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STOCHASTIC DIFFERENTIAL EQUATIONS WITH ANTICIPATING INITIAL CONDITIONS

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Abstract. In this paper, we study the solutions of a stochastic differential equation with various anticipating initial conditions. We show that the conditional expectation of the solution of such a stochastic differential equation is not simply the solution of the corresponding stochastic differential equation with initial condition taken as the conditional expectation of the anticipating initial condition. We derive the conditional expectation of the solution in general, and apply it to the special case of anticipating initial condition given by Hermite polynomials. We also extend the class of initial conditions to functions of Wiener integrals.

1. Introduction

In 1942 [7], Kiyosi Itô published his pioneering paper on stochastic integration, which enabled integration of stochastic processes with respect to a Brownian motion. In 1944 [8], his efforts to model Markov processes led him to construct stochastic differential equations of the form $dX_t = \alpha(X_t)\, dB(t) + \beta(X_t)\, dt, X_0 = x$, which subsequently led him to publish what is now known as the Itô formula. In 1973, Black and Scholes [4], and Merton [12] used Itô's calculus to give a framework for option pricing, which rapidly expanded the interest of stochastic calculus to practitioners in other fields.

Even though it is extremely useful, the Itô calculus cannot handle *anticipating* conditions. For example, consider the following simple stochastic differential equation with anticipating initial condition

$$\begin{cases} dX_t = X_t\, dB(t), \quad t \in [0,1], \\ X_0 = B(1). \end{cases} \tag{1.1}$$

To solve the equation analytically, we have to assign a meaning to the integral

$$\int_0^t B(1)\, dB(s), \quad t \in [0,1], \tag{1.2}$$

which is outside the theory of Itô calculus since the integral is not adapted with respect to the filtration generated by the Brownian motion $B(t)$. This is the primary motivation for extending the Itô integral to anticipating integrands.

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Anticipating stochastic integrals are useful for modeling phenomena where one has prior knowledge of the future. A primary example is in mathematical finance, where one uses anticipation to model insider trading.

The outline of the paper is as follows: In Section 2, we mention some of the preexisting theories, and recall the ideas behind the general stochastic integration. In Section 3, we give a brief introduction to the general theory of stochastic integration and the solution of a linear stochastic differential equation with an anticipating initial condition. Sections 4 and 5 contain the main results of this paper. In Section 4, we discuss the conditional expectation of the solution of an anticipating stochastic differential equatio, and also elucidate the essential connection between the result and the Hermite polynomials. In Section 5, we give the solution of a linear stochastic differential equation with initial condition in a larger class of anticipating processes than those previously discussed.

2. Brief Review of the Background

Several generalizations of the Itô integral have been introduced to deal with anticipating integrands. A few approaches are those by Nualart [13], Pardoux and Protter [14], Biagini and Øksendal [3], and white noise distribution theory [10], among others. In particular, in the white noise distribution theory, the stochastic integral given in equation (1.2) is defined as

$$\int_0^t B(1)\, dB(s) = \int_0^t \partial_s^* B(1)\, ds,$$

where ∂_t^* is the adjoint of the white noise differentiation operator ∂_t. This extension of the Itô integral assigns a meaning to the integral in equation (1.2) and consequently gives a solution to equation (1.1). Nevertheless, its definition using the white noise distribution theory has several difficulties, e.g., the lack of probabilistic interpretation and conditional expectation.

In 2008 [1], W. Ayed and H.-H. Kuo introduced a general stochastic integral (see also [2]). The authors use adapted and instantly independent decomposition of an anticipating process to define its stochastic integral. In [5], it was proved that the definition of the generalized stochastic integral is well-defined. This enables us to define the stochastic integral like equation (1.2) and solve stochastic differential equation with anticipating initial condition like equation (1.1).

At this point, it is worth noting that these results obtained from the generalized stochastic integral coincide with the corresponding results obtained using white noise distribution theory. Essentially, the general stochastic integral addresses the shortcomings of both the Itô integration theory and the white noise distribution theory by allowing anticipating integrands and giving suggestive interpretation of the integral, respectively. Thus, the general stochastic integral serves as an ideal link between Itô calculus and white noise distribution theory.

An analytic expression of the solution of the stochastic differential equation with anticipating initial condition

$$\begin{cases} dX_t = \alpha(t) X_t\, dB(t) + \beta(t) X_t\, dt, \quad t \in [a,b] \\ X_a = \psi(B(b) - B(a)), \end{cases} \tag{2.1}$$

is given in [5] and [9]. The precise expression allows us to study its analytic properties, understand its probabilistic meaning, and apply it to solve problems pertaining to the real world.

It is natural for one to expect that the conditional expectation of the solution of equation (2.1) is the solution of the adapted version of the same equation. However, we show that this is not the case and its conditional expectation satisfies another adapted stochastic differential equation. This equation has a correction term that comes from the accumulative impact of the anticipating initial condition.

3. General Stochastic Integral

The bedrock of the definition of the general stochastic integral depends on instantly independent processes, which we define here.

Definition 3.1. A stochastic process $f(t)$ is called *instantly independent* of a filtration $\{\mathcal{F}_t\}$ if $f(t)$ and $\mathcal{F}_t$ are independent for each $t \in [a,b]$.

Definition 3.2 ([1])**.** Let f be adapted and φ be instantly independent, each being continuous stochastic processes. Then the *stochastic integral* of $f(t)\varphi(t)$ is defined by

$$\int_a^b f(t)\varphi(t)\,dB(t) = \lim_{\|\Delta_n\|\to 0} \sum_{j=1}^{n} f(t_{j-1})\varphi(t_j)\big(B(t_j) - B(t_{j-1})\big), \tag{3.1}$$

provided that the limit exists in probability.

In [5], the definition was extended to a more general case.

Definition 3.3 ([5])**.** Suppose $\Phi(t), t \in [a,b]$, is a stochastic process of the form

$$\Phi(t) = \sum_{i=1}^{m} f_i(t)\varphi_i(t), \tag{3.2}$$

where $f_i(t)$'s are $\{\mathcal{F}_t\}$-adapted continuous stochastic processes and $\varphi_i(t)$'s are continuous stochastic processes being instantly independent of $\{\mathcal{F}_t\}$. The *stochastic integral* of $\Phi(t)$ is defined by

$$\int_a^b \Phi(t)\,dB(t) = \sum_{i=1}^{m} \int_a^b f_i(t)\varphi_i(t)\,dB(t),$$

where the integrals $\int_a^b f_i(t)\varphi_i(t)\,dB(t)$ are defined using equation (3.1) for each i, provided that the limit exists in probability.

Intuitively, for the general stochastic integral, the adapted part of the integrand is evaluated at the left endpoints and the instantly independent part is evaluated at the right endpoints of each interval, respectively. Note that the restriction of the general stochastic integral to adapted processes gives the Itô integral, so the stochastic integral defined by equation (3.1) is an extension of the Itô integral.

The following lemma shows that the integral is indeed well-defined.

Lemma 3.4 ([5])**.** *Let $f_i(t), 1 \le i \le m$, $g_j(t), 1 \le j \le n$, be $\{\mathcal{F}_t\}$-adapted continuous stochastic processes and let $\varphi_i(t), 1 \le i \le m$, $\xi_j(t), 1 \le j \le n$, be continuous stochastic processes being instantly independent of $\{\mathcal{F}_t\}$. Suppose the stochastic integrals $\int_a^b f_i(t)\varphi_i(t)\,dB(t)$ and $\int_a^b g_j(t)\xi_j(t)\,dB(t)$ exist for $1 \le i \le m, 1 \le j \le n$. Assume that*

$$\sum_{i=1}^{m} f_i(t)\varphi_i(t) = \sum_{j=1}^{n} g_j(t)\xi_j(t), \quad t \in [a,b].$$

Then the following equality holds:

$$\sum_{i=1}^{m} \int_a^b f_i(t)\varphi_i(t)\,dB(t) = \sum_{j=1}^{n} \int_a^b g_j(t)\xi_j(t)\,dB(t). \tag{3.3}$$

In general, if a stochastic process $\Phi(t)$ can be written as a series of the form $f(t)\varphi(t)$ in $L^2(\Omega)$, we define the stochastic integral of $\Phi(t)$ to be the sum of the series in probability.

Definition 3.5 ([5])**.** Suppose $\Phi(t), t \in [a,b]$, is a stochastic process and there exists a sequence $\{\Phi_n(t)\}_{n=1}^{\infty}$ of stochastic processes of the form in equation (3.2) satisfying the conditions:

(1) $\int_a^b |\Phi(t) - \Phi_n(t)|^2\,dt \to 0$ almost surely.
(2) $\int_a^b \Phi_n(t)\,dB(t)$ converges in probability.

Then the *stochastic integral* of $\Phi(t)$ is defined by

$$\int_a^b \Phi(t)\,dB(t) = \lim_{n\to\infty} \int_a^b \Phi_n(t)\,dB(t), \quad \text{in probability.} \tag{3.4}$$

In order to study the solution of the stochastic differential equation, we need the Itô formula for the general stochastic integral, whose proof can be found in [5]. Consider the following stochastic processes

$$X_t = X_a + \int_a^t g(s)\,dB(s) + \int_a^t h(s)\,ds, \tag{3.5}$$

$$Y^{(t)} = Y^{(b)} + \int_t^b \xi(s)\,dB(s) + \int_t^b \eta(s)\,ds, \tag{3.6}$$

where $g(t)$ and $h(t)$ are $\{\mathcal{F}_t\}$-adapted so that X_t is an Itô process, and $\xi(t)$ and $\eta(t)$ are instantly independent of $\{\mathcal{F}_t\}$ such that $Y^{(t)}$ is also instantly independent of $\{\mathcal{F}_t\}$.

Theorem 3.6 (Itô formula for general stochastic integral [5])**.** *Let $X_t, t \in [a,b]$, be an Itô process given by equation* (3.5) *and $Y^{(t)}, t \in [a,b]$, an instantly independent process given by equation* (3.6)*. Suppose $\theta(x,y)$ is a real-valued C^2-function on $\mathbb{R}^2$. Then the following equality holds for $t \in [a,b]$:*

$$\begin{aligned}\theta(X_t, Y^{(t)}) =& \theta(X_a, Y^{(a)}) + \int_a^t \theta_x(X_s, Y^{(s)})\,dX_s + \frac{1}{2}\int_a^t \theta_{xx}(X_s, Y^{(s)})(dX_s)^2 \\ &+ \int_a^t \theta_y(X_s, Y^{(s)})\,dY^{(s)} - \frac{1}{2}\int_a^t \theta_{yy}(X_s, Y^{(s)})(dY^{(s)})^2,\end{aligned}$$

which can be expressed symbolically in terms of stochastic differentials as

$$d\theta(X_t, Y^{(t)}) = \theta_x\, dX_t + \frac{1}{2}\theta_{xx}(dX_t)^2 + \theta_y\, dY^{(t)} - \frac{1}{2}\theta_{yy}(dY^{(t)})^2. \tag{3.7}$$

The general Itô formula above can be used to solve a linear stochastic differential equations with certain anticipating initial condition, which is the result of the following theorem. Before we proceed, we define exponential processes.

Definition 3.7. The *exponential process* associated with the adapted stochastic processes α and β is defined as

$$\mathcal{E}_{\alpha,\beta}(t) = \exp\bigg(\int_a^t \alpha(s)\, dB(s) + \int_a^t \Big(\beta(s) - \frac{1}{2}\alpha(s)^2\Big)\, ds\bigg).$$

Remark 3.8. The exponential process $\mathcal{E}_{\alpha,\beta}(t)$ is an Itô process satisfying the stochastic differential equation

$$\begin{cases} d\mathcal{E}_{\alpha,\beta}(t) = \alpha(t)\mathcal{E}_{\alpha,\beta}(t)\, dB(t) + \beta(t)\mathcal{E}_{\alpha,\beta}(t)\, dt, \quad t \in [a,b] \\ \mathcal{E}_{\alpha,\beta}(a) = 1. \end{cases} \tag{3.8}$$

Theorem 3.9 ([5])**.** *Let $\alpha(t)$ be a deterministic function in $L^2[a,b]$, $\beta(t)$ an adapted stochastic process such that $E\int_a^b |\beta(t)|^2\, dt < \infty$, and ψ a continuous function on $\mathbb{R}$. Then the stochastic differential equation*

$$\begin{cases} dX_t = \alpha(t)X_t\, dB(t) + \beta(t)X_t\, dt, \quad t \in [a,b] \\ X_a = \psi\big(B(b) - B(a)\big). \end{cases} \tag{3.9}$$

has a unique solution given by

$$X_t = \psi\Big(B(b) - B(a) - \int_a^t \alpha(s)\, ds\Big)\mathcal{E}_{\alpha,\beta}(t), \tag{3.10}$$

where $\mathcal{E}_{\alpha,\beta}(t)$ is the exponential process associated with α and β.

4. Conditional Expectation of the Solution of a Stochastic Differential Equation

Consider the process $Y_t = E[X_t|\mathcal{F}_t]$ defined by the conditional expectation of the solution process X_t in equation (3.10). Then Y_t is adapted to the filtration $\{\mathcal{F}_t\}$. It is natural to hypothesize that Y_t is the solution to the adapted version of the stochastic differential equation (3.9), namely,

$$\begin{cases} dX_t = \alpha(t)X_t\, dB(t) + \beta(t)X_t\, dt, \quad t \in [a,b] \\ X_a = E\big(\psi(B(b) - B(a))\big), \end{cases}$$

However, this is not the case and we have the following result:

Theorem 4.1. *Let $\alpha(t)$ be a deterministic function in $L^2[a,b]$, $\beta(t)$ an adapted stochastic process such that $E\int_a^b |\beta(t)|^2\, dt < \infty$, and assume that $\psi(t)$ is a function on $\mathbb{R}$ having power series expansion at $t = 0$ with infinite radius of convergence.*

Suppose that $X_1(t)$ and $X_2(t)$ are the solutions of the same linear stochastic differential equation

$$dX_t = \alpha(t)X_t\,dB(t) + \beta(t)X_t\,dt, \quad t \in [a,b],$$

with different initial conditions

$$X_1(a) = \psi(B(b) - B(a)) \quad \textit{and} \quad X_2(a) = \psi'(B(b) - B(a)),$$

where ψ' is the derivative of ψ. Let $Y_1(t) = E[X_1(t)|\mathcal{F}_t]$ and $Y_2(t) = E[X_2(t)|\mathcal{F}_t]$. Then $Y_1(t)$ satisfies the following stochastic differential equation

$$\begin{cases} dY_1(t) = \alpha(t)Y_1(t)\,dB(t) + \beta(t)Y_1(t)\,dt + Y_2(t)\,dB(t), \quad t \in [a,b], \\ Y_1(a) = E\big(\psi(B(b) - B(a))\big). \end{cases} \tag{4.1}$$

Proof. By the assumption and Theorem 3.9, the solution processes $X_1(t)$ and $X_2(t)$ can be written as

$$\begin{aligned} X_1(t) &= \psi\Big(B(b) - B(a) - \int_a^t \alpha(s)\,ds\Big)\mathcal{E}_{\alpha,\beta}(t) \\ &= \mathcal{E}_{\alpha,\beta}(t)\sum_{k=0}^{\infty}\frac{1}{k!}\psi^{(k)}\Big(B(t) - B(a) - \int_a^t \alpha(s)\,ds\Big)(B(b) - B(t))^k. \end{aligned}$$

and

$$\begin{aligned} X_2(t) &= \psi'\Big(B(b) - B(a) - \int_a^t \alpha(s)\,ds\Big)\mathcal{E}_{\alpha,\beta}(t) \\ &= \mathcal{E}_{\alpha,\beta}(t)\sum_{k=0}^{\infty}\frac{1}{k!}\psi^{(k+1)}\Big(B(t) - B(a) - \int_a^t \alpha(s)\,ds\Big)(B(b) - B(t))^k. \end{aligned}$$

Take the conditional expectation to get

$$\begin{aligned} Y_1(t) &= E[X_1(t)|\mathcal{F}_t] \\ &= \mathcal{E}_{\alpha,\beta}(t)\sum_{k=0}^{\infty}\frac{1}{k!}\psi^{(k)}\Big(B(t) - B(a) - \int_a^t \alpha(s)\,ds\Big)E\left[(B(b) - B(t))^k|\mathcal{F}_t\right] \\ &= \mathcal{E}_{\alpha,\beta}(t)\sum_{k=0}^{\infty}\frac{1}{(2k)!}\psi^{(2k)}\Big(B(t) - B(a) - \int_a^t \alpha(s)\,ds\Big)(b-t)^k(2k-1)!! \\ &= \mathcal{E}_{\alpha,\beta}(t)\sum_{k=0}^{\infty}\frac{1}{(2k)!!}\psi^{(2k)}\Big(B(t) - B(a) - \int_a^t \alpha(s)\,ds\Big)(b-t)^k, \end{aligned}$$

and

$$Y_2(t) = \mathcal{E}_{\alpha,\beta}(t)\sum_{k=0}^{\infty}\frac{1}{(2k)!!}\psi^{(2k+1)}\Big(B(t) - B(a) - \int_a^t \alpha(s)\,ds\Big)(b-t)^k,$$

where $(2k-1)!! = (2k-1)(2k-3)\cdots 3\cdot 1$ for $k \geq 1$ and $(-1)!! = 1$ by convention, and $(2k)!! = (2k)(2k-2)\cdots 4\cdot 2$ for $k \geq 1$ and $0!! = 1$ by convention.

For convenience, denote the processes

$$V_t^k = \psi^{(2k)}\Big(B(t) - B(a) - \int_a^t \alpha(s)\,ds\Big)(b-t)^k.$$

Then we have

$$Y_1(t) = \mathcal{E}_{\alpha,\beta}(t) \sum_{k=0}^{\infty} \frac{1}{(2k)!!} V_t^k.$$

Now, both V_t^k and $\mathcal{E}_{\alpha,\beta}(t)$ are adapted. Using a slightly general case of Itô's formula given in equation (3.7) and equation (3.8), we have

$$\begin{aligned} dV_t^k =& \psi^{(2k+1)}\Big(B(t) - B(a) - \int_a^t \alpha(s)\, ds\Big)(b-t)^k (dB(t) - \alpha(t)\, dt) \\ &+ \frac{1}{2}\psi^{(2k+2)}\Big(B(t) - B(a) - \int_a^t \alpha(s)\, ds\Big)(b-t)^k\, dt \\ &- \psi^{(2k)}\Big(B(t) - B(a) - \int_a^t \alpha(s)\, ds\Big)k(b-t)^{k-1}\, dt. \end{aligned}$$

For simplicity, denote $\psi^{(n)}\big(B(t) - B(a) - \int_a^t \alpha(s)\, ds\big)$ by ψ_n. Now,

$$\begin{aligned} &d\big(Y_1(t)^k\big) \\ &= \sum_{k=0}^{\infty} \frac{1}{(2k)!!}\left[V_t^k\, d\mathcal{E}_{\alpha,\beta}(t) + \mathcal{E}_{\alpha,\beta}(t)\, dV_t^k + (dV_t^k)(d\mathcal{E}_{\alpha,\beta}(t))\right] \\ &= \sum_{k=0}^{\infty} \frac{1}{(2k)!!}\psi_{2k}(b-t)^k\big(\alpha(t)\mathcal{E}_{\alpha,\beta}(t)\, dB(t) + \beta(t)\mathcal{E}_{\alpha,\beta}(t)\, dt\big) \\ &\quad + \sum_{k=0}^{\infty} \frac{1}{(2k)!!}\mathcal{E}_{\alpha,\beta}(t)\psi_{2k+1}(b-t)^k(dB(t) - \alpha(t)\, dt) \\ &\quad + \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{(2k)!!}\mathcal{E}_{\alpha,\beta}(t)\psi_{2k+2}(b-t)^k\, dt \\ &\quad - \sum_{k=0}^{\infty} \frac{1}{(2k)!!}\mathcal{E}_{\alpha,\beta}(t)\psi_{2k}k(b-t)^{k-1}\, dt + \sum_{k=0}^{\infty} \frac{1}{(2k)!!}\alpha(t)\mathcal{E}_{\alpha,\beta}(t)\psi_{2k+1}(b-t)^k\, dt. \end{aligned}$$

Continuing, we get $d\big(Y_1(t)^k\big)$ equals

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{1}{(2k)!!}\psi_{2k}(b-t)^k\alpha(t)\mathcal{E}_{\alpha,\beta}(t)\, dB(t) \\ &+ \sum_{k=0}^{\infty} \frac{1}{(2k)!!}\psi_{2k}(b-t)^k\beta(t)\mathcal{E}_{\alpha,\beta}(t)\, dt + \sum_{k=0}^{\infty} \frac{1}{(2k)!!}\mathcal{E}_{\alpha,\beta}(t)\psi_{2k+1}(b-t)^k\, dB(t) \\ &- \cancel{\sum_{k=0}^{\infty} \frac{1}{(2k)!!}\mathcal{E}_{\alpha,\beta}(t)\psi_{2k+1}(b-t)^k\alpha(t)\, dt} + \cancel{\frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{(2k)!!}\mathcal{E}_{\alpha,\beta}(t)\psi_{2k+2}(b-t)^k\, dt} \\ &- \cancel{\frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{(2(k-1))!!}\mathcal{E}_{\alpha,\beta}(t)\psi_{2k}(b-t)^{k-1}dt} + \cancel{\sum_{k=0}^{\infty} \frac{1}{(2k)!!}\alpha(t)\mathcal{E}_{\alpha,\beta}(t)\psi_{2k+1}(b-t)^k dt}. \end{aligned}$$

Finally, using the expressions for $Y_1(t)$ and $Y_2(t)$, we get

$$d\big(Y_1(t)^k\big) = \alpha(t)Y_1(t)\, dB(t) + \beta(t)Y_1(t)\, dt + Y_2(t)\, dB(t). \qquad \square$$

The appearance of the extra term $Y_2(t)\,dB(t)$ in the stochastic differential equation (4.1) is quite natural. Because of the existence of the anticipating part, it has accumulative impact on its conditional expectation through the dynamics of the stochastic differential equation. The extra term $Y_2(t)\,dB(t)$ can be seen as a correction of the current information by accumulatively adding the initial knowledge about the future. Moreover, the derivative ψ' in the initial condition satisfied by $X_2(t)$ essentially comes from the Itô formula (3.7) for the general stochastic integral.

An important example of Theorem 4.1 is when we choose ψ to be the Hermite polynomial. Recall that the Hermite polynomial of degree n with parameter η is defined by (see, for example, [11])

$$H_n(x;\eta) = (-\eta)^n e^{x^2/2\eta} D_x^n e^{-x^2/2\eta}, \tag{4.2}$$

where D_x is the differential operator with respect to the variable x. The Hermite polynomials have the following properties:

$$D_x H_n(x;\eta) = nH_{n-1}(x;\eta), \tag{4.3}$$

$$D_\eta H_n(x;\eta) = -\frac{1}{2} D_x^2 H_n(x;\eta), \tag{4.4}$$

$$H_n(x+y;\eta) = \sum_{k=0}^{n} \binom{n}{k} H_{n-k}(x;\eta) y^k. \tag{4.5}$$

Lemma 4.2 ([11])**.** *The stochastic process $X_t = H_n(B(t)-B(a);t-a), t \in [a,b]$ is a martingale and*

$$dX_t = nH_{n-1}(B(t)-B(a);t-a)\,dB(t). \tag{4.6}$$

Theorem 4.3 ([6])**.** *Let $\alpha(t)$ be a deterministic function in $L^2[a,b]$, $\beta(t)$ an adapted stochastic process such that $E\int_a^b |\beta(t)|^2\,dt < \infty$, and n a fixed natural number. Let X_t be the solution of the stochastic differential equation*

$$\begin{cases} dX_t = \alpha(t)X_t\,dB(t) + \beta(t)X_t\,dt, \quad t \in [a,b], \\ X_a = H_n\big(B(b)-B(a);b-a\big), \end{cases} \tag{4.7}$$

and $Y_t = E[X_t|\mathcal{F}_t]$. Then we have

$$Y_t = H_n\Big(B(t)-B(a) - \int_a^t \alpha(s)\,ds;\ t-a\Big)\mathcal{E}_{\alpha,\beta}(t), \quad t \in [a,b]. \tag{4.8}$$

Moreover, Y_t satisfies the stochastic differential equation

$$\begin{aligned} dY_t =& \Big[\alpha(t)Y_t + nH_{n-1}\Big(B(t)-B(a) - \int_a^t \alpha(s)\,ds;\, t-a\Big)\mathcal{E}_{\alpha,\beta}(t)\Big]\,dB(t) \\ &+ \beta(t)Y_t\,dt, \end{aligned} \tag{4.9}$$

for $t \in [a,b]$, with initial condition $Y_a = 0$.

Proof. First, we prove equation (4.8). By Theorem 3.9, we have for $t \in [a,b]$,

$$X_t = H_n\big(B(b)-B(a) - \int_a^t \alpha(s)\,ds;\ b-a\big)\mathcal{E}_{\alpha,\beta}(t).$$

Note that $\mathcal{E}_{\alpha,\beta}(t)$ is adapted to $\{\mathcal{F}_t\}$. So

$$Y_t = E[X_t|\mathcal{F}_t] = E\big[H_n\big(B(b) - B(a) - \int_a^t \alpha(s)\,ds;\ b-a\big)\big|\mathcal{F}_t\big]\mathcal{E}_{\alpha,\beta}(t). \tag{4.10}$$

By equation (4.5) with $x = B(b) - B(a), y = -\int_a^t \alpha(s)\,ds$, and $\eta = b - a$, we get

$$H_n\Big(B(b){-}B(a){-}\int_a^t \alpha(s)\,ds; b{-}a\Big) = \sum_{k=0}^n \binom{n}{k} H_{n-k}(B(b){-}B(a); b{-}a)\Big(-\int_a^t \alpha(s)\,ds\Big)^k.$$

Taking conditional expectation with respect to $\mathcal{F}_t$, we obtain

$$\begin{aligned}
&E\big[H_n\big(B(b) - B(a) - \int_a^t \alpha(s)\,ds;\ b-a\big)\big|\mathcal{F}_t\big] \\
&= \sum_{k=0}^n \binom{n}{k} E\big[H_{n-k}(B(b) - B(a);\, b-a)\Big(-\int_a^t \alpha(s)\,ds\Big)^k\big|\mathcal{F}_t\big] \\
&= \sum_{k=0}^n \binom{n}{k} E\big[H_{n-k}(B(b) - B(a);\, b-a)\big|\mathcal{F}_t\big]\Big(-\int_a^t \alpha(s)\,ds\Big)^k \\
&= \sum_{k=0}^n \binom{n}{k} H_{n-k}(B(t) - B(a); t-a)\Big(-\int_a^t \alpha(s)\,ds\Big)^k \\
&= H_n\Big(B(t) - B(a) - \int_a^t \alpha(s)\,ds;\ t-a\Big),
\end{aligned} \tag{4.11}$$

since $\int_a^t \alpha(s)\,ds$ is adapted to $\{\mathcal{F}_t\}$ and we have used Lemma 4.2. Combining equations (4.10) and (4.11), we get equation (4.8).

By Itô's formula and equation (4.6), we have

$$\begin{aligned}
&dH_n\Big(B(t) - B(a) - \int_a^t \alpha(s)\,ds;\ t-a\Big) \\
&= D_xH_n\,dB(t) - D_xH_n\alpha(t)\,dt + \frac{1}{2}D_x^2H_n\,dt + D_\eta H_n\,dt \\
&= nH_{n-1}\,dB(t) - nH_{n-1}\alpha(t)\,dt + \frac{1}{2}D_x^2H_n\,dt - \frac{1}{2}D_x^2H_n\,dt \\
&= nH_{n-1}\Big(B(t) - B(a) - \int_a^t \alpha(s)\,ds;\ t-a\Big)\,dB(t) \\
&\quad - nH_{n-1}\Big(B(t) - B(a) - \int_a^t \alpha(s)\,ds;\ t-a\Big)\alpha(t)\,dt.
\end{aligned}$$

Now, using equation (3.8), we get

$$\begin{aligned}
dY_t =& H_n\,d\mathcal{E}_{\alpha,\beta}(t) + \mathcal{E}_{\alpha,\beta}(t)\,dH_n + (dH_n)(d\mathcal{E}_{\alpha,\beta}(t)) \\
=& H_n\alpha(t)\mathcal{E}_{\alpha,\beta}(t)\,dB(t) + H_n\beta(t)\mathcal{E}_{\alpha,\beta}(t)\,dt + nH_{n-1}\alpha(t)\mathcal{E}_{\alpha,\beta}(t)\,dB(t) \\
=& \Big[\alpha(t)Y_t + nH_{n-1}\Big(B(t) - B(a) - \int_a^t \alpha(s)\,ds;\ t-a\Big)\mathcal{E}_{\alpha,\beta}(t)\Big]\,dB(t) \\
&\quad + \beta(t)Y_t\,dt,
\end{aligned}$$

which proves equation (4.9). □

It is well-known that the Hermite polynomials form an orthonormal basis for the Hilbert space $L^2(\mathbb{R}, \gamma)$, where γ is the Gaussian measure with mean 0 and variance equal to the parameter η as defined in equation (4.2). So we can naturally extend the result of Theorem 4.3 from Hermite polynomials to certain functions in $L^2(\mathbb{R}, \gamma)$.

Theorem 4.4. *Suppose $\alpha(t)$ and $\beta(t)$ are as in Theorem 3.9, and ψ is a differentiable function in $L^2(\mathbb{R}, \gamma)$ which can be written as a Hermite series*

$$\psi(x) = \sum_{n=0}^{\infty} c_n H_n(x; b - a).$$

Assume that $X_1(t)$ and $X_2(t)$ are the solutions of the same linear stochastic differential equation

$$dX_t = \alpha(t) X_t \, dB(t) + \beta(t) X_t \, dt, \quad t \in [a, b],$$

with different initial conditions

$$X_1(a) = \psi(B(b) - B(a)) \quad \text{and} \quad X_2(a) = \psi'(B(b) - B(a)).$$

Let $Y_1(t) = E[X_1(t)|\mathcal{F}_t]$ and $Y_2(t) = E[X_2(t)|\mathcal{F}_t]$. Then

$$Y_1(t) = \sum_{n=0}^{\infty} c_n H_n\Big(B(t) - B(a) - \int_a^t \alpha(s) \, ds; \; t - a\Big) \mathcal{E}_{\alpha,\beta}(t), \quad t \in [a, b].$$

Moreover, it satisfies the following stochastic differential equation

$$\begin{cases} dX_t = \alpha(t) Y_1(t) \, dB(t) + \beta(t) Y_1(t) \, dt + Y_2(t) \, dB(t), \quad t \in [a, b] \\ Y_1(a) = E\big(\psi(B(b) - B(a))\big), \end{cases} \tag{4.12}$$

5. A Larger Class of Initial Conditions

The solution of a linear stochastic differential equation with anticipating initial condition of the form $\psi(B(b) - B(a))$ is given in [5]. In this section, we extend the result to the case when the initial condition is a function of a Wiener integral.

Theorem 5.1. *Let $\alpha(t) \in L^2[0, 1], \beta(t) \in L^1[0, 1], h(t) \in L^2[0, 1]$ and $\psi(t)$ is a C^2 function. Then the (unique) solution of the stochastic differential equation*

$$\begin{cases} dX_t = \alpha(t) X_t \, dB(t) + \beta(t) X_t \, dt, \quad t \in [0, 1] \\ X_0 = \psi\Big(\int_0^1 h(s) \, dB(s)\Big), \end{cases} \tag{5.1}$$

is given by

$$X_t = \psi\Big(\int_0^1 h(s) \, dB(s) - \int_0^t \alpha(s) h(s) \, ds\Big) \mathcal{E}_{\alpha,\beta}(t).$$

Proof. Suppose $X_t = \psi\big(\int_0^1 h(s) \, dB(s) - p(t)\big) \mathcal{E}_{\alpha,\beta}(t)$, where $p(t)$ has to be determined. In order to apply the generalized Itô formula, we write

$$X_t = \psi\Big(\int_0^t h(s) \, dB(s) + \int_t^1 h(s) \, dB(s) - p(t)\Big) \mathcal{E}_{\alpha,\beta}(t). \tag{5.2}$$

Motivated by this, we define

$$\begin{cases} X_t^{(1)} = \int_0^t h(s)\,dB(s), \\ X_t^{(2)} = \mathcal{E}_{\alpha,\beta}(t), \\ Y^{(t)} = \int_t^1 h(s)\,dB(s), \text{ and} \end{cases} \tag{5.3}$$
$$\theta(t, x_1, x_2, y) = \psi(x_1 + y - p(t))x_2,$$

so that $X_t = \theta(t, X_t^{(1)}, X_t^{(2)}, Y^{(t)})$. From the definition of θ, we get the partial derivatives

$$\begin{aligned} \theta_t &= -\psi' p'(t) x_2, & \theta_{x_1} &= \psi' x_2, & \theta_{x_2} &= \psi, \\ \theta_{x_1 x_1} &= \psi'' x_2, & \theta_{x_2 x_2} &= 0, & \theta_{x_1 x_2} &= \psi', \\ \theta_y &= \psi' x_2, & \theta_{yy} &= \psi'' x_2. \end{aligned}$$

From the definitions in equation (5.3), we have

$$\begin{aligned} dX_t^{(1)} &= h(t)\,dB(t), & dX_t^{(2)} &= \alpha(t)\mathcal{E}_{\alpha,\beta}(t)\,dB(t) + \beta(t)\mathcal{E}_{\alpha,\beta}(t)dt, \\ (dX_t^{(1)})^2 &= h(t)^2 dt, & (dX_t^{(2)})^2 &= \alpha(t)^2 \mathcal{E}_{\alpha,\beta}(t)^2 dt, \\ dX_t^{(1)} dX_t^{(2)} &= h(t)\alpha(t)\mathcal{E}_{\alpha,\beta}(t)dt, \\ dY^{(t)} &= -h(t)\,dB(t), & (dY^{(t)})^2 &= h(t)^2 dt. \end{aligned}$$

Applying the differential formula and putting everything together, we have

$$\begin{aligned} dX_t &= d\theta(t, X_t^{(1)}, X_t^{(2)}, Y^{(t)}) \\ &= \theta_t dt + \theta_{x_1} h(t)\,dB(t) + \theta_{x_2} dX_t^{(2)} \\ &\quad + \frac{1}{2}\theta_{x_1 x_1}(dX_t^{(1)})^2 + \frac{1}{2}\theta_{x_2 x_2}(dX_t^{(2)})^2 + \theta_{x_1 x_2}(dX_t^{(1)})(dX_t^{(1)}) \\ &\quad + \theta_y dY^{(t)} - \frac{1}{2}\theta_{yy} dY^t \\ &= -\psi' p'(t) X_t^{(2)} dt + \cancel{\psi' X_t^{(2)} h(t)}\,dB(t) + \psi \cdot \big[\alpha(t)\mathcal{E}_{\alpha,\beta}(t)\,dB(t) + \beta(t)\mathcal{E}_{\alpha,\beta}(t)\,dt\big] \\ &\quad + \cancel{\frac{1}{2}\,\psi''\,X_t^{(2)}\,h(t)^2}\,dt + \frac{1}{2}\cdot 0 \cdot \big\{\alpha(t)^2\,\mathcal{E}_{\alpha,\beta}(t)^2\,dt\big\} + \psi'\,h(t)\,\alpha(t)\,\mathcal{E}_{\alpha,\beta}(t)\,dt \\ &\quad - \cancel{\psi'\,X_t^{(2)}\,h(t)}\,dB(t) - \cancel{\frac{1}{2}\psi''\,X_t^{(2)}\,h(t)^2}\,dt \\ &= -\psi'\,p'(t)\,X_t^{(2)}\,dt + \alpha(t) X_t\,dB(t) + \beta(t) X_t\,dt + \psi'\,h(t)\,\alpha(t)\,\mathcal{E}_{\alpha,\beta}(t)\,dt \\ &= \alpha(t) X_t\,dB(t) + \beta(t)\,X_t\,dt + \big(\alpha(t)\,h(t) - p'(t)\big)\psi' \mathcal{E}_{\alpha,\beta}(t)\,dt, \end{aligned}$$

where in the fourth equality we used $X_t = \psi \cdot \mathcal{E}_{\alpha,\beta}(t)$.

Therefore, in order for X_t to be the solution of equation (5.1), we need the condition $p'(t) = \alpha(t)h(t)$. On the other hand, if we put $t = 0$ in equation (5.2), we get $X_0 = \psi\big(\int_0^1 h(s)\,dB(s) - p(0)\big)$. Since X_t is the solution of equation (5.1), comparing this with the initial condition gives us $p(0) = 0$. Thus we have the following ordinary differential equation

$$\begin{cases} p'(t) = \alpha(t)h(t), & t \in [0,1] \\ p(0) = 0, \end{cases}$$

whose solution is $p(t) = \int_0^t \alpha(s)h(s)ds$. Therefore

$$X_t = \psi\Big(\int_0^1 h(s)\,dB(s) - \int_0^t \alpha(s)h(s)\,ds\Big)\mathcal{E}_{\alpha,\beta}(t).$$

Uniqueness is now obvious. □

Example 5.2. Consider the stochastic differential equation

$$\begin{cases} dX_t = X_t\,dB(t) & t \in [0,1] \\ X_0 = \displaystyle\int_0^1 B(s)\,ds. \end{cases}$$

To solve this, we reformulate the initial condition as

$$\int_0^1 B(s)\,ds = -B(s)(1-s)\Big|_0^1 + \int_0^1 (1-s)\,dB(s) = \int_0^1 (1-s)\,dB(s).$$

Thus we have a special case of Theorem 5.1 with $\alpha(t) \equiv 1$, $\beta \equiv 0$, $h(t) = 1-t$, and $\psi(x) = x$. Therefore, the solution is given by

$$X_t = \left(\int_0^1 B(s)ds - \Big(t - \frac{1}{2}t^2\Big)\right)e^{B(t)-\frac{1}{2}t}.$$

6. Concluding Remarks

From the aspect of the dynamics of stochastic processes, the connection between the solution of the stochastic differential equation and the Hermite polynomials bridges the diverse ideas of the classic Itô calculus, the white noise distribution theory, the Hermite polynomials and renormalization. This reconfirms the statement that that the general stochastic integral stands as a bridge between the general Itô calculus and the white noise integration theory.

It is also worth noting that the white noise distribution theory starts with using Hermite polynomials as renormalizations. This observation reminds us to consider the white noise version of equation (3.9). If we can express our results in the white noise distribution theory, this would provide us a deeper understanding of the link between the general stochastic integral and white noise integral and give the latter a clear probability explanation. We plan to continue this study in further research papers.

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