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ON A STOCHASTIC 2D SIMPLIFIED LIQUID CRYSTAL MODEL DRIVEN BY JUMP NOISE

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Abstract. We investigate a stochastic 2D simplified liquid crystal system with a multiplicative noise of Lévy type, which model the dynamic of nematic liquid crystals under the influence of stochastic external forces of jump type. We prove that the system has a unique global strong solution and we derive some a priori estimates for the solution.

1. Introduction

Stochastic partial differential equations (SPDE) are used to model physical systems subjected to influence of internal, external or environmental noises or to describe systems that are too complex to be described deterministically, e.g. a flow of a chemical substance in a river subjected by wind and rain, an airflow around an airplane wing perturbed by the random state of the atmosphere and weather, a laser beam subjected to turbulent movement of the atmosphere, spread of an epidemic in some regions and the spatial spread of infectious diseases. SPDEs are also used in the physical sciences (e.g. in plasmas turbulence, physics of growth phenomena such as molecular beam epitaxy and fluid flow in porous media with applications to the production of semiconductors and to the oil industry) and biology (e.g. bacteria growth and DNA structure). Models related to the so called passive scalar equations have potential applications to the understanding of waste (e.g. nuclear) convection under the earths surface, [3, 5, 41, 42].

The presence of noise can lead to new and important phenomena. For example, the 2-dimensional Navier-Stokes equations with sufficiently degenerate noise have a unique invariant measure and hence exhibit ergodic behavior in the sense that the time average of a solution is equal to the average over all possible initial data. Despite continuous efforts in the last thirty years, such a property has so far not been found for the deterministic counterpart of these equations. This property could lead to profound understanding of the nature of turbulence. The aforementioned Navier-Stokes Equations (NSE) are now a widely accepted model of fluid motion, see for instance the well known monograph [52, 53]. The theory of NSE is reasonable well understood. For instance, in the case of 2-dimensional domains, it is known since the pioneering works of Lions and Prodi in the 1960s

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(see for instance [40]) that the solutions exist for all times and are unique. In the 3-dimensional case it is known that the weak solutions exist for all times, see celebrated work of Leray [29], and that the strong solutions are unique. However, despite many efforts in the recent years the questions whether the weak solutions are unique or strong solutions exist for all times, remain unresolved, see for instance [54]. To our best knowledge, the first work on the stochastic NSE (SNSE) written from the mathematical point of view is a paper [1]. Later the motivation for the large deviations paper of Faris and Jona-Lasinio [17] was clearly the stochastic fluid dynamics as they wrote, roughly speaking, the motion of a viscous incompressible fluid is described by the Navier-Stokes equations. However, these equations are only approximate. In particular, they take into account only the macroscopic nature of the fluid motion.

Liquid crystal is often viewed as the fourth state of matter besides gas, liquid and solid, or as an intermediate state between liquids and solids. It possesses either no or partial positional order but displays an orientational order at the same time. The nematic phase is the simplest among all liquid crystal phases and is close to the liquid phase. The molecules float around as in a liquid phase, but have the tendency of aligning along a preferred direction due to their orientation. The hydrodynamic theory of liquid crystals due to Ericken and Leslie was developed around the 1960s, [14, 15, 30, 31]. Their theory, now referred to as the Ericksen-Leslie (EL) dynamic theory, is one of the most successful theories used to model many dynamic phenomena in nematic liquid crystals, [21, 57].

As recalled in [22], the mathematical studies on the dynamical liquid crystal systems started with the work of [35, 36, 37, 38], where the authors established the global existence of weak solutions, in both 2D and 3D, to the Ginzburg-Landau approximation of the liquid crystal system, see [8, 50] for some generalizations to the general liquid crystal systems. Global existence of weak solutions to the original liquid crystal systems in 2D, without the Ginzburg-Landau approximation, was established in [22, 24, 26, 28, 39, 55]. In particular, it was shown that global weak solutions to liquid crystal system in 2D have at most finite many singular times, while the uniqueness of weak solutions to liquid crystal system in 2D was proved in [32, 33, 56, 60]; global existence (but without uniqueness) of weak solutions to the liquid crystal system in 3D was recently established in [34], under the assumption that the initial director filed takes value from the upper half unit sphere. If the initial data are suitably smooth, then the liquid crystal system has a unique local strong solution, see [25, 28, 55, 57, 58]. Moreover, if the initial data is suitably small, or the initial director filed satisfies some geometrical condition in 2D, then the local strong solution to the liquid crystal system can be extended to be a global one, see [21, 33]. It is worth to mention that some mathematical analysis concerning the global existence of weak solutions and local or global well-posedness of strong solutions of the non-isothermal liquid crystal systems were addressed in [18, 19].

We borrow from [6] the following motivations for the problem studied this article. Most of the physical systems confront dynamical instabilities. The instability befalls at some critical value of the control parameter (which is in our case some random external noise) of the system. In our predicament the dynamics are quite intricate because the evolution of the director field $\mathbf{d}$ is coupled to the velocity field v. In [49], the author has studied the stationary orientational correlations of the director field of a nematic liquid crystal near the Fréedericksz transition. In this transition the molecules tend to reorient due to some random external perturbations. It has been studied by [48] that the decay time, required for the system is shortened by the field fluctuations to leave an unstable state, which is built by switching on the field to a value beyond instability point. See also [27] and references there in, for more details. A nematic drifts very much like a typical organic liquid with molecules of indistinguishable size. Since, the transitional motions are coupled to inner, orientational motions of the molecules, in most cases the flow muddles the alignment. Conversely, by implementation of an external field, a change in the alignment will generate a flow in the nematic. This is an important motivation for studying the flows of nematic liquid crystals, effected by altering external forces.

In recent years, introducing a jump-type noises as Lévy-type or Poisson-type pertubations has become extremely popular for modeling natural phenomena, because these noises are very nice choice to reproduce the performance of some natural phenomena in real world models, such as some large moves and unpredictable events. There is a large amount of literature on the existence and uniqueness solutions for stochastic partial differential equations driven by jump-type noises. We refer the reader to [3, 11, 12, 13, 43, 44, 45, 46, 47, 59, 61]. However, the existing results in the literature do not cover the situation considered in this paper.

The aim of this article is to study a class of stochastic simplified nematic liquid crystal model driven by jump noise of Lévy type. The model includes an abstract and general form of random external forces depending eventually on the velocity v of the fluid and the director field $\mathbf{d}$. We prove the existence and uniqueness of strong solutions. The proof of the existence of solution is based on a Galerkin scheme similar to that of [5, 23] in the case of the 2D Navier-Stokes and the 3D Lagrangian averaged Navier-Stokes equations. In [10, 51], the authors investigated a stochastic 2D Cahn-Hilliard-Navier-Stokes and Allen-Cahn-Navier-Stokes system with a multiplicative noise of Lévy type and prove the existence and uniqueness of strong solutions. Let us recall that some difficulties associated with the mathematical analysis of the EL model (2.1) are related to the presence of the quadratically increasing term $|\nabla \mathbf{d}|^2\mathbf{d}$ as well as the the constraints $|\mathbf{d}| = 1$, which bring extra technical difficulties to proving the existence of solutions, compared to the 2D Navier-Stokes, the 3D Lagrangian averaged Navier-Stokes equations or the 2D CH-NSE. In [4], the authors proved several results (including the existence and uniqueness of weak solutions) of a simplified stochastic EL model with Ginzburg-Landau approximation. The main difference of our work and that of [4] is that the model considered in [4] is a simplified EL model with a Ginzburg-Landau type approximation in which the term $|\nabla \mathbf{d}|^2\mathbf{d}$ is replaced by a polynomial $f(\mathbf{d})$ that satisfies some reasonable growth assumptions.

The article is divided as follows. In the next section we present the stochastic the simplified nematic liquid crystal model model and its mathematical setting. We also give most of the notations and necessary preliminary used throughout this work. The main results appear in the third section, where we use a Galerkin approximation to prove the existence of strong solution. In the fourth section, we prove the pathwise uniqueness and the convergence of the whole Galerkin approximate solution.

2. The stochastic EL and its Mathematical Setting

2.1. Governing equations. In this article, we consider a stochastic version of a simplified Ericksen-Leslie model in a two-dimensional domain. More precisely, we assume that the domain $\mathcal{M}$ of the fluid is a bounded domain in $\Re^2$. Then, we consider the system

$$\begin{cases} dv + [-\nu_1 \Delta v + (v\cdot\nabla)v + \nabla p]\, dt = -\lambda_1 \nabla\cdot[\nabla\mathbf{d}\otimes\nabla\mathbf{d}]\, dt \\ \quad +g_1(t,v,\mathbf{d})dt + \displaystyle\int_Z \sigma(t,v,\mathbf{d},z)\tilde{\eta}(dt,dz) \ \text{ in } (0,T)\times\mathcal{M}, \\ \text{div } v = 0 \ \text{ in } (0,T)\times\mathcal{M}, \\ \mathbf{d}_t + v\cdot\nabla\mathbf{d} = \nu_2(\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}), \quad |\mathbf{d}| = 1 \ \text{ in } (0,T)\times\mathcal{M}. \end{cases} \tag{2.1}$$

In (2.1), the unknown functions are the velocity $v = (v_1, v_2)^T$ of the fluid, $\mathbf{d} = (\mathbf{d}^1, \mathbf{d}^2)^T$ is the director field, which stands for the averaged macroscopic/continuum molecular orientation in $\Re^2$. Finally, p denotes the pressure of the fluid. The positive constants ν_1, λ_1, ν_2 are the viscosity of the fluid, the competition between the kinetic and the potential energy and the microscopic elastic relaxation time, respectively. The symbol $\otimes$ is the usual Kronecker product, e.g. $(\mathbf{a}\otimes\mathbf{b})_{ij} = \mathbf{a}^i\mathbf{b}^j$ for $\mathbf{a},\mathbf{b}\in\Re^2$. The notation $\nabla\mathbf{d}\otimes\nabla\mathbf{d}$ denotes the 2×2 matrix, whose $(i,j)^{th}$ entry is given by $\partial_i\mathbf{d}\cdot\partial_j\mathbf{d}$. The terms $g_1(t,v,\mathbf{d})$ and $\displaystyle\int_Z \sigma(t,v,\mathbf{d},z)\tilde{\eta}(dt,dz)$ respectively represent the deterministic and the random external forces that eventually depend on $(v,\mathbf{d})$, and $\tilde{\eta}$ is a compensated Poisson measure on a measurable space $(Z,\mathcal{Z})$ endowed with a fixed $\Sigma-$finite measure ν. Precise assumption on the data are given below. The model (2.1) describes the motion of a nematic liquid crystal fluid exited by random forces.

We endow (2.1) with the boundary condition

$$v = 0,\ \mathbf{d} = \mathbf{d}^0 \text{ on } \partial\mathcal{M},\ |\mathbf{d}^0| = 1, \tag{2.2}$$

where $\partial\mathcal{M}$ is the boundary of $\mathcal{M}$.

The initial condition is given by

$$(v,\mathbf{d})(0) = (v^0,\mathbf{d}^0) \text{ in } \mathcal{M}. \tag{2.3}$$

Let us recall that one of the main difficulties associated with the mathematical analysis of the EL model (2.1) is the constraints $|\mathbf{d}| = 1$, which brings extra technical difficulties to proving the existence of solutions, compared to the model considered in [4].

2.2. The deterministic case and its reformulation. In this part, we recall from [21, 16] a reformulation of the deterministic simplified EL model, in which the constraints $|\mathbf{d}| = 1$ is automatically satisfied. We start with the following simplified deterministic EL model

$$\begin{cases} v_t - \nu_1 \Delta v + (v\cdot\nabla)v + \nabla p = -\lambda_1 \nabla\cdot[\nabla \mathbf{d}\otimes\nabla\mathbf{d}], \\ \text{div } v = 0, \\ \mathbf{d}_t + v\cdot\nabla\mathbf{d} = \nu_2(\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}), \ |\mathbf{d}| = 1. \end{cases} \tag{2.4}$$

Let us set $\mathbf{d} = (\cos\theta, \sin\theta)^T$. It follows that (see [21] for details)

$$\begin{aligned} &\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d} = \Delta\theta\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}, \\ &\nabla\mathbf{d}\otimes\nabla\mathbf{d} = \nabla\theta\otimes\nabla\theta. \end{aligned} \tag{2.5}$$

Thus we derive the following reformulation to (2.4) (see [21, 16] for details)

$$\begin{cases} v_t - \nu_1 \Delta v + (v\cdot\nabla)v + \nabla p = -\lambda_1 \nabla\cdot[\nabla\theta\otimes\nabla\theta], \\ \text{div } v = 0, \\ \theta_t + v\cdot\nabla\theta - \nu_2\Delta\theta = 0. \end{cases} \tag{2.6}$$

Note that the constraints $|\mathbf{d}| = 1$ is automatically satisfied since $\mathbf{d} = (\cos\theta, \sin\theta)^T$. The equivalence between the systems (2.4) and (2.6) is discussed in [16], where the authors studied the large-time behavior of (2.4) under some regularity conditions (referred to as a trigonometric condition) on the initial direction field. To simplify the notations, hereafter we set $\nu_1 = \lambda_1 = \nu_2 = 1$.

We also recall from [21] some basic energy laws for (2.4) and (2.6). By multiply $(2.4)_1$ by v, $(2.4)_3$ by $\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}$ and adding the resulting equalities gives

$$\frac{1}{2}\frac{d}{dt}(|v|_{L^2}^2 + |\nabla\mathbf{d}|_{L^2}^2) + \|v\|^2 + |A_1\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|_{L^2}^2 = 0, \tag{2.7}$$

which is a basic energy law for (2.4).

Substituting $\mathbf{d} = (\cos\theta, \sin\theta)^T$ into (2.7), we derive

$$\frac{1}{2}\frac{d}{dt}(|v|_{L^2}^2 + |\nabla\theta|_{L^2}^2) + \|v\|^2 + |A_1\theta|_{L^2}^2 = 0, \tag{2.8}$$

which is a basic energy law for (2.6).

Note that (2.8) can also be derived by multiply $(2.6)_1$ by v, $(2.6)_3$ by $A_1\theta$ and adding the resulting equalities.

Using similar idea as in (2.4), we rewrite (2.1) in the following form:

$$\begin{cases} dv + [-\Delta v + (v\cdot\nabla)v + \nabla p]\,dt = -\nabla\cdot[\nabla\theta\otimes\nabla\theta]\,dt \\ \qquad + g_1(t, v, \theta)dt + \int_Z \sigma(t, v, \theta, z)\tilde{\eta}(dt, dz), \\ \text{div } v = 0, \\ \theta_t + v\cdot\nabla\theta - \Delta\theta = 0. \end{cases} \tag{2.9}$$

For simplicity, we associate to (2.9) the following initial and boundary conditions

$$(v, \theta) = (0, 0) \text{ on } \partial\mathcal{M}, \quad (v, \theta)(0) = (v^0, \theta^0) \text{ in } \mathcal{M}. \tag{2.10}$$

Remark 2.1. The boundary condition

$$\frac{\partial\mathbf{d}}{\partial\eta} = 0 \text{ on } \partial\mathcal{M}, \tag{2.11}$$

where η is the outward normal to $\partial\mathcal{M}$, is often used for $\mathbf{d}$ (see [4]). If we set $\mathbf{d} = (\cos\theta, \sin\theta)^T$, it follows from (2.11) that

$$(-\sin\theta, \cos\theta)^T \frac{\partial\theta}{\partial\eta} = 0 \text{ on } \partial\mathcal{M}. \tag{2.12}$$

Taking the scalar product in $\Re^2$ of (2.12) with $(-\sin\theta, \cos\theta)^T$ gives

$$\frac{\partial\theta}{\partial\eta} = 0 \text{ on } \partial\mathcal{M}. \tag{2.13}$$

The results presented in this paper are also valid if we used (2.13) for the boundary condition for θ instead.

2.3. Mathematical setting. We first introduce a weak formulation of (2.9)-(2.10). Hereafter, we assume that the domain $\mathcal{M}$ is bounded with a smooth boundary $\partial\mathcal{M}$ (e.g., of class $\mathcal{C}^2$).

Hereafter, if X is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle_X$, we will denote the induced norm by $|\cdot|_X$, while X^* will indicate its dual. If X is a Banach space, we will denote by X^* the dual space of X. To simplify the notations, the duality paring between X and X^* will be denoted $\langle\cdot,\cdot\rangle$ and the norm in X^* will be denoted $\|\cdot\|_*$.

We set

$$\mathcal{V}_1 = \{u \in \mathcal{C}_c^\infty(\mathcal{M}) : \quad \text{div } u = 0 \text{ in } \mathcal{M}\}.$$

We denote by H_1 and V_1 the closure of $\mathcal{V}_1$ in $(L^2(\mathcal{M}))^2$ and $(H_0^1(\mathcal{M}))^2$ respectively. The scalar product in H_1 is denoted by $(\cdot,\cdot)_{L^2}$ and the associated norm by $|\cdot|_{L^2}$. Moreover, the space V_1 is endowed with the scalar product

$$((u,v)) = \sum_{i=1}^{2} (\partial_{x_i} u, \partial_{x_i} v)_{L^2}, \quad \|u\| = ((u,u))^{1/2}.$$

We now define the operator A_0 by

$$A_0 u = -\mathcal{P}_1 \Delta u, \ \forall u \in D(A_0) = (H^2(\mathcal{M}))^2 \cap V_1,$$

where $\mathcal{P}_1$ is the Leray-Helmotz projector in $L^2(\mathcal{M})$ onto H_1. Then, A_0 is a self-adjoint positive unbounded operator in H_1 which is associated with the scalar product defined above. Furthermore, A_0^{-1} is a compact linear operator on H_1 and $|A_0 \cdot|_{L^2}$ is a norm on $D(A_0)$ that is equivalent to the H^2-norm.

We denote by A_1 the Dirichlet Laplacian on $\Re^2$, that is

$$D(A_1) = \{\theta \in H^2(\mathcal{M}), \ \theta = 0 \text{ on } \partial\mathcal{M}\}, \ A_1\theta = -\Delta\theta, \ \forall\theta \in D(A_1).$$

Recall that

$$\langle A_1\theta_1, \theta_2\rangle = \int_{\mathcal{M}} \nabla\theta_1 \nabla\theta_2 dx, \ \forall\theta_1, \theta_2 \in H_0^1(\mathcal{M}).$$

Now we define the Hilbert spaces $\mathcal{H}$ and $\mathcal{U}$ by

$$\mathcal{H} = H_1 \times H_0^1(\mathcal{M}), \ \ \mathcal{U} = V_1 \times D(A_1), \tag{2.14}$$

endowed with the scalar products whose associated norms are respectively

$$|(v,\theta)|_{\mathcal{H}}^2 = |v|_{L^2}^2 + |\nabla\theta|_{L^2}^2 \ \text{ and } \ \|(v,\theta)\|_{\mathcal{U}}^2 = \|v\|^2 + |A_1\theta|_{L^2}^2. \tag{2.15}$$

Hereafter, we set

$$H_2 = H_0^1(\mathcal{M}),\ V_2 = D(A_1). \tag{2.16}$$

We introduce the bilinear operators B_0 and B_1 (and their associated trilinear forms b_0, b_1) defined from $V_1 \times V_1$ into V_1^* and $V_1 \times D(A_1)$ into $L^2(\mathcal{M})$ respectively by:

$$\begin{aligned}(B_0(u,v),w) &= \int_{\mathcal{M}} [(u\cdot\nabla)v]\cdot w dx = b_0(u,v,w),\ \forall u,v,w \in V_1,\\ (B_1(u,\theta),\rho) &= \int_{\mathcal{M}} [(u\cdot\nabla)\theta]\rho dx = b_1(u,\theta,\rho),\ \forall u \in V_1,\ \theta,\rho \in D(A_1).\end{aligned} \tag{2.17}$$

We recall that B_0 and B_1 satisfy the following estimates

$$\begin{aligned}|B_0(u,v)|_{V_1^*} &\le c|u|_{L^2}^{1/2}\|u\|^{1/2}|v|_{L^2}^{1/2}\|v\|^{1/2},\ \forall u,v \in V_1,\\ |B_1(u,\theta)|_{V_2^*} &\le c|u|_{L^2}^{1/2}\|u\|^{1/2}|\theta|_{L^2}^{1/2}\|\theta\|^{1/2},\ \forall u \in V_1, \theta \in V_2.\end{aligned} \tag{2.18}$$

We introduce the trilinear form r_0 defined by:

$$r_0(\theta,\psi,v) = -\sum_{i,j}^{2}\int_{\mathcal{M}} \frac{\partial\theta}{\partial x_i}\frac{\partial\psi}{\partial x_j}\frac{\partial v_i}{\partial x_j}dx,\ \forall\theta,\psi \in W_0^{1,4},\ v \in V_1. \tag{2.19}$$

Proposition 2.2. *There exists a constant $c > 0$ such that*

$$|r_0(\theta,\psi,v)| \le c\|\theta\|^{1/2}|A_1\theta|_{L^2}^{1/2}\|\psi\|^{1/2}|A_1\psi|_{L^2}^{1/2}\|v\|,\ \forall\theta,\psi \in D(A_1),\ v \in V_1. \tag{2.20}$$

Proof. The proof is given in [4, 6]. For the reader convenience, we repeat it. From (2.19), we easily derive that for any $\theta,\psi \in D(A_1),\ v \in V_1$, we have

$$\begin{aligned}|r_0(\theta,\psi,v)| &\le c|\nabla\theta|_{L^4}|\nabla\psi|_{L^4}|\nabla v|_{L^2}\\ &\le c\|\theta\|^{1/2}|A_1\theta|_{L^2}^{1/2}\|\psi\|^{1/2}|A_1\psi|_{L^2}^{1/2}\|v\|,\end{aligned} \tag{2.21}$$

and (2.20) is proved. □

Proposition 2.3. *There exists a bilinear operator R_0 defined on $D(A_1)$ with values in V_1^* such that*

$$\langle R_0(\theta,\psi),v\rangle = r_0(\theta,\psi,v),\ \forall\theta,\psi \in D(A_1),\ v \in V_1. \tag{2.22}$$

Moreover, there exists a constant $c > 0$ such that

$$\|R_0(\theta,\psi)\|_{V_1^*} \le c\|\theta\|^{1/2}|A_1\theta|_{L^2}^{1/2}\|\psi\|^{1/2}|A_1\psi|_{L^2}^{1/2},\ \forall\theta,\psi \in D(A_1). \tag{2.23}$$

Proof. The first part of the proposition follows from the fact that for any $v \in V_1$, the mapping $r_0(\cdot,\cdot,v)$ defined on $D(A_1)$ with values in $\Re$ is continuous. The estimate (2.23) also follows directly from (2.20). □

Proposition 2.4. *For any $v \in V_1$ and $\theta \in D(A_1)$, we have*

$$\langle B_1(v,\theta), A_1\theta\rangle = -\langle R_0(\theta,\theta),v\rangle. \tag{2.24}$$

Proof. To simplify the notations, we assume the summation over repeated indexes. Taking into account the fact that $\mathrm{div} v = 0$ as well as the boundary conditions, as in [4] we derive that

$$\begin{aligned}
-\langle B_1(v,\theta), A_1\theta\rangle &= \int_{\mathcal{M}} v_i \frac{\partial\theta}{\partial x_i}\frac{\partial^2\theta}{\partial x_j\partial x_j}dx \\
&= -\int_{\mathcal{M}} \frac{\partial v_i}{\partial x_j}\frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_j}dx - \int_{\mathcal{M}} v_i \frac{\partial^2\theta}{\partial x_i\partial x_j}\frac{\partial\theta}{\partial x_j}dx \\
&= -\int_{\mathcal{M}} \frac{\partial v_i}{\partial x_j}\frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_j}dx - \frac{1}{2}\int_{\mathcal{M}} v_i \frac{\partial|\nabla\theta|^2}{\partial x_i}dx \\
&= -\int_{\mathcal{M}} \frac{\partial v_i}{\partial x_j}\frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_j}dx \\
&= r_0(\theta,\theta,v) = \langle R_0(\theta,\theta), v\rangle.
\end{aligned} \tag{2.25}$$

□

Remark 2.5. We recall from [20] that

$$\mathrm{div}\,(\nabla\theta\otimes\nabla\theta) = \nabla\left(\frac{1}{2}|\nabla\theta|^2\right) - A_1\theta(\nabla\theta)^T. \tag{2.26}$$

It is clear that $\mathrm{div}(\nabla\theta\otimes\nabla\psi)\in L^2(\mathcal{M})$ for $\theta,\psi\in D(A_1)$, therefore

$$R_0(\theta,\psi) = \mathcal{P}_1\left[\mathrm{div}\,(\nabla\theta\otimes\nabla\psi)\right],\ \forall\theta,\psi\in D(A_1). \tag{2.27}$$

It follows from (2.26)-(2.27) that

$$\begin{aligned}
R_0(\theta,\theta) &= \mathcal{P}_1\left[\mathrm{div}\,(\nabla\theta\otimes\nabla\theta)\right] \\
&= \mathcal{P}_1\left[\nabla\left(\tfrac{1}{2}|\nabla\theta|^2\right)\right] - \mathcal{P}_1\left[A_1\theta(\nabla\theta)^T\right] = -\mathcal{P}_1\left[A_1\theta(\nabla\theta)^T\right].
\end{aligned} \tag{2.28}$$

Using the notations above, we rewrite (2.9)-(2.10) as

$$\begin{cases}
dv + [A_0 v + B_0(v,v) + R_0(\theta,\theta)]\,dt = g_1(t,v,\theta)dt \\
\qquad\qquad + \displaystyle\int_Z \sigma(t,v,\theta,z)\tilde{\eta}(dt,dz) \text{ in } V_1^*, \\
\theta_t + A_1\theta + B_1(v,\theta) = 0, \\
(v,\theta)(0) = (v^0,\theta^0)\in\mathcal{H}.
\end{cases} \tag{2.29}$$

We will denote by c a generic positive constant that depends on the domain $\mathcal{M}$.

Notations. We first recall from [23, 5] some notations and stochastic preliminaries.

Hereafter, by $\aleph$ we denote the set of nonnegative integers, i.e. $\aleph = \{0,1,2,\cdots\}$ and by $\bar{\aleph}$ we denote the set $\aleph\cup\{+\infty\}$. Whenever we speak about $\aleph$ (or $\bar{\aleph}$)$-$valued measurable functions we implicitly assume that the set is equipped with the trivial Σ-field $2^{\aleph}$ (or $2^{\bar{\aleph}}$). By $\Re_+$ we will denote the interval $[0,\infty)$ and by $\Re_*$ the set $\Re\setminus\{0\}$. If X is a topological space, then by $\mathcal{B}(X)$ we will denote the Borel $\Sigma-$field on X. By λ_d we will denote the Lebesgue measure on $(\Re^d,\mathcal{B}(\Re^d))$, by λ the Lebesgue measure on $(\Re,\mathcal{B}(\Re))$.

If $(S,\mathfrak{S})$ is a measurable space then by $M(S)$ we denote the set of all real valued measures on $(S,\mathfrak{S})$, and by $M(S)$ the $\Sigma-$field on $M(S)$ generated by the functions $i_B : M(S)\ni\varsigma\mapsto\varsigma(B)\in\Re, B\in S$. By $M_+(S)$ we denote the set of all nonnegative measures on S, and by $M(S)$ the $\Sigma-$field on $M_+(S)$ generated by the functions

$i_B : M_+(S) \ni \varsigma \mapsto \varsigma(B) \in \Re_+, B \in S$. Finally, by $M_I(S)$ we denote the family of all $\bar{\aleph}$valued measures on $(S, \mathfrak{S})$, and by $M_I(S)$ the $\Sigma-$field on $M_I(S)$ generated by functions $i_B : M(S) \ni \varsigma \mapsto \varsigma(B) \in \bar{\aleph}$, $B \in S$. If $(S, \mathfrak{S})$ is a measurable space then we will denote by $\mathfrak{S} \otimes \mathcal{B}(\Re_+)$ the product $\Sigma-$field on $S \times \Re_+$ and by $\nu \otimes \lambda$ the product measure of ν and the Lebesgue measure λ.

Preliminaries. As mentioned earlier we will study a stochastic model for a AC-NSE excited by random forces. We first describe the forces acting on the fluids. Let $(Z, \mathcal{Z})$ be a separable metric space and let ν be a $\Sigma-$finite positive measure on it. Suppose that $\mathfrak{P} = (\Omega, \mathfrak{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space, where $\mathbb{F} = (\mathfrak{F})_{t\geq 0}$ is a filtration satisfying the usual conditions, and $\eta : \Omega \times \mathcal{B}(\Re_+) \times Z \to \bar{\aleph}$ is a time homogeneous Poisson random measure, with intensity measure ν, defined over the filtered probability space $\mathfrak{P}$. A time homogeneous Poisson random measure defined over $\mathfrak{P}$ is given in the following definition.

Definition 2.6. Let Z be a metric space and $\mathcal{Z}$ its Borel $\Sigma-$algebra, ν a positive $\Sigma-$finite measure on $(Z, \mathcal{Z})$. A *Poisson random measure*, with intensity measure ν defined on $(Z, \mathbb{Z})$ over $\mathfrak{P}$, is a measurable map $\eta : (\Omega, \mathfrak{F}) \to (M_I(Z \times \Re_+), M_I(Z \times \Re+))$ satisfying the following conditions:

(i) for all $B \in \mathcal{B}(Z \otimes \Re_+), \eta(B) : \Omega \to \bar{\aleph}$ is a Poisson random measure with parameter $\mathbb{E}[\eta(B)]$;

(ii) η is independently scattered, i.e., if the sets $B_j \in \mathcal{B}(Z \otimes \Re_+), j = 1, ..., n$, are disjoint then the random variables $\eta(B_j), j = 1, \cdots, n$, are independent;

(iii) for all $U \in \mathcal{Z}$ and $I \in \mathcal{B}(\Re_+)$

$$\mathbb{E}[\eta(U \times I)] = \lambda(I)\nu(U);$$

(iv) for all $U \in \mathcal{Z}$ the $\bar{\aleph}-$valued process $(N(U, t))_{t\geq 0}$ defined by $N(U, t) := \eta(U \times (0, t]), t \geq 0$, is $\mathbb{F}-$adapted and its increments are independent of the past, i.e., if $t > s \geq 0$, then the random variable $N(U, t) - N(U, s) = \eta(U \times (s, t])$ is independent of $\mathfrak{F}_s$.

We will denote by $\tilde{\eta}$ the compensated Poisson random measure defined by

$$\tilde{\eta} := \eta - \gamma,$$

where the compensator $\gamma : \mathcal{B}(\mathcal{Z} \times \Re_+) \to \Re_+$ is defined by

$$\gamma(A \times I) = \lambda(I)\nu(A), I \in \mathcal{B}(\Re_+), A \in \mathcal{Z}.$$

As noted in [23], while items (i) and (ii) are the classical definition, see for e.g. Definition 6.1 in [46], of a Poisson Random measure η, the remaining items implicitly indicate that η is associated to a certain Lévy process $\tilde{L}$; see, for instance [[46], Proposition 4.16].

Let $\mathcal{M}^2(\Re_+, L^2(Z, \nu, H_1))$ be the class of all progressively measurable processes $\xi : \Re_+ \times Z \times \Omega \to H_1$ satisfying the condition

$$\mathbb{E}\int_0^T \int_Z |\xi(r, z)|^2_{L^2} \nu(dz)dr < \infty, \ \forall T > 0. \tag{2.30}$$

If $T > 0$, the class of all progressively measurable processes $\xi : [0, T] \times Z \times \Omega \to H_1$ satisfying the condition (2.30) just for this one T, will be denoted by

$\mathcal{M}^2(0,T,L^2(Z,\nu,H_1))$. Also, let $\mathcal{M}^2_{step}(\Re_+,L^2(Z,\nu,H_1))$ be the space of all processes $\xi\in\mathcal{M}^2(\Re_+,L^2(Z,\nu,H_1))$ such that

$$\xi(r)=\sum_{j=1}^{n}1_{(t_{j-1},t_j]}(r)\xi_j,\ 0\le r, \tag{2.31}$$

where $\{0=t_0<t_1<...<t_n<\infty\}$ is a partition of $[0,\infty)$, and for all $j,\ \xi_j$ is an $\mathfrak{F}_{t_{j-1}}-$ measurable random variable.

For any $\xi\in\mathcal{M}^2_{step}(\Re_+,L^2(Z,\nu,H_1))$, we set

$$\tilde{I}(\xi)=\sum_{j=1}^{n}\int_Z\xi_j(z)\tilde{\eta}(dz,(t_{j-1},t_j]).$$

This is basically the definition of stochastic integral of a random step process ξ with respect to the compound random Poisson measure $\tilde{\eta}$. The extension of this integral on $\mathcal{M}^2(\Re_+,L^2(Z,\nu,H_1))$ is possible thanks to the following result which is taken from [46], Theorem C.1.

Theorem 2.7. *There exists a unique bounded linear operator*

$$I:\mathcal{M}^2(\Re_+,L^2(Z,\nu;H_1))\to L^2(\Omega,\mathfrak{F};H_1)$$

such that for $\xi\in\mathcal{M}^2_{step}(\Re_+,L^2(Z,\nu,H_1))$ we have $\tilde{I}(\xi)=I(\xi)$. In particular, there exists a constant $C=C(H_1)$ such that for any $\xi\in\mathcal{M}^2(\Re_+,L^2(Z,\nu,H_1))$,

$$\mathbb{E}\left|\int_0^t\int_Z\xi(r,z)\tilde{\eta}(dz,dr)\right|^2_{L^2}\le C\mathbb{E}\int_0^t\int_Z|\xi(r,z)|^2_{L^2}\eta(dz)dr,\ t>0.$$

Moreover, for each $\xi\in\mathcal{M}^2(\Re_+,L^2(Z,\nu,H_1))$, the process $I(1_{[0,t]}\xi),\ t\ge0$, is an H_1-valued càdlàg martingale. The process $1_{[0,t]}\xi$ is defined by $[1_{[0,t]}\xi](r,z,\omega):=1_{[0,t]}(r)(r,z,\omega),\ t\ge0,r\in\Re_+,z\in Z$ and $\omega\in\Omega$.

As usual we will write

$$\int_0^t\int_Z\xi(r,z)\tilde{\eta}(dz,dr):=I(\xi)(t),\ t\ge0.$$

If $T>0$, we denote by $\mathbb{D}(0,T;H_1)$ the space of all càdlàg paths from $[0,T]$ into H_1.

Now we introduce the main set of hypotheses used in this article. As in [23, 5], we suppose that we are given a function σ satisfying the following set of constraints:

Condition 1. There exist nonnegative constants l_0,l_1,l_2 such that, for any $t\in[0,T]$ and all $(v_1,\theta_1),(v_2,\theta_2)\in\mathcal{H}$, we have

$$\begin{array}{l}|\sigma(t,v_1,\theta_1)|^p_{L^2(Z,\nu;H_1)}\le l_0+l_1|(v_1,\theta_1)|^p_{\mathcal{H}};\quad\text{for any } p\ge2,\\ |\sigma(t,v_1,\theta_1)-\sigma(t,v_2,\theta_2)|^2_{L^2(Z,\nu;H_1)}\le l_2|(v_1,\theta_1)-(v_2,\theta_2)|^2_{\mathcal{H}}.\end{array}\tag{2.32}$$

We assume that the external forcing g_1 is a measurable Lipschitz and sublinear mappings from $\Omega\times(0,T)\times H_1$ into V_1^*. More precisely, for all $(v_1,\theta_1),(v_2,\theta_2)\in V_1$,

$g_1(\cdot, v_1, \theta_1)$ is $\mathfrak{F}_t-$progressively measurable, and $d\mathbb{P} \times dt-$a.e. in $\Omega \times (0,T)$

$$\begin{aligned} &\|g_1(t,v_1,\theta_1) - g_1(t,v_2,\theta_2)\|_{V_1^*} \le L_1 |v_1 - v_2|_{L^2}, \\ &g_1(t,0,0) \in M^2_{\mathfrak{F}_t}(0,T;V_1^*). \end{aligned} \tag{2.33}$$

Finally, we assume that

$$(v_0, \theta_0) \in L^2(\Omega, \mathfrak{F}_0, \mathbb{P}; \mathcal{H}). \tag{2.34}$$

Hereafter, for any $(w, \psi) \in \mathcal{H}$, we set

$$\mathcal{E}(w, \psi) = |w|^2_{L^2} + \|\psi\|^2. \tag{2.35}$$

Definition 2.8. Let $(Z, \mathcal{Z})$ be a separable metric space on which is defined a $\Sigma-$finite measure ν and $(v_0, \theta_0) \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathcal{H})$. A *strong solution* to the problem (2.29) is a stochastic process (v, θ) such that

(1) $(v, \theta) = \{(v, \theta)(t),\ t \ge 0\}$ is a $\mathbb{F}-$progressively measurable process such that

$$\mathbb{E} \sup_{s \in [0,T]} \mathcal{E}(v(s), \theta(s)) + \mathbb{E} \int_0^T \|(v,\theta)(s)\|^2_{\mathcal{U}} dt < \infty,$$

(2) the following holds

$$\begin{aligned} (v(t), w) = (v_0, w) - &\int_0^t \langle A_0 v + B_0(v,v) + R_0(\theta,\theta) - g_1(s,v,\theta), w\rangle ds \\ + &\int_0^t \int_Z \langle \sigma(s,v,\theta,z), w\rangle \tilde{\eta}(dz,ds), \quad \forall w \in V_1, \\ (\theta(t), \psi) = (\theta_0, \psi) - &\int_0^t \langle A_1\theta + B_1(v,\theta), \psi\rangle ds = 0,\ \forall \psi \in V_2, \end{aligned} \tag{2.36}$$

for almost all $t \in [0,T]$ and $\mathbb{P}-$almost surely.

In the deterministic case, the weak formulation of (2.1) was proposed and studied in [36, 38, 35, 37], where the existence and uniqueness results for weak and strong solutions were proved.

Before we prove this result let us recall an important statement which is borrowed from [9].

Lemma 2.9. *Let X, Y, I and φ be non-negative processes and Z_1 be a non-negative integrable random variable. Assume that I is non-decreasing and that there exist non-negative constants $C, \alpha_1, \beta, \gamma_1, \delta_1$ and T satisfying first*

$$\int_0^T \varphi(s)ds \le C, \quad a.s.,\ \ 2\beta_1 e^C \le 1,\ \ 2\delta_1 e^C \le \alpha_1,$$

and secondly for all $t \in [0,T]$ there exists a constant $C_1 > 0$ such that

$$X(t) + \alpha_1 Y(t) \le Z_1 + \int_0^t \varphi(r)X(r)dr + I(t),\ \ a.s.,$$

$$\mathbb{E}I(t) \le \beta \mathbb{E}X(t) + \gamma_1 \int_0^t \mathbb{E}X(s)ds + \delta_1 \mathbb{E}Y(t) + C_1.$$

If $X \in L^\infty([0,T] \times \Omega)$, then we have

$$\mathbb{E}[X(t) + \alpha_1 Y(t)] \le 2\exp(C + 2t\gamma_1 e^C)(\mathbb{E}Z + C_1),\ \ t \in [0,T].$$

3. Existence and Uniqueness of Solutions

In this section, we prove the existence and the pathwise uniqueness of variational solution to (2.29).

Proposition 3.1. *If (v,θ) is a variational solution to (2.29), then (v,θ) satisfies*

$$\begin{aligned}\mathcal{E}(v,\theta)(t) \quad & +2\int_0^t \left(\|v(s)\|^2+|A_1\theta(s)|_{L^2}^2\right) ds = \mathcal{E}(v_0,\theta_0)\\ & +2\int_0^t \langle g_1(s,v(s),\theta(s)),v(s)\rangle ds+\int_0^t\int_Z \Upsilon(s,z)\eta(dz,ds)\\ & +2\int_0^t\int_Z (v(s-),\sigma(s,v(s),\theta(s),z))\tilde{\eta}(dz,ds),\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}\Upsilon(s,z) \quad & = |v(s-)+\sigma(s,v(s),\theta(s),z)|_{L^2}^2\\ & -|v(s-)|_{L^2}^2-(v(-s),\sigma(s,v(s),\theta(s),z)).\end{aligned} \tag{3.2}$$

Proof. Let us set

$$\mathcal{E}(v,\theta)=|v|_{L^2}^2+\|\theta\|^2.$$

Applying Itô's formula to $\mathcal{E}(v,\theta)$ and using (2.29), we derive that

$$\begin{aligned}\mathcal{E}(v,\theta)(t) \quad & = \mathcal{E}(v_0,\theta_0)+\int_0^t\int_Z \Upsilon(s,z)\eta(dz,ds)\\ & -2\int_0^t (\|v\|^2+|A_1\theta|_{L^2}^2)ds+2\int_0^t \langle g_1(s,v(s),\theta(s)),v(s)\rangle ds\\ & +2\int_0^t\int_Z (v(s-),\sigma(s,v(s),\theta(s),z))\tilde{\eta}(dz,ds),\end{aligned} \tag{3.3}$$

where $\Upsilon(s,z)$ is given by (3.2).

Note that we use the properties of B_0, B_1 and R_0 given in (2.18). In particular, we used the fact that (see [20])

$$\langle B_0(v,v),v\rangle=0,\ \langle R_0(\theta,\theta),v\rangle=-\langle B_1(v,\theta),A_1\theta\rangle.$$

□

Proposition 3.2. *We assume all the above hypotheses. Moreover, we suppose that $g_1(\cdot,0,0)\in L^4(\Omega,L^2(0,T;L^2(0,T;V_1^*))$, and $(v_0,\theta_0)\in L^2(\Omega,\mathfrak{F}_0,\mathbb{P};\mathcal{H})$ satisfies $\mathbb{E}\left[\mathcal{E}(v_0,\theta_0)\right]^2<\infty$. Then, there exists a unique solution $(v,\theta)\in L^4(\Omega,\mathbb{D}(0,T;H_1)\times\mathcal{C}(0,T;V_2))\cap L^4(\Omega,L^2(0,T;\mathcal{U}))$.*

Furthermore, the following estimate holds:

$$\mathbb{E}\sup_{t\in[0,T]}\mathcal{E}(v(t),\theta(t))+\mathbb{E}\int_0^T \|(v(s),\theta(s))\|_{\mathcal{U}}^2 ds\le C, \tag{3.4}$$

provided that $\mathbb{E}\mathcal{E}(v_0,\theta_0)<\infty$,

$$\mathbb{E}\sup_{t\in[0,T]}[\mathcal{E}(v(t),\theta(t))]^p+\mathbb{E}\left(\int_0^T \|(v(s),\theta(s))\|_{\mathcal{U}}^2 ds\right)^p\le C, \tag{3.5}$$

for any positive integer $p\ge 2$, provided that $\mathbb{E}[\mathcal{E}(v_0,\theta_0)]^p<\infty$.

Proof. Let $\{(w_i,\psi_i), i=1,2,3,\cdots\}\subset\mathcal{U}$ be an orthonormal basis of $\mathcal{H}$, where $\{w_i, i=1,2\cdots\}$, $\{\psi_i, i=1,2\cdots\}$ are eigenvectors of A_0 and A_1 respectively. We set $\mathcal{U}_m=\mathcal{H}_m=\mathrm{span}\{(w_1,\psi_1),\cdots(w_m,\psi_m)\}$. We look for $(v_m,\theta_m)\in\mathcal{H}_m$ solution to

$$\begin{aligned} dv_m(t) &= -\Pi_m^1\left[A_0v_m+B_0(v_m,v_m)+R_0(\theta_m,\theta_m)-g_1(s,v_m,\theta_m)\right]dt\\ &\quad+\int_Z \Pi_m^1\sigma(t,v_m(t-),\theta_m(t-),z)\tilde{\eta}(dt,dz),\\ d\theta_m(t) &= -\Pi_m^2\left[A_1\theta_m+B_1(v_m,\theta_m)\right]dt, \end{aligned} \tag{3.6}$$

where $\Pi_m\equiv(\Pi_m^1,\Pi_m^2)$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_m$.
As in the proof of Theorem 1.2.1 of [2], we can obtain the existence and uniqueness of a solution $(v_m,\theta_m)\in L^2(\Omega\times[0,T];\mathcal{U}_m)$ of (3.6) on an interval $[0,T_m]$.

For each $n\geq 1$, we consider the $\mathfrak{F}_t-$stopping time τ_n defined by:

$$\tau_n = T\wedge\inf_{t\in[0,T]}\left\{\ \mathcal{E}(v_m,\theta_m)(t)+\int_0^t(\|v_m(s)\|^2+|A_1\theta_m(s)|_{L^2}^2)ds\geq n^2\right\}, \tag{3.7}$$

where hereafter $a\wedge b=\min(a,b)$.

For fixed m, the sequence $\{\tau_n; n\geq 1\}$ is increasing to T. Throughout we fix $r\in[0,T]$ and $0\leq t\leq r\wedge\tau_n$. Now using Itô's formula, we derive that as in the proof of (3.1) that

$$\begin{aligned} \mathcal{E}(v_m,\theta_m)(t) &+2\int_0^t\left(\|v_m(s)\|^2+|A_1\theta_m(s)|_{L^2}^2\right)ds\\ &=\mathcal{E}(v_0,\theta_0)+2\int_0^t\langle g_1(s,v_m(s),\theta_m(s)),v_m(s)\rangle ds\\ &+\int_0^t\int_Z\Upsilon(s,z)\tilde{\eta}(dz,ds)\\ &+2\int_0^t\int_Z(v_m(s-),\sigma(s,v_m(s),\theta_m(s),z))\tilde{\eta}(dz,ds), \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} \Upsilon(s,z) &= |v_m(s-)+\sigma(s,v_m(s),\theta_m(s),z)|_{L^2}^2\\ &-|v_m(s-)|_{L^2}^2-(v_m(-s),\sigma(s,v_m(s),\theta_m(s),z)). \end{aligned}$$

From the fact that $|x|^2-|y|^2+|x-y|^2=2\langle x-y,x\rangle$, it follows that

$$\begin{aligned} \mathcal{E}(v_m(t),\theta_m(t)) &+2\int_0^t\left(\|v_m(s)\|^2+|A_1\theta_m(s)|_{L^2}^2\right)ds=\mathcal{E}(v_0,\theta_0)\\ &+2\int_0^t\langle g_1(s,v_m(s),\theta_m(s)),v_m(s)\rangle ds\\ &+\int_0^t\int_Z|\sigma(s,v_m(s),\theta_m(s),z)|_{L^2}^2\eta(dz,ds)\\ &+2\int_0^t\int_Z(v_m(s-),\sigma(s,v_m(s),\theta_m(s),z))\tilde{\eta}(dz,ds), \end{aligned} \tag{3.9}$$

We define the following stochastic processes

$$
\begin{aligned}
X(t) &= \sup_{s\in[0,t]} \mathcal{E}(v_m(s),\theta_m(s)),\\
Y(t) &= 2\int_0^t \left(\|v_m(s)\|^2 + |A_1\theta_m(s)|_{L^2}^2\right) ds,\\
I(t) &= 2\left|\int_0^t \int_Z (v_m(s-),\sigma(s,v_m(s),\theta_m(s),z))\tilde{\eta}(dz,ds)\right|\\
&\quad + \int_0^t \int_Z |\sigma(s,v_m(s),\theta_m(s),z)|_{L^2}^2 \eta(dz,ds)\\
&= \sup_{s\in[0,t]} |I_1(s)| + I_2(t),
\end{aligned}
$$

where

$$
\begin{aligned}
I_1(t) &= 2\int_0^t \int_Z (v_m(s-),\sigma(s,v_m(s),\theta_m(s),z))\tilde{\eta}(dz,ds),\\
I_2(t) &= \sup_{s\in[0,t]} \int_0^s \int_Z |\sigma(s,v_m(s),\theta_m(s),z)|_{L^2}^2 \eta(dz,ds).
\end{aligned}
\tag{3.10}
$$

Since $I_1(t)$ is a local martingale we can apply Burkholder-Davis-Gundy's inequality to derive that

$$
\begin{aligned}
&\mathbb{E} \sup_{s\in[0,r\wedge\tau_n]} |I_1(s)| \le \\
&\quad C\mathbb{E}\left(\int_0^{r\wedge\tau_n} \int_Z (v_m(s-),\sigma(s,v_m(s),\theta_m(s),z))^2 \nu(dz)ds\right)^{1/2}.
\end{aligned}
\tag{3.11}
$$

Thanks to Hölder's and Young's inequalities we have

$$
\begin{aligned}
\mathbb{E}\sup_{s\in[0,t]} |I_1(s)| \le\ & C\left[\epsilon\mathbb{E}\sup_{s\in[0,t]} |v_m(s)|_{L^2}^2\right]^{1/2} \times\\
&\left[\epsilon^{-1}\mathbb{E}\int_0^t\int_Z |\sigma(s,v_m(s),\theta_m(s),z)|_{L^2}^2 \nu(dz)ds\right]^{1/2}\\
&\le C\epsilon\mathbb{E}\sup_{s\in[0,t]} |v_m(s)|_{L^2}^2\\
&+C\epsilon^{-1}\mathbb{E}\int_0^t\int_Z |\sigma(s,v_m(s),\theta_m(s),z)|_{L^2}^2 \nu(dz)ds\\
&\le C\epsilon\mathbb{E}\sup_{s\in[0,t]} \mathcal{E}(v_m(s),\theta_m(s))\\
&+C\epsilon^{-1}\mathbb{E}\int_0^t\int_Z |\sigma(s,v_m(s),\theta_m(s),z)|_{L^2}^2 \nu(dz)ds.
\end{aligned}
\tag{3.12}
$$

Using (2.32), we derive that

$$
\mathbb{E}\sup_{s\in[0,t]} |I_1(s)| \le C\epsilon X(t) + C\epsilon^{-1} l_0 t + C\epsilon^{-1}\int_0^t \mathbb{E}X(s)ds. \tag{3.13}
$$

Next, we will deal with the second term of $I(t)$. Taking into account that the process

$$
\int_0^t \int_Z |\sigma(s,v_m(s),\theta_m(s),z)|^2 \eta(dz,ds)
$$

has only positive jumps, we derive from (2.32) that

$$\begin{aligned}\mathbb{E}I_2(t) \quad &\leq \mathbb{E}\int_0^t \int_Z |\sigma(s, v_m(s), \theta_m(s), z)|^2_{L^2}\nu(dz)ds \\ &\leq l_0 t + l_1 \int_0^t \mathbb{E}|(v_m(s), \theta_m(s)|^2_{\mathcal{H}} ds \\ &\leq l_0 t + l_1 \int_0^t \mathbb{E}X(s)ds.\end{aligned} \tag{3.14}$$

We also have

$$\begin{aligned}|2\langle g_1(s, v_m, \theta_m), v_m\rangle| \quad &\leq 2L_1|(v_m, \theta_m)|_{\mathcal{H}}\|v_m\| + 2\|g_1(s,0,0)\|_{V_1^*}\|v_m\| \\ &\leq \frac{1}{8}\|v_m\|^2 + cL_1^2|(v_m, \theta_m)|^2_{\mathcal{H}} + c\|g_1(s,0,0)\|^2_{V_1^*}.\end{aligned} \tag{3.15}$$

It follows from (3.8)-(3.15) that

$$\begin{aligned}\mathbb{E}\mathcal{E}(v_m, \theta_m)(t) \quad &+\mathbb{E}\int_0^t \left(\|v_m(s)\|^2 + |A_1\theta_m(s)|^2_{L^2}\right) ds \leq \mathbb{E}\mathcal{E}(v_0, \theta_0) \\ &+c\mathbb{E}\int_0^t \mathcal{E}(v_m, \theta_m)(s)ds + c\mathbb{E}\int_0^t \|g_1(s,0,0)\|^2_{V_1^*}ds.\end{aligned} \tag{3.16}$$

Therefore from Lemma 2.3, we derive that there exist a positive constant C such that

$$\mathbb{E}\mathcal{E}(v_m, \theta_m)(t) + \mathbb{E}\int_0^t \left(\|v_m(s)\|^2 + |A_1\theta_m(s)|^2_{L^2}\right) ds \leq C, \tag{3.17}$$

for any $m \in \aleph$ and $t \in [0, r \wedge \tau_n], r \in [0, T]$.

We have just shown that

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,t\wedge\tau_n]} \mathcal{E}(v_m, \theta_m)(t) \quad &+\mathbb{E}\int_0^t \left(\|v_m(s)\|^2 + |A_1\theta_m(s)|^2_{L^2}\right) ds \\ &\leq C, \ \forall t \in [0, T],\end{aligned} \tag{3.18}$$

from which we can infer that

$$\mathbb{P}(\tau_n < t) \leq Cn^{-2}, \ \forall t \in [0, T], \ \forall n > 0.$$

Hence, $\lim_{n\to+\infty} \mathbb{P}(\tau_n < t) = 0$, for all $t \in [0, T]$. That is, $\tau_n \to +\infty$ in probability. Therefore, there exists a subsequence τ_{n_k} such that $\tau_{n_k} \to +\infty$, a.s. Since the sequence $(\tau_n)_n$ is increasing, we infer that $\tau_{n_k} \nearrow +\infty$ a.s.. Now we use Fatou's lemma and pass to the limit in (3.18) and derive that

$$\mathbb{E}\sup_{s\in[0,t]} \mathcal{E}(v_m(s), \theta_m(s)) + \mathbb{E}\int_0^t \left(\|v_m(s)\|^2 + |A_1\theta_m(s)|^2_{L^2}\right) ds \leq C, \tag{3.19}$$

and (3.4) is proved.

To prove (3.5), we proceed as follows. By raising both sides of (3.8) to the power of $p \geq 2$, we derive that

$$\begin{aligned}\mathbb{E} \quad &\sup_{s\in[0,t]} [\mathcal{E}(v_m(t), \theta_m(t))]^p + 2\mathbb{E}\left[\int_0^t \left(\|v_m(s)\|^2 + |A_1\theta_m(s)|^2_{L^2}\right) ds\right]^p \\ &\leq \mathbb{E}[\mathcal{E}(v_0, \theta_0)]^p + c\mathbb{E}\sup_{s\in[0,t]} |I_3(s)|^p + c\mathbb{E}\sup_{s\in[0,t]} |I_4(s)|^p \\ &+ c\mathbb{E}\left[\int_0^t \|g_1(s,0,0)\|^2_{V_1^*}ds\right]^p,\end{aligned} \tag{3.20}$$

where

$$
\begin{aligned}
&I_3(t) = \\
&\int_0^t \int_Z \left\{ |v_m(s-) + \Pi_m^1 \sigma(s, v_m(s), \theta_m(s), t)|_{L^2}^2 - |v_m(s-)|_{L^2}^2 \right\} \tilde{\eta}(dz, ds),
\end{aligned}
\tag{3.21}
$$

$$
\begin{aligned}
I_4(t) \quad &= \\
&\int_0^t \int_Z \left\{ |v_m(s-) + \Pi_m^1 \sigma(s, v_m(s), \theta_m(s), t)|_{L^2}^2 - |v_m(s-)|_{L^2}^2 \nu(dz) ds \right\} \\
&- \int_0^t \int_Z (v_m(-s), \Pi_m^1 \sigma(s, v_m(s), \theta_m(s), z)) \nu(dz) ds \\
&\leq c \int_0^t \int_Z |\Pi_m^1 \sigma(s, v_m(s), \theta_m(s), z))|_{L^2}^2 \nu(dz) ds \\
&\leq c \int_0^t (1 + |(v_m, \theta_m)|_{\mathcal{H}}^2) ds.
\end{aligned}
\tag{3.22}
$$

As in [4, 6], we note that

$$
\begin{aligned}
\int_Z \quad & \left\{ |v_m(s-) + \Pi_m^1 \sigma(s, v_m(s), \theta_m(s), z)|_{L^2}^2 - |v_m(s-)|_{L^2}^2 \right\}^2 \nu(dz) \\
&\leq |v_m(s-)|_{L^2}^2 \int_Z |\sigma(s, v_m(s), \theta_m(s), z)|_{L^2}^2 \nu(dz) \\
&+ c \int_Z |\sigma(s, v_m(s), \theta_m(s), z)|_{L^2}^4 \nu(dz) \\
&\leq c_0 + c_1 |v_m(s)|_{L^2}^2 + c_2 |v_m(s)|_{L^2}^4 \\
&\leq k_1 + k_4 |v_m(s)|_{L^2}^4.
\end{aligned}
\tag{3.23}
$$

It follows that

$$
\begin{aligned}
&\left(\int_0^t \int_Z \left\{ |v_m(s-) + \Pi_m^1 \sigma(s, v_m(s), \theta_m(s), t)|_{L^2}^2 - |v_m(s-)|_{L^2}^2 \right\}^2 \nu(dz) ds \right)^{p/2} \\
&\qquad \leq c(k_1 T)^{p/2} + c(k_2)^{p/2} \left(\int_0^t |(v_m, \theta_m)(s)|_{\mathcal{H}}^4 ds \right)^{p/2}.
\end{aligned}
\tag{3.24}
$$

We derive that

$$
\begin{aligned}
\mathbb{E} \sup_{s \in [0,t]} \quad & |I_3(s)|^p \leq c_p (k_1 T)^{p/2} \\
&+ c_p (k_2)^{p/2} \mathbb{E} \left[\left(\int_0^t |(v_m, \theta_m)(s)|_{\mathcal{H}}^4 ds \right)^{p/2} \right] \\
&\leq c + \tfrac{1}{2} \mathbb{E} \left(\sup_{s \in [0,t]} |v_m(s)|_{L^2}^2 \right)^p + c \mathbb{E} \left(\int_0^t |v_m(s)|_{L^2}^2 ds \right)^p.
\end{aligned}
\tag{3.25}
$$

From Hölder's inequality, we have

$$
\begin{aligned}
\int_0^t |(v_m, \theta_m)(s)|_{\mathcal{H}}^2 ds \quad & \leq \left(\int_0^t |(v_m, \theta_m)(s)|_{\mathcal{H}}^{2p} ds \right)^{1/p} \left(\int_0^t 1 ds \right)^{\frac{p-1}{p}} \\
&\leq T^{\frac{p-1}{p}} \left(\int_0^t |(v_m, \theta_m)(s)|_{\mathcal{H}}^{2p} ds \right)^{1/p},
\end{aligned}
\tag{3.26}
$$

which gives

$$\left(\int_0^t |(v_m,\theta_m)(s)|_{\mathcal{H}}^2 ds\right)^p \le cT^{P-1}\int_0^t |(v_m,\theta_m)(s)|_{\mathcal{H}}^{2p} ds. \tag{3.27}$$

From (3.25), (3.26), we get

$$\begin{aligned}\mathbb{E} \quad & \sup_{s\in[0,t]} |I_3(s)|^p \le \frac{1}{2}\mathbb{E}\left(\sup_{s\in[0,t]} |(v_m,\theta_m)(s)|_{\mathcal{H}}^2\right)^p \\ & + c_{p,T}\int_0^t \mathbb{E}|(v_m,\theta_m)(s)|_{\mathcal{H}}^{2p} ds \\ & \le \tfrac{1}{2}\mathbb{E}\sup_{s\in[0,t]} [\mathcal{E}(v_m(s),\theta_m(s))]^p + c_{p,T}\int_0^t \mathbb{E}[\mathcal{E}(v_m(s),\theta_m(s))]^p ds.\end{aligned} \tag{3.28}$$

From (2.32) and (3.22), we also have

$$\begin{aligned}\mathbb{E}|I_4(t)|^p \quad & \le c\mathbb{E}\left(\int_0^t (1+|(v_m,\theta_m)(s)|_{\mathcal{H}}^2) ds\right)^p \\ & \le c_p + c_p\mathbb{E}\left(\int_0^t |(v_m,\theta_m)(s)|_{\mathcal{H}}^2) ds\right)^p.\end{aligned} \tag{3.29}$$

It follows that

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,t]} |I_2(s)|^p \quad & \le c_{p,T} + c_{p,T}\int_0^t |(v_m,\theta_m)(s)|_{\mathcal{H}}^{2p} ds \\ & \le c_{p,T} + c_{p,T}\int_0^t \mathbb{E}[\mathcal{E}(v_m(s),\theta_m(s))]^p ds.\end{aligned} \tag{3.30}$$

It follows from (3.20)-(3.30)

$$\begin{aligned}\mathbb{E} \quad & \sup_{s\in[0,t]} [\mathcal{E}(v_m(s),\theta_m(s))]^p \le c_{p,T} \\ & + c_{p,T}\int_0^t \mathbb{E}[\mathcal{E}(v_m(s),\theta_m(s))]^p ds + c\left(\int_0^t \|g_1(t,0,0)\|_{V_1^*}^2\right)^p,\end{aligned} \tag{3.31}$$

and Gronwall's lemma and (3.20) give

$$\mathbb{E}\sup_{s\in[0,t]} [\mathcal{E}(v_m(s),\theta_m(s))]^p + 2\mathbb{E}\left[\int_0^t \left(\|v_m(s)\|^2 + |A_1\theta_m(s)|_{L^2}^2\right) ds\right]^p \le C, \tag{3.32}$$

and (3.5) follows. □

Proposition 3.3. *We can extract from (v_m,θ_m) a subsequence still labeled the same and there exists a stochastic process (v,θ) such that*

$$\begin{aligned}&(v_m,\theta_m) \rightharpoonup (v,\theta) \text{ in } L^4(\Omega, L^\infty([0,T];\mathcal{H})),\\ &(v_m,\theta_m) \rightharpoonup (v,\theta) \text{ in } L^2(\Omega, L^2([0,T];\mathcal{U})),\\ &B_0(v_m,v_m) \rightharpoonup \beta_0^\flat,\ R_0(\theta_m,\theta_m) \rightharpoonup r_0^\flat \text{ in } L^2(\Omega\times[0,T];V_1^*),\\ &g_1(t,v_m,\theta_m) \rightharpoonup g_1^\flat \text{ in } L^2(\Omega\times[0,T];V_1^*),\\ &B_1(v_m,\theta_m) \rightharpoonup \beta_1^\flat \text{ in } L^2(\Omega\times[0,T];V_2^*),\\ &\sigma(t,v_m,\theta_m\cdot) \rightharpoonup \sigma^\flat \text{ in } L^2(\Omega\times[0,T];L^2(Z,\nu;H_1)).\end{aligned} \tag{3.33}$$

Proof. We note that

$$\begin{aligned}
&|B_0(v_m,v_m)|_{V_1^*}\le c|v_m|_{L^2}\|v_m\|,\\
&|R_0(\theta_m,\theta_m)|_{V_1^*}\le c\|\theta_m\||A_1\theta_m|_{L^2},\\
&|B_1(v_m,\theta_m)|_{V_2^*}\le c|v_m|_{L^2}^{1/2}\|v_m\|^{1/2}\|\theta_m\|^{1/2}|A_1\theta_m|_{L^2}^{1/2}.
\end{aligned}\tag{3.34}$$

It follows from (3.32),(3.34) that

$$\mathbb{E}\sup_{[0,T]}|(v_m,\theta_m)|_{\mathcal{H}}^2\le C,\quad \mathbb{E}\int_0^T\|(v_m(s),\theta_m(s)\|_{\mathcal{U}}^2ds\le C,\tag{3.35}$$

$$\mathbb{E}\int_0^T\Big[|B_0(v_m,v_m)|_{V_1^*}^2+|R_0(\theta_m,\theta_m)|_{V_1^*}^2+|B_1(v_m,\theta_m)|_{V_2^*}^2\Big]\,ds\le C\tag{3.36}$$

$$\mathbb{E}\sup_{[0,T]}|(v_m,\theta_m)|_{\mathcal{H}}^4\le C,\ \mathbb{E}\left[\int_0^T\|(v_m(s),\theta_m(s)\|_{\mathcal{U}}^2ds\right]^2\le C,\tag{3.37}$$

$$\begin{aligned}
\mathbb{E}\int_0^T &\|\sigma(s,v_m(s),\theta_m(s),z)\|_{L^2(Z,\nu,H_1)}^2\le l_0T\\
&+l_1\mathbb{E}\int_0^T|(v_m(s),\theta_m(s))|_{\mathcal{H}}^2ds\le C.
\end{aligned}\tag{3.38}$$

From (3.38), we can find a subsequence still denoted $\{(v_m,\theta_m)\}$ such that

$$\begin{aligned}
&(v_m,\theta_m)\rightharpoonup(v,\theta)\text{ in }L^4(\Omega,L^\infty([0,T];\mathcal{H})),\\
&(v_m,\theta_m)\rightharpoonup(v,\theta)\text{ in }L^2(\Omega\times[0,T];\mathcal{U}),\\
&B_0(v_m,v_m)\rightharpoonup\beta_0^\flat,\ R_0(\theta_m,\theta_m)\rightharpoonup r_0^\flat,\ \text{ in }L^2(\Omega\times[0,T];V_1^*),\\
&g_1(t,v_m,\theta_m)\rightharpoonup g_1^\flat,\ \text{ in }L^2(\Omega\times[0,T];V_1^*),\\
&B_1(v_m,\theta_m)\rightharpoonup\beta_1^\flat\text{ in }L^2(\Omega\times[0,T];V_2^*),\\
&\sigma(t,v_m,\theta_m\cdot)\rightharpoonup\sigma^\flat\text{ in }L^2(\Omega\times[0,T];L^2(Z,\nu;H_1)).
\end{aligned}\tag{3.39}$$

□

As in [5, 23], we can check that v is an H_1-valued càdlàg and $\mathbb{F}$-progressively measurable process, and θ is an V_2-valued continuous and $\mathbb{F}$-progressively measurable process. Moreover (v,θ) satisfies for all $0\le t\le T$

$$\begin{aligned}
v(t)+\int_0^t A_0v\,ds+\int_0^t(\beta_0^\flat(s)+r_0^\flat(s))ds&=v_0+\int_0^t g_1^\flat(s)ds\\
&+\int_0^t\int_Z\sigma^\flat(s,z)\tilde\eta(dz,ds),\\
\theta(t)+\int_0^t A_1\theta ds+\int_0^t\beta_1^\flat(s)ds=\theta_0,&
\end{aligned}\tag{3.40}$$

$\mathbb{P}-$a.s. as a equality in $\mathcal{U}^*$.

Proposition 3.4. *We have the following identities*

$$\beta_0^\flat=B_0(v,v),\ r_0^\flat=R_0(\theta,\theta),\ \beta_1^\flat=B_1(v,\theta),\ \sigma(t,v,\theta,z)=\sigma^\flat.\tag{3.41}$$

Proposition 3.5. *For any $n\ge1$ we have that as $m\longrightarrow+\infty$,*

$$1_{[0,\tau_n]}((v_m,\theta_m)-(v,\theta))\longrightarrow(0,0)\quad in\ L^2(\Omega\times[0,T];\mathcal{U}),\tag{3.42}$$

and

$$\mathbb{E}|(v_m,\theta_m(\tau_n))-(v,\theta)(\tau_n)|_{\mathcal{H}}\longrightarrow0\ as\ n\longrightarrow+\infty.\tag{3.43}$$

Proof. Let

$$(\tilde{v}_m, \tilde{\theta}_m) = \Pi_m(v, \theta)$$

where $\Pi_m \equiv (\Pi_m^1, \Pi_m^2)$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_m$. It follows that

$$\begin{aligned}
&|(\tilde{v}_m, \tilde{\theta}_m)|_{\mathcal{H}} \le |(v, \theta)|_{\mathcal{H}} \\
&\|(\tilde{v}_m, \tilde{\theta}_m)\|_{\mathcal{U}} \le c\|(v, \theta)\|_{\mathcal{U}}, \\
&(\tilde{v}_m, \tilde{\theta}_m) \to (v, \theta) \text{ in } \mathcal{U} \text{ for almost every } (\omega, t) \in \Omega \times [0, T], \\
&(\tilde{v}_m, \tilde{\theta}_m) \to (v, \theta) \text{ in } L^2(\Omega \times [0, T]; \mathcal{U}), \\
&\mathbb{E}(|(\tilde{v}_m, \tilde{\theta}_m)(\tau_n) - (v, \theta)(\tau_n)|_{\mathcal{H}}) \to 0 \text{ as } n \to +\infty.
\end{aligned} \tag{3.44}$$

From (3.6) and (3.40), we derive that

$$\begin{aligned}
&\langle \tilde{v}_m(t) - v_m(t), w_k\rangle + \int_0^t \langle A_0(\tilde{v}_m - v_m), w_k\rangle ds \\
&\quad + \int_0^t \langle \beta_0^\flat - B_0(v_m, v_m), w_k\rangle ds \\
&\quad + \int_0^t \langle r_0^\flat - R_0(\theta_m, \theta_m), w_k\rangle ds = \int_0^t \langle g_1^\flat - g_1(s, v_m, \theta_m), w_k\rangle ds \\
&\quad + \int_0^t \int_Z \Big[\sigma(s, v_m(s), \theta_m(s), z) - \sigma^\flat(s, z), w_k\Big] \tilde{\eta}(dz, ds), \\
&\langle \tilde{\theta}_m(t) - \theta_m(t), A_1\psi_k\rangle + \int_0^t \langle A_1(\tilde{\theta}_m - \theta_m), A_1\psi_k\rangle ds \\
&\qquad + \int_0^t \langle \beta_1^\flat - B_1(v_m, \theta_m), A_1\psi_k\rangle ds = 0, \ 1 \le k \le m.
\end{aligned} \tag{3.45}$$

Note that

$$\begin{aligned}
\beta_0^\flat - B_0(v_m, v_m) &= \beta_0^\flat - B_0(\tilde{v}_m, \tilde{v}_m) + B_0(\tilde{v}_m - v_m, \tilde{v}_m) + B_0(v_m, \tilde{v}_m - v_m), \\
r_0^\flat - R_0(\theta_m, \theta_m) &= r_0^\flat - R_0(\tilde{\theta}_m, \tilde{\theta}_m) + R_0(\tilde{\theta}_m - \theta_m, \tilde{\theta}_m) + R_0(\theta_m, \tilde{\theta}_m - \theta_m), \\
\beta_1^\flat - B_1(v_m, \theta_m) &= \beta_1^\flat - B_1(\tilde{v}_m, \tilde{\theta}_m) + B_1(\tilde{v}_m - v_m, \tilde{\theta}_m) + B_1(v_m, \tilde{\theta}_m - \theta_m).
\end{aligned}$$

Let us set $\vartheta_m = \tilde{v}_m - v_m, \ \rho_m = \tilde{\theta}_m - \theta_m$.

From the Itô's formula, we have

$$\begin{aligned}
&|\vartheta_m(t)|_{L^2}^2 + 2\int_0^t (\|\vartheta_m\|^2 + \langle \beta_0^\flat - B_0(v_m, v_m), \vartheta_m\rangle) ds \\
&\quad + 2\int_0^t \langle r_0^\flat - R_0(\theta_m, \theta_m), \vartheta_m\rangle ds \\
&\quad = 2\int_0^t \langle g_1^\flat - g_1(s, v_m, \theta_m), \vartheta_m\rangle ds + \int_0^t \int_Z \Upsilon(s, z)\eta(dz, ds) \\
&\quad + 2\int_0^t \int_Z \Big[\sigma(sv_m(s), \theta_m(s), z) - \Pi_m^1\sigma^\flat(s, z)\Big] \tilde{\eta}(dz, ds),
\end{aligned} \tag{3.46}$$

where

$$\begin{aligned}
\Upsilon(s, z) &= |v_m(s-) + \sigma_m(s, v_m(s), \theta_m(s), z) - \Pi_m^1\sigma^\flat(s, z)|_{L^2}^2 - |v_m(s-)|_{L^2}^2 \\
&\quad - 2\langle \sigma_m(s, v_m(s), \theta_m(s), z) - \Pi_m^1\sigma^\flat(s, z), v_m(s-)\rangle \\
&= |\sigma(s, v_m(s), \theta_m(s), z) - \Pi_m^1\sigma^\flat(s, z)|_{L^2}^2.
\end{aligned}$$

Replacing ψ_k in $(3.45)_2$ respectively by ρ_m gives

$$\begin{aligned}&\frac{d}{dt}\|\rho_m\|^2 + 2|A_1\rho_m|^2_{L^2} + 2\langle \beta_1^\flat - B_1(\tilde{v}_m, \tilde{\theta}_m), A_1\rho_m\rangle \\ &\quad +2\langle B_1(\vartheta_m, \tilde{\theta}_m), A_1\rho_m\rangle + 2\langle B_1(v_m, \rho_m), A_1\rho_m\rangle = 0.\end{aligned} \tag{3.47}$$

Note that

$$\begin{aligned}&\langle \beta_0^\flat - B_0(v_m, v_m), \vartheta_m\rangle = \langle \beta_0^\flat - B_0(\tilde{v}_m, \tilde{v}_m), \vartheta_m\rangle + \langle B_0(\vartheta_m, \tilde{v}_m), \vartheta_m\rangle \\ &\quad \leq \langle \beta_0^\flat - B_0(\tilde{v}_m, \tilde{v}_m), \vartheta_m\rangle + \tfrac{1}{4}\|\vartheta_m\|^2 + c\|\tilde{v}_m\|^2|\vartheta_m|^2_{L^2},\end{aligned} \tag{3.48}$$

$$\begin{aligned}\langle \quad &\beta_1^\flat - B_1(v_m, \theta_m), A_1\rho_m\rangle = \langle \beta_1^\flat - B_1(\tilde{v}_m, \tilde{\theta}_m), A_1\rho_m\rangle \\ &+\langle B_1(\vartheta_m, \tilde{\theta}_m), A_1\rho_m\rangle + \langle B_1(v_m, \rho_m), A_1\rho_m\rangle \\ &\leq \langle \beta_1^\flat - B_1(\tilde{v}_m, \tilde{\theta}_m), A_1\rho_m\rangle + \tfrac{1}{4}(\|\vartheta_m\|^2 + \tfrac{\xi}{4}|A_1\rho_m|^2_{L^2}) \\ &+c\|\theta_m\|^2|A_1\theta_m|^2_{L^2}|\vartheta_m|^2_{L^2} + c|\tilde{v}_m|^2_{L^2}\|\tilde{v}_m\|^2\|\rho_m\|^2,\end{aligned} \tag{3.49}$$

$$\begin{aligned}\langle \quad &r_0^\flat - R_0(\theta_m, \theta_m), \vartheta_m\rangle = \langle r_0^\flat - R_0(\tilde{\theta}_m, \tilde{\theta}_m), \vartheta_m\rangle \\ &+\langle R_0(\rho_m, \tilde{\theta}_m), \vartheta_m\rangle + \langle R_0(\theta_m, \rho_m), \vartheta_m\rangle \\ &\leq \langle r_0^\flat - R_0(\tilde{\theta}_m, \tilde{\theta}_m), \vartheta_m\rangle + \tfrac{1}{4}(\|\vartheta_m\|^2 + \tfrac{\xi}{4}|A_1\rho_m|^2_{L^2}) \\ &+c|A_1\theta_m|^2_{L^2}(|\vartheta_m|^2_{L^2} + \|\rho_m\|^2) + c\|\tilde{\theta}_m\|^2|A_1\tilde{\theta}_m|^2_{L^2}|\vartheta_m|^2_{L^2},\end{aligned} \tag{3.50}$$

$$\begin{aligned}|\Upsilon(s,z)|^2_{L^2} \quad &= |\sigma_m(s, v_m(s), \theta_m(s), z) - \Pi^1_m\sigma^\flat(s,z)|^2_{L^2} \\ &= |\Pi^1_m[\sigma(s, v_m(s), \theta_m(s), z) - \sigma(s, v(s), \theta(s), z)|^2_{L^2} \\ &-|\Pi^1_m[\sigma(s, v(s), \theta(s), z) - \sigma^\flat(s,z)|^2_{L^2} + S_1(s,z),\end{aligned} \tag{3.51}$$

where

$$S_1(s,z) = 2\langle \Pi^1_m[\sigma(s, v_m(s), \theta_m(s), z) - \sigma^\flat(s,z)], \Pi^1_m[\sigma(s, v(s), \theta(s), z) - \sigma^\flat(s,z)]\rangle.$$

From (2.32) and (3.51), we derive that

$$\begin{aligned}|\Upsilon(s,z)|^2_{L^2} \quad &\leq l_2|(v_m, \theta_m)(s)|^2_{\mathcal{H}} + l_2|(v_m, \theta_m)(s) - (v,\theta)(s)|^2_{\mathcal{H}} \\ &-|\Pi^1_m[\sigma(s, v(s), \theta(s), z) - \sigma^\flat(s,z)|^2_{L^2} + S_1.\end{aligned} \tag{3.52}$$

We also have

$$\begin{aligned}\langle g_1(s, \tilde{v}_m, \tilde{\theta}_m) - g_1(s, v_m, \theta_m), \vartheta_m\rangle \quad &\leq L_1|(\vartheta_m, \rho_m)|_{\mathcal{H}}\|\vartheta_m\| \\ &\leq \tfrac{1}{4}\|\vartheta_m\|^2 + cL_1^2|(\vartheta_m, \rho_m)|^2_{\mathcal{H}}.\end{aligned} \tag{3.53}$$

Let

$$\begin{aligned}Z(t) \quad &= |\vartheta_m(t)|^2_{L^2} + \|\rho_m(t)\|^2 = |(\tilde{v}_m - v_m)(t)|^2_{L^2} + \|(\tilde{\theta}_m - \theta_m)(t)\|^2, \\ Y_1(t) \quad &= c\|\tilde{v}_m\|^2 + c\|\theta_m\|^2|A_1\theta_m|^2_{L^2} + c|\tilde{v}_m|^2_{L^2}\|\tilde{v}_m\|^2 + c|A_1\theta_m|^2_{L^2} \\ &+c\|\tilde{\theta}_m\|^2|A_1\tilde{\theta}_m|^2_{L^2} + cL_2^2 + cL_1^2, \\ K_2(t) \quad &= \|\vartheta_m\|^2 + |A_1\rho_m|^2_{L^2}.\end{aligned}$$

Let

$$\delta(t) = \exp\left(-\int_0^t Y_1(s)ds\right),$$

Using (3.46)-(3.53), it followsfrom Itôs formula that

$$\begin{aligned}\mathbb{E}\delta(t)Z(t) \quad &+\mathbb{E}\int_0^t \delta(s)(K_2(s) + |\Pi_m^1[\sigma(s,v(s),\theta(s),z) - \sigma^\flat(s,z)|_{L^2}^2)ds \\ &\leq \mathbb{E}\int_0^t \delta(s)\langle -\beta_0^\flat + B_0(\tilde{v}_m,\tilde{v}_m),\vartheta_m\rangle ds \\ &+\mathbb{E}\int_0^t \delta(s)\langle -\beta_1^\flat + B_1(\tilde{v}_m,\tilde{\theta}_m), A_1\rho_m\rangle ds \\ &+\mathbb{E}\int_0^t \delta(s)\langle r_0^\flat - R_0(\tilde{\theta}_m,\tilde{\theta}_m),\vartheta_m\rangle ds \\ &+2\mathbb{E}\int_0^t\int_Z \delta(s)S_1(s,z)\eta(dz,ds) \\ &+\mathbb{E}\int_0^t \delta(s)\langle g_1^\flat - g_1(s,\tilde{v}_m,\tilde{\theta}),\vartheta_m\rangle ds.\end{aligned} \tag{3.54}$$

For each $n \geq 1$, we consider the $\mathfrak{F}_t-$stopping time τ_n defined by:

$$\tau_n = \min\left(T,\ \inf\left\{t\in[0,T];\ |(v,\theta)|_{\mathcal{H}}^2 + \int_0^t \|(v,\theta)\|_{\mathcal{U}}^2 ds \geq n^2\right\}\right).$$

We derive from (3.54) that

$$\begin{aligned}\mathbb{E}(\tau_n) \quad & \delta(s)Z(\tau_n) \\ &+c\mathbb{E}\int_0^{\tau_n} \delta(s)(K_2(s) + |\Pi_m^1[\sigma(s,v(s),\theta(s),z) - \sigma^\flat(s,z)|_{L^2}^2)ds \\ &\leq 2\mathbb{E}\int_0^{\tau_n} \delta(s)\langle -\beta_0^\flat + B_0(\tilde{v}_m,\tilde{v}_m) + g_1^\flat - g_1(s,\tilde{v}_m,\tilde{\theta}_m),\vartheta_m\rangle ds \\ &+2\mathbb{E}\int_0^{\tau_n} \delta(s)\langle r_0^\flat - R_0(\tilde{\theta}_m,\tilde{\theta}_m),\vartheta_m\rangle ds \\ &+2\mathbb{E}\int_0^{\tau_n} \delta(s)\langle -\beta_1^\flat + B_1(\tilde{v}_m,\tilde{\theta}_m), A_1\rho_m\rangle ds \\ &+2\mathbb{E}\int_0^{\tau_n}\int_Z \delta(s)S_1(s,z)\eta(dz,ds).\end{aligned} \tag{3.55}$$

Claim 1. The right side of (3.55) goes to 0 as m goes to $+\infty$.

(**i**). Since $\Pi_m^1\circ\Pi_m^1 = \Pi_m^1$ and $\|\Pi_m^1\| \leq 1$, it follows that
$1_{[0,\tau_n]}\delta(s)\Pi_m^1[\sigma(s,v(s),\theta(s),z) - \sigma^\flat(s,z)]$
is bounded in $L^2(\Omega\times[0,T];L^2(Z,\nu;H_1))$. Therefore, from $(3.39)_6$ we see that

$$\lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n}\int_Z \delta(s)S_1(s,z)\eta(dz,ds) = 0.$$

(**ii**). Let us now prove that

$$\lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n} \delta(s)\langle -\beta_0^\flat + B_0(\tilde{v}_m,\tilde{v}_m),\vartheta_m\rangle ds = 0. \tag{3.56}$$

We recall that

$$\begin{aligned}&(v_m,\theta_m) \rightharpoonup (v,\theta),\ (\tilde{v}_m,\tilde{\theta}_m) \to (v,\theta), \\ &(\tilde{v}_m,\tilde{\theta}_m) - (v_m,\theta_m) \rightharpoonup (0,0) \text{ in } L^2(\Omega\times[0,T];\mathcal{U}).\end{aligned} \tag{3.57}$$

We also have

$$\begin{aligned}\lim_{m\to\infty} \quad & \mathbb{E}\int_0^{\tau_n} \delta(s)\langle -\beta_0^\flat + B_0(\tilde{v}_m,\tilde{v}_m),\vartheta_m\rangle ds \\ &= \lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n} \delta(s)\langle -\beta_0^\flat + B_0(v,v),\vartheta_m\rangle ds \\ &+ \lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n} \delta(s)\langle -B_0(v,v) + B_0(\tilde{v}_m,\tilde{v}_m),\vartheta_m\rangle ds.\end{aligned} \tag{3.58}$$

From (3.57) and the fact that

$$1|_{[0,\tau_n]}\delta(t)(-\beta_0^\flat + B_0(v,v)) \in L^2(\Omega\times[0,T];V_1^*),$$

it follows that

$$\lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n} \delta(s)\langle -\beta_0^\flat + B_0(v,v),\vartheta_m\rangle ds = 0. \tag{3.59}$$

We also note that

$$\begin{aligned}&\| -B_0(v,v) + B_0(\tilde{v}_m,\tilde{v}_m)\|_{V_1^*} \\ &\quad\le c|\tilde{v}_m - v|_{L^2}^{1/2}\|\tilde{v}_m - v\|^{1/2}(\|\tilde{v}_m\|^{1/2}|\tilde{v}_m|_{L^2}^{1/2} + \|v\|^{1/2}|v|_{L^2}^{1/2}),\end{aligned}$$

which implies that

$$\begin{aligned}&\|1|_{[0,\tau_n]}(-B_0(v,v) + B_0(\tilde{v}_m,\tilde{v}_m))\|_{V_1^*} \to 0 \ \text{ as } m\to\infty,\ dt\times d\mathbb{P}-\text{a.e.},\\ &\|1|_{[0,\tau_n]}(-B_0(v,v) + B_0(\tilde{v}_m,\tilde{v}_m))\|_{V_1^*} \le cn\|v\| \in L^2(\Omega\times[0,T];\Re).\end{aligned}$$

It follows that

$$\lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n} \delta(s)\langle -B_0(v,v) + B_0(\tilde{v}_m,\tilde{v}_m),\vartheta_m\rangle ds = 0. \tag{3.60}$$

We conclude from (3.59) and (3.60) that

$$\lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n} \delta(s)\langle -\beta_0^\flat + B_0(\tilde{v}_m,\tilde{v}_m),\vartheta_m\rangle ds = 0, \tag{3.61}$$

which proves (3.56).

(**iii**). Next we will prove that

$$\lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n} \delta(s)\langle r_0^\flat - R_0(\tilde{\theta}_m,\tilde{\theta}_m),\vartheta_m\rangle ds = 0. \tag{3.62}$$

From (3.57) and the fact that

$$1|_{[0,\tau_n]}\delta(t)(r_0^\flat - R_0(\theta,\theta)) \in L^2(\Omega\times[0,T];V_1^*),$$

we also have

$$\lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n} \delta(s)\langle r_0^\flat - R_0(\theta,\theta),\vartheta_m\rangle ds = 0. \tag{3.63}$$

We also note that

$$\begin{aligned}\|R_0(\tilde{\theta}_m,\tilde{\theta}_m) - R_0(\theta,\theta)\|_{V_1^*} &= \|R_1(\tilde{\theta}_m,\tilde{\theta}_m) - R_1(\theta,\theta)\|_{V_1^*} \\ &\le c|A_1(\tilde{\theta}_m-\theta)|_{L^2}^{1/2}\|\tilde{\theta}_m-\theta\|^{1/2}\|\theta\|^{1/2}|A_1\theta|_{L^2}^{1/2} \\ &+c\|\tilde{\theta}_m-\theta\|^{1/2}|A_1(\tilde{\theta}_m-\theta)|_{L^2}^{1/2}|A_1\tilde{\theta}_m|_{L^2}^{1/2}\|\tilde{\theta}_m\|^{1/2},\end{aligned}$$

which implies that

$$\begin{aligned}&\|1|_{[0,\tau_n]}(R_0(\tilde{\theta}_m,\tilde{\theta}_m) - R_0(\theta,\theta))\|_{V_1^*} \to 0 \ \text{ as } m\to\infty,\ dt\times d\mathbb{P}-\text{a.e.},\\ &\|1|_{[0,\tau_n]}(R_0(\tilde{\theta}_m,\tilde{\theta}_m) - R_0(\theta,\theta))\|_{V_1^*} \le cn\|v\| \in L^2(\Omega\times[0,T];\Re).\end{aligned}$$

It follows that

$$\lim_{m\to\infty} \mathbb{E}\int_0^{\tau_n} \delta(s)\langle R_0(\tilde{\theta}_m,\tilde{\theta}_m) - R_0(\theta,\theta), \vartheta_m\rangle ds = 0. \tag{3.64}$$

We conclude from (3.63) and (3.64) that

$$\lim_{m\to\infty} \mathbb{E}\int_0^{\tau_n} \delta(s)\langle r_0^\flat - R_0(\tilde{\theta}_m,\tilde{\theta}_m), \vartheta_m\rangle ds = 0, \tag{3.65}$$

which proves (3.62).

(**iv**). Let us now prove that

$$\lim_{m\to\infty} \mathbb{E}\int_0^{\tau_n} \delta(s)\langle -\beta_1^\flat + B_1(\tilde{v}_m,\tilde{\theta}_m), A_1\rho_m\rangle ds = 0.$$

Following similar steps as in (3.61) and (3.65), can check that

$$\begin{aligned}\lim_{m\to\infty} \quad & \mathbb{E}\int_0^{\tau_n} \delta(s)\langle -\beta_1^\flat + B_1(\tilde{v}_m,\tilde{\theta}_m), A_1\rho_m\rangle ds \\ &= \lim_{m\to\infty} \mathbb{E}\int_0^{\tau_n} \delta(s)\langle -\beta_1^\flat + B_1(v,\theta), A_1\rho_m\rangle ds \\ &+ \lim_{m\to\infty} \mathbb{E}\int_0^{\tau_n} \delta(s)\langle -B_1(v,\theta) + B_1(\tilde{v}_m,\tilde{\theta}_m), A_1\rho_m\rangle ds = 0.\end{aligned}$$

(**v**). Let us also prove that

$$\lim_{m\to\infty} \mathbb{E}\int_0^{\tau_n} \delta(s)\langle g_1^\flat(s) - g_1(s,\tilde{v}_m,\tilde{\theta}_m), \vartheta_m\rangle ds = 0.$$

From (3.57) and the fact that

$1|[0,\tau_n]\delta(t)(g_1^\flat(t) - g_1(t,v,\theta)) \in L^2(\Omega\times[0,T];V_1^*)$ and
$1|[0,\tau_n]\delta(t)(g_1(t,v,\theta) - g_1(t,\tilde{v}_m,\tilde{\theta}_m)) \to 0$ in $L^2(\Omega\times[0,T];V_1^*)$ as $m\to\infty$,

we derive that

$$\begin{aligned}&\lim_{m\to\infty} \mathbb{E}\int_0^{\tau_n} \delta(s)\langle g_1^\flat(s) - g_1(s,v,\theta), \vartheta_m\rangle ds = 0,\\ &\lim_{m\to\infty} \mathbb{E}\int_0^{\tau_n} \delta(s)\langle g_1(s,v,\theta) - g_1(s,\tilde{v}_m,\tilde{\theta}_m), \vartheta_m\rangle ds = 0.\end{aligned} \tag{3.66}$$

Therefore, we derive that

$$\begin{aligned}\lim_{m\to\infty} \quad & \mathbb{E}\int_0^{\tau_n} \delta(s)\langle g_1^\flat(s) - g_1(s,\tilde{v}_m,\tilde{\theta}_m), \vartheta_m\rangle ds \\ &= \lim_{m\to\infty} \mathbb{E}\int_0^{\tau_n} \delta(s)\langle g_1^\flat(s) - g_1(s,v,\theta), \vartheta_m\rangle ds \\ &+ \lim_{m\to\infty} \mathbb{E}\int_0^{\tau_n} \delta(s)\langle g_1(s,v,\theta) - g_1(s,\tilde{v}_m,\tilde{\theta}_m), \vartheta_m\rangle ds = 0,\end{aligned}$$

The proof of the convergence of the other terms is similar.

Finally we conclude that the right side of (3.55) goes to 0 as m goes to $+\infty$.

Now using the fact that $1|[0,\tau_n]\delta(t) \leq 1$, we derive from (3.55) that

$$\begin{aligned} &\lim_{m\to\infty} \mathbb{E}|(\vartheta_m, \psi_m)(\tau_n)|^2_{\mathcal{H}} = \lim_{m\to\infty} \mathbb{E}\left(|(\tilde{v}_m, \tilde{\theta}_m)(\tau_n) - (v_m, \theta_m)(\tau_n)|^2_{\mathcal{H}}\right) \\ &\quad = 0, \\ &\lim_{m\to\infty} \mathbb{E}\int_0^{\tau_n} K_2(s)ds = \lim_{m\to\infty} \mathbb{E}\int_0^{\tau_n} \left(\|\vartheta_m\|^2 + |A_1\rho_m|^2_{L^2}\right) ds \\ &\quad = \lim_{m\to\infty} \mathbb{E}\int_0^{\tau_n} \left(\|\tilde{v}_m - v_m\|^2 + |A_1(\tilde{\theta}_m - \theta_m)|^2_{L^2}\right) ds = 0. \end{aligned} \tag{3.67}$$

□

We now give the proof of Proposition 3.4.

Proof of Proposition 3.4 Our goal is to prove that the following hold true.

$$\begin{aligned} &\sigma(s, v, \theta, z) = \sigma^\flat(s, z) \text{ in } L^2(\Omega \times [0,T]; L^2(Z, \nu, H_1)), \\ &B_0(v, v) = \beta_0^\flat \text{ in } L^2(\Omega \times [0,T]; V_1^*), \\ &R_0(\theta, \theta) = r_0^\flat \text{ in } L^2(\Omega \times [0,T]; V_1^*), \\ &B_1(v, \theta) = \beta_1^\flat \text{ in } L^2(\Omega \times [0,T]; V_2^*), \\ &g_1(t, v, \theta) = g_1^\flat(t) \text{ in } L^2(\Omega \times [0,T]; V_1^*). \end{aligned} \tag{3.68}$$

It is clear that $(3.68)_1$, follows from $(3.33)_6$.

To prove $(3.68)_2$, we proceed as follows. We note that from $(3.67)_2$ and (3.44), we also have

$$(v_m, \theta_m)|_{[0,\tau_n]} \to (v, \theta)|_{[0,\tau_n]} \text{ in } L^2(\Omega \times [0,T]; \mathcal{U}).$$

Therefore, for any $w \in L^\infty(\Omega \times [0,T]; V_1)$, we have

$$\begin{aligned} &\mathbb{E}\int_0^{\tau_n} \langle B_0(v, v) - B_0(v_m, v_m), w\rangle ds \\ &\quad \leq c\|w\|_{L^\infty(\Omega\times[0,T];V_1)} \times \mathbb{E}\int_0^{\tau_n} \|v_m - v\|^{1/2}|v_m - v|^{1/2}_{L^2}(\|v\| + \|v_m\|)ds, \end{aligned}$$

which gives

$$\lim_{m\to\infty} \mathbb{E}\int_0^{\tau_n} \langle B_0(v, v) - B_0(v_m, v_m), w\rangle ds = 0. \tag{3.69}$$

From $(3.39)_3$ and (3.69), we derive that

$$\mathbb{E}\int_0^{\tau_n} \langle B_0(v, v) - \beta_0^\flat, w\rangle = 0, \ \forall w \in L^\infty(\Omega \times [0,T]; V_1).$$

Since $\tau_n \uparrow T$ and $L^\infty(\Omega\times[0,T]; V_1)$ is dense in $L^2(\Omega\times[0,T]; V_1)$, we conclude that

$$B_0(v, v) = \beta_0^\flat \text{ in } L^2(\Omega \times [0,T]; V_1^*).$$

This proves $(3.68)_2$.

To prove $(3.68)_3$, we note that

$$\begin{aligned}\mathbb{E}\int_0^{\tau_n} \langle R_0(\theta,\theta)-R_0(\theta_m,\theta_m),w\rangle ds \leq c\|w\|_{L^\infty(\Omega\times[0,T];V_1)}\times\\ \mathbb{E}\int_0^{\tau_n} |A_1(\theta_m-\theta)|_{L^2}^{1/2}\|\theta\|^{1/2}|A_1\theta|_{L^2}^{1/2}ds\\ +c\|w\|_{L^\infty(\Omega\times[0,T];V_1)}\times\\ \mathbb{E}\int_0^{\tau_n} \|\theta_m-\theta\|^{1/2}|A_1(\theta_m-\theta)|_{L^2}^{1/2}|A_1\theta_m|_{L^2}ds,\end{aligned}$$

which gives

$$\lim_{m\to\infty}\mathbb{E}\int_0^{\tau_n} \langle R_0(\theta,\theta)-R_0(\theta_m,\theta_m),w\rangle ds = 0. \tag{3.70}$$

From $(3.39)_4$ and (3.70), we derive that

$$\mathbb{E}\int_0^{\tau_n} \langle R_0(\theta,\theta)-r_0^\flat,w\rangle = 0,\ \forall w\in L^2(\Omega\times[0,T];V_1),$$

which gives

$$R_0(\theta,\theta)=r_0^\flat \text{ in } L^2(\Omega\times[0,T];V_1^*),$$

and $(3.68)_3$ is proved.

Similarly, we can prove that

$$\begin{aligned}&B_1(v,\theta)=\beta_1^\flat \text{ in } L^2(\Omega\times[0,T];V_2^*),\\ &g_1(t,v,\theta)=g_1^\flat(t) \text{ in } L^2(\Omega\times[0,T];V_1^*).\end{aligned}$$

□

4. Pathwise Uniqueness and Convergence of the Whole Sequence of the Galerkin Approximation

In this part, we show the pathwise uniqueness and the convergence of the whole sequence of the Galerkin approximation to the solution (v,θ) of (2.29).

Proposition 4.1. *Let $(v_0^1,\theta_0^1),(v_0^2,\theta_0^2)$ be two $\mathfrak{F}_0-$measurable and square integrable $\mathcal{H}-$valued random variables. Let $(v_1,\theta_1),(v_2,\theta_2)$ be the strong solution to (2.29) corresponding to $(v_0^1,\theta_0^1),(v_0^2,\theta_0^2)$ respectively. Then there exists a constant $C>0$ such that*

$$\mathbb{E}\delta(t)|(v_1,\theta_1)-(v_2,\theta_2)|_{\mathcal{H}}^2\leq C\mathbb{E}|(v_0^1,\theta_0^1)-(v_0^2,\theta_0^2)|_{\mathcal{H}}^2, \tag{4.1}$$

for all $t\in[0,T]$, where $\delta(t)$ is defined by (4.13).

Moreover, if $(v_0^1,\theta_0^1)=(v_0^2,\theta_0^2)$ almost surely, then for any $t\in[0,T]$,

$$\mathbb{P}((v_1,\theta_1)(t)=(v_2,\theta_2)(t))=1. \tag{4.2}$$

Proof. Let $(v_1,\theta_1), (v_2.\theta_2)$ be variational solutions to (2.29). Let $(w,\psi) = (v_1,\theta_1) - (v_2,\theta_2)$. Then (w,ψ) satisfies

$$\begin{cases} dw + [A_0 w + B_0(v_2, w) + B_0(w, v_1)]dt + [R_0(\theta_2,\psi) + R_0(\psi,\theta_1)]dt \\ \qquad = [g_1(t, v_1, \theta_1) - g_1(t, v_2, \theta_2)]dt \\ \qquad + \displaystyle\int_Z (\sigma(t, v_1, \theta_1, z) - \sigma(t, v_2, \theta_2, z))\tilde{\eta}(dz, dt), \\ \dfrac{d\psi}{dt} + A_1\psi + B_1(v_2,\psi) + B_1(w,\theta_1) = 0, \\ (w,\psi)(0) = (0,0). \end{cases} \tag{4.3}$$

Reasoning as in the proof of Proposition 3.1, applying Itô's formula to $|w|^2_{L^2}$ and using $(4.3)_1$, we derive that

$$\begin{aligned} &|w|^2_{L^2} + 2\int_0^t (\|w\|^2 + b_0(w, v_1, w))ds + 2\int_0^t \langle R_0(\theta_2,\psi) + R_0(\psi,\theta_1), w\rangle ds \\ &\quad = 2\int_0^t \langle g_1(t, v_1, \theta_1) - g_1(t, v_2, \theta_2), w\rangle ds \\ &\quad +2\int_0^t\int_Z (w(s-), \sigma(s, v_1(s), \theta_1(s), z) - \sigma(s, v_2(s), \theta_2(s), z))\tilde{\eta}(dz, ds) \\ &\quad + \int_0^t\int_Z |\sigma(s, v_1(s), \theta_1(s), z) - \sigma(s, v_2(s), \theta_2(s), z))|^2_{L^2}\eta(ds, dz). \end{aligned} \tag{4.4}$$

Now we take the duality of $(4.3)_2$ with $A_1\psi$. Adding the resulting equality to (4.4), we derive that

$$\begin{aligned} &|w(t)|^2 + \|\psi(t)\|^2 + 2\int_0^t (\|w\|^2 + |A_1\psi|^2_{L^2})ds = -2\int_0^t b_0(w, v_1, w)ds \\ &\quad +2\int_0^t (\langle R_0(\theta_2,\psi), w\rangle + \langle R_0(\psi,\theta_1), w\rangle ds \\ &\quad -2\int_0^t (b_1(w,\theta, A_1\psi) + b_1(v,\psi, A_1\psi))ds \\ &\quad +2\int_0^t \langle g_1(t, v_1, \theta_1) - g_1(t, v_2, \theta_2), w\rangle ds \\ &\quad +2\int_0^t\int_Z (w(s-), \sigma(s, v_1(s), \theta_1(s), z) - \sigma(s, v_2(s), \theta_2(s), z))\tilde{\eta}(dz, ds) \\ &\quad + \int_0^t\int_Z |\sigma(s, v_1(s), \theta_1(s), z) - \sigma(s, v_2(s), \theta_2(s), z))|^2_{L^2}\eta(dz, ds). \end{aligned} \tag{4.5}$$

Note that

$$|b_0(w, v_1, w)| \le \frac{1}{8}\|w\|^2 + c\|v_1\|^2|w|^2_{L^2}, \tag{4.6}$$

$$\begin{aligned} |\langle R_0(\psi,\theta_1), w\rangle| \quad &= |b_1(w,\theta_1, A_1\psi)| \\ &\le \tfrac{1}{8}(\|w\|^2 + |A_1\psi|^2_{L^2}) + c|w|^2_{L^2}\|\theta_1\|^2|A_1\theta_1|^2_{L^2}, \end{aligned} \tag{4.7}$$

$$\begin{aligned} |\langle R_0(\theta_2,\psi), w\rangle| \quad &= |b_1(w,\psi, A_1\theta_2)| \\ &\le \tfrac{1}{8}(\|w\|^2 + |A_1\psi_2|^2_{L^2}) + c(|w|^2_{L^2} + |\nabla\psi|^2_{L^2})\|\theta_2\|^2|\theta_2|^2_{H^2}, \end{aligned} \tag{4.8}$$

$$|b_1(v_2,\psi, A_1\psi)| \le \frac{1}{8}|A_1\psi|^2_{L^2} + c|v_2|^2_{L^2}\|v_2\|^2\|\psi\|^2, \tag{4.9}$$

$$\begin{aligned} |\langle g_1(t,v_1,\theta_1)-g_1(t,v_2,\theta_2),w\rangle| &\le L_1\|w\||(w,\psi)|_{\mathcal{H}} \\ &\le \tfrac{1}{8}\|w\|^2+cL_1^2|(w,\psi)|^2_{\mathcal{H}} \end{aligned} \tag{4.10}$$

$$\|\sigma(s,v_1,\theta_1)-\sigma(s,v_2,\theta_2)\|^2_{L^2(Z,\mu,V_1)} \le l_2^2|(w,\psi)|^2_{\mathcal{H}}. \tag{4.11}$$

Let

$$\mathcal{Y}_2(t)=|w(t)|^2_{L^2}+\|\psi(t)\|^2,$$

and

$$\begin{aligned} K_1(t)=c(&\|v_1\|^2+\|\theta_1\|^2|A_1\theta_1|^2_{L^2}+\|\theta_2\|^2|A_1\theta_2|^2_{L^2}+|v_2|^2_{L^2}\|v_2\|^2) \\ &+Q_1(|\theta_1|_{H^1},|\theta_2|_{H^1})(|A_1\theta_1|^2_{L^2}+|A_1\theta_2|^2_{L^2})+cL_1^2+l_2^2, \end{aligned} \tag{4.12}$$

$$\delta(t)=\exp\left(-\int_0^t K_1(s)ds\right). \tag{4.13}$$

Applying Itô's formula to the process $\delta(t)\mathcal{Y}_2(t)$ and using (4.5)-(4.11), we derive that

$$\begin{aligned} \mathbb{E}\delta(t)\mathcal{Y}_2(t) \quad &+\mathbb{E}\int_0^t \delta(s)(\|w\|^2+|A_1\psi|^2_{L^2})ds \le \mathbb{E}\mathcal{Y}_2(0) \\ &+\mathbb{E}\int_0^t \delta(s)\mathcal{Y}_2(s)ds. \end{aligned} \tag{4.14}$$

Note that since $0<\delta(t)\le 1$, the expectation of the stochastic integral in (4.5) vanishes. Therefore we obtain

$$\mathbb{E}\delta(t)\mathcal{Y}_2(t)\le \mathbb{E}\mathcal{Y}_2(0)+\mathbb{E}\int_0^t \delta(s)\mathcal{Y}_2(s)ds,\ 0\le t\le T.$$

It follows from the Gronwall lemma that there exists a constant $C>0$ such that

$$\mathbb{E}\delta(t)\mathcal{Y}_2(t)\le C\mathbb{E}\mathcal{Y}_2(0),$$

for any $t\in[0,T]$, which proves the first part of the Proposition. Since $\delta(t)$ is bounded and positive $\mathbb{P}-$a.s., we conclude that the second part of the Proposition follows from the last estimate. □

Next we will show that the whole sequence of solutions to the Galerkin approximation (3.6) converges in mean square to the exact strong solution of (2.29).

Theorem 4.2. *The whole sequence of solutions to the Galerkin approximation* $\{(v_m,\theta_m);\ m\in\aleph\}$ *defined by (3.6) satisfies*

$$\begin{aligned} &\lim_{m\to\infty}\mathbb{E}|(v_m,\theta_m)(T-)-(v,\theta)(T-)|^2_{\mathcal{H}}=0, \\ &\lim_{m\to\infty}\mathbb{E}\|(v_m,\theta_m)-(v,\theta)\|^2_{\mathcal{U}}=0. \end{aligned} \tag{4.15}$$

Proof. For the proof, we first recall from [2, 7] the following lemma.

Lemma 4.3. *Let* $\{Q_m;\ m\ge 1\}\subset L^2(\Omega\times[0,T];\Re)$ *be a sequence of continuous real processes, and let* $\{\tau_n; n\ge 1\}$ *be a sequence of* $\mathfrak{F}_t-$*stopping times such that* $\tau_n\uparrow T$; $\sup_{m\ge1}\mathbb{E}|Q_m(T)|^2<\infty$, *and* $\lim_{m\to\infty}\mathbb{E}|Q_m(\tau_n)|=0$, *for* $n\ge 1$. *Then* $\lim_{m\to\infty}\mathbb{E}|Q_m(T)|=0$.

Applying Lemma 4.3 to $Q_m(t) = |(v,\theta) - (v_m,\theta_m)|^2_{\mathcal{H}}$ and $\delta_n = \tau_n$ and using (3.5), $(3.67)_1$ and the uniqueness of (v,θ), we conclude that the whole sequence given by (3.6) satisfies

$$\lim_{m\to\infty} \mathbb{E}|(v,\theta) - (v_m,\theta_m)|^2_{\mathcal{H}} = 0, \ \forall t \in [0,T].$$

Similarly, applying Lemma 4.3 to

$$Q_m(t) = \int_0^t \|(v,\theta)(s) - (v_m,\theta_m)(s)\|^2_{\mathcal{U}} ds$$

and using (3.5), $(3.67)_2$, we conclude that the whole sequence (v_m,θ_m) converges to (v,θ) strongly in $L^2(\Omega \times [0,T];\mathcal{U})$, i.e.,

$$\lim_{m\to\infty} \mathbb{E} \int_0^t \|(v,\theta)(s) - (v_m,\theta_m)(s)\|^2_{\mathcal{U}} ds = 0, \ \forall t \in [0,T].$$

$\square$

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