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NON-NESTED MONTE CARLO DUAL BOUNDS FOR MULTI-EXERCISABLE OPTIONS

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Abstract. We study the optimal marginal value of discrete-time optimal multiple stopping problems and find that it can be formulated as a single optimal stopping optimization as well. Based on this result propose a marginal-value-based lower bound method to achieve a small bound on the iterative error. We further introduce a non-nested upper bound method. The convergence of both methods is analysed. The implementation details and enhancement techniques are discussed as well. Overall, our methods make a good trade-off between the time-efficiency and the tightness in dual bounds.

1. Introduction

The main difficulty in pricing derivatives is to price high-dimensional early-exercisable option in a quick and accurate way. The classic lattice-based methods usually suffer from the curse of conditionality and inevitable consume an affordable amount of time, when problems become large. Monte Carlo methods have unique power at solving problems with features of path-dependence and high dimension, but it is not clear how to implement Monte Carlo simulation, as going backwards along with the simulated path is meaningless for locating the exercise boundary. Currently, there are two main research streams to address backward dependence, i.e. lower bound methods and upper bound methods.

Lower bound, also known as 'buyer's price', relies on finding a good approximation of the optimal exercise strategy. The classic Least-Squares Monte Carlo pricing algorithm is introduced by Carrière [9], who adopted non-parametric regression to obtain a sequence of approximate continuation values and then exercise accordingly. Similar works can be found in [17] and [22]. Clément [11] proved the convergence of these lower bounds, and Stentoft [21] showed the order of convergence. Due to the complexity of the contract and the curse of dimensionality, the above method may not perform well in complicated cases. Broadie and Cao [8] suggested using double regression to discard samples far away from the exercise boundary in order to estimate a more accurate exercise boundary. Additionally, Kolodko and Schoenmakers [16] iterated the sub-optimal exercise strategy by sub-simulation. Although this iteration yields better lower bounds, the time consumption for high-dimensional cases is generally unappealing in practice.

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Upper bound, also known as 'seller's price', is based on constructing a good martingale hedging strategy. It is mathematically constructed on duality theory, and the fundamental work can be found in [12]. Rogers [19] and Haugh and Kogan [14] advised finding such a martingale by conducting an optimization over Monte Carlo simulations. The primal-dual method introduced by Andersen and Broadie [2] focuses on hedging the lower bound. They separated the martingale part from a lower bound process by introducing sub-simulations to compute conditional expectations. The quality of the upper bound is highly dependent on the goodness of the exercise strategy. Although this method has shown its great power at getting a tight upper bound and judging the goodness of the exercise strategy, the time consumption of this method prevents itself from being routinely used, just like the policy iteration lower bound.

Monte Carlo methods have been used to price multiple exercise options as well. Bender and Schoenmakers [5] proposed an iterative Monte Carlo method to price multiple exercise options and analyzed its convergence. Bender [4] derived the continuous-time multiple stopping dual representation, and Schoenmakers [20] studied the discrete-time case. Bender [3] and Aleksandrov and Hambly [1] studied multiple pricing problem with volume constraints. Bender, Schoenmakers, and Zhang [6] obtained a more general dual representation by considering both refraction periods and volume constraints.

The aim of this paper is to put forward a tighter lower bound method to boost the accuracy and a non-nested upper bound method to enhance the time-efficiency. We propose a marginal-based lower bound method, where the marginal continuation value serves as an agent to link the single stopping problem of computing the marginal value and the multiple stopping problem of computing the option value. In detail, we directly approximate the marginal continuation values by an iterative extension of classic Least-Squares Monte Carlo algorithm, and then use them to construct the exercise strategy for multiple exercise options. Compared to classic Least-Squares lower bounds, our method improves the tightness of lower bounds from two aspects.

The rest of this paper is arranged as follows. In section 2, we present the mathematical formulation of the optimal price, where it can be formulated from a nested optimal stopping problem. In section 3, we first review the classic Least-Squares lower bound method, and then prove the marginal value of multiple exercise options is still a single stopping time problem, and finally propose the new marginal-based lower bound method to build improved exercise strategies. In section 4, we review dual representation of the marginal value and its corresponding extension of primal-dual upper bound method in multiple exercise setting, and then put forward a new regression-based non-nested upper bound method to avoid nested sub-simulations. The convergence of both marginal-based lower bounds and non-nested upper bounds are analyzed in section 5. After conducting theoretical analysis, we proceed with the numerical examination of proposed methods in section 6 and then conclude in section 7.

2. Mathematical Setup

We work with a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_{t_j})_{0\le j\le L}, \mathbb{P})$. The driving $(\mathcal{F}_{t_j})$-Markov chain W_{t_j} is in $L^2(\Omega)$. The Borel function $z(t_j, \cdot)$, used to define the numeraire-adjusted payoff $Z_{t_j} = z(t_j, W_{t_j})$ at time t_j, is in $L^2(\Omega)$ and $\mathcal{F}_{t_j}$-measurable. Measure $\mathbb{P}$ is the risk-neutral measure corresponding to the numeraire. Multiple exercise options allow contract holders to exercise positive integer number, $n \le L$, of rights, which occur sequentially at a time sequence, $\{t_j\}_{1\le j\le L}$.

We assume that all rights are going to be exercised, as abandoning an exercise right is equivalent to exercising the right out of the money for zero payment. Denote a set of stopping times $\{\tau^{n,n}, \tau^{n-,n1}, \cdots, \tau^{1,n}\}$ with $t_1 \le \tau^{n,n} < \tau^{n-1,n} < \cdots < \tau^{1,n} \le t_L$ as an exercise strategy π^n, where τ^i is the exercising time of the i-th remaining right. Then the non-arbitrage price is given by:

$$V_0^{n,*} \triangleq \sup_{\pi^n} \mathbb{E}\left[\sum_{i=1}^{n} Z_{\tau^{i,n}} \,\middle|\, \mathcal{F}_0\right]. \tag{2.1}$$

Denote the exercise strategy for n remaining rights at time t_j by π_j^n, where $t_j \le \tau_j^{n,n} < \tau_j^{n-1,n} < \cdots < \tau_j^{1,n} \le t_L$. Then define the optimal value and the optimal continuation value as:

$$V_{t_j}^{n,*} \triangleq \sup_{\pi_j^n} \mathbb{E}\left[\sum_{i=1}^{n} Z_{\tau^i} \,\middle|\, \mathcal{F}_{t_j}\right],\ j \le L-n+1; \tag{2.2}$$

$$C_{t_j}^{n,*} \triangleq \mathbb{E}\left[V_{t_{j+1}}^{n,*} \middle| \mathcal{F}_{t_j}\right],\ j \le L-n. \tag{2.3}$$

Without loss of generality, we further assume that $V_{t_j}^{0,*} = C_{t_j}^{0,*} = 0$ for $1 \le j \le L$, $V_{t_j}^{n,*} = V_{t_j}^{n-1,*}$ for $j > L-n+1$, and $C_{t_j}^{n,*} = C_{t_j}^{n-1,*}$ for $j > L-n$. Define the optimal marginal value $\Delta V_{t_j}^{n,*}$ and the marginal continuation value $\Delta C_{t_j}^{n,*}$ as:

$$\Delta V_{t_j}^{n,*} \triangleq V_{t_j}^{n,*} - V_{t_j}^{n-1,*}; \tag{2.4}$$

$$\Delta C_{t_j}^{n,*} \triangleq C_{t_j}^{n,*} - C_{t_j}^{n-1,*}. \tag{2.5}$$

Then the optimal exercise strategy $\tau^{i,n,*}$ in (2.1) is given by:

$$\tau_j^{i,n,*} \triangleq \min\left\{t_{L-i+1} \ge t_j > \tau_j^{i+1,n,*} \,\middle|\, \Delta C_{t_j}^{i,*} \le Z_{t_k}\right\},\ i = n, \cdots, 1,$$

where n indicates the total number of rights. Besides, we also know the exercise strategy for i-th right with different number of remaining rights holds the following relation:

$$\tau_j^{i,n,*} = \left\{\tau_k^{i,i,*} \middle| t_{k-1} = \tau_j^{i+1,n,*}\right\}.$$

Remark 2.1. Most multiple exercise options allow contract holders to trade goods for an amount no more than defined limit specified in the contract. In the above formulation, we assume that no more than one right can be exercised at each exercise time. This assumption seems impractical as it appears, however, it comes from the fact that a rational contract holder will always exercise maximal number of rights at an exercise time, if an exercise decision is made. According to [18], we know $\Delta V_{t_j}^{n,*} \le \Delta V_{t_j}^{n-1,*}$, which implies that the exercise right is of less marginally valuable with more rights in hands. In other words, we should always exercise as more rights as possible to make them equally valuable as the marginal value with minimal possible number of rights remaining. Thus, in the above formulation, we pack the maximal number of rights that can be exercised on a single exercise time into one exercise right.

3. Lower Bounds

Due to the complexity of nested expectation and iterative optimal stopping, there hardly exits a method to compute the optimal price of multiple exercise option directly. In the section, we review the classic extension of the Least-Squares lower bound method, and then propose our new marginal-based lower bound method.

3.1. Review of classic least-squares lower bounds. This method is an extension of the classic Least-Squares lower bound algorithm, and is based on following theorem that multiple stopping optimization problem is an nested single stopping time optimization problem.

Theorem 3.1. *Suppose Z_{t_j} is numeriare-adjusted payoff, and $C_{t_j}^{n-1,*}$ is its continuation value with $n-1$ remaining rights defined as (2.3). Then the optimal value with n remaining rights, $V_{t_j}^{n,*}$, is given by optimal stopping theory as:*

$$V_{t_j}^{n,*} = \sup_{t_j \le \tau \le L-n+1} \mathbb{E}\left[Z_\tau + C_\tau^{n-1,*} \middle| \mathcal{F}_{t_j} \right].$$

This theorem immediately yields an algorithm to construct sub-optimal exercise strategies that we can use to compute lower bounds. Precisely, suppose we have an approximation of the continuation value with $n-1$ rights as $\hat{C}_{t_j}^{n-1}$, then the approximation of the continuation value with n rights can be obtained from the classic Least-Squares Monte Carlo algorithm with $\hat{C}_{t_j}^{n-1} + Z_{t_j}$ being inputted as the exercise payoff. The optimal price is given by exercising the option if and only if the marginal continuation value $\Delta C_{t_j}^{n,*}$ is greater than the exercise payoff Z_{t_j}, thus we can easily replace $\Delta C_{t_j}^{n,*}$ with $\hat{C}_{t_j}^{n} - \hat{C}_{t_j}^{n-1}$ to obtain a sub-optimal exercise strategy.

3.2. Marginal-based lower bounds. Our lower bound method can be viewed as another extension of the Least-Squares Monte Carlo algorithm, where it is iteratively applied to approximate the marginal continuation value $\Delta C_{t_j}^{n,*}$ rather than the continuation value $C_{t_j}^{n,*}$. First, we show that the optimal marginal value is given by a single stopping time optimization problem as well.

Lemma 3.2. *Supposing $\Delta C_{t_j}^{n-1,*}$ is define as (2.5), it holds:*

$$\Delta C_{t_j}^{n,*} \le \Delta C_{t_j}^{n-1,*}.$$

According to [18], we have $\Delta V_{t_j}^{n,*} \le \Delta V_{t_j}^{n-1,*}$, which, by definition, implies lemma 3.2. Now, we give the main theorem.

Theorem 3.3. *Suppose Z_{t_j} is numeriare-adjusted payoff, and $\Delta C_{t_j}^{n-1,*}$ is the marginal continuation value with $n-1$ remaining rights defined as (2.5). Then the optimal marginal value with n remaining rights, $\Delta V_{t_j}^{n,*}$, is given by optimal stopping theory as:*

$$\Delta V_{t_j}^{n,*} = \sup_{t_j \leq \tau \leq L-n+1} \mathbb{E}\left[\min\left\{Z_\tau, \Delta C_\tau^{n-1,*}\right\}\middle|\mathcal{F}_{t_j}\right]. \tag{3.1}$$

The equality is achieved, when stopping time is

$$\tau_j^{n,*} = \tau_j^{n,n,*} = \min\left\{t >= t_j | Z_t \geq \Delta C_\tau^{n,*}\right\}.$$

Proof. According to dynamic programming principle, it is enough to prove the following equivalent recursive equation holds:

$$\Delta V_{t_j}^{n,*} = \max\left\{\min\left\{Z_{t_j}, \Delta C_{t_j}^{n-1,*}\right\}, \Delta C_{t_j}^{n,*}\right\}. \tag{3.2}$$

Now we focus on how to prove (3.2). As we need to deal with the recursive equations of $V_{t_j}^{n,*}$ and $V_{t_j}^{n-1,*}$ at same time, and their exercise strategies are different, we simplify the analysis into following two cases according to the exercise decision for $(n-1)$-th right.

First, we consider the case where $\Delta C_{t_j}^{n-1,*} \leq Z_{t_j}$. By lemma 3.2, we have $\Delta C_{t_j}^{n,*} \leq Z_{t_j}$. By definition of marginal value, $\Delta V_{t_j}^{n,*}$ can be written as:

$$\Delta V_{t_j}^{n,*} = \max\left\{\min\left\{Z_{t_j}, \Delta C_{t_j}^{n-1,*}\right\}, \Delta C_{t_j}^{n,*}\right\}.$$

Next, we look at the case where $\Delta C_{t_j}^{n-1,*} > Z_{t_j}$. Then $\Delta V_{t_j}^{n,*}$ can be expressed as:

$$\Delta V_{t_j}^{n,*} = \max\left\{\min\left\{Z_{t_j}, \Delta C_{t_j}^{n-1,*}\right\}, \Delta C_{t_j}^{n,*}\right\}.$$

Combining above two cases, the proof is completed. ☐

Remark 3.4. If $\tau_j^{n-1,n-1,*}$ is accessible, we have the following special case of theorem 3.3:

$$\Delta V_{t_j}^{n,*} = \sup_{t_j \leq \tau \leq \tau_j^{n-1,n-1,*}} \mathbb{E}\left[Z_\tau 1_{\left\{\tau < \tau_j^{n-1,n-1,*}\right\}} + \Delta C_\tau^{n-1,*} 1_{\left\{\tau = \tau_j^{n-1,n-1,*}\right\}}\middle|\mathcal{F}_{t_j}\right],$$

where the maximal holds for $\tau_j^{n,*} = \tau_j^{n,n,*}$. This result comes from $\tau_j^{n,n,*} \leq \tau_j^{n-1,n-1,*}$, which is implied by $\Delta C_{t_j}^{n,*} \leq \Delta C_{t_j}^{n-1,*}$.

Based on theorem 3.3, we can describe our lower bound algorithm as following two steps. In the first step, marginal continuation value is approximated by iteratively implementing Least-Squares Monte Carlo method for standard single stopping problem with updating exercise values every time. Suppose the approximation of marginal continuation value with $n-1$ rights is given as $\Delta C_{t_j}^{n-1}$, then the approximation of that with n rights can be obtained from Least-Squares Monte Carlo algorithm with $\min\left\{\Delta C_{t_j}^{n-1}, Z_{t_j}\right\}$ being inputted as new exercise payoff.

In detail, denote basis functions for the continuation value with n remaining rights by $y^n_{j,k},\ k \geq 1$, which are $\mathcal{F}_{t_j}$-measurable and satisfy following conditions:

$$\forall m > 0,\ \forall n > 0,\ \forall 0 < j < L,\ \sum_{k=1}^{m} b_k y^n_{j,k} = 0\ a.s. \implies b_k = 0,\ k = 1, ..., m; \quad (3.3)$$

$$\left\{ y^n_{j,k} \right\}_{k \geq 1} \text{ is total in } L^2\left(\sigma\left(W_{t_j}\right)\right) \text{ for } n > 0,\ 0 < j < L. \quad (3.4)$$

We denote the orthogonal projection into the space generated by these basis functions by $P^{n,[m]}_j(\cdot)$. Then the stopping time with 1 remaining right can be expressed by:

$$\begin{cases} \tau^{1,[m]}_L & = t_L, \\ \tau^{1,[m]}_j & = 1_{\left\{Z_{t_j} \geq \Delta C^{1,[m]}_{t_j}\right\}} t_j + 1_{\left\{Z_{t_j} < \Delta C^{1,[m]}_{t_j}\right\}} \tau^{1,[m]}_{j+1},\ 0 \leq j \leq L-1, \end{cases}$$

where $\Delta C^{1,[m]}_{t_j}$ is the corresponding approximation of the continuation value as:

$$\Delta C^{1,[m]}_{t_j} = P^{[m],1}_j \left(Z_{\tau^{1,[m]}_{j+1}} \right) \text{ in } L^2.$$

Then sub-optimal stopping rules for n remaining rights are constructed as follows:

$$\tau^{n,[m]}_j = \begin{cases} t_{L-n+1}, \quad j = L-n-1, \\ \begin{array}{l} 1_{\left\{\min\left\{Z_{t_j}, \Delta C^{n-1,[m]}_{t_j}\right\} \geq \Delta C^{n,[m]}_{t_j}\right\}} t_j \\ +1_{\left\{\min\left\{Z_{t_j}, \Delta C^{n-1,[m]}_{t_j}\right\} < \Delta C^{n,[m]}_{t_j}\right\}} \tau^{n,m}_{j+1}, \end{array}, \quad 0 < j < L-n-1, \end{cases} \quad (3.5)$$

where $\Delta C^{n,[m]}_{t_j}$ is the corresponding approximation continuation value as:

$$\Delta C^{n,[m]}_{t_j} = P^{n,[m]}_j \left(\min \left\{ Z_{\tau^{n,[m]}_{j+1}}, \Delta C^{n-1,[m]}_{\tau^{n,[m]}_{j+1}} \right\} \right).$$

In the next step, the lower bound of multiple exercise option is evaluated using the exercise strategy generated by using approximate marginal continuation values obtained from step one. The optimal value is given by exercising the option if and only if the marginal continuation value $\Delta C^{n,*}_{t_j}$ is greater than exercise payoff Z_{t_j}. We can easily replace $\Delta C^{n,*}_{t_j}$ with $\Delta C^n_{t_j}$ to yield a lower bound. Precisely, the approximate lower exercise strategy $\pi^{n,[m]}$ can be expressed as:

$$\tau^{i,n,[m]}_j \triangleq \min \left\{ t_{L-i+1} \geq t_j > \tau^{i+1,n,*}_j \,\middle|\, \Delta C^{i,[m]}_{t_j} \leq Z_{t_k} \right\},\ i = n, \cdots, 1, \quad (3.6)$$

with assuming $\tau^{n+1,n,[m]}_j = t_{j-1}$, and its corresponding lower bound $L^{n,[m]}_0$ is

$$L^{n,[m]}_{t_j} \triangleq \mathbb{E}\left[\sum_{i=1}^{n} Z_{\tau^{i,n,[m]}_j} \,\middle|\, \mathcal{F}_{t_j} \right]. \quad (3.7)$$

Remark 3.5. Compared to the classic Least-Squares lower bound, our method improves the tightness of lower bound in two aspects. Primarily, when iteratively running Least-Squares strategy construction algorithm to the marginal value, the iterative exercise payoff is the minimal of the exercise payoff and the previous iterative marginal continuation value, which is bounded by the exercise payoff, instead of the exercise payoff plus the previous iterative continuation, which is unbounded. The bounded error will accumulate much slower in the iterative process than the unbounded error. Moreover, as our method approximates marginal continuation values directly, which can be used directly to yield a sub-optimal exercise strategy, the error will be introduced only one time, while the classic Least-Squares method will introduce the error from both two continuation values and the error from computing their difference.

4. Upper Bounds

After discussing how to construct exercise strategies for buyers, we now focus on how to help sellers build hedging strategy. In this section, we will first review the dual representation of the marginal value, and then put forward our non-nested upper bound method.

4.1. Dual representation for single stopping problem. Here, we briefly review the dual representation of single stopping problem, which we discussed in chapter 3. Assuming that M_t is a martingale with $M_0 = 0$, we have:

$$\begin{aligned} V^*_{t_j} &= \sup_{t_j \le \tau \le t_n} \mathbb{E}\left[Z_\tau \middle| \mathcal{F}_{t_j}\right] \\ &= \sup_{t_j \le \tau \le t_n} \mathbb{E}\left[Z_\tau - \left(M_\tau - M_{t_j}\right) \middle| \mathcal{F}_{t_j}\right] \\ &\le \mathbb{E}\left[\max_{j \le i \le L} \left(Z_{t_i} - \left(M_{t_i} - M_{t_j}\right)\right)\right]. \end{aligned} \tag{4.1}$$

The equality in (4.1) is achieved, if M_t is the martingale part of $V^*_{t_j}$'s Doob decomposition. That is:

$$V^*_{t_j} = V^*_{t_0} + M^*_{t_j} - A^*_{t_j},$$

where $M^*_{t_j}$ is a martingale with $M^*_{t_0} = 0$, and $A^*_{t_j}$ is a previsible non-decreasing process with with $A^*_{t_0} = 0$. Precisely, they are of following forms:

$$M^*_{t_{j+1}} - M^*_{t_j} \triangleq V^*_{t_{j+1}} - \mathbb{E}\left[V^*_{t_{j+1}} \middle| \mathcal{F}_{t_j}\right], \ A^*_{t_{j+1}} - A^*_{t_j} \triangleq V^*_{t_j} - \mathbb{E}\left[V^*_{t_{j+1}} \middle| \mathcal{F}_{t_j}\right].$$

We also have:

$$V^*_{t_j} = \max_{j \le i \le L} \left(Z_{t_i} - \left(M^*_{t_i} - M^*_{t_j}\right)\right) \ a.s.. \tag{4.2}$$

4.2. Dual representation and sub-simulation upper bounds. In this part, we will review the marginal dual representation for multiple stopping problems, and show the sub-simulation upper bound method for multiple exercise options, which is an extension of single stopping primal-dual upper bound method. The following theorem, put forward by Meinshausen and Hambly [18], gives the duality expression of the marginal value $\Delta V^{n,*}_{t_j}$.

Theorem 4.1. *Suppose π^n is an exercise strategy with n remaining rights, and $\left\{M_{t_j}\right\}$ is a martingale starting from 0. Then the optimal marginal value $\Delta V^{n,*}_{t_j}$ is obtained from the following minimization:*

$$\Delta V^{n,*}_{t_j} = \inf_{\pi^{n-1}} \inf_{M} \mathbb{E}\left[\max_{t_k \ge t_j \text{ and } t_k \ne \tau^{i,n-1},\ i=1,\cdots,n-1} \left(Z_{t_k} - M_{t_k}\right) \middle| \mathcal{F}_{t_j}\right].$$

Note the equality is achieved when π^n *is the optimal exercise strategy and* $\{M_{t_j}\}$ *is characterized by martingale increments of optimal marginal value processes:*

$$M^*_{t_{j+1}} - M^*_{t_j} = \Delta V^{l+1,*}_{t_{j+1}} - \Delta C^{l+1,*}_{t_j}, \text{ if } \tau^{l+1,n,*} < t_{j+1} <= \tau^{l,n,*},$$

where we assume $\tau^{n+1,n,*} = 0$.

For the proof of theorem 4.1, please refer to [18]. From theorem 4.1, we know, for any given exercise strategy π^n and for any given martingale process M, we can compute upper bound $\Delta V^{n,\pi^n,M}_{t_{j+1}}$ as:

$$\Delta U^{n,\pi^n,M}_{t_j} = \mathbb{E}\left[\max_{t_k \geq t_j \text{ and } t_k \neq \tau^{i,n-1},\ i=1,\cdots,n-1} (Z_{t_k} - M_{t_k}) \middle| \mathcal{F}_{t_j} \right]. \tag{4.3}$$

Next, we look at how to compute upper bounds through sub-simulations. Without loss of generality, suppose exercise strategies for different numbers of rights have been obtained. Then the difficulty will be about the computation of inputted martingales. Here, we refer to the idea in the primal-dual upper bound method, where martingale components of lower bound processes separated by sub-simulation are used as replacements of optimal martingale processes. Denote the sub-optimal exercised strategy we have obtained at t_{j+1} by π^n_{j+1}. Starting from state W_{t_j}, we simulate N independent paths denoted as:

$$\left(W^{(k)}_{t_j}, \cdots, W^{(k)}_{t_j}\right)_{k=1,\cdots,N}.$$

The Monte Carlo approximation of the continuation value is computed as:

$$\tilde{C}^n_{t_j} \triangleq \frac{1}{N} \sum_{k=1}^{N} \left(\sum_{i=1}^{n} Z_{\tau^{i,n}_{j+1}} \right).$$

The Monte Carlo approximation of the option value is expressed:

$$\tilde{V}^n_{t_j} = \left(Z_j + \tilde{C}^{n-1}_{t_j}\right) 1_{\left\{\tau^{n,n}_j = t_j\right\}} + \tilde{C}^n_{t_j} 1_{\left\{\tau^{n,n}_j != t_j\right\}}.$$

The Monte Carlo approximation of the martingale process is expressed as:

$$\Delta \tilde{V}^m_{t_{j+1}} - \Delta \tilde{C}^m_{t_j} = \tilde{V}^n_{t_{j+1}} - \tilde{V}^{n-1}_{t_{j+1}} - \tilde{C}^n_{t_j} - \tilde{C}^{n-1}_{t_j}.$$

Then the sub-simulation upper bound is just an Monte Carlo estimator of (4.3).

4.3. Non-nested upper bounds. Upper bound method is initially used to judge the goodness of lower bounds, but nowadays industry is seeking to apply upper bound methods in locating the range of the optimal price and constructing hedging portfolios, which requires upper bound methods to be of both high time-efficiency and good accuracy. In this part, we propose our non-nested upper bounds to meet such requirements. The intuitive idea is to approximate the martingale component of the lower bound by orthogonal projection.

Let $\Delta C^{n-1}_{t_j}$, $j \leq L - n + 1$ be the approximate marginal continuation values with $n-1$ rights, and τ^n_j, $j \leq L - n + 1$ be stopping times corresponding to a given exercise strategy with n rights. Then define a martingale process $M_t^{n,L}$ by following increments:

$$M_{t_{j+1}}^{n,\mathcal{L}} - M_{t_j}^{n,\mathcal{L}} = \mathbb{E}\left[\min\left\{Z_{\tau_{j+1}^n}, \Delta C_{\tau_{j+1}^n}^{n-1}\right\}\middle|\mathcal{F}_{t_{j+1}}\right] - \mathbb{E}\left[\min\left\{Z_{\tau_{j+1}^n}, \Delta C_{\tau_{j+1}^n}^{n-1}\right\}\middle|\mathcal{F}_{t_j}\right]. \tag{4.4}$$

This is the martingale component of the lower bound process with the payoff format $\min\left\{Z, \Delta C^{n-1}\right\}$ and the exercise strategy τ_{j+1}^n, which can be viewed as an approximation to $\Delta V_{t_{j+1}}^{n,*} - \Delta C_{t_j}^{n,*}$.

Next, we formulate the orthogonal project of $M_{t_{j+1}}^{n,\mathcal{L}} - M_{t_j}^{n,\mathcal{L}}$. For any $t_j,\ j = 0, ..., L-n+1$, there are m basis functions $x_{j+1,k}^n,\ k = 1, \cdots, m$, $\mathcal{F}_{t_{j+1}}$-measurable, satisfying following conditions:

$$\forall m \geq 1,\ \sum_{k=1}^{m} a_k x_{j+1,k}^n = 0\ a.s. \implies a_k = 0,\ k = 1, \cdots, m; \tag{4.5}$$

$$\mathbb{E}\left[x_{j+1,k}^n \middle| \mathcal{F}_{t_j}\right] = 0,\ k = 1, \cdots, m. \tag{4.6}$$

Denote the orthogonal projection of the martingale increment into the space generated by these basis functions by $Q_j^{n,[m]}(M_{t_{j+1}}^{n,\mathcal{L}} - M_{t_j}^{n,\mathcal{L}})$. Denote the m-dimensional row vector $\begin{bmatrix} x_{j+1,1}^n & x_{j+1,2}^n & \cdots & x_{j+1,m}^n \end{bmatrix}$ by $X_{j+1}^{n,[m]}$. Define the m-dimensional column vector $\alpha_j^{n,[m]} = \begin{bmatrix} \alpha_{j,1}^n & \alpha_{j,2}^n & \cdots & \alpha_{j,m}^n \end{bmatrix}^T$ to be the unique solution of:

$$Q_j^{n,[m]}\left(M_{t_{j+1}}^{n,\mathcal{L}} - M_{t_j}^{n,\mathcal{L}}\right) = X_{j+1}^{n,[m]} a_j^{n,[m]}.$$

The coefficients α_j^m minimize the following objective function:

$$\min_{a_j^{[m]}} \mathbb{E}\left[\left(M_{t_{j+1}}^{n,\mathcal{L}} - M_{t_j}^{n,\mathcal{L}} - X_{j+1}^{n,[m]} a_j^{[m]}\right)^2\right]. \tag{4.7}$$

The condition (4.5) guarantees the absence of multicollinearity, and make sure that the coefficients are unique. Precisely, the solutions of the minimization problem are obtained by taking the first order derivatives of objective function and given by:

$$\alpha_j^{n,[m]} = \mathbb{E}\left[{X_{j+1}^{n,[m]}}^T X_{j+1}^{n,[m]}\right]^{-1} \mathbb{E}\left[{X_{j+1}^{n,[m]}}^T \left(M_{t_{j+1}}^{n,\mathcal{L}} - M_{t_j}^{n,\mathcal{L}}\right)\right],$$

and its corresponding martingale increment is denoted by:

$$H_{t_{j+1}}^{n,[m]}\left(M^{n,\mathcal{L}}\right) = X_{j+1}^{n,[m]} a_j^{n,[m]} = Q_j^{n,[m]}\left(M_{t_{j+1}}^{n,\mathcal{L}} - M_{t_j}^{n,\mathcal{L}}\right). \tag{4.8}$$

Repeating this procedure for every j, we can approximate the martingale component of the optimal marginal value process $\Delta V_{t_j}^{n,*}$. We further repeat this process for every n as well, then we can obtain all martingale increments that are used along with the exercise strategy π^n to generate a hedge for upper bound computation:

$$H_{t_{j+1}}^{[m],\pi^n}\left(M^{\mathcal{L}}\right) = \Delta H_{t_{j+1}}^{l+1,[m]},\ \text{if}\ \tau^{l+1,n} < t_{j+1} <= \tau^{l,n}. \tag{4.9}$$

Before considering how to implement our method by Monte Carlo, there still exists an issue that $M^{n,\mathcal{L}}_{t_{j+1}} - M^{n,\mathcal{L}}_{t_j}$ is a conditional expectation which is not available directly. To deal with this problem, we transform our minimization problem without changing the solution.

Lemma 4.2. *The minimization problem (4.7) has the same solution as:*

$$\min_{a_j^{n,[m]}} \mathbb{E}\left[\left(\min\left\{Z_{\tau^n_{j+1}}, \Delta C^{n-1}_{\tau^n_{j+1}}\right\} - \Delta C^n_{t_j} - X^{n,[m]}_{j+1} a^{n,[m]}_j\right)^2\right].$$

Proof. Denote following terms:

$$\begin{aligned}
e_{t_{j+1}} &\triangleq \min\left\{Z_{\tau^n_{j+1}}, \Delta C^{n-1}_{\tau^n_{j+1}}\right\} - \mathbb{E}\left[\min\left\{Z_{\tau^n_{j+1}}, \Delta C^{n-1}_{\tau^n_{j+1}}\right\}\middle|\mathcal{F}_{t_{j+1}}\right],\\
D_{t_j} &\triangleq \Delta V^n_{t_j} - \mathbb{E}\left[\min\left\{Z_{\tau^n_{j+1}}, \Delta C^{n-1}_{\tau^n_{j+1}}\right\}\middle|\mathcal{F}_{t_j}\right],\\
R_{t_{j+1}} &\triangleq \min\left\{Z_{\tau^n_{j+1}}, \Delta C^{n-1}_{\tau^n_{j+1}}\right\} - \Delta V^n_{t_j},
\end{aligned}$$

where $e_{t_{j+1}}$ and $R_{t_{j+1}}$ are $\mathcal{F}_{t_{j+1}}$-measurable, and D_{t_j} is $\mathcal{F}_{t_j}$-measurable. Then we have:

$$\begin{aligned}
&\mathbb{E}\left[\left(M^{n,\mathcal{L}}_{t_{j+1}} - M^{n,\mathcal{L}}_{t_j} - X^{n,[m]}_{j+1} a^{n,[m]}_j\right)^2\right]\\
=&\mathbb{E}\left[\left(R_{t_{j+1}} - X^{n,[m]}_{j+1} a^{n,[m]}_j\right)^2\right] + \mathbb{E}\left[\left(D_{t_j} - e_{t_{j+1}}\right)\left(D_{t_j} - e_{t_{j+1}} - 2R_{t_{j+1}}\right)\right]\\
&+ 2\mathbb{E}\left[\left(D_{t_j} - e_{t_{j+1}}\right) X^{n,[m]}_{j+1} a^{n,[m]}_j\right].
\end{aligned}$$

As $\mathbb{E}\left[\left(D_{t_j} - e_{t_{j+1}}\right)\left(D_{t_j} - e_{t_{j+1}} - 2R_{t_{j+1}}\right)\right]$ is independent with coefficients $a^{n,[m]}_j$, we focus on the last term $\mathbb{E}\left[\left(D_{t_j} - e_{t_{j+1}}\right) X^{n,[m]}_{j+1} a^{n,[m]}_j\right]$, which follows

$$\mathbb{E}\left[\left(D_{t_j} - e_{t_{j+1}}\right) X^{n,[m]}_{j} a^{n,[m]}_j\right] = \mathbb{E}\left[D_{t_j} 0\right] - \mathbb{E}\left[0 X^{n,[m]}_{j+1} a^{n,[m]}_j\right] = 0. \tag{4.10}$$

So $\mathbb{E}\left[\left(D_{t_j} - e_{t_{j+1}}\right) X^{n,[m]}_{j+1} a^{n,[m]}_j\right]$ is independent with coefficients $a^{n,[m]}_j$ as well, which completes the proof. $\square$

Now, we look at the detailed Monte Carlo algorithm. As usual, assume N path are simulated and denoted as:

$$\left(W^{(i)}_0 = W_0, W^{(i)}_{t_1}, \cdots, W^{(i)}_{t_j}, \cdots, W^{(i)}_{t_j}\right)_{i=0,\cdots,N}.$$

Then we have the following pathwise indicators and pathwise payoffs:

$$\begin{aligned}
E^{n,(i)}_{t_j} &= 1_{\left\{\tau^{n,(i)}_j = t_j\right\}}, \ j = 1, \cdots, L-n+1, \ i = 1, \cdots, N;\\
U^{n,(i)}_{t_j} &= \min\left\{Z^{(i)}_{t_j}, \Delta C^{n-1,(i)}_{t_j}\right\}, \ j = 1, \cdots, L-n+1, \ i = 1, \cdots, N.
\end{aligned}$$

For each path, evolve pathwise values of $\min\left\{Z^{(i)}_{\tau^n_j}, \Delta C^{n-1,(i)}_{\tau^n_j}\right\}$ backwards as follows:

$$\min\left\{Z^{(i)}_{\tau^n_j}, \Delta C^{n-1,(i)}_{\tau^n_j}\right\} = \begin{cases} E^{n,(i)}_{t_j} U^{n,(i)}_{t_j} + (1 - E^{n,(i)}_{t_j}) \\ \qquad \min\left\{Z^{(i)}_{\tau^n_{j+1}}, \Delta C^{n-1,(i)}_{\tau^n_{j+1}}\right\}, & , j = 1, \cdots, L-1, \\ U^{n,(i)}_{t_L - n+1}, j = L - n + 1. \end{cases}$$

Now all information required for conducting Least Squares regression has been obtained, thus the coefficients can be estimated by solving the following regression:

$$\min_{a^{[m]}_j} \sum_{i=1}^{N} \left(\min\left\{Z^{(i)}_{\tau^n_{j+1}}, \Delta C^{n-1,(i)}_{\tau^n_j}\right\} - \Delta C^{n,(i)}_{\tau^n_{j+1}} - X^{n,[m],(i)}_{j+1} a^{[m]}_j \right)^2,$$

whose explicit solution is as:

$$\begin{aligned} \hat{\alpha}^{n,[m],N}_j &= \left(\sum_{i=1}^{N} X^{n,[m],(i)}_{j+1}{}^T X^{n,[m],(i)}_{j+1} \right)^{-1} \\ &\quad \times \left(\sum_{i=1}^{N} X^{n,[m],(i)}_{j+1}{}^T \left(\min\left\{Z^{(i)}_{\tau^n_{j+1}}, \Delta C^{n-1,(i)}_{\tau^n_j}\right\} - \Delta C^{n,(i)}_{\tau^n_{j+1}} \right) \right). \end{aligned}$$

After repeating this procedure for each time frame (t_j, t_{j+1}), and each number of remaining rights n, we have everything required to compute hedge martingales without sub-simulation.

Remark 4.3. In the above construction, $M^{n,\mathcal{L}}_{t_{j+1}} - M^{n,\mathcal{L}}_{t_j}$ is not accessible, and then replaced by $M^{n,\mathcal{L}}_{t_{j+1}} - M^{n,\mathcal{L}}_{t_j} + e_{t_{j+1}} - D_{t_j}$. Note that $e_{t_{j+1}}$ carries information after time t_{j+1}, which, for regression, is pure noise along with pathwise values, and D_{t_j} represents the difference between the approximate continuation value and the optimal continuation value, which only depends on the information up to time t_j. Therefore both $e_{t_{j+1}}$ and D_{t_j} are irrelevant with basis functions $X^{n,[m]}_{j+1}$, since $X^{n,[m]}_{j+1}$ are proxy of information flowing from t_j to t_{j+1}. In other words, basis functions $X^{n,[m]}_{j+1}$ are used to filter the change not driven by information between t_j and t_{j+1}, and the rest is exactly the martingale increment $M^{n,\mathcal{L}}_{t_{j+1}} - M^{n,\mathcal{L}}_{t_j}$.

Remark 4.4. From (4.10), we know that both $e_{t_{j+1}}$ and D_{t_j} won't affect coefficients $\alpha^{n,[m]}_j$. However, with limited number of simulated paths, their effect cannot be ignored, and can be formulated as:

$$\left(\frac{1}{N} \sum_{i=1}^{N} X^{n,[m],(i)}_{j+1}{}^T X^{n,[m],(i)}_{j+1} \right)^{-1} \left(\frac{1}{N} \sum_{i=1}^{N} X^{n,[m],(i)}_{j+1}{}^T \left(e_{t_{j+1},(i)} - D_{t_j,(i)} \right) \right).$$

This effect will vanish as more paths are used, and this follows the Law of Large Numbers as:

$$\frac{1}{N} \sum_{i=1}^{N} X^{n,[m],(i)}_{j+1}{}^T \left(e_{t_{j+1},(i)} - D_{t_j,(i)} \right) \to \mathbb{E}\left[X^{n,[m]}_{j+1} \left(e_{t_{j+1}} - D_{t_j} \right) \right] = 0 \ a.s..$$

The convergence speed is judged by the the variance of $X_{j+1}^{n,[m]}\left(e_{t_{j+1}} - D_{t_j}\right)$. Thus better approximation of the marginal continuation value will help improve the convergence of coefficients. Meanwhile, feasible variance reduction techniques can also help accelerate the convergence (See [13] for detail). Fox example, using hedged pathwise value, whose variance is smaller, is a good choice. Cheng and Joshi [10] suggested to do this by using a small number of paths to construct an initial hedge.

5. Convergence

In this section, we will prove that both our marginal-based lower bounds and non-nested upper bounds converge to the optimal value. As both upper bound and lower bound methods rely on iterative construction along with the number of exercise rights, the logic behind these proof is to build iterative convergences of marginal continuation values and marginal martingale increments.

5.1. Convergence of lower bounds. Here we will prove that our lower bound will converge to the optimal price under L^2-measure. To simplify derivation, we introduce following lemmas before presenting the main result.

Lemma 5.1. *Given basis functions of the continuation value satisfying conditions (3.3) and (3.4), we have*

$$\lim_{m\to\infty} L_{t_j}^{1,[m]} = V_{t_j}^{1,*} \text{ in } L^2.$$

Regarding the proof of lemma 5.1, please see [11] for detail.

Lemma 5.2. *Suppose X_n is a sequence of random variables converging to X in L^2, and $P^{[m]}$ is the orthogonal projection generated by m basis functions. Then we have:*

$$\lim_{n\to\infty} P^{[m]}\left(X_n\right) = P^{[m]}\left(X\right).$$

This lemma is implied by $\mathbb{E}\left[\left(P^{[m]}\left(X_n\right) - P^{[m]}\left(X\right)\right)^2\right] \leq \mathbb{E}\left[\left(X_n - X\right)^2\right]$.

Lemma 5.3. *Assuming basis functions of the continuation value satisfy conditions (3.3) and (3.4), we have:*

$$\lim_{m\to\infty} \Delta C_{t_j}^{1,[m]} = \Delta C_{t_j}^{1,*} \text{ in } L^2.$$

Proof. Since $P_j^{[m],1}\left(Z_{\tau_{j+1}^{1,[m]}}\right) = P_j^{[m],1}\left(\mathbb{E}\left[L_{t_{j+1}}^{1,[m]}\middle|\mathcal{F}_{t_j}\right]\right)$, we have:

$$\begin{aligned}
&\left|\Delta C_{t_j}^{1,[m]} - \Delta C_{t_j}^{1,*}\right| \\
&= \left|P_j^{[m],1}\left(\mathbb{E}\left[L_{t_{j+1}}^{1,[m]}\middle|\mathcal{F}_{t_j}\right]\right) - C_{t_j}^{1,*}\right| \\
&\leq \left|P_j^{[m],1}\left(\mathbb{E}\left[L_{t_{j+1}}^{1,[m]}\middle|\mathcal{F}_{t_j}\right]\right) - \mathbb{E}\left[L_{t_{j+1}}^{1,[m]}\middle|\mathcal{F}_{t_j}\right]\right| + \left|\mathbb{E}\left[L_{t_{j+1}}^{1,[m]} - V_{t_{j+1}}^{1,*}\middle|\mathcal{F}_{t_j}\right]\right|.
\end{aligned}$$

The first term converges to 0 due to the assumption that basis functions are total, and the zero convergence of the second term comes from lemma 5.1, so the proof is completed. ∎

Remark 5.4. Lemma 5.3 can be generalized to the fact that, for any form of exercise payoff, e.g. $\min\left\{Z_{\tau_{j+1}^{1,1,[m]}}, \Delta C^{n-1,[m]}_{\tau_{j+1}^{1,1,[m]}}\right\}$, the corresponding approximate continuation converges to its optimal continuation value under condition that basis functions are total.

Lemma 5.5. *Suppose Z_{t_j} and Y_{t_j} are two different forms of payoff, then the difference between their optimal option prices is bounded as follows:*

$$\left|\sup_{t_j\le\tau\le t_L}\mathbb{E}\left[Z_\tau\middle|\mathcal{F}_{t_j}\right]-\sup_{t_j\le\tau\le t_L}\mathbb{E}\left[Y_\tau\middle|\mathcal{F}_{t_j}\right]\right|\le\sum_{l=j}^{L}\mathbb{E}\left[\left|Z_{t_l}-Y_{t_l}\right|\middle|\mathcal{F}_{t_j}\right].$$

Proof. Lemma 5.5 obviously holds for $j = L$. Now, we look at j with assuming the result is true for $j + 1$:

$$\begin{aligned}&\left|\sup_{t_j\le\tau\le t_L}\mathbb{E}\left[Z_\tau\middle|\mathcal{F}_{t_j}\right]-\sup_{t_j\le\tau\le t_L}\mathbb{E}\left[Y_\tau\middle|\mathcal{F}_{t_j}\right]\right|\\ &\le\left|Z_{t_j}-Z_{t_j}\right|+\mathbb{E}\left[\left|\sup_{t_{j+1}\le\tau\le t_L}\mathbb{E}\left[Z_\tau\middle|\mathcal{F}_{t_j}\right]-\sup_{t_{j+1}\le\tau\le t_L}\mathbb{E}\left[Y_\tau\middle|\mathcal{F}_{t_j}\right]\right|\middle|\mathcal{F}_{t_j}\right].\end{aligned}$$

By mathematical induction, the proof is completed. □

Lemma 5.6. *Given the basis functions of continuation value satisfying conditions (3.3) and (3.4), we have, for $n > 1$,*

$$\lim_{m\to\infty}\Delta C^{n,[m]}_{t_j}=\Delta C^{n,*}_{t_j}\ \textit{in}\ L^2.$$

Proof. The proof is conducted by induction on n. From lemma 5.1. we know that, the result holds for $n = 1$. Assuming this is true for $n - 1$, we now proceed to n. Denoting the number of basis functions for n-th marginal continuation value by $m^{'}$, and the maximal number of basis functions for previous $n - 1$ marginal continuation values by m, we have:

$$\begin{aligned}&\left|\Delta C^{n,\left[m^{'}\right]}_{t_j}-\Delta C^{n,*}_{t_j}\right|\\ &\le\left|\Delta C^{n,\left[m^{'}\right]}_{t_j}-\mathbb{E}\left[\sup_{t_{j+1}<\tau\le t_{L-n+1}}\mathbb{E}\left[\min\left\{Z_\tau,\Delta C^{n-1,[m]}_\tau\right\}\middle|\mathcal{F}_{t_{j+1}}\right]\middle|\mathcal{F}_{t_j}\right]\right|\\ &\quad+\left|\mathbb{E}\left[\sup_{t_{j+1}<\tau\le t_{L-n+1}}\mathbb{E}\left[\min\left\{Z_\tau,\Delta C^{n-1,[m]}_\tau\right\}\middle|\mathcal{F}_{t_{j+1}}\right]\middle|\mathcal{F}_{t_j}\right]-\mathbb{E}\left[\Delta V^{n,*}_{t_{j+1}}\middle|\mathcal{F}_{t_j}\right]\right|.\end{aligned}$$

Then we focus on the second term, which, by lemma 5.5, follows:

$$\begin{aligned}&\left|\mathbb{E}\left[\sup_{t_{j+1}<\tau\le t_{L-n+1}}\mathbb{E}\left[\min\left\{Z_\tau,\Delta C^{n-1,[m]}_\tau\right\}\middle|\mathcal{F}_{t_{j+1}}\right]\middle|\mathcal{F}_{t_j}\right]-\mathbb{E}\left[\Delta V^{n,*}_{t_{j+1}}\middle|\mathcal{F}_{t_j}\right]\right|\\ &\le\sum_{l=j+1}^{L-n+1}\mathbb{E}\left[\left|\Delta C^{n-1,[m]}_{t_l}-\Delta C^{n-1,*}_{t_l}\right|\middle|\mathcal{F}_{t_j}\right].\end{aligned}$$

By mathematical induction, the proof is completed. □

Lemma 5.7. *An exercise strategy π^n is generated by approximating marginal continuation values $\left\{\Delta C_{t_l}^{i,[m]}\right\}_{m>0}$ in recursive form defined as (3.6), and its corresponding lower bound $L_{t_j}^{n,[m]}$ is defined as (3.7). If $\Delta C_{t_l}^{i,[m]}$ converges to $\Delta C_{t_l}^{i,*}$ in L^2 for any $0 < i \leq n$ and $0 < l \leq L - i$, we have:*

$$\lim_{m\to\infty} L_{t_j}^{n,[m]} = V_{t_j}^{n,*} \text{ in } L^2.$$

Proof. The proof is conducted by induction on t_j and n. The lower bound can be expressed as follows:

$$L_{t_j}^{n,[m]} = 1_{\left\{\tau_j^{n,n,[m]}=t_j\right\}} \left(Z_{t_j} + \mathbb{E}\left[L_{t_{j+1}}^{n-1,[m]} \middle| \mathcal{F}_{t_j}\right]\right) + 1_{\left\{\tau_j^{n,n,[m]}>t_j\right\}} \mathbb{E}\left[L_{t_{j+1}}^{n,[m]} \middle| \mathcal{F}_{t_j}\right].$$

Expressing $V_{t_j}^{n,*}$ in a similar recursive form, we have:

$$\begin{aligned}
L_{t_j}^{n,[m]} - V_{t_j}^{n,*} = &\left(Z_{t_j} + \mathbb{E}\left[V_{t_{j+1}}^{n-1,*} - V_{t_{j+1}}^{n,*} \middle| \mathcal{F}_{t_j}\right]\right)\left(1_{\left\{\tau_j^{n,n,[m]}=t_j\right\}} - 1_{\left\{\tau_j^{n,n,*}=t_j\right\}}\right) \\
&- 1_{\left\{\tau_j^{n,n,[m]}=t_j\right\}} \mathbb{E}\left[L_{t_{j+1}}^{n-1,[m]} - V_{t_{j+1}}^{n-1,*} \middle| \mathcal{F}_{t_j}\right] \\
&+ 1_{\left\{\tau_j^{n,n,[m]}>t_j\right\}} \mathbb{E}\left[L_{t_{j+1}}^{n,[m]} - V_{t_{j+1}}^{n,*} \middle| \mathcal{F}_{t_j}\right].
\end{aligned}$$

By inductive assumption, the last two terms of the right side converge to 0, so we focus on the first term as follows:

$$\begin{aligned}
&\left|\left(Z_{t_j} + \mathbb{E}\left[V_{t_{j+1}}^{n-1,*} - V_{t_{j+1}}^{n,*} \middle| \mathcal{F}_{t_j}\right]\right)\left(1_{\left\{\tau_j^{n,n,[m]}=t_j\right\}} - 1_{\left\{\tau_j^{n,n,*}=t_j\right\}}\right)\right| \\
&= \left|\left(Z_{t_j} - \Delta C_{t_j}^{n,*}\right)\right| \left|\left(1_{\left\{\Delta C_{t_j}^{n,[m]} \leq Z_{t_j} < \Delta C_{t_j}^{n,*}\right\}} - 1_{\left\{\Delta C_{t_j}^{n,[m]} > Z_{t_j} \geq \Delta C_{t_j}^{n,*}\right\}}\right)\right| \\
&\leq \left|\Delta C_{t_j}^{n,[m]} - \Delta C_{t_j}^{n,*}\right|.
\end{aligned}$$

This term converges to 0 due to lemma 5.6, which therefore completes the proof. □

Theorem 5.8. *Suppose that basis functions of the continuation value satisfy conditions (3.3) and (3.4), and that lower bound $L_{t_j}^{n,[m]}$ is defined as (3.7). Then we have:*

$$\lim_{m\to\infty} L_{t_j}^{n,[m]} = V_{t_j}^{n,*} \text{ in } L^2.$$

The proof of theorem 5.8 is a combination of the convergence of approximate marginal continuation values and the convergence of corresponding lower bounds, which are implied by lemma 5.6 and 5.7 respectively.

5.2. Convergence of upper bounds. In this part, we will prove the convergence of the upper bound method. To simplify the proof, for any given approximate marginal values $\Delta C_{t_j}^{n,[m]}$ and its corresponding stopping time $\tau_j^{n,[m]}$ generated as (3.5), we define following terms:

$$\Delta L_{t_j}^{n,[m]} \triangleq \mathbb{E}\left[\min\left\{Z_{\tau_j^{n,[m]}}, \Delta C_{\tau_j^{n,[m]}}^{n-1,[m]}\right\} \middle| \mathcal{F}_{t_j}\right];$$

$$M^{n,\mathcal{L},[m]}_{t_{j+1}} - M^{n,\mathcal{L},[m]}_{t_j} \triangleq \Delta L^{n,[m]}_{t_j} - \mathbb{E}\left[\Delta L^{n,[m]}_{t_j} \middle| \mathcal{F}_{t_j}\right].$$

Given strategy π^{n-1}_j at time t_j and a martingale process M, define the pathwise upper bound as:

$$G^n_{t_j}\left(\pi^{n-1}, M\right) \triangleq \max_{t_{L-n+1}\geq t_k \geq t_j \text{ and } t_k \neq \tau^{i,n-1}_j,\ i=1,\cdots,n-1} \left(Z_{t_k} - M_{t_k}\right).$$

Lemma 5.9. *Assuming M and $M^{'}$ are two martingales, for any given strategy π^{n-1}, for any j, we have:*

$$\left|G^n_{t_j}\left(\pi^{n-1}, M\right) - G^n_{t_j}\left(\pi^{n-1}, M^{'}\right)\right| \leq \sum_{k=j}^{L-1} \left|\left(M_{t_{k+1}} - M_{t_k}\right) - \left(M^{'}_{t_{k+1}} - M^{'}_{t_k}\right)\right|.$$

The proof is straightforward by mathematical induction.

Lemma 5.10. *For every n, there are a sequence of approximate marginal continuation values $\left\{\Delta C^{i,[m_c]}_{t_l}\right\}_{m_c>0, 0<i<n}$, which are indexed by m_c and satisfy condition that, for $i>0$,*

$$\lim_{m_c\to\infty} \Delta C^{i,[m_c]}_{t_j} = \Delta C^{i,*}_{t_j} \text{ in } L^2, \tag{5.1}$$

and basis functions for martingale increment, which are indexed by m_m and satisfy conditions (4.5) and (4.6). Then we have:

$$\lim_{m_c\to\infty}\lim_{m_m\to\infty} H^{n,[m_m]}_{t_{j+1}}\left(M^{n,\mathcal{L},[m]}\right) = \Delta V^{n,*}_{t_{j+1}} - \Delta C^{n,*}_{t_j} \text{ in } L^2.$$

Proof. We have:

$$\begin{aligned}
&\left|H^{n,[m_m]}_{t_{j+1}}\left(M^{n,\mathcal{L},[m]}\right) - \left(\Delta V^{n,*}_{t_{j+1}} - \Delta C^{n,*}_{t_j}\right)\right| \\
\leq &\left|Q^{n,[m]}_j\left(M^{n,\mathcal{L},[m]}_{t_{j+1}} - M^{n,\mathcal{L},[m]}_{t_j}\right) - \left(M^{n,\mathcal{L},[m]}_{t_{j+1}} - M^{n,\mathcal{L},[m]}_{t_j}\right)\right| \\
&+ \left|\left(M^{n,\mathcal{L},[m]}_{t_{j+1}} - M^{n,\mathcal{L},[m]}_{t_j}\right) - \left(\Delta V^{n,*}_{t_{j+1}} - \Delta C^{n,*}_{t_j}\right)\right|.
\end{aligned}$$

The first term of the right side converges to 0 in L^2 due to condition (4.6), so we focus on the second term, whose second moment can be expressed as:

$$\begin{aligned}
&\mathbb{E}\left[\left(\left(M^{n,\mathcal{L},[m]}_{t_{j+1}} - M^{n,\mathcal{L},[m]}_{t_j}\right) - \left(\Delta V^{n,*}_{t_{j+1}} - \Delta C^{n,*}_{t_j}\right)\right)^2\right] \\
=&\mathbb{E}\left[\left(\left(\Delta L^{n,[m]}_{t_{j+1}} - \Delta V^{n,*}_{t_{j+1}}\right) - \mathbb{E}\left[\left(\Delta L^{n,[m]}_{t_{j+1}} - \Delta V^{n,*}_{t_{j+1}}\right)\middle|\mathcal{F}_{t_j}\right]\right)^2\right] \\
\leq&\mathbb{E}\left[\left(\Delta L^{n,[m]}_{t_{j+1}} - \Delta V^{n,*}_{t_{j+1}}\right)^2\right].
\end{aligned}$$

Now, it is enough to prove the convergence of $\Delta L^{n,[m]}_{t_{j+1}}$ to $\Delta V^{n,*}_{t_{j+1}}$ in L^2, which follows:

$$\begin{aligned}
&\left|\Delta L^{n,[m]}_{t_{j+1}} - \Delta V^{n,*}_{t_{j+1}}\right| \\
&\leq \left| \begin{array}{l} \mathbb{E}\left[\min\left\{Z_{\tau_j^{n,[m]}}, \Delta C^{n-1,[m]}_{\tau_j^{n,[m]}}\right\}\middle|\mathcal{F}_{t_j}\right] \\ \qquad - \sup\limits_{t_j<\tau\leq t_{L-n+1}} \mathbb{E}\left[\min\left\{Z_\tau, \Delta C^{n-1,[m]}_\tau\right\}\middle|\mathcal{F}_{t_j}\right] \end{array}\right| \\
&+ \left| \begin{array}{l} \sup\limits_{t_j<\tau\leq t_{L-n+1}} \mathbb{E}\left[\min\left\{Z_\tau, \Delta C^{n-1,[m]}_\tau\right\}\middle|\mathcal{F}_{t_j}\right] \\ \qquad - \sup\limits_{t_j<\tau\leq t_{L-n+1}} \mathbb{E}\left[\min\left\{Z_\tau, \Delta C^{n-1,*}_\tau\right\}\middle|\mathcal{F}_{t_j}\right] \end{array}\right|.
\end{aligned}$$

The first term of the right side converges to 0 by lemma 5.1, as well as the second term by condition (5.1) and lemma 5.5. Thus, the proof is completed. □

Lemma 5.11. *Let $M^{1,[m]}$ be an martingale satisfying the condition that:*

$$\lim_{m\to\infty}\left(M^{1,[m]}_{t_{j+1}} - M^{1,[m]}_{t_j}\right) = \Delta V^{1,*}_{t_{j+1}} - \Delta C^{1,*}_{t_j} \text{ in } L^2. \tag{5.2}$$

Then we have:

$$\lim_{m\to\infty}\max_{j\leq i\leq L}\left(Z_{t_i} - \left(M^{1,[m]}_{t_i} - M^{1,[m]}_{t_j}\right)\right) = \Delta V^{1,*}_{t_j} \text{ in } L^2.$$

Proof. From equation (4.2), there exists:

$$\Delta V^{1,*}_{t_j} = \max_{j\leq i\leq L}\left(Z_{t_i} - \sum_{k=j}^{i-1}\left(\Delta V^{1,*}_{t_{k+1}} - \Delta C^{1,*}_{t_k}\right)\right) \text{ a.s..}$$

Along with lemma 5.9, we have:

$$\begin{aligned}
&\left|\max_{j\leq i\leq L}\left(Z_{t_i} - \left(M^{1,[m]}_{t_i} - M^{1,[m]}_{t_j}\right)\right) - \Delta V^{1,*}_{t_j}\right| \\
&\leq \sum_{k=j}^{L-1}\left|\left(\left(M^{1,[m]}_{t_{j+1}} - M^{1,[m]}_{t_j}\right)\right) - \left(\Delta V^{1,*}_{t_{k+1}} - \Delta C^{1,*}_{t_k}\right)\right|.
\end{aligned}$$

The condition (5.2) will guarantee the convergence of right side towards 0. □

Lemma 5.12. *Given Z_{t_j} and Y_{t_j} are two different forms of payoff and M is a martingale process, we have:*

$$\left|\max_{L\geq k\geq j}\left(Z_{t_k} - \left(M_{t_k} - M_{t_j}\right)\right) - \max_{L\geq k\geq j}\left(Y_{t_k} - \left(M_{t_k} - M_{t_j}\right)\right)\right| \leq \sum_{k=j}^{L}\left|Z_{t_k} - Y_{t_k}\right|.$$

The proof is straight forward form mathematical induction.

Lemma 5.13. *For every n, there is a sequence of approximate marginal continuation values $\Delta C^{n-1,[m_c]}_{t_j}$, which are indexed by m_c and satisfy condition that:*

$$\lim_{m_c\to\infty}\Delta C^{n-1,[m_c]}_{t_j} = \Delta C^{n-1,*}_{t_j} \text{ in } L^2. \tag{5.3}$$

Meanwhile, there is a sequence of approximate marginal values $\Delta V_{t_j}^{n-1,[m_m]}$*, which are indexed by* m_m *and satisfy the condition that:*

$$\lim_{m_m\to\infty} \Delta V_{t_j}^{n-1,[m_m]} = \Delta V_{t_j}^{n-1,*} \ \text{in}\ L^2.$$

Then the following convergence holds:

$$\lim_{m_c\to\infty}\lim_{m_m\to\infty}\begin{pmatrix} Z_{t_j}1_{\left\{Z_{t_j}\geq \Delta C_{t_j}^{n-1,[m_c]}\right\}} \\ +\left(\Delta V_{t_j+1}^{n-1,[m_m]} - \left(\Delta V_{t_{j+1}}^{n-1,*} - \Delta C_{t_j}^{n-1,*}\right)\right)1_{\left\{Z_{t_j}<\Delta C_{t_j}^{n-1,[m_c]}\right\}}\end{pmatrix}$$
$$= \min\left\{Z_{t_j}, \Delta C_{t_j}^{n-1,*}\right\} \ \text{in}\ L^2.$$

Proof. We have:

$$\begin{vmatrix} Z_{t_j}1_{\left\{Z_{t_j}\geq \Delta C_{t_j}^{n,[m_c]}\right\}} + \left(\Delta V_{t_j+1}^{n-1,[m_m]} - \left(\Delta V_{t_{j+1}}^{n,*} - \Delta C_{t_j}^{n,*}\right)\right)1_{\left\{Z_{t_j}<\Delta C_{t_j}^{n,[m_c]}\right\}} \\ \quad - \min\left\{Z_{t_j}, \Delta C_{t_j}^{n-1,*}\right\}\end{vmatrix}$$
$$\leq \left|\left(Z_{t_j} - \Delta C_{t_j}^{n-1,*}\right)\left(1_{\left\{\Delta C_{t_j}^{n,[m_c]}\leq Z_{t_j}<\Delta C_{t_j}^{n-1,*}\right\}} - 1_{\left\{\Delta C_{t_j}^{n,[m_c]}> Z_{t_j}\geq\Delta C_{t_j}^{n-1,*}\right\}}\right)\right|$$
$$+ \left|\left(\Delta V_{t_j+1}^{n-1,[m_m]} - \Delta V_{t_{j+1}}^{n-1,*}\right)1_{\left\{Z_{t_j}<\Delta C_{t_j}^{n-1,[m_c]}\right\}}\right|.$$

The first term of the right side converges to 0 like we have proved before in lemma 5.7, while the second term converges to 0 due to condition (5.3). □

Lemma 5.14. *For every* n*, there is a sequence of approximate marginal continuation values* $\Delta C_{t_j}^{n-1,[m_c]}$*, which are indexed by* m_c *and satisfy the condition that:*

$$\lim_{m_c\to\infty} \Delta C_{t_j}^{n-1,[m_c]} = \Delta C_{t_j}^{n-1,*} \ \text{in}\ L^2.$$

Meanwhile, there is a sequence of approximate marginal values $\Delta V_{t_j}^{n-1,[m_m]}$*, which are indexed by* m_m *and satisfy the condition that:*

$$\lim_{m_c\to\infty} \Delta V_{t_j}^{n-1,[m_m]} = \Delta V_{t_j}^{n-1,*} \ \text{in}\ L^2.$$

Then the following holds:

$$\lim_{m_c\to\infty}\lim_{m_m\to\infty}\max_{j\leq i\leq L-n+1}\begin{pmatrix} Z_{t_i}1_{\left\{Z_{t_i}\geq \Delta C_{t_i}^{n,[m_c]}\right\}} \\ +\begin{pmatrix}\Delta V_{t_i+1}^{n-1,[m_m]} \\ -\left(\Delta V_{t_{i+1}}^{n,*} - \Delta C_{t_i}^{n,*}\right)\end{pmatrix}1_{\left\{Z_{t_i}<\Delta C_{t_i}^{n,[m_c]}\right\}} \\ -\sum_{k=j}^{i-1}\left(\Delta V_{t_{k+1}}^{n,*} - \Delta C_{t_k}^{n,*}\right)\end{pmatrix}$$
$$= \Delta V_{t_j}^{n,*} \ \text{in}\ L^2.$$

The proof is straightforward following the results in lemma 5.13.

Lemma 5.15. *For every n, there is a sequence of approximate marginal continuation values $\{\Delta C_{t_l}^{i,[m_c]}\}_{m_c>0,\ 0<i<n}$, which are indexed by m_c and satisfy the condition that, for $i>0$,*

$$\lim_{m_c\to\infty} \Delta C_{t_j}^{i,[m_c]} = \Delta C_{t_j}^{i,*} \text{ in } L^2. \tag{5.4}$$

and they generate exercise strategy $\pi^{n,[m_c]}$, i.e. $\tau_j^{i,n,[m_c]}$, $0<i\le n$. Denoting by $H^{*,\pi^{n-1,[m_c]}}$ the martingale synthesized as (4.9) using $\Delta V_{t_{j+1}}^{n,*} - \Delta C_{t_j}^{n*}$ and $\pi^{n-1,[m_c]}$, then the following holds:

$$\lim_{m_c\to\infty} G_{t_j}^n\left(\pi^{n-1,[m_c]}, H^{*,\pi^{n-1,[m_c]}}\right) = \Delta V_{t_{j+1}}^{n,*} \text{ in } L^2.$$

The proof follows directly from the results in lemma 5.12, lemma 5.13, and lemma 5.14.

Theorem 5.16. *For every n, we have a sequence of approximate marginal continuation values $\left\{\Delta C_{t_l}^{i,[m_c]}\right\}_{m_c>0,0<i<n}$, which are indexed by m_c and satisfy condition that, for $i>0$,*

$$\lim_{m_c\to\infty} \Delta C_{t_j}^{i,[m_c]} = \Delta C_{t_j}^{i,*} \text{ in } L^2.$$

and the basis functions for the martingale increment, which are indexed by m_m and satisfy conditions (4.5) and (4.6). Martingale increments $H_{t_{j+1}}^{n,[m_m]}\left(M^{n,\mathcal{L},[m_c]}\right)$ are computed from (4.4) and (4.8) with approximate marginal continuation values $\Delta C_{t_l}^{i,[m_c]}$ and the stopping time $\tau_{j+1}^{n,[m_c]}$ generated by $\Delta C_{t_l}^{i,[m_c]}$ as (3.5). Exercise strategy $\pi^{n,[m]}$, i.e. $\tau_j^{i,n,[m_c]}$ $0<i\le n$, generated by $\Delta C_{t_l}^{i,[m_c]}$ as (3.6), and martingale increments $H_{t_{j+1}}^{n,[m_m]}\left(M^{n,\mathcal{L},[m_c]}\right)$ are used to construct hedge processes in (4.9) as $H_{t_{j+1}}^{[m_m],\pi^{n,[m_c]}}\left(M^{\mathcal{L},[m_c]}\right)$, which is then used to compute upper bound defined according to (4.3) as $\Delta U_{t_j}^{n,\pi^{n,[m_c]},H^{[m_m],\pi^{n,[m_c]}}\left(M^{\mathcal{L},[m_c]}\right)}$. Then we have

$$\lim_{m_c\to\infty}\lim_{m_m\to\infty} \Delta U_{t_j}^{n,\pi^{n,[m_c]},H^{[m_m],\pi^{n,[m_c]}}\left(M^{\mathcal{L},[m_c]}\right)} = \Delta V_{t_j}^{n,*} \text{ in } L^2.$$

Proof. We have the following convergence:

$$\lim_{m_m\to\infty} \mathbb{E}\left[G_{t_j}^n\left(\pi^{n-1,[m_c]}, H^{*,\pi^{n-1,[m_c]}}\right)\Big|\mathcal{F}_{t_j}\right] = \Delta V_{t_{j+1}}^{n,*} \text{ in } L^2.$$

Then we have

$$\begin{aligned}&\mathbb{E}\left[\left(\Delta U_{t_j}^{n,\pi^{n,[m_c]},H^{[m_m],\pi^{n,[m_c]}}\left(M^{\mathcal{L},[m_c]}\right)} - \Delta V_{t_{j+1}}^{n,*}\right)^2\right]\\ \le&\mathbb{E}\left[\left(G_{t_j}^n\left(\pi^{n-1,[m_c]}, H^{[m_c],\pi^{n-1,[m_c]}}\right) - \Delta V_{t_{j+1}}^{n,*}\right)^2\right].\end{aligned}$$

Thus it is enough to prove that $G^n_{t_j}\left(\pi^{n-1,[m_c]}, H^{[m_c],\pi^{n-1,[m_c]}}\right)$ converges to $\Delta V^{n,*}_{t_{j+1}}$ in L^2. Then it follows:

$$\begin{aligned}
&\left|G^n_{t_j}\left(\pi^{n-1,[m_c]}, H^{[m_c],\pi^{n-1,[m_c]}}\right) - \Delta V^{n,*}_{t_{j+1}}\right| \\
&\leq \left|G^n_{t_j}\left(\pi^{n-1,[m_c]}, H^{[m_c],\pi^{n-1,[m_c]}}\right) - G^n_{t_j}\left(\pi^{n-1,[m_c]}, H^{*,\pi^{n-1,[m_c]}}\right)\right| \\
&\quad + \left|G^n_{t_j}\left(\pi^{n-1,[m_c]}, H^{*,\pi^{n-1,[m_c]}}\right) - \Delta V^{n,*}_{t_{j+1}}\right|.
\end{aligned}$$

Combining lemma 5.9 and 5.10, we know the first term of the right side converges to 0. From lemma 5.15, we have the convergence of the second term towards 0 as well. The proof is completed. □

6. Numerical Results

In this section, we show the effectiveness of proposed methods by applying them to price derivatives in two different markets, i.e. chooser's flexible cap in interest rate market and swing option in energy market.

6.1. Chooser's flexible cap. Chooser's flexible cap is an interest rate derivative which enables contract holders to lower the risk subject to adverse movements in financial market. A cap is a sequence of caplets corresponding to each of L sequential time periods, where the i-th caplets will pay contract holders the nominal amount times the excess of the current interest rate to the fixed strike. Chooser's flexible cap is similar to cap, but, rather than cover every forward rate, it offers contract holders greater flexibility to choose maximal $n < T$ forward rates over the lifetime of the policy. For each exercise time, the contract holder can choose to exercise the right or spare it for the future.

Here we assume the dynamic of interest rate R_t follows the two-factor additive Gaussian model as below:

$$\begin{aligned}
R_t &\triangleq \phi_t + S_t + U_t, \\
dS_t &\triangleq -aS_t + \sigma dW^S_t, \\
dU_t &\triangleq -bU_t + \eta W^U_t,
\end{aligned}$$

where ϕ_t is a deterministic time-varying rate, and W^S_t and W^U_t are Brownian motions with correlation ρ. There is a set of tenor times $\{t_j\}$, $j = 0, \cdots, L$, with $t_i < t_j$, $\forall i < j$, and a strike K. Then the exercise payoff at time t_j will be:

$$\max\left\{R_{t_j} - K, 0\right\}.$$

The price of zero coupon bond maturing at t_L is used as numeraire. The dynamic of interest rate under the corresponding T-forward measure is given by numeraire changing technique (See [7] for detail).

Specifically, the parameters are set as follows:

- $a = 5$, $b = 2$, $\sigma = 0.05$, $\eta = 0.02$, $\rho = 0.2$;
- $\phi_t = 0.05$;
- $S_0 = 0$, $U_0 = 0$;
- $L = 40$, $t_{j+1} - t_j = 0.25$;
- $K = 0.05$, $n = 40$.

6.1.1. *Basis Functions for the Continuation Value.* Considering the complexity of driving dynamic and the form of payoff function, basis functions are chosen as the polynomials of S and U as:

$$1,\ S,\ U,\ SU$$

6.1.2. *Basis Functions for the Martingale Increment.* Selection of basis function for martingale increment follows the suggestion in [10] that coefficients obtained from LS regression should be state-dependent to allow for dynamic hedging. First, we decompose W^S and W^U into two independent Brownian motions W^1 and W^2 as follows:

$$W^S = W^1;$$
$$W^U = \rho W^1 + \sqrt{1-\rho}W^2.$$

Denote the martingale increments of W^1 and W^2 by:

$$\kappa^1_{j+1} = W^1_{t_{j+1}} - W^1_{t_j};$$
$$\kappa^2_{j+1} = W^1_{t_{j+1}} - W^2_{t_j}.$$

The basis functions of martingale increments are chosen to be the product of basis functions of continuation value and powers of uncorrelated driving Brownian motion increments. Denote the basis functions of continuation value by:

$$\mathrm{Y} \triangleq [y_1, y_2, \cdots, y_n] \triangleq \left[S^{i_S} U^{i_U}\right]_{i_S + i_U <= 2},$$

and powers of driving uncorrelated Brownian Motion increments by:

$$\Phi \triangleq \left[\kappa^{1\ i_1}_{j+1} \kappa^{2\ i_2}_{j+1} - \mathbb{E}\left[\kappa^{1\ i_1}_{j+1}\right]\mathbb{E}\left[\kappa^{2\ i_2}_{j+1}\right]\right]_{0<i_1+i_2\leq 10}.$$

Basis functions of martingale increment are expressed as:

$$\begin{bmatrix} y_1\Phi & y_2\Phi & \cdots & y_n\Phi \end{bmatrix}$$

The above way to expand the set of basis functions will lead to a big increase in the number of basis functions in regression, and thus will slow down the convergence of estimated coefficients. We now present a decomposition method to handle this problem. The covariance matrix of $\begin{bmatrix} y_1\Phi & y_2\Phi & \cdots & y_n\Phi \end{bmatrix}$ is given by:

$$\begin{bmatrix} \mathbb{E}[y_1 y_1]\,\mathbb{E}\left[\Phi\Phi^T\right] & \cdots & \mathbb{E}[y_1 y_n]\,\mathbb{E}\left[\Phi\Phi^T\right] \\ \cdots & & \cdots \\ \mathbb{E}[y_1 y_1]\,\mathbb{E}\left[\Phi\Phi^T\right] & \cdots & \mathbb{E}[y_1 y_n]\,\mathbb{E}\left[\Phi\Phi^T\right] \end{bmatrix}. \tag{6.1}$$

The inverse of above covariance matrix is easily computed from:

$$\begin{bmatrix} A_{11}\mathbb{E}\left[\Phi\Phi^T\right]^{-1} & \cdots & A_{1n}\mathbb{E}\left[\Phi\Phi^T\right]^{-1} \\ \cdots & & \cdots \\ A_{n1}\mathbb{E}\left[\Phi\Phi^T\right]^{-1} & \cdots & A_{nn}\mathbb{E}\left[\Phi\Phi^T\right]^{-1} \end{bmatrix},$$

where A_{ij} is the element of inverse matrix $\mathbb{E}\left[\mathrm{YY}^T\right]^{-1}$. Here we can estimate the $\mathbb{E}\left[\mathrm{YY}^T\right]$ and $\mathbb{E}\left[\Phi\Phi^T\right]$ separately rather than do that for matrix (6.1) as a whole to accelerate the convergence of the inverse of estimated covariance matrix. The inverse of the covariance matrix can then be computed from individual inverses of

$\mathbb{E}\left[\mathrm{YY}^T\right]$ and $\mathbb{E}\left[\Phi\Phi^T\right]$, and this can help lower the time consumed on computing the inverse of high dimensional square matrix as well.

6.1.3. *Lower Bounds and Upper Bounds Computed.* In this part, we present numerical results of different bounds to demonstrate the effectiveness of our methods.

Regarding the lower bound, we compute classic Least-Squares lower bounds based iterative construction of continuation described in section 3.1 and marginal-based lower bounds in section 3.2. All lower bound strategies are developed using 524 288 paths. A second independent pass with 524 288 paths is used to evaluate independent lower bound estimates.

For the upper bound, we first compute non-nested upper bounds proposed in section 4.3. To further examine the effectiveness of sub-simulation upper bounds, we also introduce another extension of primal-dual upper bound, where the marginal martingale process is obtained from running sub-simulation for $M^{n,\mathcal{L}}_{t_{j+1}} - M^{n,\mathcal{L}}_{t_j}$ defined in (4.4). The continuation values used in (4.4) is the approximation of the marginal continuation value from the marginal-based lower bound method. For non-nested upper bounds, 524 288 paths are used to construct parametric martingale processes, and a second pass with independent 524 288 paths is used to develop independent upper bound estimates. For primal-dual upper bounds, we use 8 192 paths for each sub-simulation and 2 048 paths for the outer simulation.

In summary, all benchmarks lower bounds and upper bounds are listed as follows:

- $\mathrm{LB}^{\mathrm{LS}}$ - Least-Squares Lower Bounds;
- $\mathrm{LB}^{\mathrm{MB}}$ - Marginal-Based Lower Bounds;
- $\mathrm{UB}^{\mathrm{NN}}$ - Non-Nested Upper Bounds;
- $\mathrm{UB}^{\mathrm{EPD}}$ - Extended Primal-Dual Upper Bounds.

The numerical results presented in Table 1. All values are multiplied by 10^4 to match one basis point. The first column represents the number of exercise rights. Observing from results, the non-nested upper bound method works well, and some trade-off between accuracy and time-efficiency needs to be made for practical implementation. For lower bound methods, since the contract is not very long-dated, the marginal-based lower bounds are almost same as those from classic Least-Squares lower bound method.

6.2. Swing option. In this part, we will apply the proposed methods to price a commonly traded energy market derivative, swing option. It allows contract holders to buy a certain amount of energy, but will limit the maximal amount of energy that can be purchased over the lifetime of policy. Here we assume that the log of energy price S_t follows AR(1) model as:

$$\log S_{t_{j+1}} = (1-k)\left(\log S_{t_{j+1}} - \mu\right) + \mu + \sigma N_{t_{j+1}},$$

where N_{t_j} are an independent standard Gaussian random variables. Denoting the strike by K, the payoff is

$$\max\left\{S_{t_j} - K, 0\right\}.$$

The parameters are set as follows:

- $\sigma = 0.5$, $k = 0.9$, $\mu = 0$

n	$\mathrm{LB}^{\mathrm{LS}}$		$\mathrm{LB}^{\mathrm{MB}}$		$\mathrm{UB}^{\mathrm{EPD}}$		$\mathrm{UB}^{\mathrm{NN}}$	
1	302.09	0.15	302.09	0.15	302.43	0.65	303.31	0.03
2	556.18	0.23	556.18	0.23	556.67	0.86	559.14	0.04
3	777.98	0.30	777.98	0.30	778.76	0.99	783.16	0.05
4	975.31	0.36	975.32	0.36	976.23	1.09	982.87	0.06
5	1152.59	0.42	1152.58	0.42	1153.68	1.17	1162.73	0.07
6	1312.73	0.48	1312.72	0.48	1313.87	1.23	1325.57	0.08
7	1457.95	0.53	1457.94	0.53	1459.04	1.28	1473.44	0.09
8	1589.46	0.58	1589.46	0.58	1590.75	1.31	1607.82	0.10
9	1708.88	0.63	1708.88	0.63	1710.15	1.34	1729.93	0.10
10	1816.90	0.68	1816.92	0.68	1818.25	1.37	1840.74	0.11
15	2212.57	0.89	2212.55	0.89	2214.19	1.45	2249.11	0.13
20	2415.82	1.06	2415.82	1.06	2417.81	1.48	2461.99	0.15
25	2489.77	1.16	2489.78	1.16	2492.08	1.49	2541.14	0.16
30	2503.48	1.19	2503.48	1.19	2505.90	1.49	2556.73	0.16
35	2504.27	1.19	2504.27	1.19	2506.70	1.49	2557.75	0.16
40	2504.27	1.19	2504.27	1.19	2506.71	1.49	2557.76	0.16

TABLE 1. Lower Bounds and Upper Bounds of Chooser's Flexible Cap

- $S_0 = 1$
- $L = 200$, $t_{j+1} - t_j = 1$
- $K = 0$, $n = 100$

The basis functions for marginal continuation values are chosen as:

$$1, \ S.$$

The basis functions for martingale increments are chosen as:

$$S_{t_j}^{i_S}\left(N_{t_{j+1}}^{i_N} - \mathbb{E}\left[N_{t_{j+1}}^{i_N}\right]\right), \ 0 \leq i_S \leq 2 \text{ and } 1 \leq i_N \leq 2.$$

Benchmark upper bounds and lowers bounds are computed as same as those for chooser's flexible cap. Numbers of paths for lower bounds and upper bounds share same setup as those of chooser's flexible cap as well. The numerical results are presented in Table 2. We can observe from the table that the marginal-based lower bound method tends to outperform the classic Least-Square lower bound method along with the increase of the number of exercise rights.

7. Conclusion

Our lower bound method helps lower the error in the approximation of the marginal continuation value, and therefore offers a better exercise strategy. This improvement makes the practical pricing more accurate and assists contract holders to make higher profit by adopting better trading strategy. The marginal-based lower bound method outperforms the classic Least-Squares lower bound method by lowering the error in the approximate marginal continuation value, especially when the contract is long-dated and owns a large number of exercise rights. We also extend the sub-simulation-free upper bound method into the multiple exercise

n	$\mathrm{LB}^{\mathrm{LS}}$		$\mathrm{LB}^{\mathrm{MB}}$		$\mathrm{UB}^{\mathrm{EPD}}$		$\mathrm{UB}^{\mathrm{NN}}$	
1	3.690	0.001	3.690	0.001	3.691	0.000	3.717	0.000
2	6.936	0.002	6.936	0.002	6.936	0.000	7.005	0.000
3	9.934	0.002	9.934	0.002	9.932	0.000	10.053	0.000
4	12.758	0.002	12.758	0.002	12.756	0.000	12.932	0.001
5	15.450	0.003	15.450	0.003	15.449	0.000	15.683	0.001
10	27.609	0.004	27.609	0.004	27.611	0.000	28.141	0.001
15	38.378	0.004	38.378	0.004	38.381	0.000	39.203	0.001
20	48.242	0.005	48.242	0.005	48.246	0.000	49.352	0.002
30	66.081	0.006	66.082	0.006	66.091	0.000	67.743	0.002
40	82.102	0.007	82.102	0.007	82.116	0.000	84.288	0.002
50	96.753	0.008	96.754	0.008	96.775	0.000	99.441	0.003
60	110.298	0.008	110.300	0.008	110.328	0.000	113.463	0.003
70	122.906	0.009	122.910	0.009	122.946	0.000	126.522	0.003
80	134.695	0.009	134.699	0.009	134.748	0.000	138.738	0.004
90	145.749	0.010	145.754	0.010	145.816	0.000	150.197	0.004
100	156.129	0.010	156.136	0.010	156.215	0.000	160.964	0.004

Table 2. Lower Bounds and Upper Bounds of Swing Option

derivative pricing problem. This is an effective tool for sellers to quickly construct dynamic hedging portfolios. Meanwhile, our non-nested upper bound method is also free from diffusive assumption of underlying dynamics, which will ensure its compatibility with general stochastic models. Moreover, the enhancement and acceleration techniques for sub-simulation-free upper bounds can be used to improve the non-nested upper bound method as well. In future studies, we may consider extend the problem to solve control problems in complex stochastic systems as in [15].

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