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GLOBAL STRONG SOLUTIONS OF THE STOCHASTIC THREE DIMENSIONAL INVISCID SIMPLIFIED BARDINA TURBULENCE MODEL

MANIL T. MOHAN*

Abstract. In this article, we consider the stochastic three dimensional inviscid simplified Bardina model, arising from the turbulent flows of fluids. We examine the global well-posedness of such models subject to additive and multiplicative Gaussian noise. Using the Banach fixed point theorem (or contraction mapping principle), we show that the stochastic 3D inviscid simplified Bardina turbulence model has a unique global pathwise strong solution.

1. Introduction

Turbulent fluid motion can be considered as an irregular condition of flow in which several fluid parameters such as velocity, pressure etc, exhibit a random variation with time and space (or the particle trajectories vary randomly in time) in such a way that the statistical average of those quantities can be quantitatively expressed. Turbulence is inevitably connected to the important dimensionless quantity, namely Reynolds number. The Reynolds number is defined as $\mathrm{Re} = \frac{\mathbf{u}L}{\nu}$, where $\mathbf{u}$ is the velocity of the fluid, L is a characteristic linear dimension (traveled length of the fluid) and ν is the co-efficient of kinematic viscosity of the fluid. At a Reynolds number less than the critical value (that is if $\mathbf{u}$ or L (or both) are small and the viscosity is large), the kinetic energy of fluid flow is not enough to sustain the random fluctuations against the viscous damping and in such cases laminar or streamline flow continues to exist. At higher Reynolds number than the critical value, the kinetic energy of flow supports the growth of fluctuations and transition to turbulence takes place. The well known Navier-Stokes equations explain both laminar and turbulent flows in great detail (cf. [15]). When a flow is turbulent, the Navier-Stokes equations do not provide amenable mathematical models that can authentically predict the properties of turbulent flows. By approximating the Reynolds stress tensor, a particular closure model called the *Bardina model* is introduced in [1]. Later, [17] considered a simpler approximation of the Reynolds stress tensor for Bardina model and is named as *simplified Bardina model.* In this article, we consider the stochastic three dimensional inviscid simplified Bardina model subject to periodic boundary conditions, which is given below.

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Let $\mathcal{O} = [0, 2\pi]^3$ be a periodic domain. Let $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ denotes the large-scale (or averaged) velocity of the fluid with constant density, the scalar valued function $p = p(x,t)$ denotes the pressure field and is determined by the incompressibility constraint, and $\mathbf{f}$ represents the external forcing. Let T be an arbitrary but fixed positive number. For $t \in [0,T]$ and $x \in \mathcal{O}$, let us consider the simplified Bardina turbulence model of inviscid incompressible flows, subject to periodic boundary condition, written in expanded form as [3]:

$$\left\{\begin{aligned} \frac{\partial}{\partial t}(\mathbf{u}(x,t) - \alpha\Delta\mathbf{u}(x,t)) + (\mathbf{u}(x,t)\cdot\nabla)\mathbf{u}(x,t) &= -\nabla p(x,t) + \mathbf{f}(x,t), \\ &\text{in } \ \mathcal{O}\times(0,T), \\ \nabla\cdot\mathbf{u}(x,t) = 0, \ \ \text{in } \ \mathcal{O}\times(0,T)&, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x), \ \ \text{in } \ \mathcal{O}&, \end{aligned}\right. \tag{1.1}$$

where $\alpha > 0$ is the square of the spatial scale at which fluid motion is filtered, i.e., spatial scales smaller than α are averaged out. Also, for $\mathbf{v} := \mathbf{u} - \alpha\Delta\mathbf{u}$, the simplified Bardina model can be written as:

$$\left\{\begin{aligned} \frac{\partial}{\partial t}\mathbf{v}(x,t) + (\mathbf{u}(x,t)\cdot\nabla)\mathbf{u}(x,t) + \nabla p(x,t) = \mathbf{f}(x,t), \ \ \text{in } \ \mathcal{O}\times(0,T), \\ \nabla\cdot\mathbf{u}(x,t) = \nabla\cdot\mathbf{v}(x,t) = 0, \ \ \text{in } \ \mathcal{O}\times(0,T), \\ \mathbf{v}(x,t) = (\mathrm{I} - \alpha\Delta)\mathbf{u}(x,t), \\ \mathbf{u}(0) = \mathbf{u}_0, \ \mathbf{v}(0,x) = \mathbf{v}_0(x) = (\mathrm{I}-\alpha\Delta)\mathbf{u}_0(x), \ \ \text{in } \ \mathcal{O}, \end{aligned}\right. \tag{1.2}$$

where $\mathbf{u}$ and $\mathbf{v}$ are periodic.

Let us now discuss about the solvability results available in the literature for the system (1.1) (or equivalently (1.2)). The global existence and uniqueness of strong solutions to the viscous simplified Bardina turbulence model has been established in [17]. Analytical studies of three-dimensional viscous and inviscid simplified Bardina turbulence models with periodic boundary conditions has been conducted in [3]. The authors in [3] established the global well-posedness of the viscous model for weaker initial conditions. The global existence and uniqueness of weak solutions to the inviscid model is also proved in [3]. The authors in [28] considered the stochastic version of the 3D Bardina model arising from the turbulent flows of fluids and obtained the existence of probabilistic weak solution for the model with the non-Lipschitz conditions. The alpha models, such as Lagrangian averaged Navier-Stokes equations [6] (also known as the Navier-Stokes-α or viscous Camassa-Holm equations), Leray-α model [7], etc, are also related to the simplified Bardina models (see [28] for more details). The stochastic versions of alpha models, such as stochastic Lagrangian averaged Navier-Stokes equations [5, 10], stochastic Leray-alpha model [11], etc, are also available in the literature. For a sample of literature on stochastic Navier-Stokes equations, we refer the readers to [2, 4, 9, 13, 14, 18, 20, 23, 24, 29, 31], etc.

In this work, we consider the 3D inviscid simplified Bardina turbulence model perturbed by additive and multiplicative Gaussian noise subject to periodic boundary conditions and examine global solvability results. The Banach fixed point theorem (or contraction mapping principle) is used to establish the existence of a global unique pathwise strong solution to the system (1.1). One can also obtain the global solvability results using a vanishing viscosity method and Galerkin approximation techniques (see Remark 3.4). The results obtained in this paper has an important application in computational fluid dynamics also. The inviscid simplified Bardina model can be considered as a regularizing model of the three-dimensional stochastic Euler equations (see [22]). This is also a motivation for us to consider such a problem. We also remark that the results obtained in this paper are still valid for some unbounded domains like Poncaré domains (see Remark 3.5). We now state the main result obtained in this work.

Theorem 1.1. *Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ be a given probability space. Let the $\mathscr{F}_0$-measurable initial data $\mathbf{u}_0 \in \mathrm{L}^2(\Omega;\mathbb{V})$ be given. Then, there exists a strong solution $\mathbf{u} \in \mathrm{L}^2(\Omega;\mathrm{L}^\infty([0,T];\mathbb{V}))$ to the problem (2.12) (see below) satisfying*

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t (\mathrm{I}+\alpha\mathrm{A})^{-1}\mathrm{B}(\mathbf{u}(s))\mathrm{d}s + \int_0^t (\mathrm{I}+\alpha\mathrm{A})^{-1}\sqrt{\mathrm{Q}}\mathrm{dW}(s),$$

for all $t \in [0,T]$, as an element of $\mathbb{V}'$, $\mathbb{P}$-a.s., that is,

$$\langle(\mathrm{I}+\alpha\mathrm{A})\mathbf{u}(t),\mathbf{w}\rangle = \langle\mathbf{v}_0,\mathbf{w}\rangle + \int_0^t \langle\mathrm{B}(\mathbf{u}(s)),\mathbf{w}\rangle\mathrm{d}s + \int_0^t \langle\sqrt{\mathrm{Q}}\mathrm{dW}(s),\mathbf{w}\rangle,$$

for all $\mathbf{w} \in \mathbb{V}$, and

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathbf{u}(t)\|_{\mathbb{V}}^2\right] \leq \left(\frac{1}{\lambda_1}+\alpha\right)\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathbb{V}}^2\big] + \big[\mathrm{Tr}((\mathrm{I}+\alpha\mathrm{A})^{-1}\mathrm{Q}) + 6\,\mathrm{Tr}(\mathrm{Q})\big]T,$$

for arbitrary $T > 0$. Also, the $(\mathscr{F}_t)_{t\geq 0}$-adapted paths of the strong solution has continuous trajectories in $\mathrm{C}([0,T];\mathbb{V})$, $\mathbb{P}$-a.s., and the solution is pathwise unique.

The rest of the paper is organized as follows: In the next section, we describe an abstract formulation of the problem (1.1) and explain the necessary function spaces needed to obtain the global solvability results of the system (1.1). Existence and uniqueness of a global pathwise strong solution to the problem (1.1) is obtained in section 3. The methodology of establishing such a result is as follows. First we consider a cut-off problem (see (3.2) below) and obtain the unique solvability results to the system (3.2), using Picard's iteration and contraction mapping principle (see Proposition 3.2). Then using a uniform energy bound, we extend this solution to a unique global strong solution to the problem (2.12) (see Theorem 3.3). In the final section, we consider the 3D inviscid simplified Bardina turbulence model subject to multiplicative Gaussian noise and establish the global solvability results (see Theorem 4.3).

2. Stochastic Inviscid Simplified Bardina Model

In this section, we give an abstract formulation of the system (1.1) (or equivalently (1.2)) and explain the necessary functional settings required to obtain the global solvability results.

2.1. Functional setting. Let us now explain the function spaces needed to establish the global solvability results of the system (1.1). Let $\mathcal{O} := [0, 2\pi] \times [0, 2\pi] \times [0, 2\pi]$ be a periodic domain, and we define the spaces

$$\mathbb{H} := \left\{ \mathbf{u} \in \mathbb{L}^2(\mathcal{O}; \mathbb{R}^3),\ \text{div } \mathbf{u} = 0,\ \int_{\mathcal{O}} \mathbf{u}(x)\mathrm{d}x = 0, \mathbf{u} \cdot \mathbf{n} \text{ is periodic} \right\},$$
$$\mathbb{V} := \left\{ \mathbf{u} \in \mathbb{H}^1(\mathcal{O}; \mathbb{R}^3),\ \text{div } \mathbf{u} = 0, \int_{\mathcal{O}} \mathbf{u}(x)\mathrm{d}x = 0, \mathbf{u} \cdot \mathbf{n} \text{ is periodic} \right\},$$

where $\mathbf{n}$ is the unit outward normal, and for an integer $k \geq 1$, $\mathbb{H}^k(\mathcal{O}; \mathbb{R}^3)$ is the space of $\mathbb{R}^3$-valued measurable functions $\mathbf{u}$ that are in $\mathbb{H}^k_{\text{loc}}(\mathbb{R}^3; \mathbb{R}^3)$ and such that $\mathbf{u}(x + 2\pi\mathbf{e}_i) = \mathbf{u}(x)$ for every $x \in \mathbb{R}^3$ and $i = 1, 2, 3$. Here $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the canonical basis of $\mathbb{R}^3$. We denote by $(\cdot, \cdot)$ and $\|\cdot\|_{\mathbb{H}}$, the usual $\mathbb{L}^2$-inner product and norm in $\mathbb{H}$ with

$$\|\mathbf{u}\|_{\mathbb{H}}^2 := \int_{\mathcal{O}} |\mathbf{u}(x)|^2 \mathrm{d}x.$$

Using the zero mean condition, we also have the *Poincaré-Wirtinger inequality*, $\|\mathbf{u}\|_{\mathbb{H}} \leq \frac{1}{\lambda}\|\nabla \mathbf{u}\|_{\mathbb{H}}$, where λ is defined to be the smallest constant for which this inequality holds (see [15]). Using the Poincaré-Wirtinger inequality, we may endow $\mathbb{V}$ with the norm

$$\|\mathbf{u}\|_{\mathbb{V}}^2 := \int_{\mathcal{O}} |\nabla \mathbf{u}(x)|^2 \mathrm{d}x.$$

The induced duality pairing, for instance between the spaces $\mathbb{V}$ and $\mathbb{V}'$, is denoted by $\langle \cdot, \cdot \rangle$. For any $\mathbf{u} \in \mathbb{H}$ and $\mathbf{v} \in \mathbb{V}$, there exists a $\mathbf{u}' \in \mathbb{V}'$, such that $(\mathbf{u}, \mathbf{v}) = \langle \mathbf{u}', \mathbf{v} \rangle$.

2.2. Linear operator. Let $\mathrm{P}_{\mathbb{H}} : \mathbb{L}^2(\mathcal{O}) \to \mathbb{H}$ be the Helmholtz-Hodge orthogonal projection operator. We define the Stokes operator

$$\mathrm{A} : \mathrm{D}(\mathrm{A}) \to \mathbb{H} \quad \text{with} \quad \mathrm{A}\mathbf{u} := -\mathrm{P}_{\mathbb{H}} \Delta \mathbf{u}, \tag{2.1}$$

where $\mathrm{D}(\mathrm{A}) = \mathbb{V} \cap \mathbb{H}^2(\mathcal{O}) = \left\{ \mathbf{u} \in \mathbb{H}_0^1(\mathcal{O}) \cap \mathbb{H}^2(\mathcal{O}) : \nabla \cdot \mathbf{u} = 0 \right\}$ is the domain of the operator A. The Stokes operator is a positive selfadjoint operator with compact resolvent and if $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ are the eigenvalues of A, then we have $\|\mathbf{u}\|_{\mathbb{V}}^2 \geq \lambda_1 \|\mathbf{u}\|_{\mathbb{H}}^2$, for all $\mathbf{u} \in \mathbb{V}$. This can be shown in the following way. Let $\{e_1, e_2, \ldots\}$ be the orthonormal eigenvectors of A corresponding to the eigenvalues $\{\lambda_1, \lambda_2, \ldots\}$ such that $0 < \lambda_1 \leq \lambda_2 \leq \ldots$. We know that any $\mathbf{u} \in \mathbb{V}$ can be expressed as

$$\mathbf{u} = \sum_{j=1}^{\infty} \langle \mathbf{u}, e_j \rangle e_j \quad \text{and hence} \quad \mathrm{A}\mathbf{u} = \sum_{j=1}^{\infty} \lambda_j \langle \mathbf{u}, e_j \rangle e_j.$$

Thus, it is immediate that

$$\|\nabla \mathbf{u}\|_{\mathbb{H}}^2 = \langle \mathrm{A}\mathbf{u}, \mathbf{u} \rangle = \sum_{j=1}^{\infty} \lambda_j |\langle \mathbf{u}, e_j \rangle|^2 \geq \lambda_1 \sum_{j=1}^{\infty} |\langle \mathbf{u}, e_j \rangle|^2 = \lambda_1 \|\mathbf{u}\|_{\mathbb{H}}^2.$$

Remark 2.1. Let us now show that the norms $\|\mathbf{u}\|_{\mathbb{V}}$ and $\|\mathbf{v}\|_{\mathbb{V}'}$ are equivalent. Note that $\sqrt{\lambda_1}\|\mathbf{u}\|_{\mathbb{V}'} \leq \|\mathbf{u}\|_{\mathbb{H}} \leq \frac{1}{\sqrt{\lambda_1}}\|\mathbf{u}\|_{\mathbb{V}}$ and hence we have

$$\langle (\mathrm{I} + \alpha\mathrm{A})\mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \alpha \langle \mathrm{A}\mathbf{u}, \mathbf{w} \rangle \leq \|\mathbf{u}\|_{\mathbb{V}'} \|\mathbf{w}\|_{\mathbb{V}} + \alpha \|\mathrm{A}^{1/2}\mathbf{u}\|_{\mathbb{H}} \|\mathrm{A}^{1/2}\mathbf{w}\|_{\mathbb{H}}$$

$$\leq \left(\frac{1}{\lambda_1}+\alpha\right)\|\mathbf{u}\|_{\mathbb{V}}\|\mathbf{w}\|_{\mathbb{V}},$$

for all $\mathbf{u},\mathbf{w}\in\mathbb{V}$. Thus, it is immediate that

$$\|(\mathrm{I}+\alpha\mathrm{A})\mathbf{u}\|_{\mathbb{V}'}\leq \left(\frac{1}{\lambda_1}+\alpha\right)\|\mathbf{u}\|_{\mathbb{V}}, \text{ so that } \|\mathbf{v}\|_{\mathbb{V}'}\leq \left(\frac{1}{\lambda_1}+\alpha\right)\|\mathbf{u}\|_{\mathbb{V}}, \tag{2.2}$$

where $\mathbf{v}=(\mathrm{I}+\alpha\mathrm{A})\mathbf{u}$. We also know that

$$\begin{aligned}\|(\mathrm{I}+\alpha\mathrm{A})^{-1}\mathbf{v}\|_{\mathbb{V}}^2 &= \|\mathrm{A}^{1/2}(\mathrm{I}+\alpha\mathrm{A})^{-1}\mathbf{v}\|_{\mathbb{H}}^2=\sum_{j=1}^{\infty}\left|\langle \mathrm{A}^{1/2}(\mathrm{I}+\alpha\mathrm{A})^{-1}\mathbf{v},e_j\rangle\right|^2\\
&=\sum_{j=1}^{\infty}\left|\langle \mathrm{A}^{-1/2}\mathbf{v},\mathrm{A}(\mathrm{I}+\alpha\mathrm{A})^{-1}e_j\rangle\right|^2=\sum_{j=1}^{\infty}\left|\frac{\lambda_j}{(1+\alpha\lambda_j)}\langle \mathrm{A}^{-1/2}\mathbf{v},e_j\rangle\right|^2\\
&\leq \frac{1}{\left(\frac{1}{\lambda_1}+\alpha\right)^2}\sum_{j=1}^{\infty}\left|\langle \mathrm{A}^{-1/2}\mathbf{v},e_j\rangle\right|^2=\frac{1}{\left(\frac{1}{\lambda_1}+\alpha\right)^2}\|\mathrm{A}^{-1/2}\mathbf{v}\|_{\mathbb{H}}^2,\end{aligned}$$

so that we have

$$\|\mathbf{u}\|_{\mathbb{V}}\leq \frac{1}{\left(\frac{1}{\lambda_1}+\alpha\right)}\|\mathbf{v}\|_{\mathbb{V}'}. \tag{2.3}$$

Combining (2.2) and (2.3), we find

$$\frac{1}{\left(\frac{1}{\lambda_1}+\alpha\right)}\|\mathbf{u}\|_{\mathbb{V}}\leq\|\mathbf{v}\|_{\mathbb{V}'}\leq\left(\frac{1}{\lambda_1}+\alpha\right)\|\mathbf{u}\|_{\mathbb{V}}, \tag{2.4}$$

and hence the norms $\|\mathbf{u}\|_{\mathbb{V}}$ and $\|\mathbf{v}\|_{\mathbb{V}'}$ are equivalent.

2.3. Nonlinear operator. For $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{V}$, we define the trilinear operator $b(\cdot,\cdot,\cdot)$ as

$$b(\mathbf{u},\mathbf{v},\mathbf{w}):=\int_{\mathcal{O}}(\mathbf{u}(x)\cdot\nabla)\mathbf{v}(x)\cdot\mathbf{w}(x)\mathrm{d}x=\sum_{i,j=1}^{3}\int_{\mathcal{O}}u_i(x)\frac{\partial v_j(x)}{\partial x_i}w_j(x)\mathrm{d}x,$$

and the bilinear operator $\mathrm{B}:\mathbb{V}\times\mathbb{V}\to\mathbb{V}'$ is defined by,

$$\langle \mathrm{B}(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle:=b(\mathbf{u},\mathbf{v},\mathbf{w}), \text{ for all } \mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{V}.$$

An integration by parts yields,

$$\begin{cases} b(\mathbf{u},\mathbf{v},\mathbf{v})=0, \text{ for all } \mathbf{u},\mathbf{v}\in\mathbb{V},\\ b(\mathbf{u},\mathbf{v},\mathbf{w})=-b(\mathbf{u},\mathbf{w},\mathbf{v}), \text{ for all } \mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{V}.\end{cases} \tag{2.5}$$

For more details about the linear and nonlinear operators, we refer the readers to [8, 32].

Now we provide an important inequality due to Gagliardo-Nirenberg-Sobolev, which is used to estimate the trilinear form and hence bilinear operator. Even though the inequality given below is stated in bounded domains, it is valid in periodic domains also.

Lemma 2.2 (Gagliardo-Nirenberg-Sobolev inequality, Theorem 2.1, [27], Theorem 2.1, [12])**.** *Let $\mathcal{O} \subset \mathbb{R}^n$ be bounded and $\mathbf{u} \in \mathrm{W}_0^{1,p}(\mathcal{O};\mathbb{R}^n), p \geq 1$. Then for any fixed number $q, r \geq 1$, there exists a constant $C > 0$ depending only on n, p, q such that*

$$\|\mathbf{u}\|_{\mathbb{L}^r} \leq C\|\nabla \mathbf{u}\|_{\mathbb{L}^p}^{\theta}\|\mathbf{u}\|_{\mathbb{L}^q}^{1-\theta},\ \theta \in [0,1], \tag{2.6}$$

where the numbers p, q, r and θ satisfy the relation

$$\theta = \left(\frac{1}{q} - \frac{1}{r}\right)\left(\frac{1}{n} - \frac{1}{p} + \frac{1}{q}\right)^{-1}.$$

Let us take $r = n = 3$ and $p = q = 2$ in (2.6) to get $\theta = \frac{1}{2}$ and

$$\|\mathbf{u}\|_{\mathbb{L}^3} \leq C\|\nabla \mathbf{u}\|_{\mathbb{L}^2}^{1/2}\|\mathbf{u}\|_{\mathbb{L}^2}^{1/2}. \tag{2.7}$$

Now if we take $r = 4$, $n = 3$ and $p = q = 2$ in (2.6), we find $\theta = \frac{3}{4}$ and

$$\|\mathbf{u}\|_{\mathbb{L}^4} \leq C\|\nabla \mathbf{u}\|_{\mathbb{L}^2}^{3/4}\|\mathbf{u}\|_{\mathbb{L}^2}^{1/4}, \tag{2.8}$$

where the constant $C = \sqrt{2}$ (see Lemma 2, Chapter 1[16]). We also take $r = 6$, $n = 3$ and $p = q = 2$ in (2.6) to obtain $\theta = 1$ and

$$\|\mathbf{u}\|_{\mathbb{L}^6} \leq C\|\nabla \mathbf{u}\|_{\mathbb{L}^2}, \tag{2.9}$$

where the constant $C = 48^{\frac{1}{6}}$ (see Lemma 2, Chapter 1, [16]).

Using Hölder's and Gagliardo-Nirenberg-Sobolev inequalities, we find

$$|\langle \mathrm{B}(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle| = |\langle \mathrm{B}(\mathbf{u},\mathbf{w}),\mathbf{v}\rangle| \leq \|\mathbf{u}\|_{\mathbb{L}^3}\|\mathbf{w}\|_{\mathbb{V}}\|\mathbf{v}\|_{\mathbb{L}^6} \leq C\|\mathbf{u}\|_{\mathbb{H}}^{1/2}\|\mathbf{u}\|_{\mathbb{V}}^{1/2}\|\mathbf{v}\|_{\mathbb{V}}\|\mathbf{w}\|_{\mathbb{V}}. \tag{2.10}$$

Thus, we have

$$\|\mathrm{B}(\mathbf{u},\mathbf{v})\|_{\mathbb{V}'} \leq C\|\mathbf{u}\|_{\mathbb{H}}^{1/2}\|\mathbf{u}\|_{\mathbb{V}}^{1/2}\|\mathbf{v}\|_{\mathbb{V}} \leq \frac{C}{\lambda_1^{1/4}}\|\mathbf{u}\|_{\mathbb{V}}\|\mathbf{v}\|_{\mathbb{V}}, \tag{2.11}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{V}$.

2.4. Abstract formulation. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a given complete probability space equipped with an increasing family of sub-sigma fields $\{\mathscr{F}_t\}_{0\leq t\leq T}$ of $\mathscr{F}$ satisfying usual conditions. Let us consider the external forcing to be random (additive Gaussian noise) in (1.1) adapted to the filtration $\{\mathscr{F}_t\}_{0\leq t\leq T}$. We apply the orthogonal projection $\mathrm{P}_{\mathbb{H}}$ to the system (1.1) to obtain an abstract version of the stochastic inviscid simplified Bardina model (1.1) as

$$\begin{cases} \mathrm{d}\mathbf{v}(t) = -\mathrm{B}(\mathbf{u}(t))\mathrm{d}t + \sqrt{\mathrm{Q}}\mathrm{d}\mathrm{W}(t), \\ \mathbf{v}(t) = (\mathrm{I} + \alpha\mathrm{A})\mathbf{u}(t) = \mathbf{u}(t) + \alpha\mathrm{A}\mathbf{u}(t), \\ \mathbf{u}(0) = \mathbf{u}_0, \mathbf{v}(0) = \mathbf{v}_0 = \mathbf{u}_0 + \alpha\mathrm{A}\mathbf{u}_0. \end{cases} \tag{2.12}$$

Since the projection $\mathrm{P}_{\mathbb{H}}$ and $(\mathrm{I}+\alpha\mathrm{A})$ commutes, the above system is equivalent to

$$\begin{cases} \mathrm{d}\mathbf{u}(t) = -(\mathrm{I} + \alpha\mathrm{A})^{-1}\mathrm{B}(\mathbf{u}(t))\mathrm{d}t + (\mathrm{I} + \alpha\mathrm{A})^{-1}\sqrt{\mathrm{Q}}\mathrm{d}\mathrm{W}(t), \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{2.13}$$

where $\mathbf{u}_0 \in \mathrm{L}^2(\Omega;\mathbb{V})$. In (2.12), $\mathrm{W}(\cdot)$ is an $\mathbb{H}$-valued cylindrical Weiner process.

Let $\mathcal{L}(\mathbb{H},\mathbb{H})$ be the space of all bounded linear operators on $\mathbb{H}$. Let $\mathrm{Q}\in\mathcal{L}(\mathbb{H},\mathbb{H})$ be a non-negative, symmetric and trace class operator on $\mathbb{H}$. Thus there exists an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ of $\mathbb{H}$ such that $\mathrm{Q}e_k=\mu_k e_k, k\in\mathbb{N}$, where μ_k is the eigenvalue corresponding to $\{e_k\}$ which is real and non-negative satisfying

$$\mathrm{Tr}(\mathrm{Q})=\sum_{k=1}^{\infty}\mu_k<+\infty \quad\text{and}\quad \sqrt{\mathrm{Q}}\mathbf{v}=\sum_{k=1}^{\infty}\sqrt{\mu_k}(\mathbf{v},e_k)e_k, \quad \text{for all}\quad \mathbf{v}\in\mathbb{H}.$$

The stochastic process $\{\mathrm{W}(t):0\leq t\leq T\}$ is an $\mathbb{H}$-valued cylindrical Wiener process on $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geq 0},\mathbb{P})$ if and only if for arbitrary t, the process $\mathrm{W}(t)$ can be expressed as $\mathrm{W}(t)=\sum\limits_{k=1}^{\infty}\beta_k(t)e_k$, where $\beta_k(t),k\in\mathbb{N}$ are independent, one dimensional Brownian motions on the space $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geq 0},\mathbb{P})$ (see [9]). Now we give some examples of the operator Q, considered in this paper.

Example 2.3. 1. The operator $\mathrm{Q}=(\mathrm{I}+\alpha\mathrm{A})^{-\beta}$ with $\beta>3/2$ satisfies the conditions:

(i) $\mathrm{Tr}(\mathrm{Q})<+\infty$,
(ii) $\mathrm{Tr}((\mathrm{I}+\alpha\mathrm{A})^{-1}\mathrm{Q})<+\infty$.

Indeed, since the asymptotic behavior of the eigenvalues of the operator A in periodic domain is given by $\lambda_k\sim\lambda_1 k^{2/3}$ (Theorem 4.11, [8], page 54, [15]). That is, there is a dimensionless constant C_0 such that

$$\frac{k^{2/3}}{C_0}\leq\frac{\lambda_k}{\lambda_1}\leq C_0 k^{2/3}, \text{ for } k=1,2,\ldots.$$

Thus, we have

$$\begin{aligned}\mathrm{Tr}(\mathrm{Q})=\mathrm{Tr}((\mathrm{I}+\alpha\mathrm{A})^{-\beta})&=\sum_{k=1}^{\infty}((1+\alpha\lambda_k)^{-\beta}e_k,e_k)=\sum_{k=1}^{\infty}(1+\alpha\lambda_k)^{-\beta}\\&\leq C_0\sum_{k=1}^{\infty}(1+\alpha k^{2/3})^{-\beta}\leq\frac{C_0}{\alpha^{2\beta}}\sum_{k=1}^{\infty}\frac{1}{k^{2\beta/3}}<+\infty,\end{aligned}$$

for $\beta>3/2$. Similarly, we have

$$\begin{aligned}\mathrm{Tr}((\mathrm{I}+\alpha\mathrm{A})^{-1}\mathrm{Q})&=\sum_{k=1}^{\infty}((1+\alpha\lambda_k)^{-(1+\beta)}e_k,e_k)=\sum_{k=1}^{\infty}(1+\alpha\lambda_k)^{-(1+\beta)}\\&\leq C_0\sum_{k=1}^{\infty}(1+\alpha k^{2/3})^{-(1+\beta)}\leq\frac{C_0}{\alpha^{2\beta}}\sum_{k=1}^{\infty}\frac{1}{k^{2(1+\beta)/3}}<+\infty,\end{aligned}$$

for $\beta>1/2$. Hence, for $\beta>3/2$, both the conditions are satisfied.

2. One can also show that the operator $\mathrm{Q}=\mathrm{A}^{-\beta}$, $\beta>3/2$ satisfies $\mathrm{Tr}(\mathrm{Q})<+\infty$, and $\mathrm{Tr}((\mathrm{I}+\alpha\mathrm{A})^{-1}\mathrm{Q})<+\infty$.

2.5. Global strong solution. Let us now give the definition of a unique global pathwise strong solution to the system (2.12).

Definition 2.4 (Global strong solution)**.** Let the $\mathscr{F}_0$-measurable initial data $\mathbf{u}_0\in\mathrm{L}^2(\Omega;\mathbb{V})$ be given. A $\mathbb{V}$-valued $(\mathscr{F}_t)_{t\geq 0}$-adapted continuous process $\mathbf{u}(\cdot)$ is called a *strong solution* to (2.12) if the following conditions are satisfied:

(i) the process $\mathbf{u} \in \mathrm{L}^2(\Omega; \mathrm{L}^\infty([0,T]; \mathbb{V}))$ is such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|\mathbf{u}(t)\|_{\mathbb{V}}^2\right] \leq \left(\frac{1}{\lambda_1} + \alpha\right)\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathbb{V}}^2\big] + \big[\mathrm{Tr}((\mathrm{I}+\alpha\mathrm{A})^{-1}\mathrm{Q}) + 6\,\mathrm{Tr}(\mathrm{Q})\big]T, \tag{2.14}$$

(ii) the following equality holds for every $t \in [0,T]$, as an element of $\mathbb{V}'$, $\mathbb{P}$-a.s.,

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t (\mathrm{I}+\alpha\mathrm{A})^{-1}\mathrm{B}(\mathbf{u}(s))\mathrm{d}s + \int_0^t (\mathrm{I}+\alpha\mathrm{A})^{-1}\sqrt{\mathrm{Q}}\mathrm{d}\mathrm{W}(s), \tag{2.15}$$

that is,

$$\langle(\mathrm{I}+\alpha\mathrm{A})\mathbf{u}(t), \mathbf{w}\rangle = \langle\mathbf{v}_0, \mathbf{w}\rangle + \int_0^t \langle\mathrm{B}(\mathbf{u}(s)), \mathbf{w}\rangle\mathrm{d}s + \int_0^t \langle\sqrt{\mathrm{Q}}\mathrm{d}\mathrm{W}(s), \mathbf{w}\rangle, \tag{2.16}$$

for all $\mathbf{w} \in \mathbb{V}$.

Definition 2.5. A strong solution $\mathbf{u}(\cdot)$ to (2.12) is called a *unique strong solution* if $\widetilde{\mathbf{u}}(\cdot)$ is an another strong solution, then

$$\mathbb{P}\Big\{\omega \in \Omega : \mathbf{u}(t,\omega) = \widetilde{\mathbf{u}}(t,\omega), \ \text{ for all } \ t \in [0,T]\Big\} = 1.$$

3. Existence and Uniqueness of Global Strong Solution

In this section, we establish the global existence and uniqueness of pathwise strong solution to the stochastic inviscid *simplified Bardina model* (2.12). In order to do this we first consider a cut-off problem and establish the global solvability of the cut-off problem using the Banach fixed point theorem (or contraction mapping principle). Let $(\mathbb{X}, d)$ be a metric space. A map $\mathcal{F} : \mathbb{X} \to \mathbb{X}$ is called a *contraction mapping* on $\mathbb{X}$, if there exists $0 \leq \alpha < 1$ such that

$$d(\mathcal{F}(\mathbf{u}), \mathcal{F}(\mathbf{v})) \leq \alpha d(\mathbf{u}, \mathbf{v}),$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{X}$. Let $(\mathbb{X}, d)$ be a non-empty complete metric space with a contraction mapping $\mathcal{F} : \mathbb{X} \to \mathbb{X}$. Then $\mathcal{F}$ admits a *unique fixed-point* $\mathbf{u}^*$ in $\mathbb{X}$ (i.e., $\mathcal{F}(\mathbf{u}^*) = \mathbf{u}^*$). Moreover, $\mathbf{u}^*$ can be found as follows: start with an arbitrary element $\mathbf{u}_0 \in \mathbb{X}$ and define a sequence $\{\mathbf{u}_n\}$ by $\mathbf{u}_n = \mathcal{F}(\mathbf{u}_{n-1})$, then $\mathbf{u}_n \to \mathbf{u}^* \in \mathbb{X}$.

3.1. The cut-off problem.

Let us define a function $\Pi_n : [0,\infty) \to [0,1]$ by

$$\Pi_n(y) = \begin{cases} 1, & \text{for } \ 0 \leq y \leq n, \\ n+1-y, & \text{for } \ n < y \leq n+1, \\ 0, & \text{for } \ y > n+1, \end{cases} \tag{3.1}$$

where n is a positive integer. Note that the function $\Pi_n(\cdot)$ is continuous. Let us first consider the following cut-off problem:

$$\begin{cases} \mathrm{d}\mathbf{v}_n(t) = -\Pi_n(\|\mathbf{u}_n\|_{\mathbb{V}})\mathrm{B}(\mathbf{u}_n(t))\mathrm{d}t + \sqrt{\mathrm{Q}}\mathrm{d}\mathrm{W}(t), \\ \mathbf{u}_n(0) = \mathbf{u}_0, \mathbf{v}_n(0) = \mathbf{v}_0 = \mathbf{u}_0 + \alpha\mathrm{A}\mathbf{u}_0. \end{cases} \tag{3.2}$$

The above system is equivalent to

$$\begin{cases} \mathrm{d}\mathbf{u}_n(t) = -\Pi_n(\|\mathbf{u}_n\|_{\mathbb{V}})(\mathrm{I}+\alpha\mathrm{A})^{-1}\mathrm{B}(\mathbf{u}_n(t))\mathrm{d}t + (\mathrm{I}+\alpha\mathrm{A})^{-1}\sqrt{\mathrm{Q}}\mathrm{d}\mathrm{W}(t), \\ \mathbf{u}_n(0) = \mathbf{u}_0. \end{cases} \tag{3.3}$$

First we show that the system (3.2) (or equivalently (3.3)) has a unique strong solution using Banach fixed point theorem (or contraction mapping principle). Let us define $\mathbb{X} := \mathrm{L}^2(\Omega; \mathrm{L}^{\infty}([0,T];\mathbb{V}))$ and the metric

$$d(\mathbf{u},\mathbf{v}) = \mathbb{E}\left[\sup_{t\in[0,T]}\|\mathbf{u}(t)-\mathbf{v}(t)\|_{\mathbb{V}}^2\right].$$

Then one can easily show that $(\mathbb{X}, d)$ form a complete metric space. Let us define a map $\mathcal{F}(\cdot) : \mathbb{X} \to \mathbb{X}$ as

$$\mathcal{F}(\mathbf{u}_n(t)) := \mathbf{u}_0 + \int_0^t \Pi_n(\|\mathbf{u}_n(s)\|_{\mathbb{V}})(\mathrm{I}+\alpha\mathrm{A})^{-1}\mathrm{B}(\mathbf{u}_n(s))\mathrm{d}s + \int_0^t (\mathrm{I}+\alpha\mathrm{A})^{-1}\sqrt{\mathrm{Q}}\mathrm{d}\mathrm{W}(s). \tag{3.4}$$

The above map is understood in the following way:

$$\langle(\mathrm{I}+\alpha\mathrm{A})\mathcal{F}(\mathbf{u}_n(t)),\mathbf{w}\rangle = \langle\mathbf{v}_0,\mathbf{w}\rangle + \int_0^t\langle\Pi_n(\|\mathbf{u}_n(s)\|_{\mathbb{V}})\mathrm{B}(\mathbf{u}_n(s)),\mathbf{w}\rangle\mathrm{d}s + \int_0^t\langle\sqrt{\mathrm{Q}}\mathrm{d}\mathrm{W}(s),\mathbf{w}\rangle, \tag{3.5}$$

for all $\mathbf{w}\in\mathbb{V}$.

We consider the space $\mathbb{X}$ as a Banach space consisting of all $\mathbb{V}$-valued, $(\mathscr{F}_t)_{t\geq 0}$-adapted stochastic processes with the norm defined by

$$\|\mathbf{u}\|_{\mathbb{X}}^2 := \mathbb{E}\left[\sup_{t\in[0,T]}\|\mathbf{u}(t)\|_{\mathbb{V}}^2\right] < +\infty, \quad \text{for all } \mathbf{u}\in\mathbb{X}.$$

For $\mathbf{v} = (\mathrm{I}+\alpha\mathrm{A})\mathbf{u}$, remember that the norms $\|\mathbf{v}\|_{\mathbb{V}'}$ and $\|\mathbf{u}\|_{\mathbb{V}}$ are equivalent (see Remark 2.1), and we show that the map $\mathcal{F}$ is a contraction on $\mathbb{X}$. In order to establish the existence of a unique pathwise strong solution to the system (3.2), we need the following important lemma.

Lemma 3.1. *For all $\mathbf{u}_n^1,\mathbf{u}_n^2\in\mathbb{V}$, we have*

$$\|\Pi_n(\|\mathbf{u}_n^1\|_{\mathbb{V}})\mathrm{B}(\mathbf{u}_n^1) - \Pi_n(\|\mathbf{u}_n^2\|_{\mathbb{V}})\mathrm{B}(\mathbf{u}_n^2)\|_{\mathbb{V}'} \leq \frac{C_n}{\lambda_1^{1/4}}\|\mathbf{u}_n^1-\mathbf{u}_n^2\|_{\mathbb{V}}. \tag{3.6}$$

Proof. For simplicity, we take $\mathbf{u}_n^1 = \mathbf{u}_1$ and $\mathbf{u}_n^2 = \mathbf{u}_2$. Without loss of generality, we may assume that $\|\mathbf{u}_1\|_{\mathbb{V}} \leq \|\mathbf{u}_2\|_{\mathbb{V}}$. For every $\mathbf{w}\in\mathbb{V}$, we have

$$\begin{aligned}&\langle\Pi_n(\|\mathbf{u}_1\|_{\mathbb{V}})\mathrm{B}(\mathbf{u}_1) - \Pi_n(\|\mathbf{u}_2\|_{\mathbb{V}})\mathrm{B}(\mathbf{u}_2),\mathbf{w}\rangle\\ &= \langle(\Pi_n(\|\mathbf{u}_1\|_{\mathbb{V}}) - \Pi_n(\|\mathbf{u}_2\|_{\mathbb{V}}))\mathrm{B}(\mathbf{u}_1),\mathbf{w}\rangle + \langle\Pi_n(\|\mathbf{u}_2\|_{\mathbb{V}})\mathrm{B}(\mathbf{u}_1-\mathbf{u}_2,\mathbf{u}_2),\mathbf{w}\rangle\\ &\quad + \langle\Pi_n(\|\mathbf{u}_2\|_{\mathbb{V}})\mathrm{B}(\mathbf{u}_1,\mathbf{u}_1-\mathbf{u}_2),\mathbf{w}\rangle.\end{aligned}$$

From the above equality, we get

$$\begin{aligned}&\|\Pi_n(\|\mathbf{u}_1\|_{\mathbb{V}})\mathrm{B}(\mathbf{u}_1) - \Pi_n(\|\mathbf{u}_2\|_{\mathbb{V}})\mathrm{B}(\mathbf{u}_2)\|_{\mathbb{V}'}\\ &\leq \|(\Pi_n(\|\mathbf{u}_1\|_{\mathbb{V}}) - \Pi_n(\|\mathbf{u}_2\|_{\mathbb{V}}))\mathrm{B}(\mathbf{u}_1)\|_{\mathbb{V}'} + \|\Pi_n(\|\mathbf{u}_2\|_{\mathbb{V}})\mathrm{B}(\mathbf{u}_1-\mathbf{u}_2,\mathbf{u}_2)\|_{\mathbb{V}'}\\ &\quad + \|\Pi_n(\|\mathbf{u}_2\|_{\mathbb{V}})\mathrm{B}(\mathbf{u}_1,\mathbf{u}_1-\mathbf{u}_2)\|_{\mathbb{V}'} =: I,\end{aligned} \tag{3.7}$$

where I denote the right hand side of the inequality (3.7). Let us establish (3.6) in the following 6 different cases:

Case 1: $\|\mathbf{u}_1\|_{\mathbb{V}}, \|\mathbf{u}_2\|_{\mathbb{V}} \leq n$. In this case, using (2.11), we obtain

$$\begin{aligned} I &= \|\mathrm{B}(\mathbf{u}_1-\mathbf{u}_2,\mathbf{u}_2)\|_{\mathbb{V}'} + \|\mathrm{B}(\mathbf{u}_1,\mathbf{u}_1-\mathbf{u}_2)\|_{\mathbb{V}'} \\ &\leq \frac{C}{\lambda_1^{1/4}}\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbb{V}}(\|\mathbf{u}_1\|_{\mathbb{V}}+\|\mathbf{u}_2\|_{\mathbb{V}}) \leq \frac{2Cn}{\lambda_1^{1/4}}\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbb{V}}. \end{aligned}$$

Case 2: $n < \|\mathbf{u}_1\|_{\mathbb{V}}, \|\mathbf{u}_2\|_{\mathbb{V}} \leq n+1$. For this case, we have

$$\begin{aligned} I &= \|(\|\mathbf{u}_2\|_{\mathbb{V}} - \|\mathbf{u}_1\|_{\mathbb{V}})\mathrm{B}(\mathbf{u}_1)\|_{\mathbb{V}'} + (n+1-\|\mathbf{u}_2\|_{\mathbb{V}})\|\mathrm{B}(\mathbf{u}_1-\mathbf{u}_2,\mathbf{u}_2)\|_{\mathbb{V}'} \\ &\quad + (n+1-\|\mathbf{u}_2\|_{\mathbb{V}})\|\mathrm{B}(\mathbf{u}_1,\mathbf{u}_1-\mathbf{u}_2)\|_{\mathbb{V}'} \\ &\leq \|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbb{V}}\frac{C}{\lambda_1^{1/4}}\|\mathbf{u}_1\|_{\mathbb{V}}^2 + (n+1-\|\mathbf{u}_2\|_{\mathbb{V}})\frac{C}{\lambda_1^{1/4}}\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbb{V}}(\|\mathbf{u}_1\|_{\mathbb{V}}+\|\mathbf{u}_2\|_{\mathbb{V}}) \\ &\leq \frac{C}{\lambda_1^{1/4}}\big(\|\mathbf{u}_1\|_{\mathbb{V}}^2 + (n+1-\|\mathbf{u}_2\|_{\mathbb{V}})(\|\mathbf{u}_1\|_{\mathbb{V}}+\|\mathbf{u}_2\|_{\mathbb{V}})\big)\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbb{V}} \\ &\leq \frac{5C(n+1)^2}{\lambda_1^{1/4}}\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbb{V}}. \end{aligned}$$

Case 3: $\|\mathbf{u}_1\|_{\mathbb{V}}, \|\mathbf{u}_2\|_{\mathbb{V}} > n+1$. In this case, $I = 0$ and (3.6) is trivially satisfied.

Case 4: $\|\mathbf{u}_1\|_{\mathbb{V}} \leq n, n < \|\mathbf{u}_2\|_{\mathbb{V}} \leq n+1$. For this case, we get

$$\begin{aligned} I &= (\|\mathbf{u}_2\|_{\mathbb{V}} - n)\|\mathrm{B}(\mathbf{u}_1)\|_{\mathbb{V}'} + \|(n+1-\|\mathbf{u}_2\|_{\mathbb{V}})\mathrm{B}(\mathbf{u}_1-\mathbf{u}_2,\mathbf{u}_2)\|_{\mathbb{V}'} \\ &\quad + \|(n+1-\|\mathbf{u}_2\|_{\mathbb{V}})\mathrm{B}(\mathbf{u}_1,\mathbf{u}_1-\mathbf{u}_2)\|_{\mathbb{V}'} \\ &\leq (\|\mathbf{u}_2\|_{\mathbb{V}} - \|\mathbf{u}_1\|_{\mathbb{V}})\frac{C}{\lambda_1^{1/4}}\|\mathbf{u}_1\|_{\mathbb{V}}^2 \\ &\quad + (n+1+\|\mathbf{u}_2\|_{\mathbb{V}})\frac{C}{\lambda_1^{1/4}}\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbb{V}}(\|\mathbf{u}_1\|_{\mathbb{V}}+\|\mathbf{u}_2\|_{\mathbb{V}}) \\ &\leq \frac{C}{\lambda_1^{1/4}}\big(n^2+2(n+1)(2n+1)\big)\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbb{V}}. \end{aligned}$$

Case 5: $\|\mathbf{u}_1\|_{\mathbb{V}} \leq n, \|\mathbf{u}_2\|_{\mathbb{V}} > n+1$, so that

$$1 < \|\mathbf{u}_2\|_{\mathbb{V}} - n \leq \|\mathbf{u}_2\|_{\mathbb{V}} - \|\mathbf{u}_1\|_{\mathbb{V}} \leq \|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbb{V}}.$$

For this case, we find

$$I = \|\mathrm{B}(\mathbf{u}_1)\|_{\mathbb{V}'} \leq \frac{C}{\lambda_1^{1/4}}\|\mathbf{u}_1\|_{\mathbb{V}}^2 \leq \frac{Cn^2}{\lambda_1^{1/4}}\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbb{V}}.$$

Case 6: $n < \|\mathbf{u}_1\|_{\mathbb{V}} \leq n+1, \|\mathbf{u}_2\|_{\mathbb{V}} > n+1$, so that

$$n+1-\|\mathbf{u}_1\|_{\mathbb{V}} \leq \|\mathbf{u}_2\|_{\mathbb{V}} - \|\mathbf{u}_1\|_{\mathbb{V}} \leq \|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbb{V}}.$$

In this case, we infer that

$$I = \|(n+1-\|\mathbf{u}_1\|_{\mathbb{V}})\mathrm{B}(\mathbf{u}_1)\|_{\mathbb{V}'} \leq \|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbb{V}}\frac{C}{\lambda_1^{1/4}}\|\mathbf{u}_1\|_{\mathbb{V}}^2$$

$$\leq \frac{C(n+1)^2}{\lambda_1^{1/4}}\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbb{V}}.$$

Combining all these cases, we arrive at (3.6). □

Proposition 3.2. *Let the $\mathscr{F}_0$-measurable initial data $\mathbf{u}_0 \in \mathrm{L}^2(\Omega;\mathbb{V})$ be given. Then, there exists a unique $(\mathscr{F}_t)_{t\geq 0}$-adapted strong solution $\mathbf{u}(\cdot)$ to the system* (3.2) *in $\mathrm{L}^2(\Omega;\mathrm{L}^\infty([0,T];\mathbb{V}))$ with continuous trajectories in $\mathrm{C}([0,T];\mathbb{V})$, $\mathbb{P}$-a.s.*

Proof. Let $\mathcal{F}(\cdot):\mathbb{X}\to\mathbb{X}$ be the map defined in (3.4) and we show that $\mathcal{F}(\cdot)$ is a contraction on $\mathbb{X}$. We prove this in the following steps:

Step 1. *Claim: $\mathcal{F}(\mathbf{u}_n)\in\mathbb{X}$, for every $\mathbf{u}_n\in\mathbb{X}$.* We first show $\mathcal{F}(\cdot):\mathbb{X}\to\mathbb{X}$. For any $\mathbf{u}_n\in\mathbb{X}$, we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathcal{F}(\mathbf{u}_n(t))\|_{\mathbb{V}}^2\right] \qquad (3.8)\\
&\leq \left(\frac{\lambda_1}{1+\lambda_1\alpha}\right)\mathbb{E}\left[\sup_{t\in[0,T]}\|(\mathrm{I}+\alpha\mathrm{A})\mathcal{F}(\mathbf{u}_n(t))\|_{\mathbb{V}'}^2\right]\\
&= \left(\frac{\lambda_1}{1+\lambda_1\alpha}\right)\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\mathbf{v}_0+\int_0^t \Pi_n(\|\mathbf{u}_n(s)\|_{\mathbb{V}})\mathrm{B}(\mathbf{u}_n(s))\mathrm{d}s+\int_0^t\sqrt{\mathrm{Q}}\mathrm{dW}(s)\right\|_{\mathbb{V}'}^2\right]\\
&\leq 3\left(\frac{\lambda_1}{1+\lambda_1\alpha}\right)\mathbb{E}\left[\|\mathbf{v}_0\|_{\mathbb{V}'}^2+\sup_{t\in[0,T]}\left\|\int_0^t \Pi_n(\|\mathbf{u}_n(s)\|_{\mathbb{V}})\mathrm{B}(\mathbf{u}_n(s))\mathrm{d}s\right\|_{\mathbb{V}'}^2\right.\\
&\quad\left.+\sup_{t\in[0,T]}\left\|\int_0^t\sqrt{\mathrm{Q}}\mathrm{dW}(s)\right\|_{\mathbb{V}'}^2\right]\\
&\leq 3\left(\frac{\lambda_1}{1+\lambda_1\alpha}\right)\left\{\left(\frac{1}{\lambda_1}+\alpha\right)\mathbb{E}[\|\mathbf{u}_0\|_{\mathbb{V}}^2]\right.\\
&\quad\left.+\mathbb{E}\left[\sup_{t\in[0,T]}\left(\int_0^t \Pi_n(\|\mathbf{u}_n(s)\|_{\mathbb{V}})\|\mathrm{B}(\mathbf{u}_n(s))\|_{\mathbb{V}'}\mathrm{d}s\right)^2\right]+\frac{1}{\lambda_1}\mathbb{E}\left[\sup_{t\in[0,T]}\|M_t\|_{\mathbb{H}}^2\right]\right\}\\
&\leq 3\left(\frac{\lambda_1}{1+\lambda_1\alpha}\right)\left\{\left(\frac{1}{\lambda_1}+\alpha\right)\mathbb{E}[\|\mathbf{u}_0\|_{\mathbb{V}}^2]\right.\\
&\quad\left.+\frac{C}{\lambda_1^{1/4}}\mathbb{E}\left[\sup_{t\in[0,T]}\left(\int_0^t \Pi_n(\|\mathbf{u}_n(s)\|_{\mathbb{V}})\|\mathbf{u}_n(s)\|_{\mathbb{V}}^2\mathrm{d}s\right)^2\right]+\frac{C}{\lambda_1}\mathbb{E}[M,M]_T\right\}\\
&\leq 3\left(\frac{\lambda_1}{1+\lambda_1\alpha}\right)\left\{\left(\frac{1}{\lambda_1}+\alpha\right)\mathbb{E}[\|\mathbf{u}_0\|_{\mathbb{V}}^2]+\frac{CT^2(n+1)^2}{\lambda_1^{1/4}}+\frac{C}{\lambda_1}\mathrm{Tr}(\mathrm{Q})T\right\}<+\infty,
\end{aligned}$$

using (2.11) and Burkholder-Davis-Gundy inequality (see Theorem 1.1, [19]). In (3.8), $M_t=\int_0^t\sqrt{\mathrm{Q}}\mathrm{dW}(s)$ and $[M,M]_T$ denotes the quadratic variation process. Hence, we obtain $\mathcal{F}(\mathbf{u}_n)\in\mathbb{X}$.

Step 2. *Claim: $\mathcal{F}(\cdot):\mathbb{X}\to\mathbb{X}$ is a contraction.* Next our aim is to establish that the map $\mathcal{F}$ is a contraction on $\mathbb{X}$. From Lemma 3.1, it is clear that the operator $\Pi_n(\|\mathbf{u}_n\|_{\mathbb{V}})\mathrm{B}(\mathbf{u}_n)$ is a globally Lipschitz operator. Now, for $C_{\lambda_1}^{\alpha}:=\left(\frac{\lambda_1}{1+\lambda_1\alpha}\right)$, we consider

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathcal{F}(\mathbf{u}_n^1(t))-\mathcal{F}(\mathbf{u}_n^2(t))\|_{\mathbb{V}}^2\right]\\
&\leq\left(\frac{\lambda_1}{1+\lambda_1\alpha}\right)\mathbb{E}\left[\sup_{t\in[0,T]}\|(\mathrm{I}+\alpha\mathrm{A})\big(\mathcal{F}(\mathbf{u}_n^1(t))-\mathcal{F}(\mathbf{u}_n^2(t))\big)\|_{\mathbb{V}'}^2\right]\\
&=C_{\lambda_1}^{\alpha}\,\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t\big[\Pi_n(\|\mathbf{u}_n^1(s)\|_{\mathbb{V}})\mathrm{B}(\mathbf{u}_n^1(s))-\Pi_n(\|\mathbf{u}_n^2(s)\|_{\mathbb{V}})\mathrm{B}(\mathbf{u}_n^2(s))\big]\mathrm{d}s\right\|_{\mathbb{V}'}^2\right]\\
&\leq C_{\lambda_1}^{\alpha}\,\mathbb{E}\left[\sup_{t\in[0,T]}\left(\int_0^t\|\Pi_n(\|\mathbf{u}_n^1(s)\|_{\mathbb{V}})\mathrm{B}(\mathbf{u}_n^1(s))-\Pi_n(\|\mathbf{u}_n^2(s)\|_{\mathbb{V}})\mathrm{B}(\mathbf{u}_n^2(s))\|_{\mathbb{V}'}\mathrm{d}s\right)^2\right]\\
&\leq\left(\frac{\lambda_1}{1+\lambda_1\alpha}\right)\frac{C_n}{\lambda_1^{1/4}}\mathbb{E}\left[\sup_{t\in[0,T]}\left(\int_0^t\|\mathbf{u}_n^1(s)-\mathbf{u}_n^2(s)\|_{\mathbb{V}}\mathrm{d}s\right)^2\right]\\
&\leq\left(\frac{\lambda_1^{3/4}C_nT^2}{1+\lambda_1\alpha}\right)\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathbf{u}_n^1(t)-\mathbf{u}_n^2(t)\|_{\mathbb{V}}^2\right],
\end{aligned}
\tag{3.9}
$$

where we used Lemma 3.1 and the fact that the norms $\|\mathbf{v}\|_{\mathbb{V}'}$ and $\|\mathbf{u}\|_{\mathbb{V}}$ are equivalent (see Remark 2.1). Thus $\mathcal{F}(\cdot)$ is a contraction mapping on $\mathbb{X}$ if $\frac{C_nT^2\lambda_1^{3/4}}{(1+\alpha\lambda_1)}<1$.

Step 3. *Fixed point and local strong solution.* Using the *Banach contraction mapping principle*, there exists a time $0<T^*<T$ such that the map $\mathcal{F}(\cdot)$ has a unique fixed point in $\mathbb{X}$, for $0<T^*<\sqrt{\frac{(1+\alpha\lambda_1)}{C_n\lambda_1^{3/4}}}$. Since $\mathbf{v}=(\mathrm{I}+\alpha\mathrm{A})\mathbf{u}$, there exists a local strong solution $\mathbf{u}(\cdot)$ for the system (3.2) in $\mathrm{L}^2(\Omega;\mathrm{L}^\infty([0,T^*];\mathbb{V}))$ with $(\mathscr{F}_t)_{t\geq0}$-adapted, continuous trajectories in $\mathrm{C}([0,T^*];\mathbb{V})$, $\mathbb{P}$-a.s. Furthermore, a Picard's iteration scheme gives the required solvability result, that is, one can consider

$$\mathbf{u}_n^{m+1}(t)=\mathcal{F}(\mathbf{u}_n^m(t))\ \text{ with }\ \mathbf{u}_0(0)=\mathbf{u}_0,$$

for $m=0,1,2,\dots$, and finish local existence and pathwise uniqueness (using global Lipschitz property of the operator $\Pi_n(\|\mathbf{u}_n\|_{\mathbb{V}})\mathrm{B}(\mathbf{u}_n)$, see Lemma 3.1) of strong solution to the system (3.2), using standard arguments. The right continuity of $\mathbf{u}_n(\cdot)$ at 0 can be used to obtain that the initial data $\mathbf{u}_n(0)=\mathbf{u}_0$, $\mathbb{P}$-a.s.

Step 4. *Global strong solution to the system* (3.2). Let T^* be the maximal time of existence for the cut-off problem (3.2). Next, we show that $T^*=T$, where T is arbitrary. Let us assume that $T^*<T$ such that

$$\limsup_{t\uparrow T^*}\|\mathbf{u}_n(t)\|_{\mathbb{V}}=+\infty,\ \mathbb{P}\text{-a.s.}\tag{3.10}$$

That is, we also have

$$\mathbb{E}\left[\sup_{t\in[0,T^*]}\|\mathbf{u}_n(t)\|_{\mathbb{V}}^2\right]=+\infty.\tag{3.11}$$

We show that $\mathbb{E}\left[\sup_{t\in[0,T^*]}\|\mathbf{u}_n(t)\|_{\mathbb{V}}^2\right]<+\infty$ and obtain a contradiction to (3.11).

Let us first examine the energy estimate satisfied by the unique pathwise strong solution to the system (3.3). Let τ_M be a sequence of stopping times defined by

$$\tau_M := \inf_{t\geq 0}\Big\{t : \|\mathbf{u}_n(t)\|_{\mathbb{V}} \geq M\Big\}, \tag{3.12}$$

for $M \in \mathbb{N}$. Next, we use the Itô product formula (see [21]) to the process $(\mathbf{v}_n(t), \mathbf{u}_n(t))$ to obtain

$$\begin{aligned}(\mathbf{v}_n(t\wedge\tau_M), \mathbf{u}_n(t\wedge\tau_M)) &= (\mathbf{v}_0, \mathbf{u}_0) + \int_0^{t\wedge\tau_M} (\mathbf{v}_n(s), \mathrm{d}\mathbf{u}_n(s)) \\ &\quad + \int_0^{t\wedge\tau_M} (\mathbf{u}_n(s), \mathrm{d}\mathbf{v}_n(s)) + [\mathbf{u}_n(t), \mathbf{v}_n(t)]_{t\wedge\tau_M}. \end{aligned} \tag{3.13}$$

From (3.13), we obtain

$$\begin{aligned}&\|\mathbf{u}_n(t\wedge\tau_M)\|_{\mathbb{H}}^2 + \alpha\|\mathbf{u}_n(t\wedge\tau_M)\|_{\mathbb{V}}^2 \\ &= \|\mathbf{u}_0\|_{\mathbb{H}}^2 + \alpha\|\mathbf{u}_0\|_{\mathbb{V}}^2 - 2\int_0^{t\wedge\tau_M} \langle \Pi_n(\|\mathbf{u}_n\|_{\mathbb{V}})\mathrm{B}(\mathbf{u}_n(s)), \mathbf{u}_n(s)\rangle \mathrm{d}s \\ &\quad + 2\int_0^{t\wedge\tau_M} (\sqrt{\mathrm{Q}}\mathrm{d}\mathrm{W}(s), \mathbf{u}_n(s))\mathrm{d}s + \int_0^{t\wedge\tau_M} \mathrm{Tr}((\mathrm{I}+\alpha\mathrm{A})^{-1}\mathrm{Q})\mathrm{d}s. \end{aligned} \tag{3.14}$$

Let us take expectation in (3.14) to find

$$\begin{aligned}&\mathbb{E}\big[\|\mathbf{u}_n(t\wedge\tau_M)\|_{\mathbb{H}}^2 + \alpha\|\mathbf{u}_n(t\wedge\tau_M)\|_{\mathbb{V}}^2\big] \\ &= \mathbb{E}\big[\|\mathbf{u}_0\|_{\mathbb{H}}^2\big] + \alpha\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathbb{V}}^2\big] + \mathrm{Tr}((\mathrm{I}+\alpha\mathrm{A})^{-1}\mathrm{Q})\mathbb{E}[t\wedge\tau_M] \\ &\leq \left(\frac{1}{\lambda_1} + \alpha\right)\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathbb{V}}^2\big] + \mathrm{Tr}((\mathrm{I}+\alpha\mathrm{A})^{-1}\mathrm{Q})t, \end{aligned} \tag{3.15}$$

where we used the fact that $\int_0^{t\wedge\tau_M} (\sqrt{\mathrm{Q}}\mathrm{d}\mathrm{W}(s), \mathbf{u}_n(s))\mathrm{d}s$ is a martingale with zero average and $\langle \mathrm{B}(\mathbf{u}), \mathbf{u}\rangle = 0$. On the other hand, we have

$$\begin{aligned}\mathbb{E}\big[\|\mathbf{u}_n(t\wedge\tau_M)\|_{\mathbb{V}}^2\big] &= \mathbb{E}\big[\|\mathbf{u}_n(t\wedge\tau_M)\|_{\mathbb{V}}^2\chi_{\{\tau_M<t\}}\big] + \mathbb{E}\big[\|\mathbf{u}_n(t\wedge\tau_M)\|_{\mathbb{V}}^2\chi_{\{\tau_M\geq t\}}\big] \\ &= \mathbb{E}\big[\|\mathbf{u}_n(\tau_M)\|_{\mathbb{V}}^2\chi_{\{\tau_M<t\}}\big] + \mathbb{E}\big[\|\mathbf{u}_n(t)\|_{\mathbb{V}}^2\chi_{\{\tau_M\geq t\}}\big], \end{aligned} \tag{3.16}$$

where χ is the indicator function. From the continuity of the process $\mathbf{u}^n(\cdot)$ (see (3.12)), we know that $\|\mathbf{u}^n(\tau_M)\|_{\mathbb{V}} \geq M$, and remember that

$$\mathbb{E}\Big[\chi_{\{\tau_M^n<t\}}\Big] = \mathbb{P}\Big\{\omega\in\Omega : \tau_M^n(\omega) < t\Big\}.$$

Equation (3.16) gives

$$\begin{aligned}\mathbb{E}\big[\|\mathbf{u}_n(t\wedge\tau_N)\|_{\mathbb{V}}^2\big] &= \mathbb{E}\big[\|\mathbf{u}^n(\tau_M)\|_{\mathbb{V}}^2\chi_{\{\tau_M<t\}}\big] + \mathbb{E}\big[\|\mathbf{u}^n(t)\|_{\mathbb{V}}^2\chi_{\{\tau_M\geq t\}}\big] \\ &\geq \mathbb{E}\big[\|\mathbf{u}^n(\tau_M)\|_{\mathbb{V}}^2\chi_{\{\tau_M<t\}}\big] \\ &\geq M^2\mathbb{P}\Big\{\omega\in\Omega : \tau_M(\omega) < t\Big\}. \end{aligned} \tag{3.17}$$

Thus by using (3.15), we finally obtain

$$\begin{aligned}\mathbb{P}\Big\{\omega\in\Omega : \tau_M(\omega) < t\Big\} &\leq \frac{1}{M^2}\mathbb{E}\big[\|\mathbf{u}_n(t\wedge\tau_N)\|_{\mathbb{V}}^2\big] \\ &\leq \frac{1}{M^2}\left[\left(\frac{1}{\lambda_1} + \alpha\right)\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathbb{V}}^2\big] + \mathrm{Tr}((\mathrm{I}+\alpha\mathrm{A})^{-1}\mathrm{Q})t\right]. \end{aligned} \tag{3.18}$$

Hence, we have

$$\lim_{M\to\infty}\mathbb{P}\Big\{\omega\in\Omega:\tau_M(\omega)<t\Big\}=0,\quad \text{for all}\quad t\in[0,T], \tag{3.19}$$

and hence $t\wedge\tau_N\to t$ as $M\to\infty$. Then on taking limit $M\to\infty$ in (3.15) and using the dominated convergence theorem, we get

$$\mathbb{E}\big[\|\mathbf{u}_n(t)\|_{\mathbb{H}}^2+\alpha\|\mathbf{u}_n(t)\|_{\mathbb{V}}^2\big]\leq\left(\frac{1}{\lambda_1}+\alpha\right)\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathbb{V}}^2\big]+\mathrm{Tr}((\mathrm{I}+\alpha\mathrm{A})^{-1}\mathrm{Q})t, \tag{3.20}$$

for all $0\leq t\leq T^*$. Thus we also have

$$\sup_{t\in[0,T^*]}\mathbb{E}\big[\|\mathbf{u}_n(t)\|_{\mathbb{H}}^2+\alpha\|\mathbf{u}_n(t)\|_{\mathbb{V}}^2\big]\leq\left(\frac{1}{\lambda_1}+\alpha\right)\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathbb{V}}^2\big]+\mathrm{Tr}((\mathrm{I}+\alpha\mathrm{A})^{-1}\mathrm{Q})T^*. \tag{3.21}$$

Let us take the supremum over $t\in[0,T^*]$ and then take expectation in (3.14) to obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T^*\wedge\tau_M]}\big(\|\mathbf{u}_n(t)\|_{\mathbb{H}}^2+\alpha\|\mathbf{u}_n(t)\|_{\mathbb{V}}^2\big)\right]\\
&\quad\leq\mathbb{E}\big[\|\mathbf{u}_0\|^2+\alpha\|\mathbf{u}_0\|_{\mathbb{V}}^2\big]+\mathrm{Tr}((\mathrm{I}+\alpha\mathrm{A})^{-1}\mathrm{Q})T^*\\
&\qquad+2\mathbb{E}\left[\sup_{t\in[0,T^*\wedge\tau_M]}\left|\int_0^{t\wedge\tau_M}(\sqrt{\mathrm{Q}}\mathrm{d}\mathrm{W}(s),\mathbf{u}_n(s))\mathrm{d}s\right|\right]=:I_3,
\end{aligned} \tag{3.22}$$

where I_3 is the final term appearing in (3.22). Let us use the Davis, Hölder and Young's inequalities to obtain

$$\begin{aligned}
I_3&\leq 2\sqrt{3}\mathbb{E}\left[\int_0^{T^*\wedge\tau_M}\sum_{k=1}^{\infty}\mu_k\|e_k(x)\|_{\mathbb{H}}^2\|\mathbf{u}_n(t)\|_{\mathbb{H}}^2\mathrm{d}t\right]^{1/2}\\
&\leq 2\sqrt{3}\mathbb{E}\left[\sup_{t\in[0,T^*\wedge\tau_M]}\|\mathbf{u}_n(t)\|_{\mathbb{H}}\left(\int_0^{T^*\wedge\tau_M}\sum_{k=1}^{\infty}\mu_k\mathrm{d}t\right)^{1/2}\right]\\
&\leq\frac{1}{2}\mathbb{E}\left[\sup_{t\in[0,T^*\wedge\tau_M]}\|\mathbf{u}_n(t)\|_{\mathbb{H}}^2\right]+6\,\mathrm{Tr}(\mathrm{Q})T^*.
\end{aligned} \tag{3.23}$$

Let us substitute (3.23) in (3.22) to get

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T^*\wedge\tau_M]}\left(\frac{1}{2}\|\mathbf{u}_n(t)\|_{\mathbb{H}}^2+\alpha\|\mathbf{u}_n(t)\|_{\mathbb{V}}^2\right)\right]\\
&\quad\leq\mathbb{E}\big[\|\mathbf{u}_0\|^2+\alpha\|\mathbf{u}_0\|_{\mathbb{V}}^2\big]+\big[\mathrm{Tr}((\mathrm{I}+\alpha\mathrm{A})^{-1}\mathrm{Q})+6\,\mathrm{Tr}(\mathrm{Q})\big]T^*.
\end{aligned} \tag{3.24}$$

A calculation similar to (3.19) yields that as $M\to\infty$, $T^*\wedge\tau_M\to T^*$. Passing $M\to\infty$ in (3.24) and using dominated convergence theorem, we infer that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T^*]}\left(\frac{1}{2}\|\mathbf{u}_n(t)\|_{\mathbb{H}}^2+\alpha\|\mathbf{u}_n(t)\|_{\mathbb{V}}^2\right)\right]\\
&\leq\left(\frac{1}{\lambda_1}+\alpha\right)\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathbb{V}}^2\big]+\big[\mathrm{Tr}((\mathrm{I}+\alpha\mathrm{A})^{-1}\mathrm{Q})+6\,\mathrm{Tr}(\mathrm{Q})\big]T^*<+\infty,
\end{aligned} \tag{3.25}$$

which is a contradiction to (3.11) and hence $T^* = T$. Thus there exists an $(\mathscr{F}_t)_{t\geq 0}$-adapted solution $\mathbf{u}_n(\cdot)$ to the problem (3.3) with continuous trajectories in $\mathrm{C}([0,T];\mathbb{V})$, $\mathbb{P}$-a.s.. □

Let us now show that the system (2.12) has a unique pathwise strong solution by passing $n\to\infty$ in (3.3).

Theorem 3.3 (Global existence and uniqueness)**.** *Let the $\mathscr{F}_0$-measurable initial data $\mathbf{u}_0 \in \mathrm{L}^2(\Omega;\mathbb{V})$ be given. Then, there exists a unique strong solution to the system* (2.12) *with $(\mathscr{F}_t)_{t\geq 0}$-adapted, continuous trajectories in $\mathrm{C}([0,T];\mathbb{V})$, $\mathbb{P}$-a.s.*

Proof. We establish the existence and uniqueness of strong solution to the system (2.12) in the following steps.

Step 1. *Local strong solution.* For each $n\in\mathbb{N}$, let us define an $(\mathscr{F}_t)_{t\geq 0}$-adapted stopping time

$$\varrho_n := \inf_{t\geq 0}\Big\{t : \|\mathbf{u}_n(t)\|_{\mathbb{V}} \geq n\Big\}. \tag{3.26}$$

For each $m\geq n$, we have

$$\mathbf{u}_n(t) = \mathbf{u}_m(t),\ \mathbb{P}\text{-a.s.},\ \text{ for all }\ t\in[0,T\wedge\varrho_m\wedge\varrho_n),$$

by using pathwise uniqueness of the system (3.3). For each $m\geq n$, one can easily see that $\varrho_m\geq\varrho_n$, $\mathbb{P}$-a.s. Since T is arbitrary, we can also obtain

$$\mathbf{u}_n(t) = \mathbf{u}_m(t),\ \mathbb{P}\text{-a.s.},\ \text{ for all } t\in[0,\varrho_n).$$

Let us now define

$$\mathbf{u}(t) := \mathbf{u}_n(t),\ \text{ for all }\ t\in[0,\varrho_n],\ \text{ and }\ \varrho := \lim_{n\to\infty}\varrho_n,\ \mathbb{P}\text{-a.s.}, \tag{3.27}$$

where $\varrho\leq T$, $\mathbb{P}$-a.s., and T is arbitrary. Thus Proposition 3.2 ensures the existence of a unique pathwise strong solution

$$\mathbf{u}(t) = \lim_{n\to\infty}\mathbf{u}_n(t),\ \mathbb{P}\text{-a.s.},$$

to the system (2.12) in the interval $[0,\varrho]$. Hence $(\mathbf{u},\varrho)$ is a local strong solution to the system (2.12).

Step 2. *A probabilistic estimate of the stopping time.* For a given $0<\delta<1$, we now show that

$$\mathbb{P}\Big\{\omega\in\Omega : \varrho(\omega)>\delta\Big\} \geq 1 - C\delta^2\Big\{1+\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathbb{V}}^2\big]\Big\}, \tag{3.28}$$

for some positive constant C independent of $\mathbf{u}_0$ and δ. Since $\langle\mathrm{B}(\mathbf{u},\mathbf{u}),\mathbf{u}\rangle = 0$, a calculation similar to (3.25) yields

$$\begin{aligned}
&\mathbb{E}\Bigg[\sup_{t\in[0,\varrho]}\bigg(\frac{1}{2}\|\mathbf{u}(t)\|_{\mathbb{H}}^2 + \alpha\|\mathbf{u}(t)\|_{\mathbb{V}}^2\bigg)\Bigg] \qquad (3.29)\\
&\leq \bigg(\frac{1}{\lambda_1}+\alpha\bigg)\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathbb{V}}^2\big] + \big[\mathrm{Tr}((\mathrm{I}+\alpha\mathrm{A})^{-1}\mathrm{Q}) + 6\,\mathrm{Tr}(\mathrm{Q})\big]\delta\\
&\leq \max\Bigg\{\bigg(\frac{1}{\lambda_1}+\alpha\bigg), \big[\mathrm{Tr}((\mathrm{I}+\alpha\mathrm{A})^{-1}\mathrm{Q}) + 6\,\mathrm{Tr}(\mathrm{Q})\big]\Bigg\}\Big\{\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathbb{V}}^2\big]+\delta\Big\} < +\infty.
\end{aligned}$$

For the given $0<\delta<1$, there exists a positive integer n such that

$$\frac{1}{n+1}\leq\delta<\frac{1}{n}.$$

Further, using Markov's inequality, (3.29) and

$$\frac{1}{n^2}=\frac{1}{n^2}\frac{(n+1)^2}{(n+1)^2}\leq\frac{1}{n^2}(n+1)^2\delta^2=\left(1+\frac{1}{n}\right)^2\delta^2\leq 4\delta^2,$$

we have

$$\begin{aligned}\mathbb{P}\Big\{\omega\in\Omega:\varrho(\omega)>\delta\Big\}&\geq\mathbb{P}\Big\{\omega\in\Omega:\varrho_n(\omega)>\delta\Big\}\\&\geq\mathbb{P}\left\{\omega\in\Omega:\sup_{t\in[0,\delta]}\|\mathbf{u}(t,\omega)\|_{\mathbb{V}}<n\right\}\\&=\mathbb{P}\left\{\omega\in\Omega:\sup_{t\in[0,\delta]}\|\mathbf{u}(t,\omega)\|_{\mathbb{V}}^2<n^2\right\}\\&\geq 1-\frac{1}{n^2}\mathbb{E}\left(\sup_{t\in[0,\delta]}\|\mathbf{u}(t)\|_{\mathbb{V}}^2\right)\\&\geq 1-\frac{1}{n^2}C(\alpha,\lambda_1,\mathrm{Q})\big\{\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathbb{V}}^2\big]+\delta\big\}\\&\geq 1-C(\alpha,\lambda_1,\mathrm{Q})\delta^2\big(1+\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathbb{V}}^2\big]\big).\end{aligned}\tag{3.30}$$

Similar methods for proving positivity of stopping times for different models can be found in [23, 24, 25, 26], etc.

Step 3. *Global strong solution.* In order to prove that $(\mathbf{u},\varrho)$ is a global pathwise strong solution, we need to show that $\varrho=T$, $\mathbb{P}$-a.s., where T is arbitrary. Once again, a calculation similar to (3.25) yields

$$\begin{aligned}&\sup_{n\in\mathbb{N}}\mathbb{E}\left[\sup_{t\in[0,T]}\left(\frac{1}{2}\|\mathbf{u}_n(t)\|_{\mathbb{H}}^2+\alpha\|\mathbf{u}_n(t)\|_{\mathbb{V}}^2\right)\right]\\&\leq\max\left\{\left(\frac{1}{\lambda_1}+\alpha\right),\big[\mathrm{Tr}((\mathrm{I}+\alpha\mathrm{A})^{-1}\mathrm{Q})+\mathrm{Tr}(\mathrm{Q})\big]\right\}\big\{\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathbb{V}}^2\big]+T\big\}<+\infty.\end{aligned}\tag{3.31}$$

Thus the solutions $\mathbf{u}_n(\cdot)$ is uniformly bounded in $\mathrm{L}^2(\Omega;\mathrm{L}^\infty([0,T];\mathbb{V}))$. For any $T>0$, we assume that $\varrho(\omega)<T$, $\mathbb{P}$-a.s. Thus, for arbitrary $T>0$, using Markov's inequality and (3.31), we have

$$\begin{aligned}\mathbb{P}\Big\{\omega\in\Omega:\varrho_n(\omega)<T\Big\}&=\mathbb{P}\left\{\omega\in\Omega:\sup_{t\in[0,T]}\|\mathbf{u}_n(t,\omega)\|_{\mathbb{V}}\geq n\right\}\\&\leq\frac{1}{n^2}\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathbf{u}_n(t,\omega)\|_{\mathbb{V}}^2\right]\\&\leq\frac{1}{n^2}C(\alpha,\lambda_1,\mathrm{Q})\big\{\mathbb{E}\big[\|\mathbf{u}_0\|_{\mathbb{V}}^2\big]+T\big\}.\end{aligned}\tag{3.32}$$

Let us take $n \to \infty$ in (3.32) to infer that $\mathbb{P}\big\{\omega \in \Omega : \varrho_n(\omega) < T\big\} \to 0$ as $n \to \infty$ and hence $\varrho(\omega) = T$, $\mathbb{P}$-a.s., which also gives a unique global pathwise strong solution to the system (2.12). □

Remark 3.4. It should be noted that an another way to approach this problem is to consider

$$\begin{cases} \mathrm{d}\mathbf{v}(t) = -[\nu \mathrm{A}\mathbf{u}(t) + \mathrm{B}(\mathbf{u}(t))]\mathrm{d}t + \sqrt{\mathrm{Q}}\mathrm{dW}(t), \\ \mathbf{v}(0) = \mathbf{v}_0 = \mathbf{u}_0 + \alpha \mathrm{A}\mathbf{u}_0, \end{cases} \tag{3.33}$$

for $\nu \geq 0$, or equivalently

$$\begin{cases} \mathrm{d}\mathbf{u}(t) = -(\mathrm{I} + \alpha\mathrm{A})^{-1}[\nu \mathrm{A}\mathbf{u}(t) + \mathrm{B}(\mathbf{u}(t))]\mathrm{d}t + (\mathrm{I} + \alpha\mathrm{A})^{-1}\sqrt{\mathrm{Q}}\mathrm{dW}(t), \\ \mathbf{u}(0) = \mathbf{u}_0. \end{cases} \tag{3.34}$$

A calculation similar to (3.25) yields the a-priori estimate:

$$\begin{aligned} &\mathbb{E}\left[\sup_{t\in[0,T]}\left(\frac{1}{2}\|\mathbf{u}(t)\|_{\mathbb{H}}^2 + \alpha\|\mathbf{u}(t)\|_{\mathbb{V}}^2\right) + 2\nu\int_0^T \|\mathbf{u}(s)\|_{\mathbb{V}}^2 \mathrm{d}s\right] \\ &\leq \mathbb{E}\big[\|\mathbf{u}_0\|_{\mathbb{H}}^2 + \alpha\|\mathbf{u}_0\|_{\mathbb{V}}^2\big] + \big[\mathrm{Tr}((\mathrm{I} + \alpha\mathrm{A})^{-1}\mathrm{Q}) + \mathrm{Tr}(\mathrm{Q})\big]T. \end{aligned} \tag{3.35}$$

Note that the right hand side of the inequality (3.35) is independent of ν. Using a standard Galerkin approximation technique and then passing $\nu \to 0$ gives the global solvability results as in Theorem 3.3.

Remark 3.5. For simplicity, we have taken our domain $\mathcal{O}$ as periodic. One can consider Poincaré domains (so that the Poincaré inequality $\|\mathbf{u}\|_{\mathbb{H}} \leq \frac{1}{\lambda}\|\nabla\mathbf{u}\|_{\mathbb{H}}$ holds true, see [30]) also and the global solvability results obtained for the system (2.12) remains the same.

4. Inviscid Simplified Bardina Model with Multiplicative Gaussian Noise

The stochastic inviscid simplified Bardina model perturbed by multiplicative Gaussian noise in $(0, T)$ (after taking the Helmholtz-Hodge orthogonal projection $\mathrm{P}_{\mathbb{H}}$) can be written in the Itô stochastic differential equations as

$$\begin{cases} \mathrm{d}\mathbf{v}(t) = -\mathrm{B}(\mathbf{u}(t))\mathrm{d}t + \Phi(\mathbf{u}(t))\mathrm{dW}(t), \\ \mathbf{v}(t) = (\mathrm{I} + \alpha\mathrm{A})\mathbf{u}(t) = \mathbf{u}(t) + \alpha\mathrm{A}\mathbf{u}(t), \\ \mathbf{u}(0) = \mathbf{u}_0, \mathbf{v}(0) = \mathbf{v}_0 = \mathbf{u}_0 + \alpha\mathrm{A}\mathbf{u}_0. \end{cases} \tag{4.1}$$

The above system is equivalent to

$$\begin{cases} \mathrm{d}\mathbf{u}(t) = -(\mathrm{I} + \alpha\mathrm{A})^{-1}\mathrm{B}(\mathbf{u}(t))\mathrm{d}t + (\mathrm{I} + \alpha\mathrm{A})^{-1}\Phi(\mathbf{u}(t))\mathrm{dW}(t), \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{4.2}$$

where $\mathbf{u}_0 \in \mathbb{V}$. We need some additional assumptions on the noise co-efficient to prove the existence and uniqueness of global pathwise strong solution to the system (4.1).

Let $\mathscr{L}_2(\mathbb{H},\mathbb{H})$ be the space of all Hilbert-Schmidt operators from $\mathbb{H}$ to $\mathbb{H}$ (see [9]). For an orthonormal basis $\{e_j\}_{j=1}^{\infty}$ in $\mathbb{H}$, we know that

$$\begin{aligned}\mathrm{Tr}((\Phi(\mathbf{u}))^*\Phi(\mathbf{u})) &= \sum_{j=1}^{\infty}((\Phi(\mathbf{u}))^*\Phi(\mathbf{u})e_j, e_j)_{\mathbb{H}} = \sum_{j=1}^{\infty}(\Phi(\mathbf{u})e_j, \Phi(\mathbf{u})e_j)_{\mathbb{H}} \\ &= \sum_{j=1}^{\infty}\|\Phi(\mathbf{u})e_j\|_{\mathbb{H}}^2 = \|\Phi(\mathbf{u})\|_{\mathscr{L}_2(\mathbb{H},\mathbb{H})}^2. \end{aligned} \tag{4.3}$$

Also, since $\|(\mathrm{I}+\alpha\mathrm{A})^{-1/2}\mathbf{u}\|_{\mathbb{H}} \leq \frac{1}{(1+\alpha\lambda_1)}\|\mathbf{u}\|_{\mathbb{H}}$, we have

$$\begin{aligned}&\mathrm{Tr}\Big(((\mathrm{I}+\alpha\mathrm{A})^{-1/2}\Phi(\mathbf{u}))^*(\mathrm{I}+\alpha\mathrm{A})^{-1/2}\Phi(\mathbf{u})\Big) \\ &= \sum_{j=1}^{\infty}\Big((\mathrm{I}+\alpha\mathrm{A})^{-1/2}\Phi(\mathbf{u}))e_j, (\mathrm{I}+\alpha\mathrm{A})^{-1/2}\Phi(\mathbf{u})e_j\Big)_{\mathbb{H}} \\ &= \sum_{j=}^{\infty}\Big\|(\mathrm{I}+\alpha\mathrm{A})^{-1/2}\Phi(\mathbf{u})e_j\Big\|_{\mathbb{H}}^2 = \frac{1}{(1+\alpha\lambda_1)}\sum_{j=}^{\infty}\|\Phi(\mathbf{u})e_j\|_{\mathbb{H}}^2 \\ &= \frac{1}{(1+\alpha\lambda_1)}\|\Phi(\mathbf{u})\|_{\mathscr{L}_2(\mathbb{H},\mathbb{H})}^2. \end{aligned} \tag{4.4}$$

Let us assume that the noise co-efficient $\Phi(\cdot)$ satisfies the following hypothesis of continuity, linear growth and Lipschitz condition.

Hypothesis 4.1. The noise co-efficient $\Phi(\cdot) : \mathbb{H} \to \mathscr{L}_2(\mathbb{H},\mathbb{H})$ satisfies

(H.1) the function $\Phi \in \mathrm{C}(\mathbb{V};\mathscr{L}_2(\mathbb{H},\mathbb{H}))$,

(H.2) (Growth Condition) There exists a positive constant $K > 0$ such that

$$\|\Phi(\mathbf{u})\|_{\mathscr{L}_2(\mathbb{H},\mathbb{H})}^2 \leq K\big(1+\|\mathbf{u}\|_{\mathbb{V}}^2\big),$$

for all $\mathbf{u} \in \mathbb{V}$.

(H.3) (Lipschitz Condition) There exists a positive constant $L > 0$ such that

$$\|\Phi(\mathbf{u}_1) - \Phi(\mathbf{u}_2)\|_{\mathscr{L}_2(\mathbb{H},\mathbb{H})} \leq L\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbb{V}},$$

for all $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{V}$.

The existence and uniqueness of global pathwise strong solution for the stochastic inviscid simplified Bardina model with multiplicative Gaussian noise can be proved in a similar way as that of additive noise case. We explain here the major differences, when we consider the system (4.1). We first consider the following cut-off problem:

$$\begin{cases} \mathrm{d}\mathbf{v}_n(t) = -\Pi_n(\|\mathbf{u}_n\|_{\mathbb{V}})\mathrm{B}(\mathbf{u}_n(t))\mathrm{d}t + \Phi(\mathbf{u}_n(t))\mathrm{dW}(t), \\ \mathbf{u}_n(0) = \mathbf{u}_0, \mathbf{v}_n(0) = \mathbf{v}_0 = \mathbf{u}_0 + \alpha\mathrm{A}\mathbf{u}_0. \end{cases} \tag{4.5}$$

The above system is equivalent to

$$\begin{cases} \mathrm{d}\mathbf{u}_n(t) = -\Pi_n(\|\mathbf{u}_n\|_{\mathbb{V}})(\mathrm{I}+\alpha\mathrm{A})^{-1}\mathrm{B}(\mathbf{u}_n(t))\mathrm{d}t + (\mathrm{I}+\alpha\mathrm{A})^{-1}\Phi(\mathbf{u}_n(t))\mathrm{dW}(t), \\ \mathbf{u}_n(0) = \mathbf{u}_0. \end{cases} \tag{4.6}$$

The Proposition given below is similar to Proposition 3.2 and we give a sketch of the proof only.

Proposition 4.2. *Let the $\mathscr{F}_0$-measurable initial data $\mathbf{u}_0 \in \mathrm{L}^2(\Omega;\mathbb{V})$ be given. Under the Hypothesis 4.1, there exists a unique $(\mathscr{F}_t)_{t\geq 0}$-adapted strong solution $\mathbf{u}(\cdot)$ to the system (4.5) in $\mathrm{L}^2(\Omega;\mathrm{L}^\infty([0,T];\mathbb{V}))$ with continuous trajectories in $\mathrm{C}([0,T];\mathbb{V})$, $\mathbb{P}$-a.s.*

Proof. We define a map $\mathcal{F}(\cdot):\mathbb{X}\to\mathbb{X}$ as

$$\mathcal{F}(\mathbf{u}_n(t)) := \mathbf{u}_0 + \int_0^t \Pi_n(\|\mathbf{u}_n(s)\|_{\mathbb{V}})(\mathrm{I}+\alpha\mathrm{A})^{-1}\mathrm{B}(\mathbf{u}_n(s))\mathrm{d}s + \int_0^t (\mathrm{I}+\alpha\mathrm{A})^{-1}\Phi(\mathbf{u}_n(t))\mathrm{dW}(s). \tag{4.7}$$

In order to show that $\mathcal{F}(\cdot)$ is a contraction on $\mathbb{X}$, we need to establish estimates similar to (3.8) and (3.9). An application of the the Burkholder-Divis-Gundy inequality and Hypothesis 4.1 (H.2) yields

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t \Phi(\mathbf{u}_n(t))\mathrm{W}(t)\right\|_{\mathbb{H}}^2\right] &\leq \mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t \Phi(\mathbf{u}_n(t))\mathrm{W}(t)\right\|_{\mathbb{H}}^2\right]\\ &\leq C\mathbb{E}\left[\int_0^T \|\Phi(\mathbf{u}_n(t))\|^2_{\mathscr{L}_2(\mathbb{H},\mathbb{H})}\mathrm{d}t\right]\\ &\leq CK\mathbb{E}\left[\int_0^T (1+\|\mathbf{u}_n(t)\|_{\mathbb{V}}^2)\mathrm{d}t\right]\\ &\leq CKT\left\{1+\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathbf{u}_n(t)\|_{\mathbb{V}}^2\right]\right\} < +\infty,\end{aligned}$$

for all $\mathbf{u}_n\in\mathbb{X}$ and hence (3.8) holds true.

In order to establish (3.9), we need to estimate the following also. We have

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t (\Phi(\mathbf{u}_n^1(t))-\Phi(\mathbf{u}_n^2(t)))\mathrm{dW}(t)\right\|_{\mathbb{V}'}^2\right]\\ &\leq \frac{1}{\lambda_1}\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t (\Phi(\mathbf{u}_n^1(t))-\Phi(\mathbf{u}_n^2(t)))\mathrm{dW}(t)\right\|_{\mathbb{H}}^2\right]\\ &\leq \frac{C}{\lambda_1}\mathbb{E}\left[\int_0^T \|\Phi(\mathbf{u}_n^1(t))-\Phi(\mathbf{u}_n^2(t))\|^2_{\mathscr{L}_2(\mathbb{H},\mathbb{H})}\mathrm{d}t\right]\\ &\leq \frac{CL^2}{\lambda_1}\mathbb{E}\left[\int_0^T \|\mathbf{u}_n^1(t)-\mathbf{u}_n^2(t)\|_{\mathbb{V}}^2\mathrm{d}t\right] \leq \frac{CL^2T}{\lambda_1}\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathbf{u}_n^1(t)-\mathbf{u}_n^2(t)\|_{\mathbb{V}}^2\right],\end{aligned} \tag{4.8}$$

where we used the Burkholder-Divis-Gundy and Hölder inequalities, and Hypothesis 4.1 (H.3). The rest of the arguments for unique local strong solution to the system (4.5) follows similarly as in Step 2, Proposition 3.2.

Using Burkholder-Divis-Gundy inequality, one can also establish that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T^*]}\left(\frac{1}{2}\|\mathbf{u}_n(t)\|_{\mathbb{H}}^2+\alpha\|\mathbf{u}_n(t)\|_{\mathbb{V}}^2\right)\right]\\
&\leq\left(\frac{1}{\lambda_1}+\alpha\right)\mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{V}}^2\right]\\
&\quad+\mathbb{E}\left[\int_0^{T^*}\left(\|(\mathrm{I}+\alpha\mathrm{A})^{-1/2}\Phi(\mathbf{u}_n(t))\|_{\mathcal{L}_2(\mathbb{H},\mathbb{H})}^2+6\|\Phi(\mathbf{u}_n(t))\|_{\mathcal{L}_2(\mathbb{H},\mathbb{H})}^2\right)\mathrm{d}t\right]. \qquad (4.9)
\end{aligned}$$

Let us use (4.4) and Hypothesis 4.1 (H.2) to obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left(\frac{1}{2}\|\mathbf{u}_n(t)\|_{\mathbb{H}}^2+\alpha\|\mathbf{u}_n(t)\|_{\mathbb{V}}^2\right)\right]\\
&\leq\left(\frac{1}{\lambda_1}+\alpha\right)\mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{V}}^2\right]+K\left(6+\frac{1}{(1+\lambda_1\alpha)}\right)\mathbb{E}\left[\int_0^T\left(1+\|\mathbf{u}_n(t)\|_{\mathbb{V}}^2\right)\mathrm{d}t\right]. \qquad (4.10)
\end{aligned}$$

An application of Gronwall's inequality in (4.10) yields

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left(\frac{1}{2}\|\mathbf{u}_n(t)\|_{\mathbb{H}}^2+\alpha\|\mathbf{u}_n(t)\|_{\mathbb{V}}^2\right)\right]\\
&\leq\left\{\left(\frac{1}{\lambda_1}+\alpha\right)\mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{V}}^2\right]+K\left(6+\frac{1}{(1+\lambda_1\alpha)}\right)T\right\}e^{\frac{K}{\alpha}\left(6+\frac{1}{(1+\lambda_1\alpha)}\right)T}. \qquad (4.11)
\end{aligned}$$

The estimate (4.11) ensures the existence of global strong solution to the system (4.5) and the rest of the arguments can be completed as in Proposition 3.2. □

With the help of above Proposition, one can prove the following Theorem on the existence and uniqueness of global pathwise strong solution to the system (4.1) as in Theorem 3.3.

Theorem 4.3 (Global existence and uniqueness)**.** *Let the $\mathscr{F}_0$-measurable initial data $\mathbf{u}_0\in\mathrm{L}^2(\Omega;\mathbb{V})$ be given. Under the Hypothesis 4.1, there exists a unique strong solution to the problem* (4.1) *with $(\mathscr{F}_t)_{t\geq 0}$-adapted, continuous trajectories in* $\mathrm{C}([0,T];\mathbb{V})$*, $\mathbb{P}$-a.s., satisfying*

$$\mathbf{u}(t)=\mathbf{u}_0+\int_0^t(\mathrm{I}+\alpha\mathrm{A})^{-1}\mathrm{B}(\mathbf{u}(s))\mathrm{d}s+\int_0^t(\mathrm{I}+\alpha\mathrm{A})^{-1}\Phi(\mathbf{u}(s))\mathrm{dW}(s), \qquad (4.12)$$

in $\mathbb{V}'$, $\mathbb{P}$-a.s., for all $t\in[0,T]$.

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