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NON-CONTINUOUS DOUBLE BARRIER REFLECTED BSDES WITH JUMPS UNDER A STOCHASTIC LIPSCHITZ COEFFICIENT

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Abstract. We consider a doubly reflected backward stochastic differential equations with jumps where the lower barrier and the opposite of the upper barrier are assumed to be right upper-semicontinuous (not necessarily *càdlàg*). We provide existence and uniqueness result when the coefficient is stochastic Lipschitz by using an equivalent transformation which is a coupled system of one-reflected backward stochastic differential equations.

1. Introduction

Backward Stochastic Differential Equations (BSDEs in short) were introduced (in the linear case) by Bismut [5]. The non-linear case was developed by Pardoux and Peng [32]. These equations have attracted great interest due to their connections with mathematical finance [11, 12], stochastic control and stochastic games [5, 19] and partial differential equations [33].

In their seminal paper [32], Pardoux and Peng generalized such equations to the Lipschitz condition and proved existence and uniqueness results in a Brownian framework. Other extensions include BSDEs with jumps on non-Brownian filtrations, which are driven, additionally, by a compensated Poisson random measure [35, 37]. Moreover, many efforts have been made to relax the Lipschitz condition on the coefficient [23, 27]. In this context, Bender and Kohlmann [4] considered the so-called stochastic Lipschitz condition introduced by El Karoui and Huang [8]. Later, some works have investigated this extension, especially [29, 30, 38].

Further, El Karoui et al. [9] have introduced the notion of reflected BSDEs (RBSDEs in short), which is a BSDE, but the solution is forced to stay above a given process called barrier. Once more under square integrability of terminal condition and the barrier, and Lipschitz property of the coefficient, the authors have proved the existence and uniqueness results in the case of a Brownian filtration and a continuous barrier. These equations have been proven to be powerful tools in mathematical finance [10], mixed game problems [20], providing a probabilistic

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formula for the viscosity solution of an obstacle problem for a class of parabolic partial differential equations [9].

Later, there have been several extensions to the case of a RBSDEs with jumps [21, 22]. Grigorova et al. [14] is the first paper investigate a new extension of the theory to RBSDEs in the case where the barrier is not necessarily right-continuous (just right upper-semicontinuous), the authors studied the existence and uniqueness result under the Lipschitz assumption on the coefficient. Since then, some works generalize the corresponding results of [14] in several directions [1, 24]. Recently, Marzougue and El Otmani [31] discussed RBSDEs with right upper-semicontinuous barrier under stochastic Lipschitz coefficient.

Doubly reflected BSDEs (DRBSDEs in short) have been introduced by Cvitanic and Karatzas [6] in the case of continuous barriers, a Brownian filtration and a Lipschitz coefficient. The solutions of such equations are constrained to stay between a lower barrier ξ and a upper barrier ζ. Many efforts have been made to relax the assumptions on parameters [2, 16, 28, 30], and [13, 17] for DRBSDEs with jumps. In the case of discontinuous barriers, Hamadène et al. [18] show the existence of a solution when the barriers and their left limits are completely separated. Later, Grigorova et al. [15] formulate a notion of DRBSDE in the case where the barriers do not satisfy any regularity assumption, the authors show existence and uniqueness result under the so-called Mokobodski's condition (assuming the existence of two strong supermartingales whose difference is between ξ and ζ) and a general Lipschitz driver. The interpretation of solution to this equations in terms of a two-stopper game problem which has been studied in [15].

Let us have a look at the Dynkin game problem whose the terminal time of the game is given by a stopping time $\tau \wedge \nu$, and the terminal payoff of the game (at $\tau \wedge \nu$) is given by

$$J(\tau,\nu) = \xi_\tau \mathbb{1}_{\{\tau \le \nu\}} + \zeta_\nu \mathbb{1}_{\{\nu < \tau\}}.$$

We consider the following generalization of the Dynkin game problem which the criterion is defined as

$$\mathcal{E}^f_{t,\tau\wedge\nu}(J(\tau,\nu)),$$

where $\mathcal{E}^f_{t,\tau\wedge\nu}(.)$ denotes the f-stochastic expectation at time t with terminal time equal to $\tau \wedge \nu$. We refer to this generalized game problem as $\mathcal{E}^f$-Dynkin game. Grigirova et al. [15] have shown that if ξ and $-\zeta$ are right upper-semicontinuous and satisfy Mokobodski's condition, then there exists a (common) value function for the $\mathcal{E}^f$-Dynkin game, that is

$$\inf_{\nu\in\mathcal{T}_{[0,T]}} \sup_{\tau\in\mathcal{T}_{[0,T]}} \mathcal{E}^f_{0,\tau\wedge\nu}(J(\tau,\nu)) = \sup_{\tau\in\mathcal{T}_{[0,T]}} \inf_{\nu\in\mathcal{T}_{[0,T]}} \mathcal{E}^f_{0,\tau\wedge\nu}(J(\tau,\nu))$$

where $\mathcal{T}_{[0,T]}$ is the collection of all stopping times $\tau \in [0,T]$ ($T > 0$ is a fixed horizon). But, this result remains valid just in the case when f is Lipschitz driver. So who can it say in the case of stochastic Lipschitz driver?

Inside the present paper, we consider a further extension of the theory to DRBSDEs in the case where the barriers are left limited, and the generator is stochastic Lipschitz in a filtration that supports a Brownian motion and an independent Poisson random measure. We show that the solution to DRBSDEs can be written in terms of difference between the solutions of a coupled system made of one-reflected

BSDEs, we show that this system admits a solution if and only if the Mokobodski's condition holds. To prove our result, we use some tools from the optimal stopping theory [25], other tools from the general theory of process [7] such as Mertens decomposition of strong optional supermartingale, and a generalization of Itô's formula to the case of strong optional supermartingale due to Gal'chouk and Lenglart [26]. Furthermore, a comparison theorem for the solutions of DRBSDEs will be established.

The paper is organized as follows: In Section 2, we consider the case where the generator does not depend on the solution, we then establish a priori estimate for solutions, and we give the coupled system equivalent to our DRBSDEs, further, we prove the existence and uniqueness of a (minimal) solution to DRBSDEs in this particular case. Section 3 is devoted to solve our DRBSDEs in the case of a general stochastic Lipshitz driver by using fixed point theorem. In Section 4, we give the comparison theorem for the solutions of DRBSDEs.

Preliminaries

Let T strictly positive real number. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\le T}, \mathbb{P})$ be a filtered probability space where $(\mathcal{F}_t)_{t\le T}$ is the natural filtration generated by a one-dimensional Brownian motion $(B_t)_{t\le T}$ and an independent Poisson random measure $\mu(dt, de)$ with compensator $\lambda(de)dt$. We denote by $\tilde{\mu}(dt, de)$ the compensated process, i.e. $\tilde{\mu}(dt, de) := \mu(dt, de) - \lambda(de)dt$. Let $(\mathrm{U}, \mathcal{U})$ be a measurable space equipped with a σ-finite positive measure λ where $\mathrm{U} := \mathbb{R}^l \setminus \{0\}$, $(l > 1)$. We will denote by $|.|$ the Euclidian norm on $\mathbb{R}^n$, $\mathcal{T}_{[t,T]}$ the set of stopping times τ such that $\tau \in [t, T]$ and $\mathcal{P}$ (resp. $\mathcal{O}$) be the predictable (resp. optional) σ-algebra on $\Omega \times [0, T]$.

Let's introduce some spaces:

- $\mathcal{H}^2$ is the space of $\mathbb{R}$-valued and predictable processes $(Z_t)_{t\le T}$ such that
$$\|Z\|^2_{\mathcal{H}^2} = \mathbb{E}\left[\int_0^T |Z_t|^2 dt\right] < +\infty.$$
- $\mathcal{S}^2$ is the space of $\mathbb{R}$-valued and optional processes $(K_t)_{t\le T}$ such that
$$\|K\|^2_{\mathcal{S}^2} = \mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} |K_\tau|^2\right] < +\infty.$$
- $\mathcal{L}_\lambda$ is the set of $\mathbb{R}^d$-valued and $\mathcal{P}\otimes\mathcal{U}$-measurable mapping $V : \Omega \times \mathrm{U} \to \mathbb{R}$ such that
$$\|V\|^2_\lambda = \int_{\mathrm{U}} |V(e)|^2 \lambda(de) < +\infty.$$
- $\mathcal{L}^2$ is the space of $\mathbb{R}^d$-valued and $\mathcal{P}\otimes\mathcal{U}$-predictable processes $(V_t)_{t\le T}$ such that
$$\|V\|^2_{\mathcal{L}^2} = \mathbb{E}\left[\int_0^T \|V_t\|^2_\lambda dt\right] < +\infty.$$

Let $\beta > 0$ and $(a(t))_{t\le T}$ be a nonnegative $\mathcal{F}_t$-adapted process. We define the increasing continuous process $A(t) = \int_0^t a^2(s)ds$ for all $t \le T$, and we introduce the following spaces:

- $\mathcal{S}^2(\beta,a)$ is the space of $\mathbb{R}$-valued and optional processes $(Y_t)_{t\leq T}$ such that
$$\|Y\|^2_{\mathcal{S}^2_\beta}=\mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} e^{\beta A(\tau)}|Y_\tau|^2\right]<+\infty.$$
- $\mathcal{S}^{2,a}(\beta,a)$ is the space of $\mathbb{R}$-valued and optional processes $(Y_t)_{t\leq T}$ such that
$$\|aY\|^2_{\mathcal{S}^{2,a}_\beta}=\mathbb{E}\left[\int_0^T e^{\beta A(t)}|a(t)Y_t|^2dt\right]<+\infty.$$
- $\mathcal{H}^2(\beta,a)$ is the space of $\mathbb{R}^d$-valued and predictable processes $(Z_t)_{t\leq T}$ such that
$$\|Z\|^2_{\mathcal{H}^2_\beta}=\mathbb{E}\left[\int_0^T e^{\beta A(t)}|Z_t|^2dt\right]<+\infty.$$
- $\mathcal{L}^2(\beta,a)$ is the space of $\mathbb{R}^d$-valued and $\mathcal{P}\otimes\mathcal{U}$-predictable processes $(V_t)_{t\leq T}$ such that
$$\|V\|^2_{\mathcal{L}^2_\beta}=\mathbb{E}\left[\int_0^T e^{\beta A(t)}\|V_t\|^2_\lambda dt\right]<+\infty.$$
- $\mathfrak{B}^2(\beta,a)=\mathcal{S}^2(\beta,a)\cap\mathcal{S}^{2,a}(\beta,a)$.

A function f is said to be a stochastic Lipschitz driver if

(i) $f\ :\ \Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathcal{L}_\lambda\longrightarrow\mathbb{R}$, $(\omega,t,y,z,v)\mapsto f(\omega,t,y,z,v)$ is $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable.

(ii) For all $(t,y,y',z,z',v,v')\in[0,T]\times\mathbb{R}^{1+1+d+d}\times\mathcal{L}_\lambda\times\mathcal{L}_\lambda$, there exists three nonnegative $\mathcal{F}_t$-adapted processes θ, γ and η such that
$$|f(t,y,z,v)-f(t,y',z',v')|\leq\theta(t)|y-y'|+\gamma(t)|z-z'|+\eta(t)\|v-v'\|_\lambda.$$
where $\theta(t)+\gamma^2(t)+\eta^2(t)=a^2(t)\geq\epsilon>0$.

(iii)
$$\forall t\in[0,T],\qquad\frac{f(t,0,0,0)}{a}\in\mathcal{H}^2(\beta,a).$$

For a *làdlàg* process Y, we denote Y_{t+} (resp. Y_{t-}) the right-hand (resp. left-hand) limit of Y at t. We denote by $\Delta_+Y_t:=Y_{t+}-Y_t$ the size of the right jump of Y at t and by $\Delta Y_t:=Y_t-Y_{t-}$ the size of the left jump.

Let $\xi=(\xi_t)_{t\leq T}$ and $\zeta=(\zeta_t)_{t\leq T}$ be two left limited process in $\mathcal{S}^2(\beta,a)$ such that $\xi_t\leq\zeta_t$ for all $t\leq T$ a.s. and $\xi_T=\zeta_T$ a.s. We suppose moreover that the processes ξ and $-\zeta$ are right upper-semicontinuous ($r.u.s.c$ in short). A pair of process (ξ,ζ) will be called a pair of admissible barriers.

We will denote by $\mathcal{R}ef[.]$, the operator induced by a reflected BSDEs, defined as following
$$\begin{array}{cccc}\mathcal{R}ef & : & \mathcal{S}^2(\beta,a) & \longrightarrow & \mathfrak{B}^2(\beta,a)\\ & & \xi & \longmapsto & \mathcal{R}ef[\xi]=X\end{array}$$
i.e. X is the first component of solution to reflected BSDEs associated with parameters $(0,\xi)$ (the driver 0 and the lower barrier ξ).

Definition 1.1. Let f be a stochastic Lipschitz driver and $(\xi_t, \zeta_t)_{t\leq T}$ be a pair of admissible barriers. A process $(Y, Z, V, K^+, K^-, C^+, C^-)$ is said to be a *solution* to doubly reflected BSDE with parameters (f, ξ, ζ), if

(i) $(Y, Z, V, K^+, K^-, C^+, C^-) \in \mathfrak{B}^2(\beta, a) \times \mathcal{H}^2(\beta, a) \times \mathcal{L}^2(\beta, a) \times \left(\mathcal{S}^2\right)^4$,

(ii)
$$Y_t = \xi_T + \int_t^T f(s, Y_s, Z_s, V_s)ds + (K_T^+ - K_t^+) - (K_T^- - K_t^-) - \int_t^T Z_s dB_s - \int_t^T \int_{\mathrm{U}} V_s(e)\tilde{\mu}(ds, de) + (C_{T-}^+ - C_{t-}^+) - (C_{T-}^- - C_{t-}^-), \tag{1.1}$$

(iii) $\xi_t \leq Y_t \leq \zeta_t \quad \forall t \leq T$ $a.s.$

(iv) K^+ and K^- are nondecreasing right-continuous predictable processes with $K_0^+ = K_0^- = 0$,
$$\int_0^T \mathbb{1}_{\{Y_{t-} > \xi_{t-}\}} dK_t^+ = 0 \ a.s. \text{ and } \int_0^T \mathbb{1}_{\{Y_{t-} < \zeta_{t-}\}} dK_t^- = 0 \ a.s. \tag{1.2}$$

(v) $C^+$and C^- are nondecreasing right-continuous adapted purely discontinuous processes with $C_{0-}^+ = C_{0-}^- = 0$,
$$(Y_\tau - \xi_\tau)\Delta C_\tau^+ = 0 \ a.s. \text{ and } (Y_\tau - \zeta_\tau)\Delta C_\tau^- = 0 \ a.s. \quad \forall \tau \in \mathcal{T}_{[0,T]}, \tag{1.3}$$

(vi) $dK_t^+ \perp dK_t^-$ and $dC_t^+ \perp dC_t^-$.

Remark 1.2. (i) The constraints $dK_t^+ \perp dK_t^-$ and $dC_t^+ \perp dC_t^-$ will allow us to obtain the uniqueness of the nondecreasing processes K^+, K^-, C^+ and C^- without the strict separability condition ($\xi_t < \zeta_t$ and $\xi_{t-} < \zeta_{t-}$ for all $t \leq T$).

(ii) Due to equation (1.1), we have
$$\Delta C_t^+(\omega) - \Delta C_t^-(\omega) = -\Delta_+ Y_t(\omega), \quad \forall (\omega, t) \in \Omega \times [0, T].$$
This, together with the condition $dC_t^+ \perp dC_t^-$ gives $\Delta C_t^+ = (\Delta_+ Y_t)^-$ (the negative part of $\Delta_+ Y_t$) and $\Delta C_t^- = (\Delta_+ Y_t)^+$ (the positive part of $\Delta_+ Y_t$) for all $t \leq T$ a.s.

On the other hand, since in our framework the filtration is quasi-left-continuous, martingales have only totally inaccessible jumps. Hence, for each predictable stopping time $\tau \in \mathcal{T}_{[0,T]}$, $\Delta K_\tau^{+,d} - \Delta K_\tau^{-,d} = -\Delta Y_\tau$ (consequence of equation (1.1)), where $K^{\pm,d}$ is the discontinuous parts of $K^\pm$. This, together with the condition $dK_t^+ \perp dK_t^-$, ensures that for each predictable stopping time $\tau \in \mathcal{T}_{[0,T]}$, $\Delta K_\tau^{+,d} = (\Delta Y_\tau)^-$ and $\Delta K_\tau^{-,d} = (\Delta Y_\tau)^+$ a.s.

We denote also that Y can jump (on the left) at totally inaccessible stopping times; these jumps of Y come from the jumps of the stochastic integral with respect to $\tilde{\mu}$ in equation (1.1).

Proposition 1.3. *Let $(Y, Z, V) \in \mathcal{S}^2(\beta, a) \times \mathcal{H}^2(\beta, a) \times \mathcal{L}^2(\beta, a)$ where Y is a làdlàg process. Then $\left(\int_0^t e^{\beta A(s)} Y_s Z_s dB_s\right)_{t\leq T}$ and $\left(\int_0^t \int_{\mathrm{U}} e^{\beta A(s)} Y_{s-} V_s(e)\tilde{\mu}(ds, de)\right)_{t\leq T}$ are the martingales.*

Proof. For all $\nu \leq \tau \leq T$ we have

$$\begin{aligned}\int_\nu^\tau e^{2\beta A(s)}|Y_s|^2|Z_s|^2 ds &\leq \int_0^\tau e^{2\beta A(s)} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} |Y_\tau|^2|Z_s|^2 ds \\ &\leq \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} e^{\beta A(\tau)}|Y_\tau|^2 \int_0^\tau e^{\beta A(s)}|Z_s|^2 ds.\end{aligned}$$

Hence

$$\mathbb{E}\left[\sqrt{\int_\nu^\tau e^{2\beta A(s)}|Y_s|^2|Z_s|^2 ds}\right] \leq \mathbb{E}\left[\sqrt{\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} e^{\beta A(\tau)}|Y_\tau|^2}\sqrt{\int_0^T e^{\beta A(s)}|Z_s|^2 ds}\right].$$

Since $(Y,Z)\in\mathcal{S}^2(\beta,a)\times\mathcal{H}^2(\beta,a)$ then we get the finite expectation. Hence, one can derive that $\mathbb{E}\left[\int_\nu^\tau e^{\beta A(s)}Y_sZ_s dB_s|\mathcal{F}_\nu\right]=0$. Since $\left(\int_0^t e^{\beta A(s)}Y_sZ_sdB_s\right)_{t\leq T}$ is $\mathcal{F}_t$-adapted process, then it is a martingale.

Now, let's use the left continuity of trajectory of the process Y_{s-}, we have

$$|Y_{s-}(\omega)|^2 \leq \sup_{t\in[0,T]\cap\mathbb{Q}} |Y_{t-}(\omega)|^2 \qquad \forall(s,\omega)\in[0,T]\times\Omega.$$

On the other hand, we have $|Y_{t-}|^2 \leq \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}}|Y_\tau|^2$, then

$$\sup_{t\in[0,T]\cap\mathbb{Q}}|Y_{t-}|^2 \leq \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}}|Y_\tau|^2.$$

From the above equations, we obtain for all $\nu\leq\tau\leq T$

$$\begin{aligned}&\int_\nu^\tau\int_{\mathrm{U}} e^{2\beta A(s)}|Y_{s-}|^2|V_s(e)|^2\lambda(de)ds \\ &\qquad\leq \int_0^\tau e^{2\beta A(s)}\sup_{t\in[0,T]\cap\mathbb{Q}}|Y_{t-}|^2|V_s(e)|^2\lambda(de)ds \\ &\qquad\leq \int_0^\tau e^{2\beta A(s)}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}}|Y_\tau|^2|V_s(e)|^2\lambda(de)ds \\ &\qquad\leq \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} e^{\beta A(\tau)}|Y_\tau|^2\int_0^\tau e^{\beta A(s)}|V_s(e)|^2\lambda(de)ds.\end{aligned}$$

Hence

$$\begin{aligned}&\mathbb{E}\left[\sqrt{\int_\nu^\tau\int_{\mathrm{U}} e^{2\beta A(s)}|Y_{s-}|^2|V_s(e)|^2\lambda(de)ds}\right] \\ &\qquad\qquad\leq \mathbb{E}\left[\sqrt{\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} e^{\beta A(\tau)}|Y_\tau|^2}\sqrt{\int_0^T e^{\beta A(s)}\|V_s\|_\lambda^2 ds}\right].\end{aligned}$$

Since $(Y,V)\in\mathcal{S}^2(\beta,a)\times\mathcal{L}^2(\beta,a)$ then we get the finite expectation. Hence, one can derive that $\mathbb{E}\left[\int_\nu^\tau\int_{\mathrm{U}} e^{\beta A(s)}Y_{s-}V_s(e)\tilde{\mu}(ds,de)|\mathcal{F}_\nu\right]=0$, which implies that the process $\left(\int_0^t\int_{\mathrm{U}} e^{\beta A(s)}Y_{s-}V_s(e)\tilde{\mu}(ds,de)\right)_{t\leq T}$ is a martingale, since it's a $\mathcal{F}_t$-adapted process. □

2. Special Case of Solution to DRBSDEs

In this section we prove existence and uniqueness of the solution to DRBSDE with parameters (f, ξ, ζ) in the special case which is the stochastic Lipschitz driver f does not depend on (y, z, v). Let $f(t, y, z, v) = g(t)$.

We first will proved a priori estimate which is a consequently to uniqueness of solution.

2.1. A priori estimate.

Lemma 2.1. *Let $(Y^i, Z^i, V^i, K^{+,i}, K^{-,i}, C^{+,i}, C^{-,i})$ be the solution to the DRBSDE with parameters (g^i, ξ^i, ζ^i) for $i = 1, 2$. We denote $\bar{\Gamma} := \Gamma^1 - \Gamma^2$ with $\Gamma := Y, Z, V, K^{\pm}, C^{\pm}, g, \xi, \zeta$. Then there exists a constant C_β depending on β such that*

$$\|\bar{Y}\|^2_{\mathfrak{B}^2_\beta} + \|\bar{Z}\|^2_{\mathcal{H}^2_\beta} + \|\bar{V}\|^2_{\mathcal{L}^2_\beta} \quad \leq \quad C_\beta \left(\left\| \frac{\bar{g}}{a} \right\|^2_{\mathcal{H}^2_\beta} + \|\bar{\xi}\|^2_{\mathcal{S}^2_\beta} + \|\bar{\zeta}\|^2_{\mathcal{S}^2_\beta} \right).$$

Proof. Let $\tau \in \mathcal{T}_{[0,T]}$, one can derive that

$$\begin{aligned} \bar{Y}_\tau \quad = \quad & \bar{\xi}_T + \int_\tau^T \bar{g}(s)ds + (\bar{K}^+_T - \bar{K}^+_\tau) - (\bar{K}^-_T - \bar{K}^-_\tau) - \int_\tau^T \bar{Z}_s dB_s \\ & - \int_\tau^T \int_{\mathrm{U}} \bar{V}_s(e)\tilde{\mu}(ds, de) + (\bar{C}^+_{T-} - \bar{C}^+_{\tau-}) - (\bar{C}^-_{T-} - \bar{C}^-_{\tau-}). \end{aligned}$$

We denote $M_\tau = \int_0^\tau \bar{Z}_s dB_s + \int_0^\tau \int_{\mathrm{U}} \bar{V}_s(e)\tilde{\mu}(ds, de)$, $N_\tau = -\int_0^\tau \bar{g}(s)ds - \bar{K}^+_\tau + \bar{K}^-_\tau$ and $W_\tau = -\bar{C}^+_{\tau-} + \bar{C}^-_{\tau-}$. Then the process $\bar{Y}$ is an optional strong semimartingale with decomposition $\bar{Y}_\tau = \bar{Y}_0 + M_\tau + N_\tau + W_\tau$. Using the Corollary A.6, we have

$$\begin{aligned} & e^{\beta A(t)}|\bar{Y}_t|^2 \\ = \quad & |\bar{Y}_0|^2 + \int_0^t \beta e^{\beta A(s)} a^2(s)|\bar{Y}_s|^2 ds + \int_0^t 2e^{\beta A(s)}\bar{Y}_{s-}d(M + N)_s \\ & + \frac{1}{2}\int_0^t 2e^{\beta A(s)} d < M^c, M^c >_s + \sum_{0<s\leq t} e^{\beta A(s)} \left[\bar{Y}^2_s - \bar{Y}^2_{s-} - 2\bar{Y}_{s-}\Delta\bar{Y}_s\right] \\ & + \int_0^t 2e^{\beta A(s)}\bar{Y}_s d(W)_{s+} + \sum_{0\leq s<t} e^{\beta A(s)} \left[\bar{Y}^2_{s+} - \bar{Y}^2_s - 2\bar{Y}_s\Delta_+\bar{Y}_s\right]. \end{aligned}$$

Moreover, $\bar{Y}^2_s - \bar{Y}^2_{s-} - 2\bar{Y}_{s-}\Delta\bar{Y}_s = (\Delta\bar{Y}_s)^2$ and $\bar{Y}^2_{s+} - \bar{Y}^2_s - 2\bar{Y}_s\Delta_+\bar{Y}_s = (\Delta_+\bar{Y}_s)^2$, then

$$\begin{aligned} & e^{\beta A(t)}|\bar{Y}_t|^2 + \int_t^T \beta e^{\beta A(s)} a^2(s)|\bar{Y}_s|^2 ds + \int_t^T e^{\beta A(s)}|\bar{Z}_s|^2 ds \\ = \quad & e^{\beta A(T)}|\bar{\xi}_T|^2 + 2\int_t^T e^{\beta A(s)}\bar{Y}_s\bar{g}(s)ds + 2\int_t^T e^{\beta A(s)}\bar{Y}_{s-}(d\bar{K}^+_s - d\bar{K}^-_s) \\ & -2\int_t^T e^{\beta A(s)}\bar{Y}_s\bar{Z}_s dB_s - 2\int_t^T \int_{\mathrm{U}} e^{\beta A(s)}\bar{Y}_{s-}\bar{V}_s(e)\tilde{\mu}(ds, de) - \sum_{t<s\leq T} e^{\beta A(s)}(\Delta\bar{Y}_s)^2 \\ & +2\int_t^T e^{\beta A(s)}\bar{Y}_s(d\bar{C}^+_{s+} - d\bar{C}^-_{s+}) - \sum_{t\leq s<T} e^{\beta A(s)}(\Delta_+\bar{Y}_s)^2. \end{aligned} \tag{2.1}$$

Let us first show that $\bar{Y}_{s-}(d\bar{K}_s^+ - d\bar{K}_s^-) \leq 0$ and $\bar{Y}_s(\Delta\bar{C}_s^+ - \Delta\bar{C}_s^-) \leq 0$; indeed, using the property (1.2) and (1.3), cited in Definition 1.1, respectively, we have

$$\begin{aligned}\bar{Y}_{s-}(d\bar{K}_s^+ - d\bar{K}_s^-) &= -(Y_{s-}^1 - \xi_{s-})dK_s^{+,2} - (Y_{s-}^2 - \xi_{s-})dK_s^{+,1} \\ &\quad +(Y_{s-}^1 - \zeta_{s-})dK_s^{-,2} + (Y_{s-}^2 - \zeta_{s-})dK_s^{-,1} \\ &\leq 0,\end{aligned}$$

and

$$\begin{aligned}\bar{Y}_s(\Delta\bar{C}_s^+ - \Delta\bar{C}_s^-) &= -(Y_s^1 - \xi_s)\Delta C_s^{+,2} - (Y_s^2 - \xi_s)\Delta C_s^{+,1} \\ &\quad +(Y_s^1 - \zeta_s)\Delta C_s^{-,2} + (Y_s^2 - \zeta_s)\Delta C_s^{-,1} \\ &\leq 0.\end{aligned}$$

On the other hand, we have

$$2\int_t^T e^{\beta A(s)}\bar{Y}_s\bar{g}(s)ds \leq \frac{\beta}{2}\int_t^T e^{\beta A(s)}a^2(s)|\bar{Y}_s|^2ds + \frac{2}{\beta}\int_t^T e^{\beta A(s)}\left|\frac{\bar{g}(s)}{a(s)}\right|^2 ds.$$

Consequently, the equation (2.1) lead to the following inequality

$$\begin{aligned}&e^{\beta A(t)}|\bar{Y}_t|^2 + \int_t^T \frac{\beta}{2}e^{\beta A(s)}a^2(s)|\bar{Y}_s|^2ds + \int_t^T e^{\beta A(s)}|\bar{Z}_s|^2ds \\ &\leq e^{\beta A(T)}|\bar{\xi}_T|^2 + \frac{2}{\beta}\int_t^T e^{\beta A(s)}\left|\frac{\bar{g}(s)}{a(s)}\right|^2 ds - 2\int_t^T e^{\beta A(s)}\bar{Y}_s\bar{Z}_sdB_s \\ &-2\int_t^T\int_{\mathrm{U}} e^{\beta A(s)}\bar{Y}_{s-}\bar{V}_s(e)\tilde{\mu}(ds,de) - \sum_{t<s\leq T} e^{\beta A(s)}(\Delta\bar{Y}_s)^2 - \sum_{t\leq s<T} e^{\beta A(s)}(\Delta_+\bar{Y}_s)^2.\end{aligned} \tag{2.2}$$

By Remark 1.2, the processes $K^{\pm,1}$ and $K^{\pm,2}$ jumps only at predictable stopping times and $\mu(.,de)$ jumps only at totally inaccessible stopping times, then we can note that

$$\begin{aligned}&\sum_{t<s\leq T} e^{\beta A(s)}(\Delta\bar{Y}_s)^2 \\ &= \int_t^T\int_{\mathrm{U}} e^{\beta A(s)}|\bar{V}_s(e)|^2\mu(ds,de) + \sum_{t<s\leq T} e^{\beta A(s)}(\Delta\bar{K}_s^+ - \Delta\bar{K}_s^-)^2.\end{aligned}$$

Hence

$$\begin{aligned}&\int_t^T e^{\beta A(s)}\|\bar{V}_s\|_\lambda^2 ds - \sum_{t<s\leq T} e^{\beta A(s)}(\Delta\bar{Y}_s)^2 \\ &= \int_t^T e^{\beta A(s)}\|\bar{V}_s\|_\lambda^2 ds - \int_t^T\int_{\mathrm{U}} e^{\beta A(s)}|\bar{V}_s(e)|^2\mu(ds,de) \\ &\quad - \sum_{t<s\leq T} e^{\beta A(s)}(\Delta\bar{K}_s^+ - \Delta\bar{K}_s^-)^2 \\ &\leq -\int_t^T\int_{\mathrm{U}} e^{\beta A(s)}|\bar{V}_s(e)|^2\tilde{\mu}(ds,de).\end{aligned}$$

By adding the term $\int_t^T e^{\beta A(s)}\|\bar{V}_s\|_\lambda^2 ds$ on both sides of inequality (2.2), we get

$$\begin{aligned} & e^{\beta A(t)}|\bar{Y}_t|^2 + \int_t^T e^{\beta A(s)} \left\{ \frac{\beta}{2} a^2(s)|\bar{Y}_s|^2 + |\bar{Z}_s|^2 + \|\bar{V}_s\|_\lambda^2 \right\} ds \\ \leq \quad & e^{\beta A(T)}|\bar{\xi}_T|^2 + \frac{2}{\beta}\int_t^T e^{\beta A(s)} \left|\frac{\bar{g}(s)}{a(s)}\right|^2 ds - 2\int_t^T e^{\beta A(s)}\bar{Y}_s\bar{Z}_s dB_s \\ & - \int_t^T \int_{\mathrm{U}} e^{\beta A(s)} \left\{2\bar{Y}_{s-}\bar{V}_s(e) + |\bar{V}_s(e)|^2\right\} \tilde{\mu}(ds, de). \end{aligned} \tag{2.3}$$

Observe that

$$e^{\beta A(T)}|\bar{\xi}_T|^2 = \frac{1}{4}e^{\beta A(T)}|\bar{\xi}_T + \bar{\zeta}_T|^2 \leq \frac{1}{2}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} e^{\beta A(\tau)}|\bar{\xi}_\tau|^2 + \frac{1}{2}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} e^{\beta A(\tau)}|\bar{\zeta}_\tau|^2.$$

Taking expectation on the both sides of the inequality (2.3) with $t = 0$ and using the Proposition 1.3, we get

$$\frac{\beta}{2}\|a\bar{Y}\|^2_{\mathcal{S}^{2,a}_\beta} + \|\bar{Z}\|^2_{\mathcal{H}^2_\beta} + \|\bar{V}\|^2_{\mathcal{L}^2_\beta} \leq \frac{2}{\beta}\left\|\frac{\bar{g}}{a}\right\|^2_{\mathcal{H}^2_\beta} + \frac{1}{2}\|\bar{\xi}\|^2_{\mathcal{S}^2_\beta} + \frac{1}{2}\|\bar{\zeta}\|^2_{\mathcal{S}^2_\beta}. \tag{2.4}$$

On the other hand, from (2.2) we also get for all $\tau \in \mathcal{T}_{[0,T]}$

$$\begin{aligned} & e^{\beta A(\tau)}|\bar{Y}_\tau|^2 \\ \leq \quad & \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} e^{\beta A(\tau)}|\bar{\xi}_\tau|^2 + \frac{2}{\beta}\int_0^T e^{\beta A(s)}\left|\frac{\bar{g}(s)}{a(s)}\right|^2 ds - 2\int_0^T e^{\beta A(s)}\bar{Y}_s\bar{Z}_s dB_s \\ & +2\int_0^\tau e^{\beta A(s)}\bar{Y}_s\bar{Z}_s dB_s - 2\int_0^T\int_{\mathrm{U}} e^{\beta A(s)}\bar{Y}_{s-}\bar{V}_s(e)\tilde{\mu}(ds, de) \\ & +2\int_0^\tau\int_{\mathrm{U}} e^{\beta A(s)}\bar{Y}_{s-}\bar{V}_s(e)\tilde{\mu}(ds, de). \end{aligned}$$

Taking the essential supremum over $\tau \in \mathcal{T}_{[0,T]}$ and then the expectation on both sides of the above inequality, using the fact that

$$\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} X_\tau = \sup_{t\leq T} X_t$$

for all *càdlàg* process X (see Remark A.1 in Grigorova et al. [14]) and Burkholder-Davis-Gundy's inequality, we have

$$\begin{aligned} 2\mathbb{E}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}}\left|\int_0^\tau e^{\beta A(s)}\bar{Y}_s\bar{Z}_s dB_s\right| &= 2\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t e^{\beta A(s)}\bar{Y}_s\bar{Z}_s dB_s\right| \\ &\leq 2c\mathbb{E}\left[\sqrt{\int_0^T e^{2\beta A(s)}|\bar{Y}_s|^2|\bar{Z}_s|^2 ds}\right] \\ &\leq \frac{1}{4}\|\bar{Y}\|^2_{\mathcal{S}^2_\beta} + 4c^2\|\bar{Z}\|^2_{\mathcal{H}^2_\beta} \end{aligned}$$

and

$$
\begin{aligned}
2\mathbb{E}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}}&\left|\int_0^\tau\int_{\mathrm{U}} e^{\beta A(s)}\bar{Y}_{s-}\bar{V}_s(e)\tilde{\mu}(ds,de)\right| \\
&\le 2c\mathbb{E}\left[\sqrt{\int_0^T\int_{\mathrm{U}} e^{2\beta A(s)}|\bar{Y}_{s-}|^2|\bar{V}_s(e)|^2\mu(ds,de)}\right] \\
&\le \frac{1}{4}\mathbb{E}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} e^{\beta A(\tau)}|\bar{Y}_\tau|^2 + 4c^2\mathbb{E}\int_0^T\int_{\mathrm{U}} e^{\beta A(s)}|\bar{V}_s(e)|^2\mu(ds,de) \\
&= \frac{1}{4}\|\bar{Y}\|^2_{\mathcal{S}^2_\beta} + 4c^2\|\bar{V}\|^2_{\mathcal{L}^2_\beta},
\end{aligned}
$$

where c is a universal positive constant. Then

$$
\|\bar{Y}\|^2_{\mathcal{S}^2_\beta} \le \frac{1}{2}\|\bar{\xi}\|^2_{\mathcal{S}^2_\beta} + \frac{1}{2}\|\bar{\zeta}\|^2_{\mathcal{S}^2_\beta} + \frac{2}{\beta}\left\|\frac{\bar{g}}{a}\right\|^2_{\mathcal{H}^2_\beta} + \frac{1}{2}\|\bar{Y}\|^2_{\mathcal{S}^2_\beta} + 4c^2\|\bar{Z}\|^2_{\mathcal{H}^2_\beta} + 4c^2\|\bar{V}\|^2_{\mathcal{L}^2_\beta}.
$$

By (2.4), It follows that

$$
\|\bar{Y}\|^2_{\mathcal{S}^2_\beta} \le C_\beta\left(\left\|\frac{\bar{g}}{a}\right\|^2_{\mathcal{H}^2_\beta} + \|\bar{\xi}\|^2_{\mathcal{S}^2_\beta} + \|\bar{\zeta}\|^2_{\mathcal{S}^2_\beta}\right), \tag{2.5}
$$

where C_β is a constant depending on β and c. The desired result obtained by the estimates (2.4) and (2.5). □

2.2. The coupled system equivalent to DRBSDE. We first show that the existence of a solution to the DRBSDE is equivalent to the existence of a solution to a coupled system made of reflected BSDE. Let $(Y,Z,V,K^+,K^-,C^+,C^-)\in\mathfrak{B}^2(\beta,a)\times\mathcal{H}^2(\beta,a)\times\mathcal{L}^2(\beta,a)\times(\mathcal{S}^2)^4$ be a solution to DRBSDE with parameters (g,ξ,ζ).

We denote $\tilde{Y}_t = Y_t - \mathbb{E}\left[\xi_T + \int_t^T g(s)ds|\mathcal{F}_t\right]$, together with equation (1.1), we get

$$
\tilde{Y}_t = X_t - X'_t \quad \forall t\le T \quad a.s.,
$$

where the processes X and X' are defined as

$$
X_t = \mathbb{E}[K^+_T - K^+_t + C^+_{T-} - C^+_{t-}|\mathcal{F}_t] \text{ and } X'_t = \mathbb{E}[K^-_T - K^-_t + C^-_{T-} - C^-_{t-}|\mathcal{F}_t] \quad a.s.
$$

We note that $X\in\mathfrak{B}^2(\beta,a)$ and $X'\in\mathfrak{B}^2(\beta,a)$ are two nonnegative right upper-semicontinuous strong supermartingales (with $X_T = X'_T = 0$ a.s.), and they are of class (D) (i.e. the set of all random variable X_ν, for each finite stopping time ν, is uniformly integrable). Then by the Mertens decomposition (see Theorem A.4), there exists an uniformly integrable martingale (*càdlàg*) M (resp. M'), nondecreasing right-continuous predictable process $\tilde{K}^+$ (resp. $\tilde{K}^-$) (with $\tilde{K}^\pm_0 = 0$) such that $\mathbb{E}[\tilde{K}^\pm_T] < +\infty$ and nondecreasing right-continuous adapted purely discontinuous process $\tilde{C}^+$ (resp. $\tilde{C}^-$) (with $\tilde{C}^\pm_{0-} = 0$) such that $\mathbb{E}[\tilde{C}^\pm_T] < +\infty$, gives the following

$$
X_t = M_t - \tilde{K}^+_t - \tilde{C}^+_{t-} \quad (\text{ resp. } X'_t = M'_t - \tilde{K}^-_t - \tilde{C}^-_{t-}) \quad \forall t\le T.
$$

On the other hand, from the martingale representation Theorem there exists a unique pair $(\mathcal{Z}, U)$ (resp. $(\mathcal{Z}', U')$) belongs to $\mathcal{H}^2 \times \mathcal{L}^2$ such that

$$X_t = -\int_t^T \mathcal{Z}_s dB_s - \int_t^T \int_{\mathrm{U}} U_s(e)\tilde{\mu}(ds, de) + \tilde{K}_T^+ - \tilde{K}_t^+ + \tilde{C}_{T-}^+ - \tilde{C}_{t-}^+ \tag{2.6}$$

$$(\text{ resp. } X'_t = -\int_t^T \mathcal{Z}'_s dB_s - \int_t^T \int_{\mathrm{U}} U'_s(e)\tilde{\mu}(ds, de) + \tilde{K}_T^- - \tilde{K}_t^- + \tilde{C}_{T-}^- - \tilde{C}_{t-}^-). \tag{2.7}$$

Noting that $\tilde{K}^+ \equiv K^+$, $\tilde{K}^- \equiv K^-$, $\tilde{C}^+ \equiv C^+$ and $\tilde{C}^- \equiv C^-$. Now, let introduce the following optional processes

$$\tilde{\xi}_t^g := \xi_t - \mathbb{E}\left[\xi_T + \int_t^T g(s)ds|\mathcal{F}_t\right] \text{ and } \tilde{\zeta}_t^g := \zeta_t - \mathbb{E}\left[\zeta_T + \int_t^T g(s)ds|\mathcal{F}_t\right]. \tag{2.8}$$

We denote by

$$\tilde{\xi}^{X',g} = X' + \tilde{\xi}^g \qquad \text{and} \qquad \tilde{\zeta}^{X,g} = X - \tilde{\zeta}^g.$$

Remark 2.2. Since the coefficient of reflected BSDE (2.6) (resp. (2.7)) equal to zero, then thanks to Grigorova et al. [14] the solution $(X, \mathcal{Z}, U, K^+, C^+)$ (resp. $(X', \mathcal{Z}', U', K^-, C^-)$) belongs to $\mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{L}^2 \times \mathcal{S}^2 \times \mathcal{S}^2$.

Lemma 2.3. *Assuming that* $\frac{g}{a} \in \mathcal{H}^2(\beta, a)$.

- (i) *The process* $(X, \mathcal{Z}, U, K^+, C^+)$ *is solution of the reflected BSDE associated with parameters* $(0, \tilde{\xi}^{X',g})$ *belonging to* $\mathfrak{B}^2(\beta, a) \times \mathcal{H}^2(\beta, a) \times \mathcal{L}^2(\beta, a) \times \mathcal{S}^2 \times \mathcal{S}^2$.
- (ii) *The process* $(X', \mathcal{Z}', U', K^-, C^-)$ *is solution of the reflected BSDE associated with parameters* $(0, \tilde{\zeta}^{X,g})$ *belonging to* $\mathfrak{B}^2(\beta, a) \times \mathcal{H}^2(\beta, a) \times \mathcal{L}^2(\beta, a) \times \mathcal{S}^2 \times \mathcal{S}^2$.

Proof. Note that $\tilde{\xi}_T^{X',g} = \tilde{\zeta}_T^{X,g} = 0$ a.s. Let show that $\tilde{\xi}^{X',g} \in \mathcal{S}^2(\beta, a)$. By (2.8), we can write

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{[0,T]}} e^{\beta A(\tau)} |\tilde{\xi}_\tau^g|^2 \\
\leq \quad & 2 \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{[0,T]}} e^{\beta A(\tau)} |\xi_\tau|^2 + 2 \sup_{0 \leq t \leq T} \left| e^{\frac{\beta}{2} A(t)} \mathbb{E}\left[\xi_T + \int_t^T g(s)ds|\mathcal{F}_t\right]\right|^2.
\end{aligned}$$

Moreover

$$\begin{aligned}
& e^{\frac{\beta}{2} A(t)} \left| \mathbb{E}\left[\xi_T + \int_t^T g(s)ds|\mathcal{F}_t\right]\right| \\
\leq \quad & \sqrt{2}\mathbb{E}\left[\sqrt{e^{\beta A(T)}|\xi_T|^2 + e^{\beta A(t)} \left|\int_t^T g(s)ds\right|^2} |\mathcal{F}_t\right] \\
\leq \quad & \sqrt{2}\mathbb{E}\left[\sqrt{e^{\beta A(T)}|\xi_T|^2 + \frac{1}{\beta}\int_0^T e^{\beta A(s)} \left|\frac{g(s)}{a(s)}\right|^2 ds} |\mathcal{F}_t\right].
\end{aligned}$$

Thus, by Doob's martingale inequality one has

$$\mathbb{E}\sup_{0\le t\le T}\left|e^{\frac{\beta}{2}A(t)}\mathbb{E}\left[\xi_T+\int_t^T g(s)ds|\mathcal{F}_t\right]\right|^2<+\infty.$$

It follows that $\tilde{\xi}^g\in\mathcal{S}^2(\beta,a)$, since $X'\in\mathcal{S}^2(\beta,a)$ then $\tilde{\xi}^{X',g}\in\mathcal{S}^2(\beta,a)$. Similarly, it can be shown that $\tilde{\zeta}^{X,g}\in\mathcal{S}^2(\beta,a)$.

On the other hand, from $\xi\le Y\le\zeta$ and definition of $\tilde{Y}$, $\tilde{\xi}^g$ and $\tilde{\zeta}^g$, we derive that $\tilde{\xi}^g\le\tilde{Y}\le\tilde{\zeta}^g$. Since $\tilde{Y}=X-X'$ we have $X\ge\tilde{\xi}^{X',g}$ and $X'\ge\tilde{\zeta}^{X,g}$. Note that $Y-\xi=\tilde{Y}-\tilde{\xi}^g=X-\tilde{\xi}^{X',g}$. The minimality condition (1.3) satisfied by C^+ can thus be written $(X_\tau-\tilde{\xi}^{X',g}_\tau)\Delta C^+_\tau=0$ a.s. We also have $\{Y_{t-}>\xi_{t-}\}=\{X_{t-}>\tilde{\xi}^{X',g}_{t-}\}$, hence the Skorokhod condition (1.2) satisfied by K^+ can be written $\int_0^T\mathbb{1}_{\{X_{t-}>\tilde{\xi}^{X',g}_{t-}\}}dK^+_t=0$ a.s. Additionally, the process $(X,\mathcal{Z},U,K^+,C^+)$ satisfies the equation (2.6), then it is a solution of the reflected BSDE associated with the driver 0 and the barrier $\tilde{\xi}^{X',g}$. In other terms, $X=\mathcal{R}ef[\tilde{\xi}^{X',g}]$. By similar arguments, we get that $X'=\mathcal{R}ef[\tilde{\zeta}^{X,g}]$.

Now, remark that

$$\begin{aligned}X_t &= -\int_t^T\mathcal{Z}_s dB_s-\int_t^T\int_{\mathrm{U}}U_s(e)\tilde{\mu}(ds,de)+K^+_T-K^+_t+C^+_{T-}-C^+_{t-}\\ &= X_0+M_t+N_t+W_t\end{aligned}$$

where

$$M_t=\int_0^t\mathcal{Z}_s dB_s+\int_0^t\int_{\mathrm{U}}U_s(e)\tilde{\mu}(ds,de),\quad N_t=-K^+_t \text{ and } W_t=-C^+_{t-}.$$

Using the Corollary A.6, we get

$$\begin{aligned}&e^{\beta A(t)}|X_t|^2+\int_t^T\beta e^{\beta A(s)}a^2(s)|X_s|^2ds+\int_t^T e^{\beta A(s)}|\mathcal{Z}_s|^2ds\\ =\;&-2\int_t^T e^{\beta A(s)}X_s\mathcal{Z}_s dB_s-2\int_t^T\int_{\mathrm{U}}e^{\beta A(s)}X_{s-}U_s(e)\tilde{\mu}(ds,de)\\ &+2\int_t^T e^{\beta A(s)}X_{s-}dK^+_s+2\int_t^T e^{\beta A(s)}X_s d(C^+)_{s+}\\ &-\sum_{t<s\le T}e^{\beta A(s)}(\Delta X_s)^2-\sum_{t\le s<T}e^{\beta A(s)}(\Delta_+X_s)^2.\end{aligned}\tag{2.9}$$

Since the process K^+ be a jumps only at predictable stopping times and $\mu(.,de)$ jumps only at totally inaccessible stopping times, then we can note that

$$\sum_{t<s\le T}e^{\beta A(s)}(\Delta X_s)^2=\int_t^T\int_{\mathrm{U}}e^{\beta A(s)}|U_s(e)|^2\mu(ds,de)+\sum_{t<s\le T}e^{\beta A(s)}(\Delta K^+_s)^2.$$

Then one can derive that

$$\begin{aligned}&\int_t^T e^{\beta A(s)}\|U_s\|_\lambda^2 ds - \sum_{t<s\leq T} e^{\beta A(s)}(\Delta X_s)^2\\ &= \int_t^T e^{\beta A(s)}\|U_s\|_\lambda^2 ds - \int_t^T\int_{\mathrm{U}} e^{\beta A(s)}|U_s(e)|^2\mu(ds,de) - \sum_{t<s\leq T} e^{\beta A(s)}(\Delta K_s^+)^2\\ &\leq -\int_t^T\int_{\mathrm{U}} e^{\beta A(s)}|U_s(e)|^2\tilde{\mu}(ds,de).\end{aligned}$$

By adding the term $\int_t^T e^{\beta A(s)}\|U_s\|_\lambda^2 ds$ on both sides of inequality (2.9), we get

$$\begin{aligned}&\int_t^T e^{\beta A(s)}\left\{|\mathcal{Z}_s|^2 + \|U_s\|_\lambda^2\right\} ds\\ &\quad\leq -2\int_t^T e^{\beta A(s)} X_s\mathcal{Z}_s dB_s - \int_t^T\int_{\mathrm{U}} e^{\beta A(s)}\left\{2X_{s-}U_s(e) + |U_s(e)|^2\right\}\tilde{\mu}(ds,de)\\ &\qquad+2\int_t^T e^{\beta A(s)}X_{s-}dK_s^+ + 2\int_t^T e^{\beta A(s)}X_s d(C^+)_{s+}.\end{aligned}$$

Since $X_{s-}dK_s^+ = \tilde{\xi}_{s-}^{X',g}dK_s^+$ and $X_s\Delta C_s^+ = \tilde{\xi}_s^{X',g}\Delta C_s^+$, taking into consideration the Proposition 1.3, we get

$$\begin{aligned}&\mathbb{E}\int_t^T e^{\beta A(s)}\left\{|\mathcal{Z}_s|^2 + \|U_s\|_\lambda^2\right\} ds\\ &\qquad\leq 2\mathbb{E}\int_t^T e^{\beta A(s)}\tilde{\xi}_{s-}^{X',g}dK_s^+ + 2\mathbb{E}\sum_{t\leq s<T} e^{\beta A(s)}\tilde{\xi}_s^{X',g}\Delta C_s^+\\ &\qquad\leq 4\mathbb{E}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} e^{\beta A(\tau)}|\tilde{\xi}_\tau^{X',g}|^2 + \mathbb{E}|e^{\beta A(T)}K_T^+|^2 + \mathbb{E}|e^{\beta A(T)}C_T^+|^2 < +\infty.\end{aligned}$$

Here we suppose, in addition, that $\mathbb{E}\int_0^T\{\theta(t)+\gamma^2(t)+\eta^2(t)\}dt < +\infty$. It follows that $(\mathcal{Z},U)\in\mathcal{H}^2(\beta,a)\times\mathcal{L}^2(\beta,a)$.

Finally, from Remark 2.2, $(X,\mathcal{Z},U,K^+,C^+)\in\mathfrak{B}^2(\beta,a)\times\mathcal{H}^2(\beta,a)\times\mathcal{L}^2(\beta,a)\times\mathcal{S}^2\times\mathcal{S}^2$. Similarly, we can prove that $(X',\mathcal{Z}',U',K^-,C^-)\in\mathfrak{B}^2(\beta,a)\times\mathcal{H}^2(\beta,a)\times\mathcal{L}^2(\beta,a)\times\mathcal{S}^2\times\mathcal{S}^2$. The proof is complete. □

Lemma 2.4. *The following assertions are equivalent:*

(*i*) *The DRBSDE associated with parameters* (g,ξ,ζ) *has a solution belonging to* $\mathfrak{B}^2(\beta,a)\times\mathcal{H}^2(\beta,a)\times\mathcal{L}^2(\beta,a)\times(\mathcal{S}^2)^4$.

(*ii*) *There exists two processes* $\mathcal{X}\in\mathfrak{B}^2(\beta,a)$ *and* $\mathcal{X}'\in\mathfrak{B}^2(\beta,a)$ *such that*

$$\begin{cases}\mathcal{X} = \mathcal{R}ef[\tilde{\xi}^{\mathcal{X}',g}]\\ \mathcal{X}' = \mathcal{R}ef[\tilde{\zeta}^{\mathcal{X},g}].\end{cases}\tag{2.10}$$

Proof. $(i)\Rightarrow(ii)$ has been proved above. $(ii)\Rightarrow(i)$: Let $\mathcal{X}\in\mathfrak{B}^2(\beta,a)$ and $\mathcal{X}'\in\mathfrak{B}^2(\beta,a)$ be satisfying the coupled system (2.10). Let $(\mathcal{Z},U,K^+,C^+)$ (resp. $(\mathcal{Z}',U',K^-,C^-)$) be the vector of the remaining components of the solution to the reflected BSDE whose first component is $\mathcal{X}$ (resp. $\mathcal{X}'$). We note that the equations (2.6) and (2.7) hold for $\mathcal{X}$ and $\mathcal{X}'$. We define the optional process Y by

$$Y_t = \mathcal{X}_t - \mathcal{X}'_t + \mathbb{E}[\xi_T + \int_t^T g(s)ds|\mathcal{F}_t] \quad \forall t \le T \quad a.s. \tag{2.11}$$

Since $\mathcal{X}$ and $\mathcal{X}'$ are belongs to $\mathfrak{B}^2(\beta, a)$, it follows that the process Y is well-defined. From (2.11) and the property $\mathcal{X}_T = \mathcal{X}'_T = 0$ a.s., we get $Y_T = \xi_T$ a.s. Note that, he coupled system (2.10) implies that $\mathcal{X} > \tilde{\xi}^{\mathcal{X}',g}$ and $\mathcal{X}' > \tilde{\zeta}^{\mathcal{X},g}$ a.s. Then, by using the definitions of $\tilde{\xi}^g$, $\tilde{\zeta}^g$ and Y, we derive that $\xi \le Y \le \zeta$ a.s.

Moreover, the processes K^+ and C^+ satisfy the minimality condition for reflected BSDE. More precisely, for all $\tau \in \mathcal{T}_{[0,T]}$ $(\mathcal{X}_\tau - \tilde{\xi}_\tau^{\mathcal{X}',g})\Delta C_\tau^+$ a.s. and for all $t \le T$ $\int_0^T \mathbb{1}_{\{\mathcal{X}_t > \tilde{\xi}_t^{\mathcal{X}',g}\}} dK_t^+ = 0$ a.s. Now, by using the definitions of Y and $\tilde{\xi}^g$ we get $\{\mathcal{X}_\tau = \tilde{\xi}_\tau^{\mathcal{X}',g}\} = \{Y_\tau = \xi_\tau\}$, $\{\mathcal{X}_{\tau-} = \tilde{\xi}_{\tau-}^{\mathcal{X}',g}\} = \{Y_{\tau-} = \xi_{\tau-}\}$ and $\{\mathcal{X}_t > \tilde{\xi}_t^{\mathcal{X}',g}\} = \{Y_t > \xi_t\}$, then we can derive that for all $\tau \in \mathcal{T}_{[0,T]}$ $(Y_\tau - \xi_\tau)\Delta C_\tau^+$ a.s. and for all $t \le T$ $\int_0^T \mathbb{1}_{\{Y_{t-} > \xi_{t-}\}} dK_t^+ = 0$ a.s. By applying the same arguments to K^- and C^- we also get for all $\tau \in \mathcal{T}_{[0,T]}$ $(Y_\tau - \zeta_\tau)\Delta C_\tau^-$ a.s. and for all $t \le T$ $\int_0^T \mathbb{1}_{\{Y_{t-} < \zeta_{t-}\}} dK_t^- = 0$ a.s.

Now, we note that the process $\left(\mathbb{E}\left[\xi_T + \int_t^T g(s)ds|\mathcal{F}_t\right]\right)_{t\le T}$ corresponds to the first component of the solution to the non-reflected BSDE with parameters (ξ_T, g). Hence, by martingale representation theorem, there exists $\mathcal{Z}'' \in \mathcal{H}^2$ and $U'' \in \mathcal{L}^2$ such that

$$\mathbb{E}\left[\xi_T + \int_t^T g(s)ds|\mathcal{F}_t\right] = \xi_T + \int_t^T g(s)ds - \int_t^T \mathcal{Z}''_s dB_s - \int_t^T \int_{\mathrm{U}} U''_s(e)\tilde{\mu}(ds, de).$$

Further, by using Itô's formula for a semimartingale, we have also $\mathcal{Z}'' \in \mathcal{H}^2(\beta, a)$ and $U'' \in \mathcal{L}^2(\beta, a)$. Together with definition of Y and the equations (2.6) and (2.7), we obtain

$$\begin{aligned} Y_t \quad = \quad & \xi_T + \int_t^T g(s)ds - \int_t^T Z_s dB_s + (K_T^+ - K_t^+) - (K_T^- - K_t^-) \\ & - \int_t^T \int_{\mathrm{U}} V_s(e)\tilde{\mu}(ds, de) + (C_{T-}^+ - C_{t-}^+) - (C_{T-}^- - C_{t-}^-), \end{aligned}$$

where $Z = \mathcal{Z} - \mathcal{Z}' + \mathcal{Z}''$ and $V = U - U' + U''$. Moreover,

$$\|Y\|_{\mathfrak{B}_\beta^2}^2 \le 3\left(\|\mathcal{X}\|_{\mathfrak{B}_\beta^2}^2 + \|\mathcal{X}'\|_{\mathfrak{B}_\beta^2}^2 + \left\|\mathbb{E}\left[\xi_T + \int_t^T g(s)ds|\mathcal{F}_t\right]\right\|_{\mathfrak{B}_\beta^2}^2\right) < +\infty.$$

Thus, if $dK^+ \perp dK^-$ and $dC^+ \perp dC^-$, the process $(Y, Z, V, K^+, K^-, C^+, C^-)$ is a solution to the DRBSDE with parameters (g, ξ, ζ). □

2.3. Existence of a (minimal) solution to DRBSDE. Let us notice that obviously for arbitrary pair of admissible barriers (ξ, ζ), the DRBSDE with parameters (g, ξ, ζ) does not have a solution since, for example, ξ and ζ coincide and

ξ is not semimartingale then we cannot find a semimartingale which equals to ξ. However as into account the Mokobodski's condition which reads as:

(Mk)**:** There exist two nonnegative supermartingales $(H_t)_{t\leq T}$ and $(H'_t)_{t\leq T}$ such that

$$\forall t \leq T, \qquad \xi_t \leq H_t - H'_t \leq \zeta_t.$$

Then we can prove the existence of a solution for DRBSDE with driver g and a pair of admissible barriers (ξ, ζ).

Theorem 2.5. *Assume that* $\frac{g}{a} \in \mathcal{H}^2(\beta, a)$ *and* (Mk) *holds. Then the DRBSDE associated with data* (g, ξ, ζ) *admits a unique solution* $(Y, Z, V, K^+, K^-, C^+, C^-)$ *that belongs to* $\mathfrak{B}^2(\beta, a) \times \mathcal{H}^2(\beta, a) \times \mathcal{L}^2(\beta, a) \times (\mathcal{S}^2)^4$.

The idea to the proof is from establishing the existence of solution to the coupled system (2.10) which equivalent to the DRBSDE by using lemma 2.4. To do that, we use Picard's iterations, whose we define recursively the processes

$$\begin{cases} \mathcal{X}^0 = 0, & \mathcal{X}'^0 = 0; \\ \mathcal{X}^{n+1} = \mathcal{R}ef[\mathcal{X}'^n + \tilde{\xi}^g], & \mathcal{X}'^{n+1} = \mathcal{R}ef[\mathcal{X}^n - \tilde{\zeta}^g]. \end{cases} \tag{2.12}$$

Lemma 2.6. *The sequences* $(\mathcal{X}^n)_{n\geq 0}$ *and* $(\mathcal{X}'^n)_{n\geq 0}$ *are nondecreasing of optional processes. Moreover, there exists two nonnegative strong optional supermartingales* $\mathcal{X}$ *and* $\mathcal{X}'$ *in* $\mathfrak{B}^2(\beta, a)$ *satisfying the system (2.10) and*

$$\tilde{\xi}^g \leq \mathcal{X} - \mathcal{X}' \leq \tilde{\zeta}^g. \tag{2.13}$$

Proof. By induction, the processes $\mathcal{X}^n$ and $\mathcal{X}'^n$ are well-defined, moreover they are strong supermartingales in $\mathfrak{B}^2(\beta, a)$.

We first show that $\mathcal{X}^n \geq 0$ and $\mathcal{X}'^n \geq 0$, for all $n \in \mathbb{N}$. Clearly, $\mathcal{X}^n_T = \mathcal{X}'^n_T = 0$. Since $\mathcal{X}^n$ is a strong supermartingales, it follows that $\mathcal{X}^n_\tau \geq \mathbb{E}[\mathcal{X}^n_T|\mathcal{F}_\tau] = 0$ a.s. for all $\tau \in \mathcal{T}_{[0,T]}$, which implies that $\mathcal{X}^n \geq 0$. Similarly we see that $\mathcal{X}'^n \geq 0$.

We prove recursively that $(\mathcal{X}^n)_{n\geq 0}$ and $(\mathcal{X}'^n)_{n\geq 0}$ are nondecreasing sequences of processes. We have $\mathcal{X}^1 \geq 0 = \mathcal{X}^0$ and $\mathcal{X}'^1 \geq 0 = \mathcal{X}'^0$. Suppose that $\mathcal{X}^n \geq \mathcal{X}^{n-1}$ and $\mathcal{X}'^n \geq \mathcal{X}'^{n-1}$. The induction hypothesis and nondecreasingness of the operator $\mathcal{R}ef$ (see lemma A.7) implies that

$$\mathcal{R}ef[\mathcal{X}'^n + \tilde{\xi}^g] \geq \mathcal{R}ef[\mathcal{X}'^{n-1} + \tilde{\xi}^g] \text{ and } \mathcal{R}ef[\mathcal{X}^n + \tilde{\zeta}^g] \geq \mathcal{R}ef[\mathcal{X}^{n-1} + \tilde{\zeta}^g].$$

Hence $\mathcal{X}^{n+1} \geq \mathcal{X}^n$ and $\mathcal{X}'^{n+1} \geq \mathcal{X}'^n$, which is the desired result.

Now we show that $(\mathcal{X}^n)_{n\geq 0}$ and $(\mathcal{X}'^n)_{n\geq 0}$ are bounded from above by some processes H^g and H'^g respectively, which define as follows

$$\begin{cases} H^g_t = H_t + \mathbb{E}\left[\xi^-_T + \int_t^T g^-(s)ds|\mathcal{F}_t\right]; \\ H'^g_t = H'_t + \mathbb{E}\left[\xi^+_T + \int_t^T g^+(s)ds|\mathcal{F}_t\right], \end{cases}$$

where H and H' come from Mokobodski's condition (Mk) for (ξ, ζ). We note that H^g and H'^g are nonnegative strong supermartingales in $\mathcal{S}^2(\beta, a)$. From (Mk), we get

$$\tilde{\xi}^g \leq H^g - H'^g \leq \tilde{\zeta}^g. \tag{2.14}$$

By recursively, note first that $\mathcal{X}^0 = 0 \leq H^g$ and $\mathcal{X}'^0 = 0 \leq H'^g$. Suppose that $\mathcal{X}^n \leq H^g$ and $\mathcal{X}'^n \leq H'^g$. From this, together with (2.14), we get $\mathcal{X}'^n \leq$

$H'^g \leq H^g - \tilde{\xi}^g$, which implies $\mathcal{X}'^n + \tilde{\xi}^g \leq H^g$. By the nondecreasingness of the operator $\mathcal{R}ef$, we derive $\mathcal{X}^{n+1} = \mathcal{R}ef[\mathcal{X}'^n + \tilde{\xi}^g] \leq \mathcal{R}ef[H^g]$. Since H^g is a strong supermartingale, the second assertion of lemma A.7 gives $\mathcal{R}ef[H^g] = H^g$. Hence $\mathcal{X}^{n+1} \leq H^g$. Similarly we shown $\mathcal{X}'^{n+1} \leq H'^g$.

Since $(\mathcal{X}^n)_{n\geq 0}$ and $(\mathcal{X}'^n)_{n\geq 0}$ are nondecreasing sequences of processes, and they are bounded from above by H^g and H'^g respectively, then there exists two nonnegative optional strong supermartingales $\mathcal{X}$ and $\mathcal{X}'$ limits to $(\mathcal{X}^n)_{n\geq 0}$ and $(\mathcal{X}'^n)_{n\geq 0}$ respectively. This limits satisfies $0 \leq \mathcal{X} \leq H^g$ and $0 \leq \mathcal{X}' \leq H'^g$, as $H^g, H'^g \in \mathcal{S}^2(\beta, a)$, it follows that $\mathcal{X}$ and $\mathcal{X}'$ are belongs to $\mathcal{S}^2(\beta, a)$.

It remains to show that $\mathcal{X}$ and $\mathcal{X}'$ are the solutions of the coupled system (2.10). Note that the sequence $(\mathcal{X}'^n + \tilde{\xi}^g)_{n\geq 0}$ is nondecreasing and converges to $\tilde{\xi}^{\mathcal{X}',g}$. By lemma A.8, we thus derive that $\lim_{n\to+\infty} \mathcal{R}ef[\mathcal{X}'^n + \tilde{\xi}^g] = \mathcal{R}ef[\tilde{\xi}^{\mathcal{X}',g}]$ and similarly $\lim_{n\to+\infty} \mathcal{R}ef[\mathcal{X}^n - \tilde{\zeta}^g] = \mathcal{R}ef[\tilde{\zeta}^{\mathcal{X},g}]$. Hence, by letting n tend to $+\infty$ in (2.12), one can derive that $\mathcal{X}$ and $\mathcal{X}'$ solve the coupled system (2.10).

The property $\tilde{\xi}^g \leq \mathcal{X} - \mathcal{X}' \leq \tilde{\zeta}^g$ come from the definition of operator $\mathcal{R}ef$; indeed, $\mathcal{X}^n = \mathcal{R}ef[\mathcal{X}'^n + \tilde{\xi}^g]$ implies that $\mathcal{X}^n \geq \mathcal{X}'^n + \tilde{\xi}^g$, by letting n tend to $+\infty$ we get $\mathcal{X} - \mathcal{X}' \geq \tilde{\xi}^g$ and with same way $\mathcal{X} - \mathcal{X}' \leq \tilde{\zeta}^g$. Moreover, $\mathcal{X}$ and $\mathcal{X}'$ are the minimal nonnegative strong supermartingales in $\mathcal{S}^2(\beta, a)$ satisfying (2.13); indeed, if H, H' are nonnegative strong supermartingales in $\mathcal{S}^2(\beta, a)$ satisfying $\tilde{\xi}^g \leq H - H' \leq \tilde{\zeta}^g$, then by using the some arguments as above, we have always $\mathcal{X} \leq H$ and $\mathcal{X}' \leq H'$.

From this minimality property, it follows that $(\mathcal{X}, \mathcal{X}')$ is also characterized as the minimal solution of the coupled system (2.10). □

Proof of Theorem 2.5. By lemma 2.6 we have existence of two nonnegative strong optional supermartingales $\mathcal{X}$ and $\mathcal{X}'$ that belonging to $\mathfrak{B}^2(\beta, a)$ solution of the coupled system (2.10). Then, from lemma 2.4, there exists $(Y, Z, V, K^+, K^-, C^+, C^-)$ that belonging to $\mathfrak{B}^2(\beta, a) \times \mathcal{H}^2(\beta, a) \times \mathcal{L}^2(\beta, a) \times (\mathcal{S}^2)^4$ solution of DRBSDE associated with parameters (g, ξ, ζ) such that

$$Y_t = \mathcal{X}_t - \mathcal{X}'_t + \mathbb{E}\left[\xi_T + \int_t^T g(s)ds|\mathcal{F}_t\right] \quad \forall t \leq T \quad a.s.$$

The uniqueness derive from the a priori estimate which proved in lemma 2.1. □

3. Solving the DRBSDEs with General Stochastic Lipschitz Driver

By means of the fixed point theorem, we prove the existence and uniqueness of solution to the DRBSDE associated with parameters (f, ξ, ζ) where f is stochastic Lipschitz driver.

Theorem 3.1. *Let f be a stochastic Lipschitz driver and $(\xi_t, \zeta_t)_{t\leq T}$ are left limited processes satisfying (Mk). Then the DRBSDE associated with parameters (f, ξ, ζ) admits a unique solution.*

Proof. Given $(\phi,\psi,\varphi)\in\mathfrak{B}^2(\beta,a)\times\mathcal{H}^2(\beta,a)\times\mathcal{L}^2(\beta,a)$, consider the following DRBSDE:

$$\begin{aligned} Y_t &= \xi_T+\int_t^T f(s,\phi_s,\psi_s,\varphi_s)ds+(K^+_T-K^+_t)-(K^-_T-K^-_t)-\int_t^T Z_s dB_s \\ &\quad -\int_t^T\int_{\mathrm{U}} V_s(e)\tilde{\mu}(ds,de)+(C^+_{T-}-C^+_{t-})-(C^-_{T-}-C^-_{t-}) \quad t\le T. \qquad (3.1)\end{aligned}$$

From the stochastic Lipschitz assumption on f, we have

$$\begin{aligned} |f(t,\phi_t,\psi_t,\varphi_t)|^2 &\le (\theta(t)|\phi_t|+\gamma(t)|\psi_t|+\eta(t)\|\varphi_t\|_\lambda+|f(t,0,0,0)|)^2 \\ &\le 4\left(\theta^2(t)|\phi_t|^2+\gamma^2(t)|\psi_t|^2+\eta^2(t)\|\varphi_t\|^2_\lambda+|f(t,0,0,0)|^2\right). \end{aligned}$$

It follows

$$\frac{|f(t,\phi_t,\psi_t,\varphi_t)|^2}{a^2(t)}\le 4\left(a^2(t)|\phi_t|^2+|\psi_t|^2+\|\varphi_t\|^2_\lambda+\frac{|f(t,0,0,0)|^2}{a^2(t)}\right).$$

Then $\dfrac{f}{a}\in\mathcal{H}^2(\beta,a)$. It follows from Theorem 2.5 that the DRBSDE(3.1) has a unique solution (Y,Z,V,K^+,K^-,C^+,C^-). Define a mapping Φ from $\mathfrak{B}^2(\beta,a)\times\mathcal{H}^2(\beta,a)\times\mathcal{L}^2(\beta,a)$ into itself. Let (ϕ',ψ',φ') an other element of $\mathfrak{B}^2(\beta,a)\times\mathcal{H}^2(\beta,a)\times\mathcal{L}^2(\beta,a)$. We set $\Phi(\phi,\psi,\varphi)=(Y,Z,V)$ and $\Phi(\phi',\psi',\varphi')=(Y',Z',V')$. We also set

$$\delta\phi=\phi-\phi',\quad \delta\psi=\psi-\psi',\quad \delta\varphi=\varphi-\varphi',\quad \delta Y=Y-Y',\quad \delta Z=Z-Z',$$
$$\delta V=V-V',\quad \delta f_t=f(t,{\phi'}_t,{\psi'}_t,\varphi'_t)-f(t,\phi_t,\psi_t,\varphi_t).$$

With same way as to inequality (2.4) (see the proof of lemma 2.1), we get

$$\frac{\beta}{2}\|a\delta Y\|^2_{\mathcal{S}^{2,a}_\beta}+\|\delta Z\|^2_{\mathcal{H}^2_\beta}+\|\delta V\|^2_{\mathcal{L}^2_\beta}\le\frac{2}{\beta}\left\|\frac{\delta f}{a}\right\|^2_{\mathcal{H}^2_\beta}.$$

By using the stochastic Lipschitz assumption on f, we can write for $\beta>6$

$$\frac{\beta}{2}\|a\delta Y\|^2_{\mathcal{S}^{2,a}_\beta}+\|\delta Z\|^2_{\mathcal{H}^2_\beta}+\|\delta V\|^2_{\mathcal{L}^2_\beta}\le\frac{6}{\beta}\left(\|a\delta\phi\|^2_{\mathcal{S}^{2,a}_\beta}+\|\delta\psi\|^2_{\mathcal{H}^2_\beta}+\|\delta\varphi\|^2_{\mathcal{L}^2_\beta}\right).$$

It follows that Φ is a strict contraction mapping on $\mathfrak{B}^2(\beta,a)\times\mathcal{H}^2(\beta,a)\times\mathcal{L}^2(\beta,a)$. Henceforth, there exists a process (Y,Z,V) fixed point to Φ which, with $(K^\pm,C^\pm)$ is the unique solution to DRBSDE with parameters (f,ξ,ζ). □

4. Comparison Theorem

The comparison theorem is one of the principal tools in the theories of the BSDEs. But it does not hold in general for solutions of BSDEs with jumps (see the counter example in [3]). However, it's shown in special cases (see for example [36, 39]). In order to obtain the comparison theorem, in this section, we will discus the following generator

$$f(\omega,t,y,z,v)=h(\omega,t,y,z)+\int_{\mathrm{U}} c_t(\omega,e)v_t(e)\lambda(de),$$

where

- $c:\Omega\times[0,T]\times\mathrm{U}\to[-1,+\infty[$ is a $\mathcal{P}\otimes\mathcal{U}$-measurable mapp belongs in $\mathcal{L}_\lambda$.

- There exist two nonnegative $\mathcal{F}_t$-adapted processes θ_1 and γ_1 such that $\forall (y, y', z, z') \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$
$$|h(\omega, t, y, z) - h(\omega, t, y', z')| \leq \theta_1(t)|y - y'| + \gamma_1(t)|z - z'|,$$
with $\theta_1(t) + \gamma_1^2(t) = a_1^2(t) \geq \epsilon_1 > 0$.

Let $(Y^i, Z^i, V^i, K^{+,i}, K^{-,i}, C^{+,i}, C^{-,i})$ be the unique solution of the DRBSDE with data (f^i, ξ^i, ζ^i), for $i = 1, 2$. Then we have the following:

Theorem 4.1. *Assume that*
$$\xi_t^1 \leq \xi_t^2, \;\; \zeta_t^1 \leq \zeta_t^2 \;\textit{and}\; f^1(t, Y_t^2, Z_t^2, V_t^2) \leq f^2(t, Y_t^2, Z_t^2, V_t^2) \;\; \forall t \leq T \; a.s.$$
Then $Y_t^1 \leq Y_t^2 \quad \forall t \leq T \quad a.s.$

Proof. Let $\bar{\Re} = \Re^1 - \Re^2$ with $\Re = Y, Z, V, K^+, K^-, C^+, C^-, \xi, \zeta$. Then
$$\begin{aligned} \bar{Y}_t &= \bar{\xi}_T + \int_t^T \left\{ \sigma_s \bar{Y}_s + \delta_s \bar{Z}_s + \int_{\mathrm{U}} c_s(\omega, e)\bar{V}_s(e)\lambda(de) + h_s \right\} ds + (\bar{K}_T^+ - \bar{K}_t^+) \\ &\quad -(\bar{K}_T^- - \bar{K}_t^-) - \int_t^T \bar{Z}_s dB_s - \int_t^T \int_{\mathrm{U}} \bar{V}_s(e)\tilde{\mu}(ds, de) + (\bar{C}_{T-}^+ - \bar{C}_{t-}^+) \\ &\quad -(\bar{C}_{T-}^- - \bar{C}_{t-}^-), \end{aligned} \tag{4.1}$$
where
$$\sigma_t = \mathbb{1}_{\{\bar{Y}_t \neq 0\}} \frac{h(t, Y_t^1, Z_t^1) - h(t, Y_t^2, Z_t^1)}{\bar{Y}_t}, \delta_t = \mathbb{1}_{\{\bar{Z}_t \neq 0\}} \frac{h(t, Y_t^2, Z_t^1) - h(t, Y_t^2, Z_t^2)}{\bar{Z}_t}$$
and $h_t = f^1(t, Y_t^2, Z_t^2, V_t^2) - f^2(t, Y_t^2, Z_t^2, V_t^2)$.

Now, by Girsanov transformation theorem, there exists a probability measure $\mathbb{Q}$ defined on the standard measurable space $(\Omega, \mathcal{F})$ such that $\mathbb{Q} := \mathcal{E}_T(M)\mathbb{P}$ where $\mathcal{E}_T(.)$ is Doléans-Dade exponential (see Protter [34], theorem 37 pp.84), $\bar{B}_t = B_t - \int_0^t \delta_s ds$ is a Brownian motion under probability measure $\mathbb{Q}$ and $\bar{\mu}(de, ds) = \tilde{\mu}(de, ds) - c_s(\omega, e)\lambda(de)ds$ is a $\mathbb{Q}$-martingale measure. Hence the DRBSDE(4.1) can be rewritten as
$$\begin{aligned} \bar{Y}_t &= \bar{\xi}_T + \int_t^T \left(\sigma_s \bar{Y}_s + h_s\right) ds + (\bar{K}_T^+ - \bar{K}_t^+) - (\bar{K}_T^- - \bar{K}_t^-) - \int_t^T \bar{Z}_s d\bar{B}_s \\ &\quad - \int_t^T \int_{\mathrm{U}} \bar{V}_s(e)\bar{\mu}(ds, de) + (\bar{C}_{T-}^+ - \bar{C}_{t-}^+) - (\bar{C}_{T-}^- - \bar{C}_{t-}^-). \end{aligned}$$
Applying Gal'chouk formula with the convex function $x \mapsto x^+$ (see Theorem A.9), we get for all $t \leq T$
$$\begin{aligned} \bar{Y}_t^+ &= \bar{Y}_0^+ + \int_0^t \mathbb{1}_{\{\bar{Y}_{s-} > 0\}} d\bar{Y}_s + \frac{1}{2} L_t(\bar{Y}) + \sum_{0 \leq s < t} \Big[\bar{Y}_{s^+}^+ \mathbb{1}_{\{\bar{Y}_{s-} \leq 0\}} + \bar{Y}_{s^-}^- \mathbb{1}_{\{\bar{Y}_{s-} > 0\}} \\ &\quad + \bar{Y}_t^+ \mathbb{1}_{\{\bar{Y}_{t-} \leq 0\}} + \bar{Y}_t^- \mathbb{1}_{\{\bar{Y}_{t-} > 0\}} \Big] \\ &= \bar{Y}_0^+ - \int_0^t \mathbb{1}_{\{\bar{Y}_{s-} > 0\}} (\sigma_s \bar{Y}_s + h_s) ds - \bar{K}_t^+ + \bar{K}_t^- + \int_0^t \bar{Z}_s d\bar{B}_s \\ &\quad + \int_0^t \int_{\mathrm{U}} \bar{V}_s(e)\bar{\mu}(ds, de) - \bar{C}_{t-}^+ + \bar{C}_{t-}^- + \frac{1}{2} L_t(\bar{Y}) + \Sigma_t \end{aligned}$$

where $(L_t)_{t\leq T}$ is a local time (nondecreasing continuous process with $L_0 = 0$) and $\Sigma_t = \sum_{0\leq s<t} \bar{Y}^+_{s^+}\mathbb{1}_{\{\bar{Y}_{s^-}\leq 0\}} + \bar{Y}^-_{s^-}\mathbb{1}_{\{\bar{Y}_{s^-}>0\}} + \bar{Y}^+_t\mathbb{1}_{\{\bar{Y}_{t^-}\leq 0\}} + \bar{Y}^-_t\mathbb{1}_{\{\bar{Y}_{t^-}>0\}}$ is finite by Theorem A.10. $\bar{Y}^+$ is a strong optional semimartingale (see Theorem A.9) with decomposition $\bar{Y}^+_t = \bar{Y}^+_0 + M_t + N_t + W_t$ where $M_t = \int_0^t \bar{Z}_s d\bar{B}_s + \int_0^t\int_{\mathrm{U}} \bar{V}_s(e)\bar{\mu}(ds,de)$, $N_t = -\int_0^t(\sigma_s\bar{Y}_s + h_s)ds - \bar{K}^+_t + \bar{K}^-_t + \frac{1}{2}L_t(\bar{Y}) + \Sigma_t$ and $W_t = -\bar{C}^+_{t-} + \bar{C}^-_{t-}$.

Next, we denote $R_t = e^{\beta A_1(t)+2\int_0^t \sigma_s ds}$ where $A_1(t) = \int_0^t a_1^2(s)ds$ is a increasing continuous process. We apply the Corollary A.6 to obtain

$$\begin{aligned}
& R_t|\bar{Y}^+_t|^2 \\
= \quad & |\bar{Y}^+_0|^2 + \int_0^t \left\{\beta a_1^2(s) + 2\sigma_s\right\} R_s|\bar{Y}^+_s|^2 ds + 2\int_0^t R_s\bar{Y}^+_s\bar{Z}_s d\bar{B}_s \\
& +2\int_0^t\int_{\mathrm{U}} R_s\bar{Y}^+_{s-}\bar{V}_s(e)\bar{\mu}(ds,de) - 2\int_0^t R_s\bar{Y}^+_s\left\{\sigma_s\bar{Y}_s + h_s\right\}ds \\
& -2\int_0^t R_s\bar{Y}^+_{s-}(d\bar{K}^+_s - d\bar{K}^-_s) + \int_0^t R_s\bar{Y}^+_s dL_s + \int_{]0,t]} R_s|\bar{Z}_s|^2 ds \\
& + \sum_{0<s\leq t} R_s(\bar{Y}^+_s - \bar{Y}^+_{s-})^2 - 2\int_0^t R_s\bar{Y}^+_s(d\bar{C}^+_s - d\bar{C}^-_s) + \sum_{0\leq s<t} R_s(\bar{Y}^+_{s+} - \bar{Y}^+_s)^2.
\end{aligned}$$

Consequently

$$\begin{aligned}
& R_t|\bar{Y}^+_t|^2 + \int_t^T \beta R_s a_1^2(s)|\bar{Y}^+_s|^2 ds + \int_t^T R_s|\bar{Z}_s|^2 ds \\
= \quad & |\bar{\xi}^+_T|^2 - 2\int_t^T R_s\bar{Y}^+_s\bar{Z}_s d\bar{B}_s - 2\int_t^T\int_{\mathrm{U}} R_s\bar{Y}^+_{s-}\bar{V}_s(e)\bar{\mu}(ds,de) \\
& +2\int_t^T R_s\bar{Y}^+_s h_s ds + 2\int_t^T R_s\bar{Y}^+_{s-}(d\bar{K}^+_s - d\bar{K}^-_s) - \int_t^T R_s\bar{Y}^+_s dL_s \\
& - \sum_{t<s\leq T} R_s(\bar{Y}^+_s - \bar{Y}^+_{s-})^2 + 2\int_t^T R_s\bar{Y}^+_s(d\bar{C}^+_s - d\bar{C}^-_s) - \sum_{t\leq s<T} R_s(\bar{Y}^+_{s+} - \bar{Y}^+_s)^2.
\end{aligned}$$

Using the property (1.2) and (1.3) cited in Definition 1.1 to obtain that $\bar{Y}^+_{s-}(d\bar{K}^+_s - d\bar{K}^-_s) \leq 0$ and $\bar{Y}^+_s(d\bar{C}^+_s - d\bar{C}^-_s) \leq 0$ respectively. In addition $\bar{\xi}_T \leq 0$, $h_s \leq 0$ and the nondecreasingness of $(L_t)_{t\leq T}$ implies that

$$\begin{aligned}
& R_t|\bar{Y}^+_t|^2 + \int_t^T \beta R_s a_1^2(s)|\bar{Y}^+_s|^2 ds + \int_t^T R_s|\bar{Z}_s|^2 ds \\
& \qquad \leq \quad -2\int_t^T R_s\bar{Y}^+_s\bar{Z}_s d\bar{B}_s - 2\int_t^T\int_{\mathrm{U}} R_s\bar{Y}^+_{s-}\bar{V}_s(e)\bar{\mu}(ds,de).
\end{aligned}$$

Taking expectation under the measure $\mathbb{Q}$ on the both sides we get

$$\mathbb{E}[R_t|\bar{Y}^+_t|^2] \leq 0.$$

It follows that $\bar{Y}^+_t = 0$, i.e. $Y^1_t \leq Y^2_t$ for all $t \leq T$ $\mathbb{Q}$-a.s. and so $\mathbb{P}$-a.s. □

Appendix A

Definition A.1. Let $\tau \in \mathcal{T}_{[0,T]}$. An optional process $(\xi_t)_{t\leq T}$ is said to be *right upper-semicontinuous* ($r.u.s.c$) along stopping times at the stopping time τ if for all nonincreasing sequence of stopping times $(\tau_n)_{n\geq 0}$ such that $\tau_n \searrow \tau$ $a.s.$ and $\xi_\tau \geq \limsup_{n\to+\infty} \xi_{\tau_n}$ $a.s.$ The process $(\xi_t)_{t\leq T}$ is said to be $r.u.s.c$ along stopping times if it is $r.u.s.c$ along stopping times at each $\tau \in \mathcal{T}_{[0,T]}$.

Definition A.2. Let $(K_t)_{t\leq T}$ and $(C_t)_{t\leq T}$ be two $\mathbb{R}$-valued optional nondecreasing *càdlàg* processes with $K_0 = 0,\ C_0 = 0,\ \mathbb{E}[K_T] < +\infty$ and $\mathbb{E}[C_T] < +\infty$. We say that the random measures dK_t and dC_t are *mutually singular* and we write $dK_t \perp dC_t$, if there exists $D \in \mathcal{O}$ such that

$$\mathbb{E}\left[\int_0^T \mathbb{1}_{D^c} dK_t\right] = \mathbb{E}\left[\int_0^T \mathbb{1}_D dC_t\right] = 0.$$

Definition A.3. Let $(Y_t)_{t\leq T}$ be an optional process. We say that Y is a *strong (optional) supermartingale* if Y_τ is integrable for all $\tau \in \mathcal{T}_{[0,T]}$ and $Y_\nu \geq \mathbb{E}[Y_\tau|\mathcal{F}_\nu]$ a.s. for all $\nu \leq \tau \in \mathcal{T}_{[0,T]}$.

Theorem A.4 (**Mertens decomposition**). *Let $\tilde{Y}$ be a strong optional supermartingale of class(D). There exists a unique uniformly integrable martingale (càdlàg) M, a unique nondecreasing right-continuous predictable process K with $K_0 = 0$ and $\mathbb{E}[K_T] < +\infty$, and a unique nondecreasing right-continuous adapted purely discontinuous process C with $C_{0-} = 0$ and $\mathbb{E}[C_T] < +\infty$, such that*

$$\tilde{Y}_t = M_t - K_t - C_{t-} \qquad \forall t \leq T \ a.s.$$

Theorem A.5 (**Gal'chouk-Lenglart formula**). *Let $n \in \mathbb{N}$. Let Y be an n-dimensional optional semimartingale with decomposition $Y^k = Y_0^k + M^k + N^k + W^k$, for all $k = 1, \cdots, n$ where M^k is a (càdlàg) local martingale, N^k is a right-continuous process of finite variation such that $N_0^k = 0$ and W^k is a left-continuous process of finite variation which is purely discontinuous and such that $W_0^k = 0$. Let F be a twice continuously differentiable function on $\mathbb{R}^n$. Then, almost surely, for all $t \geq 0$,*

$$\begin{aligned}
F(Y_t) \quad &= \quad F(Y_0) + \sum_{k=1}^n \int_0^t D^k F(Y_{s-}) d(M^k + N^k)_s \\
&\quad + \frac{1}{2} \sum_{k,l=1}^n \int_0^t D^k D^l F(Y_{s-}) d < M^{k,c}, M^{l,c} >_s \\
&\quad + \sum_{0<s\leq t} \left[F(Y_s) - F(Y_{s-}) - \sum_{k=1}^n D^k F(Y_{s-}) \Delta Y_s^k\right] \\
&\quad + \sum_{k=1}^n \int_0^t D^k F(Y_s) d(W^k)_{s+} \\
&\quad + \sum_{0\leq s<t} \left[F(Y_{s+}) - F(Y_s) - \sum_{k=1}^n D^k F(Y_s) \Delta_+ Y_s^k\right],
\end{aligned}$$

where D^k denotes the differentiation operator with respect to the k-th coordinate, and $M^{k,c}$ denotes the continuous part of M^k.

Corollary A.6. *Let Y be an one-dimensional optional semimartingale with decomposition $Y = Y_0 + M + N + W$ where M, N and W are as in the above Theorem. Let X be a continuous process of finite variation. Then, almost surely, for all $t \geq 0$,*

$$\begin{aligned} & F(X_t, Y_t) \\ = \; & F(X_0, Y_0) + \int_0^t \partial_X F(X_s, Y_s) ds + \int_0^t \partial_Y F(X_s, Y_{s-}) d(M+N)_s \\ & + \frac{1}{2} \int_0^t \partial_Y^2 F(X_s, Y_{s-}) d < M^c, M^c >_s + \int_0^t \partial_Y F(X_s, Y_s) d(W)_{s+} \\ & + \sum_{0<s\leq t} [F(X_s, Y_s) - F(X_s, Y_{s-}) - \partial_Y F(X_s, Y_{s-}) \Delta Y_s] \\ & + \sum_{0\leq s<t} [F(X_s, Y_{s+}) - F(X_s, Y_s) - \partial_Y F(X_s, Y_s) \Delta_+ Y_s] \end{aligned}$$

where ∂_Y is the partial derivative operator with respect to Y.

We give some useful properties of the operator $\mathcal{R}ef$ in the following lemmas.

Lemma A.7. (1) *The operator $\mathcal{R}ef$ is nondecreasing, that is, for ξ and ξ' belongs to $\mathcal{S}^2(\beta, a)$ such that $\xi \leq \xi'$ we have $\mathcal{R}ef[\xi] \leq \mathcal{R}ef[\xi']$.*
(2) *If $\xi \in \mathcal{S}^2(\beta, a)$ is a strong supermartingale, then $\mathcal{R}ef[\xi] = \xi$.*
(3) *For each $\xi \in \mathcal{S}^2(\beta, a)$, $\mathcal{R}ef[\xi]$ is a strong supermartingale and satisfies $\mathcal{R}ef[\xi] \geq \xi$.*

Lemma A.8. *Let $(\xi^n)_{n\geq 0}$ be a nondecreasing sequence of processes belonging to $\mathcal{S}^2(\beta, a)$. Let $\xi = \lim_{n\to+\infty} \xi^n$. If $\xi \in \mathcal{S}^2(\beta, a)$ the we have $\mathcal{R}ef[\xi] = \lim_{n\to+\infty} \mathcal{R}ef[\xi^n]$.*

In the following we find a special case to Gal'chouk-Lenglart formula for the convex function $x \to x^+$ du in to E. Lenglart 1980 [26].

Theorem A.9. *Let Y be an one-dimensional optional semimartingale. Then, almost surely, for all $t \geq 0$,*

$$\begin{aligned} Y_t^+ = \; & Y_0^+ + \int_0^t \mathbb{1}_{\{Y_{s-}>0\}} dY_s + \frac{1}{2} L_t(Y) \\ & + \sum_{0\leq s<t} \left[Y_{s^+}^+ \mathbb{1}_{\{Y_{s-}\leq 0\}} + Y_{s^-}^- \mathbb{1}_{\{Y_{s-}>0\}} \right] + Y_t^+ \mathbb{1}_{\{Y_{t-}\leq 0\}} + Y_t^- \mathbb{1}_{\{Y_{t-}>0\}} \end{aligned}$$

where $(L_t)_{t\leq T}$ is a local time (nondecreasing continuous process). Moreover Y^+ is an optional semimartingale.

Theorem A.10. *If Y be an optional semimartingale, then $\sum_{0\leq s<t} Y_s^+ \mathbb{1}_{\{Y_{s-}\leq 0\}} + Y_s^- \mathbb{1}_{\{Y_{s-}>0\}}$ is finite a.s.*

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