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SPECTRAL THEOREM APPROACH TO THE CHARACTERISTIC FUNCTION OF QUANTUM OBSERVABLES

ANDREAS BOUKAS* AND PHILIP FEINSILVER

Abstract. Using the spectral theorem we compute the *Quantum Fourier Transform* or *Vacuum Characteristic Function* $\langle \Phi, e^{itH}\Phi\rangle$ of an observable H defined as a self-adjoint sum of the generators of a finite-dimensional Lie algebra, where Φ is a unit vector in a Hilbert space $\mathcal{H}$. We show how *Stone's formula* for computing the spectral resolution of a Hilbert space self-adjoint operator, can serve as an alternative to the traditional reliance on *splitting* or *disentanglement* formulas for the operator exponential.

1. Introduction

The simplest quantum analogue of a classical probability space (Ω, σ, μ) where Ω is a set, σ is a sigma-algebra of subsets of Ω and μ is a measure defined on σ with $\mu(\Omega) = 1$, is a *finite dimensional quantum probability space* [14] defined as a triple $(\mathcal{H}, \mathcal{P}(\mathcal{H}), \rho)$ where $\mathcal{H}$ is a finite dimensional Hilbert space, $\mathcal{P}(\mathcal{H})$ is the set of projections (called *events*) $E : \mathcal{H} \to \mathcal{H}$ and $\rho : \mathcal{H} \to \mathcal{H}$ is a *state* on $\mathcal{H}$, i.e., a positive operator of unit trace. We call $\mathrm{tr}\rho\mathrm{E}$ the *probability of the event E in the state* ρ. For a *quantum observable* H, i.e. for a symmetric or Hermitian matrix H, the *characteristic function* or *Fourier transform* of H in the state ρ is defined as $\mathrm{tr}\rho\mathrm{e}^{\mathrm{itH}}$. If ρ is a *pure state* defined in terms of a unit vector u, i.e., if $\rho = |u\rangle\langle u|$ then the characteristic function of H in the state defined by u is $\langle u, e^{itH}u\rangle$. By the spectral theorem, if $H = \sum_n \lambda_n E_n$ then for every continuous function $\phi : \mathbb{R} \to \mathbb{C}$,

$$\phi(H) = \sum_n \phi(\lambda_n) E_n.$$

Therefore, for $\phi(H) = e^{itH}$ we have

$$\langle u, e^{itH}u\rangle = \langle u, \sum_n e^{it\lambda_n} E_n u\rangle = \sum_n e^{it\lambda_n}\langle u, E_n u\rangle,$$

where we have assumed that the inner product is linear in the second and conjugate linear in the first argument. If the Hilbert space $\mathcal{H}$ is infinite dimensional then the

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above sums are replaced by spectral integrals with respect to a resolution of the identity $\{E_\lambda \ : \ \lambda \in \mathbb{R}\}$ and we have the corresponding formulas

$$H = \int_{\mathbb{R}} \lambda \, dE_\lambda \ ; \ \phi(H) = \int_{\mathbb{R}} \phi(\lambda) \, dE_\lambda$$

and

$$\langle u, e^{itH} u \rangle = \int_{\mathbb{R}} e^{it\lambda} \, d\langle u, E_\lambda u \rangle \ .$$

For compact self-adjoint operators H, the above spectral integrals are reduced to finite or infinite sums over the nonzero eigenvalues of H, see e.g. [17], Theorem 4.2.

The above probabilistic interpretation is based on Bochner's theorem (see [18] p. 346) which states that a *positive definite* continuous function $f : \mathbb{R} \mapsto \mathbb{C}$, i.e., a continuous function f such that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(t-s) \phi(t) \bar{\phi}(s) \, dt \, ds \geq 0,$$

for every continuous function $\phi : \mathbb{R} \mapsto \mathbb{C}$ with compact support, can be represented as

$$f(t) = \int_{\mathbb{R}} e^{it\lambda} \, dv(\lambda),$$

where v is a non-decreasing right-continuous bounded function. If $f(0) = 1$ then such a function v defines a probability measure on $\mathbb{R}$ and Bochner's theorem says that f is the Fourier transform of a probability measure, i.e., the characteristic function of a random variable that follows the probability distribution defined by v. Moreover, the condition of positive definiteness of f is necessary and sufficient for such a representation. The function $f(t) = \langle u, e^{itH} u \rangle$, where u is a unit vector and H is a self-adjoint operator as described above, is an example of such a positive definite function.

In this paper we use this spectral theorem based approach to compute the characteristic function of several quantum random variables H defined as self-adjoint sums of the generators of some finite dimensional Lie algebras of interest in quantum mechanics (the only reason why a Lie structure is assumed is because splitting the exponential of a sum of operators is usually done through a Campbell-Baker-Hausdorff type formula that relies on commutation relations).

If $\mathcal{H} = \mathbb{R}^n$ or $\mathcal{H} = \mathbb{C}^n$ for vectors $u = (u_1, ..., u_n)$ and $v = (v_1, ..., v_n)$ in $\mathcal{H}$ we will use the standard inner products $\langle u, v \rangle = uv^T$ and $\langle u, v \rangle = \bar{u} v^T$ respectively. The identity matrix/operator is denoted by I, while δ denotes Dirac's delta function defined, for a *test function* ϕ, by

$$\int_{\mathbb{R}} \delta(x-a) \phi(x) \, dx = \int_{\mathbb{R}} \delta(a-x) \phi(x) \, dx = \phi(a).$$

We define the Fourier transform of f by

$$\hat{f}(t) = (Uf)(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{i\lambda t} f(\lambda) d\lambda,$$

and the inverse Fourier transform of $\hat{f}$ by

$$f(\lambda) = \left(U^{-1}\hat{f}\right)(\lambda) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-i\lambda t}\hat{f}(t)dt,$$

so the inverse Fourier transform of

$$(2\pi)^{-1/2} \langle u, e^{itH}u\rangle$$

gives the probability density function $p(\lambda)$ of H.

For $a \in \mathbb{C}$, let

$$R(a; H) = (a - H)^{-1}$$

denote the resolvent of an operator H. The spectral resolution $\{E_\lambda \,|\, \lambda \in \mathbb{R}\}$ of a bounded or unbounded self-adjoint operator H in a complex separable Hilbert Space $\mathcal{H}$, is given by *Stone's formula* (see [5], Theorems X.6.1 and XII.2.10))

$$E((a,b)) = \lim_{\delta\to 0^+} \lim_{\epsilon\to 0^+} \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} \left(R(t - \epsilon i; H) - R(t + \epsilon i; H)\right)\, dt,$$

where (a, b) is the open interval $a < \lambda < b$, $R(t \pm \epsilon i; H) = (t \pm \epsilon i - H)^{-1}$, and the limit is in the strong operator topology. For $a \to -\infty$ and $b = \lambda$ we have

$$\begin{aligned} E_\lambda = E((-\infty, \lambda]) &= \lim_{\rho\to 0^+} E((-\infty, \lambda + \rho)) \\ &= \lim_{\rho\to 0^+} \lim_{\delta\to 0^+} \lim_{\epsilon\to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\lambda+\rho-\delta} \left(R(t - \epsilon i; H) - R(t + \epsilon i; H)\right)\, dt. \end{aligned}$$

In particular (see [16], Theorem 4.31), for $f, g \in \mathcal{H}$,

$$\langle f, E_\lambda g\rangle = \lim_{\rho\to 0^+} \lim_{\delta\to 0^+} \lim_{\epsilon\to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\lambda+\rho-\delta} \langle f, \left(R(t - \epsilon i; H) - R(t + \epsilon i; H)\right) g\rangle\, dt.$$

Thus, for a unit vector u, the *vacuum resolution of the identity* (terminology coming from the case when $u = \Phi$, the *vacuum vector* in a Fock space) of the operator H is given by

$$\langle u, E_\lambda u\rangle = \lim_{\rho\to 0^+} \lim_{\delta\to 0^+} \lim_{\epsilon\to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\lambda+\rho-\delta} \langle u, \left(R(t - \epsilon i; H) - R(t + \epsilon i; H)\right) u\rangle\, dt. \tag{1.1}$$

In Section 9 we will show how, using formula (1.1), we can avoid the reliance on *splitting or disentanglement lemmas*, such as Lemma 8.3 of Section 8 for the splitting of operator exponentials, in order to compute the characteristic function of a quantum random variable. In particular, Stone's formula frees us from any dependence on Lie algebraic structures. However, the difficulty of obtaining a splitting lemma, is replaced by that of computing the resolvent and the resulting spectral integrals.

2. Quantum Observables in $\mathfrak{sl}(2, \mathbb{R})$

The Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ of real (2×2) matrices of zero trace, is generated [7] by the matrices

$$\Delta = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, R = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \rho = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with commutation relations

$$[\Delta, R] = \rho, [\rho, R] = 2R, [\rho, \Delta] = -2\Delta.$$

We notice that the matrix

$$H = R - \Delta + \rho = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

is real symmetric, thus it is a quantum observable. Its eigenvalues are

$$\lambda_1 = -\sqrt{2}, \lambda_2 = \sqrt{2},$$

with corresponding eigenspaces

$$V_1 = \{xv_1 : v_1 = \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix}, x \in \mathbb{R}\},$$
$$V_2 = \{xv_1 : v_2 = \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix}, x \in \mathbb{R}\},$$

corresponding normalized basic eigenvectors

$$u_1 = \frac{v_1}{||v_1||} = \begin{pmatrix} \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \\ \\ \frac{1}{\sqrt{4-2\sqrt{2}}} \end{pmatrix}, u_2 = \frac{v_2}{||v_2||} = \begin{pmatrix} \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \\ \\ \frac{1}{\sqrt{4+2\sqrt{2}}} \end{pmatrix},$$

and eigen-projections

$$E_1 = \langle u_1, u_1 \rangle = u_1^T u_1 = \begin{pmatrix} \frac{1}{4}(2-\sqrt{2}) & -\frac{1}{2\sqrt{2}} \\ \\ -\frac{1}{2\sqrt{2}} & \frac{1}{4-2\sqrt{2}} \end{pmatrix},$$

$$E_2 = \langle u_2, u_2 \rangle = u_2^T u_2 = \begin{pmatrix} \frac{1}{4}(2+\sqrt{2}) & \frac{1}{2\sqrt{2}} \\ \\ \frac{1}{2\sqrt{2}} & \frac{1}{4+2\sqrt{2}} \end{pmatrix}.$$

We notice that E_1 and E_2 are a resolution of the identity, i.e.,

$$I = E_1 + E_2$$

and

$$H = \lambda_1 E_1 + \lambda_2 E_2.$$

Moreover

$$e^{itH} = \begin{pmatrix} \cos(\sqrt{2}t) + i\frac{1}{\sqrt{2}}\sin(\sqrt{2}t) & i\frac{1}{\sqrt{2}}\sin(\sqrt{2}t) \\ \\ i\frac{1}{\sqrt{2}}\sin(\sqrt{2}t) & \cos(\sqrt{2}t) - i\frac{1}{\sqrt{2}}\sin(\sqrt{2}t) \end{pmatrix}.$$

If $u = (a, b)$ is a unit vector in $\mathbb{R}^2$ then for $t \in \mathbb{R}$,

$$\begin{aligned}\langle u, e^{itH}u\rangle &= e^{it\lambda_1}\langle u, E_1 u\rangle + e^{it\lambda_2}\langle u, E_2 u\rangle = e^{it\lambda_1}u^T E_1 u + e^{it\lambda_2}u^T E_2 u \\ &= (a^2+b^2)\cos(\sqrt{2}t) + i\frac{a^2-b^2+2ab}{\sqrt{2}}\sin(\sqrt{2}t) \\ &= \cos(\sqrt{2}t) + i\frac{a^2-b^2+2ab}{\sqrt{2}}\sin(\sqrt{2}t),\end{aligned}$$

so H follows a Bernoulli distribution with probability density function

$$\begin{aligned}p_{a,b}(\lambda) = \frac{1}{4}\Big(&\Big(2+\sqrt{2}(a^2+2ab-b^2)\Big)\,\delta(\sqrt{2}-\lambda) \\ &+\Big(2-\sqrt{2}(a^2+2ab-b^2)\Big)\,\delta(\sqrt{2}+\lambda)\Big),\end{aligned}$$

that is, H takes the values $\lambda_1 = -\sqrt{2}$ and $\lambda_2 = \sqrt{2}$ with probabilities

$$P(H = -\sqrt{2}) = \frac{1}{4}\Big(2-\sqrt{2}(a^2+2ab-b^2)\Big)$$

and

$$P(H = \sqrt{2}) = \frac{1}{4}\Big(2+\sqrt{2}(a^2+2ab-b^2)\Big)$$

respectively. In particular if $u = (a, b)$ is a *Fock vacuum vector*, i.e. , if we require [7] that

$$\Delta u = \mathbf{0} \text{ and } \rho u = cu\,,\ c \in \mathbb{R},$$

then we find that $c = -1$ and $a = 0$ therefore $b = \pm 1$ and we obtain the characteristic function in the *vacuum state* $\Phi = (0, \pm 1)$,

$$\langle \Phi, e^{itH}\Phi\rangle = \cos(\sqrt{2}t) - i\frac{1}{\sqrt{2}}\sin(\sqrt{2}t),$$

so H follows a Bernoulli distribution with probability density function

$$p_{0,\pm1}(\lambda) = \frac{1}{4}\left(\left(2-\sqrt{2}\right)\delta(\sqrt{2}-\lambda) + \left(2+\sqrt{2}\right)\delta(\sqrt{2}+\lambda)\right).$$

The matrix

$$H_0 = R - \Delta = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

is also an observable with spectral resolution

$$H_0 = \lambda_1 E_1 + \lambda_2 E_2 = (-1)\cdot\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{pmatrix} + 1\cdot\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}$$

and

$$e^{itH_0} = \begin{pmatrix}\cos t & i\sin t\\ i\sin t & \cos t\end{pmatrix}.$$

The characteristic function of H_0 is

$$\langle u, e^{itH_0}u\rangle = (a^2+b^2)\cos t + 2iab\sin t = \cos t + 2iab\sin t,$$

so H follows a Bernoulli distribution with probability density function

$$p_{a,b}(\lambda) = \left(\frac{1}{2}+ab\right)\delta(\lambda-1) + \left(\frac{1}{2}-ab\right)\delta(\lambda+1).$$

In the Fock vacuum state $\Phi = (0, \pm 1)$ the characteristic function reduces (see also [7]) to

$$\langle \Phi, e^{itH_0}\Phi \rangle = \cos t,$$

so

$$p_{0,\pm 1}(\lambda) = \frac{1}{2}\delta(\lambda - 1) + \frac{1}{2}\delta(\lambda + 1),$$

while in the states

$$u = \pm \frac{1}{\sqrt{2}}(1, 1)$$

we have

$$\langle u, e^{itH_0}u \rangle = e^{it},$$

so H follows a discrete probability distribution with probability density function

$$p_{\pm \frac{1}{\sqrt{2}}(1,1)}(\lambda) = \delta(\lambda - 1),$$

i.e., in the state defined by $u = \pm \frac{1}{\sqrt{2}}(1, 1)$, H takes the value $\lambda = 1$ with probability 1. We remark that H_0 can also be regarded as a *Krawtchouk-Griffiths observable* (see [6], Example 5.12).

3. Pauli Matrices and $\mathfrak{su}(2)$

The *Pauli matrices* σ_j, $j = 1, 2, 3$, of quantum mechanics,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with commutation relations

$$[\sigma_1, \sigma_2] = 2i\sigma_3, [\sigma_2, \sigma_3] = 2i\sigma_1, [\sigma_3, \sigma_1] = 2i\sigma_2,$$

are Hermitian, i.e., self-adjoint. The matrices $i\sigma_1, -i\sigma_2, i\sigma_3$ generate the Lie algebra

$$\mathfrak{su}(2) = \{ \begin{pmatrix} ia & -\bar{z} \\ z & -ia \end{pmatrix} \, : \, a \in \mathbb{R}, z \in \mathbb{C} \}$$

of traceless anti-hermitian (2×2) matrices. The spectral decompositions of the quantum observables corresponding to the Pauli matrices are

$$\sigma_1 = (-1) \cdot \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} + 1 \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

$$\sigma_2 = (-1) \cdot \begin{pmatrix} \frac{1}{2} & \frac{i}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{pmatrix} + 1 \cdot \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix},$$

$$\sigma_3 = (-1) \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

with complex exponentials

$$e^{it\sigma_1} = \begin{pmatrix} \cosh(it) & \sinh(it) \\ \sinh(it) & \cosh(it) \end{pmatrix} = \begin{pmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{pmatrix},$$
$$e^{it\sigma_2} = \begin{pmatrix} \cosh(it) & -i\sinh(it) \\ i\sinh(it) & \cosh(it) \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix},$$
$$e^{it\sigma_3} = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix},$$

while, for $u = (a, b) \in \mathbb{C}^2$ with $|a|^2 + |b|^2 = 1$, using for $j = 1, 2, 3$,

$$\langle u, e^{it\sigma_j} u\rangle = e^{it\lambda_1}\langle u, E_1 u\rangle + e^{it\lambda_2}\langle u, E_2 u\rangle = e^{it\lambda_1}\bar{u}^T E_1 u + e^{it\lambda_2}\bar{u}^T E_2 u,$$

we obtain the characteristic functions of $\sigma_1, \sigma_2, \sigma_3$,

$$\langle u, e^{it\sigma_1} u\rangle = \cosh(it) + (b\bar{a} + a\bar{b})\sinh(it) = \cos t + i(b\bar{a} + a\bar{b})\sin t$$
$$\langle u, e^{it\sigma_2} u\rangle = \cosh(it) + i(a\bar{b} - b\bar{a})\sinh(it) = \cos t + (b\bar{a} - a\bar{b})\sin t$$
$$\langle u, e^{it\sigma_3} u\rangle = |a|^2 e^{it} + |b|^2 e^{-it},$$

so $\sigma_1, \sigma_2, \sigma_3$ are quantum random variables following a Bernoulli distribution with probability density function

$$p_1(\lambda) = \frac{1}{2}\left(1 + b\bar{a} + a\bar{b}\right)\delta(\lambda - 1) + \frac{1}{2}\left(1 - b\bar{a} - a\bar{b}\right)\delta(\lambda + 1),$$
$$p_2(\lambda) = \frac{1}{2}\left(1 + b\bar{a} - a\bar{b}\right)\delta(\lambda - 1) + \frac{1}{2}\left(1 - b\bar{a} + a\bar{b}\right)\delta(\lambda + 1),$$
$$p_3(\lambda) = |a|^2\delta(\lambda - 1) + |b|^2\delta(\lambda + 1),$$

respectively.

4. Pauli Matrices and $\mathfrak{su}(1,1)$

The matrices

$$K_1 = \frac{i}{2}\sigma_2 = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}, K_2 = -\frac{i}{2}\,\sigma_1 = \begin{pmatrix} 0 & -\frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix}$$

and

$$K_0 = \frac{1}{2}\sigma_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix},$$

satisfy (see [13]) the $\mathfrak{su}(1,1)$ Lie algebra commutation relations:

$$[K_1, K_2] = -iK_0, [K_0, K_1] = iK_2, [K_2, K_0] = iK_1.$$

The matrix

$$H = i(K_1 + K_2) + K_0 = \frac{1}{2}\begin{pmatrix} 1 & 1+i \\ 1-i & -1 \end{pmatrix}$$

is Hermitian so it is a quantum observable. Its eigenvalues are

$$\lambda_1 = -\frac{\sqrt{3}}{2}, \lambda_2 = \frac{\sqrt{3}}{2},$$

with corresponding normalized basic eigenvectors

$$u_1 = \frac{1}{\sqrt{3-\sqrt{3}}}\begin{pmatrix} \frac{1-\sqrt{3}}{2} + i\frac{1-\sqrt{3}}{2} \\ 1 \end{pmatrix}, u_2 = \frac{1}{\sqrt{3+\sqrt{3}}}\begin{pmatrix} \frac{1+\sqrt{3}}{2} + i\frac{1+\sqrt{3}}{2} \\ 1 \end{pmatrix},$$

and eigen-projections

$$E_1 = u_1^T u_1 = \frac{1}{2}\begin{pmatrix} \frac{3-\sqrt{3}}{3} & \frac{\sqrt{3}-1}{\sqrt{3}-3}(1+i) \\ \frac{\sqrt{3}-1}{\sqrt{3}-3}(1-i) & \frac{3+\sqrt{3}}{3} \end{pmatrix},$$

$$E_2 = u_2^T u_2 = \frac{1}{2}\begin{pmatrix} \frac{3+\sqrt{3}}{3} & \frac{\sqrt{3}+1}{\sqrt{3}+3}(1+i) \\ \frac{\sqrt{3}+1}{\sqrt{3}+3}(1-i) & \frac{3-\sqrt{3}}{3} \end{pmatrix}.$$

We have

$$I = E_1 + E_2$$

and

$$H = \lambda_1 E_1 + \lambda_2 E_2.$$

Moreover

$$e^{itH} = \begin{pmatrix} \cos\left(\frac{\sqrt{3}t}{2}\right) + i\frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}t}{2}\right) & \frac{i-1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}t}{2}\right) \\ \frac{i+1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}t}{2}\right) & \cos\left(\frac{\sqrt{3}t}{2}\right) - i\frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}t}{2}\right) \end{pmatrix}.$$

If $u = (a, b)$ is a unit vector in $\mathbb{C}^2$ then for $t \in \mathbb{R}$

$$\langle u, e^{itH}u\rangle = e^{it\lambda_1}\langle u, E_1 u\rangle + e^{it\lambda_2}\langle u, E_2 u\rangle = e^{it\lambda_1}\bar{u}^T E_1 u + e^{it\lambda_2}\bar{u}^T E_2 u$$

$$= \left(\frac{i-1}{\sqrt{3}}\, b\bar{a} + \frac{i+1}{\sqrt{3}}\, a\bar{b} + \frac{i}{\sqrt{3}}(|a|^2 - |b|^2)\right)\sin\left(\frac{\sqrt{3}t}{2}\right) + \cos\left(\frac{\sqrt{3}t}{2}\right),$$

so H follows a Bernoulli distribution with probability density function

$$p_{a,b}(\lambda) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\left((a + (1+i)b)\bar{a} + ((1-i)a - b)\bar{b}\right)\right)\delta\left(\frac{\sqrt{3}}{2} - \lambda\right)$$

$$+ \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\left((a + (1+i)b)\bar{a} + ((1-i)a - b)\bar{b}\right)\right)\delta\left(\frac{\sqrt{3}}{2} + \lambda\right).$$

In particular, for $a = 1$ and $b = 0$,

$$\langle u, e^{itH}u\rangle = \cos\left(\frac{\sqrt{3}t}{2}\right) + \frac{i}{\sqrt{3}}\sin\left(\frac{\sqrt{3}t}{2}\right),$$

i.e., H follows a Bernoulli distribution with probability density function

$$p_{1,0}(\lambda) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right)\delta\left(\frac{\sqrt{3}}{2} - \lambda\right) + \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right)\delta\left(\frac{\sqrt{3}}{2} + \lambda\right),$$

while for $a = 0$ and $b = 1$,

$$\langle u, e^{itH} u\rangle = \cos\left(\frac{\sqrt{3}t}{2}\right) - \frac{i}{\sqrt{3}}\sin\left(\frac{\sqrt{3}t}{2}\right),$$

i.e., H follows a Bernoulli distribution with probability density function

$$p_{0,1}(\lambda) = \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right)\delta\left(\frac{\sqrt{3}}{2} - \lambda\right) + \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right)\delta\left(\frac{\sqrt{3}}{2} + \lambda\right).$$

5. The Casimir Element in $\mathfrak{so}(3)$

The Lie algebra $\mathfrak{so}(3)$ of (3×3) skew-symmetric matrices is generated by the matrices

$$L_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, L_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, L_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with commutation relations

$$[L_x, L_y] = L_z, [L_y, L_z] = L_x, [L_z, L_x] = L_y.$$

Associated with L_x, L_y, L_z is the self-adjoint central *Casimir element*

$$L = L_x^2 + L_y^2 + L_z^2 = -2I_3,$$

where I_3 is the (3×3) identity matrix. For a unit vector $u = (a, b, c)$ in $\mathbb{R}^3$ we have

$$\langle u, e^{itL} u\rangle = e^{-2it},$$

so L follows a discrete probability distribution with probability density function

$$p(\lambda) = \delta(\lambda + 2),$$

i.e., in the state defined by u, H takes the value $\lambda = -2$ with probability 1.

6. Quantum Observables in $\mathfrak{h}(3, \mathbb{R})$

The *Heisenberg algebra* $\mathfrak{h}$ is the three-dimensional Lie algebra with generators D, X, h satisfying the commutation relations

$$[D, X] = h, [D, h] = [X, h] = 0.$$

A matrix representation of $\mathfrak{h}$ is provided by the 3-dimensional matrix Lie algebra $\mathfrak{h}(3, \mathbb{R})$ defined as the vector space of matrices of the form

$$A = \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}, x, y, z \in \mathbb{R},$$

spanned by the matrices

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which are easily seen to satisfy the commutation relations

$$[D, X] = h, [D, h] = [X, h] = 0.$$

Clearly, no linear combination of D, X and h can be symmetric on all of $\mathbb{R}^3$ with respect to the usual inner product. Nevertheless, the matrix

$$H = (D + X + h) + (D + X + h)^T = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = J - I,$$

where J is the all-ones (3×3) matrix, is a quantum observable with eigenvalues: $\lambda_1 = 2$, with multiplicity one and corresponding normalized basic eigenvector

$$u_1 = \frac{1}{\sqrt{3}}(1, 1, 1),$$

and $\lambda_2 = -1$ with multiplicity 2 and corresponding orthonormalized basis eigenvectors

$$u_2 = \frac{1}{\sqrt{2}}(-1, 0, 1), u_3 = \frac{1}{\sqrt{6}}(-1, 2, -1).$$

The associated eigen-projections are

$$\begin{aligned} E_1 &= \langle u_1, u_1 \rangle = u_1^T u_1 = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \\ E_2 &= \langle u_2, u_2 \rangle + \langle u_3, u_3 \rangle = u_2^T u_2 + u_3^T u_3 \\ &= \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} + \frac{1}{6} \begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \end{aligned}$$

with

$$I = E_1 + E_2,$$

and

$$H = \lambda_1 E_1 + \lambda_2 E_2.$$

If $u = (a, b, c)$ is a unit vector in $\mathbb{R}^3$ then for $t \in \mathbb{R}$:

$$\begin{aligned} \langle u, e^{itH} u \rangle &= e^{it\lambda_1} \langle u, E_1 u \rangle + e^{it\lambda_2} \langle u, E_2 u \rangle \\ &= \left(1 - \frac{(a + b + c)^2}{3}\right) e^{-it} + \frac{(a + b + c)^2}{3} e^{2it}, \end{aligned}$$

i.e., H follows a Bernoulli distribution with probability density function

$$p_{a,b,c}(\lambda) = \left(1 - \frac{(a + b + c)^2}{3}\right) \delta(\lambda + 1) + \frac{(a + b + c)^2}{3} \delta(\lambda - 2).$$

7. Multiplication and Differentiation L^2-Operators

The classical Heisenberg Lie algebra of quantum mechanics has generators P, X, I and non-zero commutation relations among generators

$$[P, X] = -i\hbar I.$$

The (self-adjoint) position, momentum and identity operators defined in $L^2(\mathbb{R}, \mathbb{C})$ with inner product

$$\langle f, g\rangle = \frac{1}{\sqrt{\hbar}} \int_{\mathbb{R}} \overline{f(x)} g(x)\, dx,$$

by

$$(X\, f)(x) = x\, f(x), (P\, f)(x) = -i\, \hbar\, f'(x), (I f)(x) = f(x),$$

realize the Heisenberg commutation relations on $\mathrm{dom}(X) \cap \mathrm{dom}(P)$, where

$$\mathrm{dom}(X) = \{f \in L^2(\mathbb{R}, \mathbb{C})) \,:\, \int_{\mathbb{R}} x^2\, |f(x)|^2\, dx < +\infty\}$$

and

$$\mathrm{dom}(P) = \left\{ f \in L^2(\mathbb{R}, \mathbb{C}) \,:\, \begin{array}{l} f \text{ is absolutely continuous,} \\ \int_{\mathbb{R}} \left| \frac{df(x)}{dx} \right|^2 dx < +\infty \end{array} \right\},$$

are respectively the, dense in $L^2(\mathbb{R}, \mathbb{C})$, domains of X and P (see [8], Sec. 2.3 and [18], Sec VII 3). Functions in the domain of P are continuous and vanish at infinity (see [15], Section 5.6). Using

$$X = \sqrt{\hbar}\, \frac{a + a^\dagger}{\sqrt{2}}\ ,\ P = \sqrt{\hbar}\, \frac{a - a^\dagger}{\sqrt{2} i}, \tag{7.1}$$

we obtain the *Boson pair*

$$a = \frac{X + iP}{\sqrt{2\hbar}}\ ,\ a^\dagger = \frac{X - iP}{\sqrt{2\hbar}}, \tag{7.2}$$

with

$$[a, a^\dagger] = \mathbf{1}\ ,\ a^* = a^\dagger\ ,\ a\Phi = 0\ , \text{where}\ \ \Phi = \Phi(x) = \pi^{-1/4}\, e^{-\frac{x^2}{2\hbar}}.$$

We notice that

$$\|\Phi\|^2 = \langle \Phi, \Phi\rangle = 1.$$

Moreover

$$\Phi'(x) = -\frac{1}{\hbar \pi^{1/4}} x e^{-\frac{x^2}{2\hbar}}\ ;\ \lim_{x \to \pm\infty} \Phi(x) = 0,$$

so Φ' is bounded on $\mathbb{R}$, which means that Φ is Lipschitz and therefore absolutely continuous on $\mathbb{R}$. Since

$$\int_{\mathbb{R}} x^2\, |\Phi(x)|^2\, dx = \frac{\hbar^{3/2}}{2} < +\infty, \int_{\mathbb{R}} \left| \frac{d\Phi(x)}{dx} \right|^2 dx = \frac{1}{\hbar^{1/2}} < +\infty,$$

Φ will be our prototype unit vector in $\mathrm{dom}(X) \cap \mathrm{dom}(P)$.

Theorem 7.1. *If units are chosen so that $\hbar = 1$, then the vacuum characteristic function of the quantum observable*

$$H = X + P$$

is

$$\langle \Phi, e^{itH}\Phi\rangle = e^{-\frac{t^2}{2}},$$

i.e., the underlying probability distribution is Gaussian.

Proof. The spectral resolutions of X and P are (see [18], Sec. XI.5 and XI.6)

$$X = \int_{\mathbb{R}} \lambda \, dE_\lambda \ ; \ P = \int_{\mathbb{R}} \lambda \, dE'_\lambda = \int_{\mathbb{R}} \lambda \, d(UE_\lambda U^{-1}),$$

where

$$E_\lambda f(t) = \begin{cases} f(t), & \text{if } t \le \lambda \\ 0, & \text{if } t > \lambda \end{cases}$$

and

$$E'_\lambda = U\, E_\lambda\, U^{-1},$$

where

$$(Uf)(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{ist} f(s) ds = \lim_{n\to+\infty} (2\pi)^{-1/2} \int_{-n}^{n} e^{ist} f(s) ds$$

is the Fourier transform of f and

$$\left(U^{-1}f\right)(t) = (U^* f)(t) = (Uf)(-t).$$

By the Campbell–Baker–Hausdorff formula [11] (see also [7], splitting formula for the Heisenberg group) we have

$$e^{itP}e^{itX} = e^{it(P+X)+\frac{1}{2}[itP,itX]} = e^{it(P+X)-\frac{t^2}{2}(-i\hbar)} = e^{it(P+X)}e^{\frac{i\hbar t^2}{2}},$$

where both sides of the above are unitary operators. Therefore,

$$e^{it(P+X)} = e^{itP}e^{itX}e^{-\frac{i\hbar t^2}{2}} = e^{itP}e^{itX}e^{-\frac{it^2}{2}}.$$

Thus

$$\begin{aligned}
\langle \Phi, e^{itH}\Phi\rangle &= e^{-\frac{it^2}{2}} \langle \Phi, e^{itP}e^{itX}\Phi\rangle = e^{-\frac{it^2}{2}} \langle e^{-itP}\Phi, e^{itX}\Phi\rangle \\
&= e^{-\frac{it^2}{2}} \langle \int_{\mathbb{R}} e^{-it\lambda'}\, d(UE_{\lambda'}U^{-1})\Phi, \int_{\mathbb{R}} e^{it\lambda}\, dE_\lambda \Phi\rangle \\
&= e^{-\frac{it^2}{2}} \int_{\mathbb{R}} e^{it\lambda}\, d_\lambda \langle \int_{\mathbb{R}} e^{-it\lambda'}\, d(UE_{\lambda'}U^{-1})\Phi, E_\lambda\Phi\rangle \\
&= e^{-\frac{it^2}{2}} \int_{\mathbb{R}}\int_{\mathbb{R}} e^{it\lambda}e^{it\lambda'}\, d_\lambda\, d_{\lambda'} \langle E_{\lambda'}U^{-1}\Phi, U^*E_\lambda\Phi\rangle.
\end{aligned}$$

Now,

$$(U^{-1}\Phi)(t) = (U\Phi)(-t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ist}\Phi(s) ds = \pi^{-1/4} e^{-\frac{t^2}{2}},$$

so

$$(E_{\lambda'}U^{-1}\Phi)(t) = \pi^{-1/4} e^{-\frac{t^2}{2}} \chi_{(-\infty,\lambda']}(t).$$

Similarly,

$$(E_\lambda\Phi)(t) = \Phi(t)\chi_{(-\infty,\lambda]}(t),$$

where

$$\chi_{(-\infty,\lambda]}(t) = \begin{cases} 1, & \text{if } t \leq \lambda \\ 0, & \text{if } t > \lambda \end{cases}$$

and

$$\begin{aligned}(U^*E_\lambda\Phi)(t) = (UE_\lambda\Phi)(-t) &= (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{-ist}\Phi(s)\chi_{(-\infty,\lambda]}(s)ds \\ &= (2\pi)^{-1/2}\int_{-\infty}^{\lambda} e^{-ist}\Phi(s)ds.\end{aligned}$$

Thus

$$\begin{aligned}d_\lambda\, d_{\lambda'}\langle E_{\lambda'}U^{-1}\Phi, U^*E_\lambda\Phi\rangle &= d_\lambda\, d_{\lambda'}\int_{-\infty}^{\infty}\overline{(E_{\lambda'}U^{-1}\Phi)(t)}\,(U^*E_\lambda\Phi)(t)\,dt \\ &= \frac{1}{\pi\sqrt{2}}\,d_\lambda\, d_{\lambda'}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\chi_{(-\infty,\lambda']}(t)\int_{-\infty}^{\lambda} e^{-ist}e^{-\frac{s^2}{2}}\,ds\,dt \\ &= \frac{1}{\pi\sqrt{2}}\,d_\lambda\, d_{\lambda'}\int_{-\infty}^{\lambda'} e^{-\frac{t^2}{2}}\int_{-\infty}^{\lambda} e^{-ist}e^{-\frac{s^2}{2}}\,ds\,dt \\ &= \frac{1}{\pi\sqrt{2}}\,d_\lambda\left(e^{-\frac{\lambda'^2}{2}}\int_{-\infty}^{\lambda} e^{-is\lambda'}e^{-\frac{s^2}{2}}\,ds\,d\lambda'\right) \\ &= \frac{1}{\pi\sqrt{2}}\left(e^{-\frac{\lambda'^2}{2}}\,e^{-i\lambda\lambda'}e^{-\frac{\lambda^2}{2}}\right)d\lambda\,d\lambda'.\end{aligned}$$

Therefore, using the integration formula

$$\int_{-\infty}^{\infty} e^{-(ax+ib)^2}\,dx = \frac{\sqrt{\pi}}{a}\ ;\ a,b\in\mathbb{R},\ a>0,$$

twice, we obtain

$$\begin{aligned}\langle\Phi, e^{itH}\Phi\rangle &= e^{-\frac{it^2}{2}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{it\lambda}e^{it\lambda'}\frac{1}{\pi\sqrt{2}}\left(e^{-\frac{\lambda'^2}{2}}\,e^{-i\lambda\lambda'}e^{-\frac{\lambda^2}{2}}\right)d_\lambda\, d_{\lambda'} \\ &= \frac{1}{\pi\sqrt{2}}e^{-\frac{it^2}{2}}\int_{\mathbb{R}} e^{it\lambda'-\frac{\lambda'^2}{2}}\left(\int_{\mathbb{R}} e^{it\lambda-\frac{\lambda^2}{2}-i\lambda\lambda'}\,d\lambda\right)d\lambda' \\ &= \frac{1}{\pi\sqrt{2}}e^{-\frac{it^2}{2}}\int_{\mathbb{R}} e^{it\lambda'-\frac{\lambda'^2}{2}}\left(e^{-2(\lambda'-t)^2}\int_{\mathbb{R}} e^{-\left(\frac{1}{\sqrt{2}}\lambda-i\sqrt{2}(\lambda'-t)\right)^2}\,d\lambda\right)d\lambda' \\ &= \frac{1}{\pi\sqrt{2}}e^{-\frac{it^2}{2}}\int_{\mathbb{R}} e^{it\lambda'-\frac{\lambda'^2}{2}}\left(e^{-2(\lambda'-t)^2}\sqrt{2\pi}\right)d\lambda' \\ &= \frac{1}{\sqrt{\pi}}e^{-\frac{(i+1)t^2}{2}}\int_{\mathbb{R}} e^{-\lambda'^2+(i+1)t\lambda'}d\lambda' \\ &= \frac{1}{\sqrt{\pi}}e^{-\frac{(i+1)t^2}{2}}\left(e^{\frac{it^2}{2}}\int_{\mathbb{R}} e^{-\left(\lambda'-\frac{t(i+1)}{2}\right)^2}d\lambda'\right) \\ &= \frac{1}{\sqrt{\pi}}e^{-\frac{(i+1)t^2}{2}}\left(e^{\frac{it^2}{2}}\sqrt{\pi}\right) \\ &= e^{-\frac{t^2}{2}}.\end{aligned}$$

□

8. Boson and X, P Form of $\mathfrak{su}(1,1)$

If K_0, K_1, K_2 satisfy the $\mathfrak{su}(1,1)$ commutation relations then [13] the change of basis operators

$$K_+ = K_1 + iK_2, K_- = K_1 - iK_2, K_0,$$

satisfy the commutation relations

$$[K_0, K_\pm] = \pm K_\pm, [K_+, K_-] = -2K_0,$$

and have the boson realization

$$K_+ = \frac{1}{2}\left(a^\dagger\right)^2, K_- = \frac{1}{2}\left(a\right)^2, K_0 = \frac{1}{4}\left(aa^\dagger + a^\dagger a\right).$$

Lemma 8.1. *In terms of the position and momentum operators X and P of Section 7, for $\hbar = 1$,*

$$K_+ + K_- + K_0 = \frac{1}{4}\left(3X^2 - P^2\right) + \frac{1}{2}I.$$

Proof. By (7.2)

$$\begin{aligned}\left(a^\dagger\right)^2 &= \frac{1}{2}\left(X - iP\right)\left(X - iP\right),\\ a^2 &= \frac{1}{2}\left(X + iP\right)\left(X + iP\right),\\ a^\dagger a &= \frac{1}{2}\left(X + iP\right)\left(X - iP\right).\end{aligned}$$

Multiplying out and using

$$PX = XP - iI,$$

we obtain

$$\begin{aligned}\left(a^\dagger\right)^2 &= \frac{1}{2}\left(X^2 - P^2 - 2iXP - I\right),\\ a^2 &= \frac{1}{2}\left(X^2 - P^2 + 2iXP + I\right),\\ a^\dagger a &= \frac{1}{2}\left(X^2 + P^2 + I\right).\end{aligned}$$

Since

$$aa^\dagger = I + a^\dagger a,$$

we have

$$\begin{aligned}K_+ + K_- + K_0 &= \frac{1}{2}\left(a^\dagger\right)^2 + \frac{1}{2}\left(a\right)^2 + \frac{1}{4}\left(aa^\dagger + a^\dagger a\right)\\ &= \frac{1}{4}\left(X^2 - P^2 - 2iXP - I\right) + \frac{1}{4}\left(X^2 - P^2 + 2iXP + I\right)\\ &+ \frac{1}{4}\left(\frac{1}{2}\left(X^2 + P^2 + I\right) + I + \frac{1}{2}\left(X^2 + P^2 + I\right)\right)\\ &= \frac{1}{4}\left(3X^2 - P^2\right) + \frac{1}{2}I.\end{aligned}$$

□

Lemma 8.2. *With X and P as in Section 7 and $\hbar = 1$,*

$$[e^{bX^2}, XP] = 2ibX^2 e^{bX^2}, \tag{8.1}$$

$$e^{ibXP} P^2 = e^{-2b} P^2 e^{ibXP}, \tag{8.2}$$

$$[e^{bX^2}, P^2] = \left(2b - 4b^2X^2 + 4ibXP\right) e^{bX^2}, \tag{8.3}$$

for all $b \in \mathbb{C}$.

Proof. The proof can be done by working directly with the X, P commutation relations. However, (8.1) and (8.3) can be obtained from formulas (vi) and (i), respectively, of Lemma 3.2 of [1] with the correspondence

$$S_2^0 = -P^2 \ , \ S_0^2 = X^2 \ , \ S_1^1 = \frac{1}{2}I + iXP.$$

Formula (8.2) is obtained, in like manner, from (2.11) of Lemma 2 of [2]. □

Lemma 8.3. *With X and P as in Section 7, and $\hbar = 1$, for all $s \in \mathbb{C}$ and $a, b \in \mathbb{R}$,*

$$e^{s(aX^2 + bP^2)} = e^{\frac{1}{2}p(s)} e^{q(s)X^2} e^{ip(s)XP} e^{r(s)P^2}, \tag{8.4}$$

where

$$q(s) = \frac{1}{2}\sqrt{\frac{a}{b}}\tanh(2\sqrt{ab}\, s), \tag{8.5}$$

$$p(s) = -4b\int_0^s q(t)\, dt = \log(\mathrm{sech}(2\sqrt{\mathrm{ab}}\,\mathrm{s})), \tag{8.6}$$

$$r(s) = b\int_0^s e^{2p(t)}\, dt = b\int_0^s \left(\mathrm{sech}(2\sqrt{\mathrm{ab}}\,\mathrm{t})\right)^2\, dt = \frac{1}{2}\sqrt{\frac{b}{a}}\tanh(2\sqrt{ab}\, s). \tag{8.7}$$

Proof. Let $R(s)$ and $L(s)$ be, respectively, the right and left hand sides of equation (8.4). Then

$$\frac{dL}{ds} = (aX^2 + bP^2)L(s) \ , \ L(0) = I. \tag{8.8}$$

We will show that

$$\frac{dR}{ds} = (aX^2 + bP^2)R(s) \ , \ R(0) = I, \tag{8.9}$$

as well. That will imply $L(s) = R(s)$ for all s. Clearly $R(0) = I$. Moreover, direct differentiation gives

$$\begin{aligned}\frac{dR}{ds} = {} & \frac{1}{2}p'(s)R(s) + q'(s)X^2R(s) + ip'(s)e^{\frac{1}{2}p(s)}e^{q(s)X^2}XPe^{ip(s)XP}e^{r(s)P^2} \\ & + r'(s)e^{\frac{1}{2}p(s)}e^{q(s)X^2}e^{ip(s)XP}P^2e^{r(s)P^2}.\end{aligned}$$

By (8.1) and (8.2),

$$e^{q(s)X^2}XP = \left(2iq(s)X^2 + XP\right)e^{q(s)X^2},$$

and

$$e^{ip(s)XP}P^2 = e^{-2p(s)}P^2e^{ip(s)XP}.$$

Thus

$$\frac{dR}{ds} = \frac{1}{2}p'(s)R(s) + q'(s)X^2R(s) - 2p'(s)q(s)X^2R(s) + ip'(s)XPR(s) \\ + r'(s)e^{-\frac{3}{2}p(s)}e^{q(s)X^2}P^2e^{ip(s)XP}e^{r(s)P^2}.$$

By (8.3),

$$e^{q(s)X^2}P^2 = P^2e^{q(s)X^2} + \left(2q(s) - 4q(s)^2X^2 + 4iq(s)XP\right)e^{q(s)X^2}.$$

Thus, using (8.5)-(8.7) to replace $p'(s)$, $q'(s)$ and $r'(s)$, we obtain

$$\frac{dR}{ds} = (aX^2 + bP^2)R(s).$$

□

Lemma 8.4. *With $\hbar = 1$ and X, P as in Section 7, for $n \in \mathbb{N} = \{1, 2, ...\}$,*

$$(XP)^n = \sum_{k=1}^{n}(-1)^{n-k}i^{n-k}S(n,k)X^kP^k,$$

where $S(n,k)$ are the Stirling numbers of the second kind.

Proof. It is known (see, for example, [4]) that, if $[a, a^\dagger] = 1$ then

$$\left(a^\dagger a\right)^n = \sum_{k=1}^{n} S(n,k)a^{\dagger k}a^k,$$

from which the result follows by taking $a = iP$ and $a^\dagger = X$. □

Notice that for each $s = it$, $t \in \mathbb{R}$, using properties of the hyperbolic functions and (8.5)-(8.7), we see that if $ab < 0$ then both sides of (8.4) consist of unitary operators (see, for example, Chapter 2 of [10]) of the form $e^{if(t)T}$ where T is an unbounded, densely defined, self-adjoint operator and $f : \mathbb{R} \to \mathbb{R}$. In particular, using the X, P commutation relations,

$$e^{ip(it)XP} = e^{if(t)(XP+PX)}e^{-f(t)I},$$

where $f(t) = \frac{p(it)}{2} \in \mathbb{R}$.

Theorem 8.5. *In the notation of Lemma 8.3, with $s = it$, $t \in \mathbb{R}$,*

$$\langle \Phi, e^{it(aX^2+bP^2)}\Phi\rangle = \frac{\sqrt{2}\, e^{\frac{1}{2}p(it)}}{\sqrt{p(it)^2 + (2\,q(it) - 1))(2\,r(it) - 1)}}.$$

Proof. By Lemma 8.3,

$$\begin{aligned}\langle \Phi, e^{it(aX^2+bP^2)}\Phi\rangle &= \langle \Phi, e^{\frac{1}{2}p(it)}e^{q(it)X^2}e^{ip(it)XP}e^{r(it)P^2}\Phi\rangle \\ &= e^{\frac{1}{2}p(it)}\langle e^{\overline{q(it)}X^2}\Phi, e^{ip(it)XP}e^{r(it)P^2}\Phi\rangle.\end{aligned}$$

Using

$$e^{\overline{q(it)}X^2} = \int_{\mathbb{R}} e^{\overline{q(it)}\lambda^2}\, dE_\lambda \;;\; e^{r(it)P^2} = \int_{\mathbb{R}} e^{r(it)\lambda^2}\, dE'_\lambda,$$

where $\{E_\lambda\}$ and $\{E'_\lambda\}$ are the spectral resolutions of X and P respectively, and the fact that the inner product is linear in the second and conjugate linear in the first argument, we have

$$\begin{aligned}\langle \Phi, e^{it(aX^2+bP^2)}\Phi\rangle &= \langle \Phi, e^{\frac{1}{2}p(it)}e^{q(it)X^2}e^{ip(it)XP}e^{r(it)P^2}\Phi\rangle \\ &= e^{\frac{1}{2}p(it)}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{q(it)\lambda^2}e^{r(it)\lambda'^2}\,d_\lambda\,d_{\lambda'}\langle E_\lambda\Phi, e^{ip(it)XP}E'_{\lambda'}\Phi\rangle.\end{aligned}$$

Using, as in the proof of Theorem 7.1,

$$(E_\lambda\Phi)(t) = \Phi(t)\chi_{(-\infty,\lambda]}(t),$$

and

$$(U^{-1}\Phi)(t) = \pi^{-1/4}e^{-\frac{t^2}{2}} = \Phi(t),$$

we obtain

$$\begin{aligned}\left(E'_\lambda\Phi\right)(t) &= \left(UE_\lambda U^{-1}\Phi\right)(t) = \left(U\chi_{(-\infty,\lambda]}\Phi\right)(t) \\ &= (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{ist}\chi_{(-\infty,\lambda]}(s)\Phi(s)\,ds = 2^{-1/2}\pi^{-3/4}\int_{-\infty}^{\lambda} e^{ist-\frac{s^2}{2}}\,ds.\end{aligned}$$

By Lemma 8.4,

$$\begin{aligned}\langle E_\lambda\Phi, e^{ip(it)XP}E'_{\lambda'}\Phi\rangle &= \sum_{n=1}^{\infty}\frac{i^n p(it)^n}{n!}\langle E_\lambda\Phi, (XP)^n E'_{\lambda'}\Phi\rangle \\ &= \sum_{n=1}^{\infty}\frac{i^n p(it)^n}{n!}\sum_{k=1}^{n}(-1)^{n-k}i^{n-k}S(n,k)\langle E_\lambda\Phi, X^kP^kE'_{\lambda'}\Phi\rangle \\ &= \sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^{n-k}\frac{i^{2n-k}p(it)^n}{n!}S(n,k)\langle E_\lambda\Phi, X^kP^kE'_{\lambda'}\Phi\rangle \\ &= \sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^{n-k}\frac{i^{2n-k}p(it)^n}{n!}S(n,k)\langle X^kE_\lambda\Phi, P^kE'_{\lambda'}\Phi\rangle \\ &= \sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{i^k p(it)^n}{n!}S(n,k)\langle X^kE_\lambda\Phi, P^kE'_{\lambda'}\Phi\rangle.\end{aligned}$$

Since

$$X^k = \int_{\mathbb{R}}\Lambda^k\,dE_\Lambda \quad ; \quad P^k = \int_{\mathbb{R}} M^k\,dE'_M,$$

we have

$$\begin{aligned}\langle X^kE_\lambda\Phi, P^kE'_{\lambda'}\Phi\rangle &= \int_{\mathbb{R}}\int_{\mathbb{R}}\Lambda^kM^k\,d_\Lambda d_M\langle E_\Lambda E_\lambda\Phi, E'_ME'_{\lambda'}\Phi\rangle \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\Lambda^kM^k\,d_\Lambda d_M\langle E_{\min(\Lambda,\lambda)}\Phi, E'_{\min(M,\lambda')}\Phi\rangle \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\Lambda^kM^k\,d_\Lambda d_M\left(\int_{\mathbb{R}}\overline{E_{\min(\Lambda,\lambda)}\Phi(t)}\,E'_{\min(M,\lambda')}\Phi(t)\,dt\right) \\ &= \frac{1}{\pi\sqrt{2}}\int_{\mathbb{R}}\int_{\mathbb{R}}\Lambda^kM^k\,d_\Lambda d_M\left(\int_{-\infty}^{\min(\Lambda,\lambda)}\int_{-\infty}^{\min(M,\lambda')}e^{ist-\frac{1}{2}(s^2+t^2)}\,dsdt\right).\end{aligned}$$

If $\min(\Lambda,\lambda)=\lambda$ and/or $\min(M,\lambda')=\lambda'$ then

$$d_\Lambda d_M\left(\int_{-\infty}^{\min(\Lambda,\lambda)}\int_{-\infty}^{\min(M,\lambda')} e^{ist-\frac{1}{2}(s^2+t^2)}\,dsdt\right)=0.$$

Thus we get a nonzero result for $\min(\Lambda,\lambda)=\Lambda$ and $\min(M,\lambda')=M$, i.e., for $\Lambda\leq\lambda$ and $M\leq\lambda'$, in which case

$$\begin{aligned}\langle X^kE_\lambda\Phi,P^kE'_{\lambda'}\Phi\rangle&=\frac{1}{\pi\sqrt{2}}\int_{-\infty}^{\lambda}\int_{-\infty}^{\lambda'}\Lambda^kM^k\,d_\Lambda d_M\left(\int_{-\infty}^{\Lambda}\int_{-\infty}^{M}e^{ist-\frac{1}{2}(s^2+t^2)}\,dsdt\right)\\&=\frac{1}{\pi\sqrt{2}}\int_{-\infty}^{\lambda}\int_{-\infty}^{\lambda'}\Lambda^kM^ke^{iM\Lambda-\frac{1}{2}(M^2+\Lambda^2)}\,dMd\Lambda,\end{aligned}$$

and

$$d_\lambda\,d_{\lambda'}\langle X^kE_\lambda\Phi,P^kE'_{\lambda'}\Phi\rangle=\frac{1}{\pi\sqrt{2}}\lambda^k\lambda'^ke^{i\lambda\lambda'-\frac{1}{2}(\lambda^2+\lambda'^2)}\,d\lambda d\lambda'.$$

Thus

$$\begin{aligned}d_\lambda\,d_{\lambda'}\langle E_\lambda\Phi,e^{ip(it)XP}E'_{\lambda'}\Phi\rangle&=\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{i^kp(it)^n}{n!}S(n,k)d_\lambda\,d_{\lambda'}\langle X^kE_\lambda\Phi,P^kE'_{\lambda'}\Phi\rangle\\&=\frac{1}{\pi\sqrt{2}}\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{i^kp(it)^n}{n!}S(n,k)\lambda^k\lambda'^ke^{i\lambda\lambda'-\frac{1}{2}(\lambda^2+\lambda'^2)}\,d\lambda d\lambda',\end{aligned}$$

and

$$\begin{aligned}\langle\Phi,e^{it(aX^2+bP^2)}\Phi\rangle&=\frac{1}{\pi\sqrt{2}}e^{\frac{1}{2}p(it)}\sum_{n=1}^{\infty}\frac{p(it)^n}{n!}\\&\quad\times\int_{\mathbb{R}}\int_{\mathbb{R}}e^{q(it)\lambda^2}e^{r(it)\lambda'^2}\sum_{k=1}^{n}i^kS(n,k)\lambda^k\lambda'^ke^{i\lambda\lambda'-\frac{1}{2}(\lambda^2+\lambda'^2)}\,d\lambda d\lambda'.\end{aligned}$$

Since $S(n,0)=0$ and $S(n,k)=0$ for $k>n$,

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}S(n,k)\frac{p(it)^n}{n!}(i\lambda\lambda')^k=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}S(n,k)\frac{p(it)^n}{n!}(i\lambda\lambda')^k=e^{i\lambda\lambda'\left(e^{p(it)}-1\right)},$$

where we have used the identity (see [9], equation (9.70)),

$$\sum_{n=0}^{\infty}\frac{x^n}{n!}S(n,k)=\frac{(e^x-1)^k}{k!}.$$

Thus

$$\langle\Phi,e^{it(aX^2+bP^2)}\Phi\rangle=\frac{1}{\pi\sqrt{2}}e^{\frac{1}{2}p(it)}\int_{\mathbb{R}}\int_{\mathbb{R}}e^{\left(q(it)-\frac{1}{2}\right)\lambda^2+\left(r(it)-\frac{1}{2}\right)\lambda'^2+i\lambda\lambda'e^{p(it)}}\,d\lambda d\lambda'.$$

Using the integration formula

$$\int_{\mathbb{R}}\int_{\mathbb{R}}e^{\alpha x^2+\beta y^2+i\gamma xy}\,dxdy=\frac{2\pi}{\sqrt{\gamma^2+4\alpha\beta}},$$

we obtain

$$\langle \Phi, e^{it(aX^2+bP^2)}\Phi\rangle = \frac{\sqrt{2}\, e^{\frac{1}{2}p(it)}}{\sqrt{p(it)^2 + (2\,q(it)-1))(2\,r(it)-1)}}.$$

□

Corollary 8.6. *For $\hbar = 1$, the vacuum characteristic function of the quantum observable*

$$H = K_+ + K_- + K_0$$

is

$$\langle \Phi, e^{itH}\Phi\rangle = \left(\frac{6\,\mathrm{sech}\left(\frac{t\sqrt{3}}{2}\right)}{3 + 3\log^2\left(\mathrm{sech}\left(\frac{t\sqrt{3}}{2}\right)\right) - 3\tanh^2\left(\frac{t\sqrt{3}}{2}\right) + 4i\sqrt{3}\tanh\left(\frac{t\sqrt{3}}{2}\right)} \right)^{\frac{1}{2}}.$$

Proof. For X, P and Φ as in Section 7,

$$H = K_+ + K_- + K_0 = \frac{1}{4}\left(3X^2 - P^2\right) + \frac{1}{2}I\ .$$

Thus, by Theorem 8.5 and Lemma 8.3 with $a = \frac{3}{4}$ and $b = -\frac{1}{4}$, using

$$\tanh(-x) = -\tanh(x)\ ;\ \mathrm{sech}(-\mathrm{x}) = \mathrm{sech}(\mathrm{x}),$$

we have

$$q(it) = -\frac{i\sqrt{3}}{2}\tanh\left(\frac{t\sqrt{3}}{2}\right),$$

$$p(it) = \log\left(\mathrm{sech}\left(\frac{t\sqrt{3}}{2}\right)\right),$$

$$r(it) = -\frac{i\sqrt{3}}{6}\tanh\left(\frac{t\sqrt{3}}{2}\right),$$

and

$$\begin{aligned}\langle \Phi, e^{itH}\Phi\rangle &= \frac{\sqrt{2}\, e^{\frac{1}{2}p(it)}}{\sqrt{p(it)^2 + (2\,q(it)-1))(2\,r(it)-1)}} \\ &= \left(\frac{6\,\mathrm{sech}\left(\frac{t\sqrt{3}}{2}\right)}{3 + 3\log^2\left(\mathrm{sech}\left(\frac{t\sqrt{3}}{2}\right)\right) - 3\tanh^2\left(\frac{t\sqrt{3}}{2}\right) + 4i\sqrt{3}\tanh\left(\frac{t\sqrt{3}}{2}\right)} \right)^{\frac{1}{2}}.\end{aligned}$$

□

9. Computing the Vacuum Resolution and the Characteristic Function with Stone's Formula

In the following we use Stone's Formula (1.1) to compute the vacuum resolution of the identity, i.e., $\langle \Phi, E_\lambda \Phi\rangle$, and the characteristic function

$$\int_{\mathbb{R}} e^{it\lambda}\, d\langle \Phi, E_\lambda \Phi\rangle,$$

of the operators X, P and $X + P$ of Section 7.

Theorem 9.1. *The vacuum spectral resolution of X is*

$$\langle \Phi, E_\lambda \Phi \rangle = \frac{1}{\pi^{1/2}} \int_{-\infty}^{\lambda} e^{-s^2}\, ds.$$

Moreover, for $t \in \mathbb{R}$,

$$\langle \Phi, e^{itX} \Phi \rangle = \sqrt{2}\, e^{-\frac{t^2}{2}}.$$

Proof. For $a \in \mathbb{C}$ with $\operatorname{Im} a \neq 0$ and for $s \in \mathbb{R}$,

$$\begin{aligned} R(a;X)g(s) = G(s) &\iff g(s) = (a - X)G(s) \\ &\iff g(s) = (a - s)G(s) \\ &\iff G(s) = \frac{g(s)}{a - s}. \end{aligned}$$

The function G is in the domain of X, since the continuous functions

$$\psi(s) = \frac{1}{|a - s|^2}\,;\, \sigma(s) = s^2 \psi(s)$$

satisfy

$$\lim_{s \to \pm\infty} \psi(s) = 0 \in \mathbb{R}, \lim_{s \to \pm\infty} \psi(s) = 1 \in \mathbb{R},$$

and are therefore bounded. Thus, since $g \in L^2(\mathbb{R}, \mathbb{C})$,

$$\int_{\mathbb{R}} |G(s)|^2\, ds < +\infty, \int_{\mathbb{R}} s^2\, |G(s)|^2\, ds < +\infty.$$

By (1.1),

$$\begin{aligned} \langle \Phi, E_\lambda \Phi \rangle &= \lim_{\rho \to 0^+} \lim_{\delta \to 0^+} \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\lambda+\rho-\delta} \langle \Phi, \left(R(t - \epsilon i; X) - R(t + \epsilon i; X)\right) \Phi \rangle\, dt \\ &= \lim_{\rho \to 0^+} \lim_{\delta \to 0^+} \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\lambda+\rho-\delta} \int_{-\infty}^{\infty} \overline{\Phi(s)}\, \left(R(t - \epsilon i; X) - R(t + \epsilon i; X)\right) \Phi(s)\, ds\, dt \\ &= \lim_{\rho \to 0^+} \lim_{\delta \to 0^+} \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\lambda+\rho-\delta} \int_{-\infty}^{\infty} \overline{\Phi(s)} \Phi(s) \left(\frac{1}{t - i\epsilon - s} - \frac{1}{t + i\epsilon - s} \right) ds\, dt \\ &= \lim_{\epsilon \to 0^+} \frac{\epsilon}{\pi^{3/2}} \int_{-\infty}^{\lambda} \int_{-\infty}^{\infty} \frac{e^{-s^2}}{(t - s)^2 + \epsilon^2}\, ds\, dt. \end{aligned}$$

Since the continuous (thus measurable) function

$$(t, s) \in (-\infty, \lambda] \times \mathbb{R} \to \frac{e^{-s^2}}{(t - s)^2 + \epsilon^2} \in \mathbb{R}$$

is nonnegative, by Fubini's theorem we can reverse the order of integration and obtain

$$\langle \Phi, E_\lambda \Phi \rangle = \lim_{\epsilon \to 0^+} \frac{\epsilon}{\pi^{3/2}} \int_{-\infty}^{\infty} e^{-s^2} \int_{-\infty}^{\lambda} \frac{1}{(t - s)^2 + \epsilon^2}\, dt\, ds,$$

which, since

$$\begin{aligned}\int_{-\infty}^{\lambda} \frac{1}{(t-s)^2+\epsilon^2}\,dt &= -\frac{1}{\epsilon}\arctan\left(\frac{s-t}{\epsilon}\right)\Big|_{t=-\infty}^{t=\lambda} \\ &= -\frac{1}{\epsilon}\left(\arctan\left(\frac{s-\lambda}{\epsilon}\right)-\frac{\pi}{2}\right),\end{aligned}$$

becomes

$$\langle \Phi, E_\lambda \Phi\rangle = \lim_{\epsilon\to 0^+}\frac{1}{\pi^{3/2}}\int_{-\infty}^{\infty} e^{-s^2}\left(\frac{\pi}{2}-\arctan\left(\frac{s-\lambda}{\epsilon}\right)\right)\,ds.$$

Splitting the integral as

$$\int_{-\infty}^{\infty} = \int_{-\infty}^{\lambda} + \int_{\lambda}^{\infty},$$

we notice that

$$\lim_{\epsilon\to 0^+}\arctan\left(\frac{s-\lambda}{\epsilon}\right) = \pm\frac{\pi}{2},$$

with the minus and plus signs corresponding to the first and second integral respectively. Thus, using the bounded convergence theorem to pass $\lim_{\epsilon\to 0^+}$ under the integral sign, we obtain

$$\langle \Phi, E_\lambda \Phi\rangle = \frac{1}{\pi^{1/2}}\int_{-\infty}^{\lambda} e^{-s^2}\,ds.$$

Thus,

$$\begin{aligned}\langle \Phi, e^{itX}\Phi\rangle &= \int_{\mathbb{R}} e^{it\lambda}\,d\langle \Phi, E_\lambda\Phi\rangle \\ &= \int_{\mathbb{R}} e^{it\lambda}\,\frac{1}{\sqrt{\pi}}e^{-\frac{\lambda^2}{2}}\,d\lambda \\ &= \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{it\lambda-\frac{\lambda^2}{2}}\,d\lambda \\ &= \sqrt{2}\,e^{-\frac{t^2}{2}}.\end{aligned}$$

□

Theorem 9.2. *The vacuum spectral resolution of P is*

$$\langle \Phi, E_\lambda \Phi\rangle = \pi^{-\frac{1}{2}}\int_{-\infty}^{\lambda} e^{-s^2}\,ds.$$

Moreover, for $t\in\mathbb{R}$,

$$\langle \Phi, e^{itP}\Phi\rangle = \sqrt{2}\,e^{-\frac{t^2}{2}}.$$

Proof. For $a \in \mathbb{C}$ with $\operatorname{Im} a \neq 0$ and for $s \in \mathbb{R}$,

$$\begin{aligned} R(a;P)g(s) = G(s) &\iff g(s) = (a-P)G(s) \\ &\iff g(s) = \left(a + i\frac{d}{ds}\right) G(s) \\ &\iff G'(s) - iaG(s) = -ig(s) \\ &\iff \left(e^{-ias}G(s)\right)' = -ie^{-ias}g(s). \end{aligned} \tag{9.1}$$

Since G is in the domain of P, it follows that $\lim_{t\to\pm\infty} G(t) = 0$. Therefore,

$$\lim_{t\to+\infty} |e^{-iat}G(t)| = \lim_{t\to+\infty} e^{t\operatorname{Im} a}|G(t)| = 0, \text{ if } \operatorname{Im} a < 0,$$

and

$$\lim_{t\to-\infty} |e^{-iat}G(t)| = \lim_{t\to-\infty} e^{t\operatorname{Im} a}|G(t)| = 0, \text{ if } \operatorname{Im} a > 0.$$

For $\operatorname{Im} a < 0$, integrating (9.1) from t to ∞ and then replacing t by s, we obtain

$$-e^{-as}G(s) = -i\int_s^\infty e^{ia(s-t)}g(t)\,dt,$$

so

$$G(s) = i\int_s^\infty e^{ia(s-t)}g(t)\,dt,$$

while, for $\operatorname{Im} a > 0$, integrating (9.1) from $-\infty$ to t and then replacing t by s, we obtain

$$G(s) = -i\int_{-\infty}^s e^{ia(s-t)}g(t)\,dt.$$

Thus,

$$R(a;P)g(s) = \begin{cases} -i\int_{-\infty}^s e^{ia(s-t)}g(t)\,dt, & \text{if } \operatorname{Im} a > 0 \\ i\int_s^\infty e^{ia(s-t)}g(t)\,dt, & \text{if } \operatorname{Im} a < 0 \end{cases}.$$

For $g(t) = \Phi(t)$ we find

$$R(a;P)\Phi(s) = \begin{cases} -i\pi^{-1/4}\int_{-\infty}^s e^{ia(s-t)-\frac{t^2}{2}}\,dt, & \text{if } \operatorname{Im} a > 0 \\ i\pi^{-1/4}\int_s^\infty e^{ia(s-t)-\frac{t^2}{2}}\,dt, & \text{if } \operatorname{Im} a < 0 \end{cases}.$$

Thus,

$$\begin{aligned} \langle \Phi, E_\lambda \Phi\rangle &= \lim_{\rho\to0^+}\lim_{\delta\to0^+}\lim_{\epsilon\to0^+}\frac{1}{2\pi i}\int_{-\infty}^{\lambda+\rho-\delta}\langle \Phi, (R(t-\epsilon i;P) - R(t+\epsilon i;P))\,\Phi\rangle\,dt \\ &= \lim_{\epsilon\to0^+}\frac{1}{2\pi i}\int_{-\infty}^{\lambda}\int_{-\infty}^{\infty}\overline{\Phi(s)}\,(R(t-\epsilon i;P) - R(t+\epsilon i;P))\,\Phi(s)\,ds\,dt \\ &= \lim_{\epsilon\to0^+}\frac{1}{2\pi^{3/2}}\int_{-\infty}^{\lambda}\int_{-\infty}^{\infty} e^{-\frac{s^2}{2}}\cdot \\ &\qquad \times\left(e^{(\epsilon+it)s}\int_s^\infty e^{-(it+\epsilon)w-\frac{w^2}{2}}\,dw + e^{(it-\epsilon)s}\int_{-\infty}^s e^{(\epsilon-it)w-\frac{w^2}{2}}\,dw\right)ds\,dt. \end{aligned}$$

For each $t \in (-\infty, \lambda]$, using the triangle inequality, the integration formulas

$$\int_{-\infty}^{\infty} e^{\pm\epsilon w - \frac{w^2}{2}}\, dw = \sqrt{2\pi} e^{\frac{\epsilon^2}{2}}, \int_{-\infty}^{\infty} e^{-\frac{s^2}{2}}\, ds = \sqrt{2\pi},$$

and the fact that, for sufficiently small ϵ, $e^{\frac{\epsilon^2}{2}} < 3$ and $e^{\epsilon s} + e^{-\epsilon s} < 3$, we see that the function

$$f_\epsilon(s;t) := e^{-\frac{s^2}{2}} \cdot \left(e^{(\epsilon+it)s} \int_s^{\infty} e^{-(it+\epsilon)w - \frac{w^2}{2}}\, dw + e^{(it-\epsilon)s} \int_{-\infty}^{s} e^{(\epsilon-it)w - \frac{w^2}{2}}\, dw \right),$$

satisfies

$$|f_\epsilon(s;t)| \leq f(s) := 9\sqrt{2\pi} e^{-\frac{s^2}{2}},$$

with

$$\int_{-\infty}^{\infty} f(s)\, ds = 18\pi < \infty.$$

Thus, by the Bounded Convergence Theorem, we may pass the limit $\epsilon \to 0^+$ inside the integral with respect to s. Moreover, for $a < 0$ and $t \in (-a, \lambda]$,

$$\left| \int_{-\infty}^{\infty} f_\epsilon(s;t)\, ds \right| \leq 18\pi,$$

with

$$\int_a^{\lambda} 18\pi\, dt = (\lambda - a)18\pi < \infty.$$

Thus, by the Bounded Convergence Theorem, we may pass the limit $\epsilon \to 0^+$ inside $\int_a^{\lambda}(...)\, dt$ as well, and we obtain

$$\begin{aligned}
\langle \Phi, E_\lambda \Phi \rangle &= \lim_{a \to -\infty} \langle \Phi, E\left((a, \lambda]\right) \Phi \rangle \\
&= \lim_{a \to -\infty} \lim_{\epsilon \to 0^+} \frac{1}{2\pi^{3/2}} \int_a^{\lambda} \int_{-\infty}^{\infty} e^{-\frac{s^2}{2}} \cdot \\
&\quad \times \left(e^{(\epsilon+it)s} \int_s^{\infty} e^{-(it+\epsilon)w - \frac{w^2}{2}}\, dw + e^{(it-\epsilon)s} \int_{-\infty}^{s} e^{(\epsilon-it)w - \frac{w^2}{2}}\, dw \right) ds\, dt \\
&= \lim_{a \to -\infty} \frac{1}{2\pi^{3/2}} \int_a^{\lambda} \int_{-\infty}^{\infty} e^{-\frac{s^2}{2}} \cdot \\
&\quad \times \left(e^{its} \int_s^{\infty} e^{-itw - \frac{w^2}{2}}\, dw + e^{its} \int_{-\infty}^{s} e^{-itw - \frac{w^2}{2}}\, dw \right) ds\, dt \\
&= \frac{1}{2\pi^{3/2}} \int_{-\infty}^{\lambda} \int_{-\infty}^{\infty} e^{-\frac{s^2}{2} + its} \int_{-\infty}^{\infty} e^{-itw - \frac{w^2}{2}}\, dw\, ds\, dt \\
&= \frac{1}{2\pi^{3/2}} \int_{-\infty}^{\lambda} \int_{-\infty}^{\infty} e^{-\frac{s^2}{2} + its}\, ds \int_{-\infty}^{\infty} e^{-itw - \frac{w^2}{2}}\, dw\, dt \\
&= \frac{1}{2\pi^{3/2}} \int_{-\infty}^{\lambda} \left(\sqrt{2\pi} e^{-\frac{t^2}{2}} \right) \left(\sqrt{2\pi} e^{-\frac{t^2}{2}} \right) dt \\
&= \pi^{-\frac{1}{2}} \int_{-\infty}^{\lambda} e^{-t^2}\, dt.
\end{aligned}$$

Thus,

$$\begin{aligned}\langle \Phi, e^{itP}\Phi\rangle &= \int_{\mathbb{R}} e^{it\lambda}\, d\langle \Phi, E_\lambda \Phi\rangle = \int_{\mathbb{R}} e^{it\lambda}\, \frac{1}{\sqrt{\pi}} e^{-\frac{\lambda^2}{2}}\, d\lambda \\ &= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{it\lambda - \frac{\lambda^2}{2}}\, d\lambda = \sqrt{2}\, e^{-\frac{t^2}{2}}.\end{aligned}$$

□

Theorem 9.3. *The vacuum spectral resolution of $X + P$ is*

$$\langle \Phi, E_\lambda \Phi\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} e^{-\frac{t^2}{2}}\, dt.$$

Moreover, for $t \in \mathbb{R}$,

$$\langle \Phi, e^{it(X+P)}\Phi\rangle = e^{-\frac{t^2}{2}}.$$

Proof. For $a \in \mathbb{C}$ with $\operatorname{Im} a \neq 0$ and for $s \in \mathbb{R}$,

$$\begin{aligned} R(a; X+P)g(s) = G(s) &\iff g(s) = (a - X - P)G(s) \\ &\iff g(s) = \left(a - s + i\frac{d}{ds}\right) G(s) \\ &\iff G'(s) + i(s-a)G(s) = -ig(s) \\ &\iff \left(e^{i\left(\frac{s^2}{2} - as\right)} G(s)\right)' = -ie^{i\left(\frac{s^2}{2} - as\right)} g(s). \qquad (9.2)\end{aligned}$$

As in the proof of Theorems 9.1 and 9.2, for $\operatorname{Im} a < 0$, integrating (9.2) from t to ∞ and then replacing t by s, we obtain

$$-e^{i\left(\frac{s^2}{2} - as\right)} G(s) = -i \int_s^{\infty} e^{i\left(\frac{t^2}{2} - at\right)} g(t)\, dt,$$

so

$$G(s) = i \int_s^{\infty} e^{i\left(\frac{t^2 - s^2}{2} - a(t-s)\right)} g(t)\, dt,$$

while, for $\operatorname{Im} a > 0$, integrating (9.2) from $-\infty$ to t and then replacing t by s, we obtain

$$G(s) = -i \int_{-\infty}^{s} e^{i\left(\frac{t^2 - s^2}{2} - a(t-s)\right)} g(t)\, dt.$$

Thus,

$$R(a; X+P)g(s) = \begin{cases} -i \int_{-\infty}^{s} e^{i\left(\frac{t^2 - s^2}{2} - a(t-s)\right)} g(t)\, dt, & \text{if } \operatorname{Im} a > 0 \\ \\ i \int_{s}^{\infty} e^{i\left(\frac{t^2 - s^2}{2} - a(t-s)\right)} g(t)\, dt, & \text{if } \operatorname{Im} a < 0 \end{cases}.$$

For $g(t) = \Phi(t)$ we find

$$R(a; X+P)\Phi(s) = \begin{cases} -i\pi^{-1/4} \int_{-\infty}^{s} e^{i\left(\frac{t^2 - s^2}{2} - a(t-s)\right) - \frac{t^2}{2}}\, dt, & \text{if } \operatorname{Im} a > 0 \\ \\ i\pi^{-1/4} \int_{s}^{\infty} e^{i\left(\frac{t^2 - s^2}{2} - a(t-s)\right) - \frac{t^2}{2}}\, dt\ , & \text{if } \operatorname{Im} a < 0 \end{cases}.$$

Thus, as in the proof of Theorems 9.1 and 9.2,

$$
\begin{aligned}
&\langle \Phi, E_\lambda \Phi\rangle \\
&= \lim_{\epsilon\to 0^+} \frac{1}{2\pi i}\int_{-\infty}^{\lambda}\int_{-\infty}^{\infty} \overline{\Phi(s)}\left(R(t-\epsilon i; X+P) - R(t+\epsilon i; X+P)\right)\Phi(s)\, ds\, dt \\
&= \lim_{\epsilon\to 0^+} \frac{1}{2\pi^{3/2}}\int_{-\infty}^{\lambda}\int_{-\infty}^{\infty} e^{-\frac{s^2}{2}}\cdot \\
&\quad \times \left(\int_s^{\infty} e^{i\left(\frac{w^2-s^2}{2} - (t-i\epsilon)(w-s)\right) - \frac{w^2}{2}}\, dw + \int_{-\infty}^{s} e^{i\left(\frac{w^2-s^2}{2} - (t+i\epsilon)(w-s)\right) - \frac{w^2}{2}}\, dw\right) ds\, dt \\
&= \frac{1}{2\pi^{3/2}}\int_{-\infty}^{\lambda}\int_{-\infty}^{\infty} e^{-\frac{s^2}{2}}\int_{-\infty}^{\infty} e^{i\left(\frac{w^2-s^2}{2} - t(w-s)\right) - \frac{w^2}{2}}\, dw\, ds\, dt \\
&= \frac{1}{2\pi^{3/2}}\int_{-\infty}^{\lambda}\left(\int_{-\infty}^{\infty} e^{-(1+i)\frac{s^2}{2} + its}\, ds\right)\left(\int_{-\infty}^{\infty} e^{-(1-i)\frac{w^2}{2} - itw}\, dw\right) dt \\
&= \frac{1}{2\pi^{3/2}}\int_{-\infty}^{\lambda}\left|\int_{-\infty}^{\infty} e^{-(1+i)\frac{s^2}{2} + its}\, ds\right|^2 dt \\
&= \frac{1}{2\pi^{3/2}}\int_{-\infty}^{\lambda}\left|\sqrt{\frac{2\pi}{1+i}} e^{-\frac{1-i}{4}t^2}\right|^2 dt \\
&= \frac{1}{2\pi^{3/2}}\int_{-\infty}^{\lambda}\sqrt{2}\pi e^{-\frac{t^2}{2}}\, dt \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} e^{-\frac{t^2}{2}}\, dt.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\langle \Phi, e^{it(X+P)}\Phi\rangle &= \int_{\mathbb{R}} e^{it\lambda}\, d\langle \Phi, E_\lambda \Phi\rangle = \int_{\mathbb{R}} e^{it\lambda}\frac{1}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{2}}\, d\lambda \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{it\lambda - \frac{\lambda^2}{2}}\, d\lambda = e^{-\frac{t^2}{2}}.
\end{aligned}
$$

□

Remark 9.4. Computing the resolvent and the resulting integral in Stone's formula for the *anti-commutator* operator $T = XP + PX$ is not easy. Not much can be found on that in the literature. The operator $T = XP + PX$ is a case where the Lie algebraic method of using an appropriate splitting lemma, seems to have an advantage over the analytic method that uses the vacuum spectral resolution. We will return to the computation of the vacuum spectral resolution of $XP + PX$, as well as of $aX^2 + bP^2$, with the use of Stone's formula, in the sequel to this paper.

The following Theorem gives an example of how the vacuum spectral resolution can be computed, once the characteristic function is known.

Theorem 9.5. *For $t \in \mathbb{R}$,*

$$\langle \Phi, e^{it(XP+PX)}\Phi\rangle = (\operatorname{sech} t)^{1/2}.$$

Moreover, the differential with respect to λ of the vacuum spectral resolution of $XP+PX$ is

$$d\left\langle \Phi, E_\lambda \Phi \right\rangle = \frac{1-i}{4\pi}\left(e^{-\frac{\pi\lambda}{2}}\, B\left(-1; \frac{1-2i\lambda}{4}, \frac{1}{2}\right) + e^{\frac{\pi\lambda}{2}}\, B\left(-1; \frac{1+2i\lambda}{4}, \frac{1}{2}\right)\right) d\lambda,$$

where

$$B(z;a,b) = \int_0^z t^{a-1}(1-t)^{b-1}\, dt$$

is the incomplete Beta function.

Proof. Using

$$X = \frac{a+a^\dagger}{\sqrt{2}}\ ,\ P = \frac{a-a^\dagger}{\sqrt{2}i}$$

and

$$[a, a^\dagger] = \mathbf{1},$$

we find that

$$XP + PX = i\left((a^\dagger)^2 - a^2\right).$$

By Proposition 3.4 of [1], (or by Proposition 3.9 of [2], see also Proposition 4.1.1 of [7] and Proposition 4 of [3], where it is shown that $XP+PX$ is a continuous binomial or Beta process), it follows that

$$\left\langle \Phi, e^{it(XP+PX)}\Phi \right\rangle = \left\langle \Phi, e^{t\left(a^2-(a^\dagger)^2\right)}\Phi \right\rangle = (\operatorname{sech} t)^{1/2}.$$

Therefore,

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{it\lambda}\, d\langle \Phi, E_\lambda \Phi\rangle = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{it\lambda}\, \frac{d}{d\lambda}\langle \Phi, E_\lambda \Phi\rangle\, d\lambda = \frac{1}{\sqrt{2\pi}}\,(\operatorname{sech} t)^{1/2},$$

which means that

$$\frac{d}{d\lambda}\langle \Phi, E_\lambda \Phi\rangle$$

is the inverse Fourier transform of

$$\frac{1}{\sqrt{2\pi}}\,(\operatorname{sech} t)^{1/2},$$

i.e.,

$$\frac{d}{d\lambda}\langle \Phi, E_\lambda \Phi\rangle = \frac{1-i}{4\pi}\left(e^{-\frac{\pi\lambda}{2}}\, B\left(-1; \frac{1-2i\lambda}{4}, \frac{1}{2}\right) + e^{\frac{\pi\lambda}{2}}\, B\left(-1; \frac{1+2i\lambda}{4}, \frac{1}{2}\right)\right).$$

□

References

1. Accardi, L. and Boukas, A.: On the characteristic function of random variables associated with Boson Lie algebras, *Communications on Stochastic Analysis* **4** (2010), no. 4 , 493–504.
2. Accardi, L. and Boukas, A.: Normally ordered disentanglement of multi-dimensional Schrödinger algebra exponentials, *Communications on Stochastic Analysis* **12** (2018), no. 3, 283–328.
3. Accardi, L. and Boukas, A.: Fock representation of the renormalized higher powers of white noise and the centerless Virasoro (or Witt) ZamolodchikovAccardi, L. and Boukas, A.: w_∞ *–Lie algebra, *J. Phys. A: Math. Theor.* **41** (2008), 1–12.

4. Blasiak, P., Horzela, A., Penson, K. A., Solomon, A. I., and Duchamp, G. H. E.: Combinatorics and Boson normal ordering: A gentle introduction, *American Journal of Physics* **75** (2007), no. 7, 639–646 .
5. Dunford, N. and Schwartz, J. T.: *Linear Operator, Part II: Spectral Theory, Self Adjoint Operators in Hilbert Space*, J. Wiley & Sons, New York, 1963.
6. Feinsilver, P. J.: Krawtchouk-Griffiths systems I: matrix approach, *Communications on Stochastic Analysis* **10** (2016), no. 3 , 297–320 .
7. Feinsilver, P. J. and Schott, R.: *Algebraic structures and operator calculus. Volumes I and III*, Kluwer, 1993.
8. Galindo, A. and Pascual, P.: *Quantum Mechanics I*, Springer-Verlag, Texts and Monographs in Physics, 1990.
9. Gould, H. W. and Quaintance, J.: *Combinatorial Identities for Stirling Numbers*, World Scientific, 2016.
10. Gustafson, S. J. and Sigal, I. M.: *Mathematical Concepts of Quantum Mechanics*, Springer-Verlag, Universitext, 2003.
11. Hall, B. C. : *Lie groups, Lie algebras, and Representations: An Elementary Introduction*, Springer, Graduate Texts in Mathematics no. 222, 2003. Second Edition, 2015.
12. Humphreys, J. E.: *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, Graduate Texts in Mathematics, vol. 9, 1972.
13. Novaes, M.: Some basics of $\mathfrak{su}(1,1)$, *Revista Brasileira de Ensino de Fisica* **26** (2004), no. 4, 351–357.
14. Parthasarathy, K. R.: *An Introduction to Quantum Stochastic Calculus*, Birkhauser Boston Inc., 1992.
15. Richtmyer, R. D.: *Principles of Advanced Mathematical Physics*, vol.1, Texts and Monographs in Physics, Springer-Verlag, 1978.
16. Roach, G. F.: *Wave Scattering by Time–Dependent Perturbations*, Princeton Series in Applied Mathematics, Princeton University Press, 2007.
17. Taylor, A. E. and Lay, D. C.: *Introduction to Functional Analysis*, Robert E. Krieger Publishing Company, 1986.
18. Yosida, K.: *Functional Analysis*, Springer-Verlag, 6th ed., 1980.

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