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ON A STOCHASTIC 2D CAHN-HILLIARD-NAVIER-STOKES SYSTEM DRIVEN BY JUMP NOISE

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Abstract. We investigate a stochastic 2D Cahn-Hilliard-Navier-Stokes system with a multiplicative noise of Lévy type. The model consists of the Navier-Stokes equations for the velocity, coupled with a Cahn-Hilliard system for the order (phase) parameter. We prove that the system has a unique global strong solution and we derive some a priori estimates for the solution.

1. Introduction

It is well accepted that the incompressible Navier-Stokes equation governs the motions of single-phase fluids such as air or water. On the other hand, we are faced with the difficult problem of understanding the motion of binary fluid mixtures, that is fluids composed by either two phases of the same chemical species or phases of different composition. Diffuse interface models are well-known tools to describe the dynamics of complex (e.g., binary) fluids, [25, 26]. For instance, this approach is used in [5] to describe cavitation phenomena in a flowing liquid. The model consists of the NSE equation coupled with the phase-field system, [14, 25, 26, 27]. In the isothermal compressible case, the existence of a global weak solution is proved in [24]. In the incompressible isothermal case, neglecting chemical reactions and other forces, the model reduces to an evolution system which governs the fluid velocity v and the order parameter ϕ. This system can be written as a NSE equation coupled with a convective Allen-Cahn equation, [25]. The associated initial and boundary value problem was studied in [25] in which the authors proved that the system generated a strongly continuous semigroup on a suitable phase space which possesses a global attractor. When the two fluids have the same constant density, the temperature differences are negligible and the diffuse interface between the two phases has a small but non-zero thickness, a well-known model is the so-called "Model H" (cf. [28, 30]). This is a system of equations where an incompressible Navier-Stokes equation for the (mean) velocity v is coupled with a convective Cahn-Hilliard equation for the order parameter ϕ, which represents the relative concentration of one of the fluids.

The purpose of this article is to study a stochastic 2D Cahn-Hilliard-Navier-Stokes equations (CH-NSE) driven by a non-Gaussian Levy noise. We recall that introducing a random term in a fluid model such as the Navier-Stokes system is now

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a well accepted tool to model the influence of internal, external and environmental noise. Adding a noise term in a fluid model can also be used to describe systems that are too complex to be described deterministically, e.g. a flow of a chemical substance in a river subjected by wind and rain, an airflow around an airplane wing perturbed by the random state of the atmosphere and weather, a laser beam subjected to turbulent movement of the atmosphere, spread of an epidemic in some regions and the spatial spread of infectious diseases, [4, 11, 12, 23, 31, 32, 33, 34, 41, 42, 43].

There are few notable works available on the stochastic CH-NSE driven by Gaussian noise. In [19], the authors considered the stochastic 3D globally modified Cahn-Hilliard-Navier-Stokes equations with multiplicative Gaussian noise. They proved the existence and uniqueness of strong solution (in the sense of partial differential equations and stochastic analysis). Moreover, they studied the asymptotic behavior of the unique solution and obtained the existence of a probabilistic weak solution for the stochastic 3D Cahn-Hilliard-Navier-Stokes equations. In [18], they also considered the asymptotic stability of the unique strong solution for the 3D globally modified Cahn-Hilliard-Navier-Stokes equations. The second author of the paper has proved the existence and uniqueness of the probabilistic strong solution for the stochastic 2D CH-NSE with multiplicative noise, [35].

In recent years, introducing a jump-type noises as Lévy-type or Poisson-type perturbations has become extremely popular for modeling natural phenomena, because these noises are very nice choice to reproduce the performance of some natural phenomena in real world models, such as some large moves and unpredictable events. There is a large amount of literature on the existence and uniqueness solutions for stochastic partial differential equations driven by jump-type noises. We refer the reader to [12, 20, 21, 22, 36, 37, 38, 39, 40, 44, 45]. However, the existing results in the literature do not cover the situation considered in this paper.

The aim of this article is to study a class of stochastic coupled CH-NSE driven by jump noise of Lévy type. To the best of our knowledge, this is the first work dealing with the stochastic version of the CH-NSE driven by jump noise. The model includes an abstract and general form of random external forces depending eventually on the velocity v of the fluid and the order parameter ϕ. We prove the existence and uniqueness of strong solutions. The proof of the existence of solution is based on a Galerkin scheme similar to that of [29, 11] in the case of the 2D Navier-Stokes and the 3D Lagrangian averaged Navier-Stokes equations. Let us note that the coupling between the Navier-Stokes and the Cahn-Hilliard systems introduces in the system a highly nonlinear coupling term that makes the analysis of the problems studied in this article more involved.

The article is divided as follows. In the next section we present the stochastic Cahn-Hilliard-Navier-Stokes model and its mathematical setting. We also give most of the notations and necessary preliminary used throughout this work. The main results appear in the third section, where we use a Galerkin approximation to prove the existence of strong solution. In the fourth section, we prove the pathwise uniqueness and the convergence of the whole Galerkin approximate solution.

2. The stochastic CH-NSE and its mathematical setting

2.1. Governing equations. We assume that the domain $\mathcal{M}$ of the fluid is a bounded domain in $\Re^2$. Then, we consider the system

$$\begin{cases} dv + [-\nu_1 \Delta v + (v \cdot \nabla)v + \nabla p - \mathcal{K}\mu\nabla\phi]\, dt = g_1(t, v, \phi)dt \\ \qquad + \displaystyle\int_Z \sigma(t, v, \phi, z)\tilde{\eta}(dt, dz) \ \text{ in } (0, T) \times \mathcal{M}, \\ \text{div } v = 0 \ \text{ in } (0, T) \times \mathcal{M}, \\ \dfrac{\partial\phi}{\partial t} + v \cdot \nabla\phi - \nu_3 \Delta\mu = 0 \ \text{ in } (0, T) \times \mathcal{M}, \\ \mu = -\nu_2 \Delta\phi + \alpha f(\phi) \ \text{ in } (0, T) \times \mathcal{M}. \end{cases} \tag{2.1}$$

In (2.1), the unknown functions are the velocity $v = (v_1, v_2)$ of the fluid, its pressure p and the order (phase) parameter ϕ.

The terms $g_1(t, v, \phi)$ and $\int_Z \sigma(t, v, \phi, z)\tilde{\eta}(dt, dz)$ respectively represent the deterministic and the random external forces that eventually depend on (v, ϕ), and $\tilde{\eta}$ is a compensated Poisson measure on a measurable space $(Z, \mathcal{Z})$ endowed with a fixed $\Sigma-$finite measure ν. Precise assumption on the data are given below. The model (2.1) describes the motion of a binary fluid exited by random forces.

The quantity μ is the variational derivative of the following free energy functional

$$\mathcal{F}_p(\phi) = \int_{\mathcal{M}} \Big(\frac{\nu_2}{2}|\nabla\phi|^2 + \alpha F(\phi)\Big)\, ds, \tag{2.2}$$

where, e.g., $F(r) = \int_0^r f(\zeta)d\zeta$. Here, the constants $\nu_1 > 0$, $\nu_3 > 0$ and $\mathcal{K} > 0$ correspond to the kinematic viscosity of the fluid, the mobility constant and the capillarity (stress) coefficient respectively. Here ν_2, $\alpha > 0$ are two physical parameters describing the interaction between the two phases. In particular, ν_2 is related with the thickness of the interface separating the two fluids.

A typical example of potential F is that of logarithmic type.

However, this potential is often replaced by a polynomial approximation of the type $F(r) = \gamma_1 r^4 - \gamma_2 r^2$, γ_1, γ_2 being positive constants. As noted in [25], $(2.1)_1$ can be replaced by

$$\begin{aligned} dv(t) + [-\nu_1 \Delta v + (v \cdot \nabla)v + \nabla\tilde{p}]\, dt = \int_Z \sigma(t, v, \phi, z)\tilde{\eta}(dt, dz) \\ + [-\mathcal{K}\text{div } (\nabla\phi \otimes \nabla\phi) + g_1(t, v, \phi)]\, dt, \end{aligned} \tag{2.3}$$

where $\tilde{p} = p - \mathcal{K}(\frac{\nu_2}{2}|\nabla\phi|^2 + \alpha F(\phi))$, since $\mathcal{K}\mu\nabla\phi = \nabla(\mathcal{K}(\frac{\nu_2}{2}|\nabla\phi|^2 + \alpha F(\phi))) - \mathcal{K}\text{div } (\nabla\phi \otimes \nabla\phi)$. The stress tensor $\nabla\phi \otimes \nabla\phi$ is considered the main contribution modeling capillary forces due to surface tension at the interface between the two phases of the fluid.

Regarding the boundary conditions for these models, we assume that the boundary conditions for ϕ are the natural no-flux condition

$$\partial_\eta \phi = \partial_\eta \Delta\phi = 0, \quad \text{on } \partial\mathcal{M} \times (0, \infty), \tag{2.4}$$

where $\partial\mathcal{M}$ is the boundary of $\mathcal{M}$ and η is the outward normal to $\partial\mathcal{M}$. These conditions ensure the mass conservation. Note that (2.4) implies that

$$\partial_\eta \mu = 0, \quad \text{on } \partial\mathcal{M} \times (0, \infty). \tag{2.5}$$

From (2.5), we deduce the conservation of the following quantity

$$\langle \phi(t) \rangle = \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} \phi(x,t)dx, \tag{2.6}$$

where $|\mathcal{M}|$ stands for the Lebesgue measure of $\mathcal{M}$. More precisely, we have

$$\langle \phi(t) \rangle = \langle \phi(0) \rangle, \ \forall t \geq 0. \tag{2.7}$$

Concerning the boundary condition for v, we assume the Dirichlet (no-slip) boundary condition

$$v = 0, \quad \text{on } \partial\mathcal{M} \times (0, \infty). \tag{2.8}$$

Therefore we assume that there is no relative motion at the fluid-solid interface.

The initial condition is given by

$$(v, \phi)(0) = (v_0, \phi_0), \ \text{in } \mathcal{M}. \tag{2.9}$$

2.2. Mathematical setting. We first recall from [25] a weak formulation of (2.1), (2.4), (2.8)-(2.9). Hereafter, we assume that the domain $\mathcal{M}$ is bounded with a smooth boundary $\partial\mathcal{M}$ (e.g., of class $\mathcal{C}^3$). We also assume that $f \in \mathcal{C}^2(\Re)$ satisfies

$$\begin{cases} \lim\limits_{|r| \to +\infty} f^{'}(r) > 0, \\ |f^{'}(r)| \leq c_f(1 + |r|^k), \ \forall r \in \Re, \end{cases} \tag{2.10}$$

where c_f is some positive constant and $k \in [1, +\infty)$ is fixed. It follows from (2.10) that

$$|f(r)| \leq c_f(1 + |r|^{k+1}), \ \forall r \in \Re. \tag{2.11}$$

Note that the derivative of the typical double-well potential f satisfies conditions similar to (2.10). Let us now recall from [25] the functional set up of the model (2.1), (2.4), (2.8),(2.9).

If X is a real Hilbert space with inner product $(\cdot, \cdot)_X$, we will denote the induced norm by $|\cdot|_X$, while X^* will indicate its dual. We set

$$\mathcal{V}_1 = \{u \in (\mathcal{C}_c^\infty(\mathcal{M}))^2 : \ \text{div } u = 0, \ \text{in } \mathcal{M}\}.$$

We denote by H_1 and V_1 the closure of $\mathcal{V}_1$ in $(L^2(\mathcal{M}))^2$ and $(H_0^1(\mathcal{M}))^2$ respectively. The scalar product in H_1 is denoted by $(\cdot, \cdot)_{L^2}$ and the associated norm by $|\cdot|_{L^2}$. Moreover, the space V_1 is endowed with the scalar product

$$((u, v)) = \sum_{i=1}^{2} (\partial_{x_i} u, \partial_{x_i} v)_{L^2}, \quad \|u\| = ((u, u))^{1/2}.$$

We now define the operator A_0 by

$$A_0 u = -\mathcal{P}_1 \Delta u, \ \forall u \in D(A_0) = (H^2(\mathcal{M}))^2 \cap V_1,$$

where $\mathcal{P}_1$ is the Leray-Helmotz projector in $(L^2(\mathcal{M}))^2$ onto H_1. Then, A_0 is a self-adjoint positive unbounded operator in H_1 which is associated with the scalar product defined above. Furthermore, A_0^{-1} is a compact linear operator on H_1 and $|A_0 \cdot|_{L^2}$ is a norm on $D(A_0)$ that is equivalent to the H^2-norm.

We introduce the linear nonnegative unbounded operator on $L^2(\mathcal{M})$

$$A_1 \phi = -\Delta\phi, \ \forall \phi \in D(A_1) = \{\phi \in H^2(\mathcal{M}), \ \partial_\eta \phi = 0, \ \text{ on } \partial\mathcal{M}\}, \tag{2.12}$$

and we endow $D(A_1)$ with the norm $|A_1 \cdot|_{L^2} + |\langle \cdot \rangle|_{L^2}$, which is equivalent to the H^2-norm. We also define the linear positive unbounded operator on the Hilbert space $L^2_0(\mathcal{M})$ of the L^2- functions with null mean

$$B_n\phi = -\Delta\phi, \ \forall \phi \in D(B_n) = D(A_1) \cap L^2_0(\mathcal{M}). \tag{2.13}$$

Note that B_n^{-1} is a compact linear operator on $L^2_0(\mathcal{M})$. More generally, we can define B_n^s, for any $s \in \Re$, noting that $|B_n^{s/2} \cdot|_{L^2}, \ s > 0$, is an equivalent norm to the canonical H^s- norm on $D(B_n^{s/2}) \subset H^s(\mathcal{M}) \cap L^2_0(\mathcal{M})$. Also note that $A_1 = B_n$ on $D(B_n)$. If ϕ is such that $\phi - \langle\phi\rangle \in D(B_n^{s/2})$, we have that $|B_n^{s/2}(\phi - \langle\phi\rangle)|_{L^2} + |\langle\phi\rangle|_{L^2}$ is equivalent to the H^s-norm. Moreover, we set $H^{-s}(\mathcal{M}) = (H^s(\mathcal{M}))^*$, whenever $s < 0$.

Finally we set

$$H_2 = D(B_n^0) = L^2_0(\mathcal{M}), \ V_2 = D(B_n^{1/2}). \tag{2.14}$$

The norms in H_2 and V_2 are denoted respectively by $|\cdot|_{L^2}$ and $\|\cdot\|$, where $\|\psi\| = |B_n^{1/2}\psi|_{L^2}$.

We introduce the bilinear operators B_0, B_1 (and their associated trilinear forms b_0, b_1) as well as the coupling mapping R_0, which are defined from $D(A_0) \times D(A_0)$ into H_1, $D(A_0) \times D(A_1)$ into $L^2(\mathcal{M})$, and $(L^2(\mathcal{M}))^2 \times (D(A_1) \cap H^3(\mathcal{M}))$ into H_1, respectively. More precisely, we set

$$\begin{aligned}
&\langle B_0(u,v), w\rangle = \int_{\mathcal{M}} [(u\cdot\nabla)v]\cdot w dx = b_0(u,v,w), \ \forall u,v,w \in D(A_0),\\
&\langle B_1(u,\phi), \rho\rangle = \int_{\mathcal{M}} [(u\cdot\nabla)\phi]\rho dx = b_1(u,\phi,\rho), \ \forall u \in D(A_0), \ \phi,\rho \in D(A_1),\\
&\langle R_0(\mu,\phi), w\rangle = \int_{\mathcal{M}} \mu[\nabla\phi\cdot w]dx = b_1(w,\phi,\mu),\\
&\qquad \forall w \in D(A_0), \ \phi \in D(A_1)\cap H^3(\mathcal{M}), \ \mu \in L^2(\mathcal{M}).
\end{aligned}$$

Note that

$$R_0(\mu,\phi) = \mathcal{P}\mu\nabla\phi.$$

We recall from [25] (see also [26, 27]) that B_0, B_1 and R_0 satisfy the following estimates

$$\begin{aligned}
&|B_0(u,v)|_{V_1^*} \le c|u|_{L^2}^{1/2}\|u\|^{1/2}\|v\|, \ \forall u,v \in V_1,\\
&|B_0(u,v)|_{L^2} \le c|u|_{L^2}^{1/2}\|u\|^{1/2}\|v\|^{1/2}|A_0 v|_{L^2}^{1/2}, \ \forall u \in V_1, \ v \in D(A_0),
\end{aligned} \tag{2.15}$$

$$\begin{aligned}
&|B_1(u,\phi)|_{V_2^*} \le c|u|_{L^2}^{1/2}\|u\|^{1/2}\|\phi\|, \ \forall u \in V_1, \phi \in V_2,\\
&|B_1(u,\phi)|_{L^2} \le c|u|_{L^2}^{1/2}\|u\|^{1/2}\|\phi\|^{1/2}|A_1\phi|_{L^2}^{1/2}, \ \forall u \in V_1, \ \phi \in D(A_1),
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
&|R_0(A_1\phi,\rho)|_{V_1^*} \le c|A_1\phi|_{L^2}^{1/2}|\phi|_{H^3}^{1/2}\|\rho\|, \ \forall \phi \in D(A_1), \ \rho \in V_2,\\
&|R_0(A_1\phi,\rho)|_{L^2} \le c\|\rho\|^{1/2}|A_1\rho|_{L^2}^{1/2}|A_1\phi|_{L^2}^{1/2}|\phi|_{H^3}^{1/2},\\
&\quad \forall \phi \in D(A_1), \rho \in D(A_1^{3/2}).
\end{aligned} \tag{2.17}$$

We recall that (due to the mass conservation) we have

$$\langle\phi(t)\rangle = \langle\phi(0)\rangle = M_0, \ \forall t > 0. \tag{2.18}$$

Thus, up to a shift of the order parameter field, we can always assume that the mean of ϕ is zero a the initial time and, therefore it will remain zero for all positive times. Hereafter, we assume that

$$\langle\phi(t)\rangle = \langle\phi(0)\rangle = 0, \ \forall t > 0. \tag{2.19}$$

We set

$$\mathcal{H} = H_1 \times D(B_n^{1/2}). \tag{2.20}$$

The space $\mathcal{H}$ is a complete metric space with respect to the norm

$$|(v,\phi)|^2_{\mathcal{H}} = \mathcal{K}^{-1}|v|^2_{L^2} + \nu_2|\nabla\phi|^2_{L^2}. \tag{2.21}$$

We define the Hilbert space $\mathcal{U}$ by

$$\mathcal{U} = V_1 \times D(B_n), \tag{2.22}$$

endowed with the scalar product whose associated norm is

$$\|(v,\phi)\|^2_{\mathcal{U}} = \|v\|^2 + |B_n\phi|^2_{L^2}. \tag{2.23}$$

We will also denote by c a generic positive constant that depends on the domain $\mathcal{M}$. To simplify the notations, we set (without loss of generality) $\nu_1 = \nu_2 = \nu_3 = \alpha = \mathcal{K} = 1$.

Using the notations above, we rewrite (2.1), (2.4), (2.8)-(2.6) as

$$\begin{cases} dv + [A_0 v + B_0(v,v) - R_0(A_1\phi,\phi)]\,dt = g_1(t,v,\phi)dt \\ \qquad + \displaystyle\int_Z \sigma(t,v,\phi,z)\tilde{\eta}(dt,dz), \\ \dfrac{d\phi}{dt} + A_1\mu + B_1(v,\phi) = 0, \\ \mu = A_1\phi + f(\phi), \\ (v,\phi)(0) = (v_0,\phi_0). \end{cases} \tag{2.24}$$

Remark 2.1. In the weak formulation (2.24), the term $\mu\nabla\phi$ is replaced by $A_1\nabla\phi$. This is justified since $f^{'}(\phi)\nabla\phi$ is the gradient $F(\phi)$ and can be incorporated into the pressure gradient, see [25] for details. For the sake of convenience, as in [25] we will replace μ in $(2.24)_3$ by $\bar{\mu} = \mu - \langle\mu\rangle$, that is $\bar{\mu} = A_1\phi + f(\phi) - \langle f(\phi)\rangle$, a.e., in $\mathcal{M}\times(0,T)$. Obviously we have $\langle\bar{\mu}(t)\rangle = 0, \ \forall t > 0$.

Notations. We first recall from [29, 11] some notations and stochastic preliminaries.

Hereafter, by $\aleph$ we denote the set of nonnegative integers, i.e. $\aleph = \{0,1,2,\cdots\}$ and by $\bar{\aleph}$ we denote the set $\aleph\cup\{+\infty\}$. Whenever we speak about $\aleph$ (or $\bar{\aleph}$)$-$valued measurable functions we implicitly assume that the set is equipped with the trivial Σ-field $2^{\aleph}$ (or $2^{\bar{\aleph}}$). By $\Re_+$ we will denote the interval $[0,\infty)$ and by $\Re_*$ the set $\Re\setminus\{0\}$. If X is a topological space, then by $\mathcal{B}(X)$ we will denote the Borel $\Sigma-$field on X. By λ_d we will denote the Lebesgue measure on $(\Re^d, \mathcal{B}(\Re^d))$, by λ the Lebesgue measure on $(\Re, \mathcal{B}(\Re))$.

If $(S,\mathfrak{S})$ is a measurable space then by $M(S)$ we denote the set of all real valued measures on $(S,\mathfrak{S})$, and by $M(S)$ the $\Sigma-$field on $M(S)$ generated by the functions $i_B : M(S) \ni \varsigma \mapsto \varsigma(B) \in \Re, B \in S$. By $M_+(S)$ we denote the set of all nonnegative measures on S, and by $M(S)$ the $\Sigma-$field on $M_+(S)$ generated by the functions

$i_B : M_+(S) \ni \varsigma \mapsto \varsigma(B) \in \Re_+, B \in S$. Finally, by $M_I(S)$ we denote the family of all $\bar{\aleph}$valued measures on $(S, \mathfrak{S})$, and by $\bar{M_I}(S)$ the $\Sigma-$field on $M_I(S)$ generated by functions $i_B : M(S) \ni \varsigma \mapsto \varsigma(B) \in \bar{\aleph}$, $B \in S$. If $(S, \mathfrak{S})$ is a measurable space then we will denote by $\mathfrak{S} \otimes \mathcal{B}(\Re_+)$ the product $\Sigma-$field on $S \times \Re_+$ and by $\nu \otimes \lambda$ the product measure of ν and the Lebesgue measure λ.

Preliminaries. As mentioned earlier we will study a stochastic model for a CH-NSE excited by random forces. We first describe the forces acting on the fluids. Let $(Z, \mathcal{Z})$ be a separable metric space and let ν be a $\Sigma-$finite positive measure on it. Suppose that $\mathfrak{P} = (\Omega, \mathfrak{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space, where $\mathbb{F} = (\mathfrak{F})_{t\geq 0}$ is a filtration satisfying the usual conditions, and $\eta : \Omega \times \mathcal{B}(\Re_+) \times Z \to \bar{\aleph}$ is a time homogeneous Poisson random measure, with intensity measure ν, defined over the filtered probability space $\mathfrak{P}$. A time homogeneous Poisson random measure defined over $\mathfrak{P}$ is given in the following definition.

Definition 2.2. Let Z be a metric space and $\mathcal{Z}$ its Borel $\Sigma-$algebra, ν a positive $\Sigma-$finite measure on $(Z, \mathcal{Z})$. A *Poisson random measure*, with intensity measure ν defined on $(Z, \mathbb{Z})$ over $\mathfrak{P}$ is a measurable map $\eta : (\Omega, \mathfrak{F}) \to (M_I(Z \times \Re_+), M_I(Z \times \Re+))$ satisfying the following conditions:

(i) for all $B \in \mathcal{B}(Z \otimes \Re_+), \eta(B) : \Omega \to \bar{\aleph}$ is a Poisson random measure with parameter $\mathbb{E}[\eta(B)]$;

(ii) η is independently scattered, i.e., if the sets $B_j \in \mathcal{B}(Z \otimes \Re_+), j = 1, ..., n$, are disjoint then the random variables $\eta(B_j), j = 1, \cdots, n$, are independent;

(iii) for all $U \in \mathcal{Z}$ and $I \in \mathcal{B}(\Re_+)$

$$\mathbb{E}[\eta(U \times I)] = \lambda(I)\nu(U);$$

(iv) for all $U \in \mathcal{Z}$ the $\bar{\aleph}-$valued process $(N(U,t))_{t\geq 0}$ defined by $N(U,t) := \eta(U \times (0,t]), t \geq 0$, is $\mathbb{F}-$adapted and its increments are independent of the past, i.e., if $t > s \geq 0$, then the random variable $N(U,t) - N(U,s) = \eta(U \times (s,t])$ is independent of $\mathfrak{F}_s$.

We will denote by $\tilde{\eta}$ the compensated Poisson random measure defined by

$$\tilde{\eta} := \eta - \gamma,$$

where the compensator $\gamma : \mathcal{B}(\mathcal{Z} \times \Re_+) \to \Re_+$ is defined by

$$\gamma(A \times I) = \lambda(I)\nu(A), I \in \mathcal{B}(\Re_+), A \in \mathcal{Z}.$$

As noted in [29], while items (i) and (ii) are the classical definition, see for e.g. Definition 6.1 in [39], of a Poisson Random measure η, the remaining items implicitly indicate that η is associated to a certain Lévy process $\tilde{L}$; see, for instance [[39], Proposition 4.16].

Let $\mathcal{M}^2(\Re_+, L^2(Z, \nu, H_1))$ be the class of all progressively measurable processes $\xi : \Re_+ \times Z \times \Omega \to H_1$ satisfying the condition

$$\mathbb{E}\int_0^T \int_Z |\xi(r,z)|^2_{L^2}\nu(dz)dr < \infty, \ \forall T > 0. \tag{2.25}$$

If $T > 0$, the class of all progressively measurable processes $\xi : [0,T] \times Z \times \Omega \to H_1$ satisfying the condition (2.25) just for this one T, will be denoted by

$\mathcal{M}^2(0,T,L^2(Z,\nu,H_1))$. Also, let $\mathcal{M}^2_{step}(\Re_+,L^2(Z,\nu,H_1))$ be the space of all processes $\xi \in \mathcal{M}^2(\Re_+,L^2(Z,\nu,H_1))$ such that

$$\xi(r) = \sum_{j=1}^{n} 1_{(t_{j-1},t_j]}(r)\xi_j,\ 0 \leq r, \tag{2.26}$$

where $\{0 = t_0 < t_1 < ... < t_n < \infty\}$ is a partition of $[0,\infty)$, and for all $j,\ \xi_j$ is an $\mathfrak{F}_{t_{j-1}}-$ measurable random variable.

For any $\xi \in \mathcal{M}^2_{step}(\Re_+,L^2(Z,\nu,H_1))$, we set

$$\tilde{I}(\xi) = \sum_{j=1}^{n} \int_Z \xi_j(z)\tilde{\eta}(dz,(t_{j-1},t_j]).$$

This is basically the definition of stochastic integral of a random step process ξ with respect to the compound random Poisson measure $\tilde{\eta}$. The extension of this integral on $\mathcal{M}^2(\Re_+,L^2(Z,\nu,H_1))$ is possible thanks to the following result which is taken from [39].

Theorem 2.3. *There exists a unique bounded linear operator*

$$I : \mathcal{M}^2(\Re_+,L^2(Z,\nu;H_1)) \to L^2(\Omega,\mathfrak{F};H_1)$$

such that for $\xi \in \mathcal{M}^2_{step}(\Re_+,L^2(Z,\nu,H_1))$ we have $\tilde{I}(\xi) = I(\xi)$. In particular, there exists a constant $C = C(H_1)$ such that for any $\xi \in \mathcal{M}^2(\Re_+,L^2(Z,\nu,H_1))$,

$$\mathbb{E}\left|\int_0^t \int_Z \xi(r,z)\tilde{\eta}(dz,dr)\right|^2_{L^2} \leq C\mathbb{E}\int_0^t \int_Z |\xi(r,z)|^2_{L^2}\eta(dz)dr,\ t > 0.$$

Moreover, for each $\xi \in \mathcal{M}^2(\Re_+,L^2(Z,\nu,H_1))$, the process $I(1_{[0,t]}\xi),\ t \geq 0$, is an H_1-valued càdlàg martingale. The process $1_{[0,t]}\xi$ is defined by $[1_{[0,t]}\xi](r,z,\omega) := 1_{[0,t]}(r)(r,z,\omega),\ t \geq 0, r \in \Re_+, z \in Z$ and $\omega \in \Omega$.

As usual we will write

$$\int_0^t \int_Z \xi(r,z)\tilde{\eta}(dz,dr) := I(\xi)(t),\ t \geq 0.$$

If $T > 0$, we denote by $\mathbb{D}(0,T;H_1)$ the space of all càdlàg paths from $[0,T]$ into H_1.

Now we introduce the main set of hypotheses used in this article. As in [29, 11], we suppose that we are given a function σ satisfying the following set of constraints:

Condition 1. There exist nonnegative constants l_0, l_1, l_2 such that, for any $t \in [0,T]$ and all $(v_1,\phi_1),(v_2,\phi_2) \in \mathcal{H}$, we have

$$\begin{aligned} &|\sigma(t,v_1,\phi_1)|^p_{L^2(Z,\nu;H_1)} \leq l_0 + l_1|(v_1,\phi_1)|^p_{\mathcal{H}}; \quad \text{for any } p \geq 2, \\ &|\sigma(t,v_1,\phi_1) - \sigma(t,v_2,\phi_2)|^2_{L^2(Z,\nu;H_1)} \leq l_2|(v_1,\phi_1) - (v_2,\phi_2)|^2_{\mathcal{H}}. \end{aligned} \tag{2.27}$$

We assume that the external forcing g_1 is a measurable Lipschitz and sublinear mappings from $\Omega\times(0,T)\times H_1$ into V_1^*. More precisely, for all $(v_1,\phi_1),(v_2,\phi_2) \in V_1$,

$g_1(\cdot, v_1, \phi_1)$ is $\mathfrak{F}_t-$progressively measurable, and $d\mathbb{P} \times dt-$a.e. in $\Omega \times (0,T)$

$$\begin{aligned} &\|g_1(t,v_1,\phi_1) - g_1(t,v_2,\phi_2)\|_{V_1^*} \leq L_1|v_1 - v_2|_{L^2}, \\ &g_1(t,0,0) \in M^2_{\mathfrak{F}_t}(0,T;V_1^*). \end{aligned} \tag{2.28}$$

Finally, we assume that

$$(v_0, \phi_0) \in L^2(\Omega, \mathfrak{F}_0, \mathbb{P}; \mathcal{H}). \tag{2.29}$$

Hereafter, for any $(w, \psi) \in \mathcal{H}$, we set

$$\mathcal{E}(w,\psi) = |w|^2_{L^2} + \|\psi\|^2 + 2\langle F(\psi), 1\rangle + c_1, \tag{2.30}$$

where $c_1 > 0$ is a constant large enough and independent on (w, ψ) such that $\mathcal{E}(w, \psi)$ is nonnegative (note that F is bounded from below).

We can check that (see [25]) there exists a monotone non-decreasing function Q_0 (independent on time and the initial condition) such that

$$|(w,\psi)|^2_{\mathcal{H}} \leq \mathcal{E}(w,\psi) \leq Q_0(|(w,\psi)|^2_{\mathcal{H}}), \ \forall (w,\psi) \in \mathcal{H}. \tag{2.31}$$

Definition 2.4. Let $(Z, \mathcal{Z})$ be a separable metric space on which is defined a $\Sigma-$finite measure ν and $(v_0, \phi_0) \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathcal{H})$. A *strong solution* to the problem (2.24) is a stochastic process (v, ϕ) such that

(1) $(v, \phi) = \{(v,\phi)(t),\ t \geq 0\}$ is a $\mathbb{F}-$progressively measurable process such that

$$\mathbb{E} \sup_{s\in[0,T]} \mathcal{E}(v(s), \phi(s)) + \mathbb{E}\int_0^T \|(v,\phi)(s)\|^2_{\mathcal{U}} dt < \infty,$$

(2) the following holds

$$\begin{aligned} &(v(t), w) = (v_0, w) + \int_0^t \int_Z \langle \sigma(s, v, \phi, z), w\rangle \tilde{\eta}(dz, ds) \\ &\quad - \int_0^t \langle A_0 v + B_0(v,v) - R_0(A_1\phi, \phi) - g_1(s, v, \phi), w\rangle ds,\ \forall w \in V_1, \\ &(\phi(t), \psi) = (\phi_0, \psi) - \int_0^t \langle A_1 \mu + B_1(v, \phi), \psi\rangle ds, \quad \forall \psi \in V_2, \\ &\mu = A_1\phi + f(\phi), \end{aligned} \tag{2.32}$$

for almost all $t \in [0,T]$ and $\mathbb{P}-$almost surely.

In the deterministic case, the weak formulation of (2.24) was proposed and studied in [8, 6, 7, 26, 25] (see also [2, 1, 15]), where the existence and uniqueness results for weak and strong solutions were proved in the deterministic case.

Before we prove this result let us recall an important statement which is borrowed from [17].

Lemma 2.5. *Let X, Y, I and φ be non-negative processes and Z_1 be a non-negative integrable random variable. Assume that I is non-decreasing and that there exist non-negative constants $C, \alpha_1, \beta, \gamma_1, \delta_1$ and T satisfying first*

$$\int_0^T \varphi(s)ds \leq C, \quad a.s., \ \ 2\beta_1 e^C \leq 1,\ 2\delta_1 e^C \leq \alpha_1,$$

and secondly for all $t \in [0,T]$ there exists a constant $C_1 > 0$ such that

$$X(t) + \alpha_1 Y(t) \le Z_1 + \int_0^t \varphi(r) X(r) dr + I(t), \ \ a.s.,$$

$$\mathbb{E} I(t) \le \beta \mathbb{E} X(t) + \gamma_1 \int_0^t \mathbb{E} X(s) ds + \delta_1 \mathbb{E} Y(t) + C_1.$$

If $X \in L^\infty([0,T] \times \Omega)$, then we have

$$\mathbb{E}[X(t) + \alpha_1 Y(t)] \le 2 \exp(C + 2t\gamma_1 e^C)(\mathbb{E} Z + C_1), \ \ t \in [0,T].$$

3. Existence and uniqueness of solutions

In this section, we prove the existence and the pathwise uniqueness of variational solution to (2.24). We first prove the following energy type equality.

Proposition 3.1. *If (v, ϕ) is a variational solution to (2.24), then (v, ϕ) satisfies*

$$\begin{aligned} &\mathcal{E}(v,\phi)(t) + \int_0^t \left(2\|v(s)\|^2 + \|\mu(s)\|^2\right) ds = \mathcal{E}(v_0, \phi_0) \\ &+2 \int_0^t \langle g_1(s, v(s), \phi(s)), v(s)\rangle ds + \int_0^t \int_Z \Upsilon(s,z) \eta(dz, ds) \\ &+2 \int_0^t \int_Z (v(s-), \sigma(s, v(s), \phi(s), z)) \tilde{\eta}(dz, ds), \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} &\Upsilon(s,z) = |v(s-) + \sigma(s, v(s), \phi(s), z)|_{L^2}^2 - |v(s-)|_{L^2}^2 \\ &-2(v(-s), \sigma(s, v(s), \phi(s), z)). \end{aligned} \tag{3.2}$$

Proof. Applying Itô's formula to $|v(t)|_{L^2}^2$, we derive that

$$\begin{aligned} &|v(t)|_{L^2}^2 = |v_0|_{L^2}^2 + 2 \int_0^t \left(-\|v(s)\|^2 + g_1(s, v(s), \phi(s)), v(s)\rangle\right) ds \\ &+2 \int_0^t \langle R_0(A_1 \phi(s), \phi(s)), v(s)\rangle ds \\ &+2 \int_0^t \int_Z (v(s-), \sigma(s, v(s), \phi(s), z)) \tilde{\eta}(dz, ds) + \int_0^t \int_Z \Upsilon(s,z) \eta(dz, ds), \end{aligned} \tag{3.3}$$

where $\Upsilon(s,z)$ is given by (3.2).

Now multiplying $(2.24)_2$ with μ, we obtain

$$\frac{d}{dt} \left(\|\phi(t)\|^2 + 2\langle F(\phi(t)), 1\rangle + c_1\right) + 2\langle B_1(v(t), \phi(t)), \mu(t)\rangle = 0. \tag{3.4}$$

Integrating (3.4) and adding the result to (3.3) gives

$$\begin{aligned} &\mathcal{E}(v,\phi)(t) = \mathcal{E}(v_0, \phi_0) - \int_0^t (2\|v(s)\|^2 + \|\mu(s)\|^2) ds \\ &+2 \int_0^t \langle g_1(s, v(s), \phi(s)), v(s)\rangle ds \\ &+2 \int_0^t \int_Z (v(s-), \sigma(s, v(s), \phi(s), z)) \tilde{\eta}(dz, ds) + \int_0^t \int_Z \Upsilon(s,z) \eta(dz, ds), \end{aligned} \tag{3.5}$$

where $\Upsilon(s,z)$ is given by (3.2).

Note that we use the properties of B_0, B_1 and R_0 given in (2.15). In particular, we used the fact that (see [25])

$$\langle B_0(v,v),v\rangle = 0,\ \langle R_0(A_1\phi,\phi),v\rangle = \langle B_1(v,\phi),\mu\rangle = \langle B_1(v,\phi),A_1\phi\rangle.$$

We also use the fact that

$$\begin{aligned}\langle\phi',\mu\rangle &= \frac{d}{dt}\left(\tfrac{1}{2}\|\phi\|^2 + \langle F(\phi(t)),1\rangle\right)\\ &= \frac{d}{dt}\left(\tfrac{1}{2}\|\phi\|^2 + \langle F(\phi(t)),1\rangle + \tfrac{1}{2}c_1\right).\end{aligned} \tag{3.6}$$

where c_1 is the the constant that appears in (2.30). □

3.1. Uniqueness of solutions. The following result implies the pathwise uniqueness of the variational solutions (v,ϕ) to (2.24).

Proposition 3.2. *Let $(v_0^1,\phi_0^1),(v_0^2,\phi_0^2)$ be two $\mathfrak{F}_0-$measurable and square integrable $\mathcal{H}-$valued random variables. Let $(v_1,\phi_1),(v_2,\phi_2)$ be the variational solution to (2.24) corresponding to $(v_0^1,\phi_0^1),(v_0^2,\phi_0^2)$ respectively. Then there exists a constant $C>0$ such that*

$$\mathbb{E}\delta(t)|(v_1,\phi_1)-(v_2,\phi_2)|^2_{\mathcal{H}} \le C\mathbb{E}|(v_0^1,\phi_0^1)-(v_0^2,\phi_0^2)|^2_{\mathcal{H}}, \tag{3.7}$$

for all $t\in[0,T]$, where $\delta(t)$ is defined by (3.22).

Moreover, if $(v_0^1,\phi_0^1)=(v_0^2,\phi_0^2)$ almost surely, then for any $t\in[0,T]$,

$$\mathbb{P}((v_1,\phi_1)(t)=(v_2,\phi_2)(t))=1. \tag{3.8}$$

Proof. Let (v_1,ϕ_1), $(v_2.\phi_2)$ be variational solutions to (2.24). Let $(w,\psi,\mu) = (v_1,\phi_1,\mu_1)-(v_2,\phi_2,\mu_2)$, $\bar\mu = \mu - \langle\mu\rangle$. Then (w,ψ) satisfies

$$\begin{cases} dw + [A_0w + B_0(v_2,w) + B_0(w,v_1)]dt = [g_1(t,v_1,\phi_1) - g_1(t,v_2,\phi_2)]dt\\ \quad +[R_0(A_1\phi_2,\psi) + R_0(A_1\psi,\phi_1)]dt\\ \quad +\displaystyle\int_Z(\sigma(t,v_1,\phi_1,z)-\sigma(t,v_2,\phi_2,z))\tilde\eta(dz,dt),\\ \dfrac{d\psi}{dt} + A_1\bar\mu + B_1(v_2,\psi) + B_1(w,\phi_1) = 0,\\ \mu = A_1\psi + f(\phi_1) - f(\phi_2),\\ (w,\psi)(0) = (0,0)\end{cases} \tag{3.9}$$

Reasoning as in the proof of Proposition (3.1) above, applying Itô's formula to $|w|^2_{L^2}$ and using $(3.9)_1$, we derive that

$$\begin{aligned}&|w|^2_{L^2} + 2\int_0^t(\|w\|^2 + b_0(w,v_1,w))ds\\ &= 2\int_0^t\langle g_1(t,v_1,\phi_1) - g_1(t,v_2,\phi_2),w\rangle ds\\ &+2\int_0^t\langle R_0(A_1\phi_2,\psi) + R_0(A_1\psi,\phi_1),w\rangle ds\\ &+2\int_0^t\int_Z(w(s-),\sigma(s,v_1(s),\phi_1(s),z) - \sigma(s,v_2(s),\phi_2(s),z))\tilde\eta(dz,ds)\\ &+\int_0^t\int_Z|\sigma(s,v_1(s),\phi_1(s),z)-\sigma(s,v_2(s),\phi_2(s),z))|^2_{L^2}\eta(ds,dz).\end{aligned} \tag{3.10}$$

Now we take the duality of $(3.9)_2$ and $(3.9)_3$ with $A_1\bar{\mu}-\zeta A_1\psi$ and $A_1\psi$ respectively, where $\zeta > 0$ is small enough and will be selected later. Adding the resulting equality to (3.10), we derive that

$$\begin{aligned}
&|w(t)|^2+\|\psi(t)\|^2+2\int_0^t(\|w\|^2+\zeta|A_1\psi|_{L^2}^2+\|\mu\|^2)ds\\
&=2\int_0^t(\langle R_0(A_1\phi_2,\psi)+R_0(A_1\psi,\phi_1),w\rangle-b_0(w,v_1,w))ds\\
&-2\int_0^t(b_1(w,\phi,A_1\psi)+b_1(v,\psi,A_1\psi))ds+\int_0^t\zeta\langle\bar{\mu},A_1\psi\rangle ds\\
&+2\int_0^t[\zeta\langle f(\phi_1)-f(\phi_2),A_1\psi\rangle-\langle f(\phi_1)-f(\phi_2),A_1\bar{\mu}\rangle]ds\\
&+2\int_0^t\langle g_1(t,v_1,\phi_1)-g_1(t,v_2,\phi_2),w\rangle ds\\
&+2\int_0^t\int_Z(w(s-),\sigma(s,v_1(s),\phi_1(s),z)-\sigma(s,v_2(s),\phi_2(s),z))\tilde{\eta}(dz,ds)\\
&+\int_0^t\int_Z|\sigma(s,v_1(s),\phi_1(s),z)-\sigma(s,v_2(s),\phi_2(s),z))|_{L^2}^2\eta(dz,ds).
\end{aligned}\tag{3.11}$$

Note that

$$|b_0(w,v_1,w)|\le\frac{1}{8}\|w\|^2+c\|v_1\|^2|w|_{L^2}^2,\tag{3.12}$$

$$\begin{aligned}|\langle R_0(A_1\psi,\phi_1),w\rangle| &= |b_1(w,\phi_1,A_1\psi)|\\ &\le\tfrac{1}{8}(\|w\|^2+\zeta|A_1\psi|_{L^2}^2)+c|w|_{L^2}^2\|\phi_1\|^2|A_1\phi_1|_{L^2}^2,\end{aligned}\tag{3.13}$$

$$\begin{aligned}&|\langle R_0(A_1\phi_2,\psi),w\rangle|=|b_1(w,\psi,A_1\phi_2)|\\ &\le\tfrac{1}{8}(\|w\|^2+\zeta|A_1\psi_2|_{L^2}^2)+c(|w|_{L^2}^2+|\nabla\psi|_{L^2}^2)\|\phi_2\|^2|\phi_2|_{H^2}^2,\end{aligned}\tag{3.14}$$

$$\begin{aligned}\zeta|\langle f(\phi_1)-f(\phi_2),A_1\psi\rangle| &\le\tfrac{\zeta}{8}|A_1\psi|_{L^2}^2\\ &+Q_1(|\phi_1|_{H^1},|\phi_2|_{H^1})\|\psi\|^2,\end{aligned}\tag{3.15}$$

$$\begin{aligned}&|\langle f(\phi_1)-f(\phi_2),A_1\bar{\mu}\rangle|\le\tfrac{1}{2}|A_1^{1/2}\bar{\mu}|_{L^2}^2\\ &+Q_1(|\phi_1|_{H^1},|\phi_2|_{H^1})(|A_1\phi_1|_{L^2}^2+|A_1\phi_2|_{L^2}^2)\|\psi\|^2,\end{aligned}\tag{3.16}$$

$$|b_1(v_2,\psi,A_1\psi)|\le\frac{\zeta}{8}|A_1\psi|_{L^2}^2+c|v_2|_{L^2}^2\|v_2\|^2\|\psi\|^2,\tag{3.17}$$

$$\zeta|(\bar{\mu},A_1\psi)_{L^2}|\le\frac{\zeta}{8}|A_1\psi|_{L^2}^2+c\zeta|\nabla\bar{\mu}|_{L^2}^2,\tag{3.18}$$

$$\begin{aligned}|\langle g_1(t,v_1,\phi_1)-g_1(t,v_2,\phi_2),w\rangle| &\le L_1\|w\||(w,\psi)|_{\mathcal{H}}\\ &\le\tfrac{1}{8}\|w\|^2+cL_1^2|(w,\psi)|_{\mathcal{H}}^2,\end{aligned}\tag{3.19}$$

$$\|\sigma(s,v_1,\phi_1)-\sigma(s,v_2,\phi_2)\|_{L^2(Z,\mu,V_1)}^2\le l_2^2|(w,\psi)|_{\mathcal{H}}^2.\tag{3.20}$$

Let

$$\mathcal{Y}_2(t)=|w(t)|_{L^2}^2+\|\psi(t)\|^2,$$

and

$$\begin{aligned}&K_1(t)=c(\|v_1\|^2+\|\phi_1\|^2|A_1\phi_1|_{L^2}^2+\|\phi_2\|^2|A_1\phi_2|_{L^2}^2+|v_2|_{L^2}^2\|v_2\|^2)\\ &+Q_1(|\phi_1|_{H^1},|\phi_2|_{H^1})(|A_1\phi_1|_{L^2}^2+|A_1\phi_2|_{L^2}^2)+cL_1^2+l_2^2,\end{aligned}\tag{3.21}$$

$$\delta(t)=\exp\left(-\int_0^t K_1(s)ds\right).\tag{3.22}$$

Applying Itô's formula to the process $\delta(t)\mathcal{Y}_2(t)$ and using (3.11)-(3.20), we derive that

$$\begin{aligned}&\mathbb{E}\delta(t)\mathcal{Y}_2(t)+\mathbb{E}\int_0^t \delta(s)(\|w\|^2+(1-c\zeta)\|\bar{\mu}\|^2+\zeta|A_1\psi|^2_{L^2})ds\\ &\quad\le \mathbb{E}\mathcal{Y}_2(0)+\mathbb{E}\int_0^t \delta(s)\mathcal{Y}_2(s)ds.\end{aligned} \tag{3.23}$$

Note that the expectation of the stochastic integral in (3.11) vanishes. Therefore we obtain

$$\mathbb{E}\delta(t)\mathcal{Y}_2(t)\le \mathbb{E}\mathcal{Y}_2(0)+\mathbb{E}\int_0^t \delta(s)\mathcal{Y}_2(s)ds,\ 0\le t\le T.$$

It follows from the Gronwall lemma that there exists a constant $C>0$ such that

$$\mathbb{E}\delta(t)\mathcal{Y}_2(t)\le C\mathbb{E}\mathcal{Y}_2(0),$$

for any $t\in[0,T]$, which proves the first part of the proposition. Since $\delta(t)$ is bounded and positive $\mathbb{P}-$a.s., we conclude that the second part of the proposition follows from the last estimate. Note that in (3.23), we choose $\zeta>0$ and small enough such that $1-c\zeta>0$. ☐

3.2. Existence of solution. In this part, we prove the existence of solution. The method relies on Galerkin approximation.

Proposition 3.3. *We assume that Condition 1 above is satisfied. Moreover, we suppose that* $g_1(\cdot,0,0)\in L^4(\Omega,L^2(0,T;L^2(0,T;V_1^*)),\ (v_0,\phi_0)\in L^2(\Omega,\mathfrak{F}_0,\mathbb{P};\mathcal{H})$ *satisfies* $\mathbb{E}\left[\mathcal{E}(v_0,\phi_0)\right]^2<\infty$. *Then, there exists a unique solution*

$$(v,\phi)\in L^4(\Omega,\mathbb{D}(0,T;H_1)\times\mathcal{C}(0,T;V_2))\cap L^4(\Omega,L^2(0,T;\mathcal{U})).$$

Furthermore, the following estimate holds:

$$\mathbb{E}\sup_{t\in[0,T]}\mathcal{E}(v(t),\phi(t))+\mathbb{E}\int_0^T\|(v(s),\phi(s))\|^2_{\mathcal{U}}ds\le C, \tag{3.24}$$

provided that $\mathbb{E}\mathcal{E}(v_0,\phi_0)<\infty$, *and*

$$\mathbb{E}\sup_{t\in[0,T]}[\mathcal{E}(v(t),\phi(t))]^p+\mathbb{E}\left(\int_0^T\|(v(s),\phi(s))\|^2_{\mathcal{U}}ds\right)^p\le C, \tag{3.25}$$

for any positive integer $p\ge 2$, *provided that* $\mathbb{E}[\mathcal{E}(v_0,\phi_0)]^p<\infty$.

Proof. Let $\{(w_i,\psi_i), i=1,2,3,\cdots\}\subset\mathcal{U}$ be an orthonormal basis of $\mathcal{H}$, where $\{w_i, i=1,2\cdots\}$, $\{\psi_i, i=1,2\cdots\}$ are eigenvectors of A_0 and A_1 respectively. We set $\mathcal{U}_m=\mathcal{H}_m=\text{span}\{(w_1,\psi_1),\cdots(w_m,\psi_m)\}$. We look for $(v_m,\phi_m)\in\mathcal{H}_m$ solution to

$$\begin{aligned}&dv_m(t)=-\Pi^1_m\left[A_0v_m+B_0(v_m,v_m)-R_0(A_1\phi_m,\phi_m)\right]dt\\ &\quad+\Pi^1_m g_1(s,v_m,\phi_m)dt+\int_Z \Pi^1_m\sigma(t,v_m(t-),\phi_m(t-),z)\tilde{\eta}(dt,dz),\\ &d\phi_m(t)=-\Pi^2_m\left[A_1\mu_m+B_1(v_m,\phi_m)\right]dt,\\ &\mu_m=A_1\phi_m+f(\phi_m),\end{aligned} \tag{3.26}$$

where $\Pi_m \equiv (\Pi_m^1, \Pi_m^2)$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_m$.
As in the proof of Theorem 1.2.1 of [3], we can obtain the existence and uniqueness of a solution $(v_m, \phi_m) \in L^2(\Omega \times [0,T]; \mathcal{U}_m)$ of (3.26) on an interval $[0, T_m]$.

For each $n \geq 1$, we consider the $\mathfrak{F}_t-$stopping time τ_n defined by:

$$\tau_n = T \wedge \inf_{t\in[0,T]} \left\{ \mathcal{E}(v_m, \phi_m)(t) + \int_0^t (\|v_m(s)\|^2 + \|\bar{\mu}_m(s)\|^2) ds \geq n^2 \right\}, \tag{3.27}$$

where hereafter $a \wedge b = \min(a,b)$.

For fixed m, the sequence $\{\tau_n; n \geq 1\}$ is increasing to T. Throughout we fix $r \in [0,T]$ and $0 \leq t \leq r \wedge \tau_n$. Now using Itô's formula, we derive that as in the proof of (3.1) that

$$\begin{aligned} \mathcal{E}(v_m, \phi_m)(t) \quad &+ \int_0^t \left(2\|v_m(s)\|^2 + \|\mu_m(s)\|^2\right) ds \\ &= \mathcal{E}(v_0, \phi_0) + 2\int_0^t \langle g_1(s, v_m(s), \phi_m(s)), v_m(s)\rangle ds \\ &+2\int_0^t \int_Z (v_m(s-), \sigma(s, v_m(s-), \phi_m(s-), z))\tilde{\eta}(dz, ds) \\ &+ \int_0^t \int_Z \Upsilon(s,z)\eta(dz, ds), \end{aligned} \tag{3.28}$$

where

$$\begin{aligned} \Upsilon(s,z) \quad &= |v_m(s-) + \sigma(s, v_m(s-), \phi_m(s-), z)|^2_{L^2} \\ &-|v_m(s-)|^2_{L^2} - (v_m(s-), \sigma(s, v_m(s-), \phi_m(s-), z)). \end{aligned}$$

From the fact that $|x|^2 - |y|^2 + |x-y|^2 = 2\langle x-y, x\rangle$, it follows that

$$\begin{aligned} \mathcal{E}(v_m(t), \phi_m(t)) \quad &+ \int_0^t \left(2\|v_m(s)\|^2 + \|\mu_m(s)\|^2\right) ds = \mathcal{E}(v_0, \phi_0) \\ &+2\int_0^t \langle g_1(s, v_m(s), \phi_m(s)), v_m(s)\rangle ds \\ &+2\int_0^t \int_Z (v_m(s-), \sigma(s, v_m(s-), \phi_m(s-), z))\tilde{\eta}(dz, ds) \\ &+ \int_0^t \int_Z |\sigma(s, v_m(s-), \phi_m(s-), z)|^2_{L^2}\eta(dz, ds), \end{aligned} \tag{3.29}$$

We define the following stochastic processes

$$\begin{aligned} &X(t) = \sup_{s\in[0,t]} \mathcal{E}(v_m(s), \phi_m(s)), \\ &Y(t) = \int_0^t \left(2\|v_m(s)\|^2 + \|\mu_m(s)\|^2\right) ds, \\ &I(t) = 2\left|\int_0^t \int_Z (v_m(s-), \sigma(s, v_m(s-), \phi_m(s-), z))\tilde{\eta}(dz, ds)\right| \\ &\quad + \int_0^t \int_Z |\sigma(s, v_m(s-), \phi_m(s-), z)|^2_{L^2}\eta(dz, ds) \equiv \sup_{s\in[0,t]} |I_1(s)| + I_2(t), \end{aligned}$$

where

$$\begin{aligned} I_1(t) &= 2\int_0^t \int_Z (v_m(s-), \sigma(s, v_m(s-), \phi_m(s-), z))\tilde{\eta}(dz, ds), \\ I_2(t) &= \sup_{s\in[0,t]} \int_0^s \int_Z |\sigma(s, v_m(s-), \phi_m(s-), z)|^2_{L^2}\eta(dz, ds). \end{aligned} \tag{3.30}$$

Since $I_1(t)$ is a local martingale we can apply Burkholder-Davis-Gundy's inequality to derive that

$$\begin{aligned} &\mathbb{E}\sup_{s\in[0,r\wedge\tau_n]} |I_1(s)| \le \\ &\quad C\mathbb{E}\left(\int_0^{r\wedge\tau_n} \int_Z (v_m(s-), \sigma(s, (v_m, \phi_m)(s-), z))^2 \nu(dz)ds\right)^{1/2}. \end{aligned} \tag{3.31}$$

Thanks to Hölder's and Young's inequalities we have

$$\begin{aligned} \mathbb{E}\sup_{s\in[0,t]} |I_1(s)| &\le C\left[\epsilon\mathbb{E}\sup_{s\in[0,t]} |v_m(s)|^2_{L^2}\right]^{1/2} \times \\ &\quad \left[\epsilon^{-1}\mathbb{E}\int_0^t \int_Z |\sigma(s, v_m(s), \phi_m(s), z)|^2_{L^2}\nu(dz)ds\right]^{1/2} \\ &\le C\epsilon\mathbb{E}\sup_{s\in[0,t]} |v_m(s)|^2_{L^2} \\ &\quad + C\epsilon^{-1}\mathbb{E}\int_0^t \int_Z |\sigma(s, v_m(s), \phi_m(s), z)|^2_{L^2}\nu(dz)ds \\ &\le C\epsilon\mathbb{E}\sup_{s\in[0,t]} \mathcal{E}(v_m(s), \phi_m(s)) \\ &\quad + C\epsilon^{-1}\mathbb{E}\int_0^t \int_Z |\sigma(s, v_m(s), \phi_m(s), z)|^2_{L^2}\nu(dz)ds. \end{aligned} \tag{3.32}$$

Using (2.27), we derive that

$$\mathbb{E}\sup_{s\in[0,t]} |I_1(s)| \le C\epsilon X(t) + C\epsilon^{-1} l_0 t + C\epsilon^{-1}\int_0^t \mathbb{E}X(s)ds. \tag{3.33}$$

Next, we will deal with the second term of $I(t)$. Taking into account that the process

$$\int_0^t \int_Z |\sigma(s, v_m(s-), \phi_m(s-), z)|^2 \eta(dz, ds)$$

has only positive jumps, we derive from (2.27) that

$$\begin{aligned} &\mathbb{E}I_2(t) \le \mathbb{E}\int_0^t \int_Z |\sigma(s, v_m(s), \phi_m(s), z)|^2_{L^2}\nu(dz)ds \\ &\le l_0 t + l_1 \int_0^t \mathbb{E}|(v_m(s), \phi_m(s)|^2_{\mathcal{H}} ds \\ &\le l_0 t + l_1 \int_0^t \mathbb{E}X(s)ds. \end{aligned} \tag{3.34}$$

We also have

$$\begin{aligned} |2\langle g_1(s, v_m, \phi_m), v_m\rangle| &\le 2L_1|(v_m, \phi_m)|_{\mathcal{H}}\|v_m\| + 2\|g_1(s, 0, 0)\|_{V_1^*}\|v_m\| \\ &\le \frac{1}{8}\|v_m\|^2 + cL_1^2|(v_m, \phi_m)|^2_{\mathcal{H}} + c\|g_1(s, 0, 0)\|^2_{V_1^*}. \end{aligned} \tag{3.35}$$

It follows from (3.28)-(3.35) that

$$\begin{aligned}\mathbb{E}\mathcal{E}(v_m,\phi_m)(t) \quad &+\mathbb{E}\int_0^t \left(\|v_m(s)\|^2+\|\mu_m(s)\|^2\right)ds \le \mathbb{E}\mathcal{E}(v_0,\phi_0)\\ &+c\mathbb{E}\int_0^t \mathcal{E}(v_m,\phi_m)(s)ds + c\mathbb{E}\int_0^t \|g_1(s,0,0)\|^2_{V_1^*}ds.\end{aligned} \tag{3.36}$$

Therefore from Lemma 2.5, we derive that there exist a positive constant C such that

$$\mathbb{E}\mathcal{E}(v_m,\phi_m)(t)+\mathbb{E}\int_0^t \left(\|v_m(s)\|^2+\|\mu_m(s)\|^2\right)ds \le C, \tag{3.37}$$

for any $m \in \aleph$ and $t \in [0, r\wedge\tau_n], r\in[0,T]$.

We have just shown that $\forall t \in [0,T]$, we have

$$\mathbb{E}\sup_{s\in[0,t\wedge\tau_n]}\mathcal{E}(v_m,\phi_m)(t)+\mathbb{E}\int_0^t \left(\|v_m(s)\|^2+\|\mu_m(s)\|^2\right)ds \le C, \tag{3.38}$$

from which we can infer that

$$\mathbb{P}(\tau_n < t) \le Cn^{-2},\ \forall t\in[0,T],\ \forall n>0.$$

Hence, $\lim_{n\to+\infty}\mathbb{P}(\tau_n<t)=0$, for all $t\in[0,T]$. That is, $\tau_n \to +\infty$ in probability. Therefore, there exists a subsequence τ_{n_k} such that $\tau_{n_k}\to+\infty$, a.s. Since the sequence $(\tau_n)_n$ is increasing, we infer that $\tau_{n_k}\nearrow+\infty$ a.s.. Now we use Fatou's lemma and pass to the limit in (3.38) and derive that

$$\mathbb{E}\sup_{s\in[0,t]}\mathcal{E}(v_m(s),\phi_m(s))+\mathbb{E}\int_0^t \left(\|v_m(s)\|^2+\|\bar{\mu}_m(s)\|^2\right)ds \le C, \tag{3.39}$$

and (3.24) is proved.

To prove (3.25), we proceed as follows. First, without loss of generality, we may assume that for any given m, the process (v_m,ϕ_m) is uniformly bounded in $[0,T]$. Otherwise, we can introduce a sequence of stopping times as before.

By raising both sides of (3.28) to the power of $p\ge 2$, we derive that

$$\begin{aligned}&\mathbb{E}\sup_{s\in[0,t]}[\mathcal{E}(v_m(t),\phi_m(t))]^p+\mathbb{E}\left[\int_0^t \left(2\|v_m(s)\|^2+\|\mu_m(s)\|^2\right)ds\right]^p\\ &\quad\le \mathbb{E}[\mathcal{E}(v_0,\phi_0)]^p + c\mathbb{E}\sup_{s\in[0,t]}|I_3(s)|^p + c\mathbb{E}\sup_{s\in[0,t]}|I_4(s)|^p\\ &\quad+c\mathbb{E}\left[\int_0^t \|g_1(s,0,0)\|^2_{V_1^*}ds\right]^p,\end{aligned} \tag{3.40}$$

where

$$\begin{aligned}&I_3(t)=\\ &\int_0^t\int_Z \left\{|v_m(s-)+\Pi_m^1\sigma(s,(v_m,\phi_m)(s-),z)|^2_{L^2}-|v_m(s-)|^2_{L^2}\right\}\tilde{\eta}(dz,ds),\end{aligned} \tag{3.41}$$

$$\begin{aligned}I_4(t) &= \\ &\int_0^t \int_Z \left\{|v_m(s) + \Pi_m^1 \sigma(s, v_m, \phi_m)(s), z)|_{L^2}^2 - |v_m(s)|_{L^2}^2\right\} \nu(dz)ds \\ &-2\int_0^t \int_Z (v_m(s), \Pi_m^1 \sigma(s, v_m(s), \phi_m(s), z))\nu(dz)ds \\ &\leq c\int_0^t \int_Z |\Pi_m^1 \sigma(s, v_m(s), \phi_m(s), z))|_{L^2}^2 \nu(dz)ds \\ &\leq c\int_0^t (1 + |(v_m, \phi_m)(s)|_{\mathcal{H}}^2)ds.\end{aligned} \tag{3.42}$$

As in [10, 13], we note that

$$\begin{aligned}&\int_Z \left\{|v_m(s-) + \Pi_m^1 \sigma(s, (v_m, \phi_m)(s), z)|_{L^2}^2 - |v_m(s-)|_{L^2}^2\right\}^2 \nu(dz) \\ &\quad\leq |v_m(s-)|_{L^2}^2 \int_Z |\sigma(s, (v_m, \phi_m)(s), z)|_{L^2}^2 \nu(dz) \\ &\quad+c\int_Z |\sigma(s, (v_m, \phi_m)(s), z)|_{L^2}^4 \nu(dz) \\ &\quad\leq c_0 + c_1|v_m(s)|_{L^2}^2 + c_2|v_m(s)|_{L^2}^4 \\ &\quad\leq k_1 + k_4|v_m(s)|_{L^2}^4.\end{aligned} \tag{3.43}$$

It follows that

$$\begin{aligned}&\left(\int_0^t \int_Z \left\{|v_m(s-) + \Pi_m^1 \sigma(s, v_m, \phi_m, z)|_{L^2}^2 - |v_m(s-)|_{L^2}^2\right\}^2 \nu(dz)ds\right)^{p/2} \\ &\quad\leq c(k_1T)^{p/2} + c(k_2)^{p/2}\left(\int_0^t |(v_m, \phi_m)(s)|_{\mathcal{H}}^4 ds\right)^{p/2}.\end{aligned} \tag{3.44}$$

We derive that

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,t]} \quad &|I_3(s)|^p \leq c_p(k_1T)^{p/2} \\ &+c_p(k_2)^{p/2}\mathbb{E}\left[\left(\int_0^t |(v_m, \phi_m)(s)|_{\mathcal{H}}^4 ds\right)^{p/2}\right] \\ &\leq c + \tfrac{1}{2}\mathbb{E}\left(\sup_{s\in[0,t]} |v_m(s)|_{L^2}^2\right)^p + c\mathbb{E}\left(\int_0^t |v_m(s)|_{L^2}^2 ds\right)^p.\end{aligned} \tag{3.45}$$

From Hölder's inequality, we have

$$\begin{aligned}\int_0^t |(v_m, \phi_m)(s)|_{\mathcal{H}}^2 ds \quad &\leq \left(\int_0^t |(v_m, \phi_m)(s)|_{\mathcal{H}}^{2p} ds\right)^{1/p} \left(\int_0^t 1ds\right)^{\frac{p-1}{p}} \\ &\leq T^{\frac{p-1}{p}}\left(\int_0^t |(v_m, \phi_m)(s)|_{\mathcal{H}}^{2p} ds\right)^{1/p},\end{aligned} \tag{3.46}$$

which gives

$$\left(\int_0^t |(v_m, \phi_m)(s)|_{\mathcal{H}}^2 ds\right)^p \leq cT^{P-1}\int_0^t |(v_m, \phi_m)(s)|_{\mathcal{H}}^{2p} ds. \tag{3.47}$$

From (3.45), (3.46), we get

$$\begin{aligned}\mathbb{E} \quad & \sup_{s\in[0,t]} |I_3(s)|^p \leq \frac{1}{2}\mathbb{E}\left(\sup_{s\in[0,t]} |(v_m,\phi_m)(s)|^2_{\mathcal{H}}\right)^p \\ & + c_{p,T}\int_0^t \mathbb{E}|(v_m,\phi_m)(s)|^{2p}_{\mathcal{H}} ds \\ & \leq \tfrac{1}{2}\mathbb{E}\sup_{s\in[0,t]}[\mathcal{E}(v_m(s),\phi_m(s))]^p + c_{p,T}\int_0^t \mathbb{E}[\mathcal{E}(v_m(s),\phi_m(s))]^p ds.\end{aligned} \tag{3.48}$$

From (2.27) and (3.42), we also have

$$\begin{aligned}\mathbb{E}|I_4(t)|^p \quad & \leq c\mathbb{E}\left(\int_0^t (1+|(v_m,\phi_m)(s)|^2_{\mathcal{H}})ds\right)^p \\ & \leq c_p + c_p\mathbb{E}\left(\int_0^t |(v_m,\phi_m)(s)|^2_{\mathcal{H}})ds\right)^p.\end{aligned} \tag{3.49}$$

It follows that

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,t]}|I_2(s)|^p \quad & \leq c_{p,T} + c_{p,T}\int_0^t |(v_m,\phi_m)(s)|^{2p}_{\mathcal{H}} ds \\ & \leq c_{p,T} + c_{p,T}\int_0^t \mathbb{E}[\mathcal{E}(v_m(s),\phi_m(s))]^p ds.\end{aligned} \tag{3.50}$$

It follows from (3.40)-(3.50)

$$\begin{aligned}\mathbb{E} \quad & \sup_{s\in[0,t]}[\mathcal{E}(v_m(s),\phi_m(s))]^p \leq c_{p,T} \\ & + c_{p,T}\int_0^t \mathbb{E}[\mathcal{E}(v_m(s),\phi_m(s))]^p ds + c\left(\int_0^t \|g_1(t,0,0)\|^2_{V_1^*}\right)^p,\end{aligned} \tag{3.51}$$

and Gronwall's lemma and (3.40) give

$$\mathbb{E}\sup_{s\in[0,t]}[\mathcal{E}(v_m(s),\phi_m(s))]^p + \mathbb{E}\left[\int_0^t \left(2\|v_m(s)\|^2 + \|\mu_m(s)\|^2\right) ds\right]^p \leq C, \tag{3.52}$$

and (3.25) follows. □

Proposition 3.4. *We can extract from (v_m,ϕ_m) a subsequence still labeled the same and there exists a stochastic process (v,ϕ) such that*

$$\begin{aligned}&(v_m,\phi_m) \rightharpoonup (v,\phi) \text{ in } L^4(\Omega, L^\infty([0,T];\mathcal{H})),\\ &(v_m,\phi_m) \rightharpoonup (v,\phi) \text{ in } L^2(\Omega, L^2([0,T];\mathcal{U})),\\ &B_0(v_m,v_m) \rightharpoonup \beta_0^\flat,\ R_0(A_1\phi_m,\phi_m) \rightharpoonup r_0^\flat \text{ in } L^2(\Omega\times[0,T];V_1^*),\\ &g_1(t,v_m,\phi_m) \rightharpoonup g_1^\flat \text{ in } L^2(\Omega\times[0,T];V_1^*),\\ &B_1(v_m,\phi_m) \rightharpoonup \beta_1^\flat,\ f(\phi_m) \rightharpoonup f^\flat \text{ in } L^2(\Omega\times[0,T];V_2^*),\\ &\sigma(t,v_m,\phi_m,\cdot) \rightharpoonup \sigma^\flat \text{ in } L^2(\Omega\times[0,T];L^2(Z,\nu;H_1)).\end{aligned} \tag{3.53}$$

We note that

$$\begin{aligned}&|B_0(v_m,v_m)|_{V_1^*} \leq c|v_m|_{L^2}\|v_m\|,\\ &|R_0(A_1\phi_m,\phi_m)|_{V_1^*} \leq c\|\phi_m\||A_1\phi_m|^{1/2}_{L^2}|\phi_m|^{1/2}_{H^3},\\ &|B_1(v_m,\phi_m)|_{V_2^*} \leq c|v_m|_{L^2}\|\phi_m\|^{1/2}|A_1\phi_m|^{1/2}_{L^2}.\end{aligned} \tag{3.54}$$

From (2.11) and $(3.26)_3$, we also have

$$\begin{aligned}
&|f(\phi_m)|_{L^2} \leq c(1+\|\phi_m\|^{k+1}) \leq c(1+\mathcal{E}(v_m,\phi_m)^{\frac{k+1}{2}}),\\
&|A_1\phi_m|_{L^2} \leq c|\mu_m|_{L^2} + c|f(\phi_m)|_{L^2}\\
&\qquad \leq c|\mu_m|_{L^2} + c(1+\mathcal{E}(v_m,\phi_m)^{\frac{k+1}{2}}),
\end{aligned} \tag{3.55}$$

$$\begin{aligned}
|\phi_m|^2_{H^3} &\leq c\|\mu_m\|^2 + c|f'(\phi_m)\nabla\phi_m|^2_{L^2}\\
&\leq c\|\mu_m\|^2 + c\|\phi_m\|^{2k+1}|A_1\phi_m|_{L^2}.
\end{aligned} \tag{3.56}$$

It follows from (3.54)-(3.55) that

$$\begin{aligned}
&\mathbb{E}\sup_{[0,T]}|(v_m,\phi_m)|^2_{\mathcal{H}} \leq C, \quad \mathbb{E}\int_0^T \|(v_m(s),\phi_m(s)\|^2_{\mathcal{U}}ds \leq C,\\
&\mathbb{E}\int_0^T |\phi_m(s)|^2_{H^3}ds \leq C, \quad \mathbb{E}\sup_{t\in[0,T]}|f(\phi_m)|_{L^2} \leq C,\\
&\mathbb{E}\int_0^T \left[|B_0(v_m,v_m)|^2_{V_1^*} + |R_0(A_1\phi_m,\phi_m)|^2_{V_1^*} + |B_1(v_m,\phi_m)|^2_{V_2^*}\right]ds \leq C,\\
&\mathbb{E}\sup_{[0,T]}|(v_m,\phi_m)|^4_{\mathcal{H}} \leq C, \quad \mathbb{E}\left[\int_0^T \|(v_m(s-),\phi_m(s-)\|^2_{\mathcal{U}}ds\right]^2 \leq C,
\end{aligned} \tag{3.57}$$

$$\begin{aligned}
\mathbb{E}\quad &\int_0^T \|\sigma(s,v_m(s),\phi_m(s),z)\|^2_{L^2(Z,\nu,H_1)}ds \leq l_0T\\
&+l_1\mathbb{E}\int_0^T |(v_m(s-),\phi_m(s-))|^2_{\mathcal{H}}ds \leq C,
\end{aligned} \tag{3.58}$$

$$\mathbb{E}\int_0^T |f(\phi_m)|^2_{L^2}ds \leq c\mathbb{E}\int_0^T (1+\mathcal{E}(v_m,\phi_m)^{k+1})ds \leq C. \tag{3.59}$$

From (3.57), we can find a subsequence still denoted $\{(v_m,\phi_m)\}$ such that

$$\begin{aligned}
&(v_m,\phi_m) \rightharpoonup (v,\phi) \text{ in } L^4(\Omega,L^\infty([0,T];\mathcal{H})),\\
&(v_m,\phi_m) \rightharpoonup (v,\phi) \text{ in } L^2(\Omega\times[0,T];\mathcal{U}),\\
&B_0(v_m,v_m) \rightharpoonup \beta_0^\flat,\ R_0(A_1\phi_m,\phi_m) \rightharpoonup r_0^\flat, \quad \text{in } L^2(\Omega\times[0,T];V_1^*),\\
&g_1(t,v_m,\phi_m) \rightharpoonup g_1^\flat,\ f(\phi_m) \rightharpoonup f^\flat \text{ in } L^2(\Omega\times[0,T];V_1^*),\\
&B_1(v_m,\phi_m) \rightharpoonup \beta_1^\flat \text{ in } L^2(\Omega\times[0,T];V_2^*),\\
&\sigma(t,v_m(s-),\phi_m(s-),\cdot) \rightharpoonup \sigma^\flat \text{ in } L^2(\Omega\times[0,T];L^2(Z,\nu;H_1)). \quad \square
\end{aligned} \tag{3.60}$$

As in [11, 29], we can check that v is an H_1-valued càdlàg and $\mathbb{F}$-progressively measurable process, and ϕ is an V_2-valued continuous and $\mathbb{F}$-progressively measurable process. Moreover (v,ϕ) satisfies for all $0\leq t\leq T$

$$\begin{aligned}
&v(t) + \int_0^t A_0v ds + \int_0^t \beta_0^\flat(s)ds = v_0 + \int_0^t (r_0^\flat(s) + g_1^\flat(s))ds\\
&\qquad + \int_0^t\int_Z \sigma^\flat(s,z)\tilde{\eta}(dz,ds),\\
&\phi(t) + \int_0^t A_1\mu^\flat ds + \int_0^t \beta_1^\flat(s)ds = \phi_0,\ \mu^\flat = A_1\phi + f^\flat,
\end{aligned} \tag{3.61}$$

$\mathbb{P}-$a.s. as a equality in $\mathcal{U}^*$.

Proposition 3.5. *We have the following identities*

$$\begin{array}{l}\beta_0^\flat = B_0(v,v),\ r_0^\flat = R_0(A_1\phi,\phi),\\ \beta_1^\flat = B_1(v,\phi),\ f^\flat = f(\phi),\ \sigma(t,v,\phi,z) = \sigma^\flat.\end{array} \tag{3.62}$$

Proposition 3.6. *For any $n \geq 1$ we have that as $m \longrightarrow +\infty$,*

$$1_{[0,\tau_n]}((v_m,\phi_m) - (v,\phi)) \longrightarrow (0,0) \quad \text{in } L^2(\Omega\times[0,T];\mathcal{U}), \tag{3.63}$$

and

$$\mathbb{E}|(v_m,\phi_m(\tau_n)) - (v,\phi)(\tau_n)|_{\mathcal{H}} \longrightarrow 0 \text{ as } n \longrightarrow +\infty. \tag{3.64}$$

Proof. Let

$$(\tilde{v}_m,\tilde{\phi}_m,\tilde{\mu}_m) = \Pi_m(v,\phi,\mu),$$

where $\Pi_m \equiv (\Pi_m^1,\Pi_m^2)$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_m$. It follows that

$$\begin{array}{l}|(\tilde{v}_m,\tilde{\phi}_m)|_{\mathcal{H}} \leq |(v,\phi)|_{\mathcal{H}}\\ \|(\tilde{v}_m,\tilde{\phi}_m)\|_{\mathcal{U}} \leq c\|v,\phi)\|_{\mathcal{U}},\\ (\tilde{v}_m,\tilde{\phi}_m) \to (v,\phi) \text{ in } \mathcal{U} \text{ for almost every } (\omega,t)\in\Omega\times[0,T],\\ (\tilde{v}_m,\tilde{\phi}_m) \to (v,\phi) \text{ in } L^2(\Omega\times[0,T];\mathcal{U}),\\ \mathbb{E}(|(\tilde{v}_m,\tilde{\phi}_m)(\tau_n) - (v,\phi)(\tau_n)|_{\mathcal{H}}) \to 0 \text{ as } n\to+\infty.\end{array} \tag{3.65}$$

From (3.26) and (3.61), we derive that for $1 \leq k \leq m$, we have

$$\begin{array}{l}\langle \tilde{v}_m(t) - v_m(t), w_k\rangle + \displaystyle\int_0^t \langle A_0(\tilde{v}_m - v_m), w_k\rangle ds\\ \quad + \displaystyle\int_0^t \langle \beta_0^\flat - B_0(v_m,v_m), w_k\rangle ds = \int_0^t \langle r_0^\flat - R_0(A_1\phi_m,\phi_m), w_k\rangle ds\\ \quad + \displaystyle\int_0^t \langle g_1^\flat - g_1(s,v_m,\phi_m), w_k\rangle ds\\ \quad + \displaystyle\int_0^t\int_Z \Big[\sigma(s,v_m(s-),\phi_m(s-),z) - \sigma^\flat(s,z), w_k\Big]\,\tilde{\eta}(dz,ds),\\ \langle \tilde{\phi}_m(t) - \phi_m(t), A_1\psi_k\rangle + \displaystyle\int_0^t \langle A_1(\tilde{\mu}_m - \mu_m), A_1\psi_k\rangle ds\\ \quad + \displaystyle\int_0^t \langle \beta_1^\flat - B_1(v_m,\phi_m), A_1\psi_k\rangle ds = 0,\\ \langle \tilde{\mu}_m - \mu_m, A_1\psi_k\rangle = \langle A_1(\tilde{\phi}_m - \phi_m), A_1\psi_k\rangle + \langle f^\flat - f(\phi_m), A_1\psi_k\rangle.\end{array} \tag{3.66}$$

Note that

$$\begin{array}{l}\beta_0^\flat - B_0(v_m,v_m) = \beta_0^\flat - B_0(\tilde{v}_m,\tilde{v}_m) + B_0(\tilde{v}_m - v_m,\tilde{v}_m)\\ \qquad + B_0(v_m,\tilde{v}_m - v_m),\\ r_0^\flat - R_0(A_1\phi_m,\phi_m) = r_0^\flat - R_0(A_1\tilde{\phi}_m,\tilde{\phi}_m) + R_0(A_1(\tilde{\phi}_m - \phi_m),\tilde{\phi}_m)\\ \qquad + R_0(A_1\phi_m,\tilde{\phi}_m - \phi_m),\\ \beta_1^\flat - B_1(v_m,\phi_m) = \beta_1^\flat - B_1(\tilde{v}_m,\tilde{\phi}_m) + B_1(\tilde{v}_m - v_m,\tilde{\phi}_m)\\ \qquad + B_1(v_m,\tilde{\phi}_m - \phi_m),\\ f^\flat - f(\phi_m) = f^\flat - f(\tilde{\phi}_m) + f(\tilde{\phi}_m) - f(\phi_m).\end{array}$$

Let us set $\theta_m = \tilde{v}_m - v_m,\ \rho_m = \tilde{\phi}_m - \phi_m,\ \zeta_m = \tilde{\mu}_m - \mu_m$. From the Itô's formula, we have

$$\begin{aligned}
&|\theta_m(t)|_{L^2}^2 + 2\int_0^t (\|\theta_m\|^2 + \langle \beta_0^\flat - B_0(v_m, v_m), \theta_m\rangle) ds \\
&= 2\int_0^t \langle r_0^\flat - R_0(A_1\phi_m, \phi_m), \theta_m\rangle ds + 2\int_0^t \langle g_1^\flat - g_1(s, v_m, \phi_m), \theta_m\rangle ds \\
&+2\int_0^t \int_Z \Big[\sigma(v_m(s-), \phi_m(s-), z) - \Pi_m^1 \sigma^\flat(s,z)\Big]\tilde{\eta}(dz, ds) \\
&+\int_0^t \int_Z \Upsilon(s,z)\eta(dz, ds),
\end{aligned} \tag{3.67}$$

where

$$\begin{aligned}
\Upsilon(s,z) &= |v_m(s-) + \sigma_m(s, (v_m,\phi_m)(s-), z) - \Pi_m^1\sigma^\flat(s,z)|_{L^2}^2 \\
&-|v_m(s-)|_{L^2}^2 - 2\langle \sigma_m(s,(v_m,\phi_m)(s-),z) - \Pi_m^1\sigma^\flat(s,z), v_m(s-)\rangle \\
&= |\sigma(s,(v_m,\phi_m)(s-),z) - \Pi_m^1\sigma^\flat(s,z)|_{L^2}^2.
\end{aligned}$$

Replacing ψ_k in $(3.66)_3$ and $(3.66)_2$ respectively by $\bar{\zeta}_m - \xi\rho_m$ and ρ_m gives

$$\begin{aligned}
&\frac{d}{dt}\|\rho_m\|^2 + 2\|\bar{\zeta}_m\|^2 + 2\xi|A_1\rho_m|_{L^2}^2 + 2\langle \beta_1^\flat - B_1(\tilde{v}_m, \tilde{\phi}_m), A_1\rho_m\rangle \\
&+2\langle B_1(\theta_m, \tilde{\phi}_m), A_1\rho_m\rangle + 2\langle B_1(v_m, \rho_m), A_1\rho_m\rangle + \xi\langle \zeta_m, A_1\rho_m\rangle \\
&+\xi\langle f^\flat - f(\phi_m), A_1\rho_m\rangle - \langle f^\flat - f(\phi_m), A_1\zeta_m\rangle = 0.
\end{aligned} \tag{3.68}$$

Note that

$$\begin{aligned}
\langle \beta_0^\flat - B_0(v_m, v_m), \theta_m\rangle &= \langle \beta_0^\flat - B_0(\tilde{v}_m, \tilde{v}_m), \theta_m\rangle + \langle B_0(\theta_m, \tilde{v}_m), \theta_m\rangle \\
&\leq \langle \beta_0^\flat - B_0(\tilde{v}_m, \tilde{v}_m), \theta_m\rangle + \tfrac{1}{4}\|\theta_m\|^2 + c\|\tilde{v}_m\|^2|\theta_m|_{L^2}^2,
\end{aligned} \tag{3.69}$$

$$\begin{aligned}
\langle \beta_1^\flat - B_1(v_m, \phi_m), A_1\rho_m\rangle &= \langle \beta_1^\flat - B_1(\tilde{v}_m, \tilde{\phi}_m), A_1\rho_m\rangle \\
&+\langle B_1(\theta_m, \tilde{\phi}_m), A_1\rho_m\rangle + \langle B_1(v_m, \rho_m), A_1\rho_m\rangle \\
&\leq \langle \beta_1^\flat - B_1(\tilde{v}_m, \tilde{\phi}_m), A_1\rho_m\rangle + \tfrac{1}{4}(\|\theta_m\|^2 + \tfrac{\xi}{4}|A_1\rho_m|_{L^2}^2) \\
&+c\|\phi_m\|^2|A_1\phi_m|_{L^2}^2|\theta_m|_{L^2}^2 + c|\tilde{v}_m|_{L^2}^2\|\tilde{v}_m\|^2\|\rho_m\|^2,
\end{aligned} \tag{3.70}$$

$$\begin{aligned}
\langle r_0^\flat - R_0(A_1\phi_m, \phi_m), \theta_m\rangle &= \langle r_0^\flat - R_0(A_1\tilde{\phi}_m, \tilde{\phi}_m), \theta_m\rangle \\
&+\langle R_0(A_1\rho_m, \tilde{\phi}_m), \theta_m\rangle + \langle R_0(A_1\phi_m, \rho_m), \theta_m\rangle \\
&\leq \langle r_0^\flat - R_0(A_1\tilde{\phi}_m, \tilde{\phi}_m), \theta_m\rangle + \tfrac{1}{4}(\|\theta_m\|^2 + \tfrac{\xi}{4}|A_1\rho_m|_{L^2}^2) \\
&+c|A_1\phi_m|_{L^2}^2(|\theta_m|_{L^2}^2 + \|\rho_m\|^2) + c\|\tilde{\phi}_m\|^2|A_1\tilde{\phi}_m|_{L^2}^2|\theta_m|_{L^2}^2,
\end{aligned} \tag{3.71}$$

$$\begin{aligned}
\langle f^\flat - f(\phi_m), A_1\zeta_m\rangle &= \langle f^\flat - f(\tilde{\phi}_m), A_1\zeta_m\rangle + \langle f(\tilde{\phi}_m) - f(\phi_m), A_1\zeta_m\rangle \\
&\leq \langle f^\flat - f(\tilde{\phi}_m), A_1\zeta_m\rangle + \tfrac{1}{2}\|\zeta_m\|^2 \\
&+Q_1(\|\tilde{\phi}_m\|, \|\phi_m\|)(|A_1\phi_m|_{L^2}^2 + |A_1\tilde{\phi}_m|_{L^2}^2)\|\rho_m\|^2,
\end{aligned} \tag{3.72}$$

$$\begin{aligned}
\xi\langle f^\flat - f(\phi_m), A_1\rho_m\rangle &= \xi\langle f^\flat - f(\tilde{\phi}_m) + f(\tilde{\phi}_m) - f(\phi_m), A_1\rho_m\rangle \\
&\leq \xi\langle f^\flat - f(\tilde{\phi}_m), A_1\zeta_m\rangle + \tfrac{\xi}{8}|A_1\rho_m|_{L^2}^2 \\
&+Q_1(\|\tilde{\phi}_m\|, \|\phi_m\|)\|\rho_m\|^2,
\end{aligned} \tag{3.73}$$

$$|\xi\langle \zeta_m, A_1\rho_m\rangle \leq \frac{\xi}{2}\|\zeta_m\|^2 + \frac{\xi}{4}|A_1\rho_m|_{L^2}^2, \tag{3.74}$$

$$\begin{aligned}
\Upsilon(s,z) &= |\sigma_m(s, v_m(s-), \phi_m(s-), z) - \Pi_m^1\sigma^\flat(s,z)|_{L^2}^2 \\
&= |\Pi_m^1[\sigma(s, v_m(s-), \phi_m(s-), z) - \sigma(s, v(s-), \phi(s-), z)|_{L^2}^2 \\
&-|\Pi_m^1[\sigma(s, v(s-), \phi(s-), z) - \sigma^\flat(s,z)|_{L^2}^2 + S_1(s,z),
\end{aligned} \tag{3.75}$$

where

$$
\begin{aligned}
&S_1(s,z) = \\
&2\langle \Pi_m^1[\sigma(s,(v_m,\phi_m)(s-),z)-\sigma^\flat(s,z)], \Pi_m^1[\sigma(s,(v,\phi)(s-),z)-\sigma^\flat(s,z)]\rangle.
\end{aligned}
$$

From (2.27) and (3.75), we derive that

$$
\begin{aligned}
\Upsilon(s,z) \quad &\le l_2|(v_m(s-),\phi_m(s-))|_{\mathcal{H}}^2 + l_2|(v_m,\phi_m)(s-)-(v,\phi)(s-)|_{\mathcal{H}}^2 \\
&-|\Pi_m^1[\sigma(s,v(s-),\phi(s-),z)-\sigma^\flat(s,z)|_{L^2}^2 + S_1(s,z).
\end{aligned} \tag{3.76}
$$

We also have

$$
\begin{aligned}
\langle g_1(s,\tilde v_m,\tilde\phi_m)-g_1(s,v_m,\phi_m),\theta_m\rangle \quad &\le L_1|(\theta_m,\rho_m)|_{\mathcal{H}}\|\theta_m\| \\
&\le \tfrac14\|\theta_m\|^2 + cL_1^2|(\theta_m,\rho_m)|_{\mathcal{H}}^2.
\end{aligned} \tag{3.77}
$$

Let

$$
\begin{aligned}
Z(t) \quad &= |\theta_m(t)|_{L^2}^2 + \|\rho_m(t)\|^2 = |(\tilde v_m - v_m)(t)|_{L^2}^2 + \|(\tilde\phi_m-\phi_m)(t)\|^2, \\
Y_1(t) \quad &= c\|\tilde v_m\|^2 + c\|\phi_m\|^2|A_1\phi_m|_{L^2}^2 + c|\tilde v_m|_{L^2}^2\|\tilde v_m\|^2 + c|A_1\phi_m|_{L^2}^2 \\
&+c\|\tilde\phi_m\|^2|A_1\tilde\phi_m|_{L^2}^2 + Q_1(\|\tilde\phi_m\|,\|\phi_m\|)(1+|A_1\phi_m|_{L^2}^2+|A_1\tilde\phi_m|_{L^2}^2) \\
&+cL_2^2+cL_1^2, \\
K_2(t) \quad &= \|\theta_m\|^2 + (1-c\xi)\|\zeta_m\|^2 + c\xi|A_1\rho_m|_{L^2}^2,
\end{aligned}
$$

where ξ is small enough such that $1-c\xi>0$.

Let

$$
\delta(t) = \exp\left(-\int_0^t Y_1(s)ds\right),
$$

Using (3.67)-(3.77), it follows from Ito's formula that

$$
\begin{aligned}
\mathbb{E}\delta(t)Z(t) \quad &+\mathbb{E}\int_0^t \delta(s)K_2(s)ds \\
&+\mathbb{E}\int_0^t \delta(s)|\Pi_m^1[\sigma(s,(v,\phi)(s),z)-\sigma^\flat(s,z)|_{L^2}^2 ds \\
&\le \mathbb{E}\int_0^t \delta(s)\langle -\beta_0^\flat + B_0(\tilde v_m,\tilde v_m),\theta_m\rangle ds \\
&+\mathbb{E}\int_0^t \delta(s)\langle -\beta_1^\flat + B_1(\tilde v_m,\tilde\phi_m),A_1\rho_m\rangle ds \\
&+\mathbb{E}\int_0^t \delta(s)\langle r_0^\flat - R_0(A_1\tilde\phi_m,\tilde\phi_m),\theta_m\rangle ds \\
&+\mathbb{E}\int_0^t \delta(s)\langle g_1^\flat - g_1(s,\tilde v_m,\tilde\phi),\theta_m\rangle ds \\
&+\mathbb{E}\int_0^t\int_Z \delta(s)S_1(s,z)\eta(dz,ds).
\end{aligned} \tag{3.78}
$$

For each $n\ge 1$, we consider the $\mathfrak{F}_t-$stopping time τ_n defined by:

$$
\tau_n = \min\left(T,\ \inf\left\{t\in[0,T];\ |(v,\phi)|_{\mathcal{H}}^2 + \int_0^t \|(v,\phi)\|_{\mathcal{U}}^2 ds \ge n^2\right\}\right).
$$

We derive from (3.78) that

$$
\begin{aligned}
\mathbb{E}(\tau_n)\delta(s) \quad & Z(\tau_n) + c\mathbb{E}\int_0^{\tau_n} \delta(s)K_2(s)ds \\
& +\mathbb{E}\int_0^{\tau_n} \delta(s)|\Pi_m^1[\sigma(s,(v,\phi)(s),z) - \sigma^\flat(s,z)|_{L^2}^2 ds \\
& \leq 2\mathbb{E}\int_0^{\tau_n} \delta(s)\langle -\beta_0^\flat + B_0(\tilde{v}_m,\tilde{v}_m),\theta_m\rangle ds \\
& +2\mathbb{E}\int_0^{\tau_n} \delta(s)\langle g_1^\flat - g_1(s,\tilde{v}_m,\tilde{\phi}_m),\theta_m\rangle ds \\
& +2\mathbb{E}\int_0^{\tau_n} \delta(s)\langle r_0^\flat - R_0(A_1\tilde{\phi}_m,\tilde{\phi}_m),\theta_m\rangle ds \\
& +2\mathbb{E}\int_0^{\tau_n} \delta(s)\langle -\beta_1^\flat + B_1(\tilde{v}_m,\tilde{\phi}_m),A_1\rho_m\rangle ds \\
& +\mathbb{E}\int_0^{\tau_n}\int_Z \delta(s)S_1(s,z)\eta(dz,ds).
\end{aligned}
\tag{3.79}
$$

Claim 1. The right side of (3.79) goes to 0 as m goes to $+\infty$.

(**i**). Since $\Pi_m^1 \circ \Pi_m^1 = \Pi_m^1$ and $\|\Pi_m^1\| \leq 1$, it follows that

$$1_{[0,\tau_n]}\delta(s)\Pi_m^1[\sigma(s,v(s),\phi(s),z) - \sigma^\flat(s,z)]$$

is bounded in $L^2(\Omega\times[0,T];L^2(Z,\nu;H_1))$. Therefore, from $(3.60)_6$ we see that

$$\lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n}\int_Z \delta(s)S_1(s,z)\eta(dz,ds) = 0.$$

(**ii**). Let us now prove that

$$\lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n}\delta(s)\langle -\beta_0^\flat + B_0(\tilde{v}_m,\tilde{v}_m),\theta_m\rangle ds = 0. \tag{3.80}$$

We recall that

$$
\begin{aligned}
&(v_m,\phi_m) \rightharpoonup (v,\phi),\ (\tilde{v}_m,\tilde{\phi}_m) \to (v,\phi),\ (\tilde{v}_m,\tilde{\phi}_m) - (v_m,\phi_m) \rightharpoonup (0,0) \\
&\text{in } L^2(\Omega\times[0,T];\mathcal{U}).
\end{aligned}
\tag{3.81}
$$

We also have

$$
\begin{aligned}
\lim_{m\to\infty}\quad & \mathbb{E}\int_0^{\tau_n}\delta(s)\langle -\beta_0^\flat + B_0(\tilde{v}_m,\tilde{v}_m),\theta_m\rangle ds \\
& = \lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n}\delta(s)\langle -\beta_0^\flat + B_0(v,v),\theta_m\rangle ds \\
& + \lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n}\delta(s)\langle -B_0(v,v) + B_0(\tilde{v}_m,\tilde{v}_m),\theta_m\rangle ds.
\end{aligned}
\tag{3.82}
$$

From (3.81) and the fact that $1|_{[0,\tau_n]}\delta(t)(-\beta_0^\flat + B_0(v,v)) \in L^2(\Omega\times[0,T];V_1^*)$, it follows that

$$\lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n}\delta(s)\langle -\beta_0^\flat + B_0(v,v),\theta_m\rangle ds = 0. \tag{3.83}$$

We also note that

$$
\begin{aligned}
&\| -B_0(v,v) + B_0(\tilde{v}_m,\tilde{v}_m)\|_{V_1^*} \\
&\quad \leq c|\tilde{v}_m - v|_{L^2}^{1/2}\|\tilde{v}_m - v\|^{1/2}(\|\tilde{v}_m\|^{1/2}|\tilde{v}_m|_{L^2}^{1/2} + \|v\|^{1/2}|v|_{L^2}^{1/2}),
\end{aligned}
$$

which implies that

$$\begin{aligned}&\|1|_{[0,\tau_n]}(-B_0(v,v)+B_0(\tilde{v}_m,\tilde{v}_m))\|_{V_1^*}\to 0 \ \text{ as } m\to\infty,\ dt\times d\mathbb{P}-\text{a.e.},\\&\|1|_{[0,\tau_n]}(-B_0(v,v)+B_0(\tilde{v}_m,\tilde{v}_m))\|_{V_1^*}\le cn\|v\|\in L^2(\Omega\times[0,T];\Re).\end{aligned}$$

It follows that

$$\lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n}\delta(s)\langle -B_0(v,v)+B_0(\tilde{v}_m,\tilde{v}_m),\theta_m\rangle ds=0. \tag{3.84}$$

We conclude from (3.83) and (3.84) that

$$\lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n}\delta(s)\langle -\beta_0^\flat+B_0(\tilde{v}_m,\tilde{v}_m),\theta_m\rangle ds=0, \tag{3.85}$$

which proves (3.80).
(**iii**). Next we will prove that

$$\lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n}\delta(s)\langle r_0^\flat-R_0(A_1\tilde{\phi}_m,\tilde{\phi}_m),\theta_m\rangle ds=0. \tag{3.86}$$

From (3.81) and the fact that

$$1|_{[0,\tau_n]}\delta(t)(r_0^\flat-R_0(A_1\phi,\phi))\in L^2(\Omega\times[0,T];V_1^*),$$

we also have

$$\lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n}\delta(s)\langle r_0^\flat-R_0(A_1\phi,\phi),\theta_m\rangle ds=0. \tag{3.87}$$

We also note that

$$\begin{aligned}\|R_0(A_1\tilde{\phi}_m,\tilde{\phi}_m)-R_0(A_1\phi,\phi)\|_{V_1^*}\quad&\le c|A_1(\tilde{\phi}_m-\phi)|_{L^2}\|\phi\|^{1/2}|A_1\phi|_{L^2}^{1/2}\\&+c\|\tilde{\phi}_m-\phi\|^{1/2}|A_1(\tilde{\phi}_m-\phi)|_{L^2}^{1/2}|A_1\tilde{\phi}_m|_{L^2},\end{aligned}$$

which implies that

$$\begin{aligned}&\|1|_{[0,\tau_n]}(R_0(A_1\tilde{\phi}_m,\tilde{\phi}_m)-R_0(A_1\phi,\phi))\|_{V_1^*}\to 0 \ \text{ as } m\to\infty,\ dt\times d\mathbb{P}-\text{a.e.},\\&\|1|_{[0,\tau_n]}(R_0(A_1\tilde{\phi}_m,\tilde{\phi}_m)-R_0(A_1\phi,\phi))\|_{V_1^*}\le cn\|v\|\in L^2(\Omega\times[0,T];\Re).\end{aligned}$$

It follows that

$$\lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n}\delta(s)\langle R_0(A_1\tilde{\phi}_m,\tilde{\phi}_m)-R_0(A_1\phi,\phi),\theta_m\rangle ds=0. \tag{3.88}$$

We conclude from (3.87) and (3.88) that

$$\lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n}\delta(s)\langle r_0^\flat-R_0(A_1\tilde{\phi}_m,\tilde{\phi}_m),\theta_m\rangle ds=0, \tag{3.89}$$

which proves (3.86).
(**iv**). Let us now prove that

$$\lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n}\delta(s)\langle -\beta_1^\flat+B_1(\tilde{v}_m,\tilde{\phi}_m),A_1\rho_m\rangle ds=0.$$

Following similar steps as in (3.85) and (3.89), can check that

$$\begin{aligned}\lim_{m\to\infty} \quad & \mathbb{E}\int_0^{\tau_n}\delta(s)\langle -\beta_1^\flat + B_1(\tilde{v}_m,\tilde{\phi}_m), A_1\rho_m\rangle ds\\ &= \lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n}\delta(s)\langle -\beta_1^\flat + B_1(v,\phi), A_1\rho_m\rangle ds\\ &+ \lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n}\delta(s)\langle -B_1(v,\phi) + B_1(\tilde{v}_m,\tilde{\phi}_m), A_1\rho_m\rangle ds = 0.\end{aligned}$$

(**v**). Let us also prove that

$$\lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n}\delta(s)\langle g_1^\flat(s) - g_1(s,\tilde{v}_m,\tilde{\phi}_m),\theta_m\rangle ds = 0.$$

From (3.81) and the fact that

$$\begin{aligned}&1|[0,\tau_n]\delta(t)(g_1^\flat(t) - g_1(t,v,\phi)) \in L^2(\Omega\times[0,T];V_1^*) \text{ and}\\ &1|[0,\tau_n]\delta(t)(g_1(t,v,\phi) - g_1(t,\tilde{v}_m,\tilde{\phi}_m)) \to 0 \text{ in } L^2(\Omega\times[0,T];V_1^*)\\ &\text{as } m\to\infty,\end{aligned}$$

we derive that

$$\begin{aligned}&\lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n}\delta(s)\langle g_1^\flat(s) - g_1(s,v,\phi),\theta_m\rangle ds = 0,\\ &\lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n}\delta(s)\langle g_1(s,v,\phi) - g_1(s,\tilde{v}_m,\tilde{\phi}_m),\theta_m\rangle ds = 0.\end{aligned} \tag{3.90}$$

Therefore, we derive that

$$\begin{aligned}\lim_{m\to\infty} \quad & \mathbb{E}\int_0^{\tau_n}\delta(s)\langle g_1^\flat(s) - g_1(s,\tilde{v}_m,\tilde{\phi}_m),\theta_m\rangle ds\\ &= \lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n}\delta(s)\langle g_1^\flat(s) - g_1(s,v,\phi),\theta_m\rangle ds\\ &+ \lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n}\delta(s)\langle g_1(s,v,\phi) - g_1(s,\tilde{v}_m,\tilde{\phi}_m),\theta_m\rangle ds = 0,\end{aligned}$$

The proof of the convergence of the other terms is similar.

Finally we conclude that the right side of (3.79) goes to 0 as m goes to $+\infty$.

Now using the fact that $1|[0,\tau_n]\delta(t)\le 1$, we derive from (3.79) that

$$\begin{aligned}&\lim_{m\to\infty}\mathbb{E}|(\theta_m,\psi_m)(\tau_n)|^2_{\mathcal{H}}\\ &\quad= \lim_{m\to\infty}\mathbb{E}\left(|(\tilde{v}_m,\tilde{\phi}_m)(\tau_n) - (v_m,\phi_m)(\tau_n)|^2_{\mathcal{H}}\right) = 0,\\ &\lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n}K_2(s)ds\\ &\quad= \lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n}\left(\|\theta_m\|^2 + (1-c\xi)\|\zeta_m\|^2 + \xi|A_1\rho_m|^2_{L^2}\right)ds\\ &\quad= \lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n}\left(\|\tilde{v}_m - v_m\|^2 + (1-c\xi)\|\tilde{\mu}_m - \mu_m\|^2\right)ds\\ &\quad+ \lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n}\xi|A_1(\tilde{\phi}_m - \phi_m)|^2_{L^2}ds = 0.\end{aligned} \tag{3.91}$$

□

We now give the proof of Proposition 3.4.

Proof of Proposition 3.4. Our goal is to prove that the following hold true.

$$\begin{aligned}
&\sigma(s,v,\phi,z)=\sigma^\flat(s,z) \text{ in } L^2(\Omega\times[0,T];L^2(Z,\nu,H_1)),\\
&B_0(v,v)=\beta_0^\flat \text{ in } L^2(\Omega\times[0,T];V_1^*),\\
&R_0(A_1\phi,\phi)=r_0^\flat \text{ in } L^2(\Omega\times[0,T];V_1^*),\\
&B_1(v,\phi)=\beta_1^\flat \text{ in } L^2(\Omega\times[0,T];V_2^*),\\
&g_1(t,v,\phi)=g_1^\flat(t) \text{ in } L^2(\Omega\times[0,T];V_1^*).
\end{aligned} \tag{3.92}$$

It is clear that $(3.92)_1$, follows from $(3.53)_6$.

To prove $(3.92)_2$, we proceed as follows. We note that from $(3.91)_2$ and (3.65), we also have

$$(v_m,\phi_m)|_{[0,\tau_n]}\to(v,\phi)|_{[0,\tau_n]} \text{ in } L^2(\Omega\times[0,T];\mathcal{U}).$$

Therefore, for any $w\in L^\infty(\Omega\times[0,T];V_1)$, we have

$$\begin{aligned}
&\mathbb{E}\int_0^{\tau_n}\langle B_0(v,v)-B_0(v_m,v_m),w\rangle ds\le\\
&c\|w\|_{L^\infty(\Omega\times[0,T];V_1)}\times\mathbb{E}\int_0^{\tau_n}\|v_m-v\|^{1/2}|v_m-v|_{L^2}^{1/2}(\|v\|+\|v_m\|)ds,
\end{aligned}$$

which gives

$$\lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n}\langle B_0(v,v)-B_0(v_m,v_m),w\rangle ds=0. \tag{3.93}$$

From $(3.60)_3$ and (3.93), we derive that

$$\mathbb{E}\int_0^{\tau_n}\langle B_0(v,v)-\beta_0^\flat,w\rangle=0,\ \forall w\in L^\infty(\Omega\times[0,T];V_1).$$

Since $\tau_n\uparrow T$ and $L^\infty(\Omega\times[0,T];V_1)$ is dense in $L^2(\Omega\times[0,T];V_1)$, we conclude that

$$B_0(v,v)=\beta_0^\flat \text{ in } L^2(\Omega\times[0,T];V_1^*).$$

This proves $(3.92)_2$.

To prove $(3.92)_3$, we note that

$$\begin{aligned}
&\mathbb{E}\int_0^{\tau_n}\langle R_0(A_1\phi,\phi)-R_0(A_1\phi_m,\phi_m),w\rangle ds\le\\
&c\|w\|_{L^\infty(\Omega\times[0,T];V_1)}\mathbb{E}\int_0^{\tau_n}|A_1(\phi_m-\phi)|_{L^2}^{1/2}\|\phi\|^{1/2}|A_1\phi|_{L^2}^{1/2}ds+\\
&c\|w\|_{L^\infty(\Omega\times[0,T];V_1)}\mathbb{E}\int_0^{\tau_n}\|\phi_m-\phi\|^{1/2}|A_1(\phi_m-\phi)|_{L^2}^{1/2}|A_1\phi_m|_{L^2}ds,
\end{aligned}$$

which gives

$$\lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n}\langle R_0(A_1\phi,\phi)-R_0(A_1\phi_m,\phi_m),w\rangle ds=0. \tag{3.94}$$

From $(3.60)_4$ and (3.94), we derive that

$$\mathbb{E}\int_0^{\tau_n}\langle R_0(A_1\phi,\phi)-r_0^\flat,w\rangle=0,\ \forall w\in L^2(\Omega\times[0,T];V_1),$$

which gives

$$R_0(A_1\phi,\phi)=r_0^\flat \text{ in } L^2(\Omega\times[0,T];V_1^*),$$

and $(3.92)_3$ is proved.

Similarly, we can prove that

$$B_1(v,\phi) = \beta_1^\flat \text{ in } L^2(\Omega \times [0,T]; V_2^*), \;\; g_1(t,v,\phi) = g_1^\flat(t) \text{ in } L^2(\Omega \times [0,T]; V_1^*). \quad \square$$

3.3. Convergence of the whole sequence of the Galerkin approximation. In this part, we prove the convergence of the whole sequence of the Galerkin approximation to the solution (v,ϕ) of (2.24).

Theorem 3.7. *The whole sequence of solutions to the Galerkin approximation* $\{(v_m,\phi_m);\; m \in \aleph\}$ *defined by (3.26) satisfies*

$$\begin{aligned} &\lim_{m\to\infty} \mathbb{E}|(v_m,\phi_m)(T-) - (v,\phi)(T-)|_{\mathcal{H}}^2 = 0, \\ &\lim_{m\to\infty} \mathbb{E}\|(v_m,\phi_m) - (v,\phi)\|_{\mathcal{U}}^2 = 0. \end{aligned} \tag{3.95}$$

Proof. For the proof, we first recall from [9, 16] the following lemma.

Lemma 3.8. *Let* $\{Q_m;\; m \geq 1\} \subset L^2(\Omega\times[0,T];\Re)$ *be a sequence of continuous real processes, and let* $\{\tau_n; n \geq 1\}$ *be a sequence of* $\mathfrak{F}_t-$*stopping times such that* $\tau_n \uparrow T$; $\sup_{m\geq 1} \mathbb{E}|Q_m(T)|^2 < \infty$, *and* $\lim_{m\to\infty} \mathbb{E}|Q_m(\tau_n)| = 0$, *for* $n \geq 1$. *Then* $\lim_{m\to\infty} \mathbb{E}|Q_m(T)| = 0$.

Applying Lemma 3.8 to $Q_m(t) = |(v,\phi) - (v_m,\phi_m)|_{\mathcal{H}}^2$ and $\delta_n = \tau_n$ and using (3.25), $(3.91)_1$ and the uniqueness of (v,ϕ), we conclude that the whole sequence given by (3.26) satisfies

$$\lim_{m\to\infty} \mathbb{E}|(v,\phi) - (v_m,\phi_m)|_{\mathcal{H}}^2 = 0, \;\; \forall t \in [0,T].$$

Similarly, applying Lemma 3.8 to $Q_m(t) = \displaystyle\int_0^t \|(v,\phi)(s) - (v_m,\phi_m)(s)\|_{\mathcal{U}}^2 ds$ and using (3.25), $(3.91)_2$, we conclude that the whole sequence (v_m,ϕ_m) converges to (v,ϕ) strongly in $L^2(\Omega \times [0,T];\mathcal{U})$, i.e.,

$$\lim_{m\to\infty} \mathbb{E} \int_0^t \|(v,\phi)(s) - (v_m,\phi_m)(s)\|_{\mathcal{U}}^2 ds = 0, \;\; \forall t \in [0,T].$$

$\square$

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