


6-2019

Strong Convergence Rate in Averaging Principle for the Heat Equation Driven by a General Stochastic Measure

Vadym Radchenko

Department of Mathematical Analysis, Taras Shevchenko National University of Kyiv, Kyiv 01601, Ukraine,
vradchenko@univ.kiev.ua

Follow this and additional works at: <https://digitalcommons.lsu.edu/cosa>

 Part of the [Analysis Commons](#), and the [Other Mathematics Commons](#)

Recommended Citation

Radchenko, Vadym (2019) "Strong Convergence Rate in Averaging Principle for the Heat Equation Driven by a General Stochastic Measure," *Communications on Stochastic Analysis*: Vol. 13 : No. 2 , Article 1.

DOI: 10.31390/cosa.13.2.01

Available at: <https://digitalcommons.lsu.edu/cosa/vol13/iss2/1>

**STRONG CONVERGENCE RATE IN AVERAGING PRINCIPLE
 FOR THE HEAT EQUATION DRIVEN BY A GENERAL
 STOCHASTIC MEASURE**

VADYM RADCHENKO*

ABSTRACT. We consider the stochastic heat equation on $[0, T] \times \mathbb{R}$ in the mild form driven by a general stochastic measure μ , for μ we assume only σ -additivity in probability. The time-averaging of the equation is studied, we estimate the rate of uniform a. s. convergence to the solution of the averaged equation.

1. Introduction

In this paper we shall establish the averaging principle for one-dimensional stochastic heat equation of the form

$$\begin{cases} du_\varepsilon(t, x) = \frac{1}{4} \frac{\partial^2 u_\varepsilon(t, x)}{\partial x^2} dt + f(t/\varepsilon, x, u_\varepsilon(t, x)) dt + \sigma(t/\varepsilon, x) d\mu(x), \\ u_\varepsilon(0, x) = u_0(x). \end{cases} \quad (t, x) \in (0, T] \times \mathbb{R}, \quad (1.1)$$

Here $\varepsilon > 0$ is a small parameter, and μ is a stochastic measure defined on Borel σ -algebra on \mathbb{R} . For μ we assume σ -additivity in probability only, assumptions for f , σ and u_0 are given in Section 3.

We will study the rate of convergence

$$\sup_{t, x} |u_\varepsilon(t, x) - \bar{u}(t, x)| \rightarrow 0, \quad \varepsilon \rightarrow 0,$$

where \bar{u} is the solution of the averaged equation

$$\begin{cases} d\bar{u}(t, x) = \frac{1}{4} \frac{\partial^2 \bar{u}(t, x)}{\partial x^2} dt + \bar{f}(x, \bar{u}(t, x)) dt + \bar{\sigma}(x) d\mu(x), \\ \bar{u}(0, x) = u_0(x). \end{cases} \quad (t, x) \in (0, T] \times \mathbb{R}, \quad (1.2)$$

We consider solutions to the formal equations (1.1) and (1.2) in the mild form (see (3.1) and (3.4) below), \bar{f} and $\bar{\sigma}$ are defined in (3.3).

A similar problem was recently studied in [19], but in that paper function f in (1.1) did not depend on the time variable, and averaging was considered for a stochastic term only. Note that the convergence rate obtained in [19] is higher than that rate in the given paper (see Remark 3.2 below).

Received 2019-7-29; Accepted 2019-8-8; Communicated by the editors.

2010 *Mathematics Subject Classification*. Primary 60H15; Secondary 60G57.

Key words and phrases. Averaging principle, stochastic heat equation, stochastic measure, mild solution, Besov space.

* Corresponding author.

The averaging principle for stochastic systems was studied by many authors. Equations driven by Wiener process were investigated, for example, in [6] and [21]. Averaging of stochastic partial differential equations mainly was studied for slow-fast systems, solutions were considered in the mild sense. Heat equation driven by Brownian motion was considered in [3], [4], and [7], by Wiener and Poisson processes – in [11] and [23]. Averaging principle for evolution equations with fractional Brownian motion was obtained in [14] and [24], with α -stable noise – in [1]. The wave equation driven by Wiener process was considered in [8] and [9], Burgers equation – in [5], Korteweg-de Vries equation – in [10].

Stochastic integrator in (1.1) is more general, but we study the model with one equation and additive noise. Averaging principle for one class of equations with multiplicative noise driven by stochastic measure with continuous paths was investigated in [20].

The rest of the paper is organized as follows. Section 2 contains the basic facts concerning stochastic measures. In Section 3 we give the exact formulation of the problem, our assumptions and formulate the main result (Theorem 3.1). Some auxiliary statements are proved in Section 4. Proof of the main result is given in Section 5.

2. Preliminaries

Let $L_0 = L_0(\Omega, \mathcal{F}, \mathbb{P})$ be the set of all real-valued random variables defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Convergence in L_0 means the convergence in probability. Let \mathcal{B} be a Borel σ -algebra on \mathbb{R} .

Definition 2.1. A σ -additive mapping $\mu : \mathcal{B} \rightarrow L_0$ is called *stochastic measure* (SM).

In other words, μ is a vector measure with values in L_0 . We do not assume any martingale properties or moment existence for SM.

For a deterministic measurable function $g : \mathbb{R} \rightarrow \mathbb{R}$, $A \in \mathcal{B}$ and SM μ , an integral of the form $\int_A g d\mu$ is defined and studied in [13, Chapter 7]. In particular, every bounded measurable g is integrable with respect to (w. r. t.) any μ . Note that this integral was constructed and studied in [13] for μ defined on arbitrary σ -algebra, but in our paper, we consider SM on Borel subsets of \mathbb{R} .

We can give the following examples of SMs. For square integrable martingale M_t , $\mu(A) = \int_0^T \mathbf{1}_A(t) dM_t$ is an SM. If W_t^H is a fractional Brownian motion with Hurst index $H > 1/2$ and $f : [0, T] \rightarrow \mathbb{R}$ is a bounded measurable function then $\mu(A) = \int_0^T f(t) \mathbf{1}_A(t) dW_t^H$ is an SM, as follows from [15, Theorem 1.1]. An α -stable random measure defined on \mathcal{B} is an SM too for $\alpha \in (0, 1) \cup (1, 2]$, see [22, Chapter 3]. Some other examples are given in [20].

In the sequel, μ denotes a SM, C and $C(\omega)$ denote positive constants and positive random constants respectively whose exact values are not important ($C < \infty$, $C(\omega) < \infty$ a. s.).

We will use the following lemma.

Lemma 2.2. (Lemma 3.1 in [17]) Let $\phi_l : \mathbb{R} \rightarrow \mathbb{R}$, $l \geq 1$ be measurable functions such that $\tilde{\phi}(x) = \sum_{l=1}^{\infty} |\phi_l(x)|$ is integrable w.r.t. μ on \mathbb{R} . Then

$$\sum_{l=1}^{\infty} \left(\int_{\mathbb{R}} \phi_l d\mu \right)^2 < \infty \quad a. s.$$

We consider the Besov spaces $B_{22}^{\alpha}([c, d])$. Recall that the norm in this classical space for $0 < \alpha < 1$ may be introduced by

$$\|g\|_{B_{22}^{\alpha}([c, d])} = \|g\|_{L_2([c, d])} + \left(\int_0^{d-c} (w_2(g, r))^2 r^{-2\alpha-1} dr \right)^{1/2}, \quad (2.1)$$

where

$$w_2(g, r) = \sup_{0 \leq h \leq r} \left(\int_c^{d-h} |g(y+h) - g(y)|^2 dy \right)^{1/2}.$$

For any $j \in \mathbb{Z}$ and all $n \geq 0$, put

$$d_{kn}^{(j)} = j + k2^{-n}, \quad 0 \leq k \leq 2^n, \quad \Delta_{kn}^{(j)} = (d_{(k-1)n}^{(j)}, d_{kn}^{(j)}], \quad 1 \leq k \leq 2^n.$$

The following lemma is a key tool for estimates of the stochastic integral.

Lemma 2.3. (Lemma 3 in [18]) Let Z be an arbitrary set, and function $q(z, s) : Z \times [j, j+1] \rightarrow \mathbb{R}$ is such that all paths $q(z, \cdot)$ are continuous on $[j, j+1]$. Denote

$$q_n(z, s) = \sum_{1 \leq k \leq 2^n} q(z, d_{(k-1)n}^{(j)}) \mathbf{1}_{\Delta_{kn}^{(j)}}(s).$$

Then the random function

$$\eta(z) = \int_{(j, j+1]} q(z, s) d\mu(s), \quad z \in Z,$$

has a version

$$\begin{aligned} \tilde{\eta}(z) &= \int_{(j, j+1]} q_0(z, s) d\mu(s) \\ &+ \sum_{n \geq 1} \left(\int_{(j, j+1]} q_n(z, s) d\mu(s) - \int_{(j, j+1]} q_{n-1}(z, s) d\mu(s) \right) \end{aligned} \quad (2.2)$$

such that for all $\beta > 0$, $\omega \in \Omega$, $z \in Z$

$$\begin{aligned} |\tilde{\eta}(z)| &\leq |q(z, j)\mu((j, j+1])| \\ &+ \left\{ \sum_{n \geq 1} 2^{n\beta} \sum_{1 \leq k \leq 2^n} |q(z, d_{kn}^{(j)}) - q(z, d_{(k-1)n}^{(j)})|^2 \right\}^{1/2} \\ &\times \left\{ \sum_{n \geq 1} 2^{-n\beta} \sum_{1 \leq k \leq 2^n} |\mu(\Delta_{kn}^{(j)})|^2 \right\}^{1/2}. \end{aligned} \quad (2.3)$$

Theorem 1.1 [12] implies that for $\alpha = (\beta + 1)/2$,

$$\left\{ \sum_{n \geq 1} 2^{n\beta} \sum_{1 \leq k \leq 2^n} |q(z, d_{kn}^{(j)}) - q(z, d_{(k-1)n}^{(j)})|^2 \right\}^{1/2} \leq C \|q(z, \cdot)\|_{B_{22}^{\alpha}([j, j+1])}. \quad (2.4)$$

From Lemma 2.2 it follows that for each $\beta > 0$, $j \in \mathbb{Z}$

$$\sum_{n \geq 1} 2^{-n\beta} \sum_{1 \leq k \leq 2^n} |\mu(\Delta_{kn}^{(j)})|^2 < +\infty \quad \text{a. s.}$$

3. Problem and Formulation of the Main Result

Consider the heat equations (1.1) in the following mild sense

$$\begin{aligned} u_\varepsilon(t, x) &= \int_{\mathbb{R}} p(t, x - y) u_0(y) dy \\ &+ \int_0^t ds \int_{\mathbb{R}} p(t - s, x - y) f(s/\varepsilon, y, u_\varepsilon(s, y)) dy \\ &+ \int_{\mathbb{R}} d\mu(y) \int_0^t p(t - s, x - y) \sigma(s/\varepsilon, y) ds. \end{aligned} \quad (3.1)$$

Here $p(t, x) = (\sqrt{\pi t})^{-1} e^{-x^2/t}$ is the Gaussian heat kernel,

$$u(t, x) = u(t, x, \omega) : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$$

is an unknown measurable random function, μ is a stochastic measure defined on Borel σ -algebra of \mathbb{R} . The integrals of random functions w.r.t. ds and dy are taken for each fixed $\omega \in \Omega$. For each pair (t, x) equality (3.1) holds a. s.

Recall that $\int_{\mathbb{R}} p(t, x) dx = 1$, and for some C , $0 < \lambda < 1$ hold

$$\left| \frac{\partial p(t, x)}{\partial x} \right| \leq Ct^{-1} e^{-\lambda x^2/t}, \quad \left| \frac{\partial p(t, x)}{\partial t} \right| \leq Ct^{-3/2} e^{-\lambda x^2/t}. \quad (3.2)$$

We will refer to the following assumptions throughout the paper.

Assumption A1. $u_0(y) = u_0(y, \omega) : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ is measurable and

$$|u_0(y)| \leq C(\omega), \quad |u_0(y_1) - u_0(y_2)| \leq L_{u_0}(\omega) |y_1 - y_2|^{\beta(u_0)}, \quad \beta(u_0) \geq 1/2.$$

Assumption A2. $f(s, y, v) : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is measurable, bounded, and

$$|f(s, y_1, v_1) - f(s, y_2, v_2)| \leq L_f (|y_1 - y_2| + |v_1 - v_2|).$$

Assumption A3. $\sigma(s, y) : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ is measurable, bounded, and

$$|\sigma(s, y_1) - \sigma(s, y_2)| \leq L_\sigma |y_1 - y_2|^{\beta(\sigma)}, \quad 1/2 < \beta(\sigma) < 1.$$

Assumption A4. $|y|^\rho$ is integrable w.r.t. μ on \mathbb{R} for some $\rho > 1/2$.

Assumption A5. There exist the following limits

$$\bar{f}(y, v) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(s, y, v) ds, \quad \bar{\sigma}(y) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \sigma(s, y) ds. \quad (3.3)$$

It is easy to see that fulfilment of A2 and A3 for $f(s, y, v)$ and $\sigma(s, y)$ imply their fulfilment for $\bar{f}(y, v)$ and $\bar{\sigma}(y)$.

We will study convergence $u_\varepsilon(t, x) \rightarrow \bar{u}(t, x)$, $\varepsilon \rightarrow 0$, where \bar{u} is the solution of the averaged equation (1.2)

$$\begin{aligned} \bar{u}(t, x) &= \int_{\mathbb{R}} p(t, x - y) u_0(y) dy + \int_0^t ds \int_{\mathbb{R}} p(t - s, x - y) \bar{f}(y, \bar{u}(s, y)) dy \\ &\quad + \int_{\mathbb{R}} d\mu(y) \int_0^t p(t - s, x - y) \bar{\sigma}(y) ds. \end{aligned} \quad (3.4)$$

Theorem of [2] and A1–A3 give that solutions of (3.1) and (3.4) exist and are unique, and have a Hölder continuous version on $[\delta, T] \times [-K, K]$ for each $\delta, K > 0$ (a similar result is given by Theorem [17]). Therefore, u_ε and \bar{u} have a continuous versions on $(0, T] \times \mathbb{R}$.

In the main theorem we will use the following assumptions.

Assumption A6. Functions

$$\begin{aligned} H_f(r, y, v) &= f(r, y, v) - \bar{f}(y, v), \\ G_f(r, y, v) &= \int_0^r (f(s, y, v) - \bar{f}(y, v)) ds, \quad r \in \mathbb{R}_+, y, v \in \mathbb{R} \end{aligned}$$

are bounded.

Assumption A7. Functions

$$\begin{aligned} H_\sigma(r, y) &= \sigma(r, y) - \bar{\sigma}(y), \\ G_\sigma(r, y) &= \int_0^r (\sigma(s, y) - \bar{\sigma}(y)) ds, \quad r \in \mathbb{R}_+, y \in \mathbb{R} \end{aligned}$$

are bounded.

Assertions A6 and A7 hold, for example, if $f(s, y, v)$ and $\sigma(s, y)$ are bounded, periodic in s for each y, v , and set of values of minimal period is bounded.

The main result of the paper is the following.

Theorem 3.1. *Assume that Assumptions A1–A7 hold. Then for continuous versions of u_ε and \bar{u} , and for any*

$$0 < \gamma_1 < \min\left\{\frac{1}{5}, \frac{1}{2}\left(1 - \frac{1}{2\beta(\sigma)}\right)\right\}, \quad (3.5)$$

we have

$$\sup_{\varepsilon > 0, t \in [0, T], x \in \mathbb{R}} \varepsilon^{-\gamma_1} |u_\varepsilon(t, x) - \bar{u}(t, x)| < +\infty \text{ a. s.} \quad (3.6)$$

Proof of the theorem will be given in Section 5.

Remark 3.2. It was assumed that $\beta(\sigma) > \frac{1}{2}$, therefore interval in (3.5) is not empty. For smooth σ we can take any $0 < \gamma_1 < \frac{1}{5}$. Also we can compare our estimate in (3.5) with results of other papers. In [19], for equations (3.1) and (3.4), for f independent of t , it was proved that we can take any $0 < \gamma_1 < \frac{1}{2}\left(1 - \frac{1}{2\beta(\sigma)}\right)$, and for smooth σ we obtain $0 < \gamma_1 < \frac{1}{4}$.

The order of strong convergence equal to $\frac{1}{4}$ was obtained in [16] for the system driven by Brownian motion and Poisson random measure. For systems driven by

Brownian motions only, in [3], were achieved strong convergence rate $\frac{1}{2}-$ and weak convergence rate $1-$.

4. Auxiliary Lemmata

At first, we will study the Hölder regularity of the stochastic integrals in our equations. The following statement is very similar to Lemma 2 [2]. But here we obtain that coefficient $C(\omega)$ in (4.1) does not depend on x .

Lemma 4.1. *Let A3 and A4 hold. Then for version (2.2) of*

$$\vartheta(x, t) = \int_{\mathbb{R}} d\mu(y) \int_0^t p(t-s, x-y)\sigma(s, y) ds, \quad t \in [0, T],$$

for any $\gamma < 1/4$ there exist a random constant $C(\omega) < \infty$ a. s. (that depends on γ , is independent of x) such that

$$|\vartheta(x, t_1) - \vartheta(x, t_2)| \leq C(\omega)|t_1 - t_2|^\gamma \quad (4.1)$$

for all $t_1, t_2 \in [0, T]$, $x \in \mathbb{R}$.

Proof. Let $x \in \mathbb{R}$, $0 \leq t_1 < t_2 \leq T$ are fixed. Set

$$q(z, y) = \int_0^{t_2} p(t_2-s, x-y)\sigma(s, y) ds - \int_0^{t_1} p(t_1-s, x-y)\sigma(s, y) ds, \quad z = (t_1, t_2, x).$$

For version (2.2) of the integral

$$\eta(z) = \int_{(j, j+1]} q(z, y) d\mu(y)$$

holds (2.3). To estimate Besov space norm of $q(z, \cdot)$, we consider

$$\begin{aligned} & q(z, y+h) - q(z, y) \\ &= \int_0^{t_1} (p(t_2-s, x-y-h) - p(t_1-s, x-y-h))(\sigma(s, y+h) - \sigma(s, y)) ds \\ &+ \int_0^{t_1} (p(t_2-s, x-y-h) - p(t_2-s, x-y))\sigma(s, y) ds \\ &- \int_0^{t_1} (p(t_1-s, x-y-h) - p(t_1-s, x-y))\sigma(s, y) ds \\ &+ \int_{t_1}^{t_2} p(t_2-s, x-y-h)(\sigma(s, y+h) - \sigma(s, y)) ds \\ &+ \int_{t_1}^{t_2} (p(t_2-s, x-y-h) - p(t_2-s, x-y))\sigma(s, y) ds \\ &:= I_{11} + I_{12} - I_{13} + I_{21} + I_{22} := I_1 + I_2. \end{aligned}$$

By A3,

$$\begin{aligned} |I_{21}| &\leq L_\sigma h^{\beta(\sigma)} \int_{t_1}^{t_2} (t_2-s)^{-1/2} e^{-\frac{(x-y-h)^2}{t_2-s}} ds \\ &\leq L_\sigma h^{\beta(\sigma)} \int_{t_1}^{t_2} (t_2-s)^{-1/2} ds = Ch^{\beta(\sigma)}(t_2-t_1)^{1/2}. \end{aligned} \quad (4.2)$$

Using boundedness of σ , obtain

$$|I_{22}| \leq C \int_{t_1}^{t_2} (t_2 - s)^{-1/2} e^{-\frac{(x-y-h)^2}{t_2-s}} ds \leq C(t_2 - t_1)^{1/2}. \quad (4.3)$$

We will use below the following simple estimate.

$$\begin{aligned} \int_0^t \frac{1}{r} e^{-\frac{b}{r}} dr \stackrel{b/r=z}{=} \int_{b/t}^{\infty} \frac{1}{z} e^{-z} dz &\leq \mathbf{1}_{\{t>b\}} \int_{b/t}^1 \frac{1}{z} dz + \int_1^{\infty} e^{-z} dz \\ &\leq \mathbf{1}_{\{t>b\}} \ln \frac{t}{b} + 1. \end{aligned} \quad (4.4)$$

We get

$$\begin{aligned} |I_{22}| &= \left| \int_{t_1}^{t_2} (p(t_2 - s, x - y - h) - p(t_2 - s, x - y)) \sigma(s, y) ds \right| \\ &\stackrel{A3}{\leq} C \int_{t_1}^{t_2} ds \int_y^{y+h} \left| \frac{\partial p(t_2 - s, x - v)}{\partial v} \right| dv \\ &\stackrel{(3.2)}{\leq} C \int_{t_1}^{t_2} ds \int_y^{y+h} (t_2 - s)^{-1} e^{-\frac{\lambda(v-x)^2}{t_2-s}} dv \\ &\stackrel{r=t_2-s}{=} C \int_0^{t_2-t_1} dr \int_y^{y+h} r^{-1} e^{-\frac{\lambda(v-x)^2}{r}} dv \\ &= C \int_y^{y+h} dv \int_0^{t_2-t_1} r^{-1} e^{-\frac{\lambda(v-x)^2}{r}} \mathbf{1}_{\{|v-x|>1\}} dr \\ &\quad + C \int_y^{y+h} dv \int_0^{t_2-t_1} r^{-1} e^{-\frac{\lambda(v-x)^2}{r}} \mathbf{1}_{\{|v-x|\leq 1\}} dr \\ &\stackrel{(4.4)}{\leq} C \int_y^{y+h} dv \int_0^{t_2-t_1} r^{-1} e^{-\frac{\lambda}{r}} dr \\ &\quad + C \int_y^{y+h} \left(\mathbf{1}_{\{t_2-t_1>|v-x|^2\}} \ln \frac{t_2-t_1}{|v-x|^2} + 1 \right) \mathbf{1}_{\{|v-x|\leq 1\}} dv \\ &\stackrel{r^{-1}e^{-\frac{\lambda}{r}} \leq C(\lambda)}{\leq} Ch(t_2 - t_1) + Ch + C \int_y^{y+h} \mathbf{1}_{\{T>|v-x|^2\}} \ln \frac{T}{|v-x|^2} \mathbf{1}_{\{|v-x|\leq 1\}} dv \\ &\leq Ch(t_2 - t_1) + Ch + C \int_y^{y+h} |\ln |v-x|| \mathbf{1}_{\{|v-x|\leq 1\}} dv \\ &\leq Ch(t_2 - t_1) + Ch + C \int_0^{h/2} |\ln r| dr \leq Ch \ln |h| \leq Ch^{\gamma_0}, \end{aligned} \quad (4.5)$$

where $0 < \gamma_0 < 1$ is arbitrary and C depends on γ_0 . (We have used that maximal value of $\int_y^{y+h} |\ln |r|| \mathbf{1}_{\{|r|\leq 1\}} dv$ is achieved for $y = -h/2$).

Multiplying (4.3) and (4.5) at power δ_0 and $1 - \delta_0$ respectively, $\delta_0 \in (0, 1)$, using (4.2), we get

$$\begin{aligned} |I_2| &\leq |I_{21}| + |I_{22}| \leq Ch^{\beta(\sigma)}(t_2 - t_1)^{1/2} + Ch^{(1-\delta_0)\gamma_0}(t_2 - t_1)^{\delta_0/2} \\ &\leq C(t_2 - t_1)^{\delta_0/2} (h^{\beta(\sigma)} + h^{(1-\delta_0)\gamma_0}). \end{aligned} \quad (4.6)$$

For $\gamma_0 \rightarrow 1-$, and $1 - \delta_0 \rightarrow 1/2+$ we get $(1 - \delta_0)\gamma_0 > 1/2$ and $\delta_0 \rightarrow 1/2-$.

By A3, we get

$$\begin{aligned}
|I_{11}| &\leq L_\sigma h^{\beta(\sigma)} \int_0^{t_1} |p(t_2 - s, x - y - h) - p(t_1 - s, x - y - h)| ds \\
&\leq L_\sigma h^{\beta(\sigma)} \int_0^{t_1} ds \int_{t_1}^{t_2} \left| \frac{\partial p(\tau - s, x - y - h)}{\partial \tau} \right| d\tau \\
&\stackrel{(3.2)}{\leq} Ch^{\beta(\sigma)} \int_0^{t_1} ds \int_{t_1}^{t_2} (\tau - s)^{-3/2} e^{-\frac{\lambda(x-y-h)^2}{\tau-s}} d\tau \\
&\leq Ch^{\beta(\sigma)} \int_0^{t_1} ds \int_{t_1}^{t_2} (\tau - s)^{-3/2} d\tau \leq Ch^{\beta(\sigma)} (t_2 - t_1)^{1/2}. \tag{4.7}
\end{aligned}$$

Further, as in (4.7), we obtain

$$\begin{aligned}
|I_{12} - I_{13}| &= \left| \int_0^{t_1} (p(t_2 - s, x - y - h) - p(t_1 - s, x - y - h)) \sigma(s, y) ds \right. \\
&\quad \left. - \int_0^{t_1} (p(t_2 - s, x - y) - p(t_1 - s, x - y)) \sigma(s, y) ds \right| \\
&\stackrel{A3}{\leq} C \int_0^{t_1} |p(t_2 - s, x - y - h) - p(t_1 - s, x - y - h)| ds \\
&\quad + C \int_0^{t_1} |p(t_2 - s, x - y) - p(t_1 - s, x - y)| ds \\
&\leq C \int_0^{t_1} ds \int_{t_1}^{t_2} \left| \frac{\partial p(\tau - s, x - y - h)}{\partial \tau} \right| d\tau \leq C(t_2 - t_1)^{1/2}. \tag{4.8}
\end{aligned}$$

Analogously to (4.5), for any $0 < \gamma_0 < 1$ we get

$$|I_{12} - I_{13}| \leq |I_{12}| + |I_{13}| \leq Ch^{\gamma_0}. \tag{4.9}$$

Multiplying (4.8) and (4.9) to the power δ_0 and $1 - \delta_0$ respectively, using (4.7), we get

$$\begin{aligned}
|I_1| &\leq |I_{11}| + |I_{12} - I_{13}| \leq Ch^{\beta(\sigma)} (t_2 - t_1)^{1/2} + Ch^{(1-\delta_0)\gamma_0} (t_2 - t_1)^{\delta_0/2} \\
&\leq C(t_2 - t_1)^{\delta_0/2} (h^{\beta(\sigma)} + h^{(1-\delta_0)\gamma_0}).
\end{aligned}$$

Thus,

$$|q(z, y + h) - q(z, y)| \leq C(t_2 - t_1)^{\delta_0/2} (h^{\beta(\sigma)} + h^{(1-\delta_0)\gamma_0}).$$

Therefore,

$$\begin{aligned}
&\left(\int_0^1 (w_2(g, r))^2 r^{-2\alpha-1} dr \right)^{1/2} \\
&\leq C(t_2 - t_1)^{\delta_0/2} \left(\int_0^1 r^{2\beta(\sigma)-2\alpha-1} dr + \int_0^1 r^{2(1-\delta_0)\gamma_0-2\alpha-1} dr \right)^{1/2} \\
&\leq C(t_2 - t_1)^{\delta_0/2}
\end{aligned}$$

for respective $1/2 < \alpha < \min\{(1 - \delta_0)\gamma_0, \beta(\sigma)\}$. As we have mentioned after (4.6), we can take any $\delta_0 < 1/2$.

Also for $y \in \mathbb{R}$ by (4.7) and (4.2)

$$\begin{aligned} |q(z, y)| &= \left| \int_0^{t_1} (p(t_2 - s, x - y) - p(t_1 - s, x - y)) \sigma(s, y) ds \right. \\ &\quad \left. + \int_{t_1}^{t_2} p(t_2 - s, x - y) \sigma(s, y) ds \right| \leq C(t_2 - t_1)^{1/2}, \end{aligned}$$

therefore

$$\|q(z, \cdot)\|_{L_2([j, j+1])} \leq C(t_2 - t_1)^{1/2}, \quad |q(z, j)| \leq C(t_2 - t_1)^{1/2}. \quad (4.10)$$

From (2.1) it follows that

$$\|q(z, \cdot)\|_{B_{22}^s([c, d])} \leq C(t_2 - t_1)^{\delta_0/2} \quad (4.11)$$

for any $\delta_0 < 1/2$. We set $\gamma = \delta_0/2 < 1/4$, take $\rho > 1/2$ from Assumption A4, and have

$$\begin{aligned} |\vartheta(t_1) - \vartheta(t_2)| &= \left| \int_{\mathbb{R}} q(z, y) d\mu(y) \right| \leq \sum_{j \in \mathbb{Z}} \left| \int_{[j, j+1]} q(z, y) d\mu(y) \right| \\ &\stackrel{(2.3), (2.4)}{\leq} \sum_{j \in \mathbb{Z}} |q(z, j) \mu((j, j+1])| \\ &\quad + C \sum_{j \in \mathbb{Z}} \|q(z, \cdot)\|_{B_{22}^s([j, j+1])} \left\{ \sum_{n \geq 1} 2^{n(1-2\alpha)} \sum_{1 \leq k \leq 2^n} |\mu(\Delta_{kn}^{(j)})|^2 \right\}^{1/2} \\ &\stackrel{(4.10), (4.11)}{\leq} C(t_2 - t_1)^\gamma \left[\sum_{j \in \mathbb{Z}} |\mu((j, j+1])| \right. \\ &\quad \left. + \sum_{j \in \mathbb{Z}} \left\{ \sum_{n \geq 1} 2^{n(1-2\alpha)} \sum_{1 \leq k \leq 2^n} |\mu(\Delta_{kn}^{(j)})|^2 \right\}^{1/2} \right] \\ &\leq C(t_2 - t_1)^\gamma \left[\left(\sum_{j \in \mathbb{Z}} (|j| + 1)^{2\rho} (\mu((j, j+1]))^2 \right)^{1/2} \left(\sum_{j \in \mathbb{Z}} (|j| + 1)^{-2\rho} \right)^{1/2} \right. \\ &\quad \left. + \left(\sum_{n \geq 1} 2^{n(1-2\alpha)} \sum_{j \in \mathbb{Z}} (|j| + 1)^{2\rho} \sum_{1 \leq k \leq 2^n} |\mu(\Delta_{kn}^{(j)})|^2 \right)^{1/2} \left(\sum_{j \in \mathbb{Z}} (|j| + 1)^{-2\rho} \right)^{1/2} \right], \end{aligned} \quad (4.12)$$

where the sums with SMs have a kind $\sum_{l=1}^{\infty} \left(\int_{\mathbb{R}} \phi_l d\mu \right)^2$,

$$\{\phi_l(y), l \geq 1\} = \{(|j| + 1)^\rho \mathbf{1}_{(j, j+1]}(y), j \in \mathbb{Z}\},$$

$$\{\phi_l(y), l \geq 1\} = \{(|j| + 1)^\rho 2^{n(1-2\alpha)/2} \mathbf{1}_{\Delta_{kn}^{(j)}}(y), j \in \mathbb{Z}, n \geq 1, 1 \leq k \leq 2^n\}.$$

From the inequalities

$$\sum_{l=1}^{\infty} |\phi_l(y)| \leq C(1 + |y|^\rho), \quad \sum_{j \in \mathbb{Z}} (|j| + 1)^{-2\rho} < \infty,$$

and Lemma 2.2 it follows that

$$\sum_{l=1}^{\infty} \left(\int_{\mathbb{R}} \phi_l d\mu \right)^2 < +\infty \text{ a. s.}$$

Thus, (4.12) implies (4.1). \square

For solutions of (3.1) and (3.4), Theorem of [2] states the Hölder regularity for $t \in [\delta, T]$ for any $\delta > 0$ (a similar result is given by Theorem [17]). But we need an estimate of increments of the solutions for $t \in (0, T]$, and will prove the following statement.

Lemma 4.2. *Let Assumptions A1–A4 hold, and u is a solution of equation*

$$\begin{aligned} u(t, x) &= \int_{\mathbb{R}} p(t, x - y) u_0(y) dy \\ &+ \int_0^t ds \int_{\mathbb{R}} p(t - s, x - y) f(s, y, u(s, y)) dy \\ &+ \int_{\mathbb{R}} d\mu(y) \int_0^t p(t - s, x - y) \sigma(s, y) ds. \end{aligned}$$

Then for continuous version of u , each $0 < \gamma < 1/4$, some $C(\omega)$, and all $0 < t_1 < t_2 \leq T$, $x \in \mathbb{R}$ holds

$$\begin{aligned} |u(t_1, x) - u(t_2, x)| &\leq C(\omega) (\ln t_2 - \ln t_1 \\ &+ t_2 \ln t_2 - t_1 \ln t_1 - (t_2 - t_1) \ln(t_2 - t_1) + (t_2 - t_1)^\gamma). \end{aligned} \quad (4.13)$$

Proof. Theorem [2] states that $u(t, x)$ has a Hölder continuous version on $[\delta, T] \times [-K, K]$ for each $\delta, K > 0$. Therefore, $u(t, x)$ has a continuous version on $(0, T] \times \mathbb{R}$. Also, we will take version (2.2) of $\vartheta(x, t)$ from Lemma 4.1.

Let $0 < t_1 < t_2 \leq T$. We have

$$\begin{aligned} |u(t_1, x) - u(t_2, x)| &\stackrel{(4.1)}{\leq} \int_{\mathbb{R}} |p(t_1, x - y) - p(t_2, x - y)| |u_0(y)| dy \\ &+ \left| \int_0^{t_1} ds \int_{\mathbb{R}} p(t_1 - s, x - y) f(s, y, u(s, y)) dy \right. \\ &\quad \left. - \int_0^{t_2} ds \int_{\mathbb{R}} p(t_2 - s, x - y) f(s, y, u(s, y)) dy \right| + C(\omega) (t_2 - t_1)^\gamma \\ &=: J_1 + J_2 + C(\omega) (t_2 - t_1)^\gamma. \end{aligned}$$

From definition of solution, it follows that the inequality holds a. s. for each pair (t, x) . We can say that it holds a. s. for all $(t, x) \in (\mathbb{Q} \cap (0, T]) \times \mathbb{Q}$, and, by continuity, obtain the estimate for $(t, x) \in (0, T] \times \mathbb{R}$.

We get

$$\begin{aligned} J_1 &\stackrel{A1}{\leq} C(\omega) \int_{\mathbb{R}} dy \int_{t_1}^{t_2} \left| \frac{\partial p(\tau, x - y)}{\partial \tau} \right| d\tau \stackrel{(3.2)}{\leq} C(\omega) \int_{\mathbb{R}} dy \int_{t_1}^{t_2} \tau^{-\frac{3}{2}} e^{-\frac{\lambda(x-y)^2}{\tau}} d\tau \\ &= C(\omega) \int_{t_1}^{t_2} \tau^{-1} d\tau \int_{\mathbb{R}} \tau^{-\frac{1}{2}} e^{-\frac{\lambda(x-y)^2}{\tau}} dy = C(\omega) (\ln t_2 - \ln t_1). \end{aligned}$$

For J_2 we obtain

$$\begin{aligned}
J_2 &\leq \int_0^{t_1} ds \int_{\mathbb{R}} |p(t_1 - s, x - y) - p(t_2 - s, x - y)| |f(s, y, u(s, y))| dy \\
&\quad + \int_{t_1}^{t_2} ds \int_{\mathbb{R}} p(t_2 - s, x - y) |f(s, y, u(s, y))| dy \\
&\stackrel{A2}{\leq} C \int_0^{t_1} ds \int_{\mathbb{R}} |p(t_1 - s, x - y) - p(t_2 - s, x - y)| dy \\
&\quad + C(t_2 - t_1) =: J_{21} + C(t_2 - t_1).
\end{aligned}$$

We get

$$\begin{aligned}
J_{21} &\leq C \int_0^{t_1} ds \int_{\mathbb{R}} dy \int_{t_1}^{t_2} \left| \frac{\partial p(\tau - s, x - y)}{\partial \tau} \right| d\tau \\
&\stackrel{(3.2)}{\leq} C \int_0^{t_1} ds \int_{\mathbb{R}} dy \int_{t_1}^{t_2} (\tau - s)^{-\frac{3}{2}} e^{-\frac{\lambda(x-y)^2}{\tau-s}} d\tau \\
&= C \int_0^{t_1} ds \int_{t_1}^{t_2} (\tau - s)^{-1} d\tau \int_{\mathbb{R}} (\tau - s)^{-\frac{1}{2}} e^{-\frac{\lambda(x-y)^2}{\tau-s}} dy \\
&= C \int_0^{t_1} (\ln(t_2 - s) - \ln(t_1 - s)) ds \\
&= C(t_2 \ln t_2 - t_1 \ln t_1 - (t_2 - t_1) \ln(t_2 - t_1)).
\end{aligned}$$

From our estimates of J_1 and J_2 we arrive at (4.13). \square

To estimate the difference of terms of (3.1) and (3.4), we will need the following result.

Lemma 4.3. *Let $h(r, y, z) : \mathbb{R}_+ \times \mathbb{R} \times Z \rightarrow \mathbb{R}$ and $\bar{h}(y, z) : \mathbb{R} \times Z \rightarrow \mathbb{R}$ be measurable for each fixed z , and functions*

$$H(r, y, z) = h(r, y, z) - \bar{h}(y, z), \quad G(r, y, z) = \int_0^r (h(v, y, z) - \bar{h}(y, z)) dv$$

are bounded on $\mathbb{R}_+ \times \mathbb{R} \times Z$. Then

$$\sup_{x \in \mathbb{R}, z \in Z, \varepsilon > 0, t \in (0, T]} \frac{1}{|\varepsilon \ln \varepsilon|} \left| \int_{\mathbb{R}} dy \int_0^t \frac{e^{-\frac{(x-y)^2}{t-s}}}{\sqrt{t-s}} (h(s/\varepsilon, y, z) - \bar{h}(y, z)) ds \right| < +\infty. \tag{4.14}$$

Proof. Let $|H(r, y, z)| \leq C_H$, $|G(r, y, z)| \leq C_G$. At first, consider the case $t \geq \varepsilon$, and use the decomposition $\int_0^t = \int_0^{t-\varepsilon} + \int_{t-\varepsilon}^t$. For the second term, we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}} dy \int_{t-\varepsilon}^t \frac{e^{-\frac{(x-y)^2}{t-s}}}{\sqrt{t-s}} (h(s/\varepsilon, y, z) - \bar{h}(y, z)) ds \right| \\
&\leq C_H \int_{t-\varepsilon}^t ds \int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{t-s}}}{\sqrt{t-s}} dy = C\varepsilon.
\end{aligned} \tag{4.15}$$

For $\int_0^{t-\varepsilon}$, we obtain

$$\begin{aligned}
& \left| \int_{\mathbb{R}} dy \int_0^{t-\varepsilon} \frac{e^{-\frac{(x-y)^2}{t-s}}}{\sqrt{t-s}} (h(s/\varepsilon, y, z) - \bar{h}(y, z)) ds \right| \\
&= \left| \int_{\mathbb{R}} dy \int_0^{t-\varepsilon} \frac{e^{-\frac{(x-y)^2}{t-s}}}{\sqrt{t-s}} \varepsilon dG(s/\varepsilon, y, z) \right| \\
&= \varepsilon \left| \int_{\mathbb{R}} dy \left(\frac{e^{-\frac{(x-y)^2}{t-s}}}{\sqrt{t-s}} G(s/\varepsilon, y, z) \Big|_0^{t-\varepsilon} \right) \right. \\
&\quad \left. - \int_{\mathbb{R}} dy \int_0^{t-\varepsilon} G(s/\varepsilon, y, z) \left(\frac{e^{-\frac{(x-y)^2}{t-s}}}{2\sqrt{(t-s)^3}} - \frac{(x-y)^2 e^{-\frac{(x-y)^2}{t-s}}}{\sqrt{(t-s)^5}} \right) ds \right| \\
&\stackrel{G(0, y, z)=0}{\leq} \varepsilon C_G \int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{\varepsilon}}}{\sqrt{\varepsilon}} dy \\
&\quad + \varepsilon C_G \int_{\mathbb{R}} dy \int_0^{t-\varepsilon} \left(\frac{e^{-\frac{(x-y)^2}{t-s}}}{2\sqrt{(t-s)^3}} + \frac{(x-y)^2 e^{-\frac{(x-y)^2}{t-s}}}{\sqrt{(t-s)^5}} \right) ds \\
&\stackrel{v=(x-y)/\sqrt{t-s}}{\leq} \varepsilon C + \varepsilon C_G \int_0^{t-\varepsilon} ds \int_{\mathbb{R}} \left(\frac{e^{-v^2}}{2\sqrt{(t-s)^3}} + \frac{v^2 e^{-v^2}}{\sqrt{(t-s)^3}} \right) \sqrt{t-s} dv \\
&= \varepsilon C + \varepsilon C \int_0^{t-\varepsilon} \frac{ds}{t-s} = \varepsilon C + \varepsilon C (\ln t - \ln \varepsilon).
\end{aligned}$$

While $\varepsilon < t \leq T$, we arrive at (4.14) in this case.

If $0 < t \leq \varepsilon$, we repeat the estimates from (4.15) for \int_0^t , and obtain the upper bound $Ct \leq C\varepsilon$. \square

5. Proof of the Main Result

Now we prove Theorem 3.1. We take the versions of stochastic integrals defined by Lemma 2.3. Consider

$$\begin{aligned}
\xi_\varepsilon &= \int_{\mathbb{R}} d\mu(y) \int_0^t p(t-s, x-y) \sigma(s/\varepsilon, y) ds \\
&\quad - \int_{\mathbb{R}} d\mu(y) \int_0^t p(t-s, x-y) \bar{\sigma}(y) ds.
\end{aligned}$$

In [19] it was proved that for given version of ξ_ε and any $0 < \gamma_1 < \frac{1}{2} \left(1 - \frac{1}{2\beta(\sigma)}\right)$

$$|\xi_\varepsilon| \leq C(\omega) \varepsilon^{\gamma_1} \text{ a. s.}, \quad (5.1)$$

where $C(\omega)$ depends on γ_1 , is independent of t , x (see Step 1 of proof of Theorem 1 [19]).

We have

$$\begin{aligned}
& |u_\varepsilon(t, x) - \bar{u}(t, x)| \\
& \leq \left| \int_0^t ds \int_{\mathbb{R}} p(t-s, x-y) (f(s/\varepsilon, y, u_\varepsilon(s, y)) - f(s/\varepsilon, y, \bar{u}(s, y))) dy \right| \\
& \quad + \left| \int_0^t ds \int_{\mathbb{R}} p(t-s, x-y) (f(s/\varepsilon, y, \bar{u}(s, y)) - \bar{f}(y, \bar{u}(s, y))) dy \right| + C(\omega)\varepsilon^{\gamma_1} \\
& =: \hat{I}_1 + \hat{I}_2 + C(\omega)\varepsilon^{\gamma_1}, \tag{5.2}
\end{aligned}$$

and for the first term obtain

$$\begin{aligned}
\hat{I}_1 & \stackrel{A_2}{\leq} L_f \int_0^t ds \int_{\mathbb{R}} p(t-s, x-y) |u_\varepsilon(s, y) - \bar{u}(s, y)| dy \\
& \leq L_f \int_0^t \sup_{y \in \mathbb{R}} |u_\varepsilon(s, y) - \bar{u}(s, y)| ds \int_{\mathbb{R}} p(t-s, x-y) dy \\
& = L_f \int_0^t \sup_{y \in \mathbb{R}} |u_\varepsilon(s, y) - \bar{u}(s, y)| ds \tag{5.3}
\end{aligned}$$

(from boundedness of f and \bar{f} it follows that $\sup_{y \in \mathbb{R}} |u_\varepsilon(s, y) - \bar{u}(s, y)| < \infty$ a. s.).

Consider \hat{I}_2 . Divide $[0, T]$ into n segments of length $\Delta = T/n$. We get that

$$\begin{aligned}
\hat{I}_2 & \leq \sum_{k=0}^{n-1} \left| \int_{(k\Delta \wedge t, (k+1)\Delta \wedge t]} ds \int_{\mathbb{R}} p(t-s, x-y) (f(s/\varepsilon, y, \bar{u}(s, y)) \right. \\
& \quad \left. - \bar{f}(y, \bar{u}(s, y))) dy \right| \\
& \leq \sum_{k=0}^{n-1} \left(\left| \int_{(k\Delta \wedge t, (k+1)\Delta \wedge t]} ds \int_{\mathbb{R}} p(t-s, x-y) (f(s/\varepsilon, y, \bar{u}(s, y)) \right. \right. \\
& \quad \left. \left. - f(s/\varepsilon, y, \bar{u}(k\Delta, y))) dy \right| \right. \\
& \quad + \left| \int_{(k\Delta \wedge t, (k+1)\Delta \wedge t]} ds \int_{\mathbb{R}} p(t-s, x-y) (f(s/\varepsilon, y, \bar{u}(k\Delta, y)) \right. \\
& \quad \left. - \bar{f}(y, \bar{u}(k\Delta, y))) dy \right| \\
& \quad + \left| \int_{(k\Delta \wedge t, (k+1)\Delta \wedge t]} ds \int_{\mathbb{R}} p(t-s, x-y) (\bar{f}(y, \bar{u}(k\Delta, y)) \right. \\
& \quad \left. - \bar{f}(y, \bar{u}(s, y))) dy \right| \Big) \\
& =: \sum_{k=0}^{n-1} (\hat{I}_{21}^{(k)} + \hat{I}_{22}^{(k)} + \hat{I}_{23}^{(k)}).
\end{aligned}$$

From Lemma 4.3 it follows that for each $1 \leq k \leq n$

$$\hat{I}_{22}^{(k)} \leq C|\varepsilon \ln \varepsilon|. \tag{5.4}$$

Consider $\sum_{k=0}^{n-1} \hat{I}_{21}^{(k)}$, where $\hat{I}_{21}^{(0)}$ will be estimated separately.

$$\begin{aligned}
\sum_{k=0}^{n-1} \hat{I}_{21}^{(k)} &= \sum_{k=0}^{n-1} \left| \int_{(k\Delta \wedge t, (k+1)\Delta \wedge t]} ds \int_{\mathbb{R}} p(t-s, x-y) (f(s/\varepsilon, y, \bar{u}(s, y)) \right. \\
&\quad \left. - f(s/\varepsilon, y, \bar{u}(k\Delta, y))) dy \right| \\
&\leq L_f \sum_{k=0}^{n-1} \int_{(k\Delta \wedge t, (k+1)\Delta \wedge t]} ds \int_{\mathbb{R}} p(t-s, x-y) |\bar{u}(s, y) - \bar{u}(k\Delta, y)| dy \\
&\stackrel{(4.13)}{\leq} L_f C(\omega) \sum_{k=1}^{n-1} \int_{\mathbb{R}} dy \int_{(k\Delta \wedge t, (k+1)\Delta \wedge t]} p(t-s, x-y) (\ln s - \ln k\Delta \\
&\quad + s \ln s - k\Delta \ln k\Delta - (s - k\Delta) \ln(s - k\Delta) + (s - k\Delta)^\gamma) ds \\
&\quad + L_f \int_{(0, \Delta \wedge t]} ds \int_{\mathbb{R}} p(t-s, x-y) |\bar{u}(s, y) - \bar{u}(0, y)| dy. \tag{5.5}
\end{aligned}$$

It is easy to check that the function

$$f_k(s) = \ln s - \ln k\Delta + s \ln s - k\Delta \ln k\Delta - (s - k\Delta) \ln(s - k\Delta) + (s - k\Delta)^\gamma$$

is increasing. Therefore, $f_k(s) \leq f_k((k+1)\Delta)$ in (5.5), and we have

$$\begin{aligned}
&\sum_{k=1}^{n-1} \int_{\mathbb{R}} dy \int_{(k\Delta \wedge t, (k+1)\Delta \wedge t]} p(t-s, x-y) f_k(s) ds \\
&\leq \sum_{k=1}^{n-1} f_k((k+1)\Delta) \int_{(k\Delta \wedge t, (k+1)\Delta \wedge t]} ds \int_{\mathbb{R}} p(t-s, x-y) dy \\
&\leq \sum_{k=1}^{n-1} f_k((k+1)\Delta) \Delta \\
&= \Delta \sum_{k=1}^{n-1} \left((\ln(k+1)\Delta - \ln k\Delta) \right. \\
&\quad \left. + ((k+1)\Delta \ln(k+1)\Delta - k\Delta \ln k\Delta) - \Delta \ln \Delta + \Delta^\gamma \right) \\
&= \Delta \left(\ln n\Delta - \ln \Delta + n\Delta \ln n\Delta - \Delta \ln \Delta - (n-1)\Delta \ln \Delta + (n-1)\Delta^\gamma \right) \\
&\stackrel{n\Delta=T}{=} T(T+1) \frac{\ln n}{n} + \frac{n-1}{n} T \left(\frac{T}{n} \right)^\gamma \leq Cn^{-\gamma}.
\end{aligned}$$

In $\hat{I}_{21}^{(0)}$ we need to estimate

$$\begin{aligned}
|\bar{u}(t, x) - \bar{u}(0, x)| &= |\bar{u}(t, x) - u_0(x)| \\
&\leq \left| \int_{\mathbb{R}} p(t, x-y) u_0(y) dy - u_0(x) \right|
\end{aligned}$$

$$\begin{aligned}
& + \left| \int_0^t ds \int_{\mathbb{R}} p(t-s, x-y) \bar{f}(y, \bar{u}(s, y)) dy \right| \\
& + \left| \int_{\mathbb{R}} d\mu(y) \int_0^t p(t-s, x-y) \bar{\sigma}(y) ds \right| =: \hat{I}_{211}^{(0)} + \hat{I}_{212}^{(0)} + \hat{I}_{213}^{(0)}, \\
\hat{I}_{211}^{(0)} & = \left| \int_{\mathbb{R}} p(t, x-y) u_0(y) dy - u_0(x) \int_{\mathbb{R}} p(t, x-y) dy \right| \\
& \stackrel{A1}{\leq} C(\omega) \int_{\mathbb{R}} p(t, x-y) |y-x|^{\beta(u_0)} dy = C(\omega) t^{-1/2} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{t}} |y-x|^{\beta(u_0)} dy \\
& \stackrel{z=(x-y)/\sqrt{t}}{=} C(\omega) t^{-1/2} \int_{\mathbb{R}} e^{-z^2} z^{\beta(u_0)} t^{\beta(u_0)+1/2} dz = C(\omega) t^{\beta(u_0)}, \\
\hat{I}_{212}^{(0)} & \leq C \int_0^t ds \int_{\mathbb{R}} p(t-s, x-y) dy = Ct, \\
\hat{I}_{213}^{(0)} & \stackrel{(4.1)}{\leq} C(\omega) t^\gamma.
\end{aligned}$$

Using these estimates for $t \leq \Delta = T/n$, taking into account inequality $\gamma < 1/4 < \beta(u_0)$, obtain

$$\sum_{k=0}^{n-1} \hat{I}_{21}^{(k)} \leq C(\omega) n^{-\gamma}.$$

Obviously, we can repeat our considerations and obtain the same estimates for $\sum_{k=0}^{n-1} \hat{I}_{23}^{(k)}$.

Using also (5.4), we get for each positive integer n

$$\hat{I}_2 \leq C(\omega)(n|\varepsilon \ln \varepsilon| + n^{-\gamma}).$$

Function $g(x) = x|\varepsilon \ln \varepsilon| + x^{-\gamma}$, $x > 0$ has minimum value

$$g(x_*) = |\varepsilon \ln \varepsilon|^{\gamma/(\gamma+1)} (\gamma+1) \gamma^{-\gamma/(\gamma+1)}, \quad x_* = (\gamma/|\varepsilon \ln \varepsilon|)^{1/(\gamma+1)}.$$

We have $\frac{g(x+1)}{g(x)} \leq C$, $x \geq x_*$ therefore there exists positive integer $n_* \in [x_*, x_* + 1)$ such that

$$g(n_*) \leq C|\varepsilon \ln \varepsilon|^{\gamma/(\gamma+1)}.$$

Recall that $\gamma < 1/4$ is arbitrary and $\gamma_1 < 1/5$. We can take $\gamma/(\gamma+1) > \gamma_1$ and obtain

$$\hat{I}_2 \leq C(\omega) \varepsilon^{\gamma_1}. \tag{5.6}$$

Therefore, using (5.1), (5.2), (5.3), and (5.6), we have

$$|u_\varepsilon(t, x) - \bar{u}(t, x)| \leq L_f \int_0^t \sup_{y \in \mathbb{R}} |u_\varepsilon(s, y) - \bar{u}(s, y)| ds + C(\omega) \varepsilon^{\gamma_1}.$$

Taking a supremum, we conclude that

$$\sup_{x \in \mathbb{R}} |u_\varepsilon(t, x) - \bar{u}(t, x)| \leq L_f \int_0^t \sup_{y \in \mathbb{R}} |u_\varepsilon(s, y) - \bar{u}(s, y)| ds + C(\omega) \varepsilon^{\gamma_1}.$$

From the Gronwall inequality obtain

$$\sup_{x \in \mathbb{R}} |u_\varepsilon(t, x) - \bar{u}(t, x)| \leq C(\omega)\varepsilon^{\gamma_1},$$

where $C(\omega)$ is independent of t and ε . Thus, we obtain (3.6).

References

1. Bao, J., Yin, G., and Yuan, C.: Two-time-scale stochastic partial differential equations driven by α -stable noises: Averaging principles, *Bernoulli* **23** (2017), no. 1, 645–669.
2. Bodnarchuk, I. M.: Regularity of the mild solution of a parabolic equation with stochastic measure, *Ukr. Math. J.* **69** (2017), no. 1, 1–18.
3. Bréhier, C.-E.: Strong and weak orders in averaging for SPDEs, *Stochastic Process. Appl.* **122** (2012), no. 7, 2553–2593.
4. Cerrai, S.: A Khasminskii type averaging principle for stochastic reaction-diffusion equations, *Ann. Appl. Probab.* **19** (2009), no. 3, 899–948.
5. Dong, Z., Sun, X., Xiao, H., and Zhai, J.: Averaging principle for one dimensional stochastic Burgers equation, *J. Differential Equations* **265** (2018), no. 10, 4749–4797.
6. Duan, J. and Wang, W.: *Effective dynamics of stochastic partial differential equations*, Elsevier, Amsterdam, 2014.
7. Fu, H. and Duan, J.: An averaging principle for two-scale stochastic partial differential equations, *Stoch. Dyn.* **11** (2011), no. 02n03, 353–367.
8. Fu, H., Wan, L., and Liu, J.: Strong convergence in averaging principle for stochastic hyperbolic–parabolic equations with two time-scales, *Stochastic Process. Appl.* **125** (2015), no. 8, 3255–3279.
9. Fu, H., Wan, L., Liu, J. and Liu, X.: Weak order in averaging principle for stochastic wave equation with a fast oscillation, *Stochastic Process. Appl.* **128** (2018), no. 8, 2557–2580.
10. Gao, P.: Averaging principle for stochastic Korteweg-de Vries equation, *J. Differential Equations* (2019), doi: 10.1016/j.jde.2019.07.012.
11. Guo, Q., Guo, P., and Wan, F.: Strong convergence in the p th-mean of an averaging principle for two-time-scales SPDEs with jumps, *Adv. Difference Equ.* **2017** (2017), no. 1, 275.
12. Kamont, A.: A discrete characterization of Besov spaces, *Approx. Theory Appl. (N.S.)* **13** (1997), no. 2, 63–77.
13. Kwapien, S. and Woyczyński, W. A.: *Random series and stochastic integrals: Single and multiple*, Birkhäuser, Boston, 1992.
14. Li, Z. and Yan, L.: Stochastic averaging for two-time-scale stochastic partial differential equations with fractional Brownian motion, *Nonlinear Analysis: Hybrid Systems* **31** (2019), 317–333.
15. Memin, T., Mishura, Yu., and Valkeila, E.: Inequalities for the moments of Wiener integrals with respect to a fractional Brownian motion, *Statist. Probab. Lett.* **51** (2001), no. 2, 197–206.
16. Pei, B., Xu, Y., and Wu, J.-L.: Two-time-scales hyperbolic-parabolic equations driven by Poisson random measures: Existence, uniqueness and averaging principles, *J. Math. Anal. Appl.* **447** (2017), no. 1, 243–268.
17. Radchenko, V.: Mild solution of the heat equation with a general stochastic measure, *Studia Math.* **194** (2009), no. 3, 231–251.
18. Radchenko, V.: Evolution equations driven by general stochastic measures in Hilbert space, *Theory Probab. Appl.* **59** (2015), no. 2, 328–339.
19. Radchenko, V.: Averaging principle for the heat equation driven by a general stochastic measure, *Statist. Probab. Lett.* **146** (2019), 224–230.
20. Radchenko, V.: Averaging principle for equation driven by a stochastic measure, *Stochastics*, **91** (2019), no. 6, 905–915.
21. Samoilenko, A. M. and Stanzhytskyi, O.: *Qualitative and asymptotic analysis of differential equations with random perturbations*, World Scientific, Singapore, 2011.
22. Samorodnitsky, G. and Taqqu, M. S.: *Stable non-Gaussian random processes: stochastic models with infinite variance*, Chapman and Hall, London, 1994.

23. Xu, J., Miao, Y., and Liu, J.: Strong averaging principle for slow-fast SPDEs with Poisson random measures, *Discrete Contin. Dyn. Syst. Ser. B* **20** (2015), no. 7, 2233–2256.
24. Xu, Y.: Stochastic averaging principle for dynamical systems with fractional Brownian motion, *Discrete Contin. Dyn. Syst. Ser. B* **19** (2014), no. 4, 1197–1212.

VADYM RADCHENKO: DEPARTMENT OF MATHEMATICAL ANALYSIS, TARAS SHEVCHENKO NATIONAL UNIVERSITY OF KYIV, KYIV 01601, UKRAINE

E-mail address: `vradchenko@univ.kiev.ua`

URL: `http://mmtest.univ.kiev.ua/eng/ppages/radchenko/`