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NEW FILTERS FOR THE CALIBRATION OF REGIME SWITCHING BETA DYNAMICS

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Abstract. In this paper we consider the estimation problem for reduced-form models that link the real economy to financial markets. Estimation is based on extending the work of R.J. Elliott and V. Krishnamurthy, who derived new recursive filters to estimate parameters of a linear Gaussian, Kalman, filter. This paper extends those works to the calibration of a model for the beta of an industry - that is, the process describing the sensitivity of an industrial sector's returns to broad market movements. In fact, the dynamics for the beta of an industry are considered where the mean reversion level depends on the state of the economy and filtered estimates for these state-dependent mean reversion levels are used in a discrete time version of the beta dynamics. The beta process is estimated using the corresponding returns process, and a new recursive filter is developed to estimate the mean reversion levels of the beta process.

1. Introduction

The link between the real economy and financial markets is an issue that has long been of interest to financial economists [14, 4]. In general, capturing this link may be made through structural models or reduced-form models. Structural models, which involve using "deep" primitive parameters and decision making to connect stock returns to macroeconomic events, usually result in general equilibrium models that do very poorly in matching empirically observed asset pricing phenomena. Reduced-form models, which are exogenously specified systems of equations, perform well in fitting empirical phenomena, but usually result in latent factors related to risk premia that are difficult to estimate, especially in the presence of structural breaks or sudden policy regime shifts in the real economy.

In this paper we consider the estimation problem for reduced-form models based on extending the work of [9, 10] who derived new recursive filters to estimate parameters in the Kalman filter framework, where a linear Gaussian signal is observed in Gaussian noise. Some of these results were applied by [7] to commodity markets. This paper presents a further application and extension to estimate the mean reverting beta process of an industry or sector which is a well documented empirical stylized fact in financial markets. That is, beginning with [2], several

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authors (for example [3, 12]) have demonstrated that the dynamics of the conditional single factor market model beta for individual stocks and portfolios of stocks, while unobservable, is generally mean reverting. As beta measures systematic risk, it is important to distinguish between conditional time variation in beta and the regime shifts or structural changes in the parameters of the beta model, which represents a macroeconomic event. Thus, a novel feature of our methodology is that the long-run mean reversion level for the beta process is structured to depend on the"state" of the economy which, in turn, is captured by a separate hidden Markov model. In fact a preliminary filter is used to estimate the state of the economy and these estimates are passed into the beta dynamics which are observed and filtered through the equity returns process. In order to be able to calibrate our extended filter to data, we also derive *filter-based* EM algorithms for parameter estimation. The calibration procedure is recursive and the estimates are updated with each new piece of data, unlike methods based on straight maximum likelihood.

The paper proceeds in a sequence of steps which will now be described:

Section 2 describes the mean reverting dynamics of the beta process which is estimated from related equity returns. The discrete-time version of the beta dynamics is introduced, and the long-run mean of the process is allowed to vary based on a filtered estimate of the state of the economy. Section 2 also introduces the Kalman filter that is used to recursively estimate the beta process from the returns.

Section 3 describes how the state of the economy is modeled by GDP growth following a finite state regime switching Markov chain. The three states of the chain correspond to the economic state being 'high', 'medium' or 'low'. Related discrete time dynamics are described together with a filter which recursively estimates the state of the economy.

Section 4 describes the change of measure procedure which enables the models for both the beta process and the state of the economy to be constructed from sequences of i.i.d. $N(0,1)$ random variables. The change of measure procedure is then extended to change the parameters of the model.

Section 5 introduces our particular expectation maximization (EM) algorithm to estimate the parameters of the model. This is a recursive procedure which re-estimates the parameters until some stopping condition is satisfied. It uses first order conditions applied to the expected value of the log-likelihood.

Section 6 derives recursions for measures associated with various component processes which arise in the EM algorithms and establishes surprisingly simple explicit forms for the densities associated with these processes. These are then used to further obtain explicit expressions for the conditional expected values of the sums, in terms of the conditional mean μ_k and variance P_k, provided by the Kalman filter. Further, recursions for the coefficients in these sums are derived.

Section 7 finally provides the updated coefficients required in the EM algorithm. Section 8 concludes.

2. The Beta Process

2.1. Mean Reverting Beta Dynamics. To provide some background to our estimation problem, we note that in economics and finance, conditional single factor asset pricing models (such as the conditional CAPM) are used in determining the cost of equity capital for firms. This cost of capital is, in turn, used in a number of different ways. In common stock valuation, it is used to determine discount rates applied to the future streams of dividends. In the performance evaluation of portfolio management, it is used as risk-adjusted benchmarks. In event studies it is used to obtain normal expected rates of return, and in the regulation of utility companies it is used to set the required rates of return. The standard conditional single factor asset pricing representation is

$$R_t = \alpha_t + \beta_t R_{m,t} + \epsilon_t .$$

where R_t is the return on the equity of a firm, $R_{m,t}$ is the return on the market, β_t is a measure of the systematic or market risk of the firm's equity, and ϵ_t is random zero mean shock. The implementation of conditional single factor models thus requires an estimate of a time-varying beta (β) and as mentioned above, it has been demonstrated that the dynamics of the conditional single factor market model beta follows a mean reverting process [2, 3, 12]. We therefore suppose that the β process has dynamics in continuous time given by:

$$d\beta_t = \alpha(\mathcal{B}_t - \beta_t)dt + \sigma d\omega_t \,. \tag{2.1}$$

Here $\alpha > 0$, $\sigma > 0$,, and $\omega = (\omega_t,\, t \geq 0)$ is a standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$. For estimation and calibration, we consider discrete time version of (2.1). For $k = 1, 2, \ldots$, and with h as the size of the time interval

$$\beta_{k+1} - \beta_k = \alpha h(\mathcal{B}_k - \beta_k) + \sigma\sqrt{h}\,\omega_{k+1} \,.$$

or

$$\beta_{k+1} = (1 - \alpha h)\beta_k + \alpha\mathcal{B}_k h + \sigma\,\sqrt{h}\,\omega_{k+1} \tag{2.2}$$

where $\omega = (\omega_1, \omega_2, \ldots)$ is a sequence of i.i.d $N(0,1)$ random variables and the long-run mean $\mathcal{B}_k$ depends on the state of the economy in a manner that will be described in Section 3 below. Now β is not observed directly but through a price$S = (S_t\,, t \geq 0)$ or equivalently a returns process $R = (R_t\,, t \geq 0)$ of the form

$$S_t = S_{t-1} \exp{(R_t)} \,. \tag{2.3}$$

The discretized returns process R_k is related to β as

$$R_k = ln(S_k) - ln(S_{k-1}) = m\beta_k h + \gamma\sqrt{h}\,v_k \tag{2.4}$$

where, again, $v = (v_1, v_2, \ldots)$ is a sequence of i.i.d $N(0,1)$ random variables. In this expression, the variable m is usually referred to as the market risk premium.

2.2. Kalman Filter. To estimate the parameters of the model, we employ a Kalman filtering approach which requires that we first take the mean reverting beta process above and re-express it as a system of an "observation" process, y, and a "signal" process, x. We take the returns as the observation process and write

$$y_k := R_k = m\beta_k h + \gamma\sqrt{h}\,v_k \,.$$

We shall also write $x_k := \beta_k$ so the signal process x and observations y have joint dynamics

$$x_k = c_k + Qx_{k-1} + G\omega_k, \tag{2.5}$$

$$y_k = Mx_k + Hv_k \tag{2.6}$$

where from (2.2) and (2.4)

$$\begin{aligned} c_k &= \alpha\mathcal{B}_k h, \quad Q = (1-\alpha h), \quad G = \sigma\sqrt{h} \\ M &= mh, \qquad\quad H = \gamma\sqrt{h}\,. \end{aligned}$$

With the dynamics for x and y given by (2.5) and (2.6) we define the information filtrations

$$\begin{aligned} \mathcal{F}_k^x &= \sigma\{x_0, x_1, x_2, \ldots, x_k\} \\ \mathcal{Y}_k &= \sigma\{y_1, y_2, \ldots, y_k\} \quad \text{and} \\ \mathcal{G}_k &= \sigma\{x_0, x_1, \ldots, x_k, y_1, y_2, \ldots, y_k\}. \end{aligned}$$

We initially assume that the coefficients c_k, Q, G, M, and H of the model (2.5) and (2.6) have been estimated, and given $\mathcal{Y}_k$ we wish to recursively estimate x_k. The optimal mean square estimate of x_k given $\mathcal{Y}_k$ is then given by the Kalman filter which we now describe. It is shown in [6] how the Kalman filter is derived from the recursions in Section 6.1 below. However, we just state the Kalman filter estimates here without proof.

Write

$$\begin{aligned} \mu_k &= \mu_{k|k} = E[x_k|\mathcal{Y}_k] \\ P_k &= P_{k|k} = E[(x_k-\mu_k)^2|\mathcal{Y}_k] \\ \mu_{k|k-1} &= E[x_k|\mathcal{Y}_{k-1}] \\ P_{k|k-1} &= E[(x_k-\mu_{k|k-1})^2|\mathcal{Y}_{k-1}]. \end{aligned}$$

Then

$$\begin{aligned} \mu_{k|k-1} &= c_k + Q\mu_{k-1} \\ P_{k|k-1} &= \sigma^2 h + Q^2 P_{k-1}\,. \end{aligned}$$

Theorem 2.1 (Kalman Filter)**.** *The Kalman filter then gives the updating equations as:*

$$\begin{aligned} \mu_k &= \mu_{k|k-1} + K_k(y_k - M)\mu_{k|k-1} \\ P_k &= P_{k|k-1} - K_k M P_{k|k-1} \\ K_k &= P_{k|k-1} M (M^2 P_{k|k-1} + \gamma^2 h)^{-1}. \end{aligned}$$

K is called the Kalman gain.

3. The Real Economy and Long-Run Mean Beta

In this section, we describe how the long-run mean of the beta process, $\mathcal{B}_k$, is obtained through a linkage with the real economy. We suppose the "state" of the economy is represented by a hidden Markov model (HMM). Depending on the type of HMM used, the state may be modeled as structurally changing or as regime switching. For example a HMM whose hidden dynamics undergo random change points (the rate matrix of the chain changes at the random time) can be used to model structural change, while a classic continuous-time, finite-state, hidden

Markov chain can be used to model regime switching. For the rest of this paper we shall assume the state of the economy undergoes regime switching.

3.1. Regime Switching State of the Economy. We suppose the state of the economy is represented by a Markov chain Z. There are three states: High, medium, low. These states may describe various aspects of the economy that affect the risk in financial markets. For example, the states may describe economic production or they may describe consumer sentiment. Without loss of generality, these states are identified with the unit vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

so the chain has values in $\{e_1, e_2, e_3\}$.

We suppose there is some economic time series (see below) from which we can estimate Z. Suppose Z is defined on $(\Omega, \mathcal{F}, P)$ and, for time period k write

$$E[Z_k] = \begin{pmatrix} p_k^1 \\ p_k^2 \\ p_k^3 \end{pmatrix} = p_k \in R^3.$$

Here $p_k^i = P(Z_k = e_i)$, and we suppose the transition probabilities are $P(Z_{k+1} = e_j | Z_k = e_i) = \pi_{ji}$. Then, with $\Pi = (\pi_{ji}\, ,\ 1 \le i,\, j \le 3)$

$$Z_{k+1} = \Pi Z_k + M_{k+1}. \tag{3.1}$$

Here M_{k+1} is a martingale increment. That is, with $\mathcal{F}_k^z = \sigma\{Z_0, Z_1, \dots, Z_k\}$

$$E[M_{k+1} | \mathcal{F}_k^z] = E[Z_{k+1} | Z_k] = 0 \in R^3.$$

We suppose the state of the economy, Z is observed through the process $Y = (Y_1, Y_2, \dots)$ where, as an example, we take Y to be the growth rate, (first difference of the logarithm) of real GDP, such as in the [13] model of the business cycle. In fact, we suppose that for $k = 1, 2, \dots$

$$Y_k = \langle g, Z_k \rangle + \langle \delta, Z_k \rangle\, \nu_k. \tag{3.2}$$

Here, $\langle \cdot \rangle$ is the scalar product, $g = (g_1, g_2, g_3)$, $\delta = (\delta_1, \delta_2, \delta_3)$, and under the measure P, $\nu = (\nu_1, \nu_2, \nu_3, \cdots)$ is a sequence of i.i.d $N(0,1)$ random variables.

We will also write $\overline{\mathcal{G}}_k = \sigma\{Y_1, \dots, Y_k, Z_1, \dots, Z_k\}$ and $\overline{\mathcal{Y}_k} = \sigma\{Y_1, \dots, Y_k\}$, and $\phi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$ will denote the $N(0,1)$ density. Given $\overline{\mathcal{Y}_k}$, we desire to recursively obtain the estimate of Z_k

$$\widehat{Z}_k := E[Z_k | \overline{\mathcal{Y}}_k].$$

This estimate of the state of the economy is then used to select a long-run mean beta which is passed to the beta process as shown next.

3.2. The Long-Run Mean Beta. With $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)' \in R^3$, $\mathcal{B}_k = \langle \mathcal{B}, Z_k \rangle$ so the long-run mean for β depends on the state of the economy represented by Z. We shall suppose that Z_k is replaced by its filtered estimate $\widehat{Z}_k$ so β has dynamics:

$$\beta_{k+1} = (1 - \alpha h)\beta_k + \alpha \widehat{\mathcal{B}}_k h + \sigma \sqrt{h}\, \omega_{k+1}$$

where $\widehat{\mathcal{B}}_k = \langle \mathcal{B}, \widehat{Z}_k \rangle$. The set of coefficients or parameters of our model are therefore

$$\theta := \{\alpha, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, Q, G^2, M, H^2\}.$$

We therefore need to obtain the filtered estimate $\widehat{Z}_k$ and use that in the Kalman filter for x_k (equivalently β_k) to derive a filter-based EM algorithm for obtaining maximum parameter likelihood estimates of the set of coefficients θ This represents a new filtering method for the calibration of regime switching beta dynamics, one that depends on the state of the economy. The main technique employed here is the introduction of a reference probability measure that simplifies the derivations of filters.

4. Change of Measure

A basic framework used in the filtering and estimation results of [6], [9] and [7] is the reference probability. This is an initial probability $\overline{P}$ under which the signal x and observations y are themselves just sequences of $N(0,1)$ i.i.d random variables. A change of measure then gives our original framework under which x and y have the dynamics (2.5) and (2.6). Similarly, we can find a reference probability measure P^* under which Z is a Markov chain with transition probabilities given by Π in (3.1) above and observations Y are sequences of $N(0,1)$ i.i.d random variables. A change of measure then recovers the original probability P such that under which Z and Y have the dynamics (3.1) and (3.2).

4.1. Reference Probability for the Beta Process ($\overline{P}$). Suppose $\overline{P}$ is a a probability measure under which $x = (x_0, x_1, \dots)$ and $y = (y_1, y_2, \dots)$ are sequences of i.i.d. $N(0,1)$ random variables. Write $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $\psi(x) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$ for two copies of the $N(0,1)$ density.

Now consider the following random variables:

$$\lambda_0 = \frac{\psi\big(H^{-1}(y_0 - Mx_0)\big)}{H\psi(y_0)} \tag{4.1}$$

and for $\ell \geq 1$

$$\lambda_\ell = \frac{\psi\big(H^{-1}(y_\ell - Mx_\ell)\big)}{H\psi(y_\ell)} \cdot \frac{\phi\big(G^{-1}(x_\ell - Qx_{\ell-1} - c_\ell)\big)}{G\phi(x_\ell)} \,. \tag{4.2}$$

Write $\Lambda_k = \prod_{\ell=0}^{k} \lambda_\ell$. A new probability measure P (the "real world" probability) can be defined by setting

$$\left.\frac{dP}{d\overline{P}}\right|_{\mathcal{G}_k} = \Lambda_k \,. \tag{4.3}$$

Notation 4.1. Define

$$\begin{aligned} w_\ell &= G^{-1}(x_\ell - Q_{\ell-1} - c_\ell), & \ell &= 1, 2, \dots \\ v_\ell &= H^{-1}(y_\ell - Mx_\ell), & \ell &= 0, 1, 2, \dots \end{aligned}$$

Theorem 4.1. *Under P defined by* (4.3)

$$\begin{aligned} w &= (w_1, w_2, w_3, \dots) \\ \textit{and} \quad v &= (v_0, v_1, v_2, \dots) \end{aligned}$$

are sequences of i.i.d. $N(0,1)$ *random variables. That is, under P*

$$\begin{aligned} x_\ell &= c_\ell + Qx_{\ell-1} + Gw_\ell \\ \textit{and} \quad y_\ell &= Mx_\ell + Hv_\ell\,. \end{aligned}$$

So under P, x and y do have the 'real world' dynamics (2.5) and (2.6). The proof follows from Lemma 3.2 of of [10]. However, as mentioned above, due to the independence of x and y, $\overline{P}$ is an easier measure under which to work.

4.2. Reference Probability for the Real Economy(P^*). We now describe the change of measure procedure for the state of the economy process Z. Suppose that there is another probability measure P^* such that under P^*, Z is a Markov chain with transition densities as above and $(Y_1, Y_2, \dots)$ is a sequence of i.i.d. $N(0,1)$ random variables.

For $\ell = 1, 2, \dots$, write

$$\xi_\ell = \frac{\phi\big(\langle \delta, Z_\ell\rangle^{-1}\big(Y_\ell - \langle g, Z_\ell\rangle\big)\big)}{\langle \delta, Z_\ell\rangle\, \phi(Y_\ell)}$$

and $\mathcal{D}_k = \prod_{\ell=0}^{k} \xi_\ell$. A new probability measure P (the "real world" probability) can be defined by setting

$$\left.\frac{dP}{dP^*}\right|_{\mathcal{G}_k} = \mathcal{D}_k\,. \tag{4.4}$$

By applying a similar approach to Theorem 4.1 above, it can be shown that under P,

$$\nu_k = \frac{Y_k - \langle g, Z_k\rangle}{\langle \delta, Z_k\rangle}$$

is a sequence of i.i.d $N(0,1)$ random variables. That is, under P, $Y_k = \langle g, Z_k\rangle + \langle \delta, Z_k\rangle\, \nu_k$ as in (3.2).

4.3. Filtering the State of the Economy (Z). Using the reference probability P^*, we wish to find a recursion for $\widehat{Z}_k := E[Z_k|\overline{\mathcal{Y}}_k]$. Write

$$\widehat{p}_k := E[Z_k|\overline{\mathcal{Y}}_k] = \big(\widehat{p}_k^1, \widehat{p}_k^2, \widehat{p}_k^3\big)'.$$

As it is easier to work under P^*, we have by an abstract form of Bayes' Theorem known as the Kallianpur-Striebel formula [6],

$$E[Z_k|\overline{\mathcal{Y}}_k] = \frac{E^*[\mathcal{D}_k Z_k|\overline{\mathcal{Y}_k}]}{E^*[\mathcal{D}_k|\overline{\mathcal{Y}_k}]}$$

where E^* denotes the expectation under P^*.

Write

$$\begin{aligned} q_k &= E^*\big[\mathcal{D}_k Z_k | \overline{\mathcal{Y}_k}\big] \\ &= \sum_{i=1}^{3} E^*\Big[\mathcal{D}_{k-1} \cdot \frac{\phi\big(\frac{Y_k - g_i}{\delta_i}\big)}{\delta_i \phi(Y_k)} \cdot \langle Z_k, e_i\rangle \,|\overline{\mathcal{Y}_k}\Big] e_i \\ &= \sum_{i=1}^{3} E^*\Big[\frac{\phi\big(\frac{Y_k - g_i}{\delta_i}\big)}{\delta_i \phi(Y_k)} \cdot \langle \mathcal{D}_{k-1} \Pi Z_{k-1}, e_i\rangle \,\overline{\mathcal{Y}_k}\Big] e_i \end{aligned}$$

With $\Phi(Y_k)$ the diagonal matrix with entries $\frac{\phi\big(\frac{Y_k - g_i}{\delta_i}\big)}{\delta_i \phi(Y_k)}$, this gives the recursion

$$q_k = \Phi(Y_k)\Pi q_{k-1} \tag{4.5}$$

Now with $\underline{1} = (1, 1, \dots 1)$

$$\langle Z_k, \underline{1}\rangle = 1$$

so

$$\begin{aligned} E^*\big[\mathcal{D}_k | \overline{\mathcal{Y}_k}\big] &= \langle E^*\big[\mathcal{D}_k Z_k | \overline{\mathcal{Y}_k}\big], \underline{1}\rangle \\ &= \langle q_k, \underline{1}\rangle \end{aligned}$$

Therefore,

$$\widehat{p}_k = \frac{q_k}{\langle q_k, \underline{1}\rangle} \in R^3$$

and q_k is updated in each period by (4.5).

5. Estimation of the Parameters

In this section, we show how the parameters of our model can be estimated using the expectation maximization (EM) algorithm. The EM algorithm was introduced by [5]. Our main focus is on the estimation of the parameters of the mean reverting beta process which has as part of its inputs, the filtered estimate $\widehat{Z}_k$. This filtered estimate will itself require estimation of the parameters g and δ in the state of the economy model (equations 3.1 and 3.2), but we do not address that here, as methods for their estimation can be found in [10].

The *filter-based* EM algorithm estimation methods that we develop, will rely on sequentially changing the parameters from one set of coefficients to another by changing the probability from one measure to another.

5.1. Change of Parameters. We have noted that the coefficients, or parameters, of our model are

$$\theta := \{\alpha, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, Q, G^2, M, H^2\}.$$

Suppose a second set of parameters is:

$$\overline{\theta} := \{\overline{\alpha}, \overline{\mathcal{B}}_1, \overline{\mathcal{B}}_2, \overline{\mathcal{B}}_3, \overline{Q}, \overline{G}^{\,2}, \overline{M}, \overline{H}^{2}\}.$$

Theorem 4.1 shows how to change the parameters from $\{0, 0, 0, 0, 1, 0, 1\}$ to θ. A similar argument proves the following result.

Notation 5.1. Write

$$\rho_0 = \frac{\overline{H}}{H} \cdot \frac{\psi\big(H^{-1}(y_0 - Mx_0)\big)}{\psi\big(\overline{H}^{-1}(y_0 - \overline{M}x_0)\big)}$$

for $\ell \geq 1$

$$\rho_\ell = \frac{\overline{H}}{H}\, \frac{\psi\big(H^{-1}(y_\ell - Mx_\ell)\big)}{\psi\big(\,\overline{H}^{\,-1}(y_\ell - \overline{M}x_\ell)\big)} \cdot \frac{\overline{G}}{G}\, \frac{\phi\big(G^{-1}(x_\ell - Qx_{\ell-1} - c_\ell)\big)}{\phi\big(\,\overline{G}^{\,-1}(x_\ell - \overline{Q}x_{\ell-1} - \overline{c}_\ell)\big)}\,.$$

Theorem 5.1. *Suppose under the probability measure $P_{\overline{\theta}}$ that x and y have dynamics*

$$x_k = \overline{c}_k + \overline{Q}_{k-1} + \overline{G}w_k \tag{5.1}$$

$$y_k = \overline{M}x_k + \overline{H}v_k\,. \tag{5.2}$$

Let

$$\Lambda_k(\overline{\theta}, \theta) = \prod_{\ell=0}^{k} \rho_\ell\,.$$

A new probability measure P_θ can be defined by setting

$$\frac{dP_\theta}{dP_{\overline{\theta}}}\Big|_{\mathcal{G}_k} = \Lambda_k(\,\overline{\theta}, \theta). \tag{5.3}$$

Then under P_θ the x and y have dynamics

$$x_k = c_k + Qx_{k-1} + Gw_k \tag{5.4}$$

$$y_k = Mx_k + Hv_k\,. \tag{5.5}$$

That is, changing the measure from $P_{\overline{\theta}}$ to P_θ using the density (5.3) *changes the dynamics from* (5.1), (5.2) *to* (5.4), (5.5).

5.2. Coefficient Updates. A finite-dimensional recursive EM algorithm will now be developed for estimating the coefficients, θ, or parameters of our model. This algorithm is recursive in the sense that the estimates at time period k are derived from the estimates at time period $(k-1)$ plus an algorithmically determined correction which is based on the new information at time period k. Essentially, at time period k, starting with an initial parameter estimate θ_0, which comes from time period $(k-1)$, the algorithm utilizes Theorem 5.1 above to generate a sequence of parameter estimates that converges to the most likely values of the parameter set θ, given the new information at time period k.

Suppose at time period k, that after j iterations, a parameter set θ_j has been obtained. This parameter set θ_j then gives a probability measure P_j defined by (4.3) relative to the reference probability measure. Consider another parameter set, which we denote by

$$\theta := \{\alpha, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, Q, G^2, M, H^2\}.$$

To change the parameters from the set θ_j to θ we use the density given in (5.3).

$$\Lambda_k(\theta_j, \theta) = \prod_{\ell=0}^{k} \rho_\ell\,. \tag{5.6}$$

The EM algorithm (see [10]) then suggests we look for the parameter set θ which maximizes the expected likelihood

$$E_{\theta_j}[\Lambda_k(\theta_j,\theta)|\mathcal{Y}_k]$$

or, equivalently, maximizes the expected log-likelihood, which from the densities $\phi(x)$ and $\psi(x)$ defined in Section 4.1, we get to be:

$$\begin{aligned} E_{\theta_j}[\log\Lambda_k(\theta_j,\theta)|\mathcal{Y}_k] = &-(k+1)\log H - k\log G \\ &- \frac{G^{-2}}{2} E_{\theta_j}\Big[\sum_{\ell=1}^{k}(x_\ell - Qx_{\ell-1} - c_\ell)^2|\mathcal{Y}_k\Big] \\ &- \frac{H^{-2}}{2} E_{\theta_j}\Big[\sum_{\ell=0}^{k}(y_\ell - Mx_\ell)^2|\mathcal{Y}_k\Big] + \Theta_j\,. \end{aligned} \tag{5.7}$$

Here Θ_j represents all terms not depending on θ, and where we recall that

$$c_\ell = \alpha\widehat{\mathcal{B}}_\ell h = \alpha h(\mathcal{B}_1\widehat{p}^{\,1}_\ell + \mathcal{B}_2\widehat{p}^{\,2}_\ell + \mathcal{B}_3\widehat{p}^{\,3}_\ell).$$

Notation 5.2. Write

$$\begin{aligned} H^0_k &= \sum_{\ell=0}^{k} x_\ell^2 & H^1_k &= \sum_{\ell=1}^{k} x_\ell x_{\ell-1} \\ H^2_k &= \sum_{\ell=1}^{k} x_{\ell-1}^2 & H^3_k &= \sum_{\ell=1}^{k} x_\ell^2 \\ L^1_k &= \sum_{\ell=1}^{k} c_\ell x_{\ell-1} & L^2_k &= \sum_{\ell=1}^{k} c_\ell x_\ell \\ J_k &= \sum_{\ell=0}^{k} y_\ell x_\ell & & \\ Y_k &= \sum_{\ell=0}^{k} y_\ell^2 & C_k &= \sum_{\ell=1}^{k} c_\ell^2\,. \end{aligned}$$

Then, for $n = 0,1,2,3$ denote $\widehat{H}^{\,n}_k = E_{\theta_j}[H^n_k|\mathcal{Y}_k]$, for $n = 1,2$ denote $\widehat{L}^{\,n}_k = E_{\theta_j}[L^n_k|\mathcal{Y}_k]$ and $\widehat{J}_k = E_{\theta_j}[J_k|\mathcal{Y}_k]$ as the corresponding filters for the processes defined in Notation 5.2 above. Consequently, from (5.7), the expected log-likelihood can be written in terms of the filters $\widehat{H}\,\widehat{L}^{\,n}_k$, and $\widehat{J}_k$, and the component processes Y_k, and C_k as

$$\begin{aligned} \Gamma_k(\theta) := &-(k+1)\log H - k\log G \\ &- \frac{G^{-2}}{2}\big[\widehat{H}^{\,3}_k + Q^2\widehat{H}^{\,2}_k + C_k - 2Q\widehat{H}^{\,1}_k - 2\widehat{L}^{\,2}_k + 2Q\widehat{L}^{\,1}_k\big] \\ &- \frac{H^{-2}}{2}\big[Y_k + M^2\widehat{H}^{\,0}_k - 2M\widehat{J}_k\big] + \Theta_j\,. \end{aligned}$$

Theorem 5.2. *Taking first order conditions of this expected log-likelihood we have the following updated (EM) parameter estimates:*

$$
\begin{aligned}
G^2 &= \frac{1}{k}\left[\widehat{H}^{\,3}_{k} + Q^2\widehat{H}^{\,2}_{k} + C_k - 2Q\widehat{H}^{\,1}_{k} - 2\widehat{L}^{\,2}_{k} + 2Q\widehat{L}^{\,1}_{k}\right]\\
H^2 &= \frac{1}{k+1}\left[Y_k + M^2\widehat{H}^{\,0}_{k} - 2M\widehat{J}_k\right]\\
Q &= \left(\widehat{H}^{\,2}_{k}\right)^{-1}\left(\widehat{L}^{\,1}_{k} - \widehat{H}^{\,1}_{k}\right)\\
M &= \left(\widehat{H}^{\,0}_{k}\right)^{-1}\widehat{J}_k\,.
\end{aligned}
$$

Theorem 5.2 provides updates for $G^2 = \sigma^2 h,\quad H^2 = \gamma^2 h,\ Q = (1-\alpha h)$ and $M = ch$. Here h is the time step, which is known, so these expressions provide updates for the parameters $\sigma^2, \gamma^2, \alpha$ and c of the original model. However, the updates for $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ require further calculations. Note that since

$$c_\ell = \alpha h\big(\mathcal{B}_1\widehat{p}^{\,1}_{\ell} + \mathcal{B}_2\widehat{p}^{\,2}_{\ell} + \mathcal{B}_3\widehat{p}^{\,3}_{\ell}\big)$$

then

$$C_k = \sum_{\ell=1}^{k} c_\ell^2 = \alpha^2 h^2\Big(\sum_{\ell=1}^{k}\big(\sum_{i=1}^{3}\mathcal{B}_i^2(\widehat{p}^{\,i}_{\ell})^2 + 2\sum_{\substack{i,\,j\\ i<j}}\mathcal{B}_i\mathcal{B}_j\widehat{p}^{\,i}_{\ell}\widehat{p}^{\,j}_{\ell}\big)\Big)$$

Notation 5.3. Write for $1 \le i,\, j \le 3$:

$$
\begin{aligned}
\mathcal{P}^{i,j}_k &= \sum_{\ell=1}^{k}\widehat{p}^{\,i}_{\ell}\widehat{p}^{\,j}_{\ell}\,,\\
\Delta^1_k &= \alpha h\sum_{\ell=1}^{k}\big(\mathcal{B}_1\widehat{p}^{\,1}_{\ell} + \mathcal{B}_2\widehat{p}^{\,2}_{\ell} + \mathcal{B}_3\widehat{p}^{\,3}_{\ell}\big)x_{\ell-1}\\
\Delta^2_k &= \alpha h\sum_{\ell=1}^{k}\big(\mathcal{B}_1\widehat{p}^{\,1}_{\ell} + \mathcal{B}_2\widehat{p}^{\,2}_{\ell} + \mathcal{B}_3\widehat{p}^{\,2}_{\ell}\big)x_{\ell}\\
\Delta^{1i}_k &= \sum_{\ell=1}^{k}\widehat{p}^{\,i}_{\ell}x_{\ell-1} \qquad \Delta^{2i}_k = \sum_{\ell=1}^{k}\widehat{p}^{\,i}_{\ell}x_{\ell}\\
\widehat{\Delta}^{\,1i}_{k} &= E_{\theta_j}\big[\Delta^{1i}_k|\mathcal{Y}_k\big], \quad \widehat{\Delta}^{\,2i}_{k} = E_{\theta_j}\big[\Delta^{2i}_k|\mathcal{Y}_k\big].
\end{aligned}
$$

Theorem 5.3. *Taking first order conditions with respect to $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ of the expected log-likelihood* (5.7) *we have the updated (EM) estimates for $\mathcal{B}_1, \mathcal{B}_2$ and $\mathcal{B}_3$ are given as solutions of the linear system:*

$$
\begin{aligned}
\mathcal{P}^{11}_k\mathcal{B}_1 + \mathcal{P}^{12}_k\mathcal{B}_2 + \mathcal{P}^{13}_k\mathcal{B}_3 &= (\alpha h)^{-1}\big(\widehat{\Delta}^{\,21}_{k} - Q\widehat{\Delta}^{\,11}_{k}\big)\\
\mathcal{P}^{21}_k\mathcal{B}_1 + \mathcal{P}^{22}_k\mathcal{B}_2 + \mathcal{P}^{23}_k\mathcal{B}_3 &= (\alpha h)^{-1}\big(\widehat{\Delta}^{\,22}_{k} - Q\widehat{\Delta}^{\,12}_{k}\big)\\
\mathcal{P}^{31}_k\mathcal{B}_1 + \mathcal{P}^{32}_k\mathcal{B}_2 + \mathcal{P}^{33}_k\mathcal{B}_3 &= (\alpha h)^{-1}\big(\widehat{\Delta}^{\,23}_{k} - Q\widehat{\Delta}^{\,13}_{k}\big).
\end{aligned}
$$

Writing $\mathcal{P}_k$ for the matrix ($\mathcal{P}^{ij}_k$, $1 \le i,\, j \le k$),

$$\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3),$$

and $\widehat{\Delta}_k$ for the vector on the right, the new estimate for $\mathcal{B}$ is given by:

$$\mathcal{P}_k\mathcal{B} = (\alpha h)^{-1}\widehat{\Delta}_k\,.$$

Theorem 5.3 provides updates for $\mathcal{B}_1$,$\mathcal{B}_2$ and $\mathcal{B}_3$. The interesting feature of this theorem is that it demonstrates the informational interdependence between financial markets and the real economy required by our sequentially updated EM algorithm parameter estimates. Write θ_{j+1} for the revised set of parameters provided by Theorems 5.2 and 5.3. As we shall demonstrate below, the filters $\widehat{H}^0_k$, $\widehat{H}^1_k$, $\widehat{H}^2_k$, $\widehat{H}^3_k$, $\widehat{L}^1_k$, $\widehat{L}^2_k$, $\widehat{J}_k$, and $\widehat{\Delta}_k$ in the theorems require μ_k and P_k from the Kalman filter updating equations (Theorem 2.1). Once the revised θ_{j+1} is computed, new values for μ_k and P_k in the Kalman filter can then be obtained using the revised θ_{j+1} parameters. This gives rise to a recursive estimation procedure which can be stopped when the total difference of the parameters at two successive steps is sufficiently small.

6. Related Measures

A key objective of this paper is to derive an EM algorithm that is *filter-based* and relatively simple to apply. In order to do this, we need recursive expressions for certain measures associated with the processes H^0, H^1, H^2, H^3, L^1, L^2, J, Δ^{1i} and Δ^{2i}. We first define these measures, then we provide a theorem to show their recursive representations, and then we show how, given our assumption of Gaussian error terms for the signal and observations processes, these recursive representations can be characterized in terms of a relatively small number of simple expressions that are straight forward to be numerically implemented.

6.1. Recursive Densities. For any bounded measurable test function g suppose there are measures (or densities) α_k, $\beta^{(n)}_k$, $\lambda^{(m)}_k$, γ_k and $\delta^{(i)j}_k$ such that:

$$\begin{aligned}
\overline{E}[\Lambda_k g(x_k)|\mathcal{Y}_k] &= \int_{-\infty}^{\infty} \alpha_k(x)g(x)dx\\
\overline{E}[\Lambda_k H^{(n)}_k g(x_k)|\mathcal{Y}_k] &= \int_{-\infty}^{\infty} \beta^{(n)}_k(x)g(x)dx, \quad n=1,2,\\
\overline{E}[\Lambda_k L^{(m)}_k g(x_k)|\mathcal{Y}_k] &= \int_{-\infty}^{\infty} \lambda^{(m)}_k(x)g(x)dx, \quad m=1,2,\\
\overline{E}[\Lambda_k J_k g(x_k)|\mathcal{Y}_k] &= \int_{-\infty}^{\infty} \gamma_k(x)g(x)dx,\\
\overline{E}[\Lambda_k \Delta^{(i)j}_k g(x_k)|\mathcal{Y}_k] &= \int_{-\infty}^{\infty} \delta^{(i)j}_k(x)g(x)dx, \quad i=1,2,\ j=1,2,3.
\end{aligned}$$

Theorem 6.1. *These measures satisfy the following recursions:*

$$\alpha_k(x) = \frac{\psi\big(H^{-1}(y_k - Mx)\big)}{GH\psi(y_k)} \int_{-\infty}^{\infty} \phi\big(G^1(x - Qz - c_k)\big)\alpha_{k-1}(z)dz \tag{6.1}$$

$$\beta_k^0(x) = \frac{\psi\big(H^{-1}(y_k - Mx)\big)}{GH\psi(y_k)} \int_{-\infty}^{\infty} \phi\big(G^{-1}(x - Qz - c_k)\big)\beta_{k-1}^0(z)dz + x^2\alpha_k(x). \tag{6.2}$$

$$\beta_k^1(x) = \frac{\psi\big(H^{-1}(y_k - Mx)\big)}{GH\psi(y_k)} \Big[\int_{-\infty}^{\infty} \phi\big(G^{-1}(x - Qz - c_k)\big)\beta_{k-1}^1(z)dz + x\int_{-\infty}^{\infty} \phi\big(G^{-1}(x - Qz - c_k)\big)z\alpha_{k-1}(z)dz\Big] \tag{6.3}$$

$$\beta_k^2(x) = \frac{\psi\big(H^{-1}(y_k - Mx)\big)}{GH\psi(y_k)} \Big[\int_{-\infty}^{\infty} \phi\big(G^{-1}(x - Qz - c_k)\big)\beta_{k-1}^2(z)dz + \int_{-\infty}^{\infty} \phi\big(G^{-1}(x - Qz - c_k)\big)z^2\alpha_{k-1}(z)dz\Big] \tag{6.4}$$

$$\beta_k^3(x) = \frac{\psi\big(H^{-1}(y_k - Mx)\big)}{GH\psi(y_k)} \int_{-\infty}^{\infty} \phi\big(G^{-1}(x - Qz - c_k)\big)\beta_{k-1}^3(z)dz + x^2\alpha_k(x) \tag{6.5}$$

$$\lambda_k^1(x) = \frac{\psi\big(H^{-1}(y_k - Mx)\big)}{GH\psi(y_k)} \Big[\int_{-\infty}^{\infty} \phi\Big(G^{-1}(x - Qz - c_k)\Big)\lambda_{k-1}^1(z)dz + c_k\int_{-\infty}^{\infty} \phi\big(G^{-1}(x - Qz - c_k)\big)z\alpha_{k-1}(z)dz\Big] \tag{6.6}$$

$$\lambda_k^2(x) = \frac{\psi\big(H^{-1}(y_k - Mx)\big)}{GH\psi(y_k)} \int_{-\infty}^{\infty} \phi\big(G^{-1}(x - Qz - c_k)\big)\lambda_{k-1}^2(z)dz + c_k x\alpha_k(x) \tag{6.7}$$

$$\gamma_k(x) = \frac{\psi\big(H^{-1}(y_k - Mx)\big)}{GH\psi(y_k)} \int_{-\infty}^{\infty} \phi\big(G^{-1}(x - Qz - c_k)\big)\gamma_{k-1}(z)dz + y_k x\alpha_k(x) \tag{6.8}$$

$$\delta_k^{1j}(x) = \frac{\psi\big(H^{-1}(y_k - Mx)\big)}{GH\psi(y_k)} \Big[\int_{-\infty}^{\infty} \phi\big(G^{-1}(x - Qz - c_k)\big)\delta_{k-1}^{1j}(z)dz + \widehat{p}_k^j\int_{-\infty}^{\infty} \phi\big(G^{-1}(x - Qz - c_k)\big)z\alpha_{k-1}(z)dz\Big] \tag{6.9}$$

$$\delta_k^{2j}(x) = \frac{\psi\big(H^{-1}(y_k - Mx)\big)}{GH\psi(y_k)} \int_{-\infty}^{\infty} \phi\big(G^{-1}(x - Qz - c_k)\big)\delta_{k-1}^{2j}(z)dz + \widehat{p}_k^j x\alpha_k(x). \tag{6.10}$$

Proof. We shall prove the recursions (6.1) and (6.2). The other results are similar. Now $\alpha_k(x)$ is defined by:

$$\overline{E}[\Lambda_k g(x_k)|\mathcal{Y}_k] = \int_{-\infty}^{\infty} \alpha_k(x) g(x) dx$$

$$= \overline{E}\Big[\Lambda_{k-1} \frac{\psi\big(H^{-1}(y_k - Mx_k)\big)}{H\psi(y_k)} \cdot \frac{\phi\big(G^{-1}(x_k - Qx_{k-1} - c_k)\big)}{G\phi(x_k)}\, g(x_k)|\mathcal{Y}_k\Big].$$

Under $\overline{P}$ all the x_k and y_k are $N(0,1)$ and i.i.d. so this is

$$= \overline{E}\Big[\Lambda_{k-1} \int_{-\infty}^{\infty} \frac{1}{HG\psi(y_k)}\, \psi\big(H^{-1}(y_k - Mx)\big) \times \phi\big(G^{-1}(x - Qx_{k-1} - c_k)\big) g(x) dx|\mathcal{Y}_{k-1}\Big]$$

$$= \frac{1}{HG\psi(y_k)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha_{k-1}(z)\psi\big(H^{-1}(y_k - Mx)\big)\phi\big(G^{-1}(x - Qz - c_k)\big) g(x) dx dz.$$

As g is arbitrary,

$$\alpha_k(x) = \frac{\psi(y_k - Mx)}{HG\psi(y_k)} \int_{-\infty}^{\infty} \alpha_{k-1}(z)\phi(x - Qz - c_k) dz.$$

□

Proof of (6.2). With $H_k^0 = \sum_{\ell=0}^{k} x_\ell^2$ the measure β_k^0 is defined by

$$\begin{aligned}
&\overline{E}[\Lambda_k H_k^0 g(x_k)|\mathcal{Y}_k]\\
&\quad = \int_{-\infty}^{\infty} \beta_k^0(x) g(x) dx\\
&\quad = \frac{1}{H\psi(y_k)}\, \overline{E}\Big[\Lambda_{k-1} H_{k-1}^0\, \psi\big(H^{-1}(y_k - Mx_k)\big)\, \frac{\phi\big(G^{-1}(x_k - Qx_{k-1} - c_k)\big)}{G\phi(x_k)} g(x_k)\\
&\qquad + \Lambda_{k-1}\, x_k^2 \psi\big(H^{-1}(y_k - Mx_k)\big) \frac{\phi\big(G^{-1}(x_k - Qx_{k-1} - c_k)\big)}{G\phi(x_k)} g(x_k) \quad |\mathcal{Y}_{k-1}\Big]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{HG\psi(y_k)}\, \overline{E}\Big[\Lambda_{k-1} H_{k-1}^0 \int_{-\infty}^{\infty} \psi\big(H^{-1}(y_k - Mx)\big)\phi\big(G^{-1}(x - Qx_{k-1} - c_k)\big) g(x) dx\\
&\qquad + \Lambda_{k-1} \int_{-\infty}^{\infty} x^2 \psi\big(H^{-1}(y_k - Mx)\big)\phi\big(G^{-1}(x - Qx_{k-1} - c_k)\big) g(x) dx \quad |\mathcal{Y}_{k-1}\Big]\\
&= \frac{1}{HG\psi(y_k)}\, \Big[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \beta_{k-1}^0(z)\psi\big(H^{-1}(y_k - Mx)\big)\phi\big(G^{-1}(x - Qz - c_k)\big) g(x) dx dz\\
&\qquad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha_{k-1}(z) x^2 \psi\big(H^{-1}(y_k - Mx)\big)\phi\big(G^{-1}(x - Qz - c_k)\big) g(x) dx dz\Big].
\end{aligned}$$

As g is arbitrary this implies

$$\beta_k^0(x) = \frac{\psi\big(H^{-1}(y_k - Mx)\big)}{HG\psi(y_k)} \Big[\int_{-\infty}^{\infty} \beta_{k-1}^0(z)\phi\big(G^{-1}(x - Qz - c_k)\big)dz + x^2 \int_{-\infty}^{\infty} \alpha_{k-1}(z)\phi\big(G^{-1}(x - Qz - c_k)\big)dz\Big].$$

Remark 6.2. These recursions have initial values: $\alpha_0(x)$ is the initial distribution of $x_0 = \beta_0$, which we assume to be Gaussian.

$$\beta_0^0(x) = x^2\alpha_0(x), \quad \beta_0^1(x) = 0, \quad \beta_0^2(x) = 0,$$

$$\beta_0^3(x) = 0, \quad \lambda_0^1(x) = 0, \quad \lambda_0^2(x) = 0, \quad \gamma_0(x) = xy\alpha_0(x),$$

$$\delta_0^{1j}(x) = 0, \qquad \delta_0^{2j}(x) = 0, \quad j = 1, 2, 3.$$

The recursive representations (6.1) to (6.10) use the Gaussian densities ϕ and ψ but, in general, are not themselves Gaussian. However, if, as we have assumed in Remark 6.2 above, $\alpha_0(x)$ is Gaussian then the recursion (6.1) gives rise to a recursive family of Gaussian densities $\alpha_k(x)$ whose mean and variance provide the Kalman filter as stated in Section 2.2. See [6]. For the other densities we derive explicit finite-dimensional expressions for them in the next section.

6.2. Explicit Expressions for Densities. The interesting fact, first described in [9] and [7], is that explicit expressions can now be found for the expected quantities required in the parameter updates of Theorems 5.2 and 5.2, and these expressions have very simple updates in terms of the conditional mean μ_k and variance P_k provided by the Kalman filter of Section 2.2.
Write:

$$\sigma_k = P_{k-1}^{-1} + Q^2G^{-2}$$
$$\Sigma_k = G^{-2}\sigma_k^{-1}Q$$
$$S_k = \sigma_{k+1}^{-1}\big(P_k^{-1}\mu_k - QG^{-2}c_k\big).$$

We first note

Theorem 6.3. *There are real constants $a_k^{(n)}, b_k^{(n)}, d_k^{(n)}$ such that for $k = 0, 1, 2, \ldots$ the measure $\beta_k^{(n)}$ has the form:*

$$\beta_k^{(n)}(x) = \big(a_k^{(n)} + b_k^{(n)}x + d_k^{(n)}x^2\big)\alpha_k(x), \quad n = 0, 1, 2, 3.$$

Proof. Consider the case $n = 0$. Then from (6.2)

$$\beta_k^{(0)}(x) = \frac{\psi\big(H^{-1}(y_k - M_k)\big)}{GH\psi(y_k)} \int_{-\infty}^{\infty} \phi\big(G^{-1}(x - Qz - c_k)\big)\beta_{k-1}^{(0)}(z)dz + x^2\alpha_k(x).$$

We prove the result by induction. When $k = 0$ $\beta_0^{(0)}(x) = x^2\alpha_0(x)$ so the result holds for $k = 0$.

Suppose the result is true up to $k = 1$. Then

$$\beta_k^{(0)}(x) = \frac{\psi\big(H^{-1}(y_k - Mx)\big)}{GH\psi(y_k)} \times \int_{-\infty}^{\infty} \phi\big(G^{-1}(x - Qz - c_k)\big)\big(a_{k-1}^{(0)} + b_{k-1}^{(0)} z + d_{k-1}^{(0)} z^2\big)\alpha_{k-1}(z)dz.$$

Now recall that $\psi(z)$ is a $N(0,1)$ density and, up to normalization, α_{k-1} is $N(\mu_{k-1}\,, P_{k-1})$. After completing the square, the integrals can be evaluated and we find that

$$\beta_k^{(0)}(x) = \big(a_k^{(0)} + b_k^{(0)} x + d_k^{(0)} x^2\big)\alpha_k(x)$$

where

$$\begin{aligned}
a_k^{(0)} &= a_{k-1}^{(0)} + b_{k-1}^{(0)} S_{k-1} + d_{k-1}^{(0)} \sigma_k^{-1} + d_{k-1}^{(0)} S_{k-1}^2\,, & a_0^{(0)} &= 0\\
b_k^{(0)} &= \Sigma_k\big(b_{k-1}^{(0)} + 2d_{k-1}^{(0)} S_{k-1}\big), & b_0^{(0)} &= 0,\\
d_k^{(0)} &= \Sigma_k^2 d_{k-1}^{(0)} + 1, & d_0^0 &= 1.
\end{aligned}$$

The proofs for $\beta^{(n)}$, $n = 1,2,3$ are similar but the recursions are:

$$\begin{aligned}
a_k^{(1)} &= a_{k-1}^{(1)} + b_{k-1}^{(1)} S_{k-1} + d_{k-1}^{(1)} \sigma_k^{-1} + d_{k-1}^{(1)} S_{k-1}^2\,, & a_0^{(1)} &= 0\\
b_k^{(1)} &= \Sigma_k\big(b_{k-1}^{(1)} + 2d_{k-1}^{(1)} S_{k-1}\big) + S_{k-1}\,, & b_0^{(1)} &= 0\\
d_k^{(1)} &= \Sigma_k^2 d_{k-1}^{(1)} + \Sigma_k\,, & d_0^{(1)} &= 0\\
a_k^{(2)} &= a_{k-1}^{(1)} + b_{k-1}^{(2)} S_{k-1} + d_{k-1}^{(2)} \sigma_k^{-1} + S_k^2(d_{k-1}^{(2)} + 1) + \sigma_{k+1}^{-1}\,, & a_0^{(2)} &= 0\\
b_k^{(2)} &= \Sigma_k\big(b_{k-1}^{(2)} + 2(d_{k-1}^{(2)} + 1)S_{k-1}\big)\,, & b_0^{(2)} &= 0\\
d_k^{(2)} &= \Sigma_k^2(d_{k-1}^{(2)} + 1), & d_0^{(2)} &= 0\\
a_k^{(3)} &= a_{k-1}^{(3)} + b_{k-1}^{(3)} S_{k-1} + d_k^{(3)} \sigma_k^{-1} + S_{k-1}^2 d_{k-1}^{(3)}\,, & a_0^{(3)} &= 0\\
b_k^{(3)} &= \Sigma_k\big(b_{k-1}^{(3)} + 2d_{k-1}^{(3)} S_{k-1}\big)\,, & b_0^{(3)} &= 0\\
d_k^{(3)} &= \Sigma_k^2 d_{k-1}^{(3)} + 1, & d_0^{(3)} &= 0.
\end{aligned}$$

□

In the same way we can establish the form of the densities

$$\lambda_k^1(x), \lambda_k^2(x), \gamma_k(x), \delta_k^{1j}(x), \delta_k^{2j}(x), \quad j = 1,2,3,$$

and the recursions for their coefficients. We state these results as a sequence of theorems.

Theorem 6.4. $\lambda_k^{(m)}(x)$ *has the form*

$$\lambda_k^{(m)}(x) = \big(\overline{a}_k^{(m)} + \overline{b}_k^{(m)} x\big)\alpha_k(x), \quad m = 1, 2,$$

where

$$\begin{aligned} \overline{a}_k^{(1)} &= \overline{a}_{k-1}^{(1)} + \big(\overline{b}_{k-1}^{(1)} + 1\big)S_{k-1}\,, & \overline{a}_0^{(1)} &= 0 \\ \overline{b}_k^{(1)} &= \Sigma_k\big(\overline{b}_{k-1}^{(1)} + 1\big), & \overline{b}_0^{(1)} &= 0 \end{aligned}$$

and

$$\begin{aligned} \overline{a}_k^{(2)} &= \overline{a}_{k-1}^{(2)} + \overline{a}_{k-1}^{(2)} S_{k-1}\,, & \overline{a}_0^{(2)} &= 0 \\ \overline{b}_k^{(2)} &= \Sigma_k\, \overline{b}_{k-1}^{(2)} + 1, & \overline{b}_0^{(2)} &= 0. \end{aligned}$$

Theorem 6.5. $\gamma_k(x)$ *has the form*

$$\gamma_k(x) = (p_k + q_k x)\alpha_k(x)$$

where

$$\begin{aligned} p_k &= p_{k-1} + q_{k-1}S_{k-1}, & p_0 &= 0 \\ q_k &= \Sigma_k q_{k-1} + y_k, & q_0 &= y_0\,. \end{aligned}$$

Theorem 6.6. *The measures* $\delta_k^{(i)j}(x), \quad i = 1, 2, \quad j = 1, 2, 3,$ *have the form*

$$\delta_k^{(i)j}(x) = \big(u_k^{(i)j} + v_k^{(i)j} x\big)\alpha_k(x)$$

where

$$\begin{aligned} u_k^{(1)j} &= u_{k-1}^{(1)j} + \big(v_{k-1}^{(1)j} + 1\big)S_{k-1}, & u_0^{(1)j} &= 0 \\ v_k^{(1)j} &= \Sigma_k\big(v_{k-1}^{(1)j} + 1\big), & v_0^{(1)j} &= 0 \end{aligned}$$

and

$$\begin{aligned} u_k^{(2)j} &= u_{k-1}^{(2)j} + v_{k-1}^{(2)j} S_{k-1}\,, & u_0^{(2)j} &= 0 \\ v_k^{(2)j} &= \Sigma_k v_{k-1}^{(2)j} + 1, & v_0^{(2)j} &= 0. \end{aligned}$$

7. Updates for the Parameter Estimates

With the explicit expressions we have derived for the densities α_k, $\beta_k^{(n)}$, $\lambda_k^{(m)}$, γ_k and $\delta_k^{(i)j}$, it is then straight forward to obtain the recursive filters $\widehat{H}_k^0$, $\widehat{H}_k^1$, $\widehat{H}_k^2$, $\widehat{H}_k^3$, $\widehat{L}_k^1$, $\widehat{L}_k^2$, $\widehat{J}_k$, and $\widehat{\Delta}_k$ which are then used directly to obtain the updates of the parameter estimates

Theorem 7.1. *For* $n = 0, 1, 2, 3$

$$\widehat{H}_k^{(n)} = a_k^{(n)} + b_k^{(n)}\mu_k + d_k^{(n)} P_k + d_k^{(n)}\mu_k^2\,.$$

Proof. Using the abstract Bayes Theorem as in Section 4 above,

$$\begin{aligned}\widehat{H}_k^{(n)} &= E\big[H_k^{(n)}|\mathcal{Y}_k\big] \\ &= \frac{\overline{E}[\Lambda_k H_k^{(n)}|\mathcal{Y}_k]}{\overline{E}[\Lambda_k|\mathcal{Y}_k]} \\ &= \frac{\int_{-\infty}^{\infty} \beta_k^{(n)}(x)dx}{\overline{\alpha}_k}\end{aligned}$$

where

$$\overline{\alpha}_k = \int_{-\infty}^{\infty} \alpha_k(x)dx = \overline{E}[\Lambda_k|\mathcal{Y}_k].$$

Now $\alpha_k(x)$ is an unnormalized $N(\mu_k, P_k)$, density so

$$\begin{aligned}\int_{-\infty}^{\infty} \beta_k^{(n)}(x)dx &= \overline{\alpha}_k E\big[a_k^{(n)} + b_k^{(n)}x + d_k^{(n)}x^2\big] \\ &= \overline{\alpha}_k\big(a_k^{(n)} + b_k^{(n)}\mu_k + d_k^{(n)}P_k + d_k^{(n)}\mu_k\big)\end{aligned}$$

and the results follow. □

Using the updated equations of Theorem 6.3 this result enables us to recursively estimate $\widehat{H}_k^{(n)}$, $n = 0, 1, 2, 3$. In the same way we can recursively update the other quantities required to calibrate the model:

$$\begin{aligned}\widehat{L}_k^{(m)} &= E\big[L_k^{(m)}|\mathcal{Y}_k\big] = \overline{a}_k^{(m)} + \overline{b}_k^{(m)}\mu_k \\ \widehat{J}_k &= E\big[J_k|\mathcal{Y}_k\big] = p_k + q_k\mu_k \\ \widehat{\Delta}_k^{(i)j} &= E\big[\Delta_k^{(i)j}|\mathcal{Y}_k\big] = u_k^{(i)j} + v_k^{(i)j}\mu_k\,.\end{aligned}$$

These are then used in Theorems 5.2 and 5.3 to update the estimates of the parameters.

8. Conclusions

In this paper we have considered reduced-form models that link the real economy to financial markets. A mean reverting beta process for a stock market industry sector is considered where the mean reversion level for the beta switches between three values depending on the state of the economy. The state of the economy is modelled as a three state Markov chain observed in the growth rate process of real GDP.

Discrete time versions for the dynamics of the state of the chain and the beta process are considered. The filtered, estimated, values of the state of the economy are used to estimate the long term mean. The beta process is estimated from observations of the returns.

Novel recursive estimates for calibrating the model are introduced which apply ideas from [10] and [7]. The filters and the filter-based EM algorithms used to estimated the mean reverting levels are new and allow for a richer information set to be used in the calibration of beta.

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