Homomorphisms and Topological Semigroups.

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HOMOMORPHISMS AND TOPOLOGICAL SEMIGROUPS

A Thesis

Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy

in

The Department of Mathematics

by

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ABSTRACT

In recent years considerable interest has been evinced in the problem of the structure of topological semigroups. In the theory of groups, the study of the character groups and the group algebra of a locally compact commutative topological group has contributed much to the knowledge of the structure of such groups. This thesis deals mainly with compact commutative topological semigroups and attempts to extend the theory of characters and algebras to these semigroups.

It is well known that a commutative semigroup with cancellation can be embedded in a group. The first section of this paper is concerned with the embedding of a commutative topological semigroup in a topological group. The construction of the group in which a commutative topological semigroup is embeddable gives rise to a natural topology on this group. Conditions that the group, with this natural topology, be a topological group are given. These conditions can be algebraic or topological. Once the group is a topological group, conditions that the embedding be a homeomorphic mapping onto its image, with the relative topology from the group, are studied. The remainder of the
first section concerns a metrization theorem for commutative topological semigroups and an extension of the theorem from groups which states that continuous homomorphisms are open mappings under suitable topological conditions.

The second section is on the character semigroup (continuous complex valued homomorphisms) of compact topological semigroups. A general homomorphism extension theorem is proved first. This theorem is used to prove part of an existence theorem for non trivial characters of compact semigroups. The extension theorem is also used in the proofs of several homomorphism and isomorphism theorems similar to the theorems for groups and their character groups. There are two natural topologies for the character semigroup. Topological properties of the character semigroup then determine algebraic properties of the original semigroup. The question, when is the character semigroup a family of functions separating the points of the semigroup, is studied and relations between this question and semigroups that are embeddable in topological groups are considered. In this manner some of the structure of commutative topological semigroups can be seen. The last theorem concerns semigroups whose character semigroups separate points. It states that two semigroups are topologically and algebraically isomorphic if and only if their character semigroups are topologically and algebraically isomorphic.
The last section contains the construction of two topological algebras (not necessarily different) for a compact topological semigroup. These algebras depend on the extension of notions from the group algebra of a finite group. One of the algebras is a Banach algebra with a convolution multiplication, but does not give the analogous results of the group algebras except in the case of finite semigroups. However, it is useful in determining some algebraic properties of the semigroup. The other algebra is a locally $m$-convex topological algebra and some results analogous to those in the theory of groups can be obtained. However, since this algebra is not a Banach algebra, the information on the topological structure of the space of continuous multiplicative linear functionals is scanty and difficult to use. A one to one correspondence is set up between the character semigroup (the semigroup of all bounded homomorphisms) of a compact semigroup and the continuous multiplicative linear functionals of the second (first) topological algebra constructed.
It is well known that a commutative semigroup with cancellation may be embedded in a group in a manner analogous to the embedding of an integral domain in a field. However, Malcev [1] has shown that a non commutative semigroup with cancellation is not necessarily embeddable in a group. He has given complicated necessary and sufficient conditions for embedding such semigroups in groups in a second paper [2].

An outline of a process for the embedding of a commutative semigroup with cancellation in a group will now be given. Let \( S \) be a commutative semigroup with cancellation. The cartesian product set \( S \times S \) can be made into a commutative semigroup with cancellation by defining the operations coordinate-wise. In \( S \times S \), define \((a,b)R(c,d)\) if and only if \( ad = bc \). Then \( R \) is a congruence relation on \( S \times S \), and the symbol \( R \) will be used to denote this relation throughout this chapter. Let \( G \) be the collection of \( R \) equivalence classes; then \( G \) is a group and is called the group generated by \( S \). Fix an element \( b \) in \( S \) and define
P:S \rightarrow G by P(s) is the equivalence class that contains the element (sb,b). Then P is a semigroup isomorphism of S into G. It should be noted that P is independent of the choice of b, since if d is any other element of S then sbd = bsd and (sb,b)R(sd,d); i.e., the elements of S uniquely determine equivalence classes.

In considering commutative semigroups with topologies, there is a natural topology induced on the group G generated by S. However, this natural topology need not be Hausdorff. When the original topology on the semigroup is related to the algebraic structure, it is natural to ask if the induced topology on the group is related to the group structure and if the group operations are continuous with respect to the induced topology. These relations and their consequences will be discussed in this chapter. It should be observed that this subject has been investigated before [3, 4, 5].

Definition (1.1): The topological semigroup S is **embeddable in** G, the group generated by S, if and only if G is a Hausdorff topological group and the function P is a homeomorphism onto P(S) with the relative topology from G.

In the construction of the group G, there is a natural mapping of S x S onto G. Let \( \pi:S \times S \rightarrow G \) be the natural mapping which takes each point of S x S into the equivalence class containing the point. The induced
topology on $G$ is obtained as a quotient topology \cite{6}; that is, $A$ is open in $G$ if and only if $\pi^{-1}(A)$ is open in the product topology of $S \times S$.

From now on the term semigroup will mean Hausdorff topological semigroup. The following definitions will be used in this section:

Definition (1.2): A semigroup $S$ has property $F$ at $x \in S$ if for $y$ in an open set $V$ of $S$, there is an open set $U$ containing $x$ such that $xy \in \bigcap \{x'V : x' \in U\}$ and $yx \in \bigcap \{Vx' : x' \in U\}$.

The semigroup $S$ has property $F$ if $S$ has property $F$ at each $x \in S$.

Definition (1.3): A semigroup $S$ has property $J$ if for $x, y \in S$, either $x \in Sy \cup SyS \cup yS$ or $y \in Sx \cup SxS \cup xS$.

Definition (1.4): $x \in S$ is a continuous canceller if for $V$, an open set containing $y$, there are open sets $U$ containing $x$ and $W$ containing $xy(yx)$ such that $x' \in U$, $y' \in S$ and $x'y' \in W(y'x' \in W)$ implies $y' \in V$.

Cancellation is continuous in $S$, or $S$ is a continuous cancellation semigroup, if each element of $S$ is a continuous canceller.

Definition (1.5): $x \in S$ is a continuous divisor if for $V$ open containing $y$, there are open sets $U$, containing $x$; $W$, containing $xy(yx)$, such that $W \subseteq \bigcap \{x'V : x' \in U\}$ ($W \subseteq \bigcap \{Vx' : x' \in U\}$).

Division is continuous in $S$ if each element of $S$ is
a continuous divisor. Definitions (1.4) and (1.5) were made by J. E. L. Peck in his dissertation [4]. Note that a continuous divisor is always a point at which $S$ has property $F$.

The relation $R$ is said to be open if $\pi$ is an open mapping. It is known that the group generated by a commutative semigroup with cancellation is a topological group if $R$ is open [3]. The proof of this fact is included for completeness as

Lemma (1.6): If $S$ is a commutative semigroup with cancellation and $R$ is open, then the group $G$ generated by $S$ is a topological group.

Proof: Inversion is continuous in $G$ if for $V$, any open set in $G$, $\pi^{-1}(V^{-1})$ is open in $S \times S$. Now, $\pi^{-1}(V^{-1}) = T[\pi^{-1}(V)]$, where $T:S \times S \rightarrow S \times S$ is the mapping such that $T(a,b) = (b,a)$, a homeomorphism. Since $\pi^{-1}(V)$ and $T[\pi^{-1}(V)]$ are open, $\pi^{-1}(V^{-1})$ is open. Hence $V^{-1}$ is open and inversion is continuous.

In order to show that multiplication is continuous in $G$, let $x, y \in G$ and $V$ be open containing $xy$. If $x = \pi(a,b)$ and $y = \pi(c,d)$ then $xy = \pi(ac, bd)$ and $\pi^{-1}(V)$ is open containing $(ac, bd)$. There are open sets $A$, $B$, $C$, and $D$ of $S$ such that $AC \times BD \subseteq \pi^{-1}(V)$, $a \in A$, $b \in B$, $c \in C$, and $d \in D$. Now $A \times B$ and $C \times D$ are open in $S \times S$, $(a, b) \in A \times B$, and $(c, d) \in C \times D$. Since $\pi$ is an open mapping, $\pi(A \times B)$ is open in $G$ containing $x$ and $\pi(C \times D)$
is open in $G$ containing $y$. Multiplication is continuous in $G$ since $\pi(A \times B) \pi(C \times D) = \pi(AC \times BD) \subset \pi[\pi^{-1}(V)]$.

Then $G$ is a topological group.

The remainder of this chapter has four major parts. In the first, relations between the definitions and some consequences of the definitions will be proved. The second part is concerned with finding conditions such that the relation $R$ will be open. Assuming $R$ is an open relation, the third part concerns the topological embedding of a semigroup with cancellation in a topological group. A summary of the results is then given. In the last part, a metrization theorem for topological semigroups is proved. This section also contains a theorem that extends several theorems which state that a continuous homomorphism is an open mapping under suitable topological hypotheses.

A partial justification for the terminology of Definition (1.4) is the

Lemma (1.7): If $S$ is a semigroup with continuous cancellation then $S$ is a cancellation semigroup.

Proof: Let $a, x, y \in S$ such that $ax = ay$. It will be shown that $x = y$. For $V$ open, containing $y$, there are open sets $U$ and $W$ such that $a \in U$, $ay \in W$, and if $x' \in U$ and $y' \in S$ with $x'y' \in W$, then $y' \in V$. Since $a \in U$ and $ax = ay$, $ax \in W$; hence $x \in V$. It has been shown that every neighborhood of $y$ contains $x$; $y \in \{x\} = \{x\}$ so $y = x$ by the Hausdorff property.
Lemma (1.8): If $S$ is a cancellation semigroup then cancellation is continuous in $S$ if division is continuous in $S$.

Proof: Let $x, y$ and $V$ open containing $y$ be given. Division is continuous in $S$ implies there are open sets $U$ and $W$ with $x \in U$ and $xy \in W$ such that $W \subset \cap \{x'y' : x' \in U\}$. If $x' \in U$ and $y' \in S$ such that $x'y' \in W$ then there is a $b \in V$ such that $x'y' = x'b$. From the cancellation, $y' = b \in V$ and cancellation is continuous.

Theorem (1.9): If $S$ is a semigroup satisfying the first axiom of countability, then cancellation is continuous in $S$ if and only if for any $a, b \in S$ and sequences $(x_i)$ and $(b_i)$ in $S$ with $b_i \to b$ and $x_i b_i \to ab$, it follows that $x_i \to a$.

Proof: Assume that cancellation is continuous in $S$. Let $a, b \in S$ and $(b_i)$ and $(x_i)$ be sequences in $S$ with $b_i \to b$ and $x_i b_i \to ab$. Let $N$ be any open set containing $a$; it will be shown that there is an integer $n_0$ such that $n > n_0$ implies $x_n \in N$. Since cancellation is continuous in $S$, there exist open sets $V$ and $W$, $b \in V$ and $ab \in W$, such that $y \in V$, $xy \in W$ implies $x \in N$. Since the sequences $x_i b_i \to ab$ and $b_i \to b$, there is an $n_0$ such that $n > n_0$ implies $b_n \in V$ and $x_n b_n \in W$ and hence $x_n \in N$. Thus, $x_i \to a$.

Conversely, if cancellation is not continuous in $S$ at some point $a \in S$, there exists an open set $N$ containing
a and a point \( b \in S \) such that for every open set \( V \) containing \( b \) and for every open set \( W \) containing \( ab \) there exists a \( y \in V \) and an \( x \in S \) such that \( xy \in W \) and \( x \notin N \).

Let \( \{V_i\} \) and \( \{W_i\} \) be countable bases at \( b \) and \( ab \) respectively with the property that \( V_{i+1} \subseteq V_i \) and \( W_{i+1} \subseteq W_i \). For each integer \( i \) there exists \( y_i \in V_i \) and \( x_i \in S \) such that \( x_iy_i \in W_i \) and \( x_i \notin N \). The sequence \( \{y_i\} \) converges to \( b \) and the sequence \( \{x_iy_i\} \) converges to \( ab \); therefore, by assumption \( x_i \to a \); but this cannot happen since the set of points of the sequence \( \{x_i\} \subseteq S \setminus N \). Thus a contradiction has been arrived at and therefore, cancellation is continuous in \( S \).

From a close observation of the proof above, there follows the

**Corollary (1.10):** Let \( S \) be a commutative semigroup in which cancellation is continuous. If \( a, b \in S \) and \( \{x_d\}, \{y_d\} \) are nets on the same directed set with \( x_d \to b \) and \( x_dy_d \to ba \), then \( y_d \to a \).

It is always interesting to know that the maximal group containing an idempotent of a semigroup is a topological group. To see that property \( F \) insures this, note the

**Lemma (1.11):** Let \( S \) be a semigroup with idempotent \( e \). If \( S \) has property \( F \) at \( e \) then the maximal group, \( H(e) \), containing \( e \) is an open topological group in \( S \).

**Proof:** Since \( S \) has property \( F \) at \( e \) and \( e^2 = e \), for
Using Lemma (1.11), it will now be shown that continuity of division is equivalent to conditions 2 and 4 of Gelbaum, Kalisch and Olmsted [3] in semigroups with identities.

Theorem (1.12): Let $S$ be a commutative semigroup with identity. Division is continuous in $S$ if and only if

1. translations are open mappings [that is, for $U$ open in $S$, $xU$ is open for all $x \in S$], and

2. there exists an open set $N$ containing the identity such that $N = N^{-1}$ and inversion is continuous in $N$.

Proof: Assume division is continuous in $S$. Then $S$ has property $F$ at the identity, and the maximal group containing the identity is an open topological group by Lemma (1.11), thus establishing (2). Let $U$ be open in $S$ and $a \in S$. To show $aU$ is open, let $x \in U$; an open set $W$ will be found such that $ax \in W \subseteq aU$. By the continuity of division, there are open sets $V$ and $W$ such that $a \in V$, 

$V$ open containing $e$, there is a $W$ open containing $e$ such that $e \in Va$ for all $a \in W$. If $a \in W$ and $e \in Va$ then for some $x \in V$, $xa = e$ and $a$ has an inverse in $S$ with respect to $e$, so $W \subseteq H(e)$. Inversion is continuous at $e$ in $H(e)$ by again applying property $F$. Since $W$ is open and contains $e$, there is a $U$ open containing $e$ such that $e \in Wx$ for all $x \in U$; that is, $U^{-1} \subseteq W$. Then $H(e)$ is a topological group and contains an open set of $S$ so $H(e)$ is open.
ax \in W \text{ and } W \subseteq aU \text{ for all } a \in V. \text{ Since } a \in V, \ ax \in W \subseteq aU \text{ and hence } aU \text{ is open.}

Conversely, let \(x, y \in S \) and \(y \in V, \) an open set in \(S. \) Now \(y = ye \) (\(e \) is the identity of \(S\)), therefore, by the continuity of multiplication, there is an \(N \) open containing the identity such that \(N = N^{-1} \) and \(yN^2 \subseteq V. \) Translations are open mappings so \(xN \) and \(xyN \) are open in \(S, \)
\(x \in xN \) and \(xy \in xyN. \) If \(a \in xyN \) and \(b \in xN \) then \(a = xyn_1 \) and \(b = xn_2 \) where \(n_1, n_2 \in N. \) Now \(a = xyn_1 = n_2^{-1}byn_1 \)
\(= byn_1n_2^{-1} \in byNN \subseteq bV; \) that is, \(xyN \subseteq \cap \{bV : b \in xN\} \) and division is continuous in \(S. \)

Conditions (1) and (2) above do not imply that \(S \) is a topological group. The multiplicative semigroup of \(p\)-adic integers of norm greater than or equal to one satisfies these conditions. This fact can be found in [3].

From (1.6), the group generated by a commutative semigroup with cancellation is a topological group when \(R \) is open. Theorems giving conditions which imply \(R \) is open will now be proved.

Theorem (1.13): If \(S \) has property \(F, \) then \(R \) is open.

Proof: Let \(U \) be an open subset of \(S \times S. \) If \((x, y) \in S \times S \) and there is an element \((a, b) \in U \) such that \((x, y)R(a, b) \) then \(xb = ya. \) Since \(U \) is open and \(S \times S \) has the product topology, there are open sets \(V, \) containing \(a, \) and \(W, \) containing \(b, \) in \(S \) such that \(V \times W \subseteq U. \) Because \(S \) has property \(F \) at \(x \) and \(y, \) there exist open sets \(M, \) con-
taining $x$, and $N$, containing $y$, such that $xb \in x'W$ for all $x' \in M$ and $ya \in y'V$ for all $y' \in N$. For $(x', y') \in M \times N$, there is an $a' \in V$ and $ab' \in W$ such that $xb = x'b'$ and $ya = y'a'$; now $xb = ya$ so $x'b' = y'a'$ and $(x', y')R(a', b')$. It has been shown that the open set $M \times N$ is contained in $\pi^{-1}([\pi(U)])$ for each point in $\pi^{-1}([\pi(U)])$; hence, $R$ is open.

From this and the remark following the definitions there follows

**Corollary (1.14):** $R$ is open if division is continuous in $S$.

**Theorem (1.15):** If $S$ is a commutative semigroup with cancellation then $R$ is open if translations are open mappings.

**Proof:** Let $U$ be an open subset of $S \times S$. If $(x, y) \in S \times S$ such that $(x, y)$ is equivalent to some $(a, b) \in U$, then $xy = ya$. The product topology on $S \times S$ is such that there are open sets $V$, containing $a$, and $W$, containing $b$, in $S$ and $V \times W \subseteq U$. Translations are open mappings so that $xW$ and $yV$ are open sets containing $xb$ and $ya$ respectively. By continuity of the multiplication in $S$ there are open sets $M$, containing $x$, and $N$, containing $y$, such that $Mb \subseteq xW$ and $Na \subseteq yV$. Let $(x', y') \in M \times N$; then $x'b = xb'$ for some $b' \in W$, and $y'a = ya'$ for some $a' \in V$. Hence, $(x'a')(ab) = (x'b)aa' = (xb')aa' = (xa)b'a' = (yb)b'a' = (ya')b'b = y'abb'$, so $x'a' = y'b'$ by cancellation, and $(x', y')R(a', b')$, an element of $U$. It follows
that $R$ is open.

Theorem (1.15) can be found in [3]. Translations are open mappings in the additive semigroup of real numbers greater than $a$, where $a \geq 0$. The semigroup of $p$-adic integers previously mentioned also has this property.

Theorem (1.16): Let $S$ be a commutative semigroup with cancellation. If for each pair of nets $(x_d), (y_d)$ in $S$ with $x_d \rightarrow a$ and $x_d y_d \rightarrow b$ it is true that $(y_d)$ converges, and if $S$ has property J, then $R$ is open.

Proof: Let $U$ be any open set in $S \times S$, and let $A = \pi^{-1}([\pi(U)])$. Suppose $A$ is not open; then there is a net $((x_d, y_d))$ in $(S \times S) \setminus A$ such that $(x_d, y_d) \rightarrow (x_0, y_0)$, an element of $A$. Let $(a, b) \in U$ such that $y_0 a = x_0 b$.

Since $S$ has property J, it can be assumed that $b = a_0 a$ for some $a_0 \in S$. Hence $y_0 a = x_0 a_0 a$ and $y_0 = x_0 a_0$.

Cofinally, it can happen that $x_d = b_d y_d$ or $y_d = a_d x_d$. Assume first that $y_d = a_d x_d$ cofinally. Now $x_d \rightarrow x_0$ and $y_d = a_d x_d \rightarrow y_0 = a_0 x_0$; thus the net $\{a_d\}$ converges to $a_0$, using the cancellation. The net $\{(a, a a_d)\}$ then converges to $(a, a a_0) = (a, b)$ by the continuity of multiplication. But there exists $a_d$ so that $(a, a a_d) \in U$ and $(x_d, y_d) R(a, a a_d)$ so $(x_d, y_d) \in A$, a contradiction. On the other hand if $x_d = b_d y_d$ cofinally then $b_d y_d = x_d \rightarrow x_0$ and $x_d \rightarrow y_0$, so the net $\{b_d\}$ converges to some $p \in S$. By continuity $b_d y_d \rightarrow py_0 = pa_0 x_0 = x_0$, but then $S$ has an
identity \( p_\alpha \), and since \( p_b = p_\alpha \alpha = \alpha \) the net \( (b_d^b, b) \) converges to \( (p_b, b) = (\alpha, b) \). Again there exists a \( d \) so that \( (b_d^b, b) \in U \), \( (x_d, y_d) \in R(b_d^b, b) \), and \( (x_d, y_d) \in A \). It follows that no net in \( (S \times S) \setminus A \) converges to a point of \( A \) and therefore that \( A \) is open. It has been shown that \( R \) is open.

Two examples of semigroups with property \( J \) are the unit interval without endpoints and the set of complex numbers of absolute value greater than zero but less than or equal to one.

**Theorem (1.17):** Let \( S \) be a commutative subsemigroup with cancellation of a compact semigroup \( T \) such that \( T \setminus S \) is a closed ideal in \( T \). Then, if \( \{x_d\} \) and \( \{y_d\} \) are nets in \( S \) with \( x_d \to x \in S \) and \( x_d^d \to y \in S \), it follows that \( \{y_d\} \) converges to a point of \( S \).

**Proof:** Since \( T \) is compact, every net in \( T \) clusters. Let \( y_0 \) be any cluster point of the net \( \{y_d\} \) in \( T \). By the continuity of multiplication in \( T \), \( x_d^d \to x y_0 = a \in S \); then \( y_0 \in S \), since \( y_0 \in T \setminus S \) implies \( x y_0 \in T \setminus S \). Every cluster point of \( \{y_d\} \) is in \( S \), and if \( y_1 \) is any other cluster point, \( x y_1 = a = x y_0 \) and \( y_1 = y_0 \). It has been shown that \( \{y_d\} \) has a unique cluster point in the compact space \( T \); hence \( y_d \to y_0 \in S \) and the conclusion follows.

In an unpublished paper [5, Theorem 5], J. E. L. Peck has proved the

**Theorem (1.18):** Let \( I \) be an ideal of the commutative
semigroup with cancellation $S$. The group $H$ generated by $I$ is isomorphic and homeomorphic to the group $G$ generated by $S$.

The above theorem shows that the group generated by a commutative semigroup with cancellation $S$ is a topological group if one can find an ideal $I$ of $S$ for which any of the preceding conditions implying $R$ is open are satisfied in $I$ as a semigroup. Conditions for embedding will now be discussed.

Theorem (1.19): Let $S$ be a commutative topological semigroup. Division is continuous in $S$ and $S$ has cancellation if and only if $S$ is embeddable as an open subsemigroup of the topological group generated by $S$.

Proof: It has already been shown that continuity of division was a sufficient condition that the relation $R$ be open. It will be shown that if $V$ is any open set in $S$ then the set of points of $S \times S$ which are $R$ equivalent to some point of $Vb \times \{b\}$, for $b$ fixed in $S$, is an open set. Then the embedding is an open mapping into $G$, the group generated by $S$, and so $S$ is homeomorphically embedded as an open set in the topological group $G$.

Let $V$ be open in $S$ and let $A = \pi^{-1}[\pi(Vb \times \{b\})]$. If $\{(x_d, y_d)\}$ is a net in $S \times S$ with $(x_d, y_d) \to (x, y) \in A$ then $(x, y)R(ab, b)$ for some $a \in V$; thus $xb = yab$ and by cancellation $x = ya$. For $V$ open containing $a$, there are open sets $U$ and $W$ with $y \in W$ and $ya \in U$ such that
Now $x_d \implies x = ya$ and $y_d \implies y$ implies there is an $d_0$ such that $d > d_0$ implies $x_d \in U$ and $y_d \in W$; hence $x_d = a_d y_d$ where $a_d \in V$. Therefore, $(x_d, y_d) R(a_d b, b)$ and $(x_d, y_d) \in A$ for all $d > d_0$, so $A$ is open as was to be shown.

Conversely, if $S$ is an open subsemigroup of the topological group $G$, $x, y \in S$ with $y \in V$, an open set in $S$, then, since $y = x^{-1}xy$, there are open sets $U$ and $W$ in $S$ with $xy \in W$ and $x \in U$ such that $U = \cap [Vx': x' \in U]$, and division is continuous in $S$. Since $S$ is a subsemigroup of a group, it has cancellation.

Combining Theorems (1.13) and (1.19), there is the Corollary (1.20): Let $S$ be a semigroup with cancellation and an identity. Then $S$ is embeddable as an open subsemigroup of a topological group if and only if translations are open mappings and there exists a neighborhood of the identity which is equal to its inverse and in which inversion is continuous.

Corollary (1.21): If $S$ is a connected commutative semigroup with cancellation and identity, then $S$ is a topological group if division is continuous in $S$.

Proof: By Lemma (1.11), the maximal group containing the identity of $S$ is open in $S$. By the theorem, $S$ is open in $G$, the group generated by $S$, so the maximal group containing the identity of $S$ is an open and hence closed subgroup of the connected group $G$ and therefore is equal
to G. Hence $S = G$ and is a topological group.

Theorem (1.22): Let $S$ be a commutative semigroup for which $R$ is open. $S$ is embeddable in the topological group $G$ generated by $S$ if and only if given any two nets $\{x_d\}$ and $\{y_d\}$ in $S$ with $x_d \rightarrow x$ and $x_d y_d \rightarrow a$ then the net $\{y_d\}$ converges in $S$.

Proof: It is clear that if $S$ is embeddable, then cancellation is continuous and the conclusion follows from (1.10).

Conversely, let $V$ be any open set in $S$. It is to be shown that $P(V)$ is open in $P(S)$ or equivalently that $\pi^{-1}[P(V)]$ is open in $\pi^{-1}[P(S)]$. Let $\{(x_d, y_d)\}$ be a net in $\pi^{-1}[P(S)]$ with $(x_d, y_d) \rightarrow (x, y) \in \pi^{-1}[P(V)]$. Fix $b \in S$; there is an $a \in V$ such that $(x, y)R(ab, b)$; that is, $x = ya$, and for each index $d$ there is an $a_d \in S$ such that $x_d = y_d a_d$. Now $y_d a_d = x_d \rightarrow x = ya$ and $y_d \rightarrow y$ implies $a_d \rightarrow a$, and there is a $d_0$ such that $d > d_0$ implies $a_d \in V$. Therefore, $(x_d, y_d)R(a_d b, b)$ and $(x_d, y_d) \in \pi^{-1}[P(V)]$ for all $d > d_0$, that is $\pi^{-1}[P(V)]$ is open in $\pi^{-1}[P(S)]$. It has been shown that $P$ is an open mapping onto $P(S)$ and therefore, $S$ is embeddable in $G$.

It is interesting to note the

Theorem (1.23): Let $S$ be a locally Euclidean semigroup. Then division is continuous in $S$, and hence $S$ is embeddable as an open subsemigroup of a Lie Group.

J. E. L. Peck [5] proved this theorem in his unpub-
lished paper. The proof involves the generalized Jordan Separation theorem for Euclidean n-space and will not be reproduced here. However, there is the interesting

Corollary (1.24): If S is a connected, locally Euclidean, commutative semigroup with cancellation and an idempotent, then S is a topological group.

Proof: The idempotent of S is an identity element because of the cancellation. By the theorem and (1.21), S is a topological group.

The question of embedding can be reduced to the embedding of factors if a semigroup is a finite cartesian product. There is the

Theorem (1.25): Let S be a locally compact commutative semigroup with cancellation. If S is a cartesian product of locally compact semigroups each of which is embeddable in a group then S is embeddable in a group.

Proof: Since S is locally compact, all but a finite number of factors are compact. The compact semigroups are groups since a compact semigroup with cancellation is a group. Let $S = \prod_{i=1}^{n} S_i \times G$ where the $S_i$ are the locally compact non compact semigroups and G is the product of the compact groups. Let $H = \prod_{i=1}^{n} G_i \times G$, where $G_i$ is the group in which $S_i$ is embeddable. Define $\theta:S \rightarrow H$ by

$\theta = [\theta_1, \theta_2, \ldots, \theta_n, \theta_0]$ where the $\theta_j$'s are the embedding functions for the $S_j$'s and $\theta_0$ is the identity on G. The
mapping $\theta$ is clearly continuous and open since the $\theta_j$'s and $\theta_0$ are continuous, and $\theta$ is an embedding of $S$ into $H$.

A summary of the results obtained is

(1.26): Let $S$ be a commutative semigroup. Consider the conditions

A. $S$ is a cancellation semigroup and

1. $S$ has property $F$ or
2. $S$ has property $J$ or
3. division is continuous in $S$ or
4. translations are open mappings in $S$ or
5. $S$ is topologically and algebraically isomorphic to a subsemigroup of a compact semigroup $T$, and the complement of the image of $S$ in $T$ is a closed ideal of $T$.

B. Cancellation is continuous in $S$.

C. $R$ is an open relation.

D. $S$ is embeddable in $G$, the group generated by $S$.

Then (i) condition C is implied by each of A(1), A(3) (see [4]), A(4) (see [3]), and A(2) and A(5); and

(ii) condition D is implied by each of B and A(1), B and A(2), B and C (see [4]), B and A(4), A(3) (see [4]), A(1) and A(5), A(2) and A(5), and A(4) and A(5).

To see that the above conditions are different, the following examples should be noted. Let $S$ be the additive semigroup of real numbers greater than or equal to zero. Let $S_1 = S \setminus \{0\}$ and $S_2 = S \setminus (0,1)$. Let $S_3 = [S \setminus (0,2)] \cup \{1\}$. 
The semigroup $S$ has property $J$ and cancellation is continuous in $S$, but division is not continuous in $S$ and $S$ does not have property $F$ at any point. The semigroup $S_1$ has property $J$ and division is continuous in $S_1$; therefore $S_1$ has property $F$ at every point and cancellation is continuous in $S_1$. The semigroup $S_2$ has property $F$ at 0, but at no other point. Cancellation is continuous in $S_2$, but division is not continuous and $S_2$ does not have property $J$. If $S$ is any semigroup with an isolated point $x$, then $S$ has property $F$ at $x$. Note that in $S_3$, 1 is not a continuous divisor, but $S_3$ has property $F$ at 1.

These examples then show that the conditions used are independent of one another except for the lemmas and theorems proved.

The paper of Gelbaum, Kalisch and Olmsted [3] deals with the embedding of commutative metric semigroups. It is of interest to consider metric semigroups and invariant metrics and to obtain a metrization theorem for commutative semigroups. Let $S$ be a semigroup whose topology is given by a metric $D$. 
Consider the

Definition (1.27): D is an invariant metric if
x,y,a ∈ S implies D(x,y) = D(ax,ay) = D(xa,ya).

A metric semigroup S which is embeddable in the group
G generated by S has induced on it an invariant metric.
Gelbaum, Kalisch and Olmsted have proved the

Theorem (1.28): If S is a commutative algebraic
semigroup with identity and with an invariant metric, then
S is a topological semigroup with cancellation and is
embeddable in the group G generated by S.

Theorem (1.29): Let S be a commutative semigroup
satisfying the first axiom of countability. Any of the
following conditions imply that S is metrizable with an
invariant metric:

(1) division is continuous in S;
(2) S is embeddable as the complement of a closed
ideal in a compact semigroup and either
(a) S has property J, or
(b) S has property F, or
(c) translations are open mappings in S.

Proof: The mapping π:S × S → G, the group gener­
ated by S, is open in all cases; hence G is a topological
group satisfying the first axiom of countability. The
Birkhoff-Kakutani Theorem [7,8] then says that G is
metrizable with an invariant metric. This metric gives
rise to the same topology on G as the original and so the
induced topology on \( S \) is a metric topology with an invariant metric.

Note that in the case where division is continuous in \( S \) and \( S \) is a locally complete metric space, then by virtue of Klee's Theorem [9] the group \( G \) generated by \( S \) is a complete metric group.

Gelbaum, Kalisch and Olmsted [3, Theorem 17] have shown that if \( S \) and \( T \) are complete separable metric semigroups with cancellation and identities in which translations are open mappings and are such that cancellation is continuous in \( S \), then if \( f \) is a continuous isomorphism of \( S \) onto \( T \), \( f \) is an open mapping. This theorem cannot be extended to any homomorphism as demonstrated by the

Example (1.30): Let \( S \) be the additive semigroup of real numbers greater than or equal to 0. Let \( T \) be the circle group and \( f:S \rightarrow T \) the mapping restricted to \( S \) of the reals to the reals modulo 1. It is easily seen that \( f \) is not an open mapping.

Let \( S \) and \( T \) be commutative semigroups with cancellation. Suppose that \( T \) is embeddable in the group \( H \) generated by \( T \) and division is continuous in \( S \). The following theorem is then an extension of the theorem of Gelbaum, Kalisch and Olmsted mentioned above and at the same time an extension of a theorem of Banach [10, Theorem 8; 11, Theorem 13], and a theorem of Bourbaki [12, page 82].
Theorem (1.31): Let $S$ be a \{locally compact, locally complete metric\} separable space and $T$ a second category subset in $H$, a \{topological, metric\} group. If $f:S \rightarrow T$ is a continuous homomorphism, then $f$ is an open mapping.

Proof: Let $\pi_1$ be the natural mapping of $S \times S$ onto $G$, the group generated by $S$, and $\pi_2: T \times T \rightarrow H$. Define $f \times f:S \times S \rightarrow T \times T$ by $f \times f(a,b) = [f(a),f(b)]$ and consider the diagram in which $f^*(g) = \pi_2(f \times f)[\pi_1^{-1}(g)]$, and $\pi_2$ is defined by $\pi_2(t_1,t_2) = t_1 t_2^{-1}$.

\[
\begin{array}{ccc}
G & \xrightarrow{f^*} & H \\
\pi_1 \uparrow & & \uparrow \pi_2 \\
S \times S & \xrightarrow{f \times f} & T \times T
\end{array}
\]

Then $f^*$ is a continuous homomorphism on the \{locally compact, complete metric\} separable group $G$ into $H$ and the image of $f^*$ contains $T$, a second category subset of $H$. By Theorems 6 and 7 [13], $f^*$ is an open homomorphism. The restriction of $f^*$ to $S$ has the same set of values as $f$ and since $S$ is open in $G$, $f$ maps open sets of $S$ into open sets of $T$. 
CHAPTER II

CHARACTERS OF COMPACT SEMIGROUPS

In order to investigate the structure of a locally compact commutative topological group, the construction of its character group [11: Section 30] is one of the principal methods employed. The character theory for such groups makes possible the reduction of questions concerning a group to corresponding questions about its character group. The character group of a locally compact commutative group is also a locally compact commutative group and the character group of a compact group is a discrete group.

Many of the theorems about character semigroups are analogous to the theorems on character groups. Schwarz [14] has shown that the non vanishing characters of a compact topological semigroup correspond to the characters of the kernel of the compact semigroup (the kernel is a group). The character theory for finite semigroups and discrete semigroups has been researched by Hewitt and Zuckerman [15, 16]. In [17], Hewitt has described the character semigroups of compact monothetic semigroups.

The purpose of this chapter is the consideration of the character semigroups of compact commutative semigroups and the attempt to establish some of the structure of such semigroups. One of the tools needed in this approach is
knowing when a homomorphism defined on an ideal of a semi-
group is extendable to a homomorphism on the entire semi-
group. This chapter will start with a general lemma whose
corollaries will be used many times in this and the
following chapter.

A Homomorphism Extension Lemma (2.1): Let $S$ and $T$
be semigroups, $T$ with a right unit $u$. Let $A$ be a subsemi-
group of $S$ and $f:A \to T$ a homomorphism. If there is an
$a \in A$ such that $f(a) \in H(u)$, $aS \cup Sa \subseteq A$, and $u$ is a left
unit for $f(Sa)$, then there is a unique extension $g$ of $f$
to $S$ such that $g$ is a homomorphism. If in addition $S$ and
$T$ are topological semigroups then $g$ is continuous whenever $f$
is continuous.

Proof: For the above $a$, define $g(x) = f(xa)[f(a)]^{-1}$;
then $f(a)^{-1}f(ax) = [f(a)]^{-1}f(ax)f(a)[f(a)]^{-1}$
$= [f(a)]^{-1}f(a)f(xa)[f(a)]^{-1} = f(xa)f(a)^{-1}$. It then
follows that $g$ is a homomorphism; for $g(xy) = f(xya)[f(a)]^{-1}$
$= [f(a)]^{-1}f(axya)[f(a)]^{-1} = [f(a)]^{-1}f(ax)f(ya)[f(a)]^{-1}$
$= f(xa)[f(a)]^{-1}f(ya)[f(a)]^{-1} = g(x)g(y)$. Now $g$ is an
extension of $f$ since $x \in A$ implies $g(x) = f(xa)[f(a)]^{-1}$
$= f(x)f(a)[f(a)]^{-1} = f(x)u = f(x)$. The homomorphism $g$
is unique since if $h$ is an extension of $f$ and $x \in S$, then
$g(x) = g(x)f(a)[f(a)]^{-1} = f(xa)[f(a)]^{-1}f(a)[f(a)]^{-1}$
$= h(xa)[f(a)]^{-1} = h(x)h(a)[f(a)]^{-1} = h(x)f(a)[f(a)]^{-1}$
$= h(x)u = h(x)$. Since $g$ is a composition of functions $g$
is continuous whenever $f$ is continuous, and $S$ and $T$ are
In the following corollaries, it could be stated that the extension is continuous if the homomorphism given is continuous.

Corollary (2.2) (S. Schwarz [18]): If I is an ideal in S and f is a continuous homomorphism defined on I to the complex numbers and for some \( x \in I \), \( f(x) \neq 0 \), then there is a unique continuous homomorphism \( g \) defined on S to the complex numbers so that the restriction of \( g \) to I is \( f \).

Corollary (2.3): If \( f \) and \( g \) are homomorphisms of S into the complex numbers and I is an ideal of S such that \( f|_I = g|_I \neq 0 \), then \( f = g \).

Corollary (2.4): Let S and T be topological semigroups, T with an identity \( v \). If I is an ideal of S and \( f: I \rightarrow T \) is a continuous homomorphism such that \( v \in f(I) \), then there is a unique continuous homomorphism \( g: S \rightarrow T \) such that \( g(x) = f(x) \) for all \( x \in I \).

It is interesting to note the

Example (2.5): Let \( S = \{(x,y): x, y \text{ real } \geq 0 \text{ and } x + y \leq 1\} \). Multiplication in S is defined by \( (x,y)(a,b) = (xa, xb + y) \). Then S is a topological semigroup [19]. Let \( p = (1/2, 1/2) \), \( r = (1/4, 0) \) and \( q = (1/8, 1/4) \). The set \( A = rSp U Sq \) is a subsemigroup of S which is not an ideal. It does not contain Sr. However, Sq \( \supseteq qS \) so that \( Sq U qS \subseteq A \) and it is seen that the conditions of the lemma
do not imply that the subsemigroup is an ideal. The projection of $A$ into the first coordinate is a homomorphism which extends to the projection of $S$ onto the unit interval, the first coordinate of $S$.

From here on, in this chapter, $S$ is a compact commutative semigroup. The definitions and remarks that follow can be found in [18].

Definition (2.6): A character on $S$ is a continuous complex valued homomorphism defined on $S$.

Remarks:

(2.7): There are always two characters on $S$: the identically zero function and the identically one function. These characters will be called the trivial characters.

Let $S^*$ denote the set of all characters.

(2.8): If $f$ is a character on $S$ and $x \in S$, then $|f(x)| \leq 1$. For if $|f(x)| > 1$ then $|f(x^n)| = |f(x)|^{n-\to} \to \infty$, but $S$ compact implies $f$ is bounded and this divergence cannot happen.

(2.9): If $e$ is an idempotent in $S$ and $f$ is a character then $f(e) = 0$ or $f(e) = 1$. This follows from the fact that $f(e) = f(e^2) = [f(e)]^2$.

Definition (2.10): For $f$ and $g \in S^*$, let $fg(x) = f(x)g(x)$.

Remark (2.11): For $f$ and $g \in S^*$, $fg \in S^*$ since $fg$ is continuous and $fg(xy) = f(xy)g(xy) = f(x)f(y)g(x)g(y) = [f(x)g(x)] [f(y)g(y)] = [fg(x)] [fg(y)]$. 
Using the above as a definition of multiplication, it is easy to see that $S^*$ is a semigroup.

**Definition (2.12):** An ideal $P$ in $S$ is a prime ideal if $S \setminus P$ is a semigroup.

**Remark (2.13):** If $f \in S^*$ and $P = \{x : f(x) = 0\}$, then $P$ is a closed prime ideal of $S$.

**Proof:** The set $P$ is an ideal since $x \in P$ and $y \in S$ imply $f(xy) = f(x)f(y) = 0$. Since it is the inverse of a closed set under a continuous function, $P$ is closed. To show $P$ is prime, let $x, y \in S \setminus P$ with $xy \in P$. Then $f(xy) = 0$, but $f(xy) = f(x)f(y)$ and therefore $f(x) = 0$ or $f(y) = 0$ since the complex numbers form an integral domain. This is a contradiction and $P$ is then a closed prime ideal.

The only paper found in the literature which discusses a topology on $S^*$ is [20]. The semigroups considered in that paper are all locally compact and embeddable in locally compact groups. There is a very natural topology to impose on $S^*$ when $S$ is a compact topological semigroup. The character semigroup $S^*$ can be endowed with the relative topology from $C(S)$, the space of all continuous complex valued functions on $S$, with its sup for norm topology. $S^*$ is then a metric semigroup with metric given by

$$d(f, g) = \sup \{ |f(x) - g(x)| : x \in S\}.$$

It will be shown that $S^*$ with the metric topology given by the metric $d$ is a topological semigroup with
uniformly continuous multiplication. To do this, note first the

**Lemma (2.14):** If $f, g, \text{ and } h \in S^*$, then

\[ d(fh, gh) \leq d(f, g). \]

**Proof:** Let $f, g, \text{ and } h \in S^*$. Then

\[ d(fh, gh) = \sup \{ |f(x)h(x) - g(x)h(x)| : x \in S\} \]

\[ = \sup \{ |h(x)| \cdot |f(x) - g(x)| : x \in S\} \]

\[ \leq \sup \{ |f(x) - g(x)| : x \in S\}, \text{ since } |h(x)| \leq 1 \text{ by (2.8)}. \]

**Theorem (2.15):** Multiplication is uniformly continuous in $S^*$.

**Proof:** Let $f, g \in S^*$. For $\varepsilon > 0$, let $\delta = \varepsilon/2$. Let $h, k \in S^*$ such that $d(h, f) < \delta$ and $d(k, g) < \delta$. Thus

\[ d(hk, fg) \leq d(hk, fk) + d(fk, fg) \leq d(h, f) + d(k, g) < 2\delta = \varepsilon, \]

and multiplication is uniformly continuous when $S^* \times S^*$ is given the product topology.

It is known that closed subgroups of compact semigroups are topological groups. It is interesting to observe that all subgroups of $S^*$ are topological groups.

**Corollary (2.16):** Each subgroup of $S^*$ is a topological group.

**Proof:** Let $G$ be a subgroup of $S^*$. Multiplication is continuous in $G$ since it is continuous in $S^*$. It is then sufficient to show that inversion is continuous at the identity in $G$. Let $e$ be the identity of $G$. If $\{x_n\}$ is a sequence in $G$ with $x_n \rightarrow e$, then $\varepsilon > 0$ implies there is an $n(\varepsilon)$ such that $n > n(\varepsilon)$ implies $d(x_n, e) < \varepsilon$. Therefore,
\[ d(x_n^{-1}, e) = d(e x_n^{-1}, x_n x_n^{-1}) \leq d(e, x_n) < \varepsilon \text{ for } n > n(\varepsilon). \]

Hence, \( x_n^{-1} \to e \) and inversion is continuous in \( G \).

Remark (2.17): Since the uniform limit of homomorphisms is a homomorphism, \( S^* \) is a closed subset of \( C(S) \) and is therefore a complete metric space.

The next few pages concern the existence of non-trivial characters on compact semigroups.

Definition (2.18) (A. D. Wallace [19]): A **closed R. S. T. subsemigroup** \( R \) of \( S \times S \) is a closed set in \( S \times S \) which is a reflexive, symmetric, and transitive relation on \( S \) and is a subsemigroup of \( S \times S \). The space \( S/R \) is again a compact semigroup.

Theorem (2.19): Let \( S \) have a zero element \( 0 \). There are non-trivial characters on \( S \) if and only if there is a closed ideal \( I \) in \( S \) and a closed R. S. T. submob \( R \) of \( I \times I \) such that \( I/R \) is topologically and algebraically isomorphic to a non-degenerate subsemigroup of the usual real unit interval.

Proof: Let \( f \) be a non-trivial character on \( S \). Note that \( S \) is an ideal in \( S \). Let \( R = \{ (x, y) \in S \times S : |f(x)| = |f(y)| \} \); \( R \) is a closed R. S. T. subsemigroup of \( S \times S \) [18]. Define \( h: S/R \to \mathbb{R} \) by \( h(a) = |f(\theta^{-1}(a))| \) where \( \theta \) is the natural mapping of \( S \) onto \( S/R \). Then \( h \) is well defined since \( x, y \in \theta^{-1}(a) \) imply \( |f(x)| = |f(y)| \). The mapping \( h \) is a homomorphism since \( h(ab) = |f(\theta^{-1}(ab))| \leq |f(\theta^{-1}(a)\theta^{-1}(b))| = |f(\theta^{-1}(a))f(\theta^{-1}(b))| = h(a)h(b) \), and since \( h(ab) \) is a
point $h(ab) = h(a)h(b)$. Since $h^{-1}(\text{closed set}) = \Theta[f^{-1}(\text{closed set})] = \Theta(\text{closed set})$ is closed in $S/R$, $h$ is continuous. To show that $h$ is one to one, let $a, b \in S/R$ with $h(a) = h(b)$; then for $x \in \Theta^{-1}(a)$ and $y \in \Theta^{-1}(b)$, $|f(x)| = |f(y)|$ and $(x,y) \in R$, that is $a = b$.

Let $T = h(S/R)$. Then $T$ is a subsemigroup of $[0,1]$ and is non-degenerate, since $f$ is non trivial and $0 \in S$ and $f(0) = 1$ implies $f = 1$; therefore, there is an $x \in S$ such that $f(x) / 0$ and $h[\Theta(x)] / 0$ and $h[\Theta(0)] = 0$.

Conversely, let $\Theta:I \rightarrow I/R$ be the natural mapping.
Let $h:I/R \rightarrow [0,1]$ be the isomorphism onto a non-degenerate subsemigroup of $[0,1]$. Then $h\Theta$ is a continuous homomorphism on $I$ into the complex numbers and is not identically zero. Hence, by the homomorphism extension lemma, there is a unique extension of $h\Theta$ to all of $S$ which is a non trivial character on $S$.

Remark (2.20): The characters of the unit interval have been completely determined by Schwarz [18]. Since the restriction of a character to a subsemigroup is still a character, $f \in [0,1]^*$ implies $f(h\Theta)$ is a character of $I$ and since it has an extension to $S$, there are many characters on $S$.

Let $H(e)$ denote the maximal group containing the idempotent $e$ in a semigroup $S$.

Lemma (2.21): $H(1) = \{f \in S^*: f(x) = 1 \text{ for all } x \in S\}$, where $1$ is the identically one function, an idempotent in $S^*$.
Proof: Let $K$ be the kernel of $S$ and $e$ the idempotent in $K$. Let $f \in H(1)$ and $x \in S$. If $|f(x)| < 1$ then $|f(x)e| = |f(xe)| < 1$. Now $xe \in K$, thus there is a $y \in K$ such that $xy = e$, and hence, $|f(xe)f(y)| = |f(e)| < 1$ and $f(e) = 0$; but there is a $g \in H(1)$ such that $gf = 1$. Now $1 = gf(e) = g(e)f(e) = 0$, a contradiction, and $|f(x)| = 1$ for all $x \in S$. If $f \in S^*$ and $|f(x)| = 1$ for all $x \in S$, then $g(x) = 1/f(x) \in S^*$ and $f(x)g(x) = 1$, so $f \in H(1)$.

Theorem (2.22): Let $K$ be the kernel of $S$. The character group $K$ of $K$ is isomorphic and homeomorphic to $H(1)$.

Proof: Define $T: K \to H(1)$ by $T(f)$ is the unique extension $f$ of $f$ to $S$. Then $f(x) = f(xe)$ and $|f(x)| = |f(xe)| = 1$ so $f \in H(1)$. Since $T$ is clearly one to one, onto, and $|f(x) - g(x)| = |f(xe) - g(xe)|$, $T$ is an isometry and the result follows.

Definition (2.23): An ideal $P$ in $S$ is called a generating ideal if there is an $f \in S^*$, so that $f^{-1}(0) = P$.

Note that a generating ideal is a closed prime ideal by (2.13). A generating ideal is a generating prime ideal in the sense of Schwarz [18].

Definition (2.24): For an ideal $I$ in $S$, let $(S^*; I) = \{f \in S^*: x \in I \implies f(x) = 0\}$ and $(S^*; I)^o = \{f \in S^*: f^{-1}(0) = 1\}$.

It is easy to see that $(S^*; I)$ is a closed ideal of $S^*$ and $(S^*; I)^o$ is a subsemigroup of $S^*$. 
Theorem (2.25): If $\mathcal{P}$ is an open and closed generating ideal in $\mathcal{S}$, then $(\mathcal{S}^*; \mathcal{P})^\circ$ is topologically and algebraically isomorphic to the character group of the kernel of the compact semigroup $\mathcal{S} \setminus \mathcal{P}$.

Proof: Since $\mathcal{P}$ is an open and closed prime ideal, the homomorphism $\psi: \mathcal{S} \to \mathbb{Z}$ defined by

$$\psi(x) = \begin{cases} 0 & \text{if } x \in \mathcal{P} \\ 1 & \text{if } x \notin \mathcal{P} \end{cases}$$

is continuous and belongs to $(\mathcal{S}^*; \mathcal{P})^\circ$. Let $\mathbb{K}$ be the character group of the kernel $\mathbb{K}$ of $\mathcal{S} \setminus \mathcal{P}$. For $\tilde{g} \in \mathbb{K}$, the function $g: \mathcal{S} \to \mathbb{Z}$ defined by

$$g(x) = \begin{cases} 0 & \text{if } x \in \mathcal{P} \\ \tilde{g}(ex) & \text{if } x \in \mathcal{S} \setminus \mathcal{P} \end{cases}$$

is a continuous homomorphism on $\mathcal{S}$ and belongs to $(\mathcal{S}^*; \mathcal{P})^\circ$. Let $T: \mathbb{K} \to (\mathcal{S}^*; \mathcal{P})^\circ$ be defined by $T(\tilde{g}) = g$; then $T$ is one to one and since each element of $(\mathcal{S}^*; \mathcal{P})^\circ$ restricted to $\mathbb{K}$ is a character on $\mathbb{K}$, $T$ is onto. For $\tilde{f}, \tilde{g} \in \mathbb{K}$,

$$|f(x) - g(x)| = |f(x)f(e) - g(x)g(e)| = |f(xe) - g(xe)| = |\tilde{f}(xe) - \tilde{g}(xe)|$$

so $d(\tilde{f}, \tilde{g})$ in $\mathbb{K}$ is equal to $d(f, g)$ in $\mathcal{S}^*$ and $T$ is an isometry. This proves the theorem.

It is readily seen from the preceding theorems that a semigroup $\mathcal{S}$ with a non zero kernel has non trivial characters and also that the existence of an open and closed prime ideal assures the existence of non trivial characters.

A group is said to have sufficiently many characters if the characters are a separating family of functions on
the group. It is known that locally compact abelian groups have sufficiently many characters. However, compact semi­
groups do not necessarily possess this property. For consider the

Example (2.26): Let $S$ be the semigroup of real numbers in the unit interval with multiplication defined by $x \cdot y = \min [x, y]$. Then $S$ is an idempotent semigroup so each character is a 0,1 valued function, but cannot take on both values since $[0,1]$ is connected. Hence, there are only two characters and they do not separate points.

At this point, it is desirable to prove some iso­
morphism theorems analogous to those for locally compact groups [11; Sections 31 and 32].

Theorem (2.27): Let $I$ be a closed ideal in $S$. Then
$(S^*;I)$ is topologically and algebraically isomorphic to
$(S/ I)^* \setminus \{1\}$.

Proof: First, it should be noted that the algebraic statement was proved for finite semigroups by S. Schwarz [21]. Consider the diagram where $\theta$ is the natural mapping to the Rees Quotient [19], $g \in (S/I)^* \setminus \{1\}$ and $f = g\theta$.

\[
\begin{array}{ccc}
S & \xrightarrow{\theta} & S/I \\
\downarrow f & \swarrow g & \\
\end{array}
\]

Note that $f \in (S^*;I)$, for if $x \in I$ then $f(x) = g\theta(x) = g(0) = 0$. Define $T:(S/I)^* \setminus \{1\} \rightarrow (S^*;I)$ by $T(g) = g\theta$. In view of the preceding, $T$ is well defined. From the
equations \( T(g_1g_2)(x) = g_1g_2[\Theta(x)] = g_1[\Theta(x)]g_2[\Theta(x)] \)
\[ = T(g_1)T(g_2)(x) \] it follows that \( T \) is a homomorphism. Now \( T \) is one to one for \( g_1 \neq g_2 \) implies \( g_1\Theta \neq g_2\Theta \), since \( \Theta \) is an onto mapping. To show \( T \) is onto, let \( f \in (S^*;I) \); then since \( I \subseteq f^{-1}(0) \) there is a homomorphism \( g \) induced on \( S/I \) such that \( g\Theta = f \) and \( g \) is continuous. Let \( g_1, g_2 \in (S/I)^* \setminus \{1\} \); then \( d(g_1g_2) \)
\[ = \sup \{|g_1(x) - g_2(x)| : x \in S/I\} = \sup \{|g_1\Theta(x) - g_2\Theta(x)| : x \in S/I\} \] since \( x \in I \)
implies \( f_1(x) = 0 = f_2(x) \). The mapping \( T \) is then an isometry and therefore \((S^*;I)\) is topologically and algebraically isomorphic to \((S;I)^* \setminus \{1\}\).

Theorem (2.28): Let \( I \) be a closed ideal of \( S \) such that \( I^2 = I \). Then \( I^* \) is isomorphic and homeomorphic to the Rees Quotient \( S^*/(S^*;I) \).

Proof: It has been noted that \((S^*;I)\) is a closed ideal of \( S^* \). The condition \( I^2 = I \) implies that the homomorphic image of \( I \) in the complex plane is a semigroup equal to its square and hence is either identically zero or contains the complex number \( 1 \). Recall that the Rees Isomorphism Theorem [19] states that \( S^*/(S^*;I) \setminus \{0\} \) is isomorphic to \( S^*/(S^*;I) \). Define \( \Theta : I^* \to S^*/(S^*;I) \) by \( \Theta(0) = 0 \) and \( \Theta(f) \) is the unique extension of \( f \) to \( S \).

Since \( f \neq 0 \), \( \|f\| \) in \( I^* \) is \( 1 \); hence \( \|f\| \) in \( S \) is \( 1 \) and \( \Theta \) is a norm preserving mapping. To show that \( \Theta \) is an isomorphism and a homeomorphism, it is then sufficient to show
$\theta$ is one to one and a homomorphism, since $\theta$ is clearly onto. To see that $\theta$ is one to one, let $f,g \in I^*$ with $f \neq g$; then the unique extensions of $f$ and $g$ are not equal. Let $f,g \in I^*$, $\overline{f}$ and $\overline{g}$ their unique extensions to $S$; $\theta$ is a homomorphism since the unique extension of $fg \in I^*$ agrees with $\overline{fg}$ on $I$.

Schwarz [21] has shown for finite semigroups, that the semigroups were isomorphic even if $I^2 \neq I$; and this is true in general. If $I$ is the ideal $[0,1/2]$ in the usual unit interval $[0,1]$, then the topological part of the theorem fails since the sequence $\{f^n\}$ converges to zero in $I^*$ if $f(x) = x$, but it does not converge in $S^*$.

The algebraic form of the next theorem was proved by Schwarz [18] for two semigroups with units. His proof easily extends to any finite number of semigroups.

Theorem (2.29): Let $\{S_i\}_{i=1}^n$ be a finite collection of semigroups with identities $\{u_i\}_{i=1}^n$. Let $S = \prod_{i=1}^n S_i$; that is, $S$ is the cartesian product semigroup with the product topology and coordinate wise multiplication. If $N = \{f \in \prod_{i=1}^n S_i^* : f = (f_1, \ldots, f_n) \text{ implies there is a } j, 1 \leq j \leq n \text{ and } f_j = 0\}$, then $S^*$ is topologically and algebraically isomorphic to the Rees Quotient $[\prod_{i=1}^n S_i^*]/N$. 
Proof: Define \( T: \bigoplus_{i=1}^{n} S_i^* \rightarrow S^* \) by \( T(f)(x_1, \ldots, x_n) = f_1(x_1)f_2(x_2) \cdots f_n(x_n) \) where \( f = (f_1, f_2, \ldots, f_n) \). \( T(f) \) is then a continuous homomorphism on \( S \). To see that \( T \) is an onto mapping, let \( g \in S^* \) and let \( f_i: S_i \rightarrow \mathbb{Z} \) be defined by \( f_i(x_i) = g(u_1, \ldots, x_i, \ldots, u_n) \); then

\[
\begin{align*}
  f &= (f_1, f_2, \ldots, f_n) \in \bigoplus_{i=1}^{n} S_i^* \quad \text{and} \quad T(f)(x_1, \ldots, x_n) \\
  &= f_1(x_1)f_2(x_2) \cdots f_n(x_n) = g(x_1, u_2, \ldots, u_n)g(u_1, x_2, u_3, \ldots, u_n) \\
  &\quad \cdots g(u_1, \ldots, u_{n-1}, x_n) = g(x_1, x_2, \ldots, x_n) \quad \text{and} \quad T(f) = g.
\end{align*}
\]

The continuity of \( T \) follows from the inequality

\[
\sup \{ |T(f)(x) - T(f_2)(x)| : x \in S \} = \sup \{ |f_1(x_1)f_2(x_2) \cdots f_n(x_n)| - |f_1(x_1)f_2(x_2) \cdots f_n(x_n)| : x = (x_1, x_2, \ldots, x_n) \in S \}
\]

\[
\leq \sup \{ \left| |f_1(x_1)f_2(x_2) \cdots f_n(x_n)| - |f_1(x_1)f_2(x_2) \cdots f_n(x_n)| \right| : x_1 \in S_1 \}
\]

\[
\leq \bigoplus_{i=1}^{n} \sup \{ |f_i(x_i) - f_i(x_i)| : x_i \in S_i \}.
\]

Since \( T \) is clearly a homomorphism, it will be shown that \( T^{-1}(0) = N \). First, if \( f \in N \), \( f = (f_1, \ldots, f_n) \), and \( f_j = 0 \), then \( T(f) = f_1f_2 \cdots f_n = 0 \), so \( N \subseteq T^{-1}(0) \). Suppose \( f \in \bigoplus_{i=1}^{n} S_i^* \) and \( T(f) = 0 \). Let \( f = (f_1, f_2, \ldots, f_n) \); then

\( T(f)(u_1, u_2, \ldots, u_n) = 0 \). That is, \( f_1(u_1)f_2(u_2) \cdots f_n(u_n) = 0 \); hence \( f_j(u_j) = 0 \) for some \( j \) and then \( f_j = 0 \) and \( f \in N \). \( N \) is then a closed ideal of \( \bigoplus_{i=1}^{n} S_i^* \). Consider the diagram
where $\theta$ is the natural map to the Rees Quotient.

$$\begin{array}{c}
T \xrightarrow{\theta} S^* \\
\downarrow \\
\left[ \bigoplus_{i=1}^n F_i \right]^*/ N \\
\uparrow \\
\bigoplus_{i=1}^n F_i
\end{array}$$

Since the kernel of $\theta$ is the kernel of $T$, the induced mapping $\overline{T}$ is a continuous isomorphism. $\overline{T}$ is a homeomorphism since $d[\overline{T}(f), \overline{T}(\tilde{f})] \leq d(f, \tilde{f})$ where $f = (f_1, \ldots, f_n)$ and $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_n)$. This follows easily since $|f_1(x) - \tilde{f}_1(x)|$ attains its supremum at some $x_0 \in S_i$ and then

$$d[\overline{T}(f), \overline{T}(\tilde{f})] \leq |[f_1(u_1), \ldots, f_i(x_0), \ldots, f_n(u_n)] - [\tilde{f}_1(u_1), \ldots, \tilde{f}_i(x_0), \ldots, \tilde{f}_n(u_n)]| = d(f, \tilde{f}).$$

The following examples should be carefully noted for their relation to the theorems that follow.

Example (2.30): Let $S_1$ be the multiplicative semigroup of real numbers of the closed unit interval $[0,1]$. Schwarz has shown that all the characters of $S$ are:

(a) $\theta_0(x) = 0$ for all $x \in S$
(b) $\theta_1(x) = 1$ for all $x \in S$
(c) $\theta_{a,b}(x) = \begin{cases} 
  x^{a+ib} & \text{if } x \neq 0 \\
  0 & \text{if } x = 0
\end{cases}$ where $a,b$ are real and $a > 0$.

From this representation, $(S_1)^* \setminus \{\theta_0, \theta_1\}$ is a semigroup isomorphic to the open right half of the complex plane under addition. A careful check of the norm on $(S_1)^*$ and absolute value on the half plane will show that the mapping is a homeomorphism.
Example (2.31): Let $S_2$ be the multiplicative semigroup on the closed interval $[1/2, 1]$ where $x \times y = \max(1/2, xy)$ [19]. This is the Calabi semigroup. Since $x < 1$ implies there is an integer $n$ so that $x^n < 1/2$, then $x \times x \times \ldots \times x = 1/2$ for sufficiently large $n$. Then $(S_2)^*$ consists of $\theta_0(x) = 0$ and $\theta_1(x) = 1$; for if $x < 1$ and $\theta(x) \neq 0$, then $\theta(1/2) = \theta(x^k) = [\theta(x)]^k \neq 0$ so $\theta(1/2) = 1$ and $\theta(x) = 1$ for all $x \in S_2$.

Example (2.32): Let $S$ be any standard thread [22]. Let $E$ be the idempotents of $S$; then if $A$ is a component of $S \in E$, $A$ closure is one of the two semigroups $S_1$ or $S_2$ from the preceding examples. Let $\theta$ be any non identically one and non identically zero character of $S_1$ or $S_2$; then if $A = S_1$ the function

$$\overline{\theta}(x) = \begin{cases} 0 & \text{if } x < A \\ \theta(x) & \text{if } x \in A \\ 1 & \text{if } x > A \end{cases}$$

is a character of $S$.

$S^*$ separates points only if there are no continua of idempotents in $S$ and each $A$ is an $S_1$.

The following result can be found in [19]. However, the commutativity of $S$ is not needed for its proof.

(2.33): If $R_1$ and $R_2$ are closed $R. S. T.$ subsemigroups of $S \times S$ and $R_1$ $R_2$, then in the diagram

$$\begin{array}{c}
\xymatrix{S/R_1 \ar@{->}[r]^-E & S/R_2 \\
S \ar@{-}[u]^\sim \ar@{-}[urr]_{f_2} \ar@{-}[uu]_\sim f_1}
\end{array}$$

where $f_1$ and $f_2$ are the natural mappings, $g$ is the unique
continuous homomorphism so that $gf_1 = f_2$.

Using this result, there follows the

Theorem (2.34): Let $S_o = \{ (x,y) \in S \times S : f \in S^* \}
implies f(x) = f(y) \}$. Then $S_o$ is a closed R. S. T. sub-
semigroup of $S \times S$ and $S^*$ is topologically and algebra-
ically isomorphic to $(S/S_o)^*$.

Proof: To show $S_o$ is a closed R. S. T. subset of
$S \times S$, note that $S_o$ is reflexive since $f(x) = f(x)$ for
all $f \in S^*$; $S_o$ is symmetric since if $(x,y) \in S_o$ then
$f(x) = f(y)$ for all $f \in S^*$ and $(y,x) \in S_o$; and $S_o$ is trans-
itive since if $(x,y)$ and $(y,z) \in S_o$ then $f(x) = f(y)
= f(z)$ and $(x,z) \in S_o$. To see that $S_o$ is a subsemigroup
of $S \times S$ let $(x,a)$ and $(y,b) \in S_o$; then $f(xy) = f(x)f(y)
= f(a)f(b) = f(ab)$ and $(xy,ab) \in S_o$. To show $S_o$ is closed,
let $(x,y) \in S \times S \setminus S_o$; then there is an $f \in S^*$ such that
$f(x) \neq f(y)$. Therefore, there exists an open set $V$ con-
taining $f(x)$ and an open set $W$ containing $f(y)$ such that
$V \cap W = \emptyset$. Since $f^{-1}(V)$ and $f^{-1}(W)$ are open in $S$,
x $\in f^{-1}(V), y \in f^{-1}(W)$ and $f^{-1}(V) \cap f^{-1}(W) = \emptyset$ , it follows
that $f^{-1}(V) \times f^{-1}(W)$ is open containing $(x,y)$ and con-
tained in $S \times S \setminus S_o$ and $S_o$ is closed.

Let $\theta : S \longrightarrow S/S_o$ be the natural mapping. Let $f \in S^*$;
then, since $S_o \subset \{ (x,y) : f(x) = f(y) \}$, in the diagram

$$
\begin{array}{ccc}
S & \xrightarrow{f} & f(S) \\
\downarrow{\theta} & & \downarrow{g} \\
S/S_o & & \\
\end{array}
$$
g is the unique continuous homomorphism such that \( g \theta = f \) and \( g \in (S/\mathcal{S}_0)^* \). Define \( T: (S/\mathcal{S}_0)^* \to (S)^* \) by \( T(g) = g \theta \).

Since for each \( f \in S^* \) there is a \( g \) such that \( f = g \theta \), \( T \) is an onto mapping. The one to oneness of \( T \) follows from the uniqueness of the \( g \)'s. Since \( T(g_1 g_2)(x) = g_1 g_2[\theta(x)] \)

\[ = g_1[\theta(x)] \cdot g_2[\theta(x)] = T(g_1)T(g_2)(x), \]

\( T \) is a homomorphism.

Now \( |T(g_1) - T(g_2)| = \sup \left| \frac{g_1[\theta(x)] - g_2[\theta(x)]}{x} : x \in S \right| \)

\[ = \sup \left| \frac{g_1(z) - g_2(z)}{z} : z \in S/S_0 \right| = |g_1 - g_2|; \] hence \( T \) is an isometric mapping and \( S^* \) and \( (S/S_0)^* \) are topologically and algebraically isomorphic.

It is now clear that given a semigroup \( S \), its character semigroup does not distinguish \( S \) from \( S/S_0 \). Only those semigroups whose character semigroups separate points will be discussed.

**Theorem (2.35):** Let \( S \) be a semigroup such that \( S^* \) is a separating family of functions. Then

(1) \( S \) contains no non-degenerate continuum of idempotents; and,

(2) If \( A \) is a closed subsemigroup of \( S \) then \( A \) contains no elements \( x \) not in its kernel such that \( x^n \) is in the kernel of \( A \) for some integer \( n \).

**Proof:** (1) Let \( N \) be a continuum of idempotents in \( S \). If \( e_1, e_2 \in N \) with \( e_1 \neq e_2 \), then there is an \( f \in S^* \) such that \( f(e_1) \neq f(e_2) \). It can be assumed that \( f(e_1) = 1 \) and \( f(e_2) = 0 \). Since for each \( n \in \mathbb{N} \), \( f(n) = 0 \) or \( 1(2.9) \), then \( N \subset f^{-1}(0) \cup f^{-1}(1) \) and is not connected; hence \( e_1 = e_2 \) and
N is degenerate.

(2) Let A be a closed subsemigroup of S and K(A) its kernel. Let $x \in A$ such that $x^n \in K(A)$ for some integer n. Let $e$ be the idempotent of $K(A)$ and $f \in S^*$; then if $f(e) = 0$, $f(x^n) = 0$ and therefore $f(x) = 0$. But if $f(e) = 1$, then $f(x) = f(x)f(e) = f(xe)$ and $f$ does not separate $x$ and $xe$; hence $x = xe \in K(A)$ and the result is established.

Remark (2.36): The above theorem does not prohibit the existence of zero divisors, for let $S$ be the semigroup

$$
\begin{array}{ccc}
1' & \rightarrow & 1 \\
\downarrow &   & \downarrow \\
0 &   & 0
\end{array}
$$

where $[0,1']$ and $[0,1]$ are the usual unit intervals, but multiplication between the intervals is always zero. $S^*$ separates points simply because the two functions

$$
h(x) = \begin{cases} 
  x & \text{if } x \in [0,1'] \\
  0 & \text{if } x \in [0,1] 
\end{cases}
$$

$$
g(x) = \begin{cases} 
  0 & \text{if } x \in [0,1'] \\
  x & \text{if } x \in [0,1] 
\end{cases}
$$

separate points of $S$ and are characters of $S$.

Theorem (2.37): Let $S^*$ be a separating family for $S$, and $S$ a semigroup with property J. If the kernel $K$ of $S$ is the only generating ideal of $S$ then $S \setminus K$ is a semigroup with cancellation and is embeddable in a locally compact topological group.
Proof: Let \( a, x, y \in S \setminus K \) with \( ax = ay \). Let \( f \in S^* \); then \( f(a)f(x) = f(ax) = f(ay) = f(a)f(y) \) and \( f(a) = 0 \) if and only if \( f = 0 \). Hence \( f(x) = f(y) \) for all \( f \in S^* \) and therefore \( x = y \) since \( S^* \) is a separating family for \( S \).
By (1.26) \( S \setminus K \) is embeddable in a locally compact topological group.

A partial converse to the above is the

Theorem (2.38): Let \( K \) be the kernel of \( S \) and a generating ideal. If \( S \setminus K \) is a semigroup with cancellation and \( S \) has property J or property F, then \( S^* \) is a separating family for \( S \).

Proof: Since \( K \) is a compact topological group, \( H(1) \) is a separating family for \( K \). \( K \) is a generating ideal implies that the elements of \( (S^*;K)^\circ \) separate points of \( K \) and \( S \setminus K \). By (1.26) \( S \setminus K \) is embeddable in a locally compact topological group \( G \) and therefore for \( x, y \in S \setminus K \), there is a character \( \theta \) of \( G \) which separates \( x \) and \( y \) as elements of \( G \). Let \( f \in (S^*;K)^\circ \); then

\[
f\theta(x) = \begin{cases} 
0 & \text{if } x \in K \\
(f(x)\theta(x) & \text{if } x \in S \setminus K
\end{cases}
\]

is an element of \( S^* \) and separates \( x \) and \( y \). Hence, \( S^* \) is a separating family for \( S \).

For the purposes of the next two theorems, let \( S \) satisfy

(1) each generating ideal of \( S \) has property J or property F; and,

(2) for \( P \) a generating ideal in \( S \), there is a gener-
ating ideal Q in S such that
(a) Q ⊆ P and
(b) if Q' is a generating ideal in S and Q' ⊇ Q, then Q' ⊇ P.

Extensions of the two preceding theorems are

Theorem (2.39): If S* separates points of S and P is a generating ideal of S, then P \ Q is embeddable in a locally compact topological group.

Proof: Let S' = P/Q. S' has kernel {0} and it is the only generating ideal of S'. By (2.37), S' \ {0} is embeddable in a locally compact topological group. From the Rees Theorem, S' \ {0} is topologically and algebraically isomorphic to P \ Q and the result is clear.

Again a partial converse is the

Theorem (2.40): If P \ Q is a cancellation semigroup with P and Q as in (1) and (2) above, then S* separates points of S.

Proof: By (2.38), (P/Q)* separates points of (P/Q) for all generating ideals P and related Q's. By the extension theorem, all homomorphisms on P can be extended to homomorphisms on S. Therefore, since S is the union of all its generating ideals the conclusion follows.

When the separation can be done with characters which never take on the value 0, there is the

Theorem (2.41): S is a topological group if and only if x, y ∈ S implies there is an f ∈ S*, f is never zero and f(x) ≠ f(y).
Proof: If $S$ is a group, then the homomorphisms in $S^* \setminus \{0\}$ never take on the value zero and are characters of the group $S$ in the usual sense. By the Pontrajagin Duality Theorem [11], $S$ is the character group of $S^* \setminus \{0\}$ and the elements of $S$ as functions on $S^* \setminus \{0\}$ are distinct functions; hence $x, y \in S$ implies there is an $f \in S^* \setminus \{0\}$ such that $f(x) \neq f(y)$.

Suppose $S$ is not a group, then the kernel $K$ of $S$ is a proper ideal of $S$. Let $x \in S \setminus K$. Let $e$ be the identity of $K$; then if $f$ is a never zero homomorphism of $S$, $f(e) = 1$ and $f(x) = f(x)f(e) = f(xe)$ so $x = xe$ and $x \in K$. This affords a contradiction and therefore $S = K$ and is a group.

The next theorem will allow the determination of a class of semigroup which has the property that the character semigroups are separating families.

Theorem (2.42): Let $\{S_i\}_{i=1}^n$ be a finite collection of semigroups with identities such that $S_i^*$ separates the points of $S_i$ for all $i$, $1 \leq i \leq n$. If $S = \bigoplus_{i=1}^n S_i$, then $S^*$ separates the points of $S$.

Proof: Let $x, y \in S$, with $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. If $x \neq y$ then there is an index $j$ so that $x_j \neq y_j$. Let $f_j$ be that element of $S_j^*$ which separates $x_j$ and $y_j$. For all $i \neq j$, let $f_i$ be the identically one homomorphism on $S_i$. The homomorphism $g$ defined on $S$ by $g(a_1, a_2, \ldots, a_n) = f_1(a_1)f_2(a_2)\ldots f_n(a_n)$ belongs to
S* by (2.29). Now \( g(x) = f_j(x_j) \) and \( g(y) = f_j(y_j) \) so \( g \) separates \( x \) and \( y \) and \( S* \) is a separating family for \( S \).

Remark (2.43): It is now clear that any semigroup which is a finite product of semigroups satisfying the conditions of (2.40) also satisfies the conclusions; that is, the character semigroup separates the points of the semigroup. Many of the common examples which can be drawn on a blackboard and contain no continua of idempotents and no nilpotent subsemigroups satisfy these conditions.

The separation of points of \( S \) by \( S* \) affects the topology in \( S* \) when \( S = S^2 \) as will now be shown.

Lemma (2.44): These are equivalent:

(1) \( 0 \) is isolated in \( S^* \)

(2) \( P \) is a generating ideal implies \( E \cap S \setminus P \neq \emptyset \)

(3) for \( f \in S^* \setminus \{0\} \), there is an \( x \in S \) such that \( f(x) = 1 \).

Proof: (1) implies (2). Suppose \( P \) is a generating ideal and \( E \setminus S \setminus P = \emptyset \). Since \( f \in (S^*;P)^\circ \) implies \( f^n \in (S^*;P)^\circ \) for all integers \( n \), and \( 0 \) is isolated, then \( f^n \) does not converge to \( 0 \). Fix \( f \in (S^*;P)^\circ \) and let \( x \in S \setminus P \); then \( |f(x)| < 1 \) for \( |f(x)| = 1 \), then \( [x; f(x)] = 1 \) is a non empty compact subsemigroup and hence contains an idempotent contrary to assumption. The compactness of \( S \) implies there is a \( y \in S \) such that \( |f(x)| \leq |f(y)| < 1 \) for all \( x \in S \). Now for \( \varepsilon > 0 \), there is an integer \( n \) so that \( |f(y)|^n < \varepsilon \), and then
\[ |f^n(x)| \leq |f^n(y)| = |f(y)|^n < \varepsilon \text{ for all } x, y \in S \text{ and } f^n \rightarrow 0 \]
a contradiction. Hence \( E \setminus S \setminus P \neq \emptyset \).

(2) implies (3). This is clear, for let \( P = f^{-1}(0) \) for some \( f \in S^* \setminus \{0\} \); then \( P \) is a generating ideal and hence there is an idempotent \( e \in S \setminus P \) and \( f(e) = 1 \).

(3) implies (1). This is also clear. Let \( f \in S^* \setminus \{0\} \); then \( |f - 0|_1 = ||f||_1 = \sup \{|f(x)| : x \in S\} = 1 \) by the hypothesis and \( O \) is isolated in \( S^* \).

Corollary (2.45): If \( S = S^2 \), then \( O \) is isolated in \( S^* \).

Proof: Let \( f \in S^* \setminus \{0\} \). Now \( f(S) = f(S^2) = f(S)f(S) \), so that \( f(S) \) is a compact semigroup contained in \( [\mathbb{Z}; \leq] \). By [23, Corollary 1], \( f(S) = f(S)E[f(S)] \).
The only idempotents in \( [\mathbb{Z}; \leq] \) are \( 0 \) and \( 1 \), and \( f(S)O = 0 \) implies that \( f = 0 \); hence \( 1 \in f(S) \) and \( O \) is isolated.

Theorem (2.46): Let \( S^* \) separate points for \( S \). If \( O \) is isolated in \( S^* \) and \( S \) has only a finite number of generating ideals then \( S = S^2 \).

Proof: Suppose there is an \( x \in S \setminus S^2 \). Since \( S^* \) separates points of \( S \), there is an \( f \in S^* \) so that \( f(x) \neq 0 \). There are a finite number of generating ideals \( \{P_i\}_{i=1}^n \) such that \( x \in P_i \) for all \( i \leq n \), and for \( P \) any other generating ideal, \( x \notin P \). Consider \( P_0 = \bigcup_{i=1}^n P_i \), a prime ideal; then by (2.42) there is an idempotent \( e \in S \setminus P_0 \). Let \( f \in S^* \). If \( f(x) = 0 \) then \( f(xe) = f(x)f(e) = 0 \) and \( f \) does not separate \( x \) and \( xe \). If \( f(x) \neq 0 \) then \( f^{-1}(0) = P_i \) for
some \( i \) and \( f(e) = 1 \); hence \( f(x) = f(x)f(e) = f(xe) \) and \( f \)
does not separate \( x \) and \( xe \). Therefore, \( x = xe \in S^2 \), a
contradiction.

Remark (2.47): Let \( S \) be the semigroup of real numbers
\([1/2, 3/4]\) where multiplication is defined by \( x \times y
= \max [1/2, xy] \). Then \( S^* \) does not separate points of \( S \) and
0 is isolated since \( S \) has only the two characters 0 and 1,
but \( S^2 \neq S \). This shows the necessity of the hypothesis
"\( S^* \) separates points of \( S \)". Further, the semigroup
\( S = [0,x] \) of real numbers with its usual multiplication
and \( 0 < x < 1 \) is not equal to its square, but \( S^* \) separates
points and there are only a finite number of generating
ideals, so the condition "0 is isolated" is needed. It
would be interesting to know if the finite number of
generating ideals condition could be dropped.

It would be nice to know whether or not \( S^* \) is a
locally compact semigroup. To attain this objective in
special cases, it is necessary to prove the

Theorem (2.48): Let \( S \) be a semigroup in which each
generating ideal \( P \) is equal to its square, \( P = P^2 \). Then
\( (S^*;P)^0 \) is a closed subsemigroup in \( S^* \).

Proof: To see that \( (S^*;P)^0 \) is closed, let \( \{f_n\} \) be any
sequence in \( (S^*;P)^0 \) with \( f_n \to f \in S^* \). To show \( (S^*;P)^0 \)
closed, it is sufficient to show that \( f \in (S^*;P)^0 \). Let
\( P_1 = f^{-1}(0) \) and assume \( P_1 \setminus P \neq \emptyset \). Since \( f_n \to f \), \( x \in P \)
implies \( f_n(x) \to f(x) \), but \( f_n(x) = 0 \) for all \( n \) and hence
\( f(x) = 0 \) and \( x \in P_1 \); that is, \( P \subseteq P_1 \). For the semigroup \( P_1 = P_1^2 \), \( 0 \) is isolated in \( P_1^* \) and there is an idempotent \( e \in P_1 \setminus P \). For all \( n \), \( f_n(e) = 1 \), but \( f(e) = 0 \) and hence \( f_n \not\rightarrow f \). This is a contradiction; hence \( P_1 = P \) and \( f \in (S^*;P)^\circ \).

Remark (2.49): If \( S \) is a semigroup in which each generating ideal \( P \) is open and closed and \( P = P_2^2 \), then, from Theorem (2.25) and (2.45), \((S^*;P)^\circ \) is a closed locally compact subgroup of \( S^* \) and each element in \( S^* \) belongs to some \((S^*;P)^\circ \), and \( S^* \) is then locally compact since the distance from \((S^*;P_1)^\circ \) to \((S^*;P_2)^\circ \) is \( > 1 \).

In a quite different situation, the desired result follows from the

Theorem (2.50): Let \( S \) be a union of groups. Then \( S^* \) is a union of groups and is locally compact.

Proof: Since \( S \) is a union of groups, each generating ideal of \( S \) is a union of groups. Let \( P \) be a non empty proper generating ideal of \( S \). If \( f \in (S^*;P)^\circ \), then \( f(x) = 0 \) if and only if \( x \in P \), and \( |f(x)| = 1 \) if and only if \( x \in S \setminus P \). Now \( |f| \in (S^*;P)^\circ \) and is an idempotent; hence \((S^*;P)^\circ \) is a group [18] and \( S^* \) is a union of groups. If \( f_1 \) and \( f_2 \) are two distinct idempotents in \( S^* \), then \(|f_1 - f_2|^2 \geq 1 \) and each \((S^*;P)^\circ \) is closed and is a locally compact group, and as in the preceding remark \( S^* \) is then locally compact.

In the theory of character groups of locally compact
commutative groups, two groups are isomorphic and homeo-
morphic under the same mapping when the character groups
are isomorphic and homeomorphic under the same mapping.
The analogous result is not true for compact semigroups
because of the

Example (2.51): Let $S$ be the idempotent semigroup
of Example (2.26) and $T$ the nilpotent semigroup
$([0,1/4] U \{1/2\})/[0,1/32]$. The character semigroups of
$S$ and $T$ are both isomorphic to the semigroup of two ele-
ments $\{0,1\}$. The spaces are not homeomorphic since $S$ is
connected and $T$ is disconnected and are clearly not
isomorphic as semigroups.

However, there is the

Theorem (2.52): Let $S$ and $T$ be semigroups such that
$S^*$ and $T^*$ separate points of $S$ and $T$ respectively. There
is a mapping $F:S \to T$ such that $F$ is a topological and
algebraic isomorphism onto $T$ if and only if there is a
mapping $G:S^* \to T^*$ which is a topological and algebraic
isomorphism onto $T^*$.

Proof: Let $F:S \to T$ be a topological and algebraic
isomorphism. Define $G:T^* \to S^*$ by $G(g)(x) = g[F(x)]$;
then $G^{-1}$ is the desired mapping. First, it will be shown
that $G$ is one to one and a homomorphism. If $G(g_1) = G(g_2)$
then $g_1[F(x)] = g_2[F(x)]$ for all $x \in S$ and therefore
$g_1 = g_2$. Since $G(g_1g_2)(x) = g_1g_2[F(x)] = g_1[F(x)]g_2[F(x)]$
$= G(g_1)G(g_2)(x)$, $G$ is a homomorphism. To see that $G$ is a
homeomorphism, note that 
\[ \|g_1 - g_2\| = \sup \{ |g_1(t) - g_2(t)| : t \in T \} \]
\[ = \sup \{ |g_1[F(x)] - g_2[F(x)]| : x \in S \} = \|G(g_1) - G(g_2)\| \]
and \( G \) is an isometry so \( G^{-1} \) is the desired function, since \( G \) is clearly onto.

Conversely, let \( G : S^* \rightarrow T^* \) be a topological and algebraic isomorphism onto \( T^* \). The mapping \( G \) will be extended to an isomorphism of \( C(S) \) onto \( C(T) \). The extension will map homomorphisms into homomorphisms. The spaces \( S \) and \( T \) are then homeomorphic and it will be shown that the homeomorphism is a homomorphism and the desired result is then established.

It has already been observed that the linear spans, \( L(S^*) \) of \( S^* \) and \( L(T^*) \) of \( T^* \) are dense subalgebras of \( C(S) \) and \( C(T) \) respectively. Define \( G_1 : L(S^*) \rightarrow L(T^*) \) by
\[ G_1( \sum_{i=1}^{n} a_i f_i ) = \sum_{i=1}^{n} a_i G(f_i) \].
It is clear that \( G_1 \) is an algebra homomorphism. To see that \( G_1 \) is an algebraic and topological isomorphism, it is sufficient to show that \( S^* \setminus \{0\} \) is a linearly independent family of functions for any semigroup \( S \). If \( f, g \in S^* \setminus \{0\} \) and \( a, b \in \mathbb{Z} \setminus \{0\} \) such that \( af + bg = 0 \) then \( f = (-b/a)g \), \( f^2 = (b/a)^2g^2 \) and \( f^2 = (-b/a)g^2 \) so that \( f = g \). Proceeding by induction; suppose that any \( n - 1 \) non zero homomorphisms are linearly independent. If \( \{f_i\}_{i=1}^{n} \subset S^* \setminus \{0\} \) and \( \{a_i\}_{i=1}^{n} \subset \mathbb{Z} \setminus \{0\} \) such that \( \sum_{i=1}^{n} a_i f_i(x) = 0 \) for all \( x \in S \), then for fixed \( y \in S \)
\[ \sum_{i=1}^{n} a_i f_i(yx) = 0 \text{ for all } x \in S. \] Since \( f_1 \) and \( f_2 \) are distinct, there is a \( y_0 \) so that \( f_1(y_0) \neq f_2(y_0). \) Then

\[ 0 = \sum_{i=1}^{n} a_i f_i(y_0) - f_1(y_0) \sum_{i=1}^{n} a_i f_i(x) \]

\[ = \sum_{i=2}^{n} a_i [f_i(y_0) - f_1(y_0)] f_i(x) \text{ for all } x \in S. \] Since

\[ a_2[f_2(y_0) - f_1(y_0)] \neq 0, \] the linear independence of the \( n - 1 \) homomorphisms \( \{f_i\}_{i=2}^{n} \) is contradicted. Therefore, \( L(S^*) \) and \( L(T^*) \) are isomorphic normed linear spaces. \( C(S) \) is the completion of \( L(S^*) \) and \( C(T) \) is the completion of \( L(T^*) \) and \( \Gamma \) can be extended to an isomorphism of \( C(S) \) onto \( C(T) \).

By a theorem of Stone [24], there is a homeomorphism \( F:S \rightarrow T \) such that \( x \in S \) and \( f \in C(S), f(x) = 0 \) if and only if \( \Gamma_1(f)(F(x)) = 0. \) Let \( f \in S^* \) and \( x \in S; \) then

\[ f(x) = G(f)[F(x)]. \] For if \( f(x) = C \neq 0 \) then \( (f - C1)(x) = 0 \) so \( [G(f) - CG(1)][F(x)] = 0 \) and \( G(f)[F(x)] = C. \)

The elements of \( T^* \) separate points of \( T, \) so that

\[ F(xy) = F(x)F(y) \text{ if } g \in T^* \text{ implies } g[F(xy)] = g[F(x)F(y)]. \] Let \( f \in S^* \) such that \( G(f) = g; \) then \( g[F(xy)] = f(xy) = f(x)f(y) = g[F(x)]g[F(y)] = g[F(x)F(y)] \) and \( F \) is also a homomorphism.

The restriction that \( G \) be a homeomorphism was necessary in order that \( L(S^*) \) and \( L(T^*) \) be equivalent normed algebras. A simple counter example shows that this restriction cannot be eliminated. Let \( S \) be the usual unit
interval and $T$ the interval $[0, 1/2]$. The character semigroups $S^*$ and $T^*$ are algebraically isomorphic since the restriction of a homomorphism to an ideal is again a homomorphism and homomorphisms can only vanish at $0$ in these semigroups. The mapping from $C(S)$ to $C(T)$ is the restriction mapping and is not one to one since $S^*$ and $T^*$ are not topologically isomorphic.
CHAPTER III

SEMIGROUP ALGEBRAS

The group algebra of a locally compact topological group [25] has been used to determine the structure of the group, multiplication being defined in the group algebra by means of a left invariant integral. In semigroups, the existence of elements with inverses makes it impossible to use this approach. However, in the case of finite groups the multiplication in the group algebra is simplified. Munn [26] and Hewitt and Zuckerman [15] have studied this type of multiplication for finite semigroups. Hewitt and Zuckerman [16] have used this method to extend these ideas and have defined convolution algebras for discrete semigroups and it is desired here to extend these concepts to compact topological semigroups. Certain inadequacies arise in a first definition of a semigroup algebra for a compact semigroup, and a second algebra will also be defined.

In all algebras considered here, the radical of the algebra is the set of those elements at which every continuous multiplicative linear functional assumes the value zero; that is, the intersection of the kernels of the continuous multiplicative linear functionals. An algebra is semisimple if its radical is the zero element. The follow-
ing two definitions are those of Hewitt and Zuckerman [15].

Let $\mathbb{Z}$ denote the complex plane with its usual topology and multiplication. Let $S$ be a semigroup and $f$ be a function defined on $S$ to $\mathbb{Z}$.

**Definition (3.1):** For $x \in S$, $f_x: S \rightarrow \mathbb{Z}$ and $f^x: S \rightarrow \mathbb{Z}$ are the functions defined by $f_x(y) = f(xy)$ and $f^x(y) = f(yx)$.

Let $S$ be a semigroup and $F$ a linear space of complex valued functions on $S$ such that $x \in S$ and $f \in F$ implies $f_x$ belongs to $F$.

**Definition (3.2):** A complex linear space $L$ of linear functionals on $F$ is a convolution algebra if

1. $B \in L$, $f \in F$ and $x \in S$, then the function $g: S \rightarrow \mathbb{Z}$ defined by $g(x) = B(f_x)$ is an element of $F$; and if

2. $A, B \in L$, then the function $N: F \rightarrow \mathbb{Z}$ defined by $N(f) = A[B(f_x)] = A(g)$, where $g$ is the function in (1), belongs to $L$. The function $N$ will be written $A \ast B$ and $\ast$ is considered as a multiplication in $L$.

The following results can be found in Hewitt and Zuckerman's paper [15].

(3.3): Every convolution algebra is associative.

(3.4): Every subalgebra of a convolution algebra is a convolution algebra.

(3.5): If $S$ is a finite commutative semigroup and $F$ is the space of all continuous complex valued functions on
S, then convolution is commutative.

For the remainder of this chapter, let \( S \) be a compact semigroup. Let \( F = C(S) \) = all continuous complex valued functions defined on \( S \).

Definition (3.6): A subset \( Q \) of \( C(S) \) is a separating family for \( S \) if \( x, y \in S \) implies there is a function \( f \in Q \) such that \( f(x) \neq f(y) \).

Remark (3.7): If \( S^* \) is a separating family for \( S \) then the linear span of \( S^* \) is dense in \( C(S) \) by virtue of the Stone-Weierstrass Theorem [25] and the fact that \( S^* \) is a semigroup with respect to the usual multiplication in \( C(S) \).

Definition (3.8): For \( x \in S \), \( \overline{x} : C(S) \rightarrow \mathbb{C} \) is defined by \( \overline{x}(f) = f(x) \). \( \overline{x} \) is a linear functional on \( C(S) \).

Definition (3.9): \( L(S) \) is the linear span of the family \( \{ \overline{x} : x \in S \} \) in the linear space of all linear functionals on \( C(S) \).

A multiplication \( * \) will now be defined on \( L(S) \), and it will be shown that \( L(S) \) with \( * \) is a convolution algebra.

Definition (3.10): If \( A, B \in L(S) \), \( A = \sum_{i=1}^{n} a_{i} \overline{x_{i}} \) and \( B = \sum_{j=1}^{m} b_{j} \overline{y_{j}} \), let \( A \ast B = \sum_{i=1}^{n} a_{i} \sum_{j=1}^{m} b_{j} \overline{x_{i}y_{j}} \).

Lemma (3.11): The set \( \{ \overline{x} : x \in S \} \) is a linearly independent subset of \( L(S) \).

Proof: Suppose \( \sum_{i=1}^{n} a_{i} \overline{x_{i}} = 0 \) and \( a_{j} \neq 0 \) for some
j, 1 \leq j \leq n. Since the \( x_i \) are distinct and finite, there is an open set \( V_j \) so that \( x_j \in V_j \) and \( x_1 \in S \setminus V_j \) for all \( i \neq j \). Let \( f:S \to \mathbb{Z} \) be that function in \( C(S) \) so that \( f(x_j) = 1 \) and \( f|_{S \setminus V_j} = 0 \). Then \( 0 = \sum_{i=1}^{n} a_i x_i(f) \)

\[ = \sum_{i=1}^{n} a_i f(x_i) = a_j f(x_j) = a_j, \] a contradiction. Hence, the linear independence follows.

**Theorem (3.12):** \( L(S) \) with \( * \) as defined in (3.10) is a convolution algebra.

**Proof:** The linear space of functions \( C(S) \) is such that \( x \in S \) and \( f \in C(S) \) implies \( f_x \) and \( f^x \in C(S) \). It will now be shown that (1) and (2) of definition (3.2) hold.

Let \( A \in L(S) \) and \( f \in C(S) \). Assume \( A = \sum_{i=1}^{n} a_i x_i \) and \( g:S \to \mathbb{Z} \) is defined by \( g(x) = A(f_x) \). Then \( A(f_x) = \sum_{i=1}^{n} a_i f(x_i) = \sum_{i=1}^{n} a_i f(xx_i) = \sum_{i=1}^{n} a_i f^x(x_i) \)

and \( g = \sum_{i=1}^{n} a_i f^x_i \), an element of \( C(S) \).

Let \( A, B \in L(S) \) with \( A = \sum_{i=1}^{n} a_i x_i \) and \( B = \sum_{j=1}^{m} b_j y_j \); then

\[ A * B(f) = \sum_{i=1}^{n} a_i \sum_{j=1}^{m} b_j [x_i y_j(f)] \]

\[ = \sum_{i=1}^{n} a_i \sum_{j=1}^{m} b_j [x_i y_j(f)] \]

\[ = \sum_{i=1}^{n} a_i \sum_{j=1}^{m} b_j f^y_j(x_i) \]

\[ = \sum_{i=1}^{n} a_i [\sum_{j=1}^{m} b_j f^y_j] \]

\[ = A \left[ \sum_{j=1}^{m} b_j f^y_j \right] = A \left[ B(f_x) \right]. \]
For each $A \in L(S)$ the norm of $A$, $\|A\|$, is the supremum $[|A(f)| : \|f\| \leq 1]$. There follows the

Theorem (3.13): If $A \in L(S)$ and $A = \sum_{i=1}^{n} a_i x_i$, then,

$\|A\| = \sum_{i=1}^{n} |a_i|.$

Proof: Let $f \in C(S)$ and $\|f\| \leq 1$. Then $|A(f)|$

$= \left| \sum_{i=1}^{n} a_i f(x_i) \right| \leq \sum_{i=1}^{n} |a_i| \|f(x_i)\| \leq \sum_{i=1}^{n} |a_i|$. Hence,

$\|A\| \leq \sum_{i=1}^{n} |a_i|$. Since $S$ is a completely regular Hausdorff
space and $(x_i)_{i=1}^{n}$ is finite, there are open sets $V_i$ with

$x_i \in V_i$ and $V_i \cap V_j = \emptyset$ for $i \neq j$, and there exists

$(f_i)_{i=1}^{n} \subseteq C(S)$ such that $f_i(x_i) = 1$ and $f_i|S \setminus V_i = 0.$

Let $f = \sum_{i=1}^{n} (\frac{a_i}{|a_i|}) f_i$; then $f \in C(S)$, and $\|f\| \leq 1$ since if

$x \in S$ then $x \in S \setminus \bigcup_{i=1}^{n} V_i$ or $x$ is in a unique $V_i$. If

$x \in S \setminus \bigcup_{i=1}^{n} V_i$ then $f_i(x) = 0$ for all $i$, and $f(x) = 0.$

If $x \in V_i$ then $f_j(x) = 0$ for all $j \neq i$, and

$|f(x)| = \left| (\frac{a_i}{|a_i|}) f_i(x_i) \right| \leq 1$. Now $A(f) = \sum_{i=1}^{n} a_i f(x_i)$ but

$f(x_i) = (\frac{a_i}{|a_i|})$, so $A(f) = \sum_{i=1}^{n} a_i (\frac{a_i}{|a_i|}) = \sum_{i=1}^{n} |a_i|$ and

therefore $\|A\| \leq \sum_{i=1}^{n} |a_i|$. Hence, $\|A\| = \sum_{i=1}^{n} |a_i|.$

Theorem (3.14): $\|A* B\| \leq \|A\| \cdot \|B\|$ for $A, B \in L(S)$.

Proof: Let $A = \sum_{i=1}^{n} a_i x_i$ and $B = \sum_{j=1}^{m} b_j y_j$. Now

$A*B = \sum_{t=1}^{\infty} \left| x_i y_j = p_t a_i b_j \right| p_t$, so that $\|A*B\| =$

$\sum_{t=1}^{\infty} \left| x_i y_j = p_t a_i b_j \right| |a_i| \cdot |b_j|$
Corollary (3.15): \( L(S) \) with \( * \) is a normed convolution algebra.

Theorem (3.16): \( L(S) \) is commutative if and only if \( S \) is commutative.

Proof: If \( L(S) \) is commutative, then for \( x, y \in S \),
\[
\overline{x} \overline{y} = \overline{y} \overline{x}.
\]
That is \( \overline{xy} = \overline{yx} \) and therefore for all \( f \in C(S) \),
\[
f(xy) = f(yx);
\]
hence, \( xy = yx \) since \( C(S) \) separates points of \( S \).

If \( S \) is commutative then \( xy = yx \) for all \( x, y \in S \).

Therefore, \( \overline{x} \overline{y} = \overline{xy} = \overline{yx} = \overline{y} \overline{x} \). If \( A = \sum_{i=1}^{n} a_i \overline{x}_i \) and
\[
B = \sum_{j=1}^{m} b_j \overline{y}_j,
\]
then \( A \ast B = \sum_{i=1}^{n} a_i \overline{y}_j \sum_{j=1}^{m} b_j \overline{x}_i \)
\[
= \sum_{i=1}^{n} a_i \overline{y}_j \sum_{j=1}^{m} b_j \overline{x}_i = \sum_{j=1}^{m} b_j \overline{a}_i \overline{y}_j \overline{x}_i = B \ast A.
\]

Theorem (3.17): If \( \theta \) is a bounded multiplicative linear functional on \( L(S) \), then there is a bounded complex valued homomorphism \( f \) on \( S \) such that if \( A \in L(S) \) and
\[
A = \sum_{i=1}^{n} a_i \overline{x}_i,
\]
then (1) \( \theta(A) = \sum_{i=1}^{n} a_i f(x_i) \). Conversely,
each such bounded homomorphism on \( S \) defines a unique bounded multiplicative linear functional on \( L(S) \) by (1) above.

Proof: Let \( \theta \) be a bounded multiplicative linear functional on \( L(S) \). Define \( f:S \rightarrow \mathbb{Z} \) by \( f(x) = \theta(\overline{x}) \); then
\[
f(xy) = \theta(\overline{xy}) = \theta(\overline{x} \overline{y}) = \theta(\overline{x})\theta(\overline{y}) = f(x)f(y)
\]
and \( f \) is a
homomorphism. Now $|f(x)| = |θ(x)| ≤ \|θ\| \|x\|$, and thus $f$ is bounded. It is clear that $f$ is unique.

If $f$ is a bounded complex valued homomorphism on $S$, then $|f(x)| ≤ 1$ for all $x \in S$. If $A ∈ L(S)$ with $A = \sum_{i=1}^{n} a_i x_i$, and $θ:L(S) \rightarrow \mathbb{Z}$ is defined by $θ(A) = \sum_{i=1}^{n} a_i f(x_i)$, then for $\|A\| = \sum_{i=1}^{n} |a_i| ≤ 1$,

$|θ(A)| ≤ \sum_{i=1}^{n} |a_i| \quad |f(x_i)| ≤ \sum_{i=1}^{n} |a_i| = \|A\|$ and $θ$ is bounded.

Since $θ$ is multiplicative $0 ≤ \sum_{i=1}^{n} |a_i| = \|A\|$ and $θ$ is bounded. linear functional, it is a bounded multiplicative linear functional.

The space $L(S)$ is not necessarily a complete space with respect to the norm given it. However, the space of all bounded linear functionals on $C(S)$ is complete relative to the supremum norm. The algebra $L(S)$ will now be extended to be a Banach algebra.

Let $L$ be the closure of $L(S)$ in the space of all bounded linear functionals on $C(S)$. Then $L$ is a Banach space and the multiplication is extended to $L$ via the

Definition (3.18): If $A, B ∈ L$, let $\{A_n\}$ and $\{B_n\}$ be sequences in $L(S)$ so that $A_n \rightarrow A$ and $B_n \rightarrow B$,

$A_nB_n$ is the limit $A_nB_n$.

Lemma (3.19): The above definition is meaningful.

Proof: The following argument is standard and is being included for completeness. First, it will be shown that
\( \{A_n \cdot B_n\} \) is a cauchy sequence and hence converges in \( L \). For \( \varepsilon > 0 \), there is an \( n(\varepsilon) \) so that \( m, n > n(\varepsilon) \) implies

(a) \( ||A_n|| < \varepsilon + ||A|| \)
(b) \( ||B_n|| < \varepsilon + ||B|| \)
(c) \( ||A_n - A_m|| < \varepsilon/2(||A|| + ||B||) \)
(d) \( ||B_n - B_m|| < \varepsilon/2(||A|| + ||A||) \).

Then for \( m \) and \( n > n(\varepsilon) \),

\[
||A_n \cdot B_n - A_m \cdot B_m|| < ||A_n \cdot B_n - A_n \cdot B_m + A_n \cdot B_m - A_m \cdot B_m||
\leq ||A_n|| \cdot ||B_n - B_m|| + ||A_n - A_m|| \cdot ||B_m||
\leq \left( \frac{\varepsilon}{2(||A|| + ||B||)} \right) + \left( \frac{\varepsilon}{2(||A|| + ||B||)} \right) = \varepsilon,
\]

and \( \{A_n \cdot B_n\} \) is cauchy.

Secondly, \( A \cdot B \) is independent of the choice of the sequences \( \{A_n\} \) and \( \{B_n\} \). Let \( \{C_n\} \) and \( \{D_n\} \) be sequences in \( L(S) \), with \( C_n \rightarrow A \) and \( D_n \rightarrow B \). It will be shown that

\[
||A_n \cdot B_n - C_n \cdot D_n|| \rightarrow 0
\]

and hence that \( A \cdot B \) is well defined.

For \( \varepsilon > 0 \), there is an \( n(\varepsilon) \) so that \( n > n(\varepsilon) \) implies

(a) \( ||D_n|| < \varepsilon + ||B|| \)
(b) \( ||A_n|| < \varepsilon + ||A|| \)
(c) \( ||D_n - B_n|| < \frac{\varepsilon}{2(||A|| + ||B||)} \)
(d) \( ||A_n - C_n|| < \frac{\varepsilon}{2(||A|| + ||B||)} \).

Then for \( n > n(\varepsilon) \),

\[
||A_n \cdot B_n - C_n \cdot D_n|| \leq ||A_n \cdot B_n - A_n \cdot D_n + A_n \cdot D_n - C_n \cdot D_n||
\leq ||A_n|| \cdot ||B_n - D_n|| + ||A_n - C_n|| \cdot ||D_n|| < \varepsilon
\]

and the conclusion follows.
Note that if $A, B \in L$ then $\|A*B\| \leq \|A\| \|B\|$ since

$$\|A*B\| = \lim \|A_n*B_n\| \leq \lim \|A_n\| \|B_n\|$$

$$= [\lim \|A_n\|][\lim \|B_n\|] = \|A\| \|B\|.$$ 

This together with the other obvious verifications proves the

Theorem (3.20): $L$ is a Banach algebra.

Theorem (3.21): $L$ is a convolution algebra.

Proof: If $A \in L$, $f \in C(S)$, and $g:S \to \mathbb{Z}$ is defined by $g(x) = A(f(x))$, then $g \in C(S)$ since $g(x) = \lim A_n(f_n(x))$ is a uniform limit of continuous functions. Further, if $A, B \in L$ then $A*B(f) = A(g)$ where $g(x) = B(f_n(x))$, since $A_n*B_n(f) = A_n(g_n)$ and $\lim A_n(g_n) = A(g)$, $L$ is a convolution algebra as in Definition (3.2).

Each bounded multiplicative linear functional $\theta$ on $L(S)$ has a unique extension to a linear functional on $L$.

This extension is multiplicative by virtue of the equations

$$\theta(A*B) = \lim \theta(A_n*B_n) = \lim \theta(A_n)\theta(B_n)$$

$$= \lim \theta(A_n) \cdot \lim \theta(B_n) = \theta(A)\theta(B).$$

From Theorem (3.17), there is a one to one correspondence between the bounded complex valued homomorphisms on $S$ and the bounded linear functionals on $L$. Since $L$ is a Banach algebra, the space of modular maximal ideals of $L$ (each is the kernel of a bounded multiplicative linear functional) is in one to one correspondence with the bounded homomorphisms on $S$. 
Let $S$ be commutative. Let $M$ denote the set of all bounded multiplicative linear functionals on $L$, so $M$ is compact in the weak topology [25]. This topology can be characterized by saying that a net $(\theta_d)$ in $M$ converges to $\theta \in M$ if and only if $\theta_d(A) \to \theta(A)$ for all $A \in L$. Let $\overline{M} = M \setminus \{0\}$. Then $\overline{M}$ is compact if no net in $\overline{M}$ converges to the zero functional.

Theorem (3.22): $L(S)$ is semisimple if and only if the bounded homomorphisms on $S$ separate points of $S$.

Proof: If $L(S)$ is semisimple and $x, y \in S$ such that $f(x) = f(y)$ for all bounded homomorphisms $f$ on $S$, then, by (3.17), for any bounded multiplicative linear functional $\theta$ on $L(S)$, $\theta(x) = \theta(y)$ and $x = y$.

Conversely, if $A \in L(S)$ such that $\theta(A) = 0$ for all bounded multiplicative linear functionals $\theta$, it will be shown that $A = 0$. In the Banach algebra $B$ of all bounded functions on $S$, the subalgebra which is the linear span of the bounded homomorphisms on $S$ is dense in $B$ by the Stone-Weierstrass Theorem. Therefore, $A$ is the zero linear functional on $B$, and since $C(S) \subset B$, $A = 0$.

It is easy to see that an element $A \in L(S)$,

$$A = \sum_{i=1}^{n} a_i x_i,$$

is in the radical of $L(S)$ if and only if

$$\sum_{i=1}^{n} a_i f(x_i) = 0$$

for all bounded homomorphism $f$ defined on $S$.

The statement "'L(S) is semisimple'" means that $0 \neq A \in L(S)$,
A = \sum_{i=1}^{n} a_i x_i$, if and only if there is a bounded homomorphism $f$ on $S$ such that $\sum_{i=1}^{n} a_i f(x_i) \neq 0$.

Theorem (3.23): If $S$ is a commutative semigroup for which $S^*$ separates points of $S$, then $L$ is a semisimple Banach algebra.

Proof: Since $S^*$ separates points of $S$, then as previously remarked, the linear space spanned by $S^*$ is a dense subalgebra of $C(S)$ by the Stone-Weierstrass Theorem.

If $A \in L$ is in the radical of $L$ then $A(f) = 0$ for all $f \in S^*$. $A$ is then identically 0 on the linear span of $S^*$, which is dense in $C(S)$, so that $A = 0$. Therefore $L$ is semisimple.

The converse of the above theorem is not true. The second corollary to the next theorem shows that if $L(S)$ is semisimple, it does not necessarily follow that $S^*$ separates points of $S$.

Theorem (3.24): If $S$ is a union of groups, then $L(S)$ is semisimple if and only if $S$ is commutative.

Proof: $L(S)$ is semisimple implies $L(S)$ is commutative, and hence by Theorem (3.16) $S$ is commutative.

Assume $S$ is commutative. It will be shown that the bounded homomorphisms on $S$ separate points of $S$ and hence, by (3.22), $L(S)$ is semisimple.

First, consider $E$, the idempotent subsemigroup of $S$. Let $e, f \in E$ with $e \neq f$; then $ef = e, ef = f$, or $e \neq ef \neq f$. 
If $ef = e$, then $e \in S_f$, a compact semigroup with identity $f$. The union of all ideals in $S_f$ not containing $f$ is a prime ideal of $S_f$ and equals $S_f \setminus H(f)$.

Define $h : S_f \to \mathbb{Z}$ by

$$h(y) = \begin{cases} 1 & \text{if } y \in H(f) \\ 0 & \text{if } y \in S_f \setminus H(f) \end{cases}.$$  

Then $h$ is a bounded homomorphism of $S_f$ into $\mathbb{Z}$ such that $h(e) = 0$, and $h(f) = 1$. The unique extension of $h$ to $S$ is a bounded homomorphism on $S$ which separates $e$ and $f$. The case $ef = f$ is exactly the same as the above.

Assume $e \neq ef \neq f$, then since $ef \in S_f$, the unique extension $\overline{h}$ of the function $h$ defined above separates $e$ and $f$ since $0 = \overline{h}(ef) = \overline{h}(e)\overline{h}(f) = \overline{h}(e)$ and $\overline{h}(f) = 1$.

Let $x, y \in S$; then either there exists an $e \in E$ such that $x, y \in H(e)$ or there exists $e$ and $f \in E$ with $e \neq f$ such that $x \in H(e)$ and $y \in H(f)$. If $x, y \in H(e)$, then there is a character $\theta$ of $H(e)$ such that $\theta(x) \neq \theta(y)$. The unique extension of the bounded homomorphism

$$\pi : S_e \to \mathbb{Z} \text{ defined by } \pi(s) = \begin{cases} \theta(s) & \text{if } s \in H(e) \\ 0 & \text{if } s \in S_e \setminus H(e) \end{cases}$$

then separates $x$ and $y$. If $x \in H(e)$, $y \in H(f)$, and $e \neq f$, then there is a bounded homomorphism $\theta$ such that $\theta(e) \neq \theta(f)$. Now $\theta(x) = \theta(x)\theta(e)$ and $\theta(y) = \theta(y)\theta(f)$, and since $\theta(f) = 0$ or $1$, it can be assumed that $\theta(e) = 0$ and thus $\theta(x) = 0$. However, $\theta(y)$ is not zero, for if $\theta(y) = 0$ then $\theta(f) = \theta(y)\theta(y^{-1}) = 0$ and $\theta$ would not separate $e$ and $f$. 

Corollary (3.25): If $S$ is an idempotent semigroup then $L(S)$ is semisimple if and only if $S$ is commutative.

Corollary (3.26): If $S$ is a connected commutative idempotent semigroup then $L(S)$ is semisimple and $S^*$ is trivial, hence it does not separate points of $S$.

Proof: To see that $S^*$ does not separate points, note that any homomorphism on $S$ takes on the values 0 and 1 only. $S$ is connected so that the only continuous homomorphisms are the identically 0 and identically 1 functions.

Theorem (3.27): Let $S$ be a finite commutative semigroup. $S^*$ separates points of $S$ if and only if $S$ is the union of groups.

Proof: Since all bounded homomorphisms on $S$ are continuous, $L(S)$ is semisimple by (3.24) and $S^*$ separates points of $S$ by (3.22).

Suppose $S^*$ separates points of $S$. Let $x \in S$ and $f \in S^*$ such that $f(x) = 0$. Then $f(x)^n = f(x^n) = 0$ for all $n$. If $S$ has $N$ elements, then $[x, x^2, \ldots, x^N] = A$ is a sub-semigroup of $S$ and therefore has an idempotent. The semigroup $A$ has only one idempotent since, if there are two, no element of $S^*$ can separate them. This idempotent is an identity for $x$ and hence for all elements of $A$, since if $x^p$ is the idempotent and $f \in S^*$ such that $f(x) \neq 0$, then $f(x^p) = 1$, $f(x) = f(x)f(x^p) = f(x^{p+1})$, and $x = xx^p$. Hence, $A$ is a compact semigroup with identity and no other idempotent, and hence is a group. Therefore, $S$ is the union of
groups, since each element belongs to a group.

Theorem (3.28): Let $S$ and $T$ be semigroups. $L(S \times T)$ is semisimple if and only if $L(S)$ and $L(T)$ are semisimple.

Proof: Let $L(S) \times L(T)$ have the norm given by $||A, B|| = ||A|| + ||B||$. Suppose $L(S)$ and $L(T)$ are semisimple. Define $G: L(S \times T) \to L(S) \times L(T)$ by $G \left[ \sum_{i=1}^{n} a_i(x, y)_i \right] = \left( \sum_{i=1}^{n} a_i \bar{x}_i, \sum_{i=1}^{n} a_i \bar{y}_i \right)$. Then $G$ is a homomorphism, since

if $A = \sum_{i=1}^{n} a_i(x, y)_i$ and $B = \sum_{j=1}^{m} b_j(x, y)_j$, $G(A \ast B) = G[ \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j(x, y)_i(x, y)_j ] = \left( \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \bar{x}_i \bar{y}_j, \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \bar{y}_i \bar{y}_j \right) = \left( \sum_{i=1}^{n} a_i \bar{x}_i, \sum_{i=1}^{n} a_i \bar{y}_i \right) \ast \left( \sum_{j=1}^{m} b_j \bar{x}_j, \sum_{j=1}^{m} b_j \bar{y}_j \right) = G(A) \ast G(B)$.

Now $|G(A)| \leq 2||A||$, so that $G$ is bounded and hence continuous. If $G(A) = G(B)$ then $A = B$, and thus $G$ is one to one.

If $A$ is in the radical of $L(S \times T)$ then $\theta(A) = 0$ for all bounded multiplicative linear functionals $\theta$ on $L(S \times T)$. Let $\theta_1 \in \mathbb{M}[L(S)]$ and $\theta_2 \in \mathbb{M}[L(T)]$; then $\theta_1 \times \theta_2: L(S) \times L(T) \to \mathbb{Z}$ defined by $\theta_1 \times \theta_2(A, B) = \theta_1(A)\theta_2(B)$ is a bounded multiplicative linear functional on $L(S) \times L(T)$, and $(\theta_1 \times \theta_2)G \in \mathbb{M}[L(S \times T)]$. If $(\theta_1 \times \theta_2)[G(A)] = 0$ and $A = \sum a_i(x, y)_i$, then $\theta_1(\sum a_i \bar{x}_i) = 0$ or $\theta_2(\sum a_i \bar{y}_i) = 0$.

In either case, all the $a_i$ are 0 since $L(S)$ and $L(T)$ are semisimple.

Conversely, if $L(S \times T)$ is semisimple, then
\((x, y), (x_0, y_0) \in S \times T\) implies there is a bounded homomorphism \(\theta : S \times T \rightarrow \mathbb{Z}\) such that \(\theta(x, y) \neq \theta(x_0, y_0)\). Let \(x, x_0 \in S\), and fix \(e \in E(T)\). Define \(\theta_e : S \rightarrow \mathbb{Z}\) by \(\theta_e(x) = \theta(x, e)\). Since \(\theta_e(ab) = \theta(ab, e) = \theta(ab, ee) = \theta((a, e)(b, e)) = \theta_e(a)\theta_e(b)\), \(\theta_e\) is a homomorphism and is bounded. If \(\theta_e(x) = \theta_e(x_0)\) then \(\theta(x_0, e) = \theta(x, e)\); and since there is a \(\theta\) separating \((x_0, e)\) and \((x, e)\), there is a \(\theta_e\) separating \(x\) and \(x_0\), and hence \(L(S)\) is semisimple. A similar argument will show that \(L(T)\) is also semisimple.

Definition (3.29): An idempotent \(e \in S\) is a maximal idempotent if \(a \in S\) and \(e \in S a S\) implies \(a \in S e S\). Let \(\mathbb{M}\) be the set of maximal idempotents of \(S\).

Theorem (3.30): Let \(S\) be a commutative semigroup such that \(S = S^2\); \(\mathbb{M}\) is compact if and only if \(S\) has only a finite number of maximal idempotents.

Proof: Suppose \(\{e_i\}_{i=1}^m\) are the finite number of maximal idempotents of \(S\). If \(\{\theta_d\}\) is a net in \(\mathbb{M}\) converging to 0, then given \(i, l \leq i \leq m\), there is a \(d_i\) such that \(d > d_i\) implies \(\theta_d(e_i) = 0\). Let \(d_0 > d_i\) for all \(i \leq m\); then \(\theta_d(e_i) = 0\) for all \(d > d_0\) and all \(i \leq m\). But \(S = S \mathbb{E}\); thus if \(x \in S\), \(\theta_d(x) = \theta_d(x e_i) = \theta_d(x)\theta_d(e_i) = 0\) for all \(x\). Then \(\theta_d = 0\) for all \(d > d_0\), a contradiction; hence \(\theta_d\) does not converge to 0 and \(\mathbb{M}\) is compact.

Conversely, suppose \(\mathbb{M}\) is compact and \(\{e_i\}_{i=1}^\infty\) is a denumerable set of distinct maximal idempotents in \(S\). Let \(J_n\) be the union of all ideals not containing \(e_i\). Now
$J_n$ is a maximal proper ideal of $S$ and $S \setminus J_n = H(e_n)$  
[27, Theorem 6 and Corollary 3]. Define 

$$ f_n:S \rightarrow \mathbb{Z} \text{ by } f_n(x) = \begin{cases} 
1 & \text{if } x \in H(e_n) \\
0 & \text{if } x \in J_n 
\end{cases}, \text{ then } f_n \text{ is a bounded homomorphism on } S. \text{ Let } \theta_n \text{ be the bounded multiplicative linear functional on } L \text{ obtained by using Theorem (3.17). If } A = \sum_{i=1}^{m} a_i x_i \in L(S), \text{ then there is an integer } k \text{ so that } r > k \text{ implies } f_r(x_i) = 0 \text{ for all } i \leq m. \text{ Hence, } \theta_r(A) = 0 \text{ for all } r > k. \text{ Therefore, } \theta_r(A) \rightarrow 0 \text{ for all } A \in L(S). \text{ If } A \in L \text{ then let } \{A_n\} \text{ be a sequence in } L(S) \text{ with } A_n \rightarrow A. \text{ It will be shown that } \theta_r(A) \rightarrow 0 \text{ and hence that } \overline{M} \text{ is not compact, a contradiction. Therefore, } \text{ } S \text{ has only a finite number of maximal idempotents. For } \varepsilon > 0, \text{ there is an } n(\varepsilon) \text{ so that } p \geq n(\varepsilon) \text{ implies } \|A_p - A\| < \varepsilon, \text{ and there is a } q(n(\varepsilon)) \text{ so that } r > q(n(\varepsilon)) \text{ implies } \theta_r(A_n(\varepsilon)) = 0. \text{ For } r > q(n(\varepsilon)), \text{ }$

$$ |\theta_r(A)| \leq |\theta_r(A) - \theta_r(A_n(\varepsilon))| + |\theta_r(A_n(\varepsilon))| 
\leq |\theta_r(A) - \theta_r(A_n(\varepsilon))| \leq \|A - A_n(\varepsilon)\| \leq \varepsilon. $$

Therefore, $\theta_r(A) \rightarrow 0$ and $\theta_r \rightarrow 0.$

Hewitt and Zuckerman [16, Theorem 7.5] have shown that if $\{e_i\}_{i=1}^{n}$ is a finite collection of maximal idempotents of a commutative semigroup $S,$ then 

$$ E = \sum_{i=1}^{n} e_i - \sum_{j<i} e_i e_j + \sum_{k} e_i e_j e_k - \cdots $$

is an identity for $L(S)$ and hence is an identity for $L.$
There follows the

Theorem (3.31): Let $S$ be a commutative semigroup such that $S = S^2$. If $M$ is compact then $L$ has an identity.

The preceding theorems show that the algebra $L$ does not play the same role in the theory of semigroups as the group algebra of a locally compact group. The primary flaw is that the bounded multiplicative linear functionals on $L$ correspond to the bounded homomorphisms on $S$ and not to the characters on $S$, whereas in groups the character group is identifiable with the space of maximal modular ideals. It is desirable to go back to $L(S)$, and retopologize it so that the continuous homomorphisms on $S$ correspond to the continuous multiplicative linear functionals on $L(S)$. It should be realized that the completion of $L(S)$ in any other topology does not give $L$ back as a Banach algebra, there being only one norm such that $L$ is a semisimple Banach algebra [25]. It then follows that the correspondence between the characters on $S$ and the modular maximal ideals on $L$ is not onto.

Let $C(S)^*$ be the space of all bounded multiplicative linear functionals on $C(S)$. For $f \in S^*$ and $A \in C(S)^*$, $|A(f)| \leq \|A\| \|f\| \leq \|f\|$ since $\|f\| \leq 1$. The restriction of the elements of $C(S)^*$ to $S^* \subset C(S)$ are then bounded continuous functions on $S^*$. The space of restrictions of elements of $C(S)^*$ to $S^*$ is a Hausdorff space because of the
Lemma (3.32): Let $S^*$ be a separating family of functions for $S$. If $A, B \in C(S)^*$ such that $f \in S^*$ implies $A(f) = B(f)$, then $A = B$.

Proof: As remarked previously, the linear space spanned by $S^*$ is a dense subalgebra of $C(S)$. Since $A$ and $B$ are linear functionals, they agree as each point of the linear span of $S^*$ and hence, agree at all points of $C(S)$.

Let $B(S^*)$ denote the set of all bounded continuous functions on $S^*$ and let $C(S)^*$ be the restrictions of the mappings of $C(S)^*$ to $S^*$. Then $C(S)^* \subseteq B(S^*)$. When $S^*$ has the norm topology, $S^*$ is a completely regular Hausdorff space; and the compact open topology on $B(S^*)$ makes $B(S^*)$ into a complete Hausdorff linear topological space which is a locally $m$-convex topological algebra when multiplication is defined by $(A * B)(f) = A(f)B(f)$, where $A, B \in B(S^*)$ and $f \in S^*$ [28]. A net $(A_d)$ in $B(S^*)$ converges to an element $A \in B(S^*)$ if and only if for any sequence $(f_n) \subseteq S^*$ with $f_n \rightarrow f_0 \in S^*$ and for $\varepsilon > 0$, there is an $d(\varepsilon)$ such that $d > d(\varepsilon)$ implies $|A_d(f_n) - A(f_n)| < \varepsilon$ for all $n \geq 0$.

The elements of $L(S)$ all belong to $C(S)^*$. Now $L(S)$ will be considered as a subset of $C(S)^* \subseteq B(S^*)$ and given the relative topology of $B(S^*)$. It is easy to see that $C(S)^*$ is complete in $B(S^*)$ when $S^*$ separates points of $S$, so it will be assumed from here on that $S^*$ separates points of $S$. The algebra that will be considered is then $L(S)$ in $B(S^*)$. First it is necessary to know the
Lemma (3.33): If A, B ∈ L(S), then A*B(f) = A(f)B(f) for all f ∈ S*.

Proof: A*B(f) = \[ \sum_{i=1}^{n} a_i \sum_{j=1}^{m} b_j x_i y_j(f) \]
= \[ \sum_{i=1}^{n} a_i \sum_{j=1}^{m} b_j f(x_i)f(y_j) \]
= \[ \sum_{i=1}^{n} a_i f(x_i) \sum_{j=1}^{m} b_j f(y_j) \]
= A(f)B(f).

Now L(S) is a subalgebra of B(S*) and its closure in B(S*) is a complete locally m-convex topological algebra [28], which is again a convolution algebra and contains the restriction of the elements of L to S*.

Lemma (3.34): The mapping h:S → L(S) defined by h(x) = \( \overline{x} \) (restricted to S*) is an isomorphism and a homeomorphism onto [\( \overline{x} : x \in S \)].

Proof: The mapping h is clearly a semigroup isomorphism. It is, therefore, sufficient to show that h is continuous, for S is compact and h is then a homeomorphism. Let \( \{x_d\} \) be a net in S with \( x_d \to x \). To show \( \overline{x}_d \to \overline{x} \) in L(S), let \( \{f_n\} \) be a sequence in S* with \( f_n \to f_0 \). For \( \varepsilon > 0 \), there is an \( d(\varepsilon) \) such that \( d > d(\varepsilon) \) implies \( |f_0(x_d) - f_0(x)| < \varepsilon/3 \); also there is an \( n(\varepsilon) \) such that \( n > n(\varepsilon) \) implies \( \|f_n - f_0\| < \varepsilon/3 \). If \( d > d(\varepsilon) \) and \( n > n(\varepsilon) \), then
\[
|\overline{x}_d(f_n) - \overline{x}_d(f_0)| + |\overline{x}_d(f_0) - \overline{x}_d(f_0)| + |\overline{x}_d(f_0) - \overline{x}_d(f_0)| < \|f_n - f_0\| + \|f_0(x_d) - f_0(x)| < \varepsilon.
\]
For each \( i \leq n(\varepsilon) \), there is a \( d_i(\varepsilon) \) such that \( d > d_i(\varepsilon) \) implies \( |\overline{x}_d(f_i) - \overline{x}(f_i)| < \varepsilon. \)
Let \( d_0 > [d(\varepsilon), d_1(\varepsilon), \ldots, d_n(\varepsilon)] \); then \( d > d_0 \) implies \( |\overline{x}_d(f_k) - \overline{x}(f_k)| < \varepsilon \) for all \( k \geq 0 \). Hence, \( \overline{x}_d \rightarrow \overline{x} \) and \( h \) is continuous.

A theorem similar to Theorem (3.17) can now be proved. It is

**Theorem (3.35):** For each continuous multiplicative linear functional \( \theta \) on \( L(S) \), there is a unique \( f \in S^* \) such that \( \theta(A) = A(f) \) for all \( A \in L(S) \). Conversely, each \( f \in S^* \) defines a continuous multiplicative linear functional on \( L(S) \) by the above formula, and any two such are distinct.

**Proof:** Let \( \theta \) be a continuous multiplicative linear functional on \( L(S) \). Define \( f : S \rightarrow \mathbb{Z} \) by \( f(x) = \theta(\overline{x}) \). From Lemma (3.34), \( f(x) = \theta h(x) \) and is a continuous homomorphism, and therefore \( f \in S^* \). Let \( A = \sum_{i=1}^{n} a_i \overline{x_i} \) be any element of \( L(S) \); then \( \theta(A) = \sum_{i=1}^{n} a_i \theta(\overline{x_i}) = \sum_{i=1}^{n} a_i f(x_i) = A(f) \). For \( A \in L(S) \), there is a net \( \{A_d\} \) in \( L(S) \) such that \( A_d \rightarrow A \).

Now \( \theta \) is continuous and so \( \theta(A_d) \rightarrow \theta(A) \); but \( \theta(A_d) = A_d(f) \), and by the definition of convergence in \( B(S^*) \), \( A_d(f) \rightarrow A(f) \) so that \( \theta(A) = A(f) \).

Conversely, for \( f \in S^* \) the function \( \theta : L(S) \rightarrow \mathbb{Z} \) defined by \( \theta(A) = A(f) \) is a continuous multiplicative linear functional. Only the multiplicative property need be checked; and \( \theta(A \cdot B) = A \cdot B(f) = A(f)B(f) = \theta(A)\theta(B) \),
so that the desired result is established. If \( f_1, f_2 \in S^* \) and \( f_1 \neq f_2 \), then \( \theta_1(\bar{x}) = \bar{x}(f_1) = f_1(x) \neq f_2(x) = \bar{x}(f_2) = \theta_2(\bar{x}) \) and the uniqueness is established.

Michael, in his monograph on locally m-convex topological algebras [28], has shown that if \( \overline{L(S)} \) is metrizable, then the space of all continuous multiplicative linear functionals has a topology in which it is hemi-compact. This topology could then be transported to \( S^* \) by use of the above theorem. However, little is known about hemi-compact spaces and a weakness of the algebra \( \overline{L(S)} \) should be noted. The space of maximal modular ideals of a Banach algebra is compact and the character group of a locally compact commutative group is again a locally compact group. It would be interesting to know if \( S^* \) is locally compact in some convenient topology.
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EXAMINATION AND THESIS REPORT

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Major Field: Mathematics
Title of Thesis: Homomorphisms and Topological Semigroups

Approved:

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July 22, 1958