
- Is SIP(S) finitely generated?
• Is SIP(S) finitely presentable?

We have also considered the symmetric Torelli group, $SI(S)$. More specifically we have looked at $\mathcal{H}(S)$, the subgroup of $SI(S)$ generated by symmetric separating twists. We have shown that the Birman-Craggs-Johnson homomorphism does not “see” the difference between $\mathcal{H}(S), SI(S)$, and $\mathcal{K}(S)$. Further we have looked at a simplicial complex, the symmetric separating curve complex $C_\mathcal{H}(S)$, that $\mathcal{H}(S)$ has a natural action on. We have shown if $g \geq 5$, then

$$\text{Aut}(C_\mathcal{H}(S)) \cong \text{SMod}(S)^{\pm}/\langle \iota \rangle.$$ 

In addition we have given a purely algebraic characterization of symmetric separating twists. One possible application of this work was mentioned in Section 3.5, namely

**Conjecture 4.0.7.** For $S$ a surface with genus $g \geq 5$, we have:

$$\text{Aut}(\mathcal{H}(S_g)) \cong \text{SMod}^\pm(S_g)/\langle \iota \rangle$$

If this conjecture proves true, then it is natural to consider the abstract commensurator of $\mathcal{H}(S)$. The abstract commensurator of a group, $G$, denoted $\text{Comm}(G)$, is the group of isomorphisms of finite index subgroups of $G$, under composition. In general, one expects $\text{Comm}(G)$ to be much larger than $\text{Aut}(G)$. For example, $\text{Aut}(\mathbb{Z}) \cong \mathbb{Z}_2$ while $\text{Comm}(\mathbb{Z}) \cong \mathbb{Q}$. Farb-Ivanov and Brendle-Margalit have shown that this is not the case for $I(S)$ and $\mathcal{K}(S)$. In particular,

**Theorem 4.0.8** (Farb-Ivanov, [14]). For $S$ a surface with genus $g \geq 3$, we have:

$$\text{Comm}(I(S)) \cong \text{Aut}(I(S)) \cong \text{Mod}^\pm(S).$$
Further, regarding $\mathcal{K}(S)$, the following is true.

**Theorem 4.0.9** (Brendle-Margalit, [7]). *For $S$ a surface with genus $g \geq 3$, we have:*

$$\text{Comm}(\mathcal{K}(S)) \cong \text{Aut}(\mathcal{K}(S)) \cong \text{Mod}^\pm(S).$$

This leads us to conjecture the following.

**Conjecture 4.0.10.** *For $S$ a surface with genus $g \geq 5$, we have:*

$$\text{Comm}(\mathcal{H}(S)) \cong \text{Aut}(\mathcal{H}(S)) \cong \text{SMod}^\pm(S)/\langle \iota \rangle.$$
Bibliography


Vita

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